A NEW GENERALIZATION OF THE KHOVANOV HOMOLOGY

by

IK JAE LEE

B.S., Inha University, Korea, 2002

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
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Abstract

In this paper we give a new generalization of the Khovanov homology. The construction begins with a Frobenius-algebra-like object in a category of graded vector-spaces with an anyonic braiding, with most of the relations weaken to hold only up to phase. The construction of Khovanov can be adapted to give a new link homology theory from such data. Both Khovanov's original theory and the odd Khovanov homology of Oszvath, Rassmusen and Szabo arise from special cases of the construction in which the braiding is a symmetry.

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Approved by:

Major Professor
David Yetter

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Acknowledgments

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Finally I wish to thank my friends, Junghoon Lee, Hong Geun Kim, JoongMin Kwon, Uk Choi, and Pearl Cha for their many support to me and my family.

Special loving thanks to Junghoon yi hyung.

Dedication

to my family and my friend, Junghoon Lee

Chapter 1

A Brief Introduction to Knots and Khovanov Homology

In this chapter, we will look at basic definitions and facts on knots and links and Khovanov homology. For more introduction we are referred to [Adams[2]], [Bar-Natan[3]], [Bar-Natan[1]], [Kauffman[4]], [Kauffman and Lins[5]], [Lauda and Pfeiffer[6]], [Rolfsen[7]], and [Yetter[8]].

1.1 Knots

What is a mathematical knot? First, let us make a knot with a piece of string. Weave a strand of string around and through itself, and merge the ends to form a single continuous strand. This is a mathematical knot. In mathematics, study of knots began in the 19^{th} century. The main approach to knots started in the early 20^{th} century by J. W. Alexander and others from the aspects of invariants from homology theory and the knot group, such as the Alexander polynomial. The discovery of the Jones, HOMFLY-PT, and Kauffman polynomials opened a new era in knot theory finding new knot and 3-manifold invariants. From that time new and surprising connections have been found between topology, algebra, and physics. In the late 20^{th} century, scientists became interested in studying physical knots in order to understand knotting phenomena in DNA and other polymers. Knot theory can be used to detect chirality in a molecule (Simon[9], 1986). Knot theory may be crucial in the construction of quantum computers, through the model of topological quantum computation

(Collins[10], 2006).

Let us recall the precise definition of knots and links:

Definition 1.1.1. *A (classical) knot is an embedding of S^1 into S^3 (or $\mathbb{R}^3$).*
A (classical) link is an embedding of $\coprod_{i=1}^{n} S^1$ *into S^3 (or $\mathbb{R}^3$), for some $n \in \mathbb{N}$.*

Note : *in order to consider an "empty link", we allow $n = 0$.*

So a mathematical knot is a simple closed loop, which is different from physical knot - what we can see daily in our shoes. And a link is a finite set of non-intersecting knots. In mathematics, knot theory is primarily studied using a notion of equivalence, which captures our intuition about manipulating a loop of string without cutting it gives the same loop of string in a different configuration, which is called **ambient isotopy**:

Definition 1.1.2. *Two knots or links K_1, K_2 are ambient isotopic or simply equivalent if there is an isotopy $H : S^3 \times \mathbb{I} \to S^3$ (or similarly for $\mathbb{R}^3$ instead of S^3) which carries one to the other.*
More precisely, H is a PL map, satisfying $H(-, 0) = Id_{S^3}$; $H(-, t)$ is a PL-homeomorphism for each t ; and

$$H(K_1(x), 1) = K_2(x)$$

(using K_i to denote the mapping, with implied domain.)

Here, "isotopy" means the deformation of the string of a knot, and "ambient" refers to the fact that the three dimensional space is deformed along with the knot. So in an ambient isotopy, we can not shrink a part of knot to a point. For the simple intuitive manipulations of a knot diagram that we correspond to knot equivalence, we use the **Reidemeister moves**.

There is a famous theorem of Reidemeister which says that two knots are equivalent if and only if any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves. But we need invariants to demonstrate non-equivalence or give evidence for their equivalence of knots because it is difficult to show a sequence of

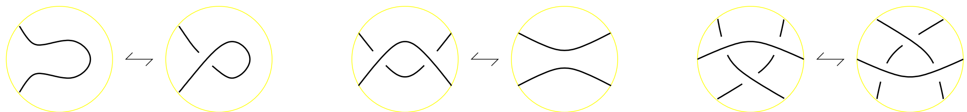


Figure 1.1: *The three Reidemeister moves $R1$, $R2$ and $R3$.*

Reidemeister moves even for simple knots that we know are equivalent to each other and not being able to give such a sequence does not prove none exists. There are many such knot invariants, such as the Alexander polynomial, and Jones polynomial. The Alexander polynomial is a knot invariant which gives a polynomial with integer coefficients to each knot type. Jones polynomial is a polynomial link invariant which is particularly appealing due to the simplicity of its combinatorial construction from the Kauffman bracket. But in the late 1990s, from the Kauffman bracket, Mikhail Khovanov developed another link invariant which is categorified version of the Jones polynomial. It is called the Khovanov homology.

1.2 Khovanov Homology

The story starts with the Kauffman bracket (bracket polynomial) [Louis H. Kauffman[11], 1987]. The (Kauffman) bracket polynomial of an oriented link diagram L with $w(L)$ the writhe of L, is defined by the following formula

$$f_L(A) = (-A^3)^{-w(L)} \langle L \rangle \Big/ \langle \bigcirc \rangle$$

with the following properties;

$\langle \emptyset \rangle = 1$;

$\langle \bigcirc L \rangle = (-A^2 - A^{-2}) \langle L \rangle$;

$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)\,(\rangle$;

$$\langle \diagup\!\!\!\!\diagdown \rangle = A \, \langle\,)\,(\,\rangle + A^{-1} \langle \asymp \rangle \,.$$

In fact, the Kauffman bracket is a polynomial invariant of framed links because it is not invariant under the first Reidemeister move. But its "normalized" version gives the famous knot invariant which is called the Jones polynomial. The Jones polynomial is a polynomial link invariant whose construction is very similar to the construction of the Khovanov homology. In the definition of the Kauffman bracket, if we set $q = -A^{-2}$, then we can get the following with normalization term, $(-1)^{n_-} \, q^{n_+ - 2n_-}$ which yields the unnormalized Jones polynomial,

$$\hat{J}(L) = (-1)^{n_-} \, q^{n_+ - 2n_-} \, \langle \, L \, \rangle.$$

with the following properties ;

$$\langle \, \emptyset \, \rangle = 1; \quad \langle \bigcirc L \rangle = (q + q^{-1}) \langle \, L \, \rangle; \quad \langle \diagdown\!\!\!\!\diagup \rangle = \langle \asymp \rangle - q \, \langle\,)\,(\,\rangle; \quad \langle \diagup\!\!\!\!\diagdown \rangle = \langle\,)\,(\,\rangle - q \, \langle \asymp \rangle.$$

And the Jones polynomial is

$$J(L) := \hat{J}(L) \Big/ (q + q^{-1}).$$

Here $\langle \, L \, \rangle$ is the Kauffman bracket of L and (n_+, n_-) are the number of positive and negative crossings in the oriented link diagram L, respectively.

Khovanov homology replaces the Kauffman bracket $\langle \, L \, \rangle$ of a link projection L by **the Khovanov bracket** $[\![L]\!]$, that is a chain complex of graded vector spaces whose graded Euler characteristic is $\langle L \rangle$. Like Jones polynomial, the definition of the Khovanov bracket can be summarized by the following axioms;

$$[\![\, \emptyset \,]\!] = 0 \to \mathbb{Z} \to 0 \;;$$

$$[\![\bigcirc L \,]\!] = V \otimes [\![\, L \,]\!] \;\;;$$

$$[\![\diagdown\!\!\!\!\diagup]\!] = Tot \Big(0 \to [\![\asymp]\!] \xrightarrow{d} [\![\,)\,(\,]\!] \{1\} \to 0 \Big) \;\;;$$

$$[\![\diagup\!\!\!\!\diagdown]\!] = Tot\left(0 \to [\![\,)\,(\,]\!] \xrightarrow{d} [\![\asymp]\!]\{1\} \to 0\right) \ .$$

Here, as the Jones polynomial associates a polynomial, $(q+q^{-1})$, with each disjoint loop, the Khovanov homology associates a graded vector space, $V = \mathrm{Span}\{v_+, v_-\}$ with degree $+1$ and -1 respcetively. Then its graded dimension is $q + q^{-1}$. The operator $\{1\}$ is the "degree shift by 1" operation, which is the appropriate replacement of "multiplication by q", Tot denotes the total complex of a double complex which forms a complex by taking direct sums along diagonals, and the differential d, is defined below.

As the (unnormalized) Jones polynomial is a minor renormalization of the Kauffman bracket, the Khovanov invariant $\mathcal{H}(L)$ is the homology of a similar renormalization

$$[\![\ L \ ]\!][-n_-]\{n_+ - 2n_-\}$$

of the Khovanov bracket. And the Khovanov invariant is indeed a link invariant and its graded Euler characteristic is $\hat{J}(L)$.

For Khovanov homology, we need several definitions.

Definition 1.2.1 (Bar-Natan[3]). *Let $W = \bigoplus_m W_m$ be a graded vector space bounded below with homogeneous components $\{W_m\}$. The graded dimension of W is the Laurent series $q\dim W := \sum_m q^m \dim W_m$.*

In practice, we will consider only graded vector spaces with finite dimensional underlying vector space, so the graded dimensions will always be a Laurent polynomial.

Definition 1.2.2 (Bar-Natan[3]). *Let $\cdot\{l\}$ be the "degree shift" operation on graded vector spaces. That is, if $W = \bigoplus_m W_m$ is a graded vector space, we set $W\{l\}_m := W_{m-l}$, so that $q\dim W\{l\} = q^l \, q\dim W$.*

Definition 1.2.3 (Bar-Natan[3]). *Likewise, let $\cdot[s]$ be the "height shift" operation on chain complexes. That is, if $\bar{\mathcal{C}}$ is a chain complex $\ldots \to \bar{\mathcal{C}}^r \xrightarrow{d^r} \bar{\mathcal{C}}^{r+1} \ldots$ of (possibly graded) vector spaces (we call r the "height" of a piece $\bar{\mathcal{C}}^r$ of that complex), and if $\mathcal{C} = \bar{\mathcal{C}}[s]$, then $\mathcal{C}^r = \bar{\mathcal{C}}^{r-s}$ (with all differentials shifted accordingly).*

Then with these definitions, let us start to make Khovanov homology with the following Figure 1.2. This part is exactly from [Bar-Natan[3]], [Bar-Natan[1]]. First we can make a commutative cube from a knot or link.

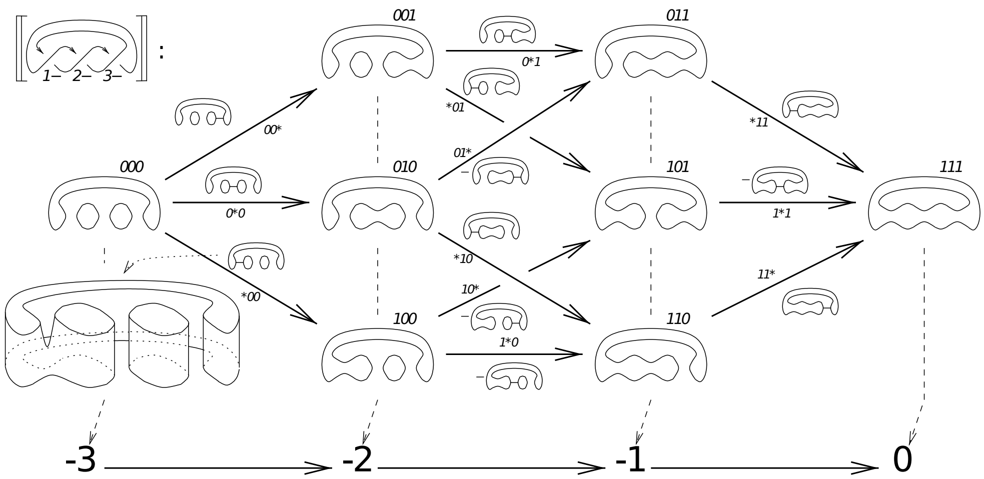
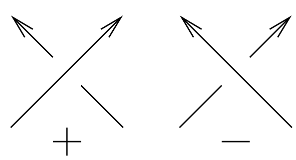


Figure 1.2: *The main picture (the left-handed trefoil knot). – (picture from [Bar-Natan][1])*

- **A knot** On the upper left of the figure we see the left-handed trefoil knot L with its $n = 3$ crossings labeled 1, 2 and 3. It is inside of double brackets ($[\![\cdot]\!]$) to denote the **formal Khovanov Bracket**.

- **Crossings**



On the figure of L we need to define the signs of its crossings ; $(+)$ for overcrossings ( ) and $(-)$ for undercrossings ( ). Let n_+ and n_- be the numbers of $(+)$ crossings and $(-)$ crossings in K, respectively. So for the left-handed trefoil knot, $(n_+, n_-) = (0, 3)$.

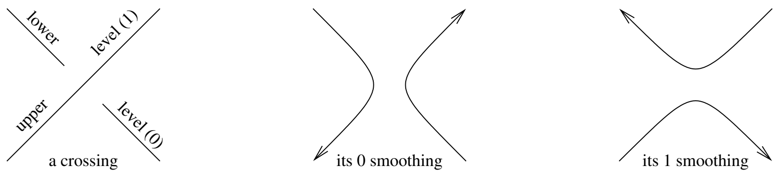
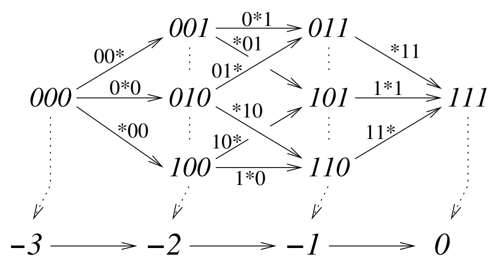


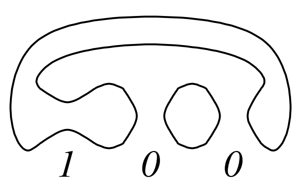
Figure 1.3: *A crossing is an interchange involving two highways. The* 0*–smoothing is when you enter on the lower level (level* 0*) and turn right at the crossing. The* 1*–smoothing is when you enter on the upper level (level* 1*) and turn right at the crossing. – (picture from [Bar-Natan]*[1]*)*

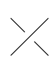
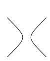
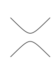
- **Cube**



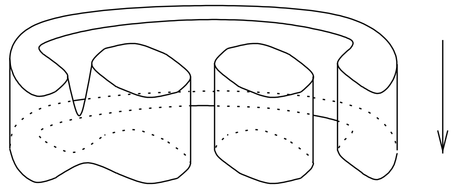
The main part of the figure is the n–dimensional cube whose vertices are all the n–letter strings with 0's and 1's. The edges of the cube are marked in the natural manner by n–letter strings of 0's, 1's and precisely one $\star$ (the $\star$ denotes the coordinate which changes from 0 to 1 along a given edge). The cube is skewed along its main diagonal, from $00\cdots0$ to $11\cdots1$. More precisely, each vertex of the cube has a "height", the sum of its coordinates, a number between 0 and n. The cube is displayed in such a way so that vertices of height k project down to the point $k - n_-$ on a line marked below the cube. We indicate these projections with dashed arrows and tilted them a bit to remind us of the $-n_-$ shift. The above picture is shown for the case of $n = 3$.

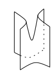
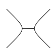
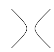
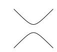
- **Vertices**



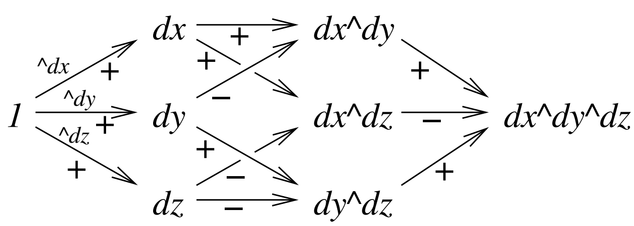
Each vertex of the cube is for a **smoothing** of L — a planar diagram obtained by resolving every crossing  in the given diagram of L into either a "0–smoothing" () or into a "1–smoothing" () (see Figure 1.3). Because our L has 3 crossings, it has $2^3 = 8$ smoothings. They are assigned to each vertices of the 3–dimensional cube $\{0,1\}^3$.

- **Edges**

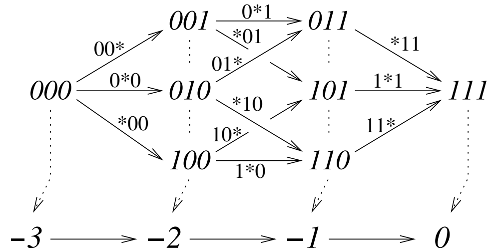


Here we can apply (1+1) – dimensional Topological Quantum Field Theory. On each vertex we have a union of 1-manifolds that are assigned to a vector spaces by TQFT, and each edge of the cube is labeled by a **cobordism** between them – the smoothing on the top of that edge and the smoothing on its bottom. Then we can replace them by the 2 - dimensional saddle-like cobordism ;  . They are displayed in Figure 1.2. Here,  denotes the saddle cobordism with top  and bottom  . And there is a famous theorem by Lowell Abrams[12], which says that there is a 1-to-1 corespondience between 2 - dimensional TQFTs and Frobenius algebras. So we can use a Frobenius algebra to define maps corresponding to the edges.

- **Signs**



In the original paper of [Khovanov[13]], he used the canonical way to construct a anti-commutative cube. We discuss it in the appendix A.1. On the other hand, Bar-Natan[3] created it in different way. Let us see the picture in the cube part above.



