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Published Version Information

Citation: Ramm, A. G. (2013). Wave scattering by many small bodies: Transmission boundary conditions. Reports on Mathematical Physics, 71(3), 279-290.

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Digital Object Identifier (DOI): doi:10.1016/S0034-4877(13)60033-6

Publisher's Link: <http://www.sciencedirect.com/science/article/pii/S0034487713600336>

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Wave scattering by many small bodies: transmission boundary conditions

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Abstract

Wave scattering by many ($M = M(a)$) small bodies, at the boundary of which transmission boundary conditions are imposed, is studied.

Smallness of the bodies means that $ka \ll 1$, where a is the characteristic dimension of the body and $k = \frac{2\pi}{\lambda}$ is the wave number in the medium in which small bodies are embedded.

Explicit asymptotic formulas are derived for the field scattered by a single small scatterer of an arbitrary shape.

Equation for the effective field is derived in the limit as $a \rightarrow 0$ while $M(a) \rightarrow \infty$ at a suitable rate.

Mathematics Subject Classification: 35J05; 74J20; 78A45;
PACS 03.04.Kf; 03.50.De; 41.20.Jb; 71.36.+c

Keywords: Transmission boundary conditions; many-body wave scattering; effective field.

1 Introduction

There is a large literature on "homogenization", which deals with the properties of the medium in which other materials is distributed. Quite often it is assumed that the medium is periodic, and homogenization is considered in the framework of G-convergence ([4],[5]). In most cases, one considers elliptic or parabolic problems with elliptic operators positive-definite and having discrete spectrum.

The author has developed a theory of wave scattering by many small particles embedded in an inhomogeneous medium ([8]-[13]). One of the practically important consequences of his theory was a derivation of the equation for the effective (self-consistent) field in the limiting medium, obtained in the limit

$a \rightarrow 0$, $M = M(a) \rightarrow \infty$, where a is the characteristic size of a small particle, and $M(a)$ is the total number of the embedded particles.

The theory was developed in [8]-[13] for boundary conditions (bc) on the surfaces of small bodies, which include the Dirichlet bc, $u|_{S_m} = 0$, where S_m is the surface of the m -th particle D_m , the impedance bc, $\zeta_m u|_{S_m} = u_N|_{S_m}$, where N is the unit normal to S_m , pointing out of D_m , ζ_m is the boundary impedance, and the Neumann bc, $u_N|_{S_m} = 0$.

The novelty in this paper is the development of a similar theory for the *transmission (interface) bc*:

$$\rho_m u_N^+ = u_N^-, \quad u^+ = u^- \quad \text{on } S_m, 1 \leq m \leq M. \quad (1)$$

Here $0 \leq \rho_m \neq 1$ is a constant, $+(-)$ denotes the limit of $\frac{\partial u}{\partial N}$, from inside (outside) of D_m .

The physical meaning of the transmission boundary conditions is the continuity of the pressure and the normal component of the velocity across the boundaries of the discontinuity of the density. One may think about problem (1)-(5) as of the problem of acoustic wave scattering by many small bodies.

The essential novelty of the theory, developed in this paper, is the asymptotically exact, as $a \rightarrow 0$, treatment of the one-body and many-body scalar wave scattering problem in the case of small scatterers on the boundaries of which the transmission boundary conditions are imposed. An analytic explicit asymptotic formula for the field scattered by one small body is derived. An integral equation for the limiting effective field in the medium, in which many small bodies are embedded, is derived in the limit $a \rightarrow 0$ and $M(a) \rightarrow \infty$, where $M(a)$ is the total number of the embedded small bodies (particles), and $M = M(a)$ tends to infinity at a suitable rate as $a \rightarrow 0$.

For the problem with the number M of particles not large, say, less than 5000, our theory gives an efficient numerical method for solving many-body wave scattering problem.

For the problem with M very large, say, larger than 10^5 , the solution to many-body wave scattering problem consists in numerical solution of the integral equation for the limiting field in the medium, in which small particles are embedded. The solution to this equation approximates the solution to the many-body wave scattering problem with high accuracy.

Our approach is quite different from the approach developed in homogenization theory, we do not assume periodicity in the location of the small scatterers. Our results are of interest also in the case when the number of scatterers is not large, so the homogenization theory is not applicable.

