

ESSAYS IN OPTIMAL MONETARY POLICY AND
STATE-SPACE ECONOMETRICS

by

C. PATRICK SCOTT

B.S., University of South Alabama, 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Economics
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Abstract

This dissertation consists of three essays relating to asymmetric preferences in optimal monetary policy models. Optimal monetary policy models are theoretical optimal control problems that seek to identify how the monetary authority makes decisions and ultimately formulate decision rules for monetary policy actions. These models are important to policy makers because they help to define expectations of policy responses by the central bank. By identifying how researchers perceive the central bank's actions over time, the monetary authority can identify how to manage those expectations better and formulate effective policy measures.

In chapter 1, using a model of an optimizing monetary authority which has preferences that weigh inflation and unemployment, Ruge-Murcia ([2003a](#); [2004](#)) finds empirical evidence that the monetary authority has asymmetric preferences for unemployment. We extend this model to weigh inflation and output and show that the empirical evidence using these series also supports an asymmetric preference hypothesis, only in our case, preferences are asymmetric for output. We also find evidence that the monetary authority targets potential output rather than some higher output level as would be the case in an extended [Barro and Gordon \(1983\)](#) model.

Chapter 2 extends the asymmetric monetary policy problem of [Surico \(2007\)](#) by relaxing the assumption that inflation and interest rate targets are constant using a time varying parameter approach. By estimating a system of equations using iterative maximum likelihood, all of the monetary planner's structural parameters are identified. Evidence indicates that the inflation and interest rate targets are not constant over time for all models estimated. Results also indicate that the Federal Reserve does exhibit asymmetric preferences toward inflationary and output gap movements for the full data sample. The results are

robust when accounting for changing monetary policy targeting behavior in an extended model. The asymmetry for both inflation and output gaps disappears over the post-Volcker subsample, as in [Surico \(2007\)](#).

In chapter 3, [Walsh \(2003b\)](#)'s speed limit objective function is generalized to allow for asymmetry of policy response. A structural model is estimated using unobserved components to account for core inflation and measure the output gap as in [Harvey, Trimbur and Van Dijk \(2007\)](#) and [Harvey \(2011\)](#). Full sample estimates provide evidence for asymmetry in changes in inflation over time, but reject asymmetry for the traditional speed limit for the output gap. Post-Volcker subsample estimates see asymmetry disappear as in a more traditional asymmetric preferences model like [Surico \(2007\)](#).

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Approved by:

Major Professor
Steven P. Cassou

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Table of Contents

Table of Contents	viii
List of Figures	x
List of Tables	xi
Acknowledgements	xii
Dedication	xiii
1 Optimal Monetary Policy with Asymmetric Preferences for Output	1
1.1 Introduction	1
1.2 The Model	2
1.3 Empirical Results	5
1.4 Conclusion	9
2 Using Time Varying Monetary Policy Parameters to Identify Asymmetric Preferences	10
2.1 Introduction	10
2.2 Model	12
2.2.1 Optimal Monetary Policy	13
2.2.2 Forward-Looking Conditional Expectations	16
2.3 Data	18
2.4 Results	18
2.5 A Time Varying Specification	22
2.5.1 Results	24
2.6 Conclusion	25
3 Estimating the Asymmetry of Speed Limits: A Structural Model Using Unobserved Components	27
3.1 Introduction	27
3.2 Model	29
3.2.1 Rational Expectations	31
3.2.2 Optimal Monetary Policy	32
3.3 Results	34
3.4 Conclusion	39
Bibliography	44

A Chapter 1 Technical Notes	45
A.1 Solving the planner’s optimization problem	45
A.2 Estimation Description	47
A.2.1 ARIMA(2,0,2)	47
A.2.2 ARIMA (1,1,2)	49
A.3 Some notes for solving an equation above.	51
A.4 Great Moderation Period Results	53
B Chapter 2 Technical Notes	54
B.1 Deriving the Benchmark Dynamic New-Keynesian Model	54
B.1.1 Households	54
B.1.2 Firms	55
B.1.3 Equilibrium	58
B.1.4 Log-Linearization	58
B.1.5 The Benchmark Model	60
B.2 State-Space Representation	62
B.2.1 Time Varying Target Rate Model	62
B.2.2 Estimation Procedure	63
B.2.3 Time Varying Aversion Parameter Model	65
B.2.4 Estimation Procedure	67
C Chapter 3 Technical Notes	69
C.1 Deriving the Modified Model	69
C.1.1 Rational expectations	69
C.1.2 The UC updated Phillips curve	69
C.1.3 The UC updated consumer’s Euler Equation	69
C.2 State-Space Formulation - Speed Limits with Asymmetry	71
C.2.1 Estimation Algorithm	73
C.2.2 Smoothed Estimates of the State Vector	74

List of Figures

1.1	Asymmetric Policy Response	4
2.1	Inflation Rate Target Estimates	21
2.2	Interest Rate Target Estimates	21
3.1	Inflation with Estimated Trend and Cycle	36
3.2	Output with Estimated Trend and Cycle	38

List of Tables

1.1	LR tests for neglected <i>ARCH</i>	5
1.2	Maximum Likelihood Estimates - <i>ARIMA</i> (1,1,2) Model	6
1.3	Maximum Likelihood Estimates - <i>ARIMA</i> (2,0,2) Model	8
2.1	Maximum Likelihood Estimates - Time Varying Policy Targets	19
2.2	Testing Joint Restrictions - Likelihood Ratio Tests	20
2.3	Maximum Likelihood Estimates - Time Varying Aversion Parameters	23
2.4	Testing Joint Restrictions - Likelihood Ratio Tests	25
3.1	Maximum Likelihood Estimates - 1960:1 to 2012:4	34
3.2	Maximum Likelihood Estimates - 1979:4 to 2012:4	37

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Dedication

For my wife, Suzanna and daughter, Elizabeth Claire.

Chapter 1

Optimal Monetary Policy with Asymmetric Preferences for Output

1.1 Introduction

The possibility that monetary policy makers may induce an upward bias in inflation was first suggested by [Barro and Gordon \(1983\)](#). They suggested that, because the monetary policy maker is unable to make long term commitments, it is possible that instead they pursue policies which create surprise inflation. This intriguing proposition has been explored in numerous empirical studies including [Ireland \(1999\)](#), [Ruge-Murcia \(2003a; 2004\)](#) and others with mixed results. Although [Ruge-Murcia \(2003a; 2004\)](#) showed that the Barro and Gordon style inflation bias is not supported by the data, these papers developed a new theory that an inflation bias may arise from asymmetric preferences on the part of the monetary authority. In the Ruge-Murcia model, the inflation bias arises because the monetary authority takes stronger action when unemployment is above the natural rate than when it is below the natural rate.

In this paper, we develop an asymmetric preference model which focuses on an output asymmetry. Such a model is consistent with many important optimal monetary policy papers, including [Cukierman \(2002\)](#), [Nobay and Peel \(2003\)](#) and [Walsh \(2003b\)](#), which have a more theoretical emphasis. The structure of our model is similar to the one in [Ruge-Murcia \(2003a; 2004\)](#). However, it includes a slightly different trend structure to handle

the growing character of the output data.¹

We find that the monetary authority targets permanent output rather than some higher level of output which would be required in a parallel Barro and Gordon type model in which output is considered instead of unemployment. Furthermore, we find that the preferences of the monetary authority are asymmetric with stronger action taken when output is below its permanent level than when it is above. For this study, we look at two different data periods, including one of the standard periods used in both Ireland (1999) and Ruge-Murcia (2003a; 2004) and a second that extends that series up to the second quarter of 2011.

1.2 The Model

The model starts with a common formulation for the short run supply curve given by

$$Y_t = Y_t^p + \alpha(P_t - P_t^e) + \eta_t,$$

where Y_t is observed output at time t , Y_t^p is permanent or potential output at time t , P_t is the price level at time t , P_t^e is the expected price level at time t based on information at time $t - 1$ and η_t is a supply disturbance.² Adding and subtracting P_{t-1} inside the parenthesis term on the right and rearranging terms gives

$$Y_t = Y_t^p + \alpha(\pi_t - \pi_t^e) + \eta_t, \tag{1.1}$$

where $\pi_t = P_t - P_{t-1}$ and $\pi_t^e = P_t^e - P_{t-1}$.

Permanent output fluctuates over time in response to a real shock ζ_t according to the autoregressive process

$$\begin{aligned} (1 - L) [Y_t^p - (1 - \delta)t] &= \psi - (1 - \delta) [Y_{t-1}^p - (1 - \delta)(t - 1)] \\ &+ \theta(1 - L) [Y_{t-1}^p - (1 - \delta)(t - 1)] + \zeta_t, \end{aligned} \tag{1.2}$$

¹Another approach taken by Surico (2007) also uses output as part of the monetary authorities objective function, but his paper differs from our paper and the Ruge-Mucia (2003a; 2004) models in that it focusses on policy rule asymmetries.

²This supply curve can be motivated in a number of ways and standard sources for it can be found in Friedman (1968) and Lucas Jr (1977).

where $-1 < \theta < 1$, $0 < \delta \leq 1$, L is the lag operator and ζ_t is serially uncorrelated and normally distributed with mean zero and standard deviation σ_ζ . As in Ruge-Murcia (2003a; 2004) we use δ to capture different types of trend possibilities in the permanent output process. To understand these different trends, rewrite (1.2) as

$$Y_t^p - Y_{t-1}^p = \psi' + (1 - \delta)^2 t - (1 - \delta)Y_{t-1}^p + \theta(Y_{t-1}^p - Y_{t-2}^p) + \zeta_t, \quad (1.3)$$

where $\psi' = \psi + (1 - \delta)[1 - \theta - (1 - \delta)]$. This formulation shows that when $\delta = 1$, the model has no deterministic trend, $\psi' = \psi$ and there is a unit root. On the other hand, when $\delta < 1$, there is a deterministic trend and no stochastic trend.³

Actual inflation for the period is then determined as the sum of a policy variable chosen by the monetary authority denoted by i_t and a control error, ε_t , so that

$$\pi_t = i_t + \varepsilon_t, \quad (1.4)$$

where ε_t is serially uncorrelated and normally distributed with mean zero and standard deviation σ_ε . Define ξ_t to be the 3×1 vector that contains the model's structural shocks at time t . We assume that ξ_t is serially uncorrelated, normally distributed with zero mean, and (possibly) conditionally heteroscedastic:

$$\xi_t | I_{t-1} = \begin{bmatrix} \eta_t \\ \zeta_t \\ \varepsilon_t \end{bmatrix} | I_{t-1} \sim N(0, \Omega_t), \quad (1.5)$$

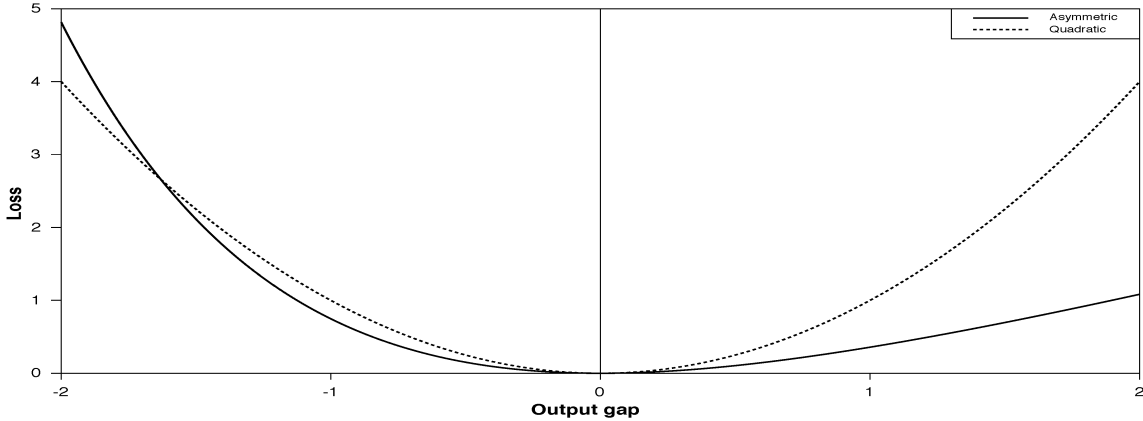
where Ω_t is a 3×3 positive-definite variance-covariance matrix. The conditional heteroscedasticity of ξ_t relaxes the more restrictive assumption of constant conditional second moments and captures temporary changes in the volatility of the structural shocks.

The policy maker selects i_t in an effort to minimize a loss function that penalizes variations of output and inflation around target values according to

$$\left(\frac{1}{2}\right) (\pi_t - \pi_t^*)^2 + \left(\frac{\phi}{\gamma^2}\right) (\exp(\gamma(Y_t^* - Y_t)) - \gamma(Y_t^* - Y_t) - 1),$$

³We empirically investigated both the integrated model, where $\delta = 1$, and a trend-stationary model where $\delta < 1$. Results for both models were similar, so only the integrated results are reported below. However, for the sake of replication, we provide some further discussion on how one could replicate our stationary model estimation results.

Figure 1.1: Asymmetric Policy Response



where $\gamma \neq 0$ and $\phi > 0$ are preference parameters, and π_t^* and Y_t^* are desired rates of inflation and output, respectively. Figure 1.1 highlights the asymmetry of the policy response to a positive and negative output gap. The symmetric (Barro-Gordon) case is shown by the dashed line while the asymmetric form is demonstrated by the solid line. The steeper slope when output is below the target (or natural) rate translates to a higher loss for the policy maker and thus a stronger policy response. The more gradual slope when output is above the target (or natural) rate yields a lower loss and a weaker policy response. In this case, since the output gap is defined as $Y_t^* - Y_t$ then γ is expected to > 0 .

As in Ireland (1999) and Ruge-Murcia (2003a), we assume π_t^* is constant and denote it by π^* . The output level targeted by the central banker is proportional to the permanent value according to

$$Y_t^* = kE_{t-1}Y_t^p, \text{ for } k \geq 1. \quad (1.6)$$

In this formulation, when $k = 1$, the authority targets permanent output, while for $k > 1$ the authority targets output beyond the permanent level. Substituting (1.1), (1.4), and (1.6) into the objective function gives

$$\min_{i_t} E_{t-1} \left\{ + \left(\frac{\phi}{\gamma^2} \right) \left(\begin{array}{c} \left(\frac{1}{2} \right) (i_t + \varepsilon_t - \pi_t^*)^2 \\ \exp(\gamma(kE_{t-1}Y_t^p - Y_t^p - \alpha(i_t + \varepsilon_t - \pi_t^e) - \eta_t)) \\ -\gamma(kE_{t-1}Y_t^p - Y_t^p - \alpha(i_t + \varepsilon_t - \pi_t^e) - \eta_t) - 1 \end{array} \right) \right\}.$$

Table 1.1: LR tests for neglected *ARCH*

Squared residuals	Sample period	No. of lags					
		1	2	3	4	5	6
Original	1960:1-1999:4	0.60	5.44 [†]	5.93	8.57 [†]	9.87 [†]	8.41
	1960:1-2011:2	1.43	7.40*	8.00*	11.29*	12.51*	10.97 [†]
Standardized	1960:1-1999:4	1.04	2.45	2.64	5.47	5.48	5.72
	1960:1-2011:2	0.49	2.59	2.68	5.75	5.83	6.08

Note: We use the convention that tests that are significant at the 10 percent level only have a † while those that are significant at the 5 percent (and 10 percent) level have an *.

1.3 Empirical Results

Solving the optimization problem and linearizing the decision rule gives the following reduced form inflation equation⁴

$$\pi_t = a + bE_{t-1}Y_t + c\sigma_{Y,t}^2 + e_t, \quad (1.7)$$

where a is a constant intercept, $b = \phi\alpha(k-1) \geq 0$, $c = \frac{\phi\alpha\gamma}{2} \gtrless 0$, and e_t is a reduced form disturbance. As in the Ruge-Murcia model, as $\gamma \rightarrow 0$ (with $k > 1$) one obtains an inflation-output version of the Barro and Gordon model. So a test of that model is, $H_0 : c = 0$. Also, when $k = 1$ the policy preferences are such that the monetary authority targets permanent output, so a test of this is, $H_0 : b = 0$.

A reduced form for the output equation is also easily derived

$$\begin{aligned} \Delta Y_t = & \psi' + (1-\delta)^2 t - (1-\delta)Y_{t-1} + \theta\Delta Y_{t-1} + \zeta_t + \eta_t + \alpha\varepsilon_t \\ & -\delta(\alpha\varepsilon_{t-1} + \eta_{t-1}) - \theta(\alpha\Delta\varepsilon_{t-1} + \Delta\eta_{t-1}). \end{aligned} \quad (1.8)$$

Equations (1.7) and (1.8) were estimated jointly using a maximum likelihood procedure. The output conditional variances were estimated first using a *GARCH*(1,1) model. Since $\sigma_{Y,t}^2$ is identified only if it is not constant, we ran some preliminary tests to see if it is time

⁴See Appendix A for a derivation of Equations (1.7) and (1.8).

