# EVERYWHERE DIFFERENTIABLE, NOWHERE MONOTONE FUNCTIONS

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# ABSTRACT



In this thesis we give a Baire Category type proof of the existence of functions $F : \mathbb{R} \to \mathbb{R}$ which are differentiable at every point but monotone on no interval having positive length and whose derivative are bounded (Theorem 3.2). We also deduce further properties that any such functions F must also have (Chapter 4) for example, F possesses both a local maximum and a local minimum on any given nonvoid open interval (Property 5). Also, the sets $Z(F') = \{ x \in \mathbb{R} \quad F'(x) = 0 \}$, $P(F') = \{ x \in \mathbb{R} : F'(x) > 0 \}$ and $N(F') = \{ x \in \mathbb{R} \quad F'(x) < 0 \}$ are each dense in $\mathbb{R}$ and for any nonvoid open interval I of $\mathbb{R}$ the sets $I \cap P(F')$, $I \cap N(F')$ both have positive Lebesgue Measure (Property 9). It follows from this that $F'$ cannot be Riemann integrable over any closed interval of positive length (Property 10).

# TABLE OF CONTENTS

# DEDICATION

This thesis is dedicated to three special people to whom I owe a lot and for whom I am very thankful  1) Father, who died a few months ago, would have been overjoyed to share in this goal - attainment  2) Mother remains supportive as she bravely faces each new challenge with hope  and 3) Husband, whose support and understanding allowed me to pursue this personal goal.

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# INTRODUCTION

The purpose of this thesis is to prove the existence of real-valued functions on $\mathbb{R}$ that are everywhere differentiable but are monotone on no interval and to examine further peculiar properties of any such function. Examples of such functions are seldom given, or even mentioned, in books on real analysis. The explicit construction of such a function was given by Köpcke (1889). An example due to Pereno (1897) is reproduced in [9]. An elegant and rigorous explicit construction of such a function was given in the American Mathematical Monthly Vol. 81 No. 4, April, 1974 (pp 349 - 354) by Y. Katznelson and Karl Stromberg. The presentation here is based on an idea of Professor Clifford Weil. It does not "construct" the desired function. Instead, it proves their existence by application of Baire's Category Theorem to a certain Banach space of bounded functions.

# Chapter 1

## The Space of Bounded Derivatives

**1.1 Definition** Let D denote the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists some differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x) \ \forall \ x \in \mathbb{R}$. Supply the set D with the uniform metric:

$$\begin{aligned} d(f,g) &= \| f - g \|_u \\ &= \text{Sup} \{ |f(x) - g(x)| : x \in \mathbb{R} \} \text{ for } f, g \in D. \end{aligned}$$

We call D the *Space of bounded derivatives.*

**1.2 Theorem** : The metric space D is complete.

**Proof** We need to prove that each Cauchy sequence in D converges to an element of D. Let $(f_n)_{n \in \mathbb{N}} \subset D$ be a Cauchy sequence. Choose $F_n$ such that $F_n' = f_n$ and $F_n(0) = 0$. Since $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$|f_n(x)-f_m(x)| < \epsilon$ whenever $m,n > N$ and $x \in \mathbb{R}$. Thus the sequence $(f_n)_{n \in N}$ is uniformly convergent on $\mathbb{R}$. So there exists $f : \mathbb{R} \to \mathbb{R}$ such that $\|f_n - f\|_u \to 0$;

i.e. $\|F_n' - f\|_u \to 0$.

Thus $F_n' \to f$ uniformly on $\mathbb{R}$. It follows from Theorem (4.56), pp. 214 of [18] that $(F_n)$ converges uniformly on $\mathbb{R}$ to some differentiable function F and $F' = f$. Thus $f \in D$ and $d(f_n, f) = \| f_n - f \|_u \to 0$. Hence, the metric space D is complete.□

**1.3 Theorem** For each $f \in D$, the set $Z(f) = \{ x \in \mathbb{R} : f(x) = 0 \}$ is a $G_\delta$-set in $\mathbb{R}$. Recall that this means that Z(f) is the intersection of some countable family of open subsets of $\mathbb{R}$. [We call Z(f) the *zero set of f.*]

**Proof** : Since $f \in D$, $\exists$ some differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$. Let

$$f_n(x) = \frac{F(x + 1/n)-F(x)}{1/n}, \text{ for } x \in \mathbb{R}.$$

Then $\lim_{n\to\infty} f_n(x) = F'(x) = f(x)$, for every $x \in \mathbb{R}$.