Let us formulate the scattering problem we are treating.

$$\text{Let } \Omega := \bigcup_{m=1}^M D_m, \quad \Omega' = \mathbb{R}^3 \setminus \Omega,$$

$$(\nabla^2 + k^2)u = 0 \quad \text{in } \Omega', \quad (2)$$

$$(\nabla^2 + k_m^2)u = 0 \quad \text{in } D_m, \quad 1 \leq m \leq M, \quad (3)$$

$$u = u_0 + v, u_0 = e^{ik\alpha \cdot x}, \quad \alpha \in S^2, S^2 \text{ is a unit sphere in } \mathbb{R}^3, \quad (4)$$

$$r \left(\frac{\partial v}{\partial r} - ikv \right) = o(1), \quad r \rightarrow \infty. \quad (5)$$

We assume that ρ_m , k and k_m^2 are fixed given positive constants, and the surfaces S_m are smooth. A sufficient smoothness condition is $S_m \in C^{1,\mu}$, $\mu \in (0, 1)$, where S_m in local coordinates is given by a continuously differentiable function whose first derivatives are Hölder-continuous with exponent μ .

We assume that $x_m \in D_m$ is a point inside D_m , $a = \frac{1}{2}\text{diam}D_m$, $d = O(a^{\frac{1}{3}})$ is the distance between the neighboring particles, $\mathcal{N}(\Delta) = \sum_{x_m \in \Delta} 1$, is the number of particles in an arbitrary open set Δ , the domains D_m are not intersecting, and

$$\mathcal{N}(\Delta) = \frac{1}{V} \int_{\Delta} N(x) dx [1 + o(1)], \quad a \rightarrow 0, \quad (6)$$

where $N(x) \geq 0$ is a function which is at our disposal, V is the volume of one small body, $V = O(a^3)$. If D_m are balls of radius a , then $V = \frac{4\pi a^3}{3}$.

It is proved in [6] that problem (1)-(5) has a unique solution.

We study wave scattering by a single small body in Section 2. In other words, we study in Section 2 problem (1)-(5) with $M = 1$. The basic results of this Section are formulated in Theorem 1.

In section 3 wave scattering by many small bodies is considered. The basic results of this Section are formulated in Theorem 2. We always assume that

$$ka \ll 1, \quad d = O(a^{\frac{1}{3}}). \quad (7)$$

2 Wave scattering by one small body

Let us look for the solution to problem (1)-(5) with $M = 1$ of the form

$$u(x) = u_0(x) + \int_S g(x, t) \sigma(t) dt + \varkappa \int_D g(x, y) u(y) dy, \quad (8)$$

where $S = S_1$, $D = D_1$, t is a point on the surface S , dt is the element of the surface area,

$$\varkappa := k_1^2 - k^2, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad (9)$$

and $\sigma(t)$ is to be found so that conditions (1) are satisfied. If the pair of functions $\sigma(t)$ and $u(x)$ can be found uniquely from equations (8) and (10), see below, then the scattering problem is solved.

For any $\sigma \in C^{0,\mu_1}$, $\mu_1 \in (0, 1]$, where C^{0,μ_1} is the set of Hölder-continuous functions with Hölder's exponent μ_1 , the solution to equation (8) satisfies equations (2) and (3) with $M = 1$, and equations (4) and (5). This is easily checked by a direct calculation. The second condition (1) is also satisfied. To satisfy the first condition in equations (1) with $\rho_1 = \rho$, one has to satisfy the following equation

$$(\rho - 1)u_{0_N} + \rho \frac{A\sigma + \sigma}{2} - \frac{A\sigma - \sigma}{2} + (\rho - 1) \frac{\partial}{\partial N_s} Bu = 0, \quad (10)$$

where

$$A\sigma = 2 \int_S \frac{\partial g(s, t)}{\partial N_S} \sigma(t) dt, \quad Bu = \varkappa \int_D g(x, y) u(y) dy, \quad (11)$$

and the well-known formulas for the limiting values of the normal derivatives of the single-layer potential $T\sigma := \int_S g(x, t) \sigma(t) dt$ on S from inside and outside D was used.

In [6] one finds a proof of the following existence and uniqueness result. Let $H^2(D)$ denote the usual Sobolev space of functions twice differentiable in L^2 -sense.

Proposition 1. *The system of equations (8) and (10) for the unknown functions σ on S and $u(x)$ in D has a solution and this solution is unique in $C^{0,\mu_1} \times H^2(D)$.*

If the solution $\{\sigma, u(x)|_{x \in D}\}$ is found, then formula (8) defines $u = u(x)$ in \mathbb{R}^3 .