Table 1.2: Maximum Likelihood Estimates - *ARIMA*(1,1,2) Model

Sample 1960:1-1999:4				
Coefficient	Model			
	Barro and Gordon	Asymmetric with		
		k free	$k \geq 1$	$k = 1$
a	3.90*	5.15	2.88*	2.88*
	(0.20)	(4.39)	(0.33)	(0.33)
b	0.0	-0.26	0.0	
	.	(0.49)	.	
c		1.29*	1.33*	1.33*
		(0.39)	(0.39)	(0.39)
log likelihood	165.29	173.40	173.28	173.28
Sample 1960:1-2011:2				
Coefficient	Model			
	Barro and Gordon	Asymmetric with		
		k free	$k \geq 1$	$k = 1$
a	3.53*	14.80*	2.68*	2.68*
	(0.17)	(3.49)	(0.27)	(0.27)
b	0.0	-1.24*	0.0	
	.	(0.38)	.	
c		0.91 [†]	1.14*	1.14*
		(0.34)	(0.36)	(0.36)
log likelihood	231.29	245.10	238.90	238.90

Note: Significance at the 5 percent level denoted by a [†] while those that are significant at the 1 percent (and 5 percent) level have an *.

varying. Table 1.1 contains the results of various neglected *ARCH* tests. The first two rows show the results using the original output series. Here the residuals from a four-lag *VAR* with a time trend were collected. These residuals were then squared and an OLS regression was run on a constant and one to six lags. The last two rows show the results using the standardized residuals from the *GARCH*(1,1) model. These test statistics have χ_q^2 distribution where q is the number of lags. These results show evidence that the original output series does have conditional heteroscedasticity, while the conditional variance series does not.

In estimating the trend-stationary model, we set $\delta = 0.991$, which is the value of the coefficient on the time trend term in a simple regression of output on a constant and a time trend. For the nonstationary model we set $\delta = 1$. The first (second) panel of Table 1.2 shows the results of the maximum likelihood estimation of the model using the sample period of 1960:1-1999:4 (1960:1-2011:2). The first sample period is one of the data periods used in Ruge-Murcia (2003a; 2004) and it is similar to the sample period of 1960:1-1997:2 considered in Ireland (1999). The table is organized so that the first column provides estimates of an output and inflation version of the Barro and Gordon model, while the next three columns provide estimates of an asymmetric preference formulation. The first asymmetric preference formulation allows k to vary freely and to possibly have negative values, contrary to the model restriction, while the second asymmetric preference formulation allows k to vary freely in a range greater than 1 and the third case constrains k to equal 1.

Focusing on the first panel, Table 1.2 shows that whenever k is allowed to vary freely, b takes on a negative, but insignificant, value as Ruge-Murcia (2003a) found in one of his estimated specifications using unemployment instead of output. Not surprisingly, constraining k to its theoretical plausible region results in it always moving to its lower bound. Furthermore, using the likelihood ratio test, the null hypothesis that b equals zero, cannot be rejected. This result is similar to results found using the inflation-unemployment model by Ruge-Murcia (2003a; 2004) and implies that policy makers target permanent income rather than some higher level of output. Moreover, Table 1.2 also allows one to test for the presence of asymmetric preferences over output by testing whether the coefficient of the conditional variance of output, c , is significant. Both the t statistic and the likelihood ratio statistic (the latter takes the value 15.98 using the theoretically consistent model with $k \geq 1$ as the unrestricted model) reject this null at any standard significance level.

The second panel shows the estimation results obtained running the same model, but considering data up through 2011:2. Estimation results are fairly robust across the samples, although b does become significantly negative in this longer sample. Overall, in both sam-

Table 1.3: Maximum Likelihood Estimates - *ARIMA*(2,0,2) Model

Sample 1960:1-1999:4				
Coefficient	Model			
	Barro and Gordon	Asymmetric with		
		k free	$k \geq 1$	$k = 1$
a	3.90* (0.20)	5.03 (4.47)	2.86* (0.32)	2.86* (0.32)
b	0.0 .	-0.25 (0.50)	0.0 .	
c		1.30* (0.39)	1.34* (0.39)	1.34* (0.39)
log likelihood	165.67	173.72	173.61	173.61
Sample 1960:1-2011:2				
Coefficient	Model			
	Barro and Gordon	Asymmetric with		
		k free	$k \geq 1$	$k = 1$
a	3.53* (0.17)	13.95* (3.48)	2.70* (0.27)	2.70* (0.27)
b	0.0 .	-1.26* (0.38)	0.0 .	
c		0.88 [†] (0.35)	1.12* (0.36)	1.12* (0.36)
log likelihood	230.26	243.99	237.58	237.59

Note: Significance at the 5 percent level denoted by a [†] while those that are significant at the 1 percent (and 5 percent) level have an *.

ples, we find robust evidence of asymmetric preferences for output on the part of an optimal monetary planner.

Table 1.3 reports estimation results for the *ARIMA*(2, 0, 2) formulation for the same two sample periods. The estimation results are almost identical under the two permanent output specifications. Again, when k is allowed to vary freely beyond its theoretical constraint b assumes a negative value. When k is limited, b is pushed to a zero lower bound implying the central bank targets natural output. In all cases we reject the hypothesis of $c = 0$ implying that the policy maker exhibits an asymmetry of preference toward output.

1.4 Conclusion

In this paper we develop an optimal monetary policy model that estimates output asymmetry for the central bank. The model is appropriately parameterized to account for the trending nature of the output data. A reduced form model is derived that depends on the conditional variance of output to identify asymmetry of monetary policy. Results indicate that output is conditionally heteroscedastic, thus asymmetry is identified. Model estimates indicate that the Federal Reserve does exhibit asymmetric preferences for output. Additionally, parameter estimates indicate that the Fed targets natural output as opposed to some nominal output level. Results are consistent across subsamples and different trend specifications.

Chapter 2

Using Time Varying Monetary Policy Parameters to Identify Asymmetric Preferences

2.1 Introduction

The optimal monetary policy literature has long focused its attention on the deviations of macroeconomic variables from their natural rates and policy variables from their target rates to analyze policy behavior. This follows a lengthy tradition in the literature first analyzed by [Kydland and Prescott \(1977\)](#) and formalized by [Barro and Gordon \(1983\)](#). One important consequence of both studies is that policy responses to positive deviations in unemployment from its natural rate are met with the same urgency as negative deviations by the central banker. In order to address this concern, some studies in the literature have turned to asymmetric specifications to model these responses in the policy maker's loss function. This allows positive and negative deviations to be weighted differently in terms of policy response by the monetary authority.

Asymmetry of optimal monetary policy has been well studied in the recent literature. [Ruge-Murcia \(2004\)](#) models an asymmetric specification of [Barro and Gordon \(1983\)](#) to show that the much studied inflation bias could arise, not by an overly ambitious central bank, but through the asymmetry of policy responses. [Surico \(2007\)](#) shows that the asymmetry of policy response appears to go away in the post-Volcker era of the Fed and thus the average

inflation bias decreases and is less important during the Great Moderation. Tangentially related to this, [Doyle and Falk \(2010\)](#) offer a meaningful critique of the asymmetric policy rule and its ability to explain the general rise and subsequent fall of inflation within the U.S. among other OECD countries.

A significant portion of the recent empirical literature impose the simplifying assumption that the policy maker's inflation target is constant over time. Closely related to this is a growing body of literature that suggests monetary policy has changed for the Federal Reserve, particularly after the appointment of Paul Volcker as Fed Chairman. The literature investigating shifting monetary policy for the U.S. is extensive and divided¹. It is not the aim of this paper to argue the evidence for or against switching monetary policy models. However, the constancy of inflation targets, interest rate targets, and policy targeting parameters may be important to testing the asymmetric preferences hypothesis.

[Ruge-Murcia \(2003a; 2003b; 2004\)](#), [Surico \(2007\)](#), and [Doyle and Falk \(2010\)](#) are not able to fully identify all of the policy maker's structural or deep parameters. [Surico \(2007\)](#) comes close by creatively identifying the asymmetry parameters. Part of this shortcoming in the literature is a direct result of a single equation estimation strategy. [Doyle and Falk \(2010\)](#) note that if we are to take the model seriously, then a single equation estimation of the linearized first order condition should suffice to capture the relationship among the structural equations of the model. The problem with this approach is that reduced form parameters are estimated as convolutions of structural parameters and time invariant monetary policy targets. So long as the only interest is in estimating and drawing inference from a reduced form parameter, they are correct. But if we are to take seriously the testing of the nonlinear hypothesis, then a multiple equation framework is needed, where all of the policy maker's structural parameters are identified.

This paper contributes to the optimal monetary policy literature by extending [Surico \(2007\)](#) to include time varying monetary policy parameters. Two models are estimated.

¹For a brief introduction to this literature see [Clarida, Galí and Gertler \(2000\)](#), [Kim and Nelson \(2006\)](#), [Lubik and Schorfheide \(2004\)](#), [Primiceri \(2005\)](#), and [Sims and Zha \(2006\)](#)

The first models time varying inflation and interest rate targets for the monetary authority. This extension, combined with the system estimation of all of the models equations, allows for identification and estimation of all deep parameters in the model. The second model relaxes the imposed assumption that policy behavior is constant over time. Evidence from these two exercises indicates a rejection of the assumption that inflation targets and interest rate targets are constant. Evidence from the second model implies that monetary policy targeting behavior is also variable. The full sample estimates from both models indicate evidence of asymmetric preferences toward inflationary and output gaps. The post-Volcker subsample estimates show that the asymmetry is no longer significant, as is the case in [Surico \(2007\)](#).

The remainder of the chapter is outlined as follows. Section 2 describes the model setup, the asymmetric preferences literature, and derives the baseline model. Section 3 describes the data and estimation procedure used. Section 4 discusses estimation results. Section 5 describes the time varying aversion parameter extension of the model and presents results. Section 6 concludes.

2.2 Model

We start with a textbook log-linearized New-Keynesian model as discussed in [Gali and Gertler \(1999\)](#), [Walsh \(2003a\)](#), [Woodford \(2003\)](#), and [Galí \(2007\)](#). This model is a familiar benchmark in the monetary policy literature and is a suitable baseline framework for models with staggered pricing features.

$$x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) \quad (2.1)$$

Equation (2.1) is the consumer's Euler equation. The output gap is denoted by x_t , more specifically $x_t = y_t - y_t^n$ where y_t is output and y_t^n is natural output. The intertemporal elasticity of substitution is given by σ^{-1} , and i_t is the interest rate. The forward looking expectations of tomorrow's output gap and inflation are conditional on the information

set, I_t , that includes all past observations of the models variables. Equation (2.2) is a New-Keynesian Phillips curve where β is the discount factor from the consumer's utility problem and ψ is a convolution of structural problem parameters that include a Calvo (1983) staggered pricing mechanism.²

$$\pi_t = \beta E_t \pi_{t+1} + \psi x_t \quad (2.2)$$

2.2.1 Optimal Monetary Policy

Monetary policy is determined by minimizing the expectational sum of future loss functions, (2.3). Loss in every period follows an asymmetric form described in Equation (2.4). Positive and negative deviations in inflation from a time varying target rate are asymmetrically weighted as are positive and negative output gaps.

$$\text{Min}_{\{i_t\}} E_{t-1} \sum_{k=0}^{\infty} l_{t+k} \quad (2.3)$$

$$l_t = \phi \left\{ \frac{\exp[\alpha(\pi_t - \pi_t^*)] - \alpha(\pi_t - \pi_t^*) - 1}{\alpha^2} \right\} + \lambda \left\{ \frac{\exp[\gamma(y_t - y_t^n)] - \gamma(y_t - y_t^n) - 1}{\gamma^2} \right\} + \left\{ \frac{(i_t - i_t^*)^2}{2} \right\} \quad (2.4)$$

This linear specification is well documented in the monetary policy literature. The asymmetric functional form was first discussed by Varian (1975) and Zellner (1986). It was later applied to the optimal monetary policy problem by Cukierman and Gerlach (2003), Nobay and Peel (2003), Ruge-Murcia (2003a; 2003b; 2004), and Surico (2003; 2007).

The parameters α and γ represent the degree of asymmetric policy response of the central banker. As α and γ approach 0, the central banker's response to positive and negative deviations in inflation and the output gap become more symmetric, like the type of symmetric policy function modeled in a Barro and Gordon (1983) type model. Defining the output gap in this way means that one would reasonably expect γ to be < 0 such that

²See Appendix (B.1) for a detailed derivation of Equations (2.1) and (2.2).

the policy maker responds more forcefully when output is below potential levels. If α is negative in this case, then the policy maker responds with greater urgency when prices do not grow as fast as the target rate. Some of the previous studies mentioned above have focused on the testable restriction that $\alpha = \gamma = 0$ with mixed results. [Surico \(2007\)](#) finds evidence that γ is statistically different from zero only for the subsample corresponding to the Great Inflation. Inflation asymmetry is rejected for all subsamples in his estimations.

The parameters ϕ and λ in Equation (2.4) are the monetary policy aversion parameters. They communicate the degree to which the central banker is averse to deviations in the inflation gap and the output gap. Regardless of whether deviations in inflation and output are asymmetrically weighted, the aversion parameters communicate targeting behavior of the policy maker. If targeting behavior changes over time, one would expect shifts in the relative sizes of the aversion parameters. [Davig and Leeper \(2007\)](#) model regime switching aversion parameters in a symmetric long-run Taylor rule specification for this model. They show that determinacy for this model is implied by parameter estimates greater than unity for both ϕ and λ . Additionally, a larger relative value implies evidence of policy targeting behavior.

There are two noticeable differences in the monetary policy specification from the model examined by [Surico \(2007\)](#). First, it differs in that the asymmetric deviations in inflation from the target inflation rate are not normalized to one. Second, the inflation and interest rate targets are modeled as time varying. The reason for the first departure is relatively self-evident. If inflation gap variation is normalized to one, then one cannot estimate an aversion parameter (ϕ) for it. The loss function is thus trivially normalized to interest rate deviations. The second departure incorporates findings in the monetary policy literature by [Schorfheide \(2005\)](#) and [Ireland \(2007\)](#) that find the inflation target for the Federal Reserve is time varying. As mentioned before, numerous monetary policy models make the simplifying assumption that the inflation target is constant. The period loss function in (2.4) highlights the trade off between inflationary gap targeting and output gap targeting. As the

inflation rate moves over time, the desire to close that gap asserts itself in the policy makers preferences. If for instance, inflation is considerably different from zero then the central bank feels more pressure to adjust it's target. The inflation rate target, π_t^* is assumed to evolve according to Equation (2.5).

$$\pi_t^* = \mu^{\pi^*} + \rho_\pi \pi_{t-1}^* + u_t^\pi \quad (2.5)$$

Here μ^{π^*} is the unconditional mean of π_t^* and $0 \leq \rho_\pi < 1$. The error process is assumed $u_t^\pi \sim N(0, \sigma^2)$. The functional form of this unobserved evolution equation is chosen because it implies a testable hypothesis of the assumption that the inflation rate target, π_t^* is constant. If the coefficient on the autoregressive component is statistically zero then $\pi_t^* = \mu^{\pi^*} + u_t^\pi$. While this does provide a testable hypothesis for constancy, the presence of the random variable, u_t^π means that the time varying model does not nest its constant parameter counterpart. More on this below.

Given that the interest rate is the primary policy tool³, then a time varying inflation target implies that the interest rate target is time varying also. This is modeled analogously in Equation (2.6).

$$i_t^* = \mu^{i^*} + \rho_i i_{t-1}^* + u_t^i \quad (2.6)$$

Again, μ^{i^*} is the unconditional mean of i_t^* and $0 \leq \rho_i < 1$. The error process is assumed to be such that $u_t^i \sim N(0, \sigma^2)$. Statistical significance of ρ_i implies a test of constancy for i_t^* . In order to test the joint restriction that $\rho_\pi = \rho_i = 0$, we must impose it in a separate estimation to use a likelihood ratio test in order to have a nested structure.

Since there are no endogenous variables in this system, discretionary optimal policy is determined by solving the minimization problem which reduces to a single period problem in the form of Equation (2.7).

$$\begin{aligned} \text{Min}_{\{i_t\}} \quad & E_{t-1} \left\{ \phi \frac{\exp[\alpha(\pi_t - \pi_t^*)] - \alpha(\pi_t - \pi_t^*) - 1}{\alpha^2} \right\} \\ & + E_{t-1} \left\{ \lambda \frac{\exp[\gamma(y_t - y_t^n)] - \gamma(y_t - y_t^n) - 1}{\gamma^2} \right\} + \left\{ \frac{(i_t - i_t^*)^2}{2} \right\} \end{aligned} \quad (2.7)$$

³By this I mean the variable being chosen in the minimization problem.

Minimizing (2.7) by choosing i_t subject to Equations (2.1) and (2.2) yields Equation (2.8).

$$i_t = \frac{\psi}{\sigma} E_{t-1} \left\{ \phi \frac{\exp[\alpha(\pi_t - \pi_t^*)] - 1}{\alpha} \right\} + \frac{1}{\sigma} E_{t-1} \left\{ \lambda \frac{\exp[\gamma(y_t - y_t^n)] - 1}{\gamma} \right\} + i_t^* \quad (2.8)$$

This nonlinear first order condition (FOC) is problematic to estimate in its current form. The statistical significance of α and γ implies a test of a linear null hypothesis versus a nonlinear alternative hypothesis. In keeping with the literature, this FOC is linearized via a second-order Taylor series expansion, Equation (2.9).

$$i_t = \frac{\psi\phi}{\sigma} \left\{ (\pi_t - \pi_t^*) + \alpha \frac{(\pi_t - \pi_t^*)^2}{2} \right\} + \frac{\lambda}{\sigma} \left\{ (y_t - y_t^n) + \gamma \frac{(y_t - y_t^n)^2}{2} \right\} + i_t^* + \zeta i_{t-1} + \delta_t \quad (2.9)$$

A lag of the interest rate is included in the linearized FOC per [Surico \(2007\)](#). The identification of ψ and σ through estimation of Equations (2.1) and (2.2), combined with the time varying inflation and interest rate targets, provides identification of the remaining deep parameters for the central banker. The inflationary gap and the output gap are still identified even if we fail to reject the null hypothesis of $\alpha = \gamma = 0$. More to the point, α now measures asymmetry of the inflationary gap, rather than just asymmetry of inflation as is the case in both [Surico \(2007\)](#) and [Doyle and Falk \(2010\)](#). The expectations are substituted with realized values. The reduced form error term, δ_t captures the rational expectations errors.