For $a \in \mathbb{R}$, we have

$$|f(a)| = \lim_{n\to\infty} |f_n(a)| = \underline{\lim}_{n\to\infty} |f_n(a)| \text{ so}$$

$$\begin{aligned} a \in Z(f) &\Leftrightarrow \lim_{n\to\infty} |f_n(a)| = 0 \\ &\Leftrightarrow \forall\, k, m \in \mathbb{N}\ \exists\, n \geq m \text{ with } |f_n(a)| < 1/k \\ &\Leftrightarrow a \in \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{k,n} \end{aligned}$$

where $A_{k,n} = \{ x \in \mathbb{R} : |f_n(x)| < 1/k \}$.

Since F is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ and hence, each $f_n$ is also continuous on $\mathbb{R}$. It follows that each $A_{k,n}$ is open in $\mathbb{R}$ and so

$$Z(f) = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{k,n}$$

is a $G_\delta$ - set in $\mathbb{R}$. □

**1.4 Definition** : Let $D_0 = \{ f \in D : Z(f) \text{ is dense in } \mathbb{R} \}$.

We next prove that $D_0$ is a real vector space.

**1.5 Theorem** : If f and g are in $D_0$ and $\alpha \in \mathbb{R}$, then $f+g$ and $\alpha f$ are in $D_0$.

**Proof** : Let f, g $\in D_0$ and let $\alpha \in \mathbb{R}$. Notice that

$$Z(\alpha f) = \begin{cases} \mathbb{R} & \text{if } \alpha = 0 \\ Z(f) & \text{if } \alpha \neq 0 \end{cases}$$

So $Z(\alpha f)$ is dense in $\mathbb{R}$ and $\alpha f \in D_0$.

One formulation of the Baire Category Theorem states that the intersection of any countable family of dense open subsets of a complete metric space is dense in that space [see Theorem (3.57) on page 110 of [18]] We apply this and theorem (1.3) in the complete metric space $\mathbb{R}$ to see that $Z(f) \cap Z(g)$ is dense in $\mathbb{R}$. Plainly $Z(f + g)$ includes this dense set so it too, is dense in $\mathbb{R}$. Thus $f + g \in D_0$. □

**1.6 Theorem** The set $D_0$ is a closed subset of D. Hence $D_0$ is also a complete metric space.

**Proof** : Let $(f_n)_{n\in N} \subset D_0$ and $f \in D$ and $d(f_n, f) \to 0$. Then

$d(f_n, f) \geq |f_n(x) - f(x)| \to 0$ for all $x \in \mathbb{R}$. It follows that

$$\bigcap_{n=1}^{\infty} Z(f_n) \subset Z(f)$$

because $x \in \bigcap_{n=1}^{\infty} Z(f_n)$

$$\begin{aligned} &\Rightarrow f_n(x) = 0 \ \forall\, n \in N \\ &\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = 0 \\ &\Rightarrow x \in Z(f). \end{aligned}$$

Since $(f_n)_{n\in N} \subset D_0$, $Z(f_n)$ is dense in $\mathbb{R}$ for each $n \in \mathbb{N}$. By Theorem (1.3), each $Z(f_n)$ is a $G_\delta$ - subset of $\mathbb{R}$. Thus, by the Baire Category Theorem,

$$\bigcap_{n=1}^{\infty} Z(f_n)$$

is dense in $\mathbb{R}$ and so its superset $Z(f)$ is also dense in $\mathbb{R}$. This proves $f \in D_0$. Thus, $D_0$ is closed in the metric space D. □

**1.7 Theorem** : If I is a nonvoid open interval of $\mathbb{R}$, then the sets

$$A = \{ f \in D_0 : f(x) > 0 \ \forall\ x \in I \}$$

and $$B = \{ f \in D_0 : f(x) \leq 0 \ \forall\ x \in I \}$$

are closed and nowhere dense in the complete metric space $D_0$.

To prove this theorem we need some preliminary results so we postpone its proof until Chapter 3.

# Chapter 2

## A Nontrivial Construction

In this chapter we prove an important theorem which has the feature that it allows us to construct a strictly increasing function belonging to $D_0$. We begin with a lemma.