Let us rewrite (10) as

$$\sigma = \lambda A\sigma + 2\lambda B_1 u + 2\lambda u_{0_N}, \quad (12)$$

where

$$\lambda = \frac{1 - \rho}{1 + \rho}, \quad B_1 u = \varkappa \frac{\partial}{\partial N_s} \int_D g(x, y) u(y) dy. \quad (13)$$

If $\rho \in (0, \infty)$ then $\lambda \in (-1, 1)$. Let us now use the first assumption (7), that

is, the smallness of a . One has:

$$g(s, t) = g_0(s, t)(1 + O(ka)), \quad a \rightarrow 0; \quad g_0(s, t) = \frac{1}{4\pi|s - t|}, \quad (14)$$

$$\frac{\partial}{\partial N_s} \frac{e^{ik|s-t|}}{4\pi|s-t|} = \frac{\partial g_0}{\partial N_s} (1 + O((ka)^2)), \quad a \rightarrow 0, \quad (15)$$

$$\text{so } A = A_0(1 + O((ka)^2)), \quad a \rightarrow 0; \quad A_0 := A|_{k=0}, \quad (16)$$

$$B = B_0(1 + O(ka)), \quad B_0 u = \varkappa \int_D g_0(x, y) u(y) dy, \quad (17)$$

$$B_1 u = \varkappa \int_D \frac{\partial g_0(s, y)}{\partial N} u(y) dy (1 + O(k^2 a^2)) := \varkappa B_{10} u (1 + O(k^2 a^2)). \quad (18)$$

It follows from equation (8) that

$$u(x) = u_0(x) + \frac{e^{ik|x-x_1|}}{|x-x_1|} \left(\frac{1}{4\pi} \int_S e^{-ik\beta \cdot t} \sigma(t) dt + \frac{\varkappa}{4\pi} u_1 V_1 \right), \quad |x-x_1| \gg a, \quad (19)$$

where V_1 is the volume of $D = D_1$, $V_1 = \text{vol}(D_1) := |D_1|$, $u_1 := u(x_1)$, $\beta := \frac{x-x_1}{|x-x_1|}$. The point $x_1 \in D$ can be chosen as we wish. For one scatterer it is convenient to choose the origin at the point x_1 so that $x_1 = 0$.

We did not keep the factor $e^{-ik\beta \cdot x}$ in the integral over D because $e^{-ik\beta \cdot x} = 1 + O(ka)$, and

$$\int_D e^{-ik\beta \cdot y} u(y) dy = u_1 V_1 (1 + O(ka)), \quad a \rightarrow 0. \quad (20)$$

However, it will be proved that *this factor under the surface integral can not be dropped* because

$$\int_S e^{-ik\beta \cdot t} \sigma(t) dt = \int_S \sigma(t) dt - ik\beta_p \int_S t_p \sigma(t) dt + O(a^4), \quad (21)$$

where over the repeated indices here and throughout this paper summation is understood, and the second integral in the right-hand side of (21) is $O(a^3)$, as $a \rightarrow 0$, that is, it is of the same order of smallness as the the first integral $Q := \int_S \sigma(t) dt$. The last statement will be proved later.

With the notations

$$Q := \int_S \sigma(t) dt, \quad Q_1 := \int_S e^{-ik\beta \cdot t} \sigma(t) dt, \quad (22)$$

the expression

$$A(\beta, \alpha) := \frac{Q_1}{4\pi} + \frac{\varkappa}{4\pi} u_1 V_1, \quad V_1 := V := |D|, \quad u_1 := u(x_1), \quad (23)$$

is the scattering amplitude, α is the unit vector in the direction of the incident wave $u_0 = e^{ik\alpha \cdot x}$, β is the unit vector in the direction of the scattered wave. Let us prove that

$$-ik\beta_p \int_S t_p \sigma(t) dt = O(a^3), \quad (24)$$

and therefore, the second integral in the right-hand side of equation (21) cannot be dropped.

It follows from equation (8) that

$$u(x) \sim u_0(x) + g(x, x_1)Q_1 + \varkappa g(x, x_1)u(x_1)V_1, \quad |x - x_1| \geq d \gg a, \quad (25)$$

where \sim means asymptotic equivalence as $a \rightarrow 0$.

