THE BUCKLING PROBLEM OF A BEAM ON AN ELASTIC FOUNDATION UNDER DISTRIBUTED AXIAL LOADS

by

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NOMENCLATURE

E	Modulus of elasticity.
I_z	Moment of inertia of a plane area with respect to z axis.
l	Length, span.
q_0, q	Intensity of distributed load.
M	Bending moment in the beam.
x, ξ	Abscissas along the span of beam.
y, η	Coordinates of the buckling mode.
β	Modulus of elastic foundation.
n	Number of divisions of the beam.
k	Spring constant.
K	A dimensionless quantity of $(q_0 l^3/EI_z)$
γ	A dimensionless quantity of $(\beta l^4/EI_z)$

INTRODUCTION

There are many buckling problems in Elastic Stability for which "exact" solutions are either impossible or impractical to obtain, using existing methods. With the development of high-speed digital computers, attempts have been made to solve such difficult problems by approximate techniques to obtain a better result.

In this report, the method using the theory of difference equations will be applied to solve the buckling problem of a beam on an elastic foundation under distributed axial loads. With the same procedure used for the beam, the problem of the stability of the upper chord of a low-truss bridge, or pony truss, can be solved. In the absence of upper chord bracing, the lateral buckling of the top chord is resisted by the elastic reactions of the vertical and diagonal members of the truss. At the supports, there are usually frames or bracing members of considerable rigidity, so that the ends of the chord can be considered as immovable in the lateral direction. Thus, the upper chord can be treated as a beam with hinged ends compressed by forces distributed along its length and elastically supported at intermediate points.

This problem has been solved by energy methods by Timoshenko[1], but no direct solution of the differential equation is known. Since the result obtained by the differential equation method would be better than the energy solution, it is worthwhile to try to solve the differential equation directly.

THEORY AND DERIVATION

As shown in Figure A, the beam is subjected to a distributed axial load q and supported by a continuous elastic foundation. The axial load q will be assumed to have the distribution shown in Figure B, that is, the intensity of distributed load at the ends is q_0 and the load is directed toward the center of the beam. The load q decreases linearly to the center, where it has zero value. The modulus of the elastic foundation β can be defined as follows: If k is the spring constant of the individual supports and a is the distance between them, the rigidity of the equivalent elastic medium is expressed by the quantity $\beta = \frac{k}{a}$. It has the dimensions of a force divided by the square of a length and, when multiplied by the deflection y, gives the reaction of the foundation per unit length of beam.

In calculating the bending moment at section mn produced by the distributed compressive load during bending, we note that the intensity of this load at any cross section, distance x from the center is

$$q = \frac{2q_0x}{1}$$

where q_0 is the intensity of load at the ends. Then the bending moment at section mn produced by the axial loading is,