Here, each edge is denoted by three digit number, ξ, which consist of $*$, 0, 1. The height $|\xi|$ of an edge ξ is defined to be the height of its starting object. Later the vertical collapse of the cube will give us a chain complex, and the differential is

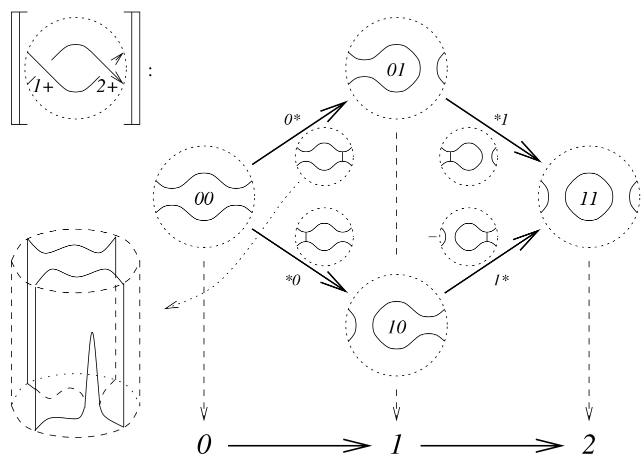
$$d^r := \sum_{|\xi|=r} (-1)^\xi \, \xi.$$

Here, we need the signs $(-1)^\xi$, for d to satisfy $d \circ d = 0$. It is enough the all square faces of the cube would be anti-commutative. For it, we construct commutative cube, and then sprinkle signs to make the faces anti-commutative. Thus, we can use

$$(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$$

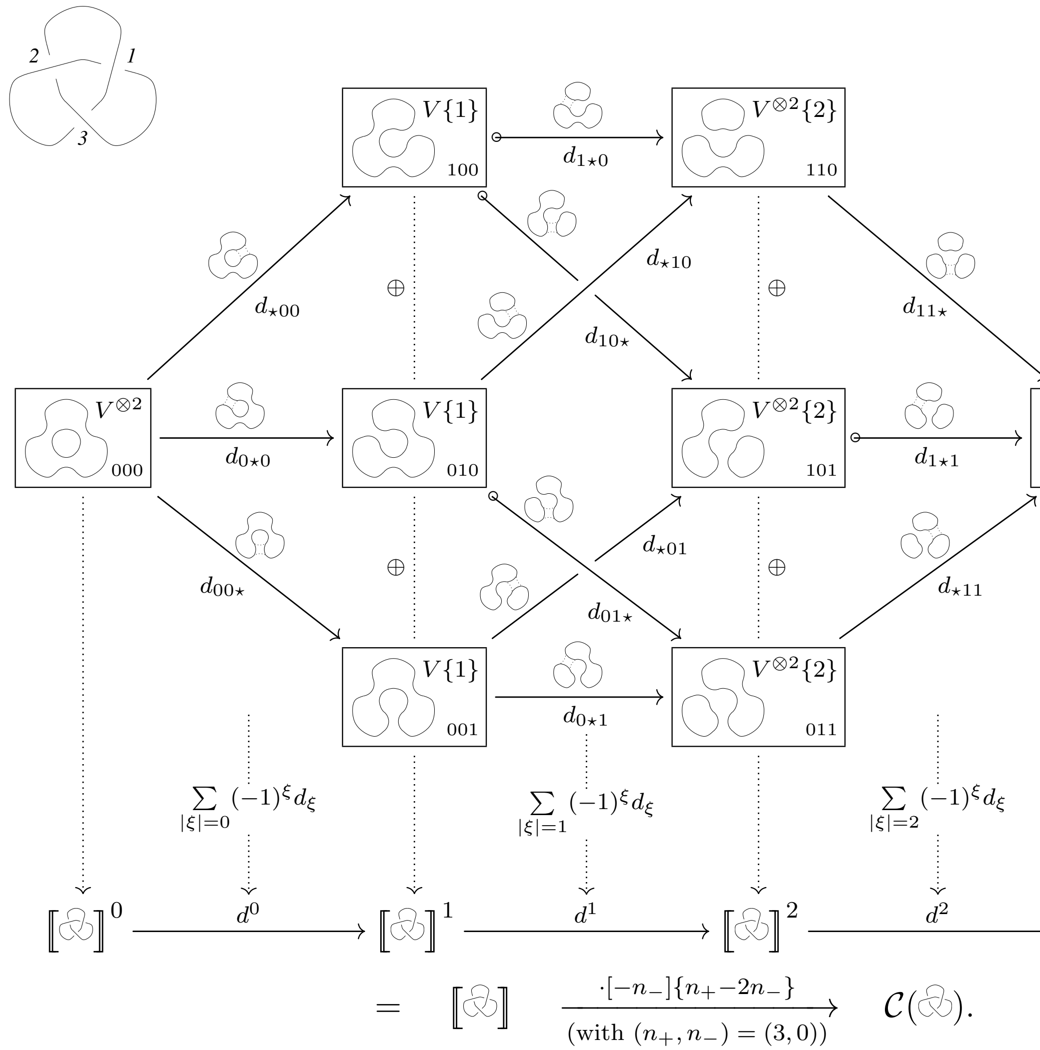
, where j is the location of the $*$ in ξ.

- **Tangles**



Now we can construct a commutative cube from an arbitrary link diagram with more crossings similar to the one in Figure 1.2. In practice, we can make a commutative cube for any **tangle**.

So we could build up the commutative cube from a knot L. Now let us consider how to construct the complex $\mathcal{C}(L)$ of a commutative cube. In this time we will use another example (right trefoil knot). Building a commutative cube is just from the previous work. For more detail, we refer to [Bar-Natan[3]] and [Bar-Natan[1]].



(1.2.1)

Then the chain groups $[\![L]\!]^r$ are direct sums of the vector spaces that appear in the vertices of the cube along the columns above each one of the $[\![L]\!]^r$ spaces.

Let $\mathcal{H}^r(L)$ denote the rth cohomology of the complex $\mathcal{C}(L)$. It is a graded vector space depending on the link projection L. Let $Kh(L)$ denote the graded Poincaré polynomial of the complex $\mathcal{C}(L)$ in the variable t; that is,

$$Kh(L) := \sum_r t^r \, q\dim \mathcal{H}^r(L).$$

Then we have the following theorem.

Theorem 1 (Khovanov[13]). *The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are link invariants, and hence $Kh(L)$, a polynomial in the variables t and q, is a link invariant that specializes to the unnormalized Jones polynomial at $t = -1$.*

So, for any planar diagram of an oriented knot K or link L, Khovanov link homology theory give a chain complex $[\![L]\!]$ of graded vector spaces whose graded Euler characteristic agrees with the Jones polynomial of the link. This construction can be thought as a categorification of the unnormalized Jones polynomial, replacing a polynomial in one indeterminate q by a chain complex of graded vector spaces. In general the homology groups contain more information about the link than the Jones polynomial. Bar-Natan[3], and Wehrli[14] had proven that there are knots and links that have the same Jones polynomial, but which can be distinguished by their Khovanov homology.

Chapter 2

TQFTs and Frobenius algebras

In this chapter we discuss topological quantum field theories (TQFTs), and Frobenius algebras. For more information, refer to [Kock[15]], [Khovanov[13]], [Lauda and Pfeiffer[6]], [MacLane[16]], and [Yetter[8]].

2.1 Topological Quantum Field Theories (TQFTs)

Topological quantum field theories (TQFTs) were introduced by Atiyah[17], and their relation to Frobenius systems were described by Abrams[12].

Let R be a commutative unital ring. A $(n+1)$–dimensional TQFT is a monoidal functor from the category of $(n+1)$–dimensional cobordisms to the category of R-modules. First, let us define the category of $(n+1)$–dimensional cobordisms.

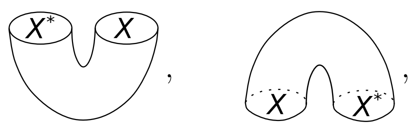
Definition 2.1.1. *The category of* $(n+1)$*–dimensional cobordisms,* $\underline{\underline{(n+1)\text{–}\boldsymbol{Cobord}}}$ *has as objects smooth compact oriented* n*-manifolds, and arrows from* X *to* Y *are named by diagrams of the form*

$$X^* \amalg Y \xrightarrow{\sim} \partial Z \hookrightarrow Z.$$

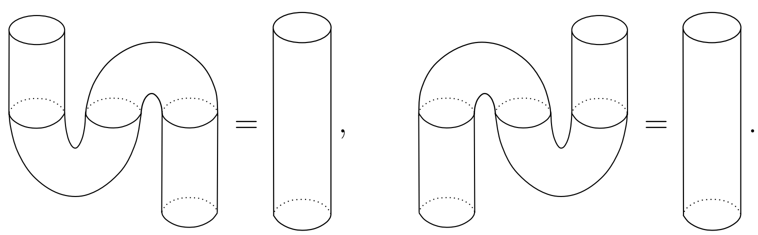
Two such maps $X^* \amalg Y \xrightarrow[\varphi]{\sim} \partial Z \hookrightarrow Z$ *, and* $X^* \amalg Y \xrightarrow[\psi]{\sim} \partial W \hookrightarrow W$ *are the* *same arrows if there exists an orientation preserving diffeomorphism $\xi : Z \to W$ such that*

$$\begin{tikzcd} & \partial Z \arrow[dd, "\sim"', "\partial\xi"] \\ X^* \amalg Y \arrow[ur, "\varphi", "\sim"'] \arrow[dr, "\psi"', "\sim"] & \\ & \partial W \end{tikzcd}$$

Then $\underline{\underline{(n+1)\text{–}\mathbf{Cobord}}}$ is a symmetric monoidal category with disjoint union as the monoidal product, and the empty n-manifold as the monoidal identity. Moreover every object X has a dual X^*, the same manifold with its orientation reversed, and the unit and counit are given by



giving the followings.



Definition 2.1.2. *A $(n+1)$–dimensional TQFT is a monoidal functor Z from $(\underline{\underline{(n+1)\text{–}\boldsymbol{Cobord}}}, \coprod, \phi, Id, Id, Id, tw)$ to $(\underline{\underline{R\text{–}\boldsymbol{mod}}}, \otimes, R, \alpha, \rho, \lambda, \sigma)$.*

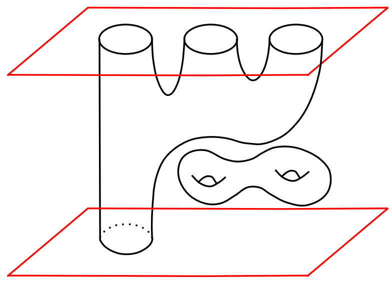
Note. $Z(X^*) = Z(X)^*$ is not an extra condition because a monoidal functors preserve duals.

Example : $(0+1) - TQFT$.

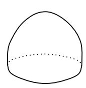
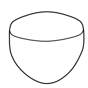
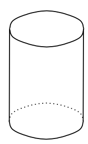
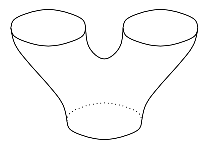
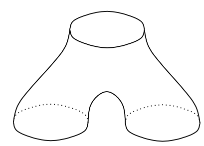
$\underline{\underline{(0+1) - Cobord}}$ has as objects compact 0–manifold; that is, signed finite sets of points, and as arrows cobordism between them. Then $\underline{\underline{(0+1)\text{–Cobord}}}$ is monoidally equivalent to a category $\underline{\underline{\textbf{FlatOTang}}}$, whose objects are sequences of $+$'s and $-$'s, and arrows are "flat oriented tangle diagrams" modulo "flat Reidemeister moves". This category was described by Kelly and Laplaza[18] as a free symmetric compact closed category. For more detail, we can see [Yetter[8]].

Example : $(1+1) - TQFT$.

$\underline{\underline{(1+1) - Cobord}}$ has as objects compact oriented 1–manifold; that is, disjoint union of oriented circles, and as arrows cobordism between them.



Note : up to diffeomorphism, any surface with 2 families of circles (one at top and one at bottom) as boundary can be obtained by gluing together copies of the 2–manifolds with ∂ indicated below:

, , , , and ;

that is, $\underline{\underline{(1+1)\text{–Cobord}}}$ is generated under composition and monoidal product by only these five cobordisms.

2.2 Frobenius Algebra

In representation theory and module theory, a Frobenius algebra is a finite dimensional unital associative algebra with a special kind of bilinear form, which gives the algebra particularly nice duality properties.

Definition 2.2.1. *A finite dimensional, unital, associative algebra A defined over a field K is called a **Frobenius algebra** if A is equipped with a nondegenerate bilinear form $\sigma : A \times A \to K$ that satisfies the following equation*

$$\sigma(a \cdot b, c) = \sigma(a, b \cdot c).$$

*This bilinear form is called the **Frobenius form** of the algebra.*

In category theory, a **Frobenius object** is a generalization of a Frobenius algebra to an arbitrary monoidal category. A Frobenius object $(A, \mu, \eta, \delta, \varepsilon)$ in a monoidal category $(\mathcal{C}, \otimes, I)$ consist of an object A of $\mathcal{C}$ together with four morphisms

$$\mu : A \otimes A \to A \quad , \quad \eta : I \to A \quad , \quad \delta : A \to A \otimes A \quad , \text{ and } \quad \varepsilon : A \to I$$

such that

- (A, μ, η) is a monoid in $\mathcal{C}$.
- (A, δ, ε) is a comonoid in $\mathcal{C}$.
- the followings diagrams commute.

$$\begin{tikzcd} A \otimes A \arrow[r, "\delta\otimes A"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "A\otimes\mu"] \\ A \arrow[r, "\delta"'] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "A\otimes\delta"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "\mu\otimes A"] \\ A \arrow[r, "\delta"'] & A \otimes A \end{tikzcd}$$

And as we have seen before, $(1+1)-TQFT$ is a monoidal functor

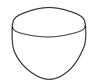
$$\mathcal{F} : \underline{\underline{(n+1) - \text{Cobord}}} \to \underline{\underline{R - \text{mod}}}.$$

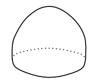
Then there is well-known correspondence described by Abrams[12] between $(1+1)-TQFT$ and Frobenius algebra. A (commutative) Frobenius system is a 4-tuple $(R, A, \varepsilon, \Delta)$, where R, A, ε, and Δ are the following objects and morphisms.

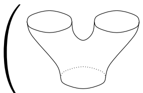
- A is a commutative unital R-algebra such that the natural R-module map $\iota : R \to A$, given by $\iota(1) = 1$, is injective. $\varepsilon : A \to R$ is a map of R-modules and Δ is a coassociative, cocommutative map $\Delta : A \to A \otimes A$ of A-bimodules such that

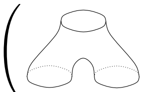
$$(\varepsilon \otimes \text{Id}) \circ \Delta = \text{Id}.$$

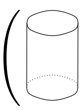
So given a commutative Frobenius algebra, we can define a $(1+1)-TQFT$, $\mathcal{F}$ by assigning R to the empty 1-manifold, A to the circle, $A \otimes_R A$ to the disjoint union of two circles, and so on. And for the generating morphisms of **<u>(n+1)–Cobord</u>**, we define $\mathcal{F}$ by

$\mathcal{F}$ () $= \varepsilon$

$\mathcal{F}$ () $= \iota$

$\mathcal{F}$ () $= m$

$\mathcal{F}$ () $= \Delta$

$\mathcal{F}$ () $= \mathrm{Id}_A$

Then algebraically $(1+1)-TQFT$ can be described in terms of (commutative) Frobenius algebra.

2.3 Graded Vector Spaces

A graded vector space is a vector space equipped with the extra structure of grading, which is a decomposition of the vector space as a direct sum of vector subspaces indexed by a ring (usually $\mathbb{Z}$, or $\mathbb{Z}/n$ for some n).

Definition 2.3.1. *An R-**grading** on a vector space W is a choice of decomposition into a direct sum of subspaces, W_m, such that $W = \bigoplus_{m \in R} W_m$. An R-**graded vector space** is a vector space equipped with an R-grading.*

In this paper, we will use the basis, v_+, v_-, for the vector space V to assign a grading. Let us give them their degree a, b , respectively. So if we set $W_a := \mathrm{Span}\{v_+\}$, and $W_b := \mathrm{Span}\{v_-\}$, then

$$V = W_a \oplus W_b.$$

If we let $W_{2a} := \mathrm{Span}\{v_+v_+\}$, $W_{a+b} := \mathrm{Span}\{v_+v_-, v_-v_+\}$, and $W_{2b} := \mathrm{Span}\{v_-v_-\}$, then

$$V^{\otimes 2} = W_{2a} \oplus W_{a+b} \oplus W_{2b}.$$

Definition 2.3.2 (Bar-Natan[3]). *Let* $W = \bigoplus_m W_m$ *be a graded vector space bounded below with homogeneous components* $\{W_m\}$*. The graded dimension of* W *is the Laurent series* $q\dim W := \sum_m q^m \dim W_m$.

Thus we see that

$$q\dim\, V \;=\; q^a\, q\dim\, W_a + q^b\, q\dim\, W_b \;=\; (q^a + q^b),$$

and

$$q\dim\,(V \otimes V) \;=\; q^{2a}\, q\dim\, W_{2a} + q^{a+b}\, q\dim\, W_{a+b} + q^{2b}\, q\dim\, W_{2b} \;=\; (q^a + q^b)^2.$$

Definition 2.3.3 (Bar-Natan[3]). *Let* $\cdot\{l\}$ *be the "degree shift" operation on graded vector spaces. That is, if* $W = \bigoplus_m W_m$ *is a graded vector space, we set* $W\{l\}_m := W_{m-l}$*, so that* $q\dim W\{l\} = q^l\, q\dim W$.

Lemma 2.3.4. *The graded dimension of a direct sum* $V \oplus W$ *is the sum of the graded dimensions of* V *and* W*,*

$$qdim\,(V \oplus W) = qdim\,V + qdim\,W$$

Lemma 2.3.5. *The graded dimension of a tensor product* $V \otimes W$ *is the product of the graded dimensions of* V *and* W*,*

$$qdim\,(V \otimes W) = (qdim\,V)\,(qdim\,W)$$

Chapter 3

Anyonic Khovanov Homology

The goal of this paper is to construct an anyoic braided version of Khovanov homology using constructions after the manner of Bar-Natan [BN][1], Scott Morrison [SM][19] for Khovanov Homology, and Beliakova and Wagner [BW][20] for Odd Khovanov Homology[21]. First we will define anyonic braided cobordisms analogously to Beliakova and Wagner's odd cobordisms and using them, we will define a cube and a complex in **AnyBraidCob**, then we will prove the invariance up to chain homotopies, so that invariant homology groups can be computed.

3.1 Anyonic Braiding

In the chapter 2, we saw the relation between Khovanov homology and $TQFT$. Here, because the circles in the (1+1)-dimensional $TQFT$ are interacting embedded in a (2+1)-dimensional background, it is natural to work in a braided setting. In physical terms, fractional statistics are possible, corresponding to the anyonic braiding.

Proposition 3.1.1. *For any $\xi \in \mathbb{C}^*$, the family of linear maps given on homogeneous elements by*

$$\sigma(a \otimes b) = \xi^{|a||b|}\, b \otimes a$$

*defines a braiding on **Gr-VS**, the category of $\mathbb{Z}$-graded complex vector spaces.*
*If $|\xi| = 1$ due to relation to the (2+1)-dimensional physics (cf. the fractional quantum Hall effect), such a braiding is called an **"anyonic braiding"**.*

Then using this definition, we can construct the basic morphisms needed to construct our generalized Khovanov homology. Because our construction covers both the original Khovanov homology (with $c = 0$) and Odd Khovanov homology as special cases, the paper [BW] [20] of Beliakova and Wagner is a good starting point, and we follow their mode of exposition.

Definition 3.1.2. *Let V be the graded vector space with two basis elements v_+, v_- whose degrees are a, b respectively, so that $qdimV = q^a + q^b$.*
Then we can define the following:

$$m = \begin{cases} v_+v_+ \to & v_+ \\ v_+v_- \to & v_- \\ v_-v_+ \to & v_- \\ v_-v_- \to & 0 \end{cases}$$

$$\Delta = \begin{cases} v_+ \to & v_-v_+ + \varphi\, v_+v_- \\ v_- \to & v_-v_- \end{cases}$$

$$e : 1 \to v_+$$

$$\varepsilon = \begin{cases} v_+ \to & 0 \\ v_- \to & 1 \end{cases}$$

These maps have degrees $-a, b, a,$ and $-b$ respectively.

