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Identification of obstacles for parabolic problems

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ABSTRACT

Obstacle identification problems for parabolic equations and systems are considered. Unique continuation property plays an important role in the arguments. These arguments are based on the idea which was first used in the paper MR0847528 (87k:35248) of one of the authors earlier. The technique is applied to inverse problems for slightly compressible fluids and to inverse problems for the heat equation.

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1 Introduction

We consider the problem of identifying an obstacle or inclusion inside a body which boundary is partly accessible. We assume that the problem can be modeled by a parabolic evolution equation. The data and the measurement are supported in a finite time interval.
In [ACFKO05] the authors considered the obstacle identifiability for the stationary and non-stationary (Navier-)Stokes equations using the unique continuation property for the stationary and the non-stationary equations. We prove that the obstacle identification problem for the parabolic equation can be solved by using the unique continuation property for the associated elliptic problem. The unique continuation property for elliptic equations and systems has been studied earlier, see, e.g., [Hoer83], [Tat95], [FL96], [AM01], [NUW04], [Wol92].

2 Optimal Control

The obstacle identification problem for elliptic and parabolic problems has been studied e.g. in [CRV02]. The ideas of this paper are based on the articles [Ram86] and [Ram08].
In order to describe the problem, let $\Omega \subset \mathbb{R}^n$ denote a domain with C^2-boundary. We assume that there is an inclusion $D \subset \overline{D} \subset \Omega$ in the domain and set $\Omega_D := \Omega \setminus \overline{D}$.
Denote by A a differential operator either in the divergence form

$$Au := -\mathrm{div}\,(a_{ij}\nabla u) + qu,$$

or in the non-divergence form

$$Au := -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + qu$$

with bounded and measurable coefficients a_{ij}, q in Ω. The operator A is elliptic if there is a constant $c > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j > c|\xi|^2, \qquad \text{for any } \xi \in \mathbb{R}^n.$$

If $A := -\sum_{i,j=1}^{n} a_{ij} D_{ij}$ is matrix-valued, i.e. $a_{ij}(x) \in \mathbb{C}^{N\times N}$ for $x \in \mathbb{R}^n$, then A is parameter-elliptic if there exist constants $\theta \in (0, \frac{\pi}{2})$ and $M > 0$ such that

$$\sigma(\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j) \subset \Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg\lambda| \le \theta\} \text{ and}$$
$$\sup_{x\in\Omega} \|(\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j)^{-1}\| \le M$$

holds for all $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Here the norm $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(\mathbb{C}^n)$.
Consider parabolic problems of the form

$$\begin{aligned} u' + Au &= F \qquad \text{in } (0,\infty)\times\Omega\setminus\overline{D}, \\ u &= 0 \qquad \text{on } (0,\infty)\times\partial(\Omega\setminus\overline{D}), \\ u(0) &= 0. \end{aligned} \tag{2.1}$$

Let $B : H^1(\partial\Omega) \to L^2(\partial\Omega)$ be a bounded operator and $\tilde{\Gamma}$ be an open part of the boundary $\partial\Omega$. In our applications below the operator B will be associated to Neumann type boundary conditions. In general, we assume that B is such that the corresponding boundary condition defines a Fredholm-type problem for the operator A. For example, the case of B which is defined by tangential derivatives is excluded.
In this work the following inverse problems are studied:

(a) *Let U denote an open subset of $\Omega\setminus\overline{D}$. Let the function F be given together with the extra data $Bu|_{\tilde{\Gamma}}$ in a finite time interval, where u is the solution of* (2.1)*, and assume that* $\operatorname{supp} F(t) \subset U$ *for all* $t > 0$.

Do these data determine the obstacle D uniquely?

(b) *Consider the initial boundary value problem*

$$\begin{aligned} u' + Au &= 0 \qquad \text{in } (0,\infty)\times\Omega\setminus\overline{D} \\ u &= \varphi \qquad \text{on } (0,\infty)\times\partial\Omega \\ u &= 0 \qquad \text{on } (0,\infty)\times\partial D \\ u(0) &= 0 \end{aligned} \tag{2.2}$$

Given the data (φ, $Bu|_{\tilde{\Gamma}}$), can one determine the obstacle D uniquely?