$$+\int_x^{1/2} \left[q(\xi)d\xi \quad (y-\eta) \right]$$

The bending moment at section mn produced by the elastic foundation is,

$$+\int_x^{1/2} (\beta\eta d\xi) \ (\xi-x)$$

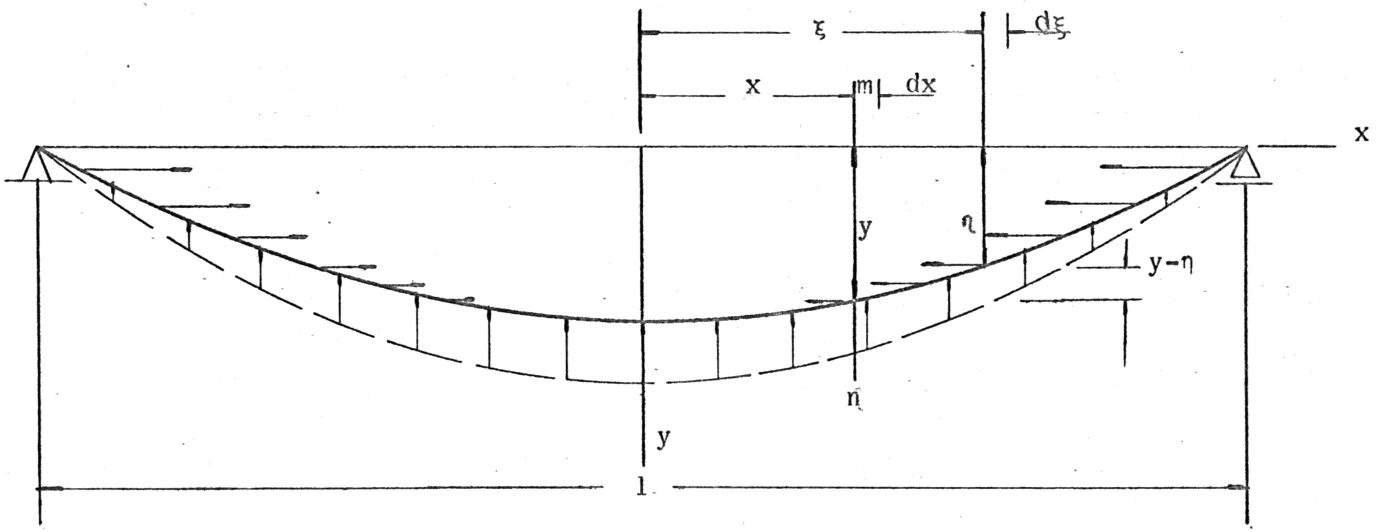

FIGURE A

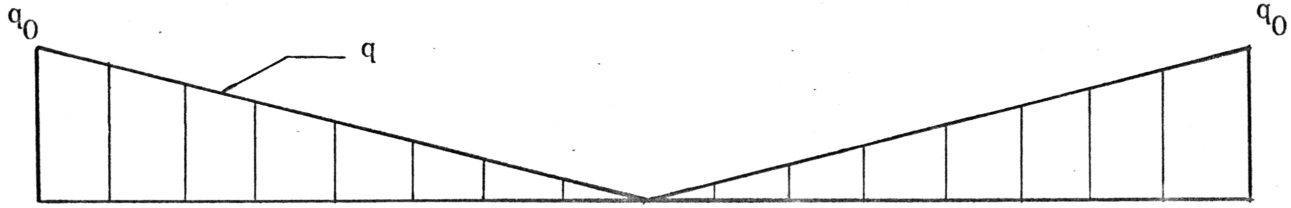

FIGURE B

and the bending moment at section mn produced by the reactions of the end supports is,

$$-\left[\int_0^{l/2} \beta y dx\right]\left(\frac{l}{2} - x\right)$$

Therefore, the total bending moment at section mn is,

$$M = \int_x^{l/2} \left(\frac{2q_0\xi}{l}\right)(y - \eta)\, d\xi + \int_x^{l/2} (\beta\eta)(\xi - x)\, d\xi - \left(\frac{l}{2} - x\right)\int_0^{l/2} \beta y dx$$

Substituting M into the formula $EI_z y'' = -M$, we obtain

$$EI_z y'' = -\int_x^{l/2} \left(\frac{2q_0\xi}{l}\right)(y - \eta)\, d\xi - \int_x^{l/2} \beta\eta(\xi - x)\, d\xi + \left(\frac{l}{2} - x\right)\int_0^{l/2} \beta y dx \qquad (1)$$

In order to solve such an integro-differential equation, it is convenient to eliminate the integrals by differentiation, thus reducing the equation to an ordinary differential equation with variable coefficients for which a solution may be easier to obtain. Differentiation of Equation 1 with respect to x, yields,

$$EI_z y''' = -\frac{d}{dx}\int_x^{l/2} \left(\frac{2q_0\xi}{l}\right)(y - \eta)\, d\xi - \frac{d}{dx}\int_0^{l/2} (\xi - x)\beta\eta\, d\xi - \int_0^{l/2} \beta y dx \qquad (2)$$

From the differentiation under integral sign(2), we obtain,

$$\frac{d}{dx}\int_x^{l/2} \left(\frac{2q_0\xi}{l}\right)(y - \eta)\, d\xi = \frac{q_0}{l}\left(\frac{l^2}{4} - x^2\right) y' \qquad (3)$$

and

$$\frac{d}{dx}\int_x^{l/2} \beta\eta\,(\xi - x)\, d\xi = -\int_x^{l/2} \beta\eta d\xi \qquad (4)$$

Substituting Equations (3) and (4) into Equation 2, we obtain,

$$EI_z y''' = - \frac{q_0}{l} \left(\frac{l^2}{4} - x^2\right) y' + \int_x^{l/2} \beta\eta d\xi - \int_0^{l/2} \beta y dx$$

Differentiating with respect to x again, Equation (5) takes the form

$$EI_z y^{iv} = - \frac{q_0}{l} \left[y'' \left(\frac{l^2}{4} - x^2\right) + y' (-2x) \right] + \beta \frac{d}{dx} \int_x^{l/2} \eta d\xi$$

But,

$$\frac{d}{dx} \int_x^{l/2} \eta d\xi = - y$$

Hence,

$$EI_z \, y^{iv} = - \frac{q_0}{l} \left[y'' \left(\frac{l^2}{4} - x^2\right) - zxy' \right] - \beta y$$

or,

$$y^{iv} + \frac{q_0}{EI_z l} \left(\frac{l^2}{4} - x^2\right) y'' - \frac{2q_0}{EI_z l} xy' + \frac{\beta y}{EI_z} = 0 \qquad (6)$$

Using the notations,

$$K = \frac{q_0 l^3}{EI_z} \quad , \gamma = \frac{\beta l^4}{EI_z}$$

Equation (6) becomes,

$$l^4 y^{iv} + K \left(\frac{l^2}{4} - x^2\right) y'' - 2Kxy' + \gamma y = 0 \qquad (7)$$

Equation (7) can be integrated by the use of infinite power series, but the series obtained is intractable and converges very slowly. The same result, however, can be obtained more easily by transforming the differential equation into a difference equation and then solving the difference equation numerically.

From the theory of central differences[3], we know the derivative operators are,

$$
\begin{aligned}
hy'_\alpha &= \frac{-1}{2} y_{\alpha-1} + \frac{1}{2} y_{\alpha+1} \\
h^2 y''_\alpha &= y_{\alpha-1} - 2y_\alpha + y_{\alpha+1} \\
h^3 y'''_\alpha &= \frac{-1}{2} y_{\alpha-2} + y_{\alpha-1} - y_{\alpha+1} + \frac{1}{2} y_{\alpha+2} \\
h^4 y^{iv}_\alpha &= y_{\alpha-2} - 4y_{\alpha-1} + 6y_\alpha - 4y_{\alpha+1} + y_{\alpha+2}
\end{aligned}
\tag{8}
$$

where h is the distance between pivotal points in the equally spaced mesh in the buckling mode. The symbols $y_{\alpha-2}$, $y_{\alpha-1}$, y_α, $y_{\alpha+1}$, and $y_{\alpha+2}$ represent the deflection of the buckling mode at points $\alpha-2$, $\alpha-1$, α, $\alpha+1$, and $\alpha+2$, respectively, as shown in Figure C.