2.2.2 Forward-Looking Conditional Expectations

Equations (2.1) and (2.2) contain forward-looking conditional expectation terms. The empirical literature on the estimation of these components is extensive with little consensus methodologically. [Nason and Smith \(2008\)](#) discuss multiple empirical techniques for the estimation of forward-looking Phillips curves. Their method is to choose instruments to capture the expectation forming process in a first stage type regression. This approach is applied here.

To see this more clearly, denote the conditional expectations of future inflation and output gap discussed above as $E_t[\pi_{t+1}|I_t]$ and $E_t[x_{t+1}|I_t]$ respectively. The econometric forecast of these expectations can be denoted by $E_t[\pi_{t+1}|z_t]$ and $E_t[x_{t+1}|z_t]$ respectively.

Here z_t is the set of instruments used to make predictions about π_{t+1} and x_{t+1} . The linear regression used to calculate this forecast is given by

$$\begin{aligned}\pi_{t+1} &= \hat{b}_1 z_t + e_{1t+1} \\ x_{t+1} &= \hat{b}_2 z_t + e_{2t+1}\end{aligned}$$

Notice that the error process e_{it+1} for $i = 1, 2$ is uncorrelated with the instrument set z_t by design. The predictions are then given by

$$\begin{aligned}E_t [\pi_{t+1}|z_t] &= \hat{\pi}_{t+1} = \hat{b}_1 z_t \\ E_t [x_{t+1}|z_t] &= \hat{x}_{t+1} = \hat{b}_2 z_t\end{aligned}$$

Applying the law of rational expectations to this implies that the expectations conditional on the information set is equal to our optimal forecast given our instruments plus some error, more formally, Equations (2.10) and (2.11).

$$E_t [\pi_{t+1}|I_t] = E_t [\pi_{t+1}|z_t] + \eta_{1t} \tag{2.10}$$

$$E_t [x_{t+1}|I_t] = E_t [x_{t+1}|z_t] + \eta_{2t} \tag{2.11}$$

As discussed in [Nason and Smith \(2008\)](#), the rational expectations errors are $\sim N(0, \sigma^2)$, and $E[\eta_{it+\tau} z_{t+s}] = 0$ for $i = 1, 2$ for all τ and s . Rational expectations imposes that nothing can be learned from the error process over time to improve our expectations. Equations (2.10) and (2.11) can be substituted into (2.1) and (2.2) to put them into a more estimable form, Equations (2.12) and (2.13) respectively.

$$x_t = [E_t x_{t+1}|z_t] - \sigma^{-1} (i_t - [E_t \pi_{t+1}|z_t]) + \varepsilon_t^d \tag{2.12}$$

$$\pi_t = \beta [E_t \pi_{t+1}|z_t] + \psi x_t + \varepsilon_t^s \tag{2.13}$$

Here ε_t^d and ε_t^s are reduced form errors that are functions of rational expectations errors. For the sake of this estimation these rational expectations errors are not individually identified. Given appropriate restrictions placed on their respective law of motion it is possible to identify them, but imposing these additional assumptions does not aid in parameter identification which is of primary interest here.

2.3 Data

For this estimation quarterly data is collected from the Federal Reserve Economic Database (FRED) and the Congressional Budget Office (CBO). The output gap series is constructed as in [Surico \(2007\)](#) as the difference between logged GDP from the FRED and logged potential output from the CBO estimates. Two inflation series are constructed by calculating the logged difference of the implicit GDP deflator and personal consumption expenditure (PCE) series from the FRED. The interest rate is the effective federal funds rate. The full quarterly data sample is from 1960:1 to 2012:2. Additionally, estimates are also presented for the post Paul Volcker period, 1982:4 to 2012:2⁴.

The instrument vector, z_t contains lagged values of the constructed output gap series, the corresponding inflation rate, and the effective federal funds rate. The initial regression forecast is constructed using lagged variables in order to account for the errors-in-variables problem that is common in the empirical rational expectations literature.

Equations (2.5), (2.6), (2.9), (2.12) and (2.13) comprise the system of equations to be estimated. These equations are arranged in state-space form and jointly estimated by iterative maximum likelihood using the Kalman filter. Parameter values are chosen by minimizing the sum of the log likelihood function. Standard errors are calculated from the hessian matrix using the optimally estimated projection matrix as described in ([Hamilton, 1994](#), Ch. 13)⁵.

2.4 Results

Table 2.1 reports the maximum likelihood estimates of the model parameters as well as standard errors over the two sample periods for both inflation rates. Initial attempts to estimate

⁴[Surico \(2007\)](#) provides a subsample estimation for 1960:1 to 1979:3. This subsample of just under 80 observations and does not provide adequate variation in the data to produce trustworthy results. Nevertheless, these results are available from the author upon request.

⁵For details on the formulation of the state-space model and the estimation algorithm please see Appendices (B.2.1) and (B.2.2).

Table 2.1: Maximum Likelihood Estimates - Time Varying Policy Targets

Parameter	GDP Deflator Inflation		PCE Inflation	
	1960:1 - 2012:2	1982:4 - 2012:2	1960:1 - 2012:2	1982:4 - 2012:2
α	-0.382 [†] (0.113)	-0.125 (0.798)	-0.288 (0.182)	-0.199 (0.108)
γ	-0.715 [†] (0.089)	-0.307 (0.201)	-0.104 [†] (0.027)	-0.273 (0.238)
ϕ	3.061 [†] (0.104)	4.755 [†] (0.352)	2.966 [†] (0.235)	4.659 [†] (0.272)
λ	4.977 [†] (0.094)	6.259 [†] (0.405)	7.295 [†] (0.031)	6.890 [†] (0.062)
β	0.886 [†] (0.002)	0.921 [†] (0.003)	0.864 [†] (0.002)	0.931 [†] (0.005)
ψ	0.009 (0.074)	0.017 (0.078)	0.040 (0.062)	0.027 [†] (0.001)
ζ	0.882 [†] (0.001)	0.867 [†] (0.005)	0.861 [†] (0.005)	0.868 [†] (0.004)
ρ_i	0.953 [†] (0.002)	0.682 [†] (0.010)	0.779 [†] (0.008)	0.681 [†] (0.008)
ρ_π	0.898 [†] (0.002)	0.590 [†] (0.008)	0.818 [†] (0.002)	0.520 [†] (0.009)
Log likelihood	-282.599	-242.871	-377.867	-345.514

[†] represents significance at the 1 percent level.

this model produced unreasonable results for the intertemporal elasticity of substitution, σ . This parameter was calibrated according to [Davig and Leeper \(2007\)](#) at 2.84. Estimates for the autoregressive components for the inflation and interest rate targets are within the unit circle and highly statistically significant for both sample periods. These results do imply a rejection that these target rates are constant for the Federal Reserve, but of more importance is the joint test that they are both equal to zero. The asymmetry parameters, α and γ are both negative and statistically significant for the full sample period for GDP deflator inflation. The full sample point estimate for α using PCE inflation is reasonable yet it falls just short of significance at the 90 percent confidence level. Generally, these results imply that the Fed reacts with more urgency to negative output fluctuations and negative inflationary gaps. Both asymmetries appear to go away in the post-Volcker subsample. This result along with the sign and size of the estimates are consistent with [Surico \(2007\)](#)'s GMM

Table 2.2: Testing Joint Restrictions - Likelihood Ratio Tests

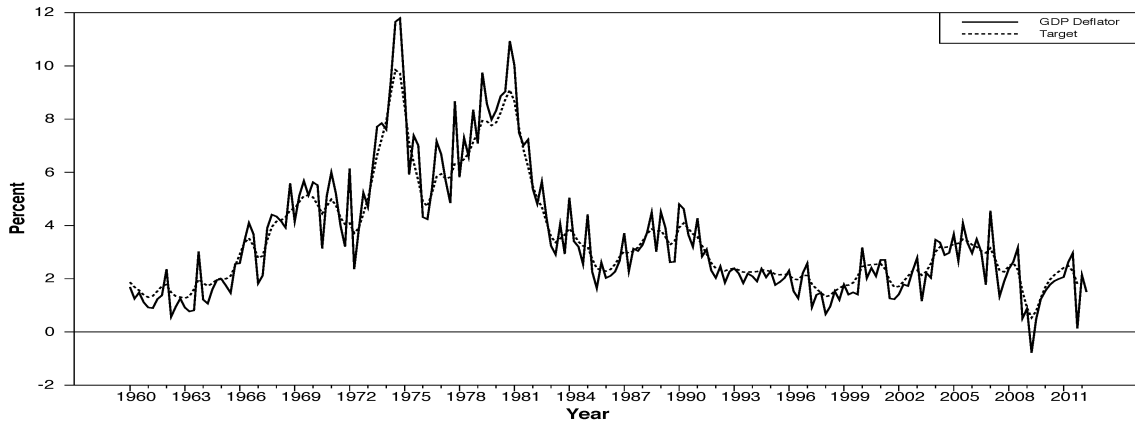
Restrictions	GDP Deflator Inflation		PCE Inflation	
	1960:1 to 2012:2	1982:4 to 2012:2	1960:1 to 2012:2	1982:4 to 2012:2
	Log Likelihood LR Test Statistic	Log Likelihood LR Test Statistic	Log Likelihood LR Test Statistic	Log Likelihood LR Test Statistic
$\rho_\pi = \rho_i = 0$	-340.738 116.278	-286.788 87.834	-447.659 139.584	-378.885 66.742
$\alpha = \gamma = 0$	-328.159 91.120	—	-422.211 88.688	—
L^r = restricted log likelihood value				
L^u = unrestricted log likelihood value				
$LR = 2(L^u - L^r)$				

estimates. The aversion parameters, ϕ and λ are statistically significant and greater than one in absolute value. The size of λ relative to ϕ suggests the Fed is more likely to target output gap fluctuations than inflationary gap fluctuations, a result that has been widely shown in the monetary policy literature. The discount parameter β is always close to one and significant. The coefficient on the interest rate lag, ζ is also within the unit circle and significant for all estimations. Again, this is consistent with previous estimates.

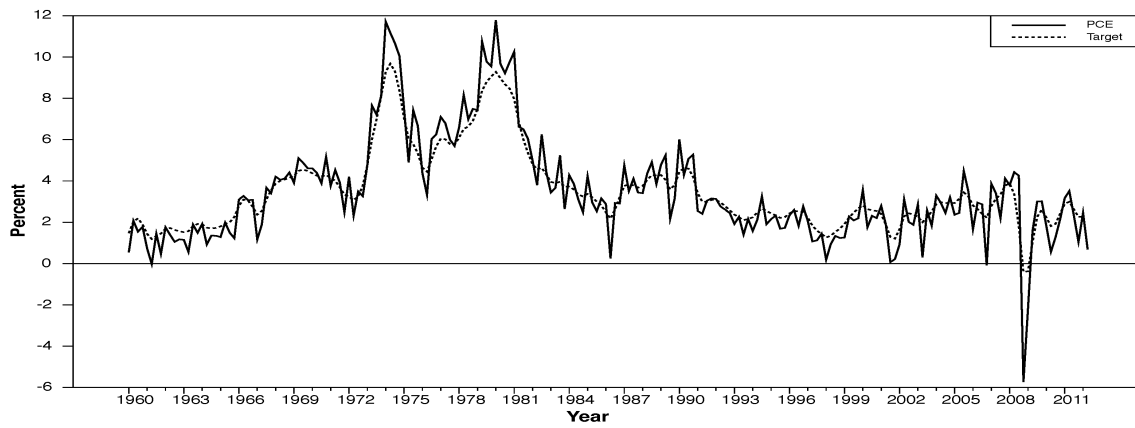
Table 2.2 shows likelihood ratio test statistics imposing two joint restrictions on the above model. The first joint restriction imposes constancy on the inflation and interest rate targets for both sample periods and both inflation series. The second joint restriction eliminates the asymmetry of policy response. This restriction is tested for the full sample only since both types of asymmetry are individually insignificant for the subsample estimations. Both tests are distributed χ^2 with two degrees of freedom. According to (Hamilton, 1994, pp. 754) the 0.001 critical value associated with this test is 13.8. This indicates a rejection of the imposed joint restrictions in all cases and provides further evidence that inflation and interest rate targets are not well represented by constant parameters. Similarly, imposing the joint restrictions of no asymmetry is also rejected for both full sample estimations.

Since this model is estimated using a Kalman filter, the estimates of our time varying conditional inflation and interest rate targets can be recursively extracted. Note, these esti-

Figure 2.1: Inflation Rate Target Estimates

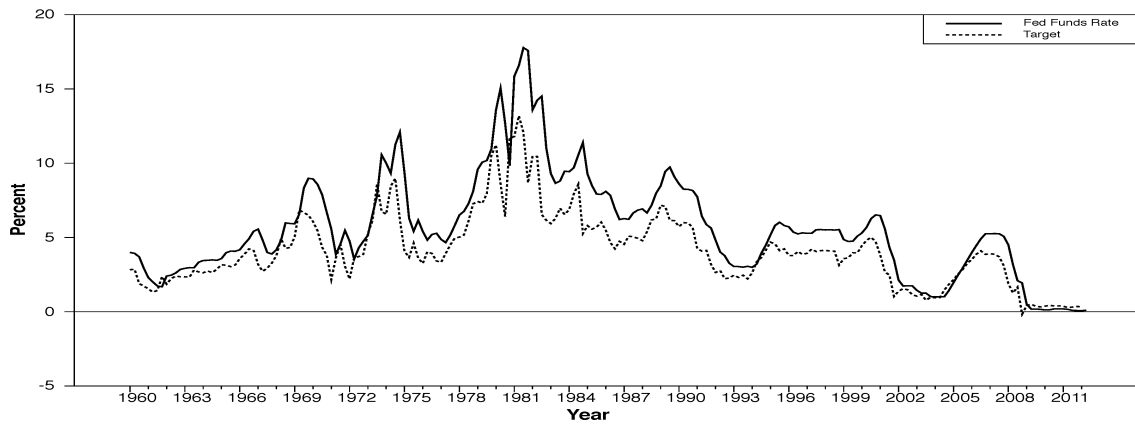


(a) GDP Deflator Inflation with Inflation Target



(b) PCE Inflation with Inflation Target

Figure 2.2: Interest Rate Target Estimates



mates are conditional on the optimal monetary policy solution characterized above. Figure 2.1 shows inflation data for both implicit GDP deflator (2.1a) and PCE inflation (2.1b) plotted in solid lines over the full sample period. The estimated conditional inflation target for each series is superimposed over the top in the dashed line. These conditional inflation targets are similar visually to those found in Ireland (2007) and embody the idea that the Fed through its preferences targets a lower rate of inflation when inflation is relatively high and a higher level of inflation when inflation is relatively low. Thus, the inflation target for both series has a lower variance than actual inflation.⁶ Similarly, Figure 2.2 shows the effective fed funds rate plotted in solid with the estimated conditional interest rate target series in the dashed line. The conditional interest rate target appears to follow the effective rate much more closely in this plot which is to be expected given the degree of control that the Fed exerts over the effective fed funds rate. Since the effective rate is the weighted average of the interest rates that banks actually charge one another to borrow funds, the difference between these two series can be interpreted as a measure of systemic risk in the banking system.

2.5 A Time Varying Specification

Implicit in the above model is the assumption that monetary policy targeting behavior does not shift over the sample period. This may be an assumption that matters to the testing of policy targets and asymmetric preferences for monetary policy. To test the robustness of the above results an alternative model is formulated that relaxes this assumption. This type of extension is briefly considered in the concluding remarks of Ruge-Murcia (2004). In this case, the aversion parameters, ϕ_t and λ_t are modeled to vary over time. Equation (2.14) is the recast FOC for the optimal monetary policy rule with the time varying aversion

⁶Summary statistics for both the inflation target series and interest rate target series are not reported here for the sake of brevity but are available from the author upon request.