**2.1 Lemma** : Let $(r_n)_{n \in N} \subset \mathbb{R}$ and put

$$g_n(t) = \frac{(t - r_n)^{1/3}}{(1 + |r_n|)^{1/3}} \quad \text{for } t \in \mathbb{R} \text{ and } n \in N.$$

Then the formula

$$(1) \qquad h(t) = t + \sum_{n=1}^{\infty} 2^{-n} g_n(t)$$

defines a function $h : \mathbb{R} \to \mathbb{R}$ that is continuous, strictly increasing and onto $\mathbb{R}$.

Moreover, at each $a \in \mathbb{R}$, h has a derivative $h'(a) > 1$ and

$$(2) \qquad h'(a) = 1 + \sum_{n=1}^{\infty} 2^{-n} g_n'(a).$$

In particular, $h'(r_n) = +\infty$, for every $n \in \mathbb{N}$.

**Proof :** For b and t in $\mathbb{R}$ with $b \geq 1$ and $|t| \leq b$, we have

$|t - r_n| \leq b + |r_n| \leq b(1 + |r_n|)$. So $|g_n(t)| \leq b^{1/3}$ for all $n \in \mathbb{N}$. Thus

$$|2^{-n} g_n(t)| \leq \frac{b^{1/3}}{2^n} \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [-b\,,\,b].$$

It follows from the Weierstrass M-test that the series (1) converges uniformly on each bounded interval of $\mathbb{R}$. Since each $g_n$ is continuous on $\mathbb{R}$, it follows that h is too. Note that if $u < v$ in $\mathbb{R}$, then $g_n(v) - g_n(u) > 0$, for all n. So $h(v) - h(u) > v - u > 0$. Thus h is a strictly increasing function on $\mathbb{R}$.

$$\begin{aligned}
\text{For } t \in \mathbb{R},\ |t| \geq 1 \quad &\Rightarrow |t - r_n| \leq |t|(1 + |r_n|) \\
&\Rightarrow |g_n(t)| \leq |t|^{1/3} \quad \forall\, n \in \mathbb{N} \\
&\Rightarrow |h(t) - t| \leq |t|^{1/3} \sum_{n=1}^{\infty} 2^{-n} = |t|^{1/3} \\
&\Rightarrow |h(t) - t| < 2|t|^{1/3}
\end{aligned}$$

and so if $t > 1$, then $h(t) > t - 2t^{1/3} = t(1 - 2t^{-2/3})$, and $h(-t) < -t + 2t^{1/3} = -t(1 - 2t^{-2/3})$.

Hence,

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \lim_{t\to-\infty} h(t) = -\infty.$$

It follows that h is neither bounded above nor bounded below. Now, an application of the Intermediate Value Theorem shows that h maps $\mathbb{R}$ onto $\mathbb{R}$.

Now we turn to the proof of (2). First notice that if $u \neq v \neq 0$ in $\mathbb{R}$, then

$$\begin{aligned} 0 < \frac{(u - v)}{(u^3 - v^3)} &= \frac{1}{(u^2 + uv + v^2)} \\ &= \frac{1}{(u + v/2)^2 + 3v^2/4} \\ &= \frac{4}{(2u + v)^2 + 3v^2} \leq \frac{4}{3v^2} \end{aligned}$$

Given $n \in \mathbb{N}$ and $t, a \in \mathbb{R}$ with $t \neq a \neq r_n$, take $u = (t - r_n)^{1/3}$ and $v = (a - r_n)^{1/3}$ to obtain

$$0 < \frac{g_n(t) - g_n(a)}{t - a} \leq \frac{4}{3}\,\frac{(a - r_n)^{-2/3}}{(1 + |r_n|)^{1/3}} = 4g_n'(a). \qquad (*)$$

$$\begin{aligned} \text{of course,} \quad g_n'(r_n) &= \lim_{t\to r_n} \frac{g_n(t) - g_n(r_n)}{t - r_n} \\ &= \lim_{t\to r_n} \frac{(t - r_n)^{-2/3}}{(1 + |r_n|)^{1/3}} \\ &= +\infty \qquad (**) \end{aligned}$$

Fix any $a \in \mathbb{R}$.

If the series in (2) diverges and $\beta > 0$ in $\mathbb{R}$, we can find $N \in \mathbb{N}$ for which the Nth partial sum of that series exceeds $\beta$ and then we can find $\delta > 0$ such that if

$0 < |t - a| < \delta$, then

$$\frac{h(t) - h(a)}{t - a} > 1 + \sum_{n=1}^{N} 2^{-n} \frac{g_n(t) - g_n(a)}{t - a} > \beta.$$

Thus, $h'(a) = \infty$ if the right-hand side of (2) equals $\infty$.