If $\xi = \varphi = 1, a = 1, b = -1$, then we can obtain the original Khovanov homology in the case where Khovanov's $c = 0$. But for different values, we can get quite a different story. For fixed ξ, let us check counital coassociativity and unital associativity.

$$(\Delta \otimes 1)\, \Delta(v_+) = (\Delta \otimes 1)\, (v_-v_+ + \varphi\, v_+v_-) = v_-v_-v_+ + \varphi\, (v_-v_+v_- + \varphi\, v_+v_-v_-)$$

$$(\Delta \otimes 1)\, \Delta(v_-) = (\Delta \otimes 1)\, (v_-v_-) = v_-v_-v_-$$

$$(1 \otimes \Delta)\, \Delta(v_+) = (1 \otimes \Delta)\, (v_-v_+ + \varphi\, v_+v_-) = \xi^{b^2}\, v_-v_-v_+ + \varphi\, \xi^{b^2}\, v_-v_+v_- + \varphi\, \xi^{ab}\, v_+v_-v_-$$

$$(1 \otimes \Delta)\, \Delta(v_-) = (1 \otimes \Delta)\, (v_-v_-) = \xi^{b^2}\, v_-v_-v_-$$

$$(\varepsilon \otimes 1)\, \Delta(v_+) = (\varepsilon \otimes 1)\, (v_-v_+ + \varphi\, v_+v_-) = v_+$$

$$(\varepsilon \otimes 1)\, \Delta(v_-) = (\varepsilon \otimes 1)\, (v_-v_-) = v_-$$

$$(1 \otimes \varepsilon)\, \Delta(v_+) = (1 \otimes \varepsilon)\, (v_- v_+ + \varphi\, v_+ v_-) = \varphi\, \xi^{-ab}\, v_+$$
$$(1 \otimes \varepsilon)\, \Delta(v_-) = (1 \otimes \varepsilon)\, (v_- v_-) = \xi^{-b^2}\, v_-$$

Note extra powers of ξ arise in these calculation due to the need to braid the operations past arguments. This braiding of operations past arguments is analogous to the extra signs that appear in the classical theory of graded algebras due to the Koszul sign rule when operations of odd degree are considered. We adopt the convention that an operation always passes in front of arguments. So, $\xi^{ab} = \varphi\, \xi^{b^2}$ must hold if Δ, ε give the structure of a counital coassociative coalgebra.

On the other hand,

$$m\,(\,(m \otimes 1)\,(v_+ v_+ v_+)\,) = m(v_+ v_+) = v_+$$
$$m\,(\,(m \otimes 1)\,(v_+ v_+ v_-)\,) = m(v_+ v_-) = v_-$$
$$m\,(\,(m \otimes 1)\,(v_+ v_- v_+)\,) = m(v_- v_+) = v_-$$
$$m\,(\,(m \otimes 1)\,(v_+ v_- v_-)\,) = m(v_- v_-) = 0$$
$$m\,(\,(m \otimes 1)\,(v_- v_+ v_+)\,) = m(v_- v_+) = v_-$$
$$m\,(\,(m \otimes 1)\,(v_- v_+ v_-)\,) = m(v_- v_-) = 0$$
$$m\,(\,(m \otimes 1)\,(v_- v_- v_+)\,) = m(0) = 0$$
$$m\,(\,(m \otimes 1)\,(v_- v_- v_-)\,) = m(0) = 0$$
$$m\,(\,(1 \otimes m)\,(v_+ v_+ v_+)\,) = \xi^{-a^2}\, m(v_+ v_+) = \xi^{-a^2}\, v_+$$
$$m\,(\,(1 \otimes m)\,(v_+ v_+ v_-)\,) = \xi^{-a^2}\, m(v_+ v_-) = \xi^{-a^2}\, v_-$$
$$m\,(\,(1 \otimes m)\,(v_+ v_- v_+)\,) = \xi^{-a^2}\, m(v_+ v_-) = \xi^{-a^2}\, v_-$$
$$m\,(\,(1 \otimes m)\,(v_+ v_- v_-)\,) = \xi^{-a^2}\, m(0) = 0$$
$$m\,(\,(1 \otimes m)\,(v_- v_+ v_+)\,) = \xi^{-ab}\, m(v_- v_+) = \xi^{-ab}\, v_-$$
$$m\,(\,(1 \otimes m)\,(v_- v_+ v_-)\,) = \xi^{-ab}\, m(v_- v_-) = 0$$
$$m\,(\,(1 \otimes m)\,(v_- v_- v_+)\,) = \xi^{-ab}\, m(v_- v_-) = 0$$
$$m\,(\,(1 \otimes m)\,(v_- v_- v_-)\,) = \xi^{-ab}\, m(0) = 0$$

$$m\,(\,(e \otimes 1)\,(v_+)\,) = m(v_+ v_+) = v_+$$
$$m\,(\,(e \otimes 1)\,(v_-)\,) = m(v_+ v_-) = v_-$$

$$m\,(\,(1\otimes e)\,(v_+)\,) = \xi^{a^2}\, m(v_+v_+) = \xi^{a^2}\, v_+$$
$$m\,(\,(1\otimes e)\,(v_-)\,) = \xi^{ab}\, m(v_-v_+) = \xi^{ab}\, v_-$$

So, $\xi^{ab} = \xi^{a^2}$ must hold for m, e to give the structure of a unital associative algebra.

We let

$$\omega := \xi^{a^2} = \xi^{ab} = \varphi\,\xi^{b^2}.$$

Now let us check the Frobenius condition.

From the right Frobenius condition,

$$(m\otimes 1)\,(\,(1\otimes\Delta)\,(v_+v_+)\,) = \xi^{ab}\, v_-v_+ + \varphi\,\xi^{ab}\, v_+v_-$$
$$(m\otimes 1)\,(\,(1\otimes\Delta)\,(v_+v_-)\,) = \xi^{ab}\, v_-v_-$$
$$(m\otimes 1)\,(\,(1\otimes\Delta)\,(v_-v_+)\,) = \varphi\,\xi^{b^2}\, v_-v_-$$
$$(m\otimes 1)\,(\,(1\otimes\Delta)\,(v_-v_-)\,) = 0$$

and the left Frobenius condition,

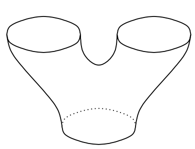
$$(\Delta\otimes 1)\,(\,(1\otimes m)\,(v_+v_+)\,) = \xi^{-ab}\, v_-v_+ + \varphi\,\xi^{-a^2}\, v_+v_-$$
$$(\Delta\otimes 1)\,(\,(1\otimes m)\,(v_+v_-)\,) = \xi^{-ab}\, v_-v_-$$
$$(\Delta\otimes 1)\,(\,(1\otimes m)\,(v_-v_+)\,) = \xi^{-ab}\, v_-v_-$$
$$(\Delta\otimes 1)\,(\,(1\otimes m)\,(v_-v_-)\,) = 0$$

we can see the Frobenius condition holds once the correct factor of ω is included. The following definition is then analogous to Beliakova and Wagner's *OddCob* in [BW] [20] :

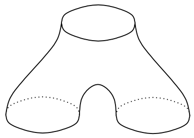
Definition 3.1.3. ***AnyBraidCob*** *is defined as follows:*

- *The objects are finite ordered set of circles*

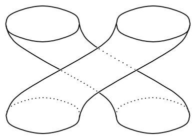
- *The morphisms are generated by*



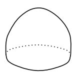
$$: \begin{cases} v_+v_+ \to & v_+ \\ v_+v_- \to & v_- \\ v_-v_+ \to & v_- \\ v_-v_- \to & 0 \end{cases}$$



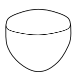
$$: \begin{cases} v_+ \to & v_- v_+ + \varphi\, v_+ v_- \\ v_- \to & v_- v_- \end{cases}$$



$$: \begin{cases} v_+ v_+ \to & \xi^{-a^2}\, v_+ v_+ \\ v_+ v_- \to & \xi^{-ab}\, v_- v_+ \\ v_- v_+ \to & \xi^{-ab}\, v_+ v_- \\ v_- v_- \to & \xi^{-b^2}\, v_- v_- \end{cases}$$



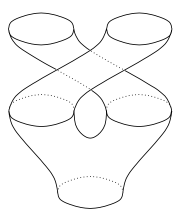
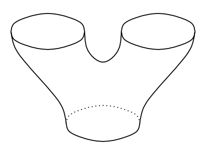
$$: \;\; 1 \to v_+$$



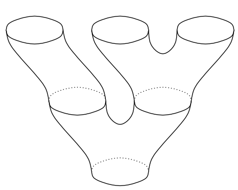
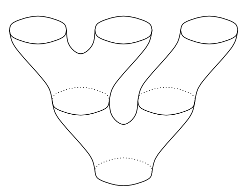
$$: \begin{cases} v_+ \to & 0 \\ v_- \to & 1 \end{cases}$$

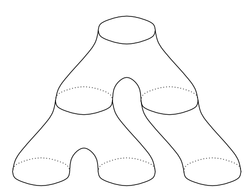
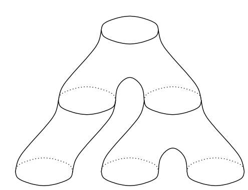
subject to the following sets of relations:

(1) Commutativity relation :

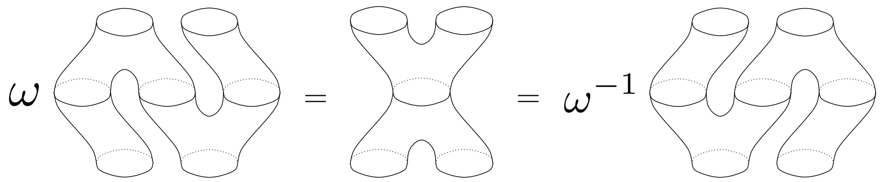
ω  $=$ 

(2) Associativity and coassociativity relations :

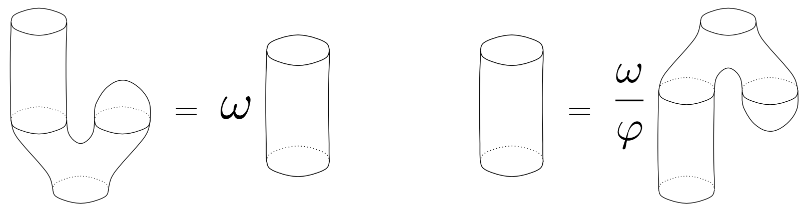
ω  $=$ 

$\dfrac{\omega}{\varphi}$  $=$ 

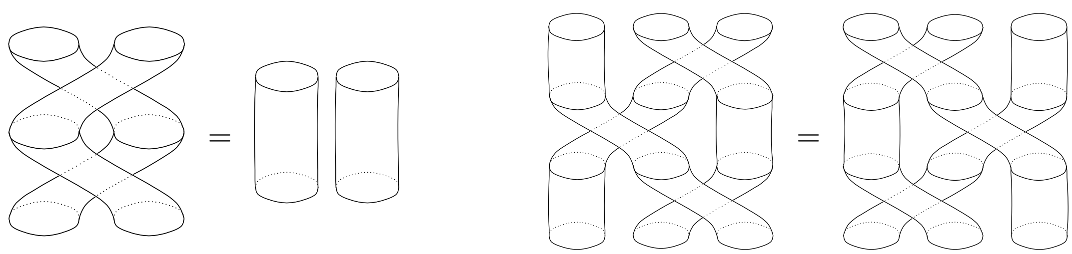
(3) Frobenius relations :



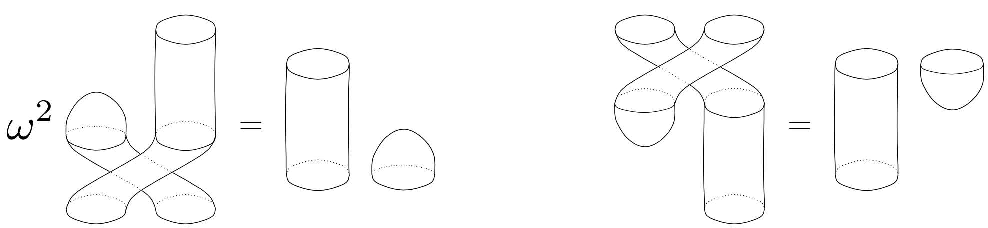
(4) Unit and counit relations :



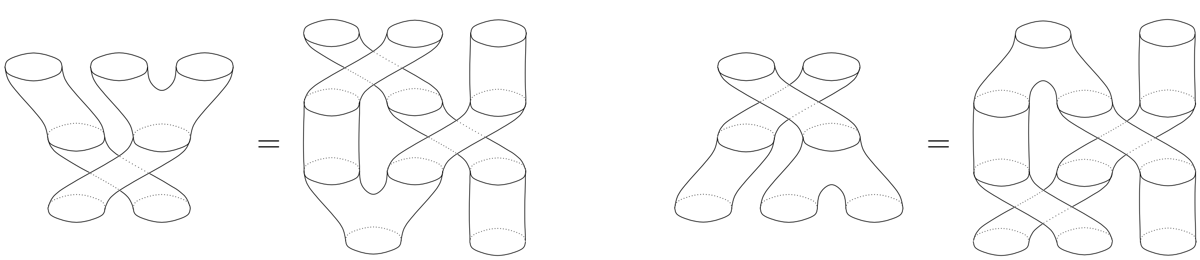
(5) Braiding relations :



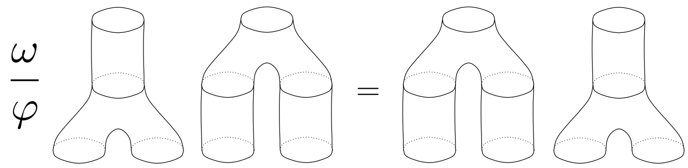
(6) Unit-braiding and counit-braiding relations :

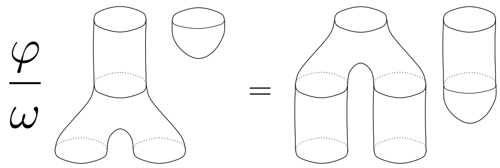


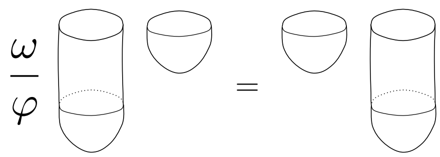
(7) Merge-braiding and split-braiding relations :

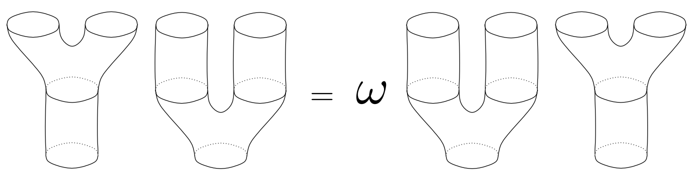


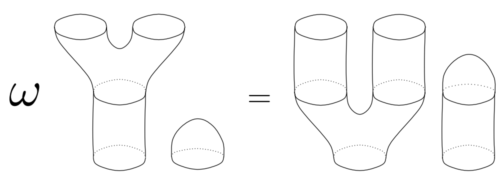
(8) Commutation relations :

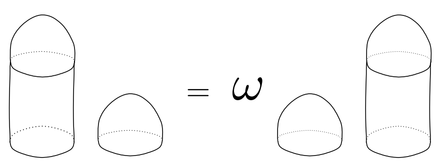


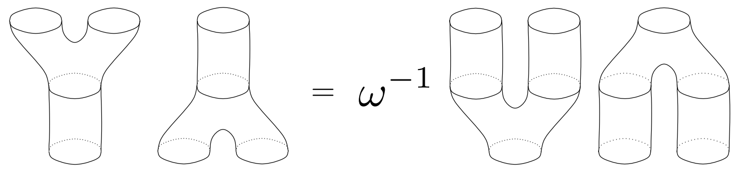


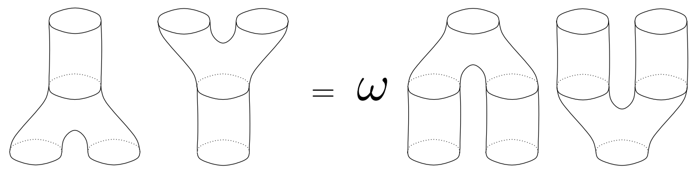






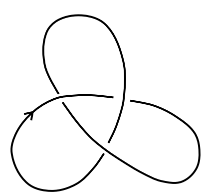




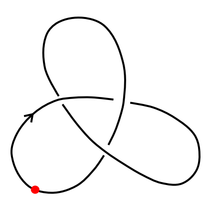


3.2 Ordering Convention

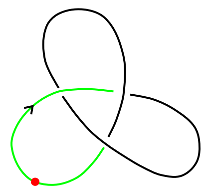
In this section we will give an ordering to our knot diagram which is used to make our cube. As we have seen in the section 3.1, the anyonic braiding is from physics, the fractional quantum Hall effect. Everything is physically instantiated in the plane. As in the definition 3.1.3, if we have an operation, it physically has to interact with them, physically go past things. Then it creates extra phase. This means that when we construct a cube from a knot diagram, according to the ordering we can get different phases. Here we define how to give an order to our knot diagram. Let us start with a knot. Here, we are using a right trefoil knot.



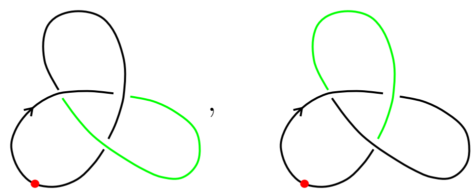
First, we need to choose a base point (starting point). It can be any point on the knot diagram.



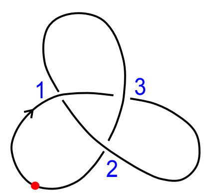
In this case, red point refers to our base point. Then to give an order, we need to chase around overcrossing. We can find the first overcrossing here,



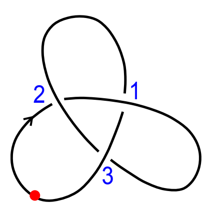
And the second and third overcrossing here,



Finally we can give an order as follows.



On the other hand, for a left trefoil knot, we can get the following.



Remark 3.2.1. The invariants of based pointed knots are the same as the invariants of knots.

If we have two knots with base points, then we choose an isotopy to move the base point to the point of infinity in each case by rotating three sphere. So we can put that base point each to the point of infinity. If they are isotopic, they are isotopic with the base point staying at the point of infinity. So choosing of base point does not affect to the invariants of knots. We are not doing anything the changes in the structure of knot. It has still same invariance.

But, that is not the case in the case of links. In the case of links, if we choose base points on them, we can not avoid possibility of doing Reidemeister moves over base points. Thus in the case of links our construction gives an invariant of string links with ordered components.

Conjecture 1. *The invariant does not depend on the order of the components.*

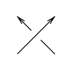
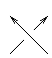
Conjecture 2. *The invariant is an invariant of the link obtained by closing the string link.*

3.3 Construction of the complex

In this section we closely follow the construction of Khovanov homology recalled in the section 1.2.

We start with Figure 3.1 at a completely descriptive level.

- **A knot :** On the above left of the figure we see the right-handed trefoil knot K with its $n = 3$ crossings labeled by 1, 2 and 3. This labels are from our ordering. It is enclosed by double brackets ($[\![\cdot]\!]$) for the **formal Khovanov Bracket** of the right-handed trefoil.

- **Crossings :** It is exactly same as Khovanov homology. We define  as $(+)$ crossing, and  as $(-)$ crossing. So for the left-handed trefoil knot, $(n_+, n_-) = (0, 3)$.