Table 2.3: Maximum Likelihood Estimates - Time Varying Aversion Parameters

Parameters	GDP Deflator Inflation 1960:1 to 2012:2	PCE Inflation 1960:1 to 2012:2
α	-0.599 [†] (0.003)	-0.362 [†] (0.012)
γ	-0.645 [†] (0.014)	-0.376 [†] (0.006)
β	0.994 (0.672)	0.9144 (0.744)
ψ	0.099 [†] (0.012)	0.198 [†] (0.008)
ζ	0.968 [†] (0.017)	0.779 [†] (0.014)
ρ_i	0.580 [†] (0.025)	0.981 [†] (0.006)
ρ_π	0.188 [†] (0.017)	0.676 [†] (0.005)
ρ_ϕ	0.697 [†] (0.004)	0.442 [†] (0.013)
ρ_λ	0.837 [†] (0.030)	0.997 [†] (0.015)
log likelihood	-585.185	-540.704

[†] represents significance at the 1 percent level.

parameters.

$$i_t = \phi_t \frac{\psi}{\sigma} \left\{ (\pi_t - \pi_t^*) + \alpha \frac{(\pi_t - \pi_t^*)^2}{2} \right\} + \lambda_t \frac{1}{\sigma} \left\{ (y_t - y_t^n) + \gamma \frac{(y_t - y_t^n)^2}{2} \right\} + i_t^* + \zeta i_{t-1} + \delta_t \quad (2.14)$$

In this alternative specification π_t^* , i_t^* , ϕ_t , and λ_t are unobserved components in the Kalman filter recursion and assumed to follow Equations (2.5), (2.6), (2.15), and (2.16), respectively.

$$\phi_t = \mu^\phi + \rho_\phi \phi_{t-1} + u_t^\phi \quad (2.15)$$

$$\lambda_t = \mu^\lambda + \rho_\lambda \lambda_{t-1} + u_t^\lambda \quad (2.16)$$

The aversion parameters are assumed to follow a stable autoregressive process implying $0 \leq \rho_\phi < 1$ and $0 \leq \rho_\lambda < 1$. The error processes are assumed to follow a multivariate Gaussian distribution. Analogously to the varying inflation and interest rate targets, the

statistically test $\rho_\phi = \rho_\lambda = 0$ implies that monetary policy targeting behavior does not shift over time. The presence of the random variables u_t^ϕ and u_t^λ has the unfortunate consequence that this model does not perfectly nest the previous model above. The individual statistical test for constancy over time is valid, but a lower log likelihood is not expected from this model. Like before, in order to test the joint restrictions, we must impose them in a separate estimation and use a likelihood ratio test. The new state space system is comprised of Equations (2.5), (2.6), (2.12), (2.13), (2.14), (2.15), and (2.16)⁷.

2.5.1 Results

Table 2.3 reports parameter values and standard errors of the maximum likelihood estimation over the full sample set for both measures of inflation. A subsample analysis is not needed in this case because the time varying aversion parameters capture any shifts in monetary policy targeting behavior that is intended to be isolated through subsample estimation. The autoregressive estimates for the unobserved processes are all statistically significant implying that individually we reject the null hypothesis of constancy for each. The estimate for ρ_π using GDP deflator inflation decreases somewhat when we account for shifting policy behavior, but it is still highly significant. The asymmetry parameters are all significant and still within expected ranges. The point estimate for the conditional expectations of inflation, β is inline with the estimates from Table 2.1, however it loses significance here. This is not uncommon in the empirical estimations of the New-Keynesian Phillips curve. The interest rate lag coefficient is still persistent and significant.

Table 2.4 imposes the same set of joint restrictions as in Table 2.2 except that it also includes another joint restriction. This new restriction is a joint test of constancy for the aversion parameters, $\rho_\phi = \rho_\lambda = 0$. This joint test serves as test of consistency of the individual hypothesis tests. As before the test statistics are distributed χ^2 with two degrees of freedom, and the 0.001 critical values is 13.8. The joint restriction of constant inflation

⁷For details on the formulation of the state-space model and estimation algorithm please see Appendices (B.2.3) and (B.2.4).

Table 2.4: Testing Joint Restrictions - Likelihood Ratio Tests

Restrictions	GDP Deflator Inflation	PCE Inflation
	1960:1 to 2012:2	1960:1 to 2012:2
	Log Likelihood LR Test Statistic	Log Likelihood LR Test Statistic
$\rho_\pi = \rho_i = 0$	-615.386 60.402	-575.614 69.820
$\alpha = \gamma = 0$	-608.652 46.934	-586.877 92.346
$\rho_\phi = \rho_\lambda = 0$	-617.140 63.910	-592.287 103.166
L^r = restricted log likelihood value		
L^u = unrestricted log likelihood value		
$LR = 2(L^u - L^r)$		

and interest rate targets, symmetric policy response, and constant aversion parameters are all rejected for both model estimations. These results indicate that the joint test of constancy for inflation and interest rate targets is robust to changing monetary policy targeting behavior.

2.6 Conclusion

This paper relaxes a commonly imposed assumption that target inflation rates and target interest rates are constant over time. In the absence of data for these processes, our best first guess is that they follow an AR(1) law of motion. [Surico \(2007\)](#)'s model is extended to incorporate this dynamic structure. Jointly estimating a system of equations for this extended model conveniently allows for identification of all of the deep parameters for the model. The model is estimated using iterative maximum likelihood. Results indicate a rejection that inflation and interest rate targets can be closely approximated by constants. Additionally, monetary policy exhibits asymmetric responses towards inflationary and output gaps over the full sample period. This asymmetry disappears over the post-Volcker subsample, like in [Surico \(2007\)](#). A more robust model that allows for changing monetary policy targeting

behavior, through time-varying aversion parameters supports these findings and provides evidence that monetary policy aversion parameters are not constant either.

Chapter 3

Estimating the Asymmetry of Speed Limits: A Structural Model Using Unobserved Components

3.1 Introduction

The benchmark New-Keynesian model, pioneered by [Yun \(1996\)](#), [Goodfriend and King \(1997\)](#), [Gali and Gertler \(1999\)](#), [Walsh \(2003a\)](#), and [Woodford \(2003\)](#), among others, has long been a useful tool for modeling monetary policy. This purely forward-looking framework has led to much discussion regarding the need to exogenously impose additional persistence on the log-linearized equations that summarize the evolution of the theoretical economy ([Gali and Gertler \(1999\)](#), [Estrella and Fuhrer \(2002\)](#) and [Walsh \(2003a\)](#)). As discussed in [Walsh \(2003a; 2003b\)](#), the inclusions of lagged dependent variables in order to build persistence into this model provides better fit and statistical significance of the estimated parameters. This has helped lend credibility to the deviation in the model that is derived from first order conditions.

Of particular relevance to this discussion are monetary policy rules that incorporate persistence in the policy maker's optimization problem. [Walsh \(2003b\)](#) proposes a relatively different objective for the monetary authority suggesting that it is bound by the speed at

which it pushes the economy. More specifically Walsh examines

$$L_t = E_t \sum_{i=0}^{\infty} \beta^i [\pi_{t+i}^2 + \lambda (x_{t+i} - x_{t+i-1})^2],$$

where the policy maker cares about the social loss incurred by inflation and the difference of the output gap, x_t over time. The policy maker seeks to target the output gap in every period in order to effect the rate of change over time. Put another way, the policy maker wishes to keep the change in the output gap small, which is interpreted as a speed limit. This so called speed limit to monetary policy has been well explored in the monetary policy literature¹. [Paez-Farrell \(2009\)](#) finds that an interest rate rule based on a speed limit objective provides a reasonable fit of the interest rate data for the United States. He attributes this to the Fed's inability to announce explicit monetary policy objectives. [Kapinos and Hanson \(2011\)](#) provide a theoretical and empirical evaluation of speed limit objectives. Their theoretical study suggests that speed limit targeting and price level targeting provide more stability than strict inflation targeting, but in their empirical evidence speed limit targeting is only significant at short forecast horizons. The authors attribute this to the relative constant Greenbook forecasts of the output gap in the near-term future.

[Harvey, Trimbur and Van Dijk \(2007\)](#) model the persistent components of inflation and output using unobserved processes for both the deterministic and stochastic components of the series. [Harvey \(2011\)](#) applies this same approach to the estimation of the New-Keynesian Phillips curve (NKPC) using an atheoretical model to obtain estimates of the output gap. More specifically, the output gap is estimated as the stochastic cycle of the output series. While the fit of this model to U.S. data is relatively good, it disregards the consumer's Euler equation which provides a theoretical specification for the output gap and does not provide a statistical test for the importance of forward-looking inflation.

Tangentially related to this, there is a portion of the optimal monetary policy literature that models the preferences of the central banker under asymmetry. [Cukierman \(2002\)](#)

¹[Blake \(2012\)](#) examines determinacy of speed limit targeting in a simple interest rule and shows that results to be not unique. However the parameterization of his representation does not apply in this case.

and [Nobay and Peel \(2003\)](#) formalize optimal monetary policy models using the type of asymmetry first explored by [Varian \(1975\)](#) and [Zellner \(1986\)](#). [Ruge-Murcia \(2003a; 2004\)](#) finds evidence of asymmetric responses to changes in the unemployment gap.

This paper adds to the empirical literature of speed limit policies in two ways. The first is to formulate a generalized version that allows asymmetry in speed limit policies, but that nests Walsh’s original model as a special case². Thus negative changes in inflation rates and the output gap over time are treated differently than positive changes of the same magnitude. The second contribution of this paper is to reconcile [Harvey \(2011\)](#) with the remainder of the forward-looking New-Keynesian framework and incorporate it into the monetary planner’s problem. Both contributions seek to expand the already vibrant discussion around persistence of inflation rates and optimal monetary policy.

The remainder of this chapter is organized intuitively. Section 2 explains the adjustments made to the New-Keynesian model to account for unobserved persistent components. The optimal monetary problem is solved and additional model details are provided. Section 3 defines the data used and relevant estimation results. Section 4 closes with brief summary remarks.

3.2 Model

The baseline forward-looking New-Keynesian model (sans monetary policy) in its log-linearized form is given by Equations (3.1) and (3.2).³

$$\pi_t = \beta E_t \pi_{t+1} + \varphi x_t + \varepsilon_t^s \tag{3.1}$$

Equation (3.1) is the NKPC. Inflation is denoted by π_t . x_t is the output gap, or $y_t - y_t^n$, where y_t^n is natural output. β is the discount factor in the consumer’s utility maximization problem and governs the degree to which today’s conditional expectations of future inflation

²Much in the same way that [Ruge-Murcia \(2003a\)](#) nests [Barro and Gordon \(1983\)](#).

³A detailed derivation of this model is located in Appendix (B.1).

effect today's inflation. ε_t^s is a supply side shock.

$$x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + \varepsilon_t^d \quad (3.2)$$

The consumer's Euler equation is given by (3.2). The interest rate is denoted by i_t . σ^{-1} is the intertemporal elasticity of substitution. Demand is shocked by ε_t^d . It is assumed that $[\varepsilon_t^d, \varepsilon_t^s] \sim N(0, \sigma^2)$. Forward-looking expectations of future inflation and the output gap are conditional on the information set, I_t which is a function of past observations and rational expectations errors.

As previously mentioned, often these equations are modified to include lagged dependent variables in order to build persistence and capture some of the stylized facts of the output and inflation data. [Harvey \(2011\)](#) shows that even when the NKPC is adjusted to incorporate lagged inflation, it cannot adequately account for nonstationarity in the data nor explain core inflation when estimated with a detrended inflation series. Rather than including lagged dependent variables, the hybrid New-Keynesian model can be modified according to [Harvey, Trimbur and Van Dijk \(2007\)](#) and [Harvey \(2011\)](#) to include unobserved deterministic and stochastic components. The adjusted NKPC is thus represented according to Equation (3.3).

$$\pi_t = \mu_t^\pi + \psi_t^\pi + \beta E_t \pi_{t+1} + \varphi x_t + \varepsilon_t^s \quad (3.3)$$

μ_t^π represents a deterministic trend that follows a random walk, Equation (3.4), while ψ_t^π is a stochastic cycle that accounts for the fluctuation around core inflation.

$$\mu_t^\pi = \mu_{t-1}^\pi + \nu_t^\pi \quad (3.4)$$

The output gap is econometrically estimated from the deconstructed output series that is modeled according to Equation (3.5).

$$y_t = \mu_t^y + \psi_t^y + \varepsilon_t^y \quad (3.5)$$

The deterministic trend of output, μ_t^y , is captured by an integrated random walk, (3.6). This is often referred to as a smooth trend model because extraction of this series will

be smoother than a traditional random walk. In this case, the smoothed estimates of μ_t^y represent estimates of natural output.

$$\begin{aligned}\mu_t^y &= \mu_{t-1}^y + \nu_{t-1}^y \\ \nu_t^y &= \nu_{t-1}^y + u_t^y\end{aligned}\tag{3.6}$$

The stochastic cycle around that deterministic trend, or the output gap, follows from Equation (3.7). Here the stochastic components of Equations (3.3) and (3.5) are modeled as similar cycles.

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos \lambda_c & \sin \lambda_c \\ -\sin \lambda_c & \cos \lambda_c \end{bmatrix} \otimes I_2 \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}\tag{3.7}$$

where $\psi_t = [\psi_t^\pi \ \psi_t^y]'$. λ_c is the frequency of the cycle expressed in radians and ρ is a dampening factor that is bounded such that $0 \leq \rho < 1$. The $E(\kappa_t \kappa_t') = E(\kappa_t^* \kappa_t^{*'}) = \Sigma_\kappa$ for both π and y . The $E(\kappa_t \kappa_t^*) = 0$. The cycles for inflation and output have the same dynamics since ρ and λ_c are the same for both series. Additionally, the errors $[\nu_t^\pi \ \epsilon_t^y \ u_t^y]$ are assumed $\sim N(0, \sigma^2)$ as well as mutually and serially uncorrelated.

3.2.1 Rational Expectations

Rational expectations implies that expected values of variables equal the sum of their realized values and a stable, mean zero error process. In other words, rational expectations errors should have no systemic errors. Rather than using instrumentation as an identification scheme as in [Nason and Smith \(2008\)](#), this will be imposed directly via Equation (3.8),

$$E_t \pi_{t+1} = \pi_{t+1} + \eta_t^\pi\tag{3.8}$$

Applying rational expectations and substituting the stochastic cycle from (3.7) for the output gap implies

$$\pi_t = \mu_t^\pi + \psi_t^\pi + \beta \pi_{t+1} + \varphi \psi_t^y + \beta \eta_t^\pi + \varepsilon_t^s\tag{3.9}$$

Since the output gap is stationary by design, there is no need to include unobserved persistence in the Euler equation. However, the forward-looking conditional expectation of the

output gap in the consumer's Euler equation can be solved for using the persistence of the stochastic cycle. Leading Equation (3.7) by the Law of Iterated Expectations implies (3.10).

$$E_t \psi_{t+1}^y = \rho \cos \lambda_c \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} + E_t \kappa_{t+1}^y \quad (3.10)$$

After some small algebra and using the intermediate result $E_t \kappa_{t+1}^y = 0$, the Euler equation can then be recast as (3.11)⁴.

$$\sigma^{-1} i_t = \sigma^{-1} \pi_{t+1} + (\rho \cos \lambda_c - 1) \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} - \sigma^{-1} \eta_t^\pi + \varepsilon_t^d \quad (3.11)$$

3.2.2 Optimal Monetary Policy

Monetary policy is based on the expectational sum of discounted period loss functions described in Equation (3.12). Here, as in Walsh (2003b), the function embodies the intertemporal constraint that the central bank faces by adjusting the economy too quickly. The change in inflation is also included in this formulation in order to provide an additional hypothesis to test as well as another source of persistence in the model.⁵

$$L_t = \sum_{i=0}^{\infty} \beta^i \left\{ \phi \frac{\exp[\alpha(\pi_{t+i} - \pi_{t+i-1})] - \alpha(\pi_{t+i} - \pi_{t+i-1}) - 1}{\alpha^2} + \lambda \frac{\exp[\gamma(x_{t+i} - x_{t+i-1})] - \gamma(x_{t+i} - x_{t+i-1}) - 1}{\gamma^2} \right\} \quad (3.12)$$

The exponentiated functional form implies that the change in inflation and the change in the output gap are asymmetrically weighted, like in Ruge-Murcia (2003a; 2004) and Surico (2003; 2007). The original specification of Walsh's model is quadratic in nature which implies that the magnitude of policy response for a positive one percent change in inflation from one period to the next is the same as a negative one percent change in inflation over the same time. This more general specification of Walsh's model nests the original model as a special case when $\alpha = \gamma = 0$. The change in the output gap and change in inflation implies that the central bank cares about fluctuations in inflation and the output gap rather

⁴See technical notes in Appendix (C.1.1) for a detailed derivation.

⁵This extension to the original framework is briefly considered Walsh.

than the actual levels of inflation and output gap. Since the velocity of the output gap and inflation over time matters in this way we expect both α and γ to > 0 . This is consistent both with the asymmetric preferences literature as well as the dual mandate of the Federal Reserve.

The parameters ϕ and λ are the aversion parameters for the policy maker. They measure the degree to which the central bank is averse to deviations in the inflation rate and output gap over time. Larger values of one with respect to the other can be interpreted as evidence of targeting behavior.