Now suppose the series (2) converges and let $\epsilon > 0$. Choose $N \in \mathbb{N}$ for which

$$\sum_{n=N}^{\infty} 2^{-n} g_n'(a) < \frac{\epsilon}{10}$$

Next find $\delta > 0$ for which

$$0 < |t - a| < \delta \quad \rightarrow \quad \left| \frac{g_n(t) - g_n(a)}{t - a} - g_n'(a) \right| < \frac{\epsilon}{2}, \quad \text{for } n = 1, 2, \ldots\ldots\ldots\ldots, N$$

By use of (*), we now see that

$$\begin{aligned} 0 < |t - a| < \delta \quad \rightarrow \quad & \left| \frac{h(t) - h(a)}{t - a} - \{ 1 + \sum_{n=1}^{\infty} 2^{-n} g_n'(a) \} \right| \\ & \leq \sum_{n=1}^{N} 2^{-n} \left| \frac{g_n(t) - g_n(a)}{t - a} - g_n'(a) \right| + \\ & \sum_{n=N+1}^{\infty} 2^{-n} \frac{g_n(t) - g_n(a)}{t - a} + \sum_{n=N+1}^{\infty} 2^{-n} g_n'(a) \\ & < \sum_{n=1}^{N} 2^{-n}\, \epsilon/2 + 5 \sum_{n=N+1}^{\infty} g_n'(a) \\ & < \epsilon/2 + 5.\epsilon/10 = \epsilon. \end{aligned}$$

This proves in this case that h is differentiable at a and (2) holds. If $k \in \mathbb{N}$, it follows from (2), (*), and (**) that

$$h'(r_k) \geq 2^{-n} g_n'(r_k) = \infty. \ \square$$

As a consequence of (2.1), we have the following remarkable fact.

**2.2 Theorem :** There exists a differentiable strictly increasing function $H : \mathbb{R} \to \mathbb{R}$ with $H(\mathbb{R}) = \mathbb{R}$ and $0 \leq H'(x) < 1$ for all $x \in \mathbb{R}$, such that both of the sets

$$Z(H') = \{ x \in \mathbb{R} : H'(x) = 0 \}$$

$$\text{and} \quad P(H') = \{ x \in \mathbb{R} : H'(x) > 0 \}$$

are dense in $\mathbb{R}$.

**Proof** : Let $\{r_n\}_{n \in N}$ be an enumeration of the set $\mathcal{Q}$ of all rational numbers and then let $h$ be as in the preceding lemma. Since $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, so is its inverse $H = h^{-1}$. Plainly, $H$ (like $h$) is strictly increasing and $H(\mathbb{R}) = \mathbb{R}$.

Let $b \in \mathbb{R}$. Choose $a \in \mathbb{R}$ with $h(a) = b$, $H(b) = a$. Then

$$\lim_{x \to b} \frac{H(x) - H(b)}{x - b} = \lim_{t \to a} \frac{t - a}{h(t) - h(a)} = \begin{cases} 1/h'(a) , & \text{if } h'(a) < \infty \\ 0 , & \text{if } h'(a) = \infty \end{cases}$$

Thus $H'(b) \in [0, 1[$ for every $b \in \mathbb{R}$. If $b = h(r_n)$, then $H'(b) = 0$ because $h'(r_n) = \infty$. Thus $h(\mathcal{Q}) \subset Z(H')$, and so $Z(H')$ is dense in $\mathbb{R}$ because $h(\mathcal{Q})$ is dense in $\mathbb{R}$. To prove $P(H')$ is dense in $\mathbb{R}$, let $I$ be any nonvoid open interval of $\mathbb{R}$. Then $I \cap P(H') \neq \phi$ [otherwise, $I \subset Z(H')$ so $H$ is constant on $I$]. This proves that $P(H')$ is dense in $\mathbb{R}$ too. □

# Chapter 3

## Closed Nowhere Dense Sets in $D_0$.

We now prove Theorem (1.7).