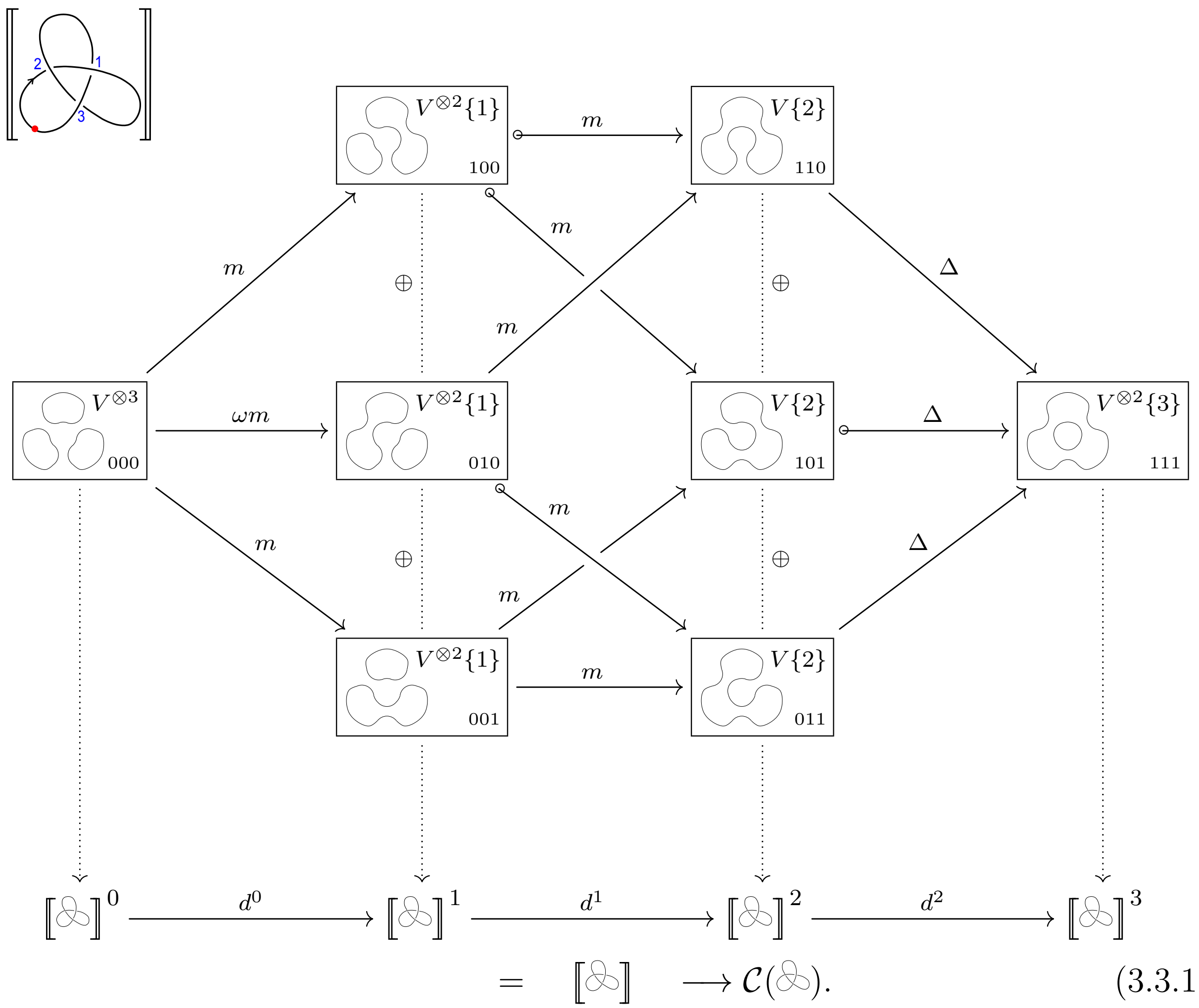
- **Vertices :** As in Khovanov homology, we can make eight vertices, labeled from 000 to 111.

- **Edges :** Here we can see all the edges consist of m and Δ.

- **Cube :** With vertices and edges, we can construct a cube for Khovanov homology.

- **Signs :** Again, it is the same as Khovanov one.

- **Commutativity :** As we have seen above everything is the same as the Khovanov homology except the commutativity. Commutativity of the cube is followed from the



(3.3.1)

Figure 3.1: *The main picture for the new generalization of Khovanov Homology (Left Trefoil Knot).*

relations in the new category defined in the definition 3.1.3. In the Figure 3.1, to make commutative cube, we need extra phase, ω, arising from ordering and associativity relation in the definition 3.1.3. On the other hand, we can construct a cube for the Right Trefoil Knot as in the Figure 3.2. The failure of the comultiplication to be cocommutative even up to phase results in a non-commutative cube.

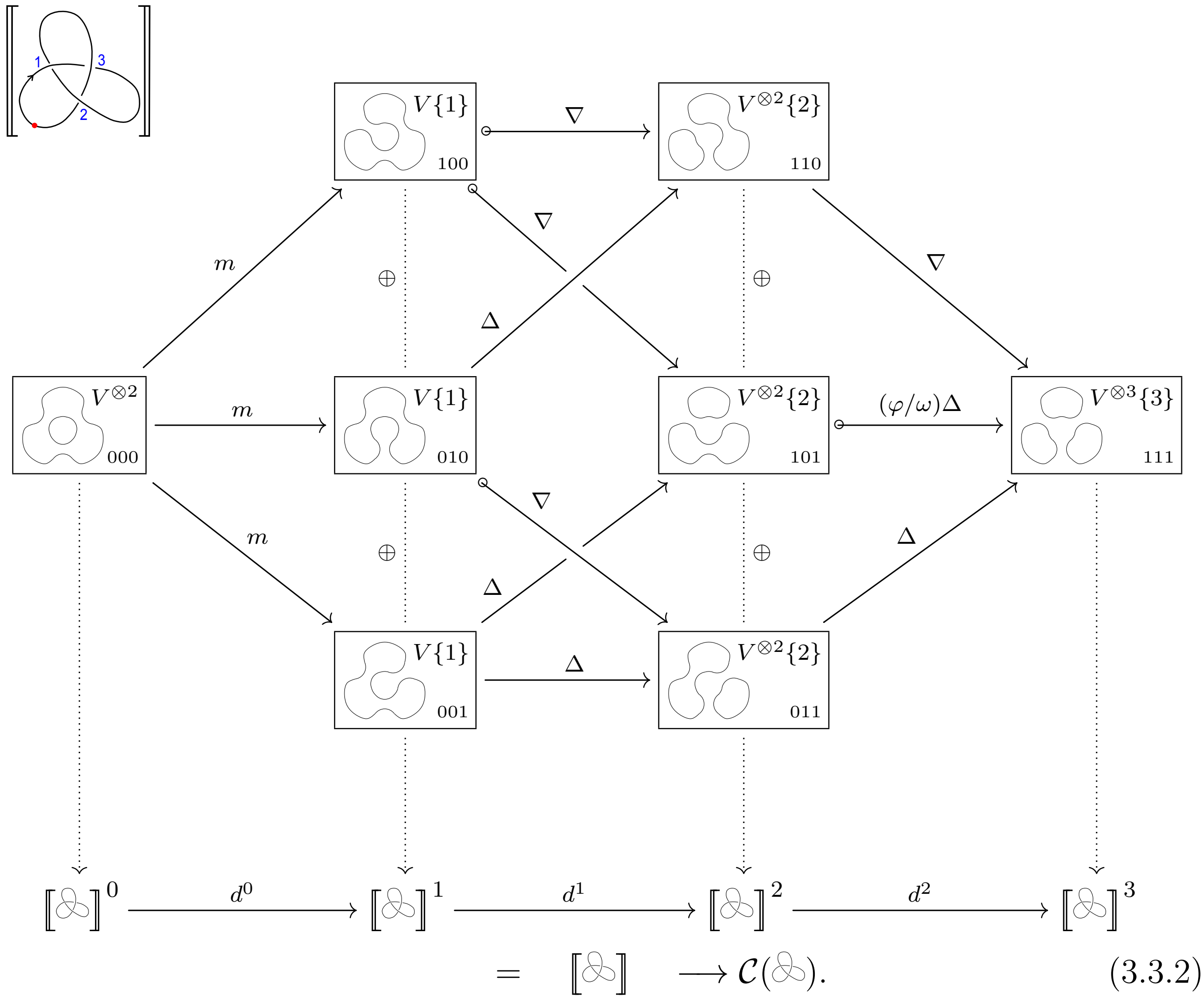


Figure 3.2: *The main picture for the new generalization of Khovanov Homology (Right Trefoil Knot).*

3.4 Homology from Arbitrary Sequence of Maps

In the section 3.3, we saw how to make a cube from a knot. Sometimes we may have a commutative cube, sometimes not. If we have a commutative cube, then we can compute the homology. But, if we do not have a commutative cube, then how can we compute the homology? The following results show us that one can compute a sort of homology from an arbitrary sequence of linear maps (or more generally of maps in any abelian category).

Let $\mathrm{Seq}(A) := A^{\cdots\to\cdot\to\cdot\to\cdots}$ be the category of all diagrams indexed by $(\mathbb{N},\geq)$ or $(\mathbb{Z},\geq)$ in an abelian category A, and $\mathrm{Chain}(A)$ be the full subcategory of chain complexes.

Theorem 3.4.1. *Chain(A) is a retract of Seq(A) with retraction functor give on objects by*

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

$$\downarrow$$

$$\cdots \longrightarrow X_{n-1}/f_{n-2}\left(f_{n-3}\left(X_{n-3}\right)\right) \xrightarrow{d_{n-1}} X_n/f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) \xrightarrow{d_n} X_{n+1}/f_n\left(f_{n-1}\left(X_{n-1}\right)\right) \longrightarrow \cdots$$

with $d_n\left(x+f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)\right) := f_n(x) + f_n\left(f_{n-1}\left(X_{n-1}\right)\right)$,
and on arrows by

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "p_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, "p_n"] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & \cdots \\
\cdots \arrow[r] & Y_{n-1} \arrow[r, "g_{n-1}"'] & Y_n \arrow[r, "g_n"'] & Y_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\longmapsto \quad \left\{\bar{p}_n : X_n/f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) \to Y_n/g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right)\right\}$$

, where $\bar{p}_n\left(x+f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)\right) := p_n(x) + g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right)$.

Proof. **I.** d_n **is well-defined :** If $x' + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) = x + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)$, then there exists a $\nu \in f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)$ such that $x' = x + \nu$.
But $f_n(\nu) = f_n\left(f_{n-1}\left(f_{n-2}(\bar{\nu})\right)\right)$, for some $\bar{\nu} \in X_{n-2}$. Thus $f_n(\nu) \in f_n\left(f_{n-1}\left(X_{n-1}\right)\right)$, and so

$$\begin{aligned}
f_n(x') + f_n\left(f_{n-1}\left(X_{n-1}\right)\right) &= f_n(x) + f_n(\nu) + f_n\left(f_{n-1}\left(X_{n-1}\right)\right) \\
&= f_n(x) + f_n\left(f_{n-1}\left(X_{n-1}\right)\right).
\end{aligned}$$

II. $d_n\left(d_{n-1}\right) = 0$ **:** We get

$$\begin{aligned} d_n\left(d_{n-1}\left(x + f_{n-2}\left(f_{n-3}\left(X_{n-3}\right)\right)\right)\right) &= d_n\left(f_{n-1}\left(x\right) + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)\right) \\ &= f_n\left(f_{n-1}\left(x\right)\right) + f_n\left(f_{n-1}\left(X_{n-1}\right)\right) \\ &= 0 \qquad \text{in } X_{n+1}/f_n\left(f_{n-1}\left(X_{n-1}\right)\right). \end{aligned}$$

Note : If $\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$ was a chain complex, then the image under the purported functor is isomorphic to

$$\cdots \longrightarrow X_{n-1}/0 \xrightarrow{f_{n-1}} X_n/0 \xrightarrow{f_n} X_{n+1}/0 \longrightarrow \cdots .$$

III. Given a map of sequences, $\{p_n : X_n \to Y_n\}$,
$\left\{\bar{p}_n : X_n/f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) \to g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right)\right\}$ is a well-defined chain map.

III-a. $\bar{p}_n$ **is well-defined :** If $x+f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) = x'+f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)$, then $x' = x + f_{n-1}\left(f_{n-2}\left(\nu\right)\right)$, for some $\nu \in X_{n-2}$. So

$$p_n\left(x'\right) = p_n\left(x\right) + p_n\left(f_{n-1}\left(f_{n-2}\left(\nu\right)\right)\right) = p_n\left(x\right) + g_{n-1}\left(g_{n-2}\left(p_{n-2}\left(\nu\right)\right)\right),$$

since $\{p_n\}$ was a map of sequences. Thus,

$$\begin{aligned} \bar{p}_n\left(x' + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)\right) &= p_n\left(x'\right) + g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right) \\ &= p_n\left(x\right) + g_{n-1}\left(g_{n-2}\left(p_{n-2}\left(\nu\right)\right)\right) + g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right) \\ &= p_n\left(x\right) + g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right) \\ &= \bar{p}_n\left(x + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right)\right). \end{aligned}$$

III-b. $\{\bar{p}_n\}_{n=-\infty}^{\infty}$ **is a chain map :** Let us consider an representatives

$$\begin{tikzcd} x + f_{n-1}\left(f_{n-2}\left(X_{n-2}\right)\right) \arrow[r, mapsto, "d"] \arrow[d, mapsto, "p_n"] & f_n\left(x\right) + f_n\left(f_{n-1}\left(X_{n-1}\right)\right) \arrow[d, mapsto, "\bar{p}_n"] \\ p_n\left(x\right) + g_{n-1}\left(g_{n-2}\left(Y_{n-2}\right)\right) \arrow[r, mapsto, "d"'] & p_{n+1}\left(f_n\left(x\right)\right) + g_n\left(g_{n-1}\left(Y_{n-1}\right)\right) \end{tikzcd}$$

Here, since $\{p_n\}$ was a map of sequences,

$$p_{n+1}\,(f_n\,(x)\,) + g_n\,(g_{n-1}\,(Y_{n-1})\,) = g_n\,(p_n\,(x)\,) + g_n\,(g_{n-1}\,(Y_{n-1})\,).$$

So the required diagram commutes.

$\square$

Definition 3.4.2. *If* $\{\,p_n : X_n \to Y_n\,\}$ *is a map of sequences from*
$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$ *to* $\cdots \longrightarrow Y_{n-1} \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} Y_{n+1} \longrightarrow \cdots$, *a* ***pseudo-contraction*** *is a sequence of maps* $\{h_n : X_n \to Y_n\}$ *such that, for any* n,

$$p_n = h_{n+1}\,(f_n) + g_{n-1}\,(h_n).$$

Depicting it as if it were a contracting homotopy :

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "p_{n-1}"'] & X_n \arrow[r, "f_n"] \arrow[d, "p_n"] \arrow[dl, dotted, "h_n"'] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] \arrow[dl, dotted, "h_{n+1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n-1} \arrow[r, "g_{n-1}"'] & Y_n \arrow[r, "g_n"'] & Y_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Figure 3.3: *Pseudo-contracting homotopy*

Theorem 3.4.3. *The retraction* $C : Seq\,(A) \to Chain\,(A)$ *described in the previous theorem is equipped with a map carrying pseudo-contractions to contracting homotopies given by* $\{h_n\}$ *a pseudo-contraction of* $p_n : (X_n, f_n) \to (Y_n, g_n)$ *maps to*

$$\bar{h}_n\,(x + f_{n-1}\,(f_{n-2}\,(X_{n-2})\,)\,) := h_n\,(x) + g_{n-2}\,(g_{n-3}\,(Y_{n-3})\,).$$

Here, this is the obvious formula. The puzzle is why it is well-defined.

Proof. The key is that by the hypothesis that $\{h_n\}$ is a pseudo-contraction, we have

$$h_n\,(f_{n-1}) = p_{n-1} - g_{n-2}\,(h_{n-1}).$$

Now let us suppose that

$$x' + f_{n-1}\,(f_{n-2}\,(X_{n-2})\,) = x + f_{n-1}\,(f_{n-2}\,(X_{n-2})\,).$$

Then there exists $\nu \in X_{n-2}$ such that $x' = x + f_{n-1}(f_{n-2}(\nu))$. So

$$\begin{aligned}
h_n(x') &= h_n(x) + h_n(f_{n-1}(f_{n-2}(\nu))) \\
&= h_n(x) + p_{n-1}(f_{n-2}(\nu)) - g_{n-2}(h_{n-1}(f_{n-2}(\nu))) \\
&= h_n(x) + g_{n-2}(p_{n-2}(\nu)) - g_{n-2}(h_{n-1}(f_{n-2}(\nu))) \\
&= h_n(x) + g_{n-2}(p_{n-2}(\nu)) - g_{n-2}(p_{n-2}(\nu) - g_{n-3}(h_{n-2}(\nu))) \\
&= h_n(x) - g_{n-2}(g_{n-3}(h_{n-2}(\nu))).
\end{aligned}$$

From which it follows that

$$h_n(x') + g_{n-2}(g_{n-3}(Y_{n-3})) = h_n(x) + g_{n-2}(g_{n-3}(Y_{n-3})).$$

Finally we check that the $\bar{h}_n$'s actually form a contracting homotopy for the $\bar{p}_n$'s. That is,

$$\bar{p}_n = \bar{h}_{n+1}\, d_n + \delta_n\, \bar{h}_n,$$

where d and δ are the differentials on $C(X^\bullet)$ and $C(Y^\bullet)$, respectively. It is from the followings.

$$\begin{aligned}
\bar{p}_n(x + f_{n-1}(f_{n-2}(X_{n-2}))) &= p_n(x) + g_{n-1}(g_{n-2}(Y_{n-2})) \\
&= h_{n+1}(f_n(x)) + g_{n-1}(h_n(x)) + g_{n-1}(g_{n-2}(Y_{n-2})),
\end{aligned}$$

since h is a pseudo-contraction. On the other hand,

$$\begin{aligned}
\bar{h}_{n+1}(d_n(x + f_{n-1}(f_{n-2}(X_{n-2})))) &= \bar{h}_{n+1}(f_n(x) + f_n(f_{n-1}(X_{n-1}))) \\
&= h_{n+1}(f_n(x)) + g_{n-1}(g_{n-2}(Y_{n-2}))\ .
\end{aligned}$$

$$\begin{aligned}
\delta_n(\bar{h}_n(x + f_{n-1}(f_{n-2}(X_{n-2}))) &= \delta_n(h_n(x) + g_{n-2}(g_{n-3}(Y_{n-3})) \\
&= g_{n-1}(h_n(x)) + g_{n-1}(g_{n-2}(Y_{n-2}))\ .
\end{aligned}$$

$\square$

Lemma 3.4.4. *The retraction functor preserves the cone construction.*

Proof. We can depict it as follow :

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "p_{n-1}"'] & X_n \arrow[r, "f_n"] \arrow[d, "p_n"] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & \cdots \\
\cdots \arrow[r] & Y_{n-1} \arrow[r, "g_{n-1}"'] & Y_n \arrow[r, "g_n"'] & Y_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now we define the cone, $\Gamma(p_n)$, to be the following sequence

$$\Gamma(p_n):\ \cdots \longrightarrow X_n \oplus Y_{n-1} \longrightarrow X_{n+1} \oplus Y_n \longrightarrow \cdots$$

with maps :

$$\varphi_{\Gamma(p_\bullet)} = \begin{pmatrix} -f_n & 0 \\ p_n & g_{n-1} \end{pmatrix}$$

Then

$$\begin{aligned}
\varphi_{\Gamma(p_\bullet)_{n+1}} \circ \varphi_{\Gamma(p_\bullet)_n}\left(\begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix}\right) &= \begin{pmatrix} -f_{n+1} & 0 \\ p_{n+1} & g_n \end{pmatrix} \begin{pmatrix} -f_n & 0 \\ p_n & g_{n-1} \end{pmatrix} \begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} f_{n+1}(f_n) & 0 \\ -p_{n+1}(f_n) + g_n(p_n) & g_n(g_{n-1}) \end{pmatrix} \begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} f_{n+1}(f_n) & 0 \\ 0 & g_n(g_{n-1}) \end{pmatrix} \begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix},
\end{aligned}$$

since $\{p_n\}$ is a map of sequences. The chain complex associated to the sequence $\{\varphi_{\Gamma(p_\bullet)}\}_n$ thus has chain groups

$$X_n/f_{n-1}(f_{n-2}(X_{n-2})) \oplus Y_{n-1}/g_{n-2}(g_{n-3}(Y_{n-3})),$$

with differentials

$$\begin{pmatrix} -\delta_n & 0 \\ \bar{p}_n & \delta_n \end{pmatrix},$$

that is, it is the cone on the induced map

$$\bar{p}_n : X_n/f_{n-1}(f_{n-2}(X_{n-2})) \to g_{n-1}(g_{n-2}(Y_{n-2}))$$

$\square$

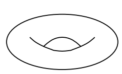
Chapter 4

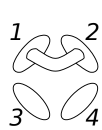
Invariance

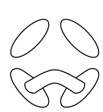
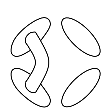
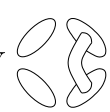
4.1 Proof of invariance

Bar-Natan's generalized Khovanov homology using an arbitrary Frobenius algebra [BN][1] is not a tangle invariant any more, but will in general depend on the underlying diagram. To be invariant under the Reidemeister moves, extra relations of the sort Bar-Natan called S, T, and $4Tu$ must be satisfied. In this section we will prove the invariance theorems of our generalized Khovanov homology. Remarkably, the transformations of cubes corresponding to Reidemeister moves always take place in portions of the cube which commute. So the proofs are not changed by the need to pass through the retraction functor.