Discretionary Equilibrium

A purely discretionary equilibrium to this problem is examined here rather than [Woodford \(1999\)](#)'s timeless perspective examined by Walsh. Given the forward-looking expectations as well as the backward-looking components, the discretionary and precommitment policy will be the same at time t . Optimal discretionary policy implies that the policy maker chooses the output gap in order to influence $(\pi_t - \pi_{t-1})$ and $(x_t - x_{t-1})$. Minimizing a single period loss function subject to [\(3.9\)](#) and [\(3.11\)](#) implies the following nonlinear first-order condition.

$$\phi \left\{ \frac{\exp[\alpha(\pi_{t+i} - \pi_{t+i-1})] - 1}{\alpha} \right\} \varphi + \lambda \left\{ \frac{\exp(\gamma(x_{t+i} - x_{t+i-1})) - 1}{\gamma} \right\} = 0 \quad (3.13)$$

Standard asymptotic theory does not apply to the estimation of equations where statistical inference of parameters represent a test of a linear null hypothesis versus a nonlinear alternative. This is especially true of exponential functional forms and was first addressed by [Luukkonen, Saikkonen and Teräsvirta \(1988\)](#). Standard convention⁶ for dealing with this is to linearize this FOC via a second-order Taylor series expansion, [\(3.14\)](#).

$$\phi \varphi \left\{ (\pi_t - \pi_{t-1}) + \alpha \frac{(\pi_t - \pi_{t-1})^2}{2} \right\} + \lambda \left\{ (x_t - x_{t-1}) + \gamma \frac{(x_t - x_{t-1})^2}{2} \right\} = 0 \quad (3.14)$$

Given joint estimation of equations above, identification of all the structural parameters in the problem are possible.

⁶Within the optimal monetary policy literature see Ruge-Murcia ([2003a](#); [2004](#)) and Surico ([2003](#); [2007](#)). For a more general treatment, [Van Dijk, Teräsvirta and Franses \(2002\)](#)

Table 3.1: Maximum Likelihood Estimates - 1960:1 to 2012:4

Parameter	PCE Inflation	CPI Inflation	GDP Deflator Inflation
α	0.488 [†] (0.108)	0.473 [†] (0.088)	0.332 [†] (0.099)
γ	0.018 (0.059)	0.017 (0.061)	0.031 (0.059)
ϕ	0.755 [†] (0.067)	0.681 [†] (0.024)	0.538 [†] (0.022)
λ	0.422 [†] (0.002)	0.421 [†] (0.002)	0.545 [†] (0.026)
β	0.468 [†] (0.016)	0.531 [†] (0.019)	0.238 [†] (0.030)
φ	0.016* (0.007)	0.013 [†] (0.002)	0.034 [†] (0.002)
σ	3.098 [†] (0.024)	3.248 [†] (0.286)	2.496 [†] (0.148)
ρ	0.703 [†] (0.049)	0.696 [†] (0.045)	0.621 [†] (0.032)
λ_c	0.203 [†] (0.043)	0.203 [†] (0.058)	0.272 [†] (0.074)
Log Likelihood	-662.851	-640.552	-873.8621

* and [†] represent significance at the 5% and 1% respectively.

3.3 Results

Equations (3.4), (3.5), (3.6), (3.7), (3.9), (3.11), and (3.14) comprise the system of equations to be estimated using maximum likelihood. The system is recast into state-space form.⁷ The state equation and the log-likelihood function are iteratively updated using a Kalman filter and the relevant parameters are chosen to minimize the sum of the individual log-likelihoods. Standard errors are calculated by taking the square root of the diagonal elements of the inverse Hessian estimated from the optimally weighted mean squared error matrix from the Kalman filter recursion.

All data sources are provided by the Federal Reserve Economic Database. Output is the

⁷For more details on the state-space formulation and estimation algorithm please see Appendices (C.2) and (C.2.1).

natural logarithm of real gross domestic product (RGDP). The interest rate is the Fed Funds rate. Because results may be sensitive to different measures of inflation, three pricing indices are examined; personal consumption expenditure (PCE), GDP deflator, and consumer price index (CPI). Each series is transformed into an inflation rate by taking the first difference of the natural log. The full sample period of the quarterly data extends from 1960 to 2012. Additional results are also provided for the corresponding time period after the appointment of Paul Volcker as Federal Reserve Chairman in order to account for structural breaks⁸.

Table 3.1 shows the maximum likelihood estimates corresponding to the three different inflation series for the full data sample. The asymmetry over the speed limit of inflation is positive and significant for all three series. This implies that the Fed responds more urgently when the difference in inflation is positive than when it is negative. Put another way, when today's inflation is higher than yesterday's inflation the response is more forceful than when the opposite is true. Interestingly, the asymmetry estimates over the change in the output gap is small and never significant; a quadratic specification for this portion of the objective function cannot be rejected. The aversion parameters ϕ and λ are always significant, but it is difficult to clearly determine targeting behavior based on the relative similarities of the estimates. Statistical significance of λ is consistent with earlier studies that find evidence for a speed limit objective. Evidence for forward-looking inflation is indicated from β . One limitation of Harvey (2011) is that it does not identify forward-looking inflation. These results indicate that future inflation expectations are still important for explaining inflation today. Estimates for ρ , the cycle dampening factor, and λ_c , the frequency, are in line with previous estimates for U.S. data.

Table 3.2 shows the model parameter estimates for the Post-Volcker subsample. For all three inflation series both types of asymmetry disappear over this time period. Surico (2007) estimates a similar asymmetric preferences model except the asymmetry in his case is over inflation (not the change in inflation) and the output gap (not the change in the output gap).

⁸Additional results for the Great Moderation subsample are available from the author upon request. These estimates are not reported here because of stability and small-sample bias concerns.

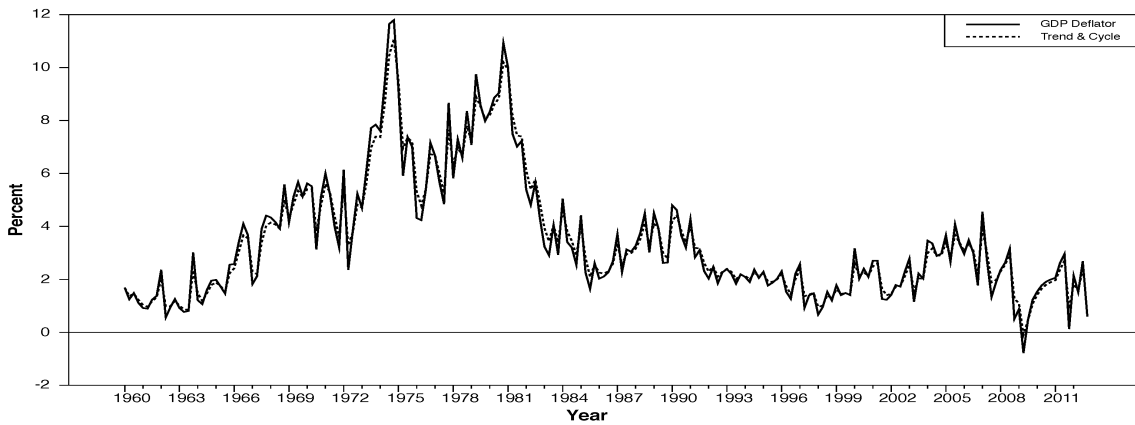
Figure 3.1: Inflation with Estimated Trend and Cycle



(a) PCE Inflation with Trend plus Cycle



(b) CPI Inflation with Trend plus Cycle



(c) GDP Deflator Inflation with Trend plus Cycle

Table 3.2: Maximum Likelihood Estimates - 1979:4 to 2012:4

Parameter	PCE Inflation	CPI Inflation	GDP Deflator Inflation
α	0.728 (1.761)	0.720 (0.770)	0.151 (0.325)
γ	0.008 (0.894)	0.002 (1.070)	0.012 (0.566)
ϕ	0.768 (0.566)	1.0189 (1.718)	0.777 (0.735)
λ	0.422 [†] (0.037)	0.300 [†] (0.026)	0.421 [†] (0.038)
β	0.3042 [†] (0.009)	0.422 [†] (0.024)	0.313 [†] (0.008)
φ	0.016 (0.024)	0.002 (0.002)	0.016 [†] (0.006)
σ	3.417 [†] (0.565)	3.599 [†] (0.724)	2.904 [†] (0.044)
ρ	0.417 [†] (0.061)	0.643 [†] (0.074)	0.031 [†] (0.010)
λ_c	0.212 (0.157)	0.201* (0.091)	0.364 [†] (0.032)
Log Likelihood	-499.929	-434.580	-632.246

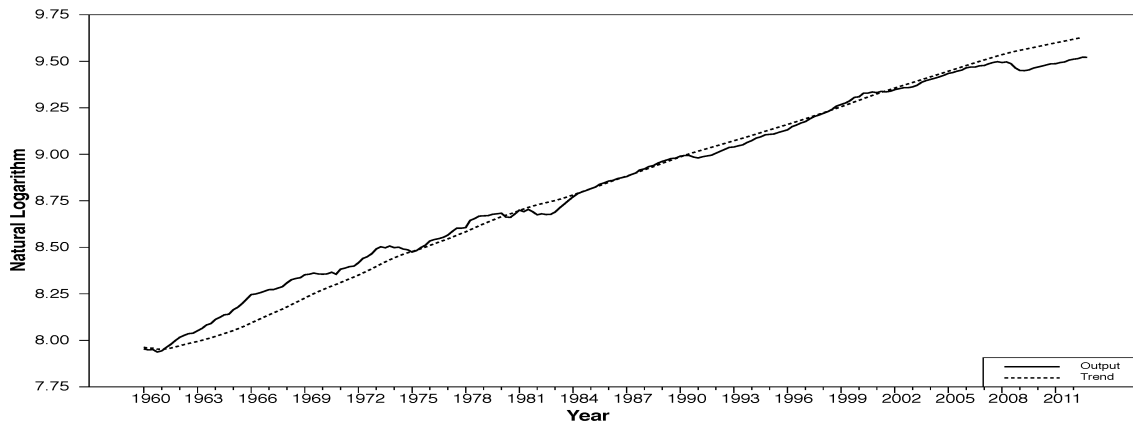
* and [†] represent significance at the 5% and 1% respectively.

This could be the result of policy leadership by the Fed or the general smoothing out of the time series data during this period,⁹ as suggested by [Doyle and Falk \(2010\)](#). Parameter estimates for ϕ , aversion toward changes in inflation increase a bit, but lose significance, while λ , aversion toward output gap changes, remains stable and significant. This can be interpreted as a possible change in policy behavior given that the loss function is correctly specified. Expectations of future inflation are still significant.

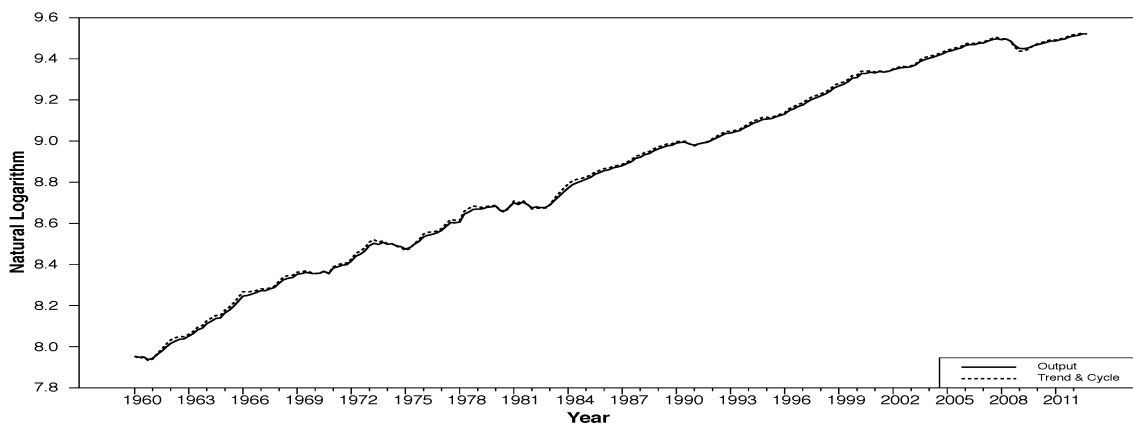
The cycle dampening factor, ρ , drops in value considerably for GDP deflator inflation. Additionally, the lower log-likelihood estimates for both sample periods is somewhat puzzling and prompted the graphs in Figures (3.1) and (3.2). Figure (3.1) shows the three inflation series, PCE, CPI, and GDP deflator inflation plotted in black in subplots (3.1a),

⁹Assuming that the two are not correlated, which they very well might be.

Figure 3.2: Output with Estimated Trend and Cycle



(a) Output with Trend



(b) Output with Trend plus Cycle

(3.1b), and (3.1c) respectively. The trend and cycles of each series are recursively extracted from the Kalman filter estimates, added together and plotted using a dashed line. For the inflation series, the deterministic trend captured by the random walk accounts for most of the variation in the data (the stochastic cycle adds little) since inflation is stationary. This is expected given the setup of the model and the loss of persistence in inflation during the Great Moderation period.

The same is also done for the log of output in Figure (3.2). Here the estimates of both trends for output are shown. Subplot (3.2a) shows output and the deterministic trend

(natural output) estimates while subplot (3.2b) shows output with the sum of both trends. One can see the effects of the stochastic trend more clearly given the nonstationarity of the output series.

Visual inspection of the plots indicate that the trend and cycles produced from the estimates do a relatively good job of accounting for the data in a backward-looking fashion. Additionally, the statistical significance of future inflation expectations reported above in the face of this fit provides additional justification for the inclusion of forward-looking expectations in the model.

3.4 Conclusion

This paper contributes to the empirical literature surrounding speed limits to monetary policy by estimating a general form specification that allows for asymmetric response of the policy maker to positive and negative changes in inflation and the output gap over time. Additionally, the model incorporates an alternative form of persistence using unobserved deterministic and stochastic trends in the data rather than lagged dependent variables. Forward-looking expectations of future inflation are still significant given the strong fit of backward-looking components. Parameter estimates indicate asymmetric responses to positive changes in inflation, but that asymmetry disappears in the Post-Volcker subsample, a well established phenomenon in a more general asymmetric preferences model.

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Appendix A

Chapter 1 Technical Notes

A.1 Solving the planner's optimization problem

Taking the derivative with respect to i_t and taking the public's inflation forecast as given yields first order condition

$$E_{t-1} \left\{ (\pi_t - \pi^*) + \left(\frac{\phi}{\gamma^2} \right) (-\gamma\alpha \exp(\gamma(kE_{t-1}Y_t^p - Y_t)) + \alpha\gamma) \right\} = 0, \quad (\text{A.1})$$

or

$$E_{t-1}\pi_t - \pi^* - \left(\frac{\phi\alpha}{\gamma} \right) E_{t-1} (\exp(\gamma(kE_{t-1}Y_t^p - Y_t)) - 1) = 0. \quad (\text{A.2})$$

As shown below, the assumption that the structural disturbances are normal implies that, conditional on the information set, output is also normally distributed. Then, $\exp(\gamma(kE_{t-1}Y_t^p - Y_t))$ is distributed log normal. Using the intermediate result

$$E_{t-1}Y_t = E_{t-1}Y_t^p, \quad (\text{A.3})$$

obtained by taking conditional expectations of both sides of (1.1) and using the assumption of rational expectations, it is possible to write the mean of this log normal distribution as $\exp\left(\gamma(k-1)E_{t-1}Y_t^p + \frac{\gamma^2\sigma_{Y,t}^2}{2}\right)$. The notation $\sigma_{Y,t}^2$ is the conditional variance of output and is derived below in terms of the elements of ξ_t . Finally, using (1.4), it is easy to show that

$$\pi_t = \pi^* + \left(\frac{\phi\alpha}{\gamma} \right) \left(\exp\left(\gamma(k-1)E_{t-1}Y_t^p + \frac{\gamma^2\sigma_{Y,t}^2}{2}\right) - 1 \right) + \mathbf{A}\xi_t, \quad (\text{A.4})$$

where $\mathbf{A} = (0, 0, 1)$.¹ Next, using (1.1), we see

$$[Y_t - E_{t-1}Y_t] = [Y_t^p - E_{t-1}Y_t^p] + [\alpha(\pi_t - \pi_t^e) - E_{t-1}(\alpha(\pi_t - \pi_t^e))] + [\eta_t - E_{t-1}\eta_t].$$

Using (1.3) and (1.4) gives

$$Y_t = E_{t-1}Y_t + \mathbf{B}\xi_t,$$

where $\mathbf{B} = (1, 1, \alpha)$. Next using (A.3) gives

$$Y_t = E_{t-1}Y_t^p + \mathbf{B}\xi_t. \tag{A.5}$$

Note that since $E_{t-1}Y_t^p$ is included in the public's information set at time $t - 1$ and the linear combination $\mathbf{B}\xi_t$ is normally distributed, so

$$Y_t|I_{t-1} \sim N(E_{t-1}Y_t^p, \sigma_{Y,t}^2) \quad \text{where} \quad \text{Var}(Y_t|I_{t-1}) \equiv \sigma_{Y,t}^2 = \mathbf{B}\Omega_t\mathbf{B}',$$

as claimed above.

¹To see this, note that (1.4) implies

$$[\pi_t - E_{t-1}\pi_t] = [i_t - E_{t-1}i_t] + [\varepsilon_t - E_{t-1}\varepsilon_t],$$

which implies

$$\pi_t = E_{t-1}\pi_t + \varepsilon_t.$$

Next using (A.2) gives the result.