Dividing the whole span l of the beam into n equal divisions and using the notations,

$$h = \frac{l}{n}\ ,\ x = \frac{\alpha}{n} l \quad (\alpha = 0, 1, 2, 3, \text{-----} \frac{n}{2})$$

the Equations (8) become

$$
\begin{aligned}
y^{iv}_\alpha &= \left(\frac{n^4}{l^4}\right) (y_{\alpha-2} - 4y_{\alpha-1} + 6y_\alpha - 4y_{\alpha+1} + y_{\alpha+2}) \\
y'''_\alpha &= \left(\frac{n^3}{l^3}\right) \left(\frac{-1}{2} y_{\alpha-2} + y_{\alpha-1} - y_{\alpha+1} + \frac{1}{2} y_{\alpha+2}\right) \\
y''_\alpha &= \left(\frac{n^2}{l^2}\right) (y_{\alpha-1} - 2y_\alpha + y_{\alpha+1}) \\
y'_\alpha &= \left(\frac{n}{l}\right) \left(\frac{-1}{2} y_{\alpha-1} + \frac{1}{2} y_{\alpha+1}\right) \\
y &= y_\alpha
\end{aligned}
\tag{9}
$$

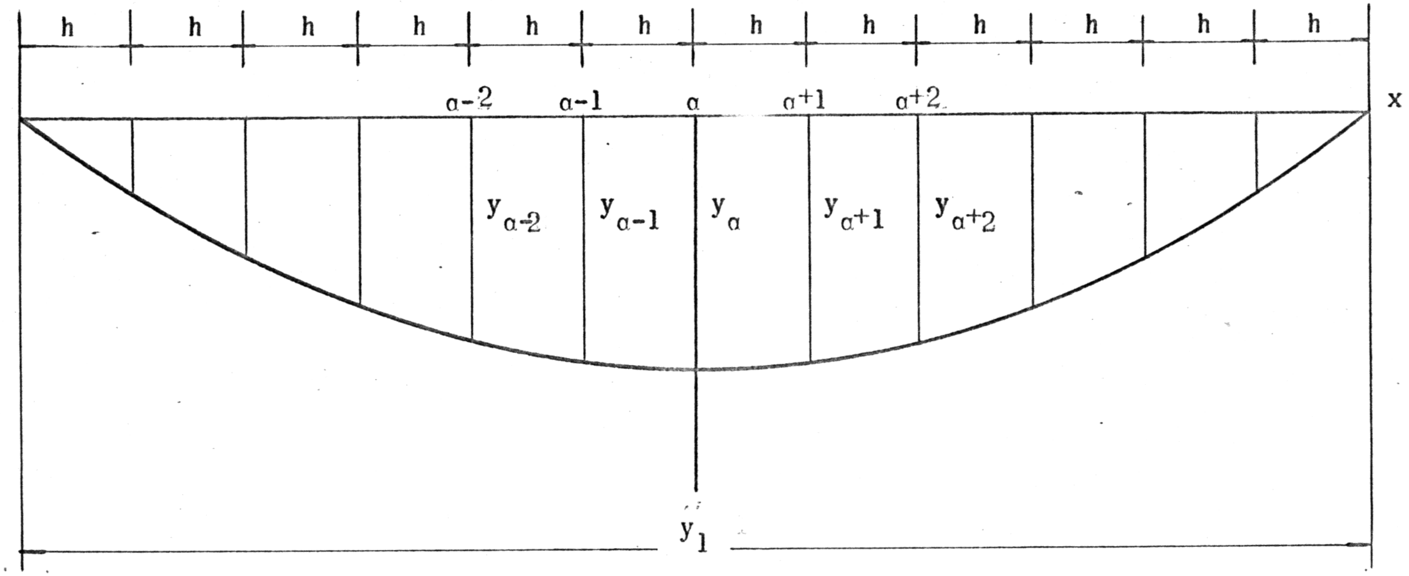

FIGURE C

Substituting Equations (9) into Equation (7), the latter takes the form

$$n^4 \,(y_{\alpha-2} - 4y_{\alpha-1} + 6y_\alpha - 4y_{\alpha+1} + y_{\alpha+2}) + K\left(\frac{l^2}{4} - \frac{\alpha^2}{n^2}\, l^2\right)\left(\frac{n^2}{l^2}\right)$$

$$(y_{\alpha-1} - 2y_\alpha + y_{\alpha+1}) - 2K\left(\frac{\alpha}{n}\, l\right)\left(\frac{n}{l}\right)\left(\frac{-1}{2}\, y_{\alpha-1} + \tfrac{1}{2} y_{\alpha+1}\right) + \gamma y_\alpha = 0$$