**Proof of Theorem (1.7)** : Let I be a nonvoid open interval of $\mathbb{R}$, and let

$$A = \{ f \in D_0 : f(x) \geq 0, \forall x \in I \}$$

and $$B = \{ f \in D_0 : f(x) \leq 0, \forall x \in I \}.$$

Let $(f_n)_{n \in N} \subset A$ and let $f \in D_0$ with $d(f_n , f) \to 0$. Then $|f_n(x) - f(x)| \leq d(f_n , f) \to 0$ and so $f_n(x) \to f(x)$, for all $x \in \mathbb{R}$. Thus $f(x) \geq 0$ for all $x \in I$ which means $f \in A$. Thus A is closed in $D_0$. Similarly, B is closed in $D_0$.

It remains to prove that both A and B have empty interiors. Assume that A has an interior point f. Then there exists $\epsilon > 0$ such that

$$g \in D_0 , \; \| f - g \|_u \leq \epsilon \Rightarrow g \in A. \qquad (*)$$

Let H be as in Theorem (2.2). Since H is differentiable on $\mathbb{R}$, $0 \leq H' < 1$, and $Z(H')$

is dense in $\mathbb{R}$, we have $H' \in D_0$. Since $P(H')$ is dense in $\mathbb{R}$, we can find $a \in \mathbb{R}$ with $H'(a) > 0$. Since $Z(f)$ is dense in $\mathbb{R}$, we can choose $b \in Z(f) \cap I$ and then define g on $\mathbb{R}$ by

$$g(x) = f(x) - \epsilon H'(x - b + a).$$

It follows from Theorem (1.5) that $g \in D_0$. The fact that $0 \leq H' < 1$ yields

$$\| f - g \|_u = \epsilon \| H' \|_u \leq \epsilon.$$

We infer from (*) that $g \in A$. But $b \in I$ and $g(b) = f(b) - \epsilon H'(a) = -\epsilon H'(a) < 0$ so $g \notin A$. This contradiction proves that the interior of A is empty. A similar argument proves that the interior of B is also empty. Thus A and B are both nowhere dense in $D_0$.

□

**3.1 Theorem** : The set of all $f \in D$ for which the three sets

$$Z(f) = \{ x \in \mathbb{R} : f(x) = 0 \},$$

$$P(f) = \{ x \in \mathbb{R} : f(x) > 0 \},$$

and $$N(f) = \{ x \in \mathbb{R} : f(x) < 0 \}$$

are all dense in $\mathbb{R}$ is a dense subset of $D_0$.

**Proof** : Let $W = \{ f \in D : P(f), Z(f), \text{ and } N(f) \text{ are each dense in } \mathbb{R} \}$. Plainly $W \subset D_0$. We must show that W is dense in $D_0$.

Let $(I_n)_{n \in N}$ be an enumeration of the family of all nonvoid open intervals of $\mathbb{R}$ having rational endpoints. For each n, define

$$A_n = \{ f \in D_0 : f(x) \geq 0, \forall x \in I_n \},$$

and $$B_n = \{ f \in D_0 : f(x) \leq 0 \ \forall x \in I_n \}$$

By Theorem (1.7) $A_n$ and $B_r$ are closed and nowhere dense in the complete metric space $D_0$. Then

$$\bigcup_{n=1}^{\infty} (A_n \cup B_n)$$

is a set of the first category in $D_0$. [*Recall* : Let X be a topological space. A set $A \subset X$ is said to be of *first category* in X if A is the union of some countable family of sets that are nowhere dense in X]. Thus the set

$$E = D_0 \setminus \bigcup_{n=1}^{\infty} (A_n \cup B_n)$$

is residual in $D_0$. [*Recall* that a subset E in a topological space X is called *residual in X* if $X \setminus E$ is of first category in X.] By the Baire Category Theorem, E is dense in $D_0$. We complete the proof by showing that $E \subset W$. Let $f \in E$ be given. To show that $f \in W$, it suffices to show that $I_n \cap P(f)$ and $I_n \cap N(f)$ are both nonvoid whenever $n \in \mathbb{N}$ (since $f \in E \Rightarrow f \in D_0 \Rightarrow Z(f)$ is dense in $\mathbb{R}$). But this is obvious since f belongs to neither $B_n$ nor $A_n$ so there exists $x$ , $y \in I_n$ with $f(x) > 0 > f(y)$. Thus $E \subset W$ and so W too is dense in $D_0$. □