A.2 Estimation Description

A.2.1 ARIMA(2,0,2)

Here I describe one state space formulation. Others probably exist, but according to Hamilton, in the middle of page 375, it does not matter which is used. The forecasts are always the same. Begin by noting that

$$\Delta Y_t = \psi' + (1-\delta)^2 t - (1-\delta)Y_{t-1} + \theta \Delta Y_{t-1} + \zeta_t + \eta_t + \alpha \varepsilon_t - \delta(\alpha \varepsilon_{t-1} + \eta_{t-1}) - \theta(\alpha \Delta \varepsilon_{t-1} + \Delta \eta_{t-1}).$$

can be written as

$$Y_t = Y_{t-1} + \psi' + (1-\delta)^2 t - (1-\delta)Y_{t-1} + \theta \Delta Y_{t-1} + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2},$$

where κ_t denotes the white noise term that builds either the ARIMA (1,1,2) or the ARIMA(2,0,2) process and β_1 and β_2 are the moving average coefficients for the MA part of the model.

First, consider the case where $\delta \neq 1$. Then this equation can be written as

$$Y_t = \psi' + (1-\delta)^2 t + [(1+\theta) - (1-\delta)] Y_{t-1} - \theta Y_{t-2} + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2}.$$

or, noting using the algebra given in the notes at the bottom we have

$$\tilde{Y}_t = [(1+\theta) - (1-\delta)] \tilde{Y}_{t-1} - \theta \tilde{Y}_{t-2} + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2}, \quad (\text{A.6})$$

where $\tilde{Y}_t = \left(Y_t - \frac{\psi}{(1-\delta)} - (1-\delta)t \right)$. Note, this is the ARIMA (2,0,2) case. This is one of the equations I will write in the state space form. The other is (1.7). For this system, write the state equation as

$$\xi_t = F \xi_{t-1} + R \varkappa_t. \quad (\text{A.7})$$

Note, I have used some of the notation from Hamilton, but I have also used some of the formulation in Harvey. There are two differences from Hamilton, the first is trivial, but it is to write the equation in terms of time t rather than time $t+1$. The second is to add a

matrix in from of the error term κ_t . Hamilton would define $v_t = R\kappa_t$. In this formulation, we have the following matrices:

$$\xi_t = \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \\ \xi_{5t} \end{bmatrix}, \quad F = \begin{bmatrix} (1+\theta) - (1-\delta) & 0 & 1 & 0 & 0 \\ (1+\theta) - (1-\delta) & 0 & 1 & 0 & 0 \\ \theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \beta_1 & 0 \\ \beta_2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \kappa_t = \begin{bmatrix} \kappa_t \\ e_t \end{bmatrix}. \quad (\text{A.8})$$

Note that multiplying elements of (A.7) starting with the last line, we get

$$\xi_{5t} = e_t,$$

$$\xi_{4t} = \beta_2 \kappa_t,$$

$$\xi_{3t} = \theta \xi_{1t-1} + \xi_{4t-1} + \beta_1 \kappa_t = \theta \xi_{1t-1} + \beta_2 \kappa_{t-1} + \beta_1 \kappa_t,$$

$$\xi_{2t} = [(1+\theta) - (1-\delta)] \xi_{1t-1} + \xi_{3t-1} = [(1+\theta) - (1-\delta)] \xi_{1t-1} + \theta \xi_{1t-2} + \beta_2 \kappa_{t-2} + \beta_1 \kappa_{t-1},$$

$$\xi_{1t} = [(1+\theta) - (1-\delta)] \xi_{1t-1} + \xi_{3t-1} + \kappa_t = [(1+\theta) - (1-\delta)] \xi_{1t-1} + \theta \xi_{1t-2} + \beta_2 \kappa_{t-2} + \beta_1 \kappa_{t-1} + \kappa_t.$$

Next define the observation equation by

$$y_t = A'x_t + H'\xi_t. \quad (\text{A.9})$$

Here I am using Hamilton's notation. In this formulation, we have the following matrices:

$$y_t = \begin{bmatrix} Y_t \\ \pi_t \end{bmatrix}, \quad A' = \begin{bmatrix} \frac{\psi}{(1-\delta)} & 0 & (1-\delta) \\ a' & c & b(1-\delta) \end{bmatrix}, \quad x_t = \begin{bmatrix} 1 \\ \sigma_{Y,t}^2 \\ t \end{bmatrix}, \quad H' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \\ \xi_{5t} \end{bmatrix}. \quad (\text{A.10})$$

Notice that a key difference between ξ_{1t} and ξ_{2t} is that $\xi_{2t} = E_{t-1}\xi_{1t}$. Since things are constructed so that $\xi_{1t} = \tilde{Y}_t$, we have $\xi_{2t} = E_{t-1}\tilde{Y}_t$. As in the earlier model this has some implications. First it implies that $a' = a + b\frac{\psi}{(1-\delta)}$ as before. Second, we need the time trend term in the inflation equation to adjust things back so that we have $bE_{t-1}Y_t$

A.2.2 ARIMA (1,1,2)

Now consider the case where $\delta = 1$. In this case,

$$\Delta Y_t = \psi - (1 - \delta)Y_{t-1} + \theta\Delta Y_{t-1} + \kappa_t + \beta_1\kappa_{t-1} + \beta_2\kappa_{t-2},$$

becomes

$$\Delta Y_t = \psi + \theta\Delta Y_{t-1} + \kappa_t + \beta_1\kappa_{t-1} + \beta_2\kappa_{t-2},$$

or, noting that $\overline{\Delta Y} \equiv E[\Delta Y_t] = \frac{\psi}{1-\theta}$, we have

$$(\Delta Y_t - \overline{\Delta Y}) = \theta(\Delta Y_{t-1} - \overline{\Delta Y}) + \kappa_t + \beta_1\kappa_{t-1} + \beta_2\kappa_{t-2}. \quad (\text{A.11})$$

This is one of the equations I will write in the state space form. The other is (1.7). For this system, write the state equation as

$$\xi_t = F\xi_{t-1} + R\boldsymbol{\varkappa}_t. \quad (\text{A.12})$$

Note, I have used some of the notation from Hamilton, but I have also used some of the formulation in Harvey. There are two differences from Hamilton, the first is trivial, but it is to write the equation in terms of time t rather than time $t + 1$. The second is to add a matrix in from of the error term κ_t . Hamilton would define $v_t = R\boldsymbol{\varkappa}_t$. In this formulation, we have the following matrices:

$$\xi_t = \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \\ \xi_{5t} \end{bmatrix}, \quad F = \begin{bmatrix} \theta & 0 & 1 & 0 & 0 \\ \theta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \beta_1 & 0 \\ \beta_2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\varkappa}_t = \begin{bmatrix} \kappa_t \\ e_t \end{bmatrix}. \quad (\text{A.13})$$

Note that multiplying elements of (A.12) starting with the last line, we get

$$\begin{aligned}
\xi_{5t} &= e_t, \\
\xi_{4t} &= \beta_2 \kappa_t, \\
\xi_{3t} &= \xi_{4t-1} + \beta_1 \kappa_t = \beta_2 \kappa_{t-1} + \beta_1 \kappa_t, \\
\xi_{2t} &= \theta \xi_{1t-1} + \xi_{3t-1} = \theta \xi_{1t-1} + \beta_2 \kappa_{t-2} + \beta_1 \kappa_{t-1}, \\
\xi_{1t} &= \theta \xi_{1t-1} + \xi_{3t-1} + \kappa_t = \theta \xi_{1t-1} + \beta_2 \kappa_{t-2} + \beta_1 \kappa_{t-1} + \kappa_t.
\end{aligned}$$

Next define the observation equation by

$$y_t = A'x_t + H'\xi_t. \quad (\text{A.14})$$

Here I am using Hamilton's notation. In this formulation, we have the following matrices:

$$y_t = \begin{bmatrix} \Delta Y_t \\ \pi_t \end{bmatrix}, \quad A' = \begin{bmatrix} \frac{\psi}{1-\theta} & 0 & 0 \\ a' & b & c \end{bmatrix}, \quad x_t = \begin{bmatrix} 1 \\ Y_{t-1} \\ \sigma_{Y,t}^2 \end{bmatrix}, \quad (\text{A.15})$$

$$H' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \\ \xi_{5t} \end{bmatrix}. \quad (\text{A.16})$$

Notice that a key difference between ξ_{1t} and ξ_{2t} is that $\xi_{2t} = E_{t-1}\xi_{1t}$. Since things are constructed so that $\xi_{1t} = (\Delta Y_t - \overline{\Delta Y})$, we have $\xi_{2t} = E_{t-1}\Delta Y_t - \overline{\Delta Y}$. Technically this means that $a' = a + b\overline{\Delta Y}$, but this does not matter since we are not interested in testing things about a anyway.

A.3 Some notes for solving an equation above.

Consider an equation of the form

$$y_t = a + bt + c_1 y_{t-1} + c_2 y_{t-2} \quad (\text{A.17})$$

which we would like to solve.

Guess the form

$$y_t = \alpha + \beta t. \quad (\text{A.18})$$

Substituting this in gives

$$\alpha + \beta t = a + bt + c_1 (\alpha + \beta(t-1)) + c_2 (\alpha + \beta(t-2)) \quad (\text{A.19})$$

$$= a + bt + c_1 \alpha + c_1 \beta t - c_1 \beta + c_2 \alpha + c_2 \beta t - 2c_2 \beta. \quad (\text{A.20})$$

Since the constants have to equal and the terms with the time trends have to equal we have the following two equations

$$\beta = b + c_1 \beta + c_2 \beta. \quad (\text{A.21})$$

$$\alpha = a + c_1 \alpha - c_1 \beta + c_2 \alpha - 2c_2 \beta, \quad (\text{A.22})$$

and these imply

$$\beta = \frac{b}{1 - c_1 - c_2}. \quad (\text{A.23})$$

$$\alpha = \frac{a - c_1 \beta - 2c_2 \beta}{1 - c_1 - c_2} = \frac{a - \beta(c_1 + 2c_2)}{1 - c_1 - c_2}. \quad (\text{A.24})$$

For our model we have

$$\beta = \frac{(1 - \delta)^2}{1 - [(1 + \theta) - (1 - \delta)] - [-\theta]} = \frac{(1 - \delta)^2}{1 - [\theta + \delta] + \theta} = (1 - \delta). \quad (\text{A.25})$$

$$\alpha = \frac{\psi' - (1 - \delta)([\theta + \delta] - 2\theta)}{1 - [\theta + \delta] + \theta} \quad (\text{A.26})$$

$$= \frac{\psi + (1 - \delta)[1 - \theta - (1 - \delta)] - (1 - \delta)(\delta - \theta)}{(1 - \delta)} \quad (\text{A.27})$$

$$= \frac{\psi + (1 - \delta)[\delta - \theta] - (1 - \delta)(\delta - \theta)}{(1 - \delta)} = \frac{\psi}{(1 - \delta)}, \quad (\text{A.28})$$

which means the solution is

$$Y_t = \frac{\psi}{(1-\delta)} + (1-\delta)t. \quad (\text{A.29})$$

If we work backwards, we can show that

$$Y_t = \psi' + (1-\delta)^2 t + [(1+\theta) - (1-\delta)] Y_{t-1} - \theta Y_{t-2} + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2}.$$

will become

$$\begin{aligned} \left(Y_t - \frac{\psi}{(1-\delta)} - (1-\delta)t \right) &= [(1+\theta) - (1-\delta)] \left(Y_{t-1} - \frac{\psi}{(1-\delta)} - (1-\delta)(t-1) \right) \\ &\quad - \theta \left(Y_{t-2} - \frac{\psi}{(1-\delta)} - (1-\delta)(t-2) \right) \\ &\quad + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2}, \end{aligned}$$

or

$$\tilde{Y}_t = [(1+\theta) - (1-\delta)] \tilde{Y}_{t-1} - \theta \tilde{Y}_{t-2} + \kappa_t + \beta_1 \kappa_{t-1} + \beta_2 \kappa_{t-2}.$$

A.4 Great Moderation Period Results

Table 4. Sample 1982:4-2003:2

<i>ARIMA</i> (2, 0, 2)			
Coefficient	Model		
	Barro and Gordon	Asymmetric with	
		$1 \leq k$	$k = 1$
<i>a</i>	2.48 (0.11)	1.59 (0.39)	1.59 (0.39)
<i>b</i>	0.0 .	0.0 .	
<i>c</i>		1.64 (0.71)	1.64 (0.71)
log likelihood	207.83864	210.30374	210.30374
<i>ARIMA</i> (1, 1, 2)			
Coefficient	Model		
	Barro and Gordon	Asymmetric with	
		$1 \leq k$	$k = 1$
<i>a</i>	2.47 (0.11)	-1.85 (1.30)	-1.85 (1.30)
<i>b</i>	0.0 .	0.0 .	
<i>c</i>		5.72 (1.72)	5.72 (1.72)
log likelihood	205.72	209.59243	209.59243

Appendix B

Chapter 2 Technical Notes

B.1 Deriving the Benchmark Dynamic New-Keynesian Model

B.1.1 Households

Households are infinitely-lived agents who are normalized to 1. They choose consumption, labor supply, bonds, and money every period to maximize utility. Assuming a simple CRRA functional form for $U_t(\cdot)$, households maximize

$$E_{t|0} \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{\eta} L_t^\eta + \nu \ln \frac{M_t}{P_t} \right]$$
$$s.t. \quad C_t + \frac{B_t}{P_t} = \frac{R_{t-1}}{P_t} B_{t-1} + \frac{W_t}{P_t} L_t + F_t + T_t - \frac{M_t - M_{t-1}}{P_t}$$

Here R_t is the nominal return on holding bonds, F_t is the dividends paid on the ownership of the intermediate goods firms, T_t is transfer payments. By solving for C_t in the budget constraint and substituting into the objective function we get

$$\max_{\{B_t, M_t, L_t\}} E_{t|0} \sum_{t=0}^{\infty} \beta^t [U_t(\cdot)]$$
$$U_t(\cdot) = \left[\frac{\left(\frac{R_{t-1}}{P_t} B_{t-1} + \frac{W_t}{P_t} L_t + F_t + T_t - \frac{M_t - M_{t-1}}{P_t} - \frac{B_t}{P_t} \right)_t^{1-\sigma}}{1-\sigma} - \frac{1}{\eta} L_t^\eta + \nu \ln \frac{M_t}{P_t} \right]$$
$$\frac{\partial U(\cdot)}{\partial B_t} = 0 \Rightarrow C_t^{-\sigma} = \beta E_t \left[R_t \left(\frac{P_{t+1}}{P_t} \right)^{-1} C_{t+1}^{-\sigma} \right] \quad (\text{B.1})$$

$$\frac{\partial U(\cdot)}{\partial L_t} = 0 \Rightarrow C_t^{-\sigma} \frac{W_t}{P_t} = L_t^{\eta-1} \quad (\text{B.2})$$

$$\frac{\partial U(\cdot)}{\partial M_t} = 0 \Rightarrow C_t^{-\sigma} = \beta E_t [C_{t+1}^{-\sigma} \Pi_{t+1}^{-1}] + \frac{\nu}{\frac{M_t}{P_t}} \quad (\text{B.3})$$

B.1.2 Firms

Output in this model is made up of final goods that are produced using intermediate goods. The final goods market is competitive while the intermediate goods market is monopolistically competitive. Y_t is the final good produced by $Y_{j,t}$ intermediate goods.

Final good firms

Final goods output is given by the CES aggregator,

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{\alpha}{\alpha-1}}$$

Final good firms seek to minimize total cost by the standard cost min problem

$$\mathcal{L} = \int_0^1 P_{jt} Y_{jt} dj + P_t \left[Y_t - \left[\int_0^1 Y_{jt}^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{\alpha}{\alpha-1}} \right] \quad (\text{B.4})$$

$$\frac{\partial \mathcal{L}}{\partial Y_{jt}} = 0 \Rightarrow P_{jt} = P_t \frac{\alpha}{\alpha-1} \left[\int_0^1 Y_{jt}^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{1}{\alpha-1}} \times \frac{\alpha-1}{\alpha} Y_{jt}^{-\frac{1}{\alpha}} \quad (\text{B.5})$$

or

$$P_{jt} = P_t Y_t^{\frac{1}{\alpha}} Y_{jt}^{-\frac{1}{\alpha}} \quad (\text{B.6})$$

This can be rearranged to solve for Y_{jt}

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\alpha} Y_t \quad (\text{B.7})$$

We can solve for the lagrangian multiplier, P_t by plugging in Y_{jt} into our original definition of Y_t

$$Y_t = \left[\int_0^1 \left(\left(\frac{P_{jt}}{P_t} \right)^{-\alpha} Y_t \right)^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{\alpha}{\alpha-1}} = Y_t \left[\int_0^1 \left(\frac{P_{jt}}{P_t} \right)^{1-\alpha} dj \right]^{\frac{\alpha}{\alpha-1}}$$

Since Y_t has constant returns to scale, it drops out and we can solve for P_t

$$P_t = \left[\int_0^1 (P_{jt})^{1-\alpha} dj \right]^{\frac{1}{1-\alpha}}$$

Intermediate good firms

The intermediate goods sector is monopolistically competitive, indexed over $j \in (0, 1)$. Each firm faces a downward sloping demand curve, Y_{jt} . Each firm uses labor to produce according to the following constraint

$$Y_{jt} = A_t L_{jt}$$

Producers choose price, P_{jt} taking the demand curve as given. Prices can only be changed when allowed to which occurs with probability $1 - \theta$.