Formula (25) can be used for calculating $u(x)$ if two quantities Q_1 and $u_1 := u(x_1)$ are found.

Let us derive asymptotic formulas for these quantities as $a \rightarrow 0$. Integrate equation (12) over S and get

$$Q = 2\lambda \int_S u_{0N} ds + \lambda \int_S A \sigma dt + 2\lambda \int_S B_1 u ds, \quad (26)$$

Use formulas (14)-(18), the following formula (see [7], p.96):

$$\int_S A_0 \sigma ds = - \int_S \sigma ds, \quad (27)$$

and the Divergence theorem, to rewrite equation (26) as

$$Q = 2\lambda \int_D \nabla^2 u_0 dx - \lambda Q + 2\lambda \varkappa \int_D dx \nabla_x^2 \int_D g(x, y) u(y) dy. \quad (28)$$

Since

$$\nabla^2 u_0 = -k^2 u_0; \quad \nabla_x^2 g(x, y) = -k^2 g(x, y) - \delta(x - y), \quad (29)$$

equation (28) takes the form

$$(1 + \lambda)Q = 2\lambda \nabla^2 u_0(x_1) V_1 - 2\lambda k^2 \varkappa \int_D dx \int_D g(x, y) u dy - 2\lambda \varkappa \int_D u(x) dx. \quad (30)$$

Let us use the following estimates:

$$\int_D u(x) dx = u_1 V_1 (1 + o(1)), \quad a \rightarrow 0; \quad u_1 := u(x_1), \quad (31)$$

$$\int_D dx \int_D g(x, y) u(y) dy = \int_D dy u(y) \int_D dx g(x, y) = O(a^5), \quad (32)$$

$$\int_D g(x, y) dx = O(a^2), \quad \forall y \in D. \quad (33)$$

From equations (30)-(33) it follows that

$$Q \sim \frac{2\lambda}{1+\lambda} V_1 \nabla^2 u_{01} - \frac{2\lambda\kappa}{1+\lambda} V_1 u_1, \quad a \rightarrow 0, \quad (34)$$

where

$$\nabla^2 u_{01} = \nabla^2 u_0(x)|_{x=x_1}. \quad (35)$$

Let us now integrate equation (8) over D and use estimate (31) to obtain

$$u_1 V_1 = u_{01} V_1 + \int_S dt \sigma(t) \int_D g(x, t) dx + \kappa \int_D dy u(y) \int_D g(x, y) dx. \quad (36)$$

If D is a ball of radius a , then one can easily check that

$$\int_D g(x, t) dx \sim \int_D g_0(x, t) dx = \frac{a^2}{3}, \quad |t| = a, \quad a \rightarrow 0. \quad (37)$$

In general, one has

$$\int_D g(x, y) dx = O(a^2), \quad y \in D, \quad a \rightarrow 0. \quad (38)$$

If D is a ball of radius a , then equations (36)-(38) imply

$$u_1 = u_{01} + Q \frac{a^2}{3 \frac{4\pi a^3}{3}} + \kappa u_1 O(a^2), \quad a \rightarrow 0. \quad (39)$$

Consequently,

$$u_1 \sim u_{01} + O(a^2), \quad a \rightarrow 0, \quad (40)$$

because $Q = O(a^3)$.

Indeed, from equations (34) and (40) one gets

$$Q \sim V_1(1 - \rho)[\nabla^2 u_{01} - \kappa u_{01}], \quad (41)$$

where we took into account that

$$\frac{2\lambda}{1+\lambda} = 1 - \rho, \quad (42)$$

the relation $u_1 \sim u_{01}$ as $a \rightarrow 0$, see equation (40), and neglected the terms of higher order of smallness. It follows from equation (41) that

$$Q = O(a^3). \quad (43)$$

From equations (40) and (41) one obtains

$$u_1 \sim u_{01}, \quad a \rightarrow 0. \quad (44)$$

Let us now estimate Q_1 . One has

$$Q_1 = \int_S \sigma(t) dt - ik\beta_p \int_S t_p \sigma(t) dt, \quad (45)$$

up to the terms of the higher order of smallness as $a \rightarrow 0$, and summation is understood over the repeated indices. It turns out that the integral

$$I := \int_S t_p \sigma(t) dt \quad (46)$$

is of the same order, namely $O(a^3)$, as $Q = \int_S \sigma(t) dt$.