Collecting terms for $y_{\alpha-2}$, $y_{\alpha-1}$, y_α, $y_{\alpha+1}$ and $y_{\alpha+2}$, and dividing the whole equation by n^4, the general expression of the difference equation of the buckling mode takes the form,

$$y_{\alpha-2} + \left[-4 + K\left(\frac{1}{4n^2} + \frac{\alpha-\alpha^2}{n^4}\right)\right] y_{\alpha-1} + \left[6 - 2K\left(\frac{1}{4n^2} - \frac{\alpha^2}{n^4}\right) + \frac{\gamma}{n^4}\right] y_\alpha$$

$$+ \left[-4 + K\left(\frac{1}{4n^2} - \frac{\alpha+\alpha^2}{n^4}\right)\right] y_{\alpha+1} + y_{\alpha+2} = 0 \qquad (10)$$

For each division point on the span of the beam, we can establish one homogeneous linear equation from Equation (10). Since the buckling mode is symmetrical about the origin, we need to set up the equations for one half of the beam only. With $n/2$ divisions in the half span, $(n/2 + 1)$ equations can be established. These equations in the $(n/2 + 5)$ unknown y's are homogeneous linear equations. Using the boundary conditions, these equations can be reduced to $(n/2 + 1)$ unknowns. Buckling of the beam becomes possible only when the deflections of the buckling mode have non-trivial solutions, that is, when the determinant Δ of the coefficients of the y's becomes zero. The coefficients of the y's are in terms of K. From $\Delta = 0$, the determinant Δ can be expanded to obtain a polynomial equation expressed in the variable K where the highest power of K is $n/2$. Using the trial and error method, the lowest positive root is determined. Since K is expressed in terms of the loading, the lowest critical load is thus determined.

The boundary conditions for a symmetrical buckling mode (First mode) are:

(1) At the ends, the deflection of the buckling mode equals zero, that is, $y = 0$ when $x = l/2$, hence $y_{n/2} = 0$.

The bending moment at the ends should be zero, thus the second derivative of the y's equals zero. From Euquation (9), we obtain ,

$$y''_{n/2} = \left(\frac{n^2}{l^2}\right)\left(y_{n/2-1} - 2y_{n/2} + y_{n/2+1}\right) = 0$$

$$\frac{n^2}{l^2} \neq 0, \text{ therefore, } y_{n/2-1} - 2y_{n/2} + y_{n/2+1} = 0 \qquad (11)$$

for $y_{n/2} = 0$ then Equation (11) becomes,

$$y_{n/2-1} + y_{n/2+1} = 0$$

Hence,

$$y_{n/2-1} = -y_{n/2+1}$$

(2) At the center, the slope of the buckling mode is zero, that is,

$$y'_o = 0 \text{ or } \frac{-1}{2} y_{-1} + \frac{1}{2} y_1 = 0$$

Hence,

$$y_{-1} = +y_1$$

Since the buckling mode is symmetrical about the origin, it is obvious that $y_{-2} = + y_2$

The boundary conditions for an anti-symmetrical buckling mode (Second mode) are:

(1) At the ends, the deflection of the buckling mode is zero, that is, $y = 0$, when $x = l/2$, hence, $y_{n/2} = 0$.

The bending moment at the ends should be zero, thus the second derivative of the y's equals zero. From Euqation (9), we obtain

$$y_{n/2 - 1} = -y_{n/2 + 1}$$

(2) At the center, the deflection of the buckling mode is zero, that is, $y_0 = 0$ when $x = 0$. The origin is the point of inflection of the buckling mode, so,

$y_{-1} = -y_1$ and $y_{-2} = -y_2$

For the case where the rigidity of the elastic medium is very small, the buckling mode of the beam has only one half-wave and is symmetrical with respect to the middle. Therefore, the boundary conditions for the symmetrical case should be used. When a greater restraint is supplied by the elastic foundation, the buckling mode of the beam may have two half-waves, and there is an inflection point at the middle of the beam. To calculate the critical load in this case, the boundary conditions for the anti-symmetrical case should be used.

CALCULATIONS

(1) Taking an n of 10, γ of 0, 80, 160, 240, 364.8, 904, and 1600 respectively, and α of 0, 1, 2, 3, 4, and 5 respectively, six difference equations were established from Equation (10). Using the boundary conditions

$$y_1 = y_{-1},\ y_2 = y_{-2},\ y_4 = -y_6,\ \text{and}\ y_5 = 0,$$

for the symmetrical buckling mode, and

$$y_o = 0,\ y_1 = -y_1\ \ y_2 = -y_{-2},\ y_4 = -y_6,\ \text{and}\ y_5 = 0$$

for the anti-symmetrical buckling mode, the six equations were expressed in the six unknowns y_o, y_1, y_2, y_3, y_4, and y_7. These corresponding equations are shown in Tables 1, 2, 3, 4, 5, 6, 7, 8, and 9. From $\Delta = 0$, the lowest positive roots of K were found, thus the critical compressive buckling loads were determined. These results for each case are shown below Tables 2, 4, 5, 6, 7, 8, and 9.

(2) Taking an n of 14, a γ of 364.8, and α of 0, 1, 2, 3, 4, 5, 6, and 7 respectively, eight difference equations were set up from Equation (10). Using the boundary conditions:

$$y_o = 0,\ y_1 = -y_1,\ y_2 = -y_{-2},\ y_6 = -y_8,\ \text{and}\ y_7 = 0,$$

the eight difference equations for the anit-symmetrical buckling mode were reduced to seven equations with unknowns y_1, y_2, y_3, y_4, y_5, y_6, and y_9. These corresponding equations are shown in Tables 10 and 11.