The S relation,  **,** stands for a sphere and says that whenever a cobordism contains a connected component which is a closed sphere, it is set equal to zero.

The T relation,  **,** stands for a torus and means that whenever a cobordism contains a connected component which is a closed torus, that component may be dropped and replaced by a numerical factor of $1 + \varphi$.

The $4Tu$ relations, X 

$+Y$  $+Z$  $+W$  $= 0$ **,** we can start from some given cobordism C and assume its intersection with a certain ball is the union of four disks D_1 through D_4 (these disks may well be on different connected components of C). Let C_{ij} denote the result of removing D_i and D_j from C and replacing them by a tube that has the same boundary. A "four tube" relation, *4Tu*, asserts that $X\,C_{12}+Y\,C_{34}+Z\,C_{13}+W\,C_{24}=0$. For some coefficients X, Y, Z, and W, *4Tu* relations are used for the proof of homotopy equivalences $(FG-I=hd+dh)$ in the first and the second Reidemeister moves. As shown in [KP][22], any relation of this form with any coefficients satisfying $FG-I=hd+dh$ suffices to construct an algebraic homotopy. Here different choices of coefficient will correspond to the choices of the map G, and the h's in the condition for a homotopy inverse.

Now we will prove the invariance theorems of our construction under the three Reidemeister moves in the Figure 1.1.

Here our proof is based on [BN][1].

Theorem 4.1.1. *[Invariance under Reidemeister Move R1]*
The chain complex $[\![\ \ ]\!]$ *is homotopy equivalent to the chain complex* $[\![\ \ ]\!]$ *on the underlying ungraded vector space.*

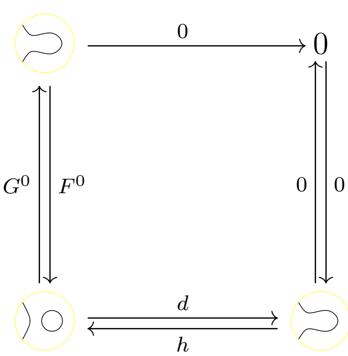
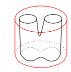


Figure 4.1: *Invariance under* $R1$

Proof. We have to show that the formal complex $[\![\ \ ]\!] = \left(0 \longrightarrow \underline{\ \ } \longrightarrow 0\right)$ is homotopy equivalent to the formal complex $[\![\ \ ]\!] = \left(0 \longrightarrow \underline{\ \ } \xrightarrow{d} \ \ \longrightarrow 0\right)$. Here, we have only one closure. So we can get the following Figure 4.2, in which $d =$  (in both complexes we have underlined the 0th term).

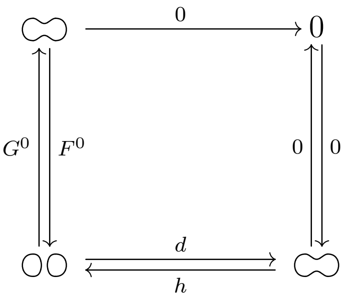
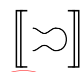
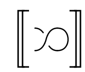
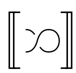
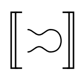
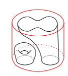
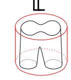
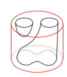
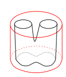
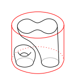
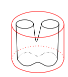
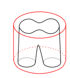
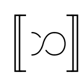
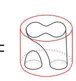
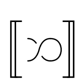
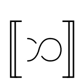


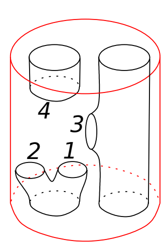
Figure 4.2: *Invariance under R1 with closure*

And we can set up (homotopically inverse) morphisms F :  $\to$  and G :  $\to$ . The morphism F is defined by $F^0 =$  $-$  (in words: a vertical curtain union a torus with a downward-facing disk removed, minus a simple saddle) and $F^{\neq 0} = 0$. The morphism G is defined by $G^0 = \frac{1}{\varphi}$ , and $G^{\neq 0} = 0$.

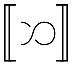
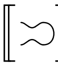
- **Commutativity :** In the Figure 4.2, the only non-trivial commutativity to verify is $dF^0 = 0$, which follows from  $\circ$  $=$  $\circ$ , and where the latter identity holds because both of its sides are the same.

- **$G \circ F = I$:** This follows from the T relation.

- **$F \circ G$ is homotopic to the identity on**  **:** Define the homopoty map $h =$  : ^{1} $=$  $\to$  $=$ ^{0}. Clearly, $F^1G^1 - I + dh = -I + dh = 0$. Now we need to see F^0G^0. Let us consider the cobordism,

  with the four distinguished disks, C_1, C_2, C_3, and C_4 marked by $1, 2, 3$, and 4 respectively. The *4Tu* relation, $\varphi^{-1} C_{12} - \varphi^{-1} C_{13} + C_{34} - C_{24} = 0$ holds. Here

$\varphi^{-1}\, C_{12} - \varphi^{-1}\, C_{13}$ is exactly the same as F^0G^0, C_{24} is the identity morphisms I, and C_{34} is hd. Of course in our map, we can see $dh = 0$. Thus our assertion $F^0G^0 - I + hd = 0$ holds. So $FG \sim I$ and we have proven that $[\![$  $]\!] \sim [\![$  $]\!]$.

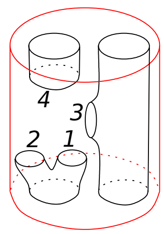
$\square$

Lemma 4.1.2. *[4Tu relation for R1]*

In the proof of the invariance under Reidemeister Move R1,

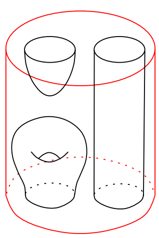
$$\varphi^{-1}\, C_{12} - \varphi^{-1}\, C_{13} + C_{34} - C_{24} = 0 \quad \textit{holds.}$$

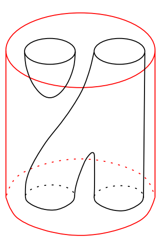
Proof. Our cobordism is

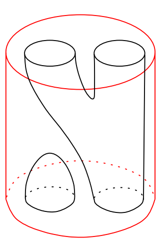


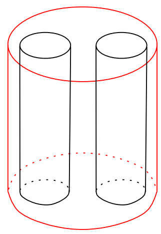
.

Then,

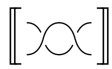
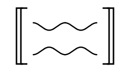
- C_{12} ,  $: \left\{ \begin{array}{lllllllll} v_+v_+ & \mapsto & 0 & & & & & & \\ v_+v_- & \mapsto & 0 & & & & & & \\ v_-v_+ & \mapsto & v_+ & \mapsto & v_+v_+ & \mapsto & v_-v_+v_+ + \varphi\, v_+v_-v_+ & \mapsto & (1+\varphi)\, v_-v_+ \\ v_-v_- & \mapsto & v_- & \mapsto & v_+v_- & \mapsto & v_-v_+v_- + \varphi\, v_+v_-v_- & \mapsto & (1+\varphi)\, v_-v_- \end{array} \right.$

- C_{13} ,  $: \left\{ \begin{array}{lllll} v_+v_+ & \mapsto & 0 & & \\ v_+v_- & \mapsto & 0 & & \\ v_-v_+ & \mapsto & v_+ & \mapsto & v_-v_+ + \varphi\, v_+v_- \\ v_-v_- & \mapsto & v_- & \mapsto & v_-v_- \end{array} \right.$

- C_{34} ,  $: \left\{ \begin{array}{lllll} v_+v_+ & \mapsto & v_+ & \mapsto & v_+v_+ \\ v_+v_- & \mapsto & v_- & \mapsto & v_+v_- \\ v_-v_+ & \mapsto & v_- & \mapsto & v_+v_- \\ v_-v_- & \mapsto & 0 & & \end{array} \right.$

• C_{24} ,  $: \begin{cases} v_+v_+ & \mapsto & v_+v_+ \\ v_+v_- & \mapsto & v_+v_- \\ v_-v_+ & \mapsto & v_-v_+ \\ v_-v_- & \mapsto & v_-v_- \end{cases}$

So $\varphi^{-1} C_{12} - \varphi^{-1} C_{13} + C_{34} - C_{24} = 0$ holds. $\square$

Theorem 4.1.3. *[Invariance under Reidemeister Move R2]*
The chain complex  *is homotopy equivalent to the chain complex*  *on the underlying ungraded vector space.*

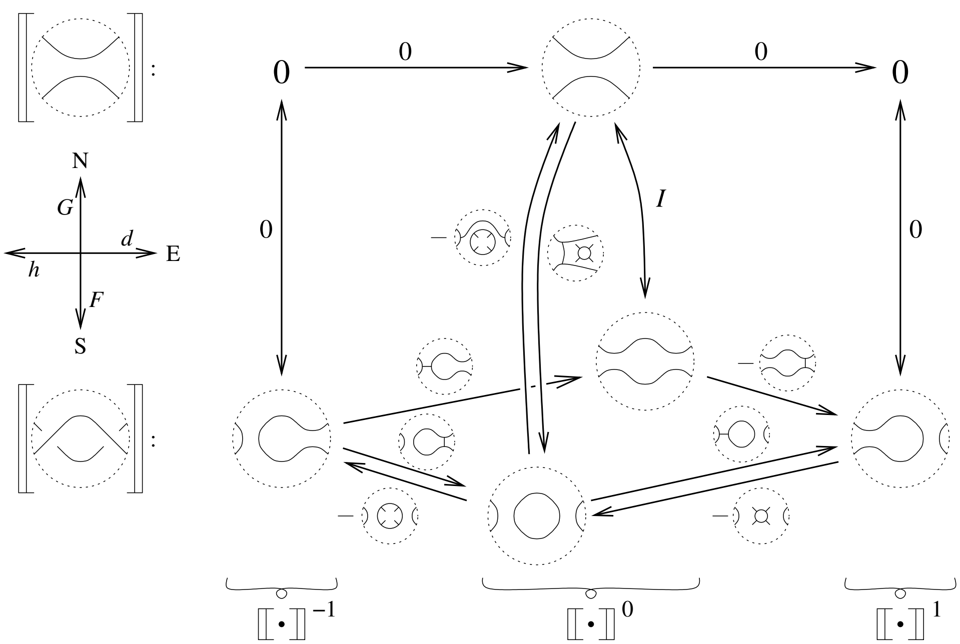
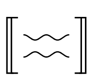
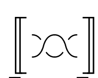
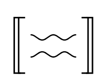
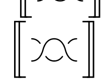
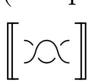
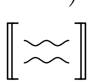
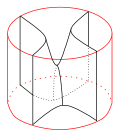
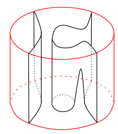
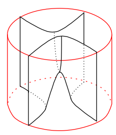
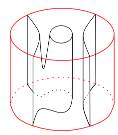


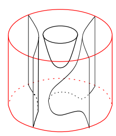
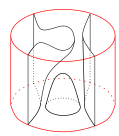
Figure 4.3: *Invariance under the Reidemeister move R2.*

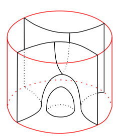
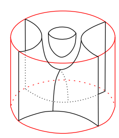
Proof. This proof appears in whole in Figure 4.3. In that figure, the top row is the formal complex  and the bottom row is the formal complex . Also, all southward arrows are the components of a morphism F :  $\to$ , the eastward arrows are (components of) differentials, the northward arrows are the components of a morphism G :  $\to$ , and the westward arrows are the non-zero components of a homotopy

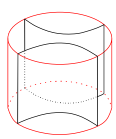
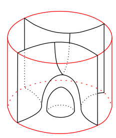
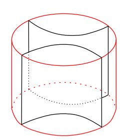
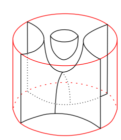
h proving that $FG \sim I$. Here, we know that F^{-1}, G^{-1}, F^0, G^0 are all zero. So we need to check at the zeroth level. Then we can set the followings.

- $d_{*0} =$ , $d_{0*} =$ 

- $d_{1*} = -$ , $d_{*1} =$ 

- $h_0 = a$ , $h_1 = b$ , where a and b are negative.

- $F^0 =$ , $G^0 = c$ , where c is negative.

- $F = \Big($ α  β  $\Big)$, $G = \Big($ γ  δ  $\Big)$

Now we can prove the second Reidemeister move in the showing the followings.

- $dF = 0$ **:** (only uses isotopies with $\alpha = 1$, $\beta = -\omega^{-1}$).

- $Gd = 0$ **:** (only uses isotopies with $\gamma = 1$, $\delta = -1$).

- $GF = I$ **:** Because $F^{\neq 0} = 0$, $G^{\neq 0} = 0$, we just need to check $G^0F^0 = I$. But it is directly from the relation S .

- $FG - I = hd + dh$ **:** Similarly, we will show that F^0G^0 is homotopic to I. From our setting, we have

$$F^0 = h_1 \circ d_{1*}$$

$$G^0 = d_{*0} \circ h_0$$

And we want to show that $F^0 \circ G^0 - I = h_1 \circ d_{*1} + d_{0*} \circ h_0$ (see Figure 4.4).

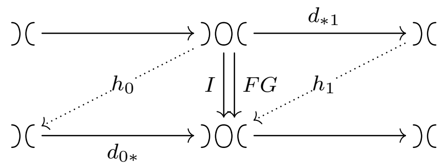
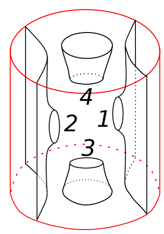
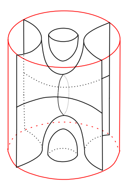
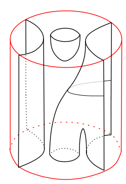
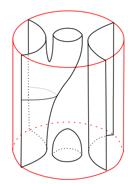
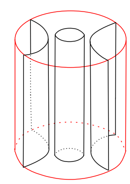


Figure 4.4: $F^0 G^0 \sim I$

As in the proof of invariance under Reidemeister Move R1, we can get it from *4Tu* relation. Consider the cobordism,



with the four distinguished disks, C_1, C_2, C_3, and C_4 marked by $1, 2, 3$, and 4 respectively. Then C_{12}, C_{13}, C_{24}, and C_{34} can be represented by followings.

$C_{12} =$ , $C_{13} =$ , $C_{24} =$ , $C_{34} =$ 

In this case, we have two different *4Tu* relations according to the closure. First, for  closure, we need to set

$$a = -1,\ b = -\omega^{-1},\ c = -1.$$

Then

$$F^0 \circ G^0 = -\,C_{12},$$

$$d_{0*} \circ h_0 = -\,C_{13},$$

$$h_1 \circ d_{*1} = -\omega^{-1}\,C_{24},$$

$$I = C_{34}$$

And we get

$$C_{12} - C_{13} - \omega^{-1}\, C_{24} + C_{34} = 0.$$

On the other hand, for the other closure,  , we need to set

$$a = -\omega\varphi^{-1},\, b = -\omega^{-1},\, c = -\varphi^{-1}.$$

Then

$$F^0 \circ G^0 = -\varphi^{-1}\, C_{12},$$

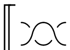
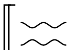
$$d_{0*} \circ h_0 = -\omega\varphi^{-1}\, C_{13},$$

$$h_1 \circ d_{*1} = -\omega^{-1}\, C_{24},$$

$$I = C_{34}$$

And

$$\varphi^{-1}\, C_{12} - \omega\varphi^{-1} C_{13} - \omega^{-1} C_{24} + C_{34} = 0 \quad \text{hold.}$$

Of course both of them give us $FG - I = hd + dh$. So $FG \sim I$ and we have proven that $\llbracket$  $\rrbracket \sim \llbracket$  $\rrbracket$.

$\square$

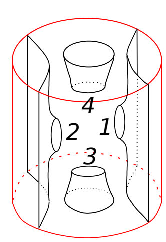
Lemma 4.1.4. *[4Tu relation for R2]*

In the proof of the invariance under Reidemeister Move R2,

$$C_{12} - C_{13} - \omega^{-1}\, C_{24} + C_{34} = 0, \quad \textit{and}$$

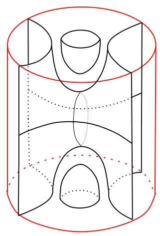
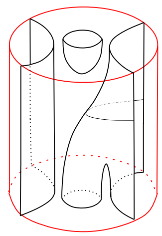
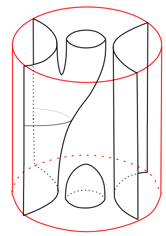
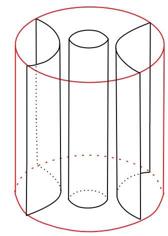
$$\varphi^{-1}\, C_{12} - \omega\varphi^{-1} C_{13} - \omega^{-1} C_{24} + C_{34} = 0 \quad \textit{hold.}$$

Proof. Our cobordism is



.

Here, C_{12}, C_{13}, C_{24}, and C_{34} can be represented by followings.