Let us check that the integral

$$J := \int_S dt t_p \frac{\partial}{\partial N} \int_D g(t, y) u(y) dy = O(a^4)$$

as $a \rightarrow 0$, and, therefore, can be neglected compared with I . Indeed, $u = O(1)$, $\int_D \frac{\partial}{\partial N} g(t, y) dy = O(a)$, and $\int_S t_p dt = O(a^3)$. Thus, $J = O(a^4)$.

Define the function σ_q , $q = 1, 2, 3$, as the unique solution to the equation

$$\sigma_q = \lambda A \sigma_q - 2\lambda N_q. \quad (47)$$

Since $\lambda = (1 - \rho)/(1 + \rho)$, and $\rho > 0$, one concludes that $\lambda \in (-1, 1)$, and it is known (see, for example, [7]) that the operator A is compact in $L^2(S)$ and does not have characteristic values in the interval $(-1, 1)$. This and the Fredholm alternative imply that equation (47) has a solution and this solution is unique.

Let us prove that $\int_S \sigma_q(t) dt = O(a^3)$. To do this, integrate equation (47) over S , take into account formula (27), the relation $(A - A_0)\sigma_q = O(a^3)$, and obtain

$$(1 + \lambda) \int_S \sigma_q(t) dt = -2\lambda \int_S N_q dt + O(a^3) = O(a^3),$$

because $\int_S N_q dt = 0$ by the Divergence theorem.

Define the tensor

$$\beta_{pq} := \beta_{pq}(\lambda) := V_1^{-1} \int_S t_p \sigma_q(t) dt, \quad p, q = 1, 2, 3. \quad (48)$$

This tensor is similar to the tensor β_{pq} defined in [7], p. 62, by a similar formula with $\lambda = 1$. In this case β_{pq} is the magnetic polarizability tensor of a superconductor D placed in a homogeneous magnetic field directed along the unit Cartesian coordinate vector e_q (see [7], p. 62). In [7] analytic formulas are given for calculating β_{pq} with an arbitrary accuracy.

One may neglect the term $B_1 u$ in equation (12) (because this term is $O(a^4)$), take into account definition (48), and get

$$\int_S t_p \sigma(t) dt = -\beta_{pq} \frac{\partial u_0}{\partial x_q} V, \quad (49)$$

where $V := V_1$, and summation is understood over q .

Consequently, one can rewrite (45) as

$$Q_1 = (1 - \rho) V_1 [\nabla^2 (u_0(x_1) - \varkappa u_0(x_1))] + ik \beta_{pq} \frac{\partial u_0}{\partial x_q} \beta_p V_1, \quad \beta := \frac{x - x_1}{|x - x_1|}, \quad (50)$$

and $(x)_p := x \cdot e_p$ is the p -th Cartesian coordinate of the vector x .

Formula (19) can be written as

$$u(x) = u_0(x) + g(x, x_1) \left((1 - \rho) [\nabla^2 u_0(x_1) - \varkappa u_0(x_1)] + ik \beta_{pq} \frac{\partial u_0(x_1)}{\partial x_{1,q}} \beta_p + \varkappa u_0(x_1) \right) V_1. \quad (51)$$

Here one sums over the repeated indices, $|x - x_1| \gg a$, and $\frac{\partial u_0(x_1)}{\partial x_{1,q}} := \frac{\partial u_0(y)}{\partial y_q} \Big|_{y=x_1}$, $q = 1, 2, 3$, $y = (y_1, y_2, y_3)$.

Formulas (41), (43), (44) are valid for small D of an arbitrary shape. Let us formulate the results of this Section in Theorem 1.

Theorem 1. *Assume that $ka \ll 1$, k_1, k , and ρ are positive constants. Then the scattering problem (1)-(5) has a unique solution. This solution has the form (8) and can be calculated by formula (51) in the region $|x - x_1| \gg a$ up to the terms of order $O(a^4)$ as $a \rightarrow 0$, where $a = 0.5 \text{diam} D$, $\varkappa = k_1^2 - k^2$, $V_1 = \text{vol} D$, $\beta = \frac{x - x_1}{|x - x_1|}$, β_{pq} is defined in equation (48), and $O(a^4)$ does not depend on x .*

3 Wave scattering by many small bodies

Assume that the distribution of small bodies is given by equation (6), and that there are $M = M(a)$ non-intersecting small bodies D_m of size a . For simplicity we assume that D_m is a ball of radius a , centered at x_m . There is an essential novel feature in the theory, developed in this paper compared with the one developed in [8], [9], [12], namely, the scattered field was much larger, as $a \rightarrow 0$ in the above papers. For example, for the impedance boundary condition, $u_N = \zeta u$ on S , the scattered field is $O(a^2)$, and for the Dirichlet boundary condition, $u = 0$ on S , the scattered field is $O(a)$.