**3.2 Main Theorem** : There exist functions $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x)$ exists with $-1 < F'(x) < 1$ at each $x \in \mathbb{R}$, but yet there is no nonvoid open interval of $\mathbb{R}$ on which F is monotone. [We might say that F is *reasonably smooth and wiggles everywhere.*]

**Proof** : Choose f as in theorem (3.1) with $\| f \|_u < 1$. Since $f \in D$, there exists a differentiable function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F' = f$. Then

$$|F'(x)| = |f(x)| \leq \|f\|_u < 1 \ \forall \ x \in \mathbb{R}.$$

Thus, $F'(x)$ exists and $-1 < F'(x) < 1$, $\forall \ x \in \mathbb{R}$. Now let I be any nonvoid open interval of $\mathbb{R}$. Since $P(F') = P(f)$ is dense in $\mathbb{R}$, $\exists \ a \in I$ with $F'(a) > 0$ and so F cannot be monotonically nonincreasing (decreasing) on I. Likewise, $N(F') = N(f)$ is dense in $\mathbb{R}$ so $\exists \ b \in I$ with $F'(b) < 0$. Thus, F cannot be monotonically nondecreasing (increasing) on I either. Thus, F is not monotone in I. □

# Chapter 4

## Additional Properties of Everywhere Differentiable, Nowhere Monotone Functions

The preceding three chapters were devoted to a proof of the existence of functions $F : \mathbb{R} \to \mathbb{R}$ having the following three properties :

($\alpha$) F is differentiable at every point of $\mathbb{R}$.

($\beta$) F is monotone on *no* nonvoid open interval of $\mathbb{R}$.

($\gamma$) $-1 < F'(x) < 1$ for every $x \in \mathbb{R}$.

In this chapter we deduce a number of other important properties that *any* $F : \mathbb{R} \to \mathbb{R}$ having properties ($\alpha$), ($\beta$), and ($\gamma$) must also have . Thus, let F be any such function.

***Property 1*** : If $a < b$ in $\mathbb{R}$, then $| F(b) - F(a) | < b - a$.

**Proof** : By the Mean Value Theorem there exists $x \in ]a, b[$ such that $F(b) - F(a) = (b - a)F'(x)$. By ($\gamma$), $|F'(x)| < 1$ so $|F(b) - F(a)| < b - a$. $\square$

***Property 2*** : If $a < b$ in $\mathbb{R}$, then F is absolutely continuous on $[a , b]$.

**Proof** : let $\epsilon > 0$ and take $\delta = \epsilon$. Suppose $\{ ]a_j , b_j[ \}^n_{j=1}$ is a finite pairwise disjoint family of subintervals of $[a , b]$, the sum of whose lengths is $< \delta$.

Then from property 1,

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \sum_{j=1}^{n} (b_j - a_j) < \delta = \epsilon.$$

Thus F is absolutely continuous on $[a , b]$. $\square$

***Property 3*** : If $a < b$ in $\mathbb{R}$, then F is of finite total variation over $[a , b]$ :

$$V_a^b F \leq b - a$$

**Proof** : Let $P = \{ a = x_0 < x_1 < \dots\dots\dots\dots < x_n = b \}$ be a subdivision of $[a , b]$.

Then

$$\begin{aligned} V(P , F) &= \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \\ &\leq \sum_{k=1}^{n} (x_k - x_{k-1}) = b - a \end{aligned}$$

Then $V_a^t F = \sup \; V(P, F) : P$ is a sub division of $[a, b] \; \leq b - a$. □

***Property 4*** The two sets $N(F') = \{ x \in \mathbb{R} \;\; F'(x) < 0 \}$ and $P(F') = \{ x \in \mathbb{R} : F'(x) > 0 \}$ are both dense in $\mathbb{R}$.

**Proof** : Let I be any nonvoid open interval of $\mathbb{R}$. Assume $N(F') \cap I = \varnothing$.Then $F'(x) \geq 0 \; \forall \; x \in I$. It follows from the Mean Value Theorem that F is monotone nondecreasing on I contrary to ($\beta$). Thus $N(F') \cap I \neq \varnothing$. Similarly, $P(F') \cap I \neq \varnothing$. □

***Property 5*** : If $a < b$ in $\mathbb{R}$, then F has a local minimum at some $u \in ]a , b[$ and a local maximum at some $v \in ]a , b[$.