$C_{12} =$  , $C_{13} =$  , $C_{24} =$  , $C_{34} =$ 

Then

$$C_{12} : \begin{cases} v_+v_+v_+ & \mapsto & 0 \\ v_+v_+v_- & \mapsto & 0 \\ v_+v_-v_+ & \mapsto & v_+v_+v_+ + \varphi\, v_+v_+v_- \\ v_+v_-v_- & \mapsto & v_-v_+v_- \\ v_-v_+v_+ & \mapsto & 0 \\ v_-v_+v_- & \mapsto & 0 \\ v_-v_-v_+ & \mapsto & \varphi\, v_-v_+v_- \\ v_-v_-v_- & \mapsto & 0 \end{cases}$$

$$C_{13} : \begin{cases} v_+v_+v_+ & \mapsto & 0 \\ v_+v_+v_- & \mapsto & 0 \\ v_+v_-v_+ & \mapsto & v_+v_-v_+ + \varphi\, v_+v_+v_- \\ v_+v_-v_- & \mapsto & v_+v_-v_- \\ v_-v_+v_+ & \mapsto & 0 \\ v_-v_+v_- & \mapsto & 0 \\ v_-v_-v_+ & \mapsto & v_-v_-v_+ + \varphi\, v_-v_+v_- \\ v_-v_-v_- & \mapsto & v_-v_-v_- \end{cases}$$

$$C_{24} : \begin{cases} v_+v_+v_+ & \mapsto & \omega\, v_+v_+v_+ \\ v_+v_+v_- & \mapsto & \omega\, v_+v_+v_- \\ v_+v_-v_+ & \mapsto & \omega\, v_-v_+v_+ \\ v_+v_-v_- & \mapsto & \omega\, v_-v_+v_- \\ v_-v_+v_+ & \mapsto & \omega\, v_-v_+v_+ \\ v_-v_+v_- & \mapsto & \omega\, v_-v_+v_- \\ v_-v_-v_+ & \mapsto & 0 \\ v_-v_-v_- & \mapsto & 0 \end{cases}$$

$$C_{34} : \begin{cases} v_+v_+v_+ & \mapsto & v_+v_+v_+ \\ v_+v_+v_- & \mapsto & v_+v_+v_- \\ v_+v_-v_+ & \mapsto & v_+v_-v_+ \\ v_+v_-v_- & \mapsto & v_+v_-v_- \\ v_-v_+v_+ & \mapsto & v_-v_+v_+ \\ v_-v_+v_- & \mapsto & v_-v_+v_- \\ v_-v_-v_+ & \mapsto & v_-v_-v_+ \\ v_-v_-v_- & \mapsto & v_-v_-v_- \end{cases}$$

So we get

$$C_{12} - C_{13} - \omega^{-1} C_{24} + C_{34} = 0.$$

For the other closure, ,  ,

$$C_{12}: \begin{cases} v_+v_+ & \mapsto & 0 \\ v_+v_- & \mapsto & (1+\varphi)\, v_-v_+ \\ v_-v_+ & \mapsto & 0 \\ v_-v_- & \mapsto & 0 \end{cases}$$

$$C_{13}: \begin{cases} v_+v_+ & \mapsto & 0 \\ v_+v_- & \mapsto & \omega^{-1}\,(v_-v_+ + \varphi\, v_+v_-) \\ v_-v_+ & \mapsto & 0 \\ v_-v_- & \mapsto & \omega^{-1}\varphi\, v_-v_- \end{cases}$$

$$C_{24}: \begin{cases} v_+v_+ & \mapsto & \omega\, v_+v_+ \\ v_+v_- & \mapsto & \omega\, v_-v_+ \\ v_-v_+ & \mapsto & \omega\, v_-v_+ \\ v_-v_- & \mapsto & 0 \end{cases}$$

$$C_{34}: \begin{cases} v_+v_+ & \mapsto & v_+v_+ \\ v_+v_- & \mapsto & v_+v_- \\ v_-v_+ & \mapsto & v_-v_+ \\ v_-v_- & \mapsto & v_-v_- \end{cases}$$

So it gives

$$\varphi^{-1}\, C_{12} - \omega\varphi^{-1} C_{13} - \omega^{-1} C_{24} + C_{34} = 0.$$

□

Remark 4.1.1. In the proof of the first, and the second Reidemeister moves, we use *4Tu* relations for the homotopy equivalence. We can see all of them are different each other in each case.

$$\varphi^{-1}\, C_{12} - \ \varphi^{-1}\, C_{13} - C_{24} + C_{34} = 0.$$

$$C_{12} - C_{13} - \omega^{-1}\, C_{24} + C_{34} = 0.$$

$$\varphi^{-1}\, C_{12} - \omega\varphi^{-1} C_{13} - \omega^{-1} C_{24} + C_{34} = 0.$$

But all of these different *4Tu* relations give us the same assertion $FG - I = hd + dh$, holding our relations in ω.

Remark 4.1.2 (Bar-Natan[1])**.** The morphism $G : [\![\ \ ]\!] \to [\![\ \ ]\!]$ in the above proof is a little more than a homotopy equivalence. Let us see the following definition.

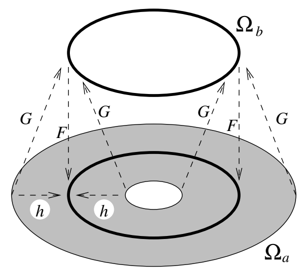


Figure 4.5: *strong deformation retract – (picture from [Bar-Natan][1])*

Now we will see several definitions and lemmas which are for the proof of invariance under the third Reidemeister move. All the difinitions and lemmas are from [BN][1].

Definition 4.1.5 (Bar-Natan[1])**.** *A morphism of complexes $G : \Omega_a \to \Omega_b$ is said to be a* ***strong deformation retract*** *if there is a morphism $F : \Omega_b \to \Omega_a$ and homotopy maps h from Ω_a to itself so that $GF = I$, $I - FG = dh + hd$ and $hF = 0$. In this case we say that F is the* ***inclusion in a strong deformation retract****. Note that a strong deformation retract is in particular a homotopy equivalence. The geometric origin of this notion is the standard notion of a strong deformation retract in homotopy theory as you can see above.*

Definition 4.1.6 (Bar-Natan[1])**.** *Let $\Psi : (\Omega_0^r, d_0) \to (\Omega_1^r, d_1)$ be a morphism of complexes. The* ***cone*** *$\Gamma(\Psi)$ of Ψ is the complex with chain spaces $\Gamma^r(\Psi) = \Omega_0^{r+1} \oplus \Omega_1^r$ and with differentials $\tilde{d}^r = \begin{pmatrix} -d_0^{r+1} & 0 \\ \Psi^{r+1} & d_1^r \end{pmatrix}$. (see Figure 4.6).*

$$\begin{tikzcd}
\Omega_0^r \arrow[r, "-d_0^r"] \arrow[d, "\Psi^r"] & \Omega_0^{r+1} \arrow[r, "-d_0^{r+1}"] \arrow[d, "\Psi^{r+1}"] \arrow[dl, dotted, no head, "\oplus"] & \Omega_0^{r+2} \arrow[d, "\Psi^{r+2}"] \arrow[dl, dotted, no head, "\oplus"] \\
\Omega_1^r \arrow[r, "d_1^r"'] & \Omega_1^{r+1} \arrow[r, "d_1^{r+1}"'] & \Omega_1^{r+2}
\end{tikzcd}$$

Figure 4.6: *Cone*

To prove the invariance under the $R3$ move, we need the following two lemmas, as well.

Lemma 4.1.7 (Bar-Natan[1])**.** *For* $[\![\,)\!-\!(\,]\!]$, $[\![\, \bowtie \,]\!]$, *where* $[\![\,)\!-\!(\,]\!]$ *is the saddle morphism* $[\![\,)\!-\!(\,]\!] : [\![\,)(\,]\!] \to [\![\, \asymp \,]\!]$ *and* $[\![\, \bowtie \,]\!] : [\![\, \asymp \,]\!] \to [\![\,)(\,]\!]$, *the following equivalences are true:*

(a) $[\![\, \times \,]\!] = \Gamma([\![\,)\!-\!(\,]\!])[-1]$

(b) $[\![\, \times \,]\!] = \Gamma([\![\, \bowtie \,]\!])$

where $\cdot[s]$ *is the operator that shifts complexes* s *units to the left:* $\Omega[s]^r := \Omega^{r+s}$.

Proof of Lemma 4.1.7. We see the Lemma A.2.1 in Appendix A $\square$

Lemma 4.1.8 (Bar-Natan[1])**.** *The cone construction is invariant up to homotopy under compositions with the inclusions in strong deformation retracts. That is,*

$$\begin{tikzcd}
\Omega_{0a} \arrow[r, shift left, "G_0"] \arrow[d, "\Psi"] & \Omega_{0b} \arrow[l, shift left, "F_0"] \\
\Omega_{1a} \arrow[r, shift left, "F_1"] & \Omega_{1b} \arrow[l, shift left, "G_1"]
\end{tikzcd}$$

Consider the complexes $(\Omega^r_{0a}, d^r_{0a})$ *and* $(\Omega^r_{0b}, d^r_{0b})$*. Suppose that* $G_0 : \Omega_{0a} \to \Omega_{0b}$ *be a strong deformation retract with corresponding inclusion,* F_0*. Similarly for the comples,* $(\Omega^r_{1a}, d^r_{1a})$ *and* $(\Omega^r_{1b}, d^r_{1b})$*, suppose that* $G_1 : \Omega_{1a} \to \Omega_{1b}$ *is a strong deformation retract with inclusion,* F_1*. Let* Ψ *be a chain homotopy from* Ω_{0a} *to* Ω_{1a}*. Then*

(a) *the cones* $\Gamma(\Psi)$ *and* $\Gamma(\Psi F_0)$ *are homotopy equivalent.*

(b) *the cones* $\Gamma(\Psi)$ *and* $\Gamma(F_1 \Psi)$ *are homotopy equivalent.*

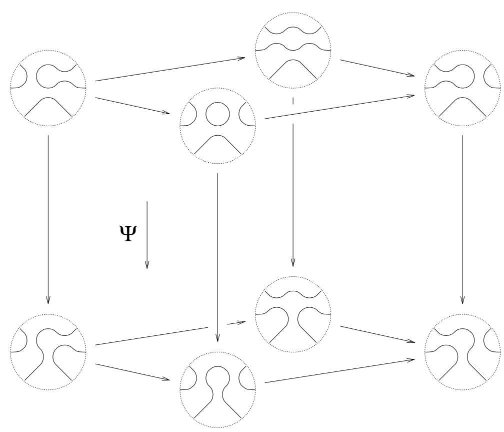
And also, it is true that the cones $\Gamma(\Psi)$ *and* $\Gamma(F_1\Psi)$ *are homotopy equivalent when* $F_1 : \Omega_{1a} \to \Omega_{1b}$ *is the strong deformation retract with the corresponding inclusion,* G_1*. But we don't need this here.*

Proof of Lemma 4.1.8. We see the Lemma A.2.2 in Appendix A $\square$

Now, we are ready to prove the third Reidemeister moves.

Theorem 4.1.9. *[Invariance under Reidemeister Move R3]*
The chain complex $[\![\quad]\!]$ *is homotopy equivalent to the chain complex* $[\![\quad]\!]$.

Proof. The proof is exactly same as one in [Bar-Natan[1]]. According to [Bar-Natan][1], this is both the easiest and hardest move. It is easy because it is just from the $R2$ move and some 'soft' algebra (just like the Kauffman bracket, whose invariance under $R3$ is for free from its invariance under $R2$). And it is hard because it consists of the most crossings and so the most complicated complexes. We note that Lemma 4.1.7 can also be readed in a "skein theoretic" sense, where each of and (or and) represents just a small area inside a bigger tangle. Thus, let us apply Lemma 4.1.7 to the bottom crossing in the tangle . Then $[\![\quad]\!]$ is the cone, $\Gamma(\Psi)$ of the morphism $\Psi = [\![\quad]\!] : [\![\quad]\!] \to [\![\quad]\!]$. In particular, Ψ is the set of four morphisms.



Here, we have five different closures for the $[\![\quad]\!]$. And we can check all Ψs make commutative cubes in each case. Thus we can apply the Lemma 4.1.8.

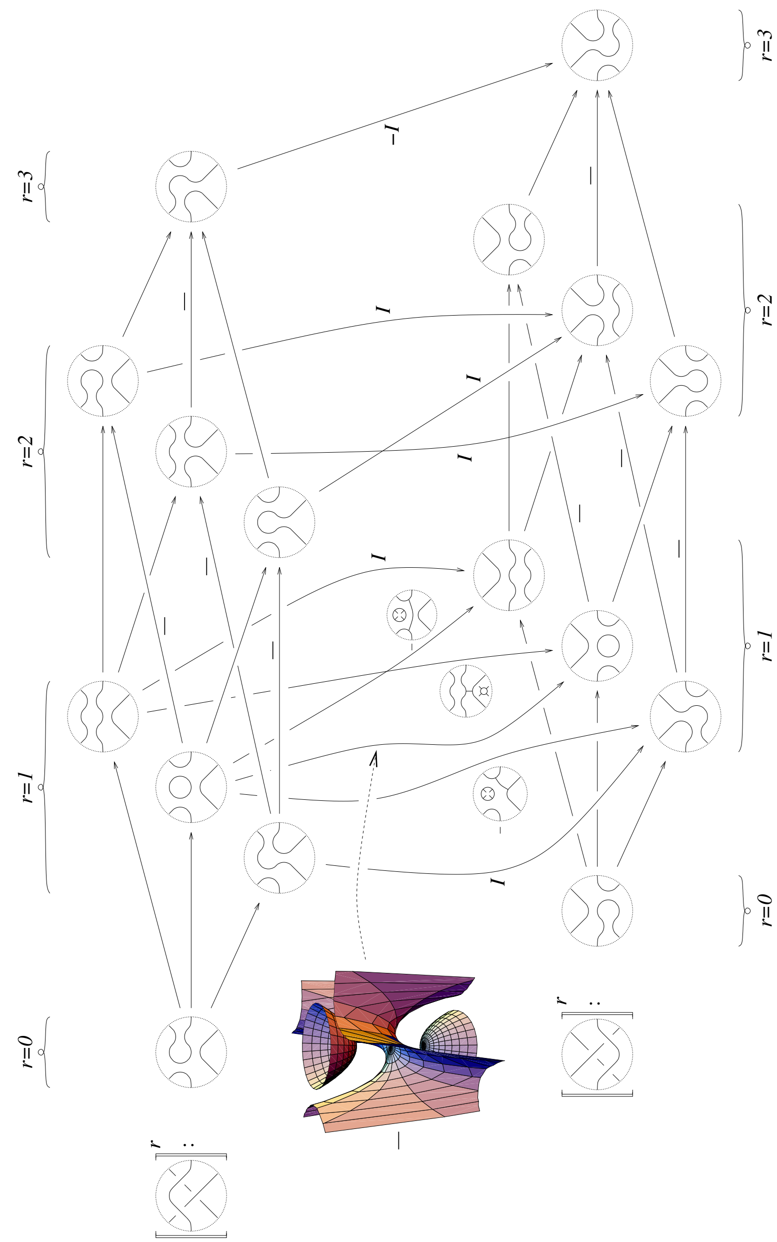
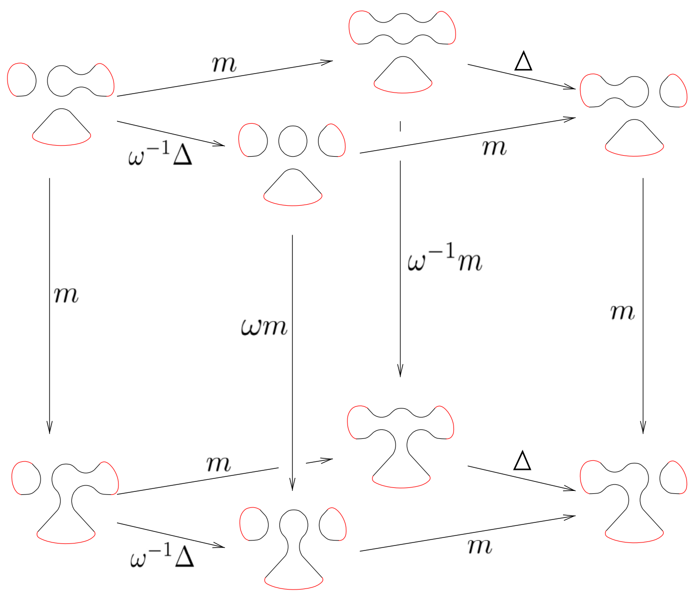
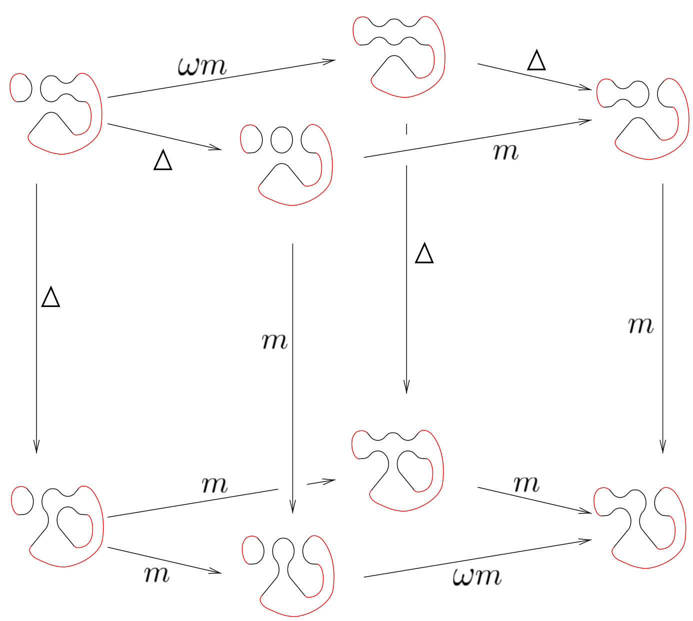


Figure 4.7: *Invariance under R3 in more detail than is strictly necessary. Notice the minus signs and consider all missing arrows between the top layer and the bottom layer as 0. – (picture from [Bar-Natan][1])*

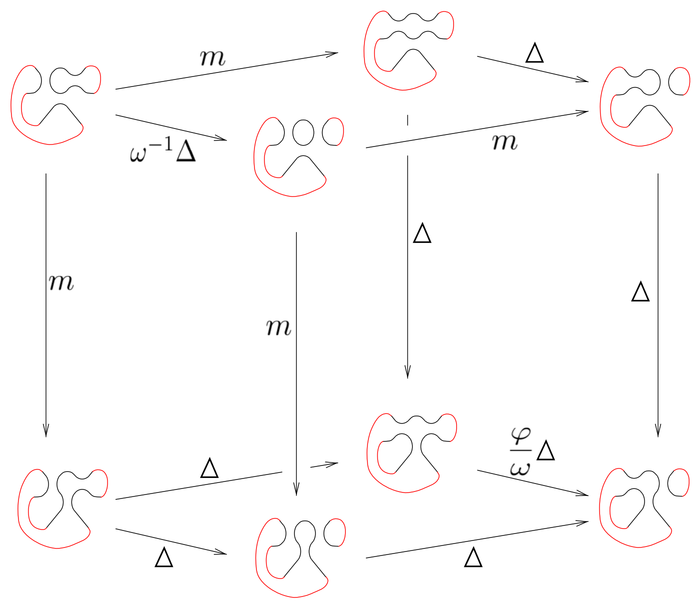
(a)



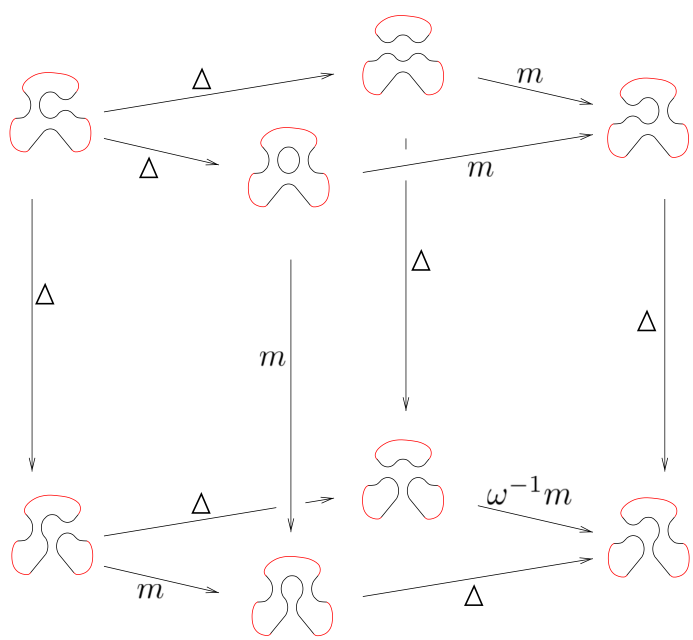
(b)



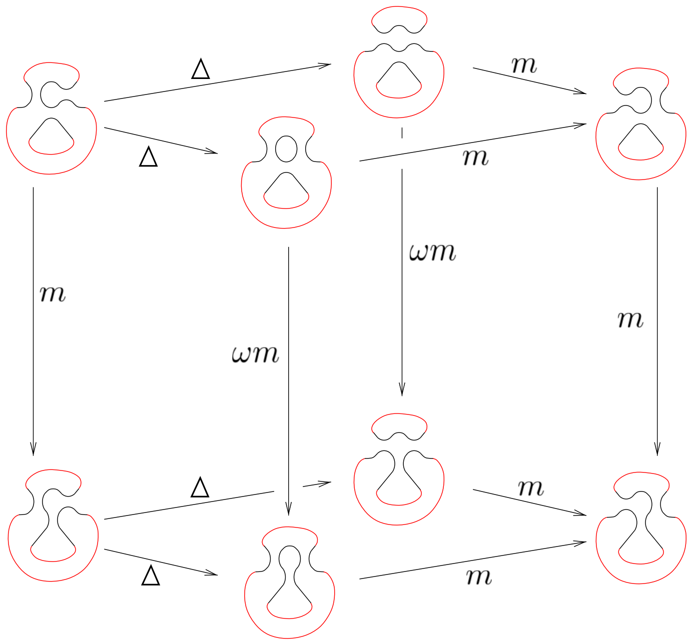
(c)


m
Δ
$\omega^{-1}\Delta$
m
Δ
m
m
Δ
Δ
$\frac{\varphi}{\omega}\Delta$
Δ
Δ

(d)


Δ
m
Δ
m
Δ
Δ
m
Δ
Δ
$\omega^{-1}m$
m
Δ

(e)



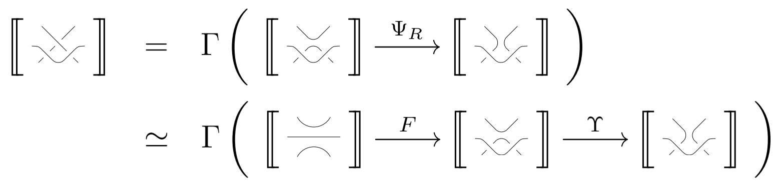
In the proof of $R2$, $G : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$ is a strong deformation retract with corresponding inclusion, $F : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$. So by the Lemma 4.1.8, we have $\Gamma(\Psi) = \Gamma(\Psi F)$.