For the Neumann boundary condition the scattered field is $O(a^3)$. We have the same order of smallness of the scattered field, $O(a^3)$, for the problem with the transmission boundary condition because $V_1 = O(a^3)$. The basic role in Section 3 is played by formula (51). We assume that the distance d between

neighboring bodies (particles) is much larger than a , $d \gg a$, but there can be many small particles on the wavelength, and the interaction of the scattered waves (multiple scattering) is essential and cannot be neglected.

This assumption effectively means that the function $N(x)$ in (6) has to be small, $N(x) \ll 1$. Indeed, if on a segment of unit length there are small particles placed at a distance d between neighboring particles, then there are $O(\frac{1}{d})$ particles on this unit segment, and $O(\frac{1}{d^3})$ in a unit cube C_1 . Since $V = O(a^3)$, by formula (6) one gets

$$\frac{1}{O(a^3)} \int_{C_1} N(x) dx = O\left(\frac{1}{d^3}\right).$$

Therefore $d \gg a$ can hold only if $(\int_{C_1} N(x) dx)^{\frac{1}{3}} = O\left(\frac{a}{d}\right) \ll 1$.

Let us look for the (unique) solution to problem (1)-(5) with $1 \leq m \leq M = M(a)$ of the form

$$u(x) = u_0(x) + \sum_{m=1}^M \int_{S_m} g(x, t) \sigma_m(t) dt + \sum_{m=1}^M \varkappa_m \int_{D_m} g(x, y) u(y) dy. \quad (52)$$

Keeping the main terms in this equation, as $a \rightarrow 0$, one gets

$$\begin{aligned} u(x) = u_0(x) + \sum_{m=1}^M g(x, x_m) \left(Q_m - ik \frac{(x - x_m)_p}{|x - x_m|} \int_{S_m} t_p \sigma_m(t) dt \right) + \\ + \sum_{m=1}^M \varkappa_m g(x, x_m) u_e(x_m) V_m, \quad Q_m := \int_{S_m} \sigma_m(t) dt, \quad a \rightarrow 0, \end{aligned} \quad (53)$$

where we have used formula (51) for the scattered field by every small particle, replaced u_0 by the effective field u_e , acting on every particle, and took into account that $\beta := \beta_m := \frac{x - x_m}{|x - x_m|}$. By $(x - x_m)_p$ the p -th component of vector $(x - x_m)$ is denoted.

The effective (self-consistent) field u_e , acting on j -th particle, is defined as:

$$\begin{aligned} u_e(x) = u_0(x) + \sum_{m=1, m \neq j}^M g(x, x_m) \left((1 - \rho_m) [\nabla^2 u_e(x_m) - \varkappa_m u_e(x_m)] + \right. \\ \left. ik \beta_{pq}^{(m)} \frac{\partial u_e}{\partial x_q} \frac{(x - x_m)_p}{|x - x_m|} \right) V_m + \sum_{m=1, m \neq j}^M \varkappa_m g(x, x_m) u_e(x_m) V_m, \quad |x - x_j| \sim a. \end{aligned} \quad (54)$$

Setting $x = x_j$ in equation (54), one gets a linear algebraic system for the unknowns $u_j := u_e(x_j)$, $1 \leq j \leq M$, $\nabla^2 u_j := \nabla^2 u_e(x_j)$, and $\frac{\partial u_e(x_j)}{\partial x_{j,p}}$. Here

$x_{j,p}$ is the p -th component of the vector x_j , $p = 1, 2, 3$. Differentiating (54) with respect to $x_{j,p}$, $p = 1, 2, 3$, and then setting $x = x_j$, one obtains a *linear algebraic system for the $5M$ unknowns u_j , $\nabla^2 u_j$, and $\frac{\partial u_e(x_j)}{\partial x_{j,p}}$, $1 \leq j \leq M$, $1 \leq p \leq 3$.*