3 Main result

We give positive answer to both questions raised above, assuming that F and φ are suitably chosen. Specifically, we assume that these functions are differentiable and supported in a compact time interval.

The proof of the uniqueness is similar to problems (2.1) and (2.2). We prove uniqueness result for problem (2.2) and outline a new point in the proof for problem (2.1).

Let $\Omega_0 \subset \mathbb{R}^n$ be a domain and $\Gamma_0 \subset \partial\Omega_0$ be a relatively open subset of the boundary. Assume that A is a second order differential operator in the divergence form or in the non-divergence form and that $B : H^1(\partial\Omega) \to L^2(\partial\Omega)$ is a bounded operator. Then we say that the pair (A, B) has the *uniqueness property for the associated Cauchy problem*, if there exists λ_0 such that for $\lambda > \lambda_0$ any solution u of the problem

$$\begin{aligned}(\lambda + A)u &= 0 \quad \text{in } \Omega_0,\\ u &= 0 \quad \text{on } \Gamma_0,\\ Bu &= 0 \quad \text{on } \Gamma_0,\end{aligned}$$

vanishes in Ω_0.

We say that A has the unique continuation property, if for $\lambda > \lambda_0$ the function u vanishes in Ω_0 whenever $(\lambda + A)u = 0$ in Ω_0 and $u = 0$ in a nonempty open subset of Ω_0.

We collect these properties in the following (unique continuation=UC) assumption:

(UC) *The pair (A, B) with operators A and B, satisfying the above assumptions, has the uniqueness property for the associated Cauchy problem and A has the unique continuation property in Ω.*

Note, that this property holds e.g. when B is the normal derivative at the boundary $Bu = \nabla u \cdot n$ and A is an elliptic differential operator with smooth scalar-valued coefficients. See [Hoer83].

The realization of the operator A in $L^2(\Omega_D)$, associated with equation (2.1), i.e., with the homogeneous Dirichlet boundary condition, is given by

$$\begin{aligned}D(A_D) &:= \{u \in H_0^1(\Omega_D) : Au \in L^2(\Omega_D)\}\\ A_D u &:= Au.\end{aligned}$$

Recall that we use the notation $\Omega_D = \Omega \setminus \overline{D}$. We assume that the operator $-A$ is the generator of a C_0-semigroup. This makes no regularity assumptions on ∂D necessary, if A is given in divergence form and q is real-valued and bounded from below. Since A is defined by a closed sectorial form, the fact that $-A$ is a generator of a C_0-semigroup is well known, see e.g. [Kat84].

For non-divergence form operators the question is more delicate. If D has C^2-boundary, the second-order coefficients of A are uniformly continuous and $q \in L^\infty$ then the realization of the operator $-A = \sum_{i,j=1}^n a_{ij} D_{ij} + q$ with boundary conditions satisfying the Shapiro-Lopatinskii condition is a generator. For generator results concerning differential operators in the non-divergence form see e.g. the monograph [Lun95] or [Theorem 8.2](DHP03). Schrödinger operators with unbounded potentials are treated, e.g., in [C87].

Let us now state our main result. We assume that each of the problems (2.1) and (2.2) has a unique global solution which grows, as time grows, not faster than an exponential. Sufficient conditions on the smoothness of the boundaries of D and Ω, on the operator A, and of the data F and φ, for this to happen are known (see, e.g., [Kat84], [P83]). We also assume that in (2.2) $\varphi = f(s)h(t)$, where $f \not\equiv 0$ is a smooth function on $\partial\Omega$ and $h \in \mathbb{C}_0^\infty(0,\infty)$, $h \not\equiv 0$, $\operatorname{supp} h \subset (c, d)$, $0 < c < d < \infty$. Similarly, we assume that in (2.1) $F = f_1(x)h_1(t)$, where $f_1 \in C_0^\infty(\omega)$, ω is a subset of $\Omega \setminus D := \Omega_D$, h_1 has the same properties as h, and $f_1 \not\equiv 0$.