Aside from the map on the zeroth complex all height chain maps of F can be approximated as zero maps. So the all height chain maps but the zeroth height map of ΨF are taken to be zero. The zeroth chain map of ΨF is $\Psi_L = \Psi \circ (F^0 \oplus I)$. Thus we have

$$\begin{aligned} \llbracket\ \rrbracket &= \Gamma\left(\llbracket\ \rrbracket \xrightarrow{\Psi_L} \llbracket\ \rrbracket \right) \\ &\simeq \Gamma\left(\llbracket\ \rrbracket \xrightarrow{F} \llbracket\ \rrbracket \xrightarrow{\Psi} \llbracket\ \rrbracket \right) \end{aligned}$$

Similarly, for $\Upsilon = \llbracket\ \rrbracket : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$, we have $G : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$ is a strong deformation retract with corresponding inclusion, $F : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$. So we get $\Gamma(\Upsilon) = \Gamma(\Upsilon F)$ by the Lemma 4.1.8.

Aside from the map on the zeroth complex all height chain maps of F can be approximated as zero maps. So the all height chain maps but the zeroth height map of ΥF are taken to be zero. The zeroth chain map of ΥF is $\Psi_R = \Upsilon \circ (F^0 \oplus I)$. Thus we have



Here, we are taking the cone of the same morphism in each case : (see Figure 4.8).

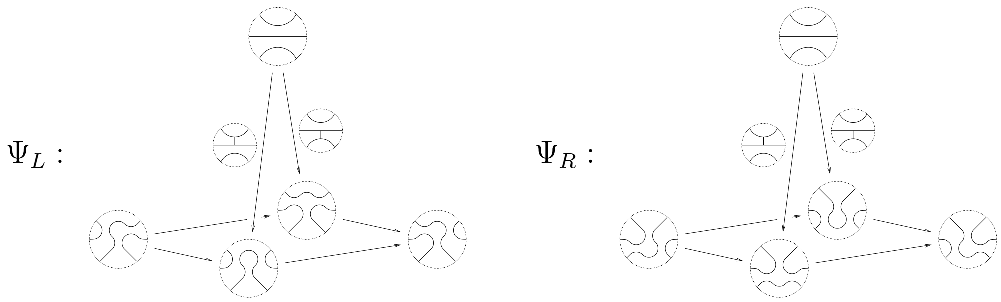


Figure 4.8: *The two sides of the Reidemeister move R3 . – (picture from [Bar-Natan][1])*

Hence, $F\Psi_L = F\Psi_R$ up to isotopies.

$\square$

As we have seen above, we always have commutative cube in the proof of invariance under three Reidemeister moves.

Remark 4.1.3. The proof of invariance still works after passing through the retraction functor.

4.2 Degree shift

We have seen the underlying ungraded vector space of our anyonic Khovanov homology is invariant under three Reidemeister moves. And we can compute their homology values in vector spaces. But our underlying spaces are actually graded vector spaces. So we need to consider their degrees so that we can compute their graded homology groups like the Khovanov homology. Here we will use a new notation and relation for the anyonic Khovanov homology. We will add this degree shift to our setting to be an invariant under the second Reidemeister move, and then either the first Reidemeister move or it framed analog. Then invariance under the third Reidemeister moves is for free. First we will define our new notation.

Let us start with a knot. Then we can build a cube from a knot. At that time each vertex is a 1-manifold decorated with a degree and a grading. Then we can replace each vertex by $S(n_+, n_-, \sigma, c)$, which denotes the degree shift which will be required for the graded vector space to be invariant under Reidemeister moves. Here n_+ and n_- be the numbers of $(+)$ crossings and $(-)$ crossings in the knot K, respectively. And let σ is the number of 1 resolutions, and c is the number of components in the state. Then our $S(n_+, n_-, \sigma, c)$ contains all the information of the vertex in the cube, except the locations of the crossing resolutions. In the proof of the second Reidemeister move, we use the Figure 4.9.

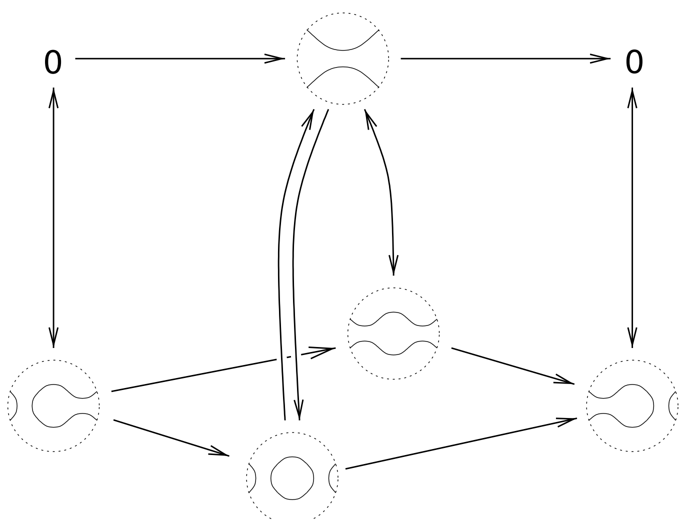


Figure 4.9: *$R2$ maps.*

As in the proof of Theorem 4.1.3, we get two different complexes according to the closure.

So if we compute $S(n_+, n_-, \sigma, c)$ in Figure 4.10, and Figure 4.11, then we get Figure 4.12, and Figure 4.13. Here $S(1, 1, 0, 2)$ means that this object has no 1 resolutions and contains two components in the knot with one $(+)$ crossing, and one $(-)$ crossing, which is .

Then from the Figure 4.12, Figure 4.13, and Definition 3.1.2, we get the following equa-

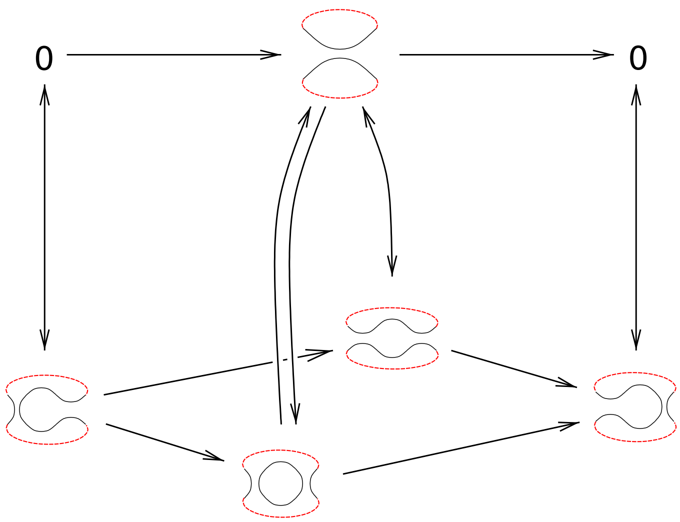


Figure 4.10: *R2 maps with a closure.*

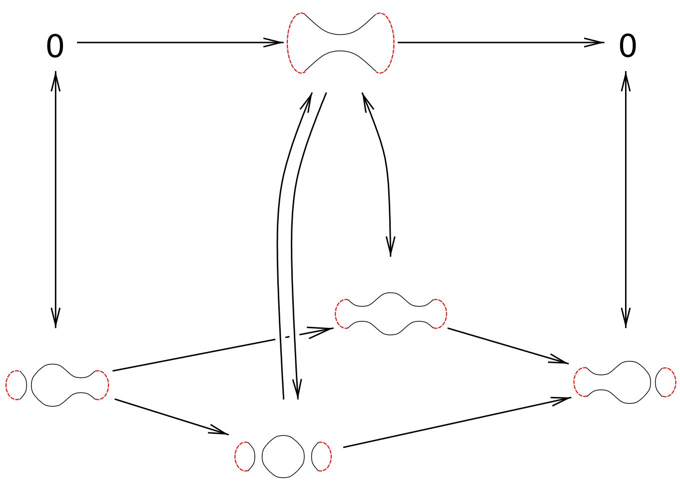


Figure 4.11: *R2 maps with a closure.*

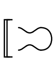
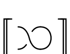
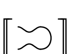
tions.

$$S\big(\vec{v}+(0,0,1,-1)\big)-S\big(\vec{v}\big)=a.$$

$$S\big(\vec{v}+(0,0,1,1)\big)-S\big(\vec{v}\big)=-b.$$

$$S\big(\vec{v}+(1,1,1,0)\big)-S\big(\vec{v}\big)=0.$$

Any degree shift satisfying these will give invariance under R2.

Now Consider R1. To make an invariant of framed links, we will show $[\![$  $]\!]$ $=$ $[\![$  $]\!]\{f\}$, and $[\![$  $]\!]$ $=$ $[\![$  $]\!]\{-f\}$.

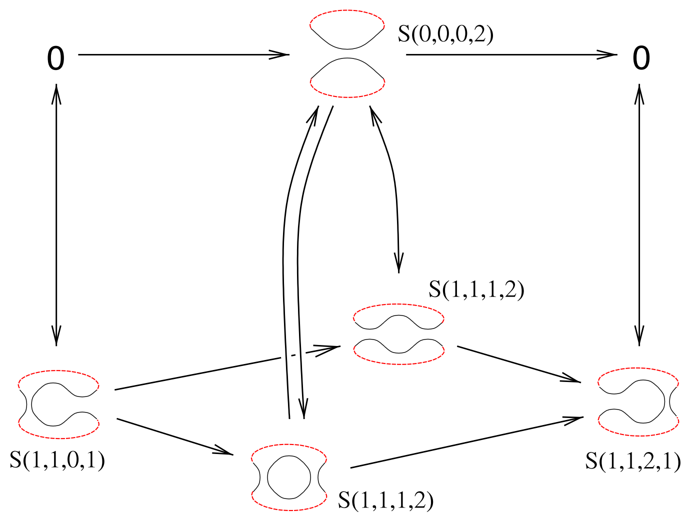


Figure 4.12: *R2 maps with a closure and* $S(n_+, n_-, \sigma, c)$.

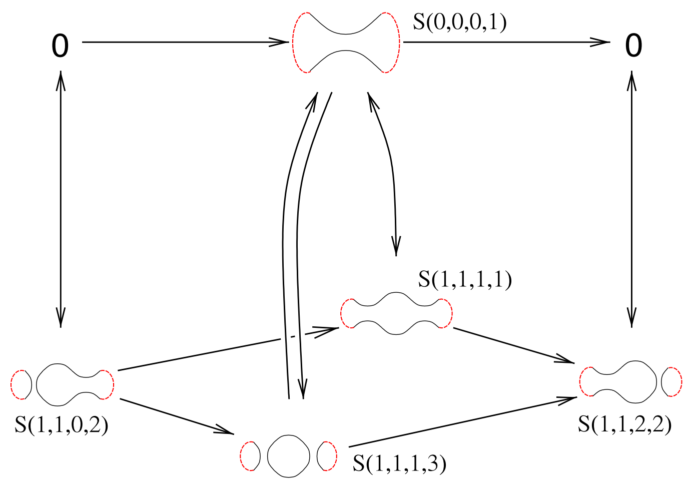


Figure 4.13: *R2 maps with a closure and* $S(n_+, n_-, \sigma, c)$.

Consider $[\![\ ]\!]$ and $[\![\ ]\!]$. From the Figure 4.1, we can get the following maps. After taking a closure, if we apply our $S(n_+, n_-, \sigma, c)$, we can get Figure 4.15. On the other hand, from $[\![\ ]\!]$ and $[\![\ ]\!]$, we have Figure 4.16. And similarly, after taking a closure and $S(n_+, n_-, \sigma, c)$, we have Figure 4.17 Then from this, we get the following equations.

$$S\big(\vec{v} + (0,1,1,1)\big) - S\big(\vec{v}\big) = -a - f.$$

$$S\big(\vec{v} + (1,0,0,1)\big) - S\big(\vec{v}\big) = f - b.$$

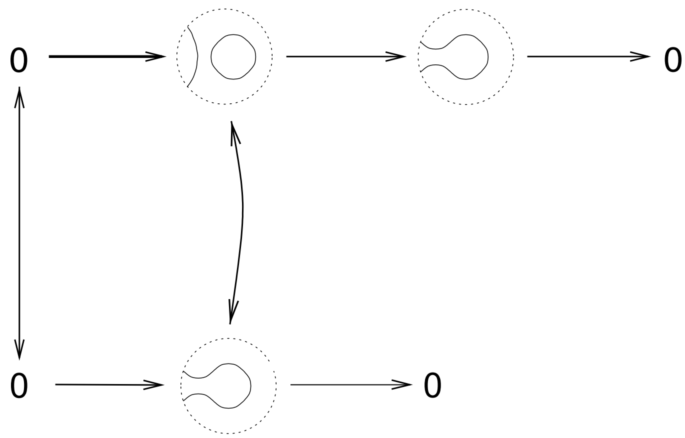


Figure 4.14: *R*1 *maps.*

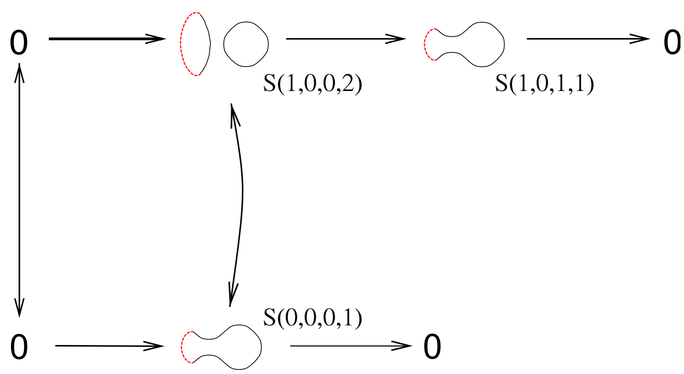


Figure 4.15: *R*1 *maps with a closure and* $S(n_+, n_-, \sigma, c)$.

So we have the following five equations.

$$\begin{cases} S\big(\vec{v} + (0,0,1,-1)\big) - S\big(\vec{v}\big) = a \\ S\big(\vec{v} + (0,0,1,1)\big) - S\big(\vec{v}\big) = -b \\ S\big(\vec{v} + (1,1,1,0)\big) - S\big(\vec{v}\big) = 0 \\ S\big(\vec{v} + (0,1,1,1)\big) - S\big(\vec{v}\big) = -a - f \\ S\big(\vec{v} + (1,0,0,1)\big) - S\big(\vec{v}\big) = f - b. \end{cases}$$

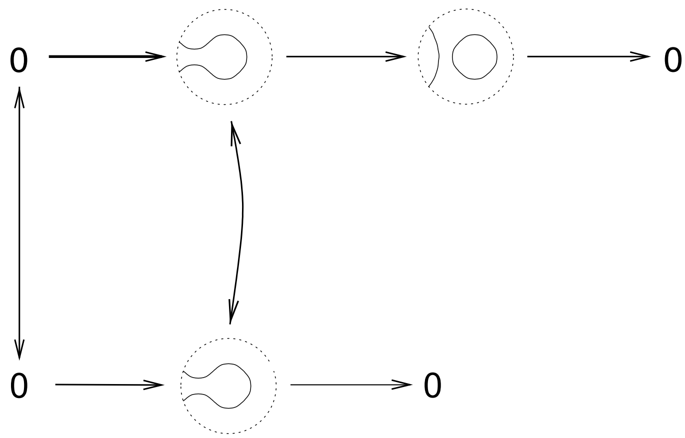


Figure 4.16: *R*1 *maps.*

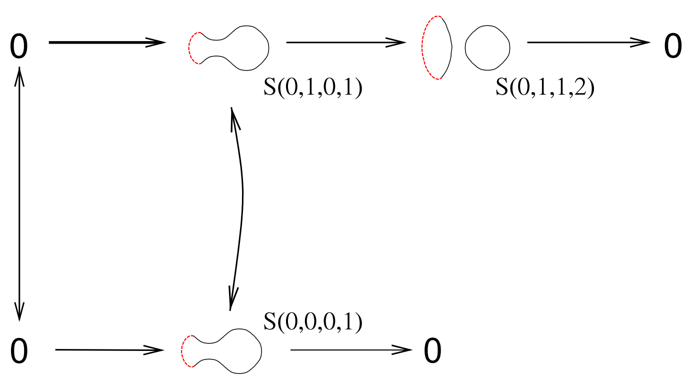


Figure 4.17: *R*1 *maps with a closure and* $S(n_+, n_-, \sigma, c)$.

, which is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & -1 & a \\ 0 & 0 & 1 & 1 & -b \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -a-f \\ 1 & 0 & 0 & 1 & f-b \end{bmatrix}$$

Then the solution is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a-b+2f}{2} \\ 0 & 1 & 0 & 0 & -a+b-f \\ 0 & 0 & 1 & 0 & \frac{a-b}{2} \\ 0 & 0 & 0 & 1 & \frac{-a-b}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we define the degree shift from the previous equations as follow.

$$S(n_+, n_-, \sigma, c) := \left\lfloor \frac{n_+ - 2n_- + \sigma - c}{2} \right\rfloor a + \left\lfloor \frac{-n_+ + 2n_- - \sigma - c}{2} \right\rfloor b + (n_+ - n_-)f.$$

Here $f = 0$ gives us an invariant for links, and any other f gives us an invariant for framed links.