This linear algebraic system one gets if one solves by a collocation method the following integral equation

$$u(x) = u_0(x) + \int_D g(x, y) \left[(1 - \rho)(\nabla^2 - K^2(y) + k^2)u(y) + ik\beta_{pq}(y, \lambda) \frac{\partial u(y)}{\partial y_q} \frac{(x - y)_p}{|x - y|} + (K^2(y) - k^2)u(y) \right] N(y) dy. \quad (55)$$

In the above equation the function $\beta_{pq}(y, \lambda)$ is defined as

$$\beta_{pq}(y, \lambda) = \lim_{a \rightarrow 0} \frac{\sum_{x_m \in \Delta_p} \beta_{pq}^{(m)}}{\mathcal{N}(\Delta_p)},$$

where $y = y_p \in \Delta_p$, and tensor $\beta_{pq}^{(m)} = \beta_{pq}^{(m)}(\lambda)$ is defined in (48). Convergence of the collocation method was proved in [12].

Equation (55) is a non-local integrodifferential equation for the limiting effective field in the medium in which many small bodies are embedded.

This is a novel result. The original scattering problem (1)-(5) has been formulated in terms of local differential operators.

In the derivation of equation (55) from equation (54) we have assumed that $\rho_m = \rho$ does not depend on m , took into account that κ_m^2 becomes in the limit $K^2(y) - k^2$, and denoted by $K^2(y)$ a continuous function in D such that $K^2(x_m) = \kappa_m^2$. As $a \rightarrow 0$ the function $K^2(y)$ is uniquely defined because the set $\{x_m\}_{m=1}^{M(a)}$ becomes dense in D as $a \rightarrow 0$.

To derive equation (55) from equation (54) we argue as follows. Consider a partition of D into a union centered at the points y_p of P non-intersecting cubes Δ_p , of size $b(a)$, $b(a) \gg d$, so that each cube contains many small bodies, $\lim_{a \rightarrow 0} b(a) = 0$. Let us demonstrate the passage to the limit $a \rightarrow 0$ in the sums in equation (54) using the first sum as an example. Write the first sum in (54) as

$$\begin{aligned} & \sum_{m \neq j} g(x, x_m) (1 - \rho_m) [\nabla^2 u_e(x_m) - \kappa_m u_e(x_m)] V_m \\ &= \sum_{p=1}^P g(x, y_p) (1 - \rho_p) [\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)] V_m \sum_{x_m \in \Delta_p} 1 \\ &= \sum_{p=1}^P g(x, y_p) (1 - \rho_p) [\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)] N(y_p) |\Delta_p| (1 + o(1)), \quad (56) \end{aligned}$$

where we have used formula (6), took into account that $\text{diam } \Delta_p \rightarrow 0$ as $a \rightarrow 0$, wrote formula (6) as

$$V \sum_{x_m \in \Delta_p} 1 = V\mathcal{N}(\Delta_p) = N(y_p)|\Delta_p|(1 + o(1)), \quad a \rightarrow 0, \quad (57)$$

and used the Riemann integrability of the functions involved, which holds, for example, if these functions are continuous. By ρ_p we denote the value $\rho(y_p)$, where $\rho(y)$ is a continuous function.

The sum in (56) is the Riemannian sum for the integral

$$\int_D g(x, y)(1 - \rho(y))[\nabla^2 u(y) - K^2(y)u(y) + k^2 u(y)]N(y)dy. \quad (58)$$

Similarly one treats the other sums in (54) and obtains in the limit $a \rightarrow 0$ equation (55).

Let us formulate the results of this Section.

Theorem 2. *Assume that conditions (6) and (7) hold. Then, as $a \rightarrow 0$, the effective field, defined by equation (54), has a limit $u(x)$. The function $u(x)$ solves equation (55).*

4 Conclusions

In this paper wave scattering by one and many small bodies of an arbitrary shape is studied in the case when the boundary conditions are transmission ones. Explicit analytic formulas are derived for the field scattered by a single body, see Theorem 1.

Equation for the effective field in the medium where many small bodies are distributed is derived. The result is formulated in Theorem 2.

One concludes that the wave scattered by a small particle subject to transmission boundary condition can be calculated by formula (51). If there are many small particles distributed in a bounded domain according to formula (6), then one concludes that the limiting effective field in such a medium satisfies equation (55) and can be calculated by solving a linear algebraic system which one obtains from formula (54) by setting $x = x_j$ in equation (54).

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