Theorem 3.1. Suppose that assumption (UC) holds and that D_1 and D_2 are bounded domains with $\overline{D_i} \subset \Omega$. Let u_1 and u_2 denote solutions to (2.2) on the domains $\Omega \setminus \overline{D_1}$ and $\Omega \setminus \overline{D_2}$, respectively. If $Bu_1 = Bu_2$ in $(c,d) \times \tilde{\Gamma}$ on an open subset $\tilde{\Gamma} \subset \partial\Omega$, then $D_1 = D_2$.

Proof. Let u_1 and u_2 denote solutions of problem (2.2) on the domains Ω_{D_1} and Ω_{D_2}, respectively, and $Bu_1 = Bu_2$. The solutions u_i satisfy the estimates

$$\|u_i(t)\|_{L^2(\Omega_{D_i})} \leq C\mathrm{e}^{wt},$$

where $w > 0$ is a constant. Taking the Laplace transform of u_1 and u_2 for sufficiently large $\lambda > 0$, one gets:

$$(\lambda + A)\hat{u}_i = 0, \quad \text{in } \Omega_{D_i},$$
$$\hat{u}_i = f(s)\hat{h} \quad \text{on } \partial\Omega,$$
$$\hat{u}_i = 0 \quad \text{on } \partial D_i,$$

where

$$\hat{u} := \int_0^\infty \mathrm{e}^{-\lambda t} u(t)\,\mathrm{d}t.$$

By our assumption, $B\hat{u}_1 = B\hat{u}_2$ for all sufficiently large λ with $\mathrm{Re}\,\lambda > 0$. Set $D := D_1 \cup D_2$ and $\hat{w} := \hat{u}_1 - \hat{u}_2$. Then $(\lambda + A)\hat{w} = 0$ in Ω_D with $\hat{w} = B\hat{w} = 0$ on $\tilde{\Gamma}$. The uniqueness of the solution to the Cauchy problem for (A, B) yields $\hat{w} \equiv 0$ in a connected component of Ω_D whose boundary contains $\tilde{\Gamma}$.
Let D_3 be a nonempty connected component of $D_2 \setminus \overline{D_1}$. Then $(\lambda + A)\hat{u}_1 = 0$ in D_3, and $\hat{u}_1 = 0$ on ∂D_3 since $\hat{w} = \hat{u}_1 - \hat{u}_2 = 0$ in Ω_D so that $\hat{u}_1 = \hat{u}_2 = 0$ on the part of the boundary of D_3 which belongs to ∂D_2. Since the spectrum of the Dirichlet operator A in D_3 is discrete, there is an open subset of positive λ-s which are regular points of this operator. Therefore, $\hat{u}_1 = 0$ in D_3 for these λ-s. By analyticity with respect to λ of $\hat{u}_1$, one concludes that $\hat{u}_1 \equiv 0$ in D_3 for all λ in a right-hand half-plane. The unique continuation property now yields $\hat{u}_1 \equiv 0$ in $\Omega \setminus \overline{D_1}$. Thus, $\hat{u}_1 = 0$ on $\tilde{\Gamma}$. On the other hand, $\hat{u}_1(s, \lambda) = f(s)\hat{h} \not\equiv 0$ on $\tilde{\Gamma}$. This contradiction proves Theorem 2.1. □
A similar proof is valid for the problem (2.1). For simplicity we sketch the proof assuming that $A = -\nabla^2$. Arguing as above, one concludes that $\hat{u}_1 = 0$ in $\Omega_{D_1} \setminus \omega$ and