Imposing a [Calvo \(1983\)](#) structure implies that if some firms are allowed to change prices then the average price in the economy will be a CES aggregate of all prices. This can be derived by imposing it directly

$$\begin{aligned} P_t &= \left[\int_0^1 (P_{jt})^{1-\alpha} dj + (1 - \theta) P_t^{*1-\alpha} \right]^{\frac{1}{1-\alpha}} \\ &= \left[\theta (P_{jt})^{1-\alpha} + (1 - \theta) P_t^{*1-\alpha} \right]^{\frac{1}{1-\alpha}} \\ P_t^{1-\alpha} &= \theta (P_{jt})^{1-\alpha} + (1 - \theta) P_t^{*1-\alpha} \end{aligned}$$

A purely forward-looking demand function for the intermediate producer is

$$Y_{jt+k}^* = \left(\frac{P_{jt}^*}{P_{t+k}} \right)^{-\alpha} Y_{t+k} \quad (\text{B.8})$$

The maximization problem that this producer faces is given by

$$\max_{\{P_{jt}^*\}} \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\left(\frac{C_t}{C_{t+k}} \right)^\sigma \left(\frac{P_{jt}^*}{P_{t+k}} - Z_{t+k} \right) Y_{jt+k}^* \right] \quad (\text{B.9})$$

Optimization yields

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \left(\frac{Y_{jt+k}^*}{P_{t+k}} + \frac{P_{jt}^*}{P_{t+k}} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - Z_{t+k} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0 \quad (\text{B.10})$$

where $\Lambda_{t,k} = \left(\frac{C_t}{C_{t+k}} \right)^\sigma$. Next factor out $\frac{Y_{jt+k}^*}{P_{t+k}}$ to get

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 + \frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - \frac{Z_{t+k} P_{t+k}}{P_{jt}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0$$

Solving for the elasticity $\frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*}$ using the forward-looking demand function (B.10) we get

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 - \alpha + \frac{Z_{t+k} P_{t+k}}{P_{jt}^*} \alpha \right) \right] = 0$$

Multiply both sides by P_{jt}^* and simplifying we get

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(P_{jt}^* - \frac{\alpha}{\alpha-1} Z_{t+k} P_{t+k} \right) \right] = 0$$

Define $X = \frac{\alpha}{\alpha-1}$, which is the steady state markup. In equilibrium the firms that do change price all choose the same price $P_{jt}^* = P_t^*$. Using these two we get

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{jt+k}^* \left(\frac{P_t^*}{P_{t+k}} - X Z_{t+k} \right) \right] = 0$$

Rearranging above we can get

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{jt+k}^* \frac{P_t^*}{P_{t+k}} \right] = X \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{jt+k}^* Z_{t+k} \right]$$

Also note that the real marginal cost, Z_{t+k} evolves following $Z_{t+k}^n = Z_{t+k} P_{t+k}$. Solving for Z_{t+k} and substituting we can solve for P_t^*

$$P_t^* = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{jt+k}^* Z_{t+k}^n P_{t+k}^{-1} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{jt+k}^* P_{t+k}^{-1} \right]} \quad (\text{B.11})$$

This expression states that the optimal price is a weighted average of current and expected future marginal costs under sticky prices. If we relax the stickiness assumption then $\theta = 0$ and the optimal price is a constant markup over marginal cost.

We have a pricing rule but we still need a labor demand rule for the producer. To get that setup the following cost minimization problem

$$\mathcal{L} = \frac{W_t}{P_t} L_{jt} + Z_t (Y_{jt} - A_t L_{jt}) \quad (\text{B.12})$$

$$\frac{\partial \mathcal{L}}{\partial L_{jt}} = 0 \Rightarrow \frac{W_t}{P_t} = Z_t A_t \quad (\text{B.13})$$

or

$$\frac{W_t}{P_t} = Z_t \frac{Y_{jt}}{L_{jt}} \quad (\text{B.14})$$

Solving for L_{jt}

$$L_{jt} = Z_t \frac{Y_{jt}}{w_t}, \quad (\text{B.15})$$

where $w_t = \frac{W_t}{P_t}$.

B.1.3 Equilibrium

Total output in the economy is characterized by

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{\alpha}{\alpha-1}} = \left[\int_0^1 (A_t L_{jt})^{\frac{\alpha-1}{\alpha}} dj \right]^{\frac{\alpha}{\alpha-1}}$$

It is not possible to simplify this further, since input usages across firms are different. The linear aggregator $Y'_t = \int_0^1 Y_{jt} dj \simeq Y_t$ within a local region of the steady state, so we can use

$$Y_t = A_t L_t$$

The goods market clears at $Y_t = C_t$ and the bond market clears at $B_t = 0$. Equilibrium is characterized by three equations

$$Y_t^{-\sigma} = \beta E_t \left[\frac{R_t P_t}{P_{t+1} Y_{t+1}^\sigma} \right] \quad (\text{B.16})$$

$$Y_t^{\eta+\sigma-1} = Z_t A_t^\eta \quad (\text{B.17})$$

$$P_t^{1-\alpha} = \theta (P_{jt})^{1-\alpha} + (1-\theta) \left[X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t [\Lambda_{t,k} Y_{jt+k}^* Z_{t+k}^\eta P_{t+k}^{-1}]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t [\Lambda_{t,k} Y_{jt+k}^* P_{t+k}^{-1}]} \right]^{1-\alpha} \quad (\text{B.18})$$

These equations represent equilibrium in the goods market, equilibrium in the labor market, and the aggregate price level respectively. We get (B.16) from imposing $Y_t = C_t$ on (B.1). Eq. (B.17) comes from setting (B.2) equal to (B.14). Eq. (B.18) comes from substituting (B.11) into our Calvo pricing structure.

B.1.4 Log-Linearization

Log-linearizing (B.16) yields

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma} (r_t - E_t \pi_{t+1}) \quad (\text{B.19})$$

Log-linearizing (B.17) yields

$$y_t = \frac{1}{\eta + \sigma - 1} z_t + \frac{\eta}{\eta + \sigma - 1} a_t \quad (\text{B.20})$$

Log-linearizing (B.18) is modestly more difficult. First start with P_t^* . Rearranging this, getting rid of $Y_{jt+k}^* = \left(\frac{P_t^*}{P_{t+k}}\right)^\alpha Y_{t+k}$, and dividing by P_t yields

$$\frac{P_t^*}{P_t} \sum_{k=0}^{\infty} (\theta\beta)^k E_t [\Lambda_{t,k} Y_{t+k} P_{t+k}^{\alpha-1}] = \frac{1}{P_t} X \sum_{k=0}^{\infty} (\theta\beta)^k E_t [\Lambda_{t,k} Y_{t+k} Z_{t+k}^n P_{t+k}^{\alpha-1}]$$

Start now with the LHS

$$\left(\hat{P}_t^* - \hat{P}_t\right) \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1}] + \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1}] E_t \left[\hat{\Lambda}_{t,k} + \hat{Y}_{t+k} + (\alpha - 1) \hat{P}_{t+k}\right]$$

Now the RHS. Notice that we split up the Z^n term

$$-\hat{P}_t \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1}] + \frac{X}{P} \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1} Z^n] E_t \left[\hat{\Lambda}_{t,k} + \hat{Y}_{t+k} + (\alpha - 1) \hat{P}_{t+k} + \hat{Z}_{t+k}^n\right]$$

Using $Z^n X = P$ or more specifically $\frac{Z^n X}{P} = 1$

$$-\hat{P}_t \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1}] + \sum_{k=0}^{\infty} (\theta\beta)^k [\Lambda Y P^{\alpha-1}] E_t \left[\hat{\Lambda}_{t,k} + \hat{Y}_{t+k} + (\alpha - 1) \hat{P}_{t+k} + \hat{Z}_{t+k}^n\right]$$

Noting also $\hat{Z}_{t+k}^n = \hat{P}_{t+k} + \hat{Z}_{t+k}$ this equation collapses to

$$\hat{P}_t^* \sum_{k=0}^{\infty} (\theta\beta)^k = \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\hat{P}_{t+k} + \hat{Z}_{t+k}\right]$$

or

$$\begin{aligned} \hat{P}_t^* &= (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\hat{P}_{t+k} + \hat{Z}_{t+k}\right] \\ &= (1 - \theta\beta) E_t \left[\left(\hat{P}_t + \hat{Z}_t\right) + \theta\beta \left(\hat{P}_{t+1} + \hat{Z}_{t+1}\right) + \theta^2 \beta^2 \left(\hat{P}_{t+2} + \hat{Z}_{t+2}\right) + \dots\right] \\ &= (1 - \theta\beta) \left(\hat{P}_t + \hat{Z}_t\right) + \theta\beta E_t \hat{P}_{t+1} \\ (1 - \theta) \hat{P}_t^* &= (1 - \theta) (1 - \theta\beta) \left(\hat{P}_t + \hat{Z}_t\right) + (1 - \theta) \theta\beta E_t \hat{P}_{t+1}^* \end{aligned}$$

Now using (B.18) we can write $\hat{P}_t - \theta\hat{P}_{t-1} = (1 - \theta)\hat{P}_t^*$ or

$$\begin{aligned}
\hat{P}_t - \theta\hat{P}_{t-1} &= (1 - \theta)(1 - \theta\beta)\left(\hat{P}_t + \hat{Z}_t\right) + (1 - \theta)\theta\beta E_t\hat{P}_{t+1}^* \\
\hat{P}_t - \theta\hat{P}_{t-1} &= (1 - \theta)(1 - \theta\beta)\left(\hat{P}_t + \hat{Z}_t\right) + \theta\beta\left(E_t\hat{P}_{t+1} - \theta\hat{P}_t\right) \\
\hat{P}_t - \hat{P}_{t-1} &= -(1 - \theta)\hat{P}_{t-1} + (1 - \theta)(1 - \theta\beta)\left(\hat{P}_t + \hat{Z}_t\right) + \theta\beta\left(E_t\hat{P}_{t+1} - \theta\hat{P}_t\right) \\
\hat{P}_t - \hat{P}_{t-1} &= (1 - \theta)\hat{P}_t - \hat{P}_{t-1} + (1 - \theta)(1 - \theta\beta)\hat{Z}_t + \theta\beta\left(E_t\hat{P}_{t+1} - \hat{P}_t\right) \\
\hat{\pi}_t &= (1 - \theta)\hat{\pi}_t + \theta\beta E_t\hat{\pi}_{t+1} + (1 - \theta)(1 - \theta\beta)\hat{Z}_t \\
\hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \frac{(1 - \theta)(1 - \theta\beta)}{\theta}\hat{Z}_t
\end{aligned}$$

Finally, we get

$$\pi_t = \beta E_t\pi_{t+1} + \frac{(1 - \theta)(1 - \theta\beta)}{\theta}z_t \quad (\text{B.21})$$

B.1.5 The Benchmark Model

We need two more things to get to the “benchmark” New-Keynesian model. First, we need some way to relate z_t , marginal cost to y_t , output. Second, we need to determine monetary policy. To relate z_t to y_t we have to recognize that some firms can price over marginal cost, and some cannot when we have sticky prices. Therefore potential output reflects output under flexible prices. Flexible prices implies $\frac{(1 - \theta)(1 - \theta\beta)}{\theta} \rightarrow \infty$ and $z_t \rightarrow 0$ so

$$y_t^n = \frac{\eta}{\eta + \sigma - 1}a_t$$

The output gap is then a constant market

$$y_t - y_t^n = \frac{1}{\eta + \sigma - 1}z_t$$

This into (B.21) implies

$$\pi_t = \beta E_t\pi_{t+1} + (\eta + \sigma - 1)\frac{(1 - \theta)(1 - \theta\beta)}{\theta}(y_t - y_t^n) \quad (\text{B.22})$$

Monetary policy is an ad hoc specification of an interest rate rule in this model. The interest rate rule chosen in most textbook examples is a Taylor (1993) rule.

$$R_t = R_{t-1}^{\phi_r} \left(\bar{r} \left(\frac{P_t}{P_{t-1}} \right)^{1 + \phi_\pi} \left(\frac{Z_t}{Z} \right)^{\phi_z} \right)^{1 - \phi_r} \epsilon_{r,t} \quad (\text{B.23})$$

Since we are not choosing an optimal monetary policy rule we can log-linearize this directly to get

$$r_t = \phi_r r_{t-1} (1 - \phi_r) ((1 + \phi_\pi) \pi_t + \phi_z z_t) + \varepsilon_{r,t} \quad (\text{B.24})$$

Substituting our earlier result for z_t is straightforward. The three equations then for the benchmark New-Keynesian model are

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \sigma^{-1} (r_t - E_t \pi_{t+1}) \quad (\text{B.25})$$

$$\pi_t = \beta E_t \pi_{t+1} + \psi \tilde{y}_t \quad (\text{B.26})$$

$$r_t = \phi_r r_{t-1} (1 - \phi_r) ((1 + \phi_\pi) \pi_t + \phi_x (y_t - y_t^n)) + \varepsilon_{r,t} \quad (\text{B.27})$$

Note, (B.27) is not needed for this paper but is included for the sake of completeness.

B.2 State-Space Representation

B.2.1 Time Varying Target Rate Model

The equations that comprise the estimated model are

$$\begin{aligned}
 x_t - [E_t x_{t+1} | z_t] + \sigma^{-1} (i_t - [E_t \pi_{t+1} | z_t]) &= \varepsilon_t^d \\
 \pi_t - \psi x_t - \beta [E_t \pi_{t+1} | z_t] &= \varepsilon_t^s \\
 \tilde{\pi}_t &= \pi_t - \pi_t^* \\
 i_t &= \frac{\psi\phi}{\sigma} \tilde{\pi}_t + \frac{\alpha\psi\phi}{2\sigma} \tilde{\pi}_t^2 + \frac{\lambda}{\sigma} x_t + \frac{\gamma\lambda}{2\sigma} x_t^2 + i_t^* + \zeta i_{t-1} + \delta_t
 \end{aligned}$$

The unobserved components i_t^* and π_t^* are assumed to follow a stable AR(1) law of motion.

$$\begin{aligned}
 i_t^* &= \mu^{i^*} + \rho_i i_{t-1}^* + u_t^i \\
 \pi_t^* &= \mu^{\pi^*} + \rho_\pi \pi_{t-1}^* + u_t^\pi
 \end{aligned}$$

One state-space form of this system is below. The state equation is given by

$$\xi_t = F \xi_{t-1} + B + Q \nu_t$$

$$\xi_t = \begin{bmatrix} i_t^* \\ \pi_t^* \\ \delta_t \\ \varepsilon_t^d \\ \varepsilon_t^s \end{bmatrix}, \quad F = \begin{bmatrix} \rho_i & 0 & 0 & 0 & 0 \\ 0 & \rho_\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \mu^{i^*} \\ \mu^{\pi^*} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Q = I_{5 \times 5}, \quad \nu_t = \begin{bmatrix} u_t^i \\ u_t^\pi \\ \delta_t \\ \varepsilon_t^d \\ \varepsilon_t^s \end{bmatrix}$$

The observation equation is given by

$$y_t = A X_t + H \xi_t$$

$$y_t = \begin{bmatrix} \sigma^{-1} (i_t - [E_t \pi_{t+1} | z_t]) - [E_t x_{t+1} | z_t] \\ \pi_t - \beta [E_t \pi_{t+1} | z_t] \\ \pi_t \\ i_t - \zeta i_{t-1} \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \psi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\lambda}{\sigma} & \frac{\gamma\lambda}{2\sigma} & \frac{\psi\phi}{\sigma} & \frac{\alpha\psi\phi}{2\sigma} \end{bmatrix}, \quad X_t = \begin{bmatrix} x_t \\ x_t^2 \\ \tilde{\pi}_t \\ \tilde{\pi}_t^2 \end{bmatrix}$$

Note $\tilde{\pi}_t$ and $\tilde{\pi}_t^2$ are functions of the unobserved vector element, ξ_{2t} . These elements must be recursively updated after the matrix $\xi_{t+1|t}$ is updated.

B.2.2 Estimation Procedure

The Kalman filter is a direct estimation of the log-likelihood function by the following algorithm. Given the initial conditions

$$\xi_{1|0} = [0 \quad \cdots \quad 0]'$$

and

$$P_{1|0} = \text{vec} [(I_{25 \times 25} - F \otimes F)^{-1} \text{vec}(Q\Sigma Q')]$$

$$\text{where } \Sigma = E(\nu_t \nu_t')$$

δ_t , ε_t^d , and ε_t^s are rational expectations error process which implies that they are mutually and serially uncorrelated with each other for all t . The covariance matrix is econometrically assumed symmetric along the diagonal.

$\xi_{t|t-1}$ and $P_{t|t-1}$ can be recursively estimated and the log-likelihood function updated by the following sequence.

$$L = -T \ln(2\pi) + \sum_{t=1}^{T-1} ll_t$$

where

$$ll_t = -\frac{1}{2} \ln [\det (HP_{t|t-1}H')] - \frac{1}{2} (y_t - Ax_t - H\xi_{t|t-1})' (HP_{t|t-1}H')^{-1} (y_t - Ax_t - H\xi_{t|t-1})$$

$$K_t = FP_{t|t-1}H' (HP_{t|t-1}H')^{-1}$$

$$\xi_{t+1|t} = F\xi_{t|t-1} + K_t (y_t - Ax_t - H\xi_{t|t-1}) + B$$

$$P_{t+1|t} = (F - K_tH) P_{t|t-1} (F' - H'K_t') + Q\Sigma Q'$$

$$X_{3t+1|t} = (\pi_{t+1} - \xi_{2t+1|t})$$

$$X_{4t+1|t} = (\pi_{t+1} - \xi_{2t+1|t})^2$$

for $t = (1, 2, \dots, T-1)$

The parameters α , γ , ϕ , λ , ζ , β , ψ , μ^{i*} , $\mu^{\pi*}$, ρ_i , ρ_π , and the relevant variance-covariance estimates are all chosen to minimize the sum of the log-likelihoods in the above algorithm.