From $\Delta = 0$, the lowest positive root of K was found, thus the critical buckling load was determined for this anti-symmetrical case. This result is shown below Table 11.

TABLE 1

General Difference Equations ($n = 10$, $\gamma = 0$)

α	y_{-2}	y_{-1}	y_o	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Const.
0	1	-4 $+2.5\times10^{-3}K$	6 $-5\times10^{-3}K$	-4 $+2.5\times10^{-3}K$	1						0
1		1	-4 $+2.5\times10^{-3}K$	6 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1					0
2			1	-4 $+2.3\times10^{-3}K$	6 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3				1	-4 $+1.9\times10^{-3}K$	6 $-3.2\times10^{-3}K$	-4 $+1.3\times10^{-3}K$	1			0
4					1	-4 $+1.3\times10^{-3}K$	6 $-1.8\times10^{-3}K$	-4 $+0.5\times10^{-3}K$	1		0
5						1	-4 $+0.5\times10^{-3}K$	6	-4 $-0.5\times10^{-3}K$	1	0

TABLE 2

Difference Equations for Symmetrical Mode ($n=10$, $\gamma = 0$)

Boundary Conditions: $y_{-1}=y_1$, $y_{-2}=y_2$, $y_4=-y_6$, and $y_5=0$

α	y_{-2}	y_{-1}	y_o	y_1	y_2	y_3	y_4	$y_5=o$	y_6	y_7	Const.
0			$6 - 5\times10^{-3}K$	$-8 + 5\times10^{-3}K$	2						0
1			$-4 + 2.5\times10^{-3}K$	$7 - 4.8\times10^{-3}K$	$-4 + 2.3\times10^{-3}K$	1					0
2			1	$-4 + 2.3\times10^{-3}K$	$6 - 4.2\times10^{-3}K$	$-4 + 1.9\times10^{-3}K$	1				0
3				1	$-4 + 1.9\times10^{-3}K$	$6 - 3.2\times10^{-3}K$	$-4 + 1.3\times10^{-3}K$				0
4					1	$-4 + 1.3\times10^{-3}K$	$5 - 1.8\times10^{-3}K$				0
5						1	$10^{-3}K$			1	0

$K = 78.6$

$$(q_o l/4)_{cr.} = 1.99 \frac{\pi^2 EI_z}{l^2}$$

TABLE 3

General Difference Equations (n=10, γ=240)

α	y_{-2}	y_{-1}	y_o	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Const.
0	1	-4 $+2.5\times10^{-3}K$	6.024 $-5\times10^{-3}K$	-4 $+2.5\times10^{-3}K$	1						0
1		1	-4 $+2.5\times10^{-3}K$	6.024 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1					0
2			1	-4 $+2.3\times10^{-3}K$	6.024 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3				1	-4 $+1.9\times10^{-3}K$	6.024 $-3.2\times10^{-3}K$	-4 $+1.3\times10^{-3}K$	1			0
4					1	-4 $+1.3\times10^{-3}K$	6.024 $-1.8\times10^{-3}K$	-4 $+0.5\times10^{-3}K$	1		0
5						1	-4 $+0.5\times10^{-3}K$	6.024	-4 $-0.5\times10^{-3}K$	1	0

TABLE 4

Difference Equations for Symmetrical mode ($n=10$, $\gamma=240$)

Boundary Conditions: $y_1=y_{-1}$, $y_2=y_{-2}$, $y_4=y_{-6}$, and $y_5=0$

	y_{-2}	y_{-1}	y_o	y_1	y_2	y_3	y_4	$y_5=o$	y_6	y_7	Const.
0			6.024 $-5\times10^{-3}K$	-8 $+5\times10^{-3}K$	2						0
1			-4 $+2.5\times10^{-3}K$	7.024 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1					0
2			1	-4 $+2.3\times10^{-3}K$	6.024 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3				1	-4 $+1.9\times10^{-3}K$	6.024 $-3.2\times10^{-3}K$	-4 $+1.3\times10^{-3}K$				0
4					1	-4 $+1.3\times10^{-3}K$	5.024 $-1.8\times10^{-3}K$				0
5						1	$10^{-3}K$			1	0
$K = 243.8$						$(q_o l/4)_{cr.} = 6.17\,\dfrac{\pi^2 EI_z}{l^2}$					

TABLE 5

Difference Equations for Symmetrical Mode ($n=10$, $\gamma=364.8$)

Boundary Conditions: $y_1=y_{-1}$, $y_2=y_{-2}$, $y_4=-y_6$, and $y_5=0$

α	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_4	$y_5=0$	y_6	y_7	Const.
0			6.03648 $-5\times10^{-3}K$	-8 $+5\times10^{-3}K$	2						0
1			-4 $+2.5\times10^{-3}K$	7.03648 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1					0
2			1	-4 $+2.3\times10^{-3}K$	6.03648 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3				1	-4 $+1.9\times10^{-3}K$	6.03648 $-3.2\times10^{-3}K$	-4 $+1.3\times10^{-3}K$				0
4					1	-4 $+1.3\times10^{-3}K$	5.03648 $-1.8\times10^{-3}K$				0
5						1	$10^{-3}K$			1	0