$$(\lambda + A)\hat{u}_1 = f_1(x)\hat{h}_1 \quad (*)$$

in ω for all $\lambda > 0$, where $f_1 \not\equiv 0$, $\hat{h}_1 \not\equiv 0$, and $\omega = \mathrm{supp}\, f_1$.
This leads to a contradiction. Indeed, since $\hat{u}_1 = 0$ in $\Omega \setminus \omega$, it follows that $\hat{u}_1 = \hat{u}_{1N} = 0$ on $\partial\omega$, where N is the unit normal to $\partial\omega$, pointing out of ω. To derive a contradiction let us show that (*) and the conditions $\hat{u}_1 = \hat{u}_{1N} = 0$ imply $f_1 = 0$. Let $k := \lambda^{1/2}$, S^{n-1} be the unit sphere, and $a \in S^{n-1}$ be an arbitrary unit vector. Multiply (*) by $exp(ka \cdot x)$, integrate over ω and then by parts. Using the conditions

$$\hat{u}_1 = \hat{u}_{1N} = 0$$

and the equation $(\lambda + A)exp(ka \cdot x) = 0$, one obtains

$$\int_\omega f_1(x) exp(ka \cdot x) dx = 0,$$

for all $k > 0$ and all $a \in S^{n-1}$. This implies that the Fourier transform of f_1 is zero, so $f_1 = 0$, which contradicts our assumption that $f_1 \neq 0$. This contradiction proves the uniqueness theorem for problem (2.1).
For a special choice of F, namely, $F = \delta(x - z)h_1(t)$, where $\delta(x - z)$ is the delta-function and $z \in \Omega_D$ is a point, the proof is shorter. In this case equation (*) takes the form $(\lambda + A)\hat{u}_1 = \delta(x - z)$ and $\hat{u}_1 = 0$ in $\Omega_D \setminus \{z\}$. This is a contradiction, since the Green's function is known to satisfy the relation $\lim_{x\to z} |\hat{u}_1| = \infty$.

4 Applications

4.1 Slightly compressible fluids

We consider the system

$$\begin{array}{rcll} u_t - \mathrm{div}\,\big(\nu(\nabla u + \nabla^T u) + \beta(\mathrm{div}\, u) Id\big) & = & f & \text{in } (0,T)\times\Omega, \\ u & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) & = & 0, & \end{array} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ can be a bounded or unbounded domain with nonempty boundary and $\nu,\ \beta > 0$ are constants. The matrix Id denotes the identity matrix in $\mathbb{R}^{n\times n}$. We set $Au := \mathrm{div}\,(\nu(\nabla u + \nabla^T u) + \beta(\mathrm{div}\, u)Id)$. The measurement at the boundary is modeled by $Bu := (\nu(\nabla u + \nabla u^T) + \beta(\mathrm{div}\, u)Id)\cdot n|_{\partial\Omega}$, where n denotes the unit outer normal to the boundary.
For large $\beta > 0$ the system (2.2) with A as above describes the motion of a slightly compressible fluid. The pressure can then be written as $p = (\beta+\nu)\mathrm{div}\, u$. This system was studied in [Tem77] in order to approximate the Navier-Stokes system. Numerically this system is easier to handle than the Stokes system since the condition $\mathrm{div}\, u = 0$ is not present. On the other hand, for very large β the resulting problem is ill-conditioned.
In view of Korn's inequality the operator $-A$ generates a holomorphic C_0-semigroup. The unique continuation property for A is exactly the unique continuation for the elasticity system (see [AM01]). The uniqueness property for the associated Cauchy problem for (A, B) follows from unique continuation. Indeed, let u denote a solution of (4.1) with $u = 0$ and $Bu = 0$ on an open part Γ of the boundary of Ω. Take $x_0 \in \Gamma$ and a small $B(x_0, r)$ such that $B(x_0, r)\cap\partial\Omega \subset \Gamma$. Extend u by zero to $B(x_0, r)\cap\Omega^c$. Then u is a solution of $Au = 0$ in $\Omega \cup B(x_0, r)$ and $u = 0$ in $B(x_0, r)\cap\Omega^c$ with $\Omega^c = \mathbb{R}^n \setminus \Omega$. Hence $u = 0$ in Ω by the unique continuation property.
Therefore, it is possible to identify an obstacle immersed in a fluid which can be described by the equation (4.1). As data, the velocity of the fluid is prescribed at a part of the boundary and the resulting Cauchy forces are measured on the same part of the boundary but in another time interval.