**Proof** From property 4 we can first find $c \in N(F')$ and then $d \in P(F')$ with $a < c < b$ and $c < d < b$. Use the Extreme Value Theorem to find $u \in [c \;\; d]$ such that $F(u) \leq F(x), \forall x \in [c \;\; d]$ Then $a < u < b$. Since $F'(c) < 0 \; \exists \; \alpha \in ]c \;\; d[$ such that

$$c < x < \alpha \qquad \Rightarrow \frac{F(x) - F(c)}{x - c} < 0$$

So $F(u) \leq F(x) < F(c) \Rightarrow u \neq c$. This proves $u \neq 0$.

Since $F'(d) > 0$, $\exists\, \beta \in\, ]c\,,\, d[$ such that

$$\beta < x < d \quad \Rightarrow \quad \frac{F(x) - F(d)}{x - d} > 0$$

so $F(u) \leq F(x) < F(d)$. Thus $u \neq d$.

Now we know $c < u < d$ and $F(u) \leq F(x)\ \forall\, x \in\, ]c\,,\, d[$.

This means F has a local minimum at u. $c < u < d$ and $f(u) \leq f(x)\ \forall\, x \in\, ]c\,,\, d[$.

Since -F also satisfies ($\alpha$), ($\beta$), and ($\gamma$), we can apply the preceding paragraph to find $p < v < q$ in $]a\,,\, b[$ such that $-F(v) \leq -F(x)\ \forall\, x \in\, ]p\,,\, q[$. Thus F has a local maximum at v. □

***Property 6*** : $Z(F')$ is a dense $G_\delta$ in $\mathbb{R}$.

**Proof** : Since $F' \in D$, Theorem (1.3) shows that $Z(F')$ is a $G_\delta$ in $\mathbb{R}$. If F has a local minimum (or maximum) at $w \in \mathbb{R}$, then $F'(w) = 0$ so $w \in Z(F')$. Thus, it follows from Property 5 that $Z(F')$ is dense in $\mathbb{R}$. □

***Property 7*** : If $F'$ is continuous at x, then $F'(x) = 0$.

**Proof** : Suppose $F'$ is continuous at x. By Property 4, $F'$ takes on both positive and negative values in every neighborhood of x so $F'(x) = 0$. □

***Property 8*** : The function $F'$ is of Baire Class 1 on $\mathbb{R}$. [*Recall* : Let X be a topological space. A function $f : X \to \mathbb{C}$ is said to be of *Baire Class 1 on X* if there exists some sequence of complex-valued functions which are continuous on X that converges to f at every point of X.]

**Proof** : For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = n[F(x + 1/n) - F(x)],\ x \in \mathbb{R}.$$

By $(\alpha)$, F is continuous on $\mathbb{R}$ so each $f_n$ is also continuous on $\mathbb{R}$.

$$\text{Obviously} \quad F'(x) = \lim_{n\to\infty} f_n(x)\ \forall\ x \in \mathbb{R}.\ \square$$

***Property 9*** : The sets $A = \{ x \in \mathbb{R} : F'(x) > 0 \}$ and $B = \{ x \in \mathbb{R} : F'(x) < 0 \}$ satisfy $\lambda(A \cap I) > 0$ and $\lambda(B \cap I) > 0$ for every nonvoid open interval $I \subset \mathbb{R}$. Here $\lambda$ is Lebesgue measure on $\mathbb{R}$.

**Proof** : Assume $\lambda(A \cap I) = 0$ for some nonvoid open interval $I \subset \mathbb{R}$. Let $a < b$ in I. By Property 2, F is absolutely continuous on [a , b]. So by the Fundamental Theorem of Calculus

$$F(b) - F(a) = \int_a^b F' \leq 0$$

Thus $F(a) \geq F(b)\ \forall\ a < b$ in I. So F is nonincreasing on I, which is a contradiction.

Thus $\lambda(A \cap I) > 0$. Similarly, $\lambda(B \cap I) > 0$. $\square$

***Property 10*** : If $a < b$ in $\mathbb{R}$, then $F'$ is not Riemann integrable over $[a , b]$.

**Proof** : Let $A = P(F')$ as in property 9 and $D = \{ x \in [a , b] : F' \text{ is discontinuous at } x \}$. By property 7, $[a , b] \cap A \subset D$ and so Property 9 shows

$$\lambda(D) \geq \lambda([a , b] \cap A) > 0.$$

Thus $F'$ is not continuous a. e. on $[a , b]$ and hence $F'$ is not Riemann integrable over $[a , b]$. $\square$

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