$K = 307.8$

$$(q_o l/4)_{cr.} = 7.8 \frac{\pi^2 EI_z}{l^2}$$

TABLE 6

Difference Equations for Symmetrical Mode ($n=10, \gamma=904$)

Boundary Conditions: $y_1 = y_{-1}$, $y_2 = y_{-2}$, $y_4 = -y_6$, and $y_5 = 0$

α	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_4	$y_5=0$	y_6	y_7	Const.
0			6.0904 $-5\text{x}10^{-3}K$	-8 $+5\text{x}10^{-3}K$	2						0
1			-4 $+2.5\text{x}10^{-3}K$	7.0904 $-4.8\text{x}10^{-3}K$	-4 $+2.3\text{x}10^{-3}K$	1					0
2			1	-4 $+2.3\text{x}10^{-3}K$	6.0904 $-4.2\text{x}10^{-3}K$	-4 $+1.9\text{x}10^{-3}K$	1				0
3				1	-4 $+1.9\text{x}10^{-3}K$	6.0904 $-3.2\text{x}10^{-3}K$	-4 $+1.3\text{x}10^{-3}K$				0
4					1	-4 $+1.3\text{x}10^{-3}K$	5.0904 $-1.8\text{x}10^{-3}K$				0
5						1	$10^{-3}K$			1	0

$K = 450.5$

$$(q_o l/4)_{cr.} = 11.41 \frac{\pi^2 EI_z}{l^2}$$

TABLE 7

Difference Equations for Anit-Symmetrical Mode ($n=10$, $\gamma=240$)

Boundary Conditions: $y_o=0$, $y_1=-y_{-1}$, $y_2=-y_{-2}$, $y_4=-y_6$, and $y_5=0$

α	y_{-2}	$y_o=0$	y_1	y_2	y_3	y_4	$y_5=0$	y_6	y_7	Const.
0			0	0	0	0				0
1			5.024 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1	0				0 0
2			-4 $+2.3\times10^{-3}K$	6.024 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3			1	-4 $+1.9\times10^{-3}K$	6.024 $-3.2\times10^{-3}K$	-4 $+1.8\times10^{-3}K$				0
4				1	-4 $+1.3\times10^{-3}K$	5.024 $-1.8\times10^{-3}K$				0
5					1	$10^{-3}K$			1	

$K=267$

$$(q_o l/4)_{cr.}=6.76\,\frac{\pi^2 EI_z}{l^2}$$

TABLE 8

Difference Equations for Anti-Symmetrical Mode ($n=10$, $\gamma=364.8$)

Boundary Conditions: $y_o=0$, $y_1=y_{-1}$, $y_2=y_{-2}$, $y_4=-y_6$, and $y_5=0$

α	y_{-2}	y_{-1}	$y_o=o$	y_1	y_2	y_3	y_4	$y_5=0$	y_6	y_7	Const.
0				0	0	0	0				0
1				5.03648 $-4.8\times10^{-3}K$	-4 $+2.3\times10^{-3}K$	1	0				0
2				-4 $+2.3\times10^{-3}K$	6.03648 $-4.2\times10^{-3}K$	-4 $+1.9\times10^{-3}K$	1				0
3				1	-4 $+1.9\times10^{-3}K$	6.03648 $-3.2\times10^{-3}K$	-4 $+1.3\times10^{-3}K$				0
4					1	-4 $+1.3\times10^{-3}K$	5.03648 $-1.8\times10^{-3}K$				0
5						1	$10^{-3}K$			1	0

$K=285.2$

$$(q_o l/4)_{cr.}=7.23\ \frac{\pi^2 EI_z}{l^2}$$

TABLE 9

Difference Equations for Anti-Symmetrical Mode ($n=10$, $\gamma=904$)

Boundary Conditions: $y_o=o$, $y_1=-y_{-1}$, $y_2=-y_{-2}$, $y_4=-y_6$, and $y_5=o$

α	y_{-2}	y_{-1}	$y_o=o$	y_1	y_2	y_3	y_4	$y_5=o$	y_6	y_7	Const.
0				0	0	0	0				0
1				$5.0904 -4.8\times10^{-3}K$	$-4 +2.3\times10^{-3}K$	1	0				0
2				$-4 +2.3\times10^{-3}K$	$6.0904 -4.2\times10^{-3}K$	$-4 +1.9\times10^{-3}K$	1				0
3				1	$-4 +1.9\times10^{-3}K$	$6.0904 -3.2\times10^{-3}K$	$-4 +1.3\times10^{-3}K$				0
4					1	$-4 +1.3\times10^{-3}K$	$5.0904 -1.8\times10^{-3}K$				0
5						1	$10^{-3}K$			1	0

$K=362.1$

$$(q_o l/4)_{cr.}=9.18\frac{\pi^2 EI_z}{l^2}$$

TABLE 10

General Difference Equations ($n=14$, $\gamma=364.8$)