4.2 The Stokes system

Although the Stokes system is not exactly of the form described above, the same technique can be used to get an identifiability result. This is due to the known fact that the Stokes operator generates a holomorphic C_0-semigroup (see e.g. [Soh01]) and that the unique continuation property holds for the Stokes system (see [FL96]).
Take a domain $\Omega \subset \mathbb{R}^n$ with non-empty boundary and a bounded obstacle immersed in the fluid. The motion of a viscous incompressible fluid can be described by the Stokes system which is given by

$$\left\{\begin{array}{rcll} u_t - \mathrm{div}\,\big(\nu(\nabla u + \nabla^T u)\big) + \nabla p & = & f & \text{in } (0,T)\times\Omega\setminus\overline{D}, \\ \mathrm{div}\, u & = & 0 & \text{in } (0,T)\times\Omega\setminus\overline{D}, \\ u & = & 0 & \text{on } (0,T)\times\partial D, \\ u & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) & = & 0, & \end{array}\right. \tag{4.2}$$

where ν denotes the constant viscosity coefficient and f is a force.
Let

$$A(u,p) := \mathrm{div}\,\big(\nu(\nabla u + \nabla^T u)\big) + \nabla p$$

and

$$B(u,p) := (\nu(\nabla u + \nabla u^T) + \mathrm{div}\,(p\cdot Id))\cdot n|_{\partial\Omega}.$$

We measure $Bu|_\Gamma$, where $\Gamma \subset \partial\Omega$ is an open part of the boundary. The identifiability of the obstacle D by the above data can be derived as follows. After applying the Laplace transform to the parabolic system (4.2) it follows that for $\lambda > 0$

$$(\lambda + A)\hat{u}_i + \nabla\hat{p}_i = \hat{f} \qquad \text{in } \Omega_{D_i}$$

where (u_i, p_i) is the unique solution of (4.2) with $D = D_i$, $i = 1, 2$.
Choose a large ball K such that $D = D_1 \cup D_2 \subset K$ and $\Gamma \subset K$. Let $w := u_1 - u_2$, $q = p_1 - p_2$. Then $(\lambda + A)\hat{w} + \nabla\hat{q} = 0$ in $K \cap \Omega_D$. The unique continuation property for the Stokes system yields $w = 0$ and $q = const$ by the same argument as in the proof of the main result. It follows that $\hat{u}_1 = 0$ in $D_2 \setminus \overline{D_1}$. Therefore, the obstacle is uniquely determined by the data.

4.3 Parabolic systems

Let us consider equations or systems of the form:

$$\begin{aligned} u_t - Au &= f, && \text{in } (0,T) \times \Omega_D, \\ u &= 0 && \text{on } (0,T) \times \partial\Omega, \\ Nu &= 0 && \text{on } (0,T) \times \partial D, \\ u(0) &= 0. \end{aligned}$$

Here A is an elliptic differential operator in divergence form or in non-divergence form. The operator N describes the boundary condition which holds at the boundary of the obstacle. One may assume, for example, the Dirichlet boundary condition as above, i.e. $Nu = u$, or the Neumann boundary condition, $Nu = \partial_n u = 0$, or the Robin boundary condition, $Nu = \partial_n u + \alpha u = 0$, where $\alpha > 0$ is a known function. If the Neumann or Robin conditions are considered, one has to assume some regularity for the boundary of the obstacle. For rather general assumptions concerning the regularity of the boundary of D see, e.g., [Ram05], [GR01], [GR05], [RS00], and [Ram04]. The arguments used in the proof of the main theorem can be used in a proof of the identifiability for obstacles with the Neumann or Robin boundary conditions.
In general, the assumed boundary condition should satisfy the Shapiro-Lopatinskii condition. This implies that the associated parabolic problem is associated to a holomorphic C_0-semigroup. For further information on this class of boundary-value problems we refer to [DHP03].
About the unique continuation property for elliptic operators see, e.g., the work by Hörmander [Hoer83] or Wolff [Wol92], and for systems see, e.g., [NUW04].

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