As discussed in (Hamilton, 1994, Ch. 13), inference using $P_{t+1|T}$ when there are transformed parameters is infeasible. An unbiased estimate $P_{t+1|T}$ must be reestimated using non-transformed optimal parameters from the algorithm described above. The optimal $P_{t+1|T}$ is used to construct the optimal Fisher information matrix (inverse hessian matrix). Standard errors are calculated by taking the square root of the diagonal elements of the inverse of the hessian matrix.

B.2.3 Time Varying Aversion Parameter Model

The equations that comprise the estimated model are

$$x_t - [E_t x_{t+1} | z_t] + \sigma^{-1} (i_t - [E_t \pi_{t+1} | z_t]) = \varepsilon_t^d$$

$$\pi_t - \psi x_t - \beta [E_t \pi_{t+1} | z_t] = \varepsilon_t^s$$

$$\tilde{\pi}_t = \pi_t - \pi_t^*$$

$$i_t = \frac{\psi}{\sigma} \phi_t \tilde{\pi}_t + \frac{\alpha \psi}{2\sigma} \phi_t \tilde{\pi}_t^2 + \frac{1}{\sigma} \lambda_t x_t + \frac{\gamma}{2\sigma} \lambda_t x_t^2 + i_t^* + \zeta i_{t-1} + \delta_t$$

Rearranging the interest rate rule by expanding $\tilde{\pi}_t$ yields

$$i_t = \frac{\psi}{\sigma} \phi_t \pi_t - \frac{\psi}{\sigma} \phi_t \pi_t^* + \frac{\alpha \psi}{2\sigma} \phi_t \pi_t^2 - \frac{\alpha \psi}{\sigma} \phi_t \pi_t \pi_t^* + \frac{\alpha \psi}{2\sigma} \phi_t (\pi_t^*)^2 + \frac{1}{\sigma} \lambda_t x_t + \frac{\gamma}{2\sigma} \lambda_t x_t^2 + i_t^* + \zeta i_{t-1} + \delta_t$$

The unobserved components i_t^* , π_t^* , ϕ_t , and λ_t are assumed to follow a stable AR(1) law of motion.

$$i_t^* = \mu^{i^*} + \rho_i i_{t-1}^* + u_t^i$$

$$\pi_t^* = \mu^{\pi^*} + \rho_\pi \pi_{t-1}^* + u_t^\pi$$

$$\phi_t = \mu^\phi + \rho_\phi \phi_{t-1} + u_t^\phi$$

$$\lambda_t = \mu^\lambda + \rho_\lambda \lambda_{t-1} + u_t^\lambda$$

One state-space form of this system is below. The state equation is given by

$$\xi_t = F \xi_{t-1} + B + Q \nu_t$$

$$\xi_t = \begin{bmatrix} i_t^* \\ \pi_t^* \\ \phi_t \\ \lambda_t \\ \delta_t \\ \varepsilon_t^d \\ \varepsilon_t^s \end{bmatrix}, \quad F = \begin{bmatrix} \rho_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \mu^{i^*} \\ \mu^{\pi^*} \\ \mu^\phi \\ \mu^\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \nu_t = \begin{bmatrix} u_t^i \\ u_t^\pi \\ u_t^\phi \\ u_t^\lambda \\ \delta_t \\ \varepsilon_t^d \\ \varepsilon_t^s \end{bmatrix}$$

and $Q = I_{7 \times 7}$. The observation equation is given by

$$y_t = A X_t + H \xi_t$$

$$y_t = \begin{bmatrix} \sigma^{-1} (i_t - [E_t \pi_{t+1} | z_t]) - [E_t \tilde{y}_{t+1} | z_t] \\ -\beta [E_t \pi_{t+1} | z_t] \\ i_t - \zeta i_{t-1} \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & a & -\frac{\psi}{\sigma} \phi_t & 0 & 1 & 0 & 0 \end{bmatrix}$$

Where $a = \frac{\alpha\psi}{\sigma} \left(\frac{1}{2} \pi_t^* - \pi_t \right)$

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & \psi & 0 \\ \frac{\psi}{\sigma} \phi_t & \frac{\alpha\psi}{2\sigma} \phi_t & \frac{1}{\sigma} \lambda_t & \frac{\gamma}{2\sigma} \lambda_t \end{bmatrix}, \quad X_t = \begin{bmatrix} \pi_t \\ \pi_t^2 \\ x_t \\ x_t^2 \end{bmatrix}$$

Note H and A are each a function of elements of ξ_t and observed data in X_t . These must be updated recursively after the matrix $\xi_{t+1|t}$ is updated.

B.2.4 Estimation Procedure

The Kalman filter is a direct estimation of the log-likelihood function by the following algorithm. Given the initial conditions

$$\xi_{1|0} = [0 \ \cdots \ 0]'$$

and

$$P_{1|0} = \text{vec} [(I_{49 \times 49} - F \otimes F)^{-1} \text{vec} (Q \Sigma Q')]$$

$$\text{where } \Sigma = E(\nu_t \nu_t')$$

δ_t , ε_t^d , and ε_t^s are rational expectations error process which implies that they are mutually and serially uncorrelated with each other for all t . The covariance matrix is econometrically assumed symmetric along the diagonal.

$\xi_{t|t-1}$, $P_{t|t-1}$, $H_{t|\xi_{t+1}}$, $A_{t|\xi_{t+1}}$, and $x_{t|\xi_{t+1}}$ can be recursively estimated and the log-likelihood function updated by the following sequence.

$$L = -T \ln(2\pi) + \sum_{t=1}^{T-1} ll_t$$

where

$$ll_t = -\frac{1}{2} \ln [\det (HP_{t|t-1}H')] - \frac{1}{2} (y_t - Ax_t - H\xi_{t|t-1})' (HP_{t|t-1}H')^{-1} (y_t - Ax_t - H\xi_{t|t-1})$$

$$K_t = FP_{t|t-1}H' (HP_{t|t-1}H')^{-1}$$

$$\xi_{t+1|t} = F\xi_{t|t-1} + K_t (y_t - Ax_t - H\xi_{t|t-1}) + B$$

$$P_{t+1|t} = (F - K_tH) P_{t|t-1} (F' - H'K_t') + Q\Sigma Q'$$

$$H(3, 2) = -\frac{\alpha\psi}{\sigma} \left(\frac{1}{2} \xi_{2|t+1} - x_{1t} \right)$$

$$H(3, 4) = -\frac{\psi}{\sigma} \times \xi_{3t|t+1}$$

$$A(3, 1) = \frac{\psi}{\sigma} \times \xi_{3t|t+1}$$

$$A(3, 2) = \frac{\alpha\psi}{2\sigma} \times \xi_{3t|t+1}$$

$$A(3, 3) = \frac{1}{\sigma} \times \xi_{4t|t+1}$$

$$A(3, 4) = \frac{\gamma}{2\sigma} \times \xi_{4t|t+1}$$

for $t = (1, 2, \dots, T - 1)$

The parameters $\alpha, \gamma, \zeta, \beta, \psi, \mu^{i^*}, \mu^{\pi^*}, \mu^\phi, \mu^\lambda, \rho_i, \rho_\pi, \rho_\phi, \rho_\lambda$, and the relevant variance-covariance estimates are all chosen to minimize the sum of the log-likelihoods in the above algorithm.

As discussed in ([Hamilton, 1994](#), Ch. 13), inference using $P_{t+1|T}$ when there are transformed parameters is infeasible. An unbiased estimate $P_{t+1|T}$ must be reestimated using non-transformed optimal parameters from the algorithm described above. The optimal $P_{t+1|T}$ is used to construct the optimal Fisher information matrix (inverse hessian matrix). Standard errors are calculated by taking the square root of the diagonal elements of the inverse of the hessian matrix.

Appendix C

Chapter 3 Technical Notes

C.1 Deriving the Modified Model

C.1.1 Rational expectations

$$E_t \pi_{t+1} = \pi_{t+1} + \eta_t^\pi \quad (\text{C.1})$$

C.1.2 The UC updated Phillips curve

$$\begin{aligned} \pi_t &= \mu_t^\pi + \psi_t^\pi + \beta E_t \pi_{t+1} + \varphi x_t + \varepsilon_t^s \\ \pi_t &= \mu_t^\pi + \psi_t^\pi + \beta \pi_{t+1} + \varphi x_t + \beta \eta_t^\pi + \varepsilon_t^s \\ \pi_t &= \mu_t^\pi + \psi_t^\pi + \beta \pi_{t+1} + \varphi \psi_t^y + \beta \eta_t^\pi + \varepsilon_t^s \end{aligned} \quad (\text{C.2})$$

C.1.3 The UC updated consumer's Euler Equation

Leading Equation (3.7) and using the law of iterated expectations leads to

$$E_t \psi_{t+1}^y = \rho \cos \lambda_c \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} + E_t \kappa_{t+1}^y \quad (\text{C.3})$$

where $E_t \kappa_{t+1}^y = 0$. Substituting this into (3.2) and using (C.1) implies

$$\psi_t^y = \rho \cos \lambda_c \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} - \sigma^{-1} (i_t - \pi_{t+1} + \eta_t^\pi) + \varepsilon_t^d \quad (\text{C.4})$$

Which can be rearranged by collecting terms to get

$$\psi_t^y = \frac{\rho \sin \lambda_c}{(1 - \rho \cos \lambda_c)} \psi_t^{y*} - \frac{1}{\sigma(1 - \rho \cos \lambda_c)} (i_t - \pi_{t+1}) - \frac{1}{\sigma(1 - \rho \cos \lambda_c)} \eta_t^\pi + \frac{1}{(1 - \rho \cos \lambda_c)} \varepsilon_t^d \quad (\text{C.5})$$

multiplying both sides by $(1 - \rho \cos \lambda_c)$ yields

$$(1 - \rho \cos \lambda_c) \psi_t^y = \rho \sin \lambda_c \psi_t^{y*} - \sigma^{-1} (i_t - \pi_{t+1}) - \sigma^{-1} \eta_t^\pi + \varepsilon_t^d \quad (\text{C.6})$$

or

$$\sigma^{-1} i_t = \sigma^{-1} \pi_{t+1} + (\rho \cos \lambda_c - 1) \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} - \sigma^{-1} \eta_t^\pi + \varepsilon_t^d \quad (\text{C.7})$$

C.2 State-Space Formulation - Speed Limits with Asymmetry

The system of equations to be estimated are

$$\mu_t^\pi = \mu_{t-1}^\pi + \nu_t^\pi \quad (\text{C.8})$$

$$\mu_t^y = \mu_{t-1}^y + \nu_{t-1}^y \quad (\text{C.9})$$

$$\nu_t^y = \nu_{t-1}^y + u_t^y \quad (\text{C.10})$$

$$y_t = \mu_t^y + \psi_t^y + \epsilon_t^y \quad (\text{C.11})$$

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos \lambda_c & \sin \lambda_c \\ -\sin \lambda_c & \cos \lambda_c \end{bmatrix} \otimes I_2 \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix} \quad (\text{C.12})$$

$$\pi_t = \mu_t^\pi + \psi_t^\pi + \beta \pi_{t+1} + \varphi \psi_t^y + \beta \eta_t^\pi + \varepsilon_t^s \quad (\text{C.13})$$

$$\sigma^{-1} i_t = \sigma^{-1} \pi_{t+1} + (\rho \cos \lambda_c - 1) \psi_t^y + \rho \sin \lambda_c \psi_t^{y*} - \sigma^{-1} \eta_t^\pi + \varepsilon_t^d \quad (\text{C.14})$$

$$\phi \varphi \left\{ (\pi_t - \pi_{t-1}) + \alpha \frac{(\pi_t - \pi_{t-1})^2}{2} \right\} + \lambda \left\{ (x_t - x_{t-1}) + \gamma \frac{(x_t - x_{t-1})^2}{2} \right\} = 0 \quad (\text{C.15})$$

The state equation is given by

$$\xi_t = F \xi_{t-1} + Q V_t \quad (\text{C.16})$$

where

$$\xi_t = \begin{bmatrix} \mu_t^\pi \\ \mu_t^y \\ \nu_t^y \\ \psi_t^\pi \\ \psi_t^y \\ \psi_t^{\pi*} \\ \psi_t^{y*} \\ \varepsilon_t^d \\ \varepsilon_t^s \\ \epsilon_t^y \\ \eta_t^\pi \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \cos \lambda_c & 0 & \rho \sin \lambda_c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho \cos \lambda_c & 0 & \rho \sin \lambda_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho \sin \lambda_c & 0 & \rho \cos \lambda_c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\rho \sin \lambda_c & 0 & \rho \cos \lambda_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad V_t = \begin{bmatrix} \nu_t^\pi \\ u_t^y \\ \kappa_t^\pi \\ \kappa_t^y \\ \kappa_t^{\pi*} \\ \kappa_t^{y*} \\ \varepsilon_t^d \\ \varepsilon_t^s \\ \epsilon_t^y \\ \eta_t^\pi \end{bmatrix}$$

The observation equation is given by

$$Y_t = AX_t + H\xi_t \tag{C.17}$$

where

$$Y_t = \begin{bmatrix} y_t \\ \pi_t \\ \sigma^{-1}i_t \\ \phi\varphi(\pi_t - \pi_{t-1}) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \sigma^{-1} & 0 & 0 & 0 \\ 0 & -\frac{\phi\varphi\alpha}{2} & -\lambda & -\frac{\lambda\gamma}{2} \end{bmatrix}, \quad X_t = \begin{bmatrix} \pi_{t+1} \\ (\pi_t - \pi_{t-1})^2 \\ (\psi_t - \psi_{t-1}) \\ (\psi_t - \psi_{t-1})^2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & \varphi & 0 & 0 & 0 & 1 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & (\rho \cos \lambda_c - 1) & 0 & \rho \sin \lambda_c & 1 & 0 & 0 & -\sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C.2.1 Estimation Algorithm

The Kalman filter is a direct estimation of the log-likelihood function by the following algorithm. Given the initial conditions

$$\xi_{1|0} = [0 \ \cdots \ 0]'$$

and

$$P_{1|0} = \text{vec} [(I_{121 \times 121} - F \otimes F)^{-1} \text{vec} (Q \Sigma Q')]$$

where $\Sigma = E(V_t V_t')$ and is symmetric. Individually, $E(\kappa_t^i \kappa_t^{i'}) = E(\kappa_t^{i*} \kappa_t^{i*'}) = \hat{\sigma}_\kappa^2$, for $i = [\pi, y]$. Also ν_t^π , u_t^y , ϵ_t^y , and η_t^π all follow a multivariate Gaussian distribution and are mutually and serially uncorrelated.

$\xi_{t|t-1}$ and $P_{t|t-1}$ can be recursively estimated and the log-likelihood function updated by the following sequence.

$$L = -T \ln(2\pi) + \sum_{t=1}^{T-1} ll_t$$

where

$$ll_t = -\frac{1}{2} \ln [\det (H P_{t|t-1} H')] - \frac{1}{2} (y_t - A x_t - H \xi_{t|t-1})' (H P_{t|t-1} H')^{-1} (y_t - A x_t - H \xi_{t|t-1})$$

$$K_t = F P_{t|t-1} H' (H P_{t|t-1} H')^{-1}$$

$$\xi_{t+1|t} = F \xi_{t|t-1} + K_t (y_t - A x_t - H \xi_{t|t-1})$$

$$P_{t+1|t} = (F - K_t H) P_{t|t-1} (F' - H' K_t') + Q \Sigma Q'$$

$$X(3, 1) = (\xi_{5,t+1|t} - \xi_{5,t|t-1})$$

$$X(4, 1) = (\xi_{5,t+1|t} - \xi_{5,t|t-1})^2$$

for $t = (1, 2, \dots, T-1)$

The parameters α , γ , ϕ , λ , φ , β , σ , ρ , λ_c and the relevant variance-covariance estimates are all chosen to minimize the sum of the log-likelihoods in the above algorithm. Standard errors are calculated by taking the square root of the diagonal elements of the inverse of the hessian matrix that is constructed from the optimal $P_{t+1|T}$ matrix in the Kalman recursion.

C.2.2 Smoothed Estimates of the State Vector

Smoothed estimates of the state vector can be recursively solved for by decomposing the Kalman Gain, $K_t = FP_{t|t-1}H' (HP_{t|t-1}H')^{-1}$ and storing the one-step updates, $\xi_{t|t}$, $\xi_{t+1|t}$, $P_{t|t}$, and $P_{t+1|t}$. After optimal estimation of the $\xi_{T|T}$ and $P_{T|T}$ matrices, the smoothed estimates can be backward solved by

$$\hat{\xi}_{T-i-1|T-i} = \hat{\xi}_{T-i-1|T-i-1} + J_{T-i-1} \left(\hat{\xi}_{T-i|T-i} - \hat{\xi}_{T-i|T-i-1} \right)$$

where $J_{T-i} = P_{t|t}F'P_{t+1|t}^{-1}$, for $i = (T - 1, \dots, 1)$.