α	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	Const.
0	1	-4 +1.2755 x10^{-3}K	6.009496 -2.5510 x10^{-3}K	-4 +1.2755 x10^{-3}K	1								0
1		1	-4 +1.2755 x10^{-3}K	6.009496 -2.4990 x10^{-3}K	-4 +1.2235 x10^{-3}K	1							0
2			1	-4 +1.2235 x10^{-3}K	6.009496 -2.3428 x10^{-3}K	-4 +1.1193 x10^{-3}K	1						0
3				1	-4 +1.1193 x10^{-3}k	6.009496 -2.0825 x10^{-3}K	-4 +0.9631 x10^{-3}K	1					0
4					1	-4 +0.9631 x10^{-3}K	6.009496 -1.7180 x10^{-3}K	-4 +0.7549 x10^{-3}K	1				0
5						1	-4 +0.7549 x10^{-3}K	6.009496 -1.2495 x10^{-3}K	-4 +0.4946 x10^{-3}K	1			0
6							1	-4 +0.4946 x10^{-3}K	6.009496 -0.6768 x10^{-3}K	-4 +0.1822 x10^{-3}K	1		0
7								1	-4 +0.1822 x10^{-3}K	6.009496	-4 -0.1822 x10^{-3}K	1	0

TABLE 11

Difference Equations for Anti-Symmetrical Mode ($n=14$ $\gamma=364.8$)

Boundary Conditions: $y_0=0$, $y_1=-y_{-1}$, $y_2=-y_{-2}$, $y_6=-y_8$, and $y_7=0$

#	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	Const.
0				0	0	0	0	0	0				0
1				5.009496 $-2.4990\times10^{-3}K$	-4 $+1.2235\times10^{-3}K$	1							0
2				-4 $+1.2235\times10^{-3}K$	6.009496 $-2.3428\times10^{-3}K$	-4 $+1.1193\times10^{-3}K$	1						0
3				1	-4 $+1.1193\times10^{-3}K$	6.009496 $-2.0825\times10^{-3}K$	-4 $+0.9631\times10^{-3}K$	1					0
4					1	-4 $+0.9631\times10^{-3}K$	6.009496 $-1.7180\times10^{-3}K$	-4 $+0.7549\times10^{-3}K$	1				0
5						1	-4 $+0.7549\times10^{-3}K$	6.009496 $-1.2495\times10^{-3}K$	-4 $+0.4946\times10^{-3}K$				0
6							1	-4 $+0.4946\times10^{-3}K$	5.009496 $-0.6768\times10^{-3}K$				0
7								1	$+0.3644\times10^{-3}K$			1	0

K=2.292.1

$$(q_0 l/4)_{cr} = \frac{7.4\pi^2 EI_z}{l^2}$$

COMPARISON OF RESULTS

From the preceeding computations, the results of the compressive buckling load at the center of the beam were tabulated in Table 12, and compared with the energy solutions derived by Timoshenko[1] in Table 13. Column 4 in Table 13 shows the difference between the two results for each value of $\gamma/16$ expressed in percentage based on the energy solutions.

These results were also plotted against different values of $\gamma/16$. One curve represents the symmetrical buckling mode, and the other the anti-symmetrical buckling mode. As shown in Figure D, the dotted line curve represents the energy solutions (See Table 13). Curve 1 shows the results for the symmetrical buckling mode, while curve 2 shows the results for the anti-symmetrical buckling mode. The point of intersection p of curves 1 and 2 means that for

$$\frac{\gamma}{16} = 18.8, \text{ and } \left(\frac{q_o l}{4}\right)_{cr.} = 7 \frac{\pi^2 E I_z}{l^2}$$

the buckling mode can either be symmetrical or anti-symmetrical. For any value of $\gamma/16$ less than 18.8, the critical buckling load will occur in the symmetrical mode. For a larger value of $\gamma/16$, the critical buckling load will occur in the anti-symmetrical mode. The heavy line represents the lowest critical buckling loads which are significant for the engineering design purposes.

TABLE 12

Compressive Buckling Loads at the Center (n=10)

$$\left(\frac{q_o l}{4}\right)_{cr.} = N \frac{\pi^2 EI_z}{l^2}$$

$\gamma/16$	N (Symmetrical mode)	N (Anti-symmetrical mode)
0	1.99	5.86
5	3.53	---
10	4.93	6.46
15	6.17	6.76
22.8	7.80	7.23
56.5	11.41	9.18
100	13.36	11.52

TABLE 13

Critical Compressive Buckling Loads at the Center

$$N \left(\frac{q_o l}{4}\right)_{cr.} = N \frac{\pi^2 EI_z}{l^2}$$

$\gamma/16$	N Energy Solutions	N Difference Equation Solutions (n=10)	Percentage of Difference Between Two Results (%)	N Difference Equation Solutions (n=14)
0	2.06	1.99	3.40	--
5	3.63	3.53	2.76	--
10	5.09	4.93	3.14	--
15	6.38	6.17	3.29	--
22.8	7.60	7.23	4.87	7.40
56.5	9.53	9.18	3.68	--
100	11.86	11.52	2.87	--

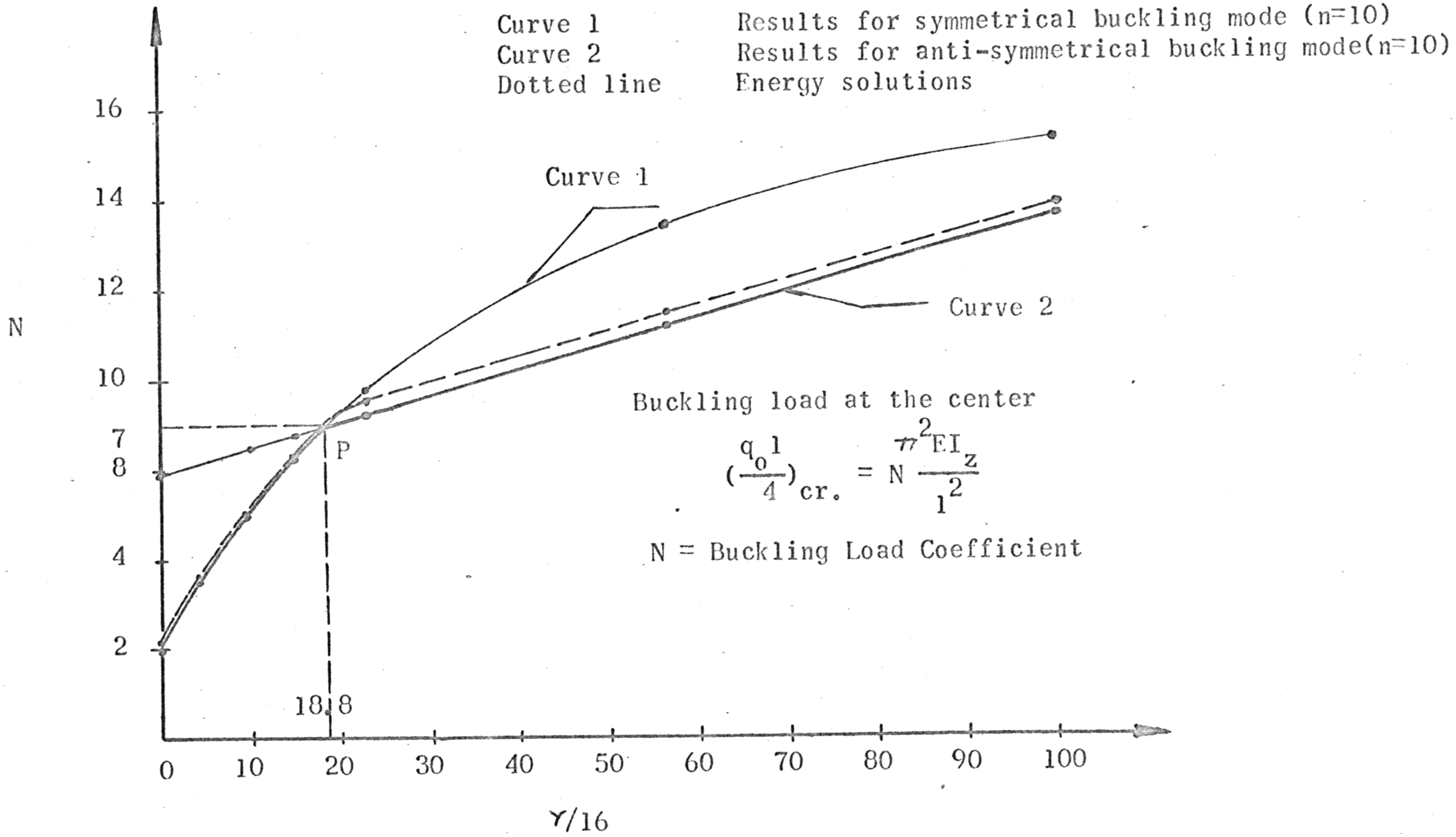
Curve 1 Results for symmetrical buckling mode (n=10)
Curve 2 Results for anti-symmetrical buckling mode(n=10)
Dotted line Energy solutions
16
14
12
10
7
8
4
2
N
Curve 1
Curve 2
P
18.8
0
10
20
30
40
50
60
70
80
90
100
$\gamma/16$
Buckling load at the center
$(\frac{q_o l}{4})_{cr.} = N \frac{\pi^2 EI_z}{l^2}$
N = Buckling Load Coefficient

FIGURE D

CONCLUSIONS

It is well known that the energy method gives an upper bound to the exact solution[4,5,6]. Comparing the results obtained by the difference equation method with the energy solutions, as in Table 13, it is seen that the greatest difference is less than five percent. The comparison is also shown graphically in Figure D, from which it is apparent that either the energy method or the difference equation method will yield satisfactory values for practical engineering use.

With a further increase of the modulus of rigidity of the elastic foundation, the buckling mode may have three or more half-waves. In general, the higher modes can be calssified into two types of buckling modes, that is, symmetrical mode for odd number of half-waves and anti-symmetrical mode for even number of half-waves so that the corresponding boundary conditions for each case should be used to obtain the critical buckling load.

The value of $\gamma/16$ in the higher modes for which the critical buckling load can occur in both types of buckling modes can be determined as mentioned on page 23 for the case of one half-wave and two half-waves.

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THE BUCKLING OF A BEAM ON AN ELASTIC FOUNDATION UNDER DISTRIBUTED AXIAL LOADS

by

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AN ABSTRACT OF MASTER'S REPORT

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There are many buckling problems in Elastic Stability for which "exact" solutions are either impossible or impractical to obtain, using existing methods. With the development of high-speed digital computers, attempts have been made to solve such difficult problems by approximate techniques to obtain a better result. In this report, the buckling of a beam on an elastic foundation under distributed axial loads has been solved by the differential equation method. In order to obtain the results more easily, the governing differential equation of the buckling mode of the beam has been transformed into a difference equation, and the difference equation has been solved numerically to obtain the critical buckling loads to compare with those results obtained by energy methods by Timoshenko.