Remark 4.2.1. The height shift is exactly the same as the one in the Khovanov homology.

As a final example we calculate the anyonic homology of the Hopf link.

Example 4.2.2. Let us consider the Hopf Link with $n_+ = 2$.

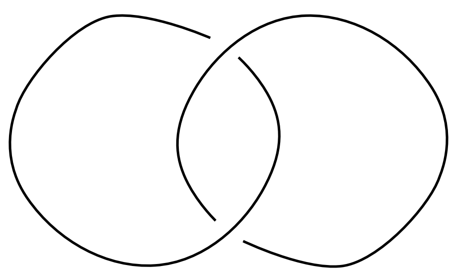


Figure 4.18: *Hopf link.*

From which we obtain

$$\begin{tikzcd} & S(2,0,1,1) \arrow[dr, "\Delta"] & \\ S(2,0,0,2) \arrow[ur, "m"] \arrow[dr, "m"'] & & S(2,0,2,2) \\ & S(2,0,1,1) \arrow[ur, "\Delta"'] & \end{tikzcd}$$

And the chain complex

$$L_0: \quad 0 \longrightarrow V^{\otimes 2}\{-2b+2f\} \longrightarrow V\{a-2b+f\} \bigoplus V\{a-2b+f\} \longrightarrow V^{\otimes 2}\{a-3b+2f\} \longrightarrow 0$$

Then, we can get

- $\mathcal{H}^0(\llbracket L_0 \rrbracket) \;= V\{-b+2f\}$.
- $\mathcal{H}^1(\llbracket L_0 \rrbracket) \;= \{0\}$.
- $\mathcal{H}^2(\llbracket L_0 \rrbracket) \;= V\{2a-3b+2f\}$.

In this case, if we set $a = 1$, $b = -1$, and $f = 0$, the we can get the same result as Khovanov homology.

We discuss the anyonic Khovanov Homology. In the work, the construction takes advantage of the fact that the “states” in Khovanov’s construction are not just unions of circles, but unions of circles in the plane, and so the cobordisms in the Bar-Natan’s approach can be regarded as embedded in $\mathbb{R}^2 \times I$. Because the circles are interacting in $(2+1)-$dimensions, it is natural to work in a braided setting. In physical terms, fractional statistics are possible, corresponding to the anyonic braiding. The anyonic braiding structure gives basic morphisms needed to construct new generalized Khovanov homology. In addition, because the operations of the Frobenius structure have degrees as in Bar-Natan, additional phases arise when braiding operations past arguments. And as in Bar-Natan, invariance under the second Reidemeister move requires the graded module assigned to a circle have underlying module of rank 2.

In practice, we constructed the key relation giving the new generalized Khovanov homology, and it gives the curious requirement that if the phase in the braiding is an n-th root of unity, the degree of the unit and the difference of degrees between the unit and the other generator must be complimentary zero-divisors (mod n). Once degrees other than 1 and -1 (or 1 and 0 (mod 2) as in odd Khovanov homology) are involved, a bit more work is involved in finding the appropriate degree shifts. In doing this one was lead to the observation, trivial in retrospect, but possibly important for the categorification of Reshetikhin-Turaev 3-manifold invariants, that link homology theories can be made into invariants of framed links by representing framed links in the blackboard framing and including a degree shift by the writhe of the diagram.

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Appendix A

More Details

A.1 Commutative/Skew-commutative Cube

In this section, we see the canonical ways to construct an anti-commutative cube from commutative cube. We follow Khovanov's work[13]. Ler $\mathcal{I}$ be a finite set. Denote by $|\mathcal{I}|$ the cardinality of $\mathcal{I}$ and by $r(\mathcal{I})$ the set of all pairs $(\mathcal{L}, a)$ where $\mathcal{L}$ is a subset of $\mathcal{I}$ and a an element of $\mathcal{I}$ that does not belong to $\mathcal{L}$. Sometimes we use the following notations.

(a) denote a one-element set $\{a\}$ by a,

(b) denote a finite set $\{a, b, \cdots, d\}$ by $ab \cdots d$,

(c) denote the disjoint union $\mathcal{L}_1 \sqcup \mathcal{L}_2$ of two sets by $\mathcal{L}_1 \mathcal{L}_2$.

In the same manner, we denote by $\mathcal{L}\, a$ the disjoint union of a set $\mathcal{L}$ and a one-element set $\{a\}$, similarly, $\mathcal{L}\, a\, b$ means $\mathcal{L} \sqcup \{a\} \sqcup \{b\}$, etc.

Definition A.1.1. *Let $\mathcal{I}$ be a finite set and $\mathcal{B}$ a category. An $\mathcal{I}-$ cube V over $\mathcal{B}$ is a collection of objects $V(\mathcal{L}) \in Ob\,(\mathcal{B})$ for each subset $\mathcal{L}$ of $\mathcal{I}$, and morphisms*

$$\xi_a^V\,(\mathcal{L}) : V\,(\mathcal{L}) \longrightarrow V\,(\mathcal{L}\, a)$$

for each $(\mathcal{L}, a) \in r\,(\mathcal{L})$. Here, $\xi_\bullet^V$ are called the structure maps of V.

A cube is commutative if for each triple $(\mathcal{L}, a, b)$, where $\mathcal{L}$ is a subset of $\mathcal{I}$ and a, b $(a \neq b)$ are two elements of $\mathcal{I}$ that do not lie in $\mathcal{L}$, there is an equality of morphisms

$$\xi_b^V(\mathcal{L}\, a)\; \xi_a^V(\mathcal{L}) = \xi_a^V(\mathcal{L}\, b)\; \xi_b^V(\mathcal{L}),$$

, that is, the following diagram commutes.

$$\begin{tikzcd}
V(\mathcal{L}) \arrow[r, "\xi_a^V(\mathcal{L})"] \arrow[d, "\xi_b^V(\mathcal{L})"] & V(\mathcal{L}\, a) \arrow[d, "\xi_b^V(\mathcal{L}\, a)"] \\
V(\mathcal{L}\, b) \arrow[r, "\xi_a^V(\mathcal{L}\, b)"'] & V(\mathcal{L}\, a\, b)
\end{tikzcd}$$

A cube is anti-commutative or skew-commutative if for each triple $(\mathcal{L}, a, b)$, where $\mathcal{L}$ is a subset of $\mathcal{I}$ and a, b $(a \neq b)$ are two elements of $\mathcal{I}$ that do not lie in $\mathcal{L}$, there is an equality of morphisms

$$\xi_b^V(\mathcal{L}\, a)\; \xi_a^V(\mathcal{L}) + \xi_a^V(\mathcal{L}\, b)\; \xi_b^V(\mathcal{L}) = 0.$$

Then, given two $\mathcal{I}-$ cubes V and W over $R-$ mod, the abelian category of graded R-modules, their tensor product, $V \otimes W$, is defined to be a commutative $\mathcal{I}-$ cube (if V and W are both commutative or both anti-commutative) or a anti-commutative $\mathcal{I}-$ cube (if one of V, W is commutative and the other is anti-commutative), given by

$$(V \otimes W)(\mathcal{L}) = V(\mathcal{L}) \otimes W(\mathcal{L}), \quad \mathcal{L} \subset \mathcal{I},$$

$$\xi_a^{V \otimes W}(\mathcal{L}) = \xi_a^V(\mathcal{L}) \otimes \xi_a^W(\mathcal{L}), \quad (\mathcal{L}, a) \in r(\mathcal{I}),$$

where the tensor products are taken over R.

For a finite set $\mathcal{L}$, let $o(\mathcal{L})$ be the set of complete orderings of elements of $\mathcal{L}$. For $s, t \in o(\mathcal{L})$, let $p(s,t)$ be the parity function, given by

$$p(s,t) = \begin{cases} 0 & \text{if } t \text{ can be obtained by from } s \text{ via an even number of transpositions} \\ & \quad \text{of two neighboring elements in the ordering,} \\ 1 & \text{otherwise .} \end{cases}$$

To a finite set $\mathcal{L}$ associate a graded R-module $E(\mathcal{L})$ defined as the quotient of the graded R-module, freely generated by elements s for all $s \in o(\mathcal{L})$, by relations

$$s = (-1)^{p(s,t)}\, t \quad \text{for all pairs } s, t \in o(\mathcal{L}).$$

Module $E(\mathcal{L})$ is a free graded R-module of rank 1. For $a \notin \mathcal{L}$, there is a canonical isomorphism of graded R-modules $E(\mathcal{L}) \longrightarrow E(\mathcal{L}\, a)$, induced by the map $o(\mathcal{L}) \longrightarrow o(\mathcal{L}\, a)$ that takes $s \in o(\mathcal{L})$ to $sa \in o(\mathcal{L}\, a)$. In addition, for a, b $(a \neq b)$, the following diagram would be anti-commutes.

$$\begin{tikzcd}
E\,(\mathcal{L}) \arrow[r] \arrow[d] & E\,(\mathcal{L}\, a) \arrow[d] \\
E\,(\mathcal{L}\, b) \arrow[r] & E\,(\mathcal{L}\, a\, b)
\end{tikzcd}$$

Denote by $E_{\mathcal{I}}$ the anti-commutative $\mathcal{I}-$ cube with

$$E_{\mathcal{I}}\,(\mathcal{L}) = E\,(\mathcal{L}) \quad \text{for } \mathcal{L} \in \mathcal{I}$$

, and the structure map $E_{\mathcal{I}}\,(\mathcal{L}) \longrightarrow E_{\mathcal{I}}\,(\mathcal{L}\, a)$ being canonical isomorphism

$$E\,(\mathcal{L}) \longrightarrow E\,(\mathcal{L}\, a).$$

For more detail, we refer to [Khovanov[13]].

A.2 Proof of Lemmas

Lemma A.2.1. *For* $\llbracket\ \rrbracket$, $\llbracket\ \rrbracket$, *where* $\llbracket\ \rrbracket$ *is the saddle morphism* $\llbracket\ \rrbracket : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$ *and* $\llbracket\ \rrbracket : \llbracket\ \rrbracket \to \llbracket\ \rrbracket$, *the following equivalences are true :*

(a) $\llbracket\ \rrbracket = \Gamma(\llbracket\ \rrbracket)$

(b) $\llbracket\ \rrbracket = \Gamma(\llbracket\ \rrbracket)[-1]$

where $\cdot[s]$ *is the operator that shifts complexes* s *units to the left:* $\Omega[s]^r := \Omega^{r+s}$.

Proof of Lemma A.2.1. **(a)** We know the complex $[\![\,)(\,]\!]$ is $0 \to)(\to 0$, and the complex $[\![\, \asymp \,]\!]$ is $0 \to \asymp \to 0$. So we have :

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] &)(\arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \asymp \arrow[r] & 0 \end{tikzcd}$$

Then the cone $\Gamma([\![\,)(\,]\!])$ is

$$0 \to)(\oplus 0 \to 0 \oplus \asymp \to 0 \ ,$$

with boundary map

$$d = \begin{pmatrix} 0 & 0 \\ [\![\,)(\,]\!] & 0 \end{pmatrix}$$

Thus $\Gamma([\![\,)(\,]\!])$ is equivalent to the complex $0 \to)(\to \asymp \to 0$.

On the other hand, the chain complex $[\![\, \times \,]\!]$ is $0 \to)(\to \asymp \to 0$, where the height of $)($ is 0. Thus $[\![\, \times \,]\!] = \Gamma([\![\,)(\,]\!])$.

(b) Now we have :

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \asymp \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] &)(\arrow[r] & 0 \end{tikzcd}$$

Then the cone $\Gamma([\![\, \asymp \,]\!])[-1]$ is

$$0 \to \asymp \oplus 0 \to 0 \oplus)(\to 0 \ ,$$

with boundary map

$$d = \begin{pmatrix} 0 & 0 \\ [\![\, \asymp \,]\!] & 0 \end{pmatrix}$$

So $\Gamma([\![\, \asymp \,]\!])[-1]$ is equivalent to the complex $0 \to \asymp \to)(\to 0$, where the complex of height -1 is $\asymp$, and its boundary map is $[\![\, \asymp \,]\!]$.

On the other hand, the chain complex $[\![\, \times \,]\!]$ is $0 \to \asymp \to)(\to 0$, where the height of $\asymp$ is -1, because of height shift. Thus $[\![\, \times \,]\!] = \Gamma([\![\, \asymp \,]\!])[-1]$.

$\square$

Lemma A.2.2. *The cone construction is invariant up to homotopy under compositions with the inclusions in strong deformation retracts. That is,*

$$\begin{tikzcd}
\Omega_{0a} \arrow[r, shift left, "G_0"] \arrow[d, "\Psi"] & \Omega_{0b} \arrow[l, shift left, "F_0"] \\
\Omega_{1a} \arrow[r, shift left, "F_1"] & \Omega_{1b} \arrow[l, shift left, "G_1"]
\end{tikzcd}$$

Consider the complexes $(\Omega^r_{0a}, d^r_{0a})$ and $(\Omega^r_{0b}, d^r_{0b})$. Suppose that $G_0 : \Omega_{0a} \to \Omega_{0b}$ be a strong deformation retract with corresponding inclusion, F_0. Similarly for the comples, $(\Omega^r_{1a}, d^r_{1a})$ and $(\Omega^r_{1b}, d^r_{1b})$, suppose that $G_1 : \Omega_{1a} \to \Omega_{1b}$ is a strong deformation retract with inclusion, F_1. Let Ψ be a chain homotopy from Ω_{0a} to Ω_{1a}. Then

- **(a)** *the cones $\Gamma(\Psi)$ and $\Gamma(\Psi F_0)$ are homotopy equivalent.*
- **(b)** *the cones $\Gamma(\Psi)$ and $\Gamma(F_1\Psi)$ are homotopy equivalent.*

And also, it is true that the cones $\Gamma(\Psi)$ and $\Gamma(F_1\Psi)$ are homotopy equivalent when $F_1 : \Omega_{1a} \to \Omega_{1b}$ is the strong deformation retract with the corresponding inclusion, G_1. But we don't need this here.

Proof of Lemma A.2.2. The cone $\Gamma(\Psi)$ has complexes, $C^r = \begin{pmatrix} \Omega^r_{0a} \\ \Omega^{r-1}_{1a} \end{pmatrix}$, $C^{r+1} = \begin{pmatrix} \Omega^{r+1}_{0a} \\ \Omega^r_{1a} \end{pmatrix}$, and boundary map,

$$\tilde{d} = \begin{pmatrix} -d & 0 \\ \Psi F_0 & d \end{pmatrix}.$$

(a) Let $h^\star_0 : \Omega^\star_{0a} \to \Omega^{\star-1}_{0a}$ be a homotopy with $I - F_0G_0 = dh_0 + h_0d$ and $h_0F_0 = 0$. Then the cone $\Gamma(\Psi F_0)$ has complexes, $C'^r = \begin{pmatrix} \Omega^r_{0b} \\ \Omega^{r-1}_{1a} \end{pmatrix}$, $C'^{r+1} = \begin{pmatrix} \Omega^{r+1}_{0b} \\ \Omega^r_{1a} \end{pmatrix}$, and boundary map,

$$\tilde{d} = \begin{pmatrix} -d^r_{0b} & 0 \\ \Psi^r F^r_0 & d^{r-1}_{1a} \end{pmatrix}.$$

So we can define the chain maps $\tilde{G}^r_0 : C^r \to C'^r$, and $\tilde{F}^r_0 : C'^r \to C^r$ with

$$\tilde{G}_0^r = \begin{pmatrix} -G_0^r & 0 \\ \\ \Psi^r h_0^r & I \end{pmatrix}, \text{and } \tilde{F}_0^r = \begin{pmatrix} -F_0^r & 0 \\ \\ 0 & I \end{pmatrix}.$$

We take the homotopy map $\tilde{h}_0^{r+1} : C^{r+1} \to C^r$ to be

$$\tilde{h}_0^r = \begin{pmatrix} -h_0^r & 0 \\ \\ 0 & 0 \end{pmatrix}.$$

Then the diagram in Figure A.1 defines morphisms $\Gamma(\Psi F_0) \underset{\tilde{G}_0}{\overset{\tilde{F}_0}{\rightleftarrows}} \Gamma(\Psi)$ and a homotopy $\tilde{h}_0^\star : \Gamma(\Psi)^\star \to \Gamma(\Psi)^{\star-1}$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\Gamma(\Psi F_0): & \begin{pmatrix} \Omega_{0b}^{r+1} \\ \Omega_{1a}^{r} \end{pmatrix} \arrow[rr, "\tilde{d}=\begin{pmatrix} -d & 0 \\ \Psi F_0 & d \end{pmatrix}"] \arrow[d, shift left, "\tilde{F}_0^r:=\begin{pmatrix} -F_0 & 0 \\ 0 & I \end{pmatrix}"] & & \begin{pmatrix} \Omega_{0b}^{r+2} \\ \Omega_{1a}^{r+1} \end{pmatrix} \arrow[d, shift left, "\tilde{F}_0^{r+1}"] \\
\Gamma(\Psi): & \begin{pmatrix} \Omega_{0a}^{r+1} \\ \Omega_{1a}^{r} \end{pmatrix} \arrow[u, shift left, "\tilde{G}_0^r:=\begin{pmatrix} -G_0 & 0 \\ \Psi h_0 & I \end{pmatrix}"] \arrow[rr, shift left, "\tilde{d}=\begin{pmatrix} -d & 0 \\ \Psi & d \end{pmatrix}"] & & \begin{pmatrix} \Omega_{0a}^{r+2} \\ \Omega_{1a}^{r+1} \end{pmatrix} \arrow[u, shift left, "\tilde{G}_0^{r+1}"] \arrow[ll, shift left, "\tilde{h}_0:=\begin{pmatrix} -h_0 & 0 \\ 0 & 0 \end{pmatrix}"]
\end{tikzcd}$$

Figure A.1: *Main diagram for* $\Gamma(\Psi)$ *and* $\Gamma(\Psi F_0)$

- $\tilde{G}_0^r \tilde{F}_0^r = \begin{pmatrix} -G_0^r & 0 \\ \\ \Psi^r h_0^r & 0 \end{pmatrix} \begin{pmatrix} -F_0^r & 0 \\ \\ 0 & I \end{pmatrix} = \begin{pmatrix} G_0^r F_0^r & 0 \\ \\ -\Psi^r h_0^r F_0^r & I \end{pmatrix}.$

  But $G_0^r F_0^r = I$, and $-\Psi^r h_0^r F_0^r = 0$, since $h_0^r F_0^r = 0$. So

$$\tilde{G}_0^r \tilde{F}_0^r = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

  , which is the identity on $\Gamma(\Psi^r F_0)$.

- $\tilde{F}_0^r \tilde{G}_0^r = \begin{pmatrix} -F_0^r & 0 \\ \\ 0 & I \end{pmatrix} \begin{pmatrix} -G_0^r & 0 \\ \\ \Psi^r h_0^r & I \end{pmatrix} = \begin{pmatrix} F_0^r G_0^r & 0 \\ \\ -\Psi^r h_0^r & I \end{pmatrix}$. So $I - \tilde{F}_0^r \tilde{G}_0^r$ is

$$I - \tilde{F}_0^r \tilde{G}_0^r = \begin{pmatrix} I - F_0^r G_0^r & 0 \\ \\ \Psi^r h_0^r & 0 \end{pmatrix}.$$

Because we have $I - F_0^r G_0^r = dh_0 + h_0 d$,

$$I - \tilde{F}_0^r \tilde{G}_0^r = \begin{pmatrix} dh_0 + h_0 d & 0 \\ \\ \Psi^r h_0^r & 0 \end{pmatrix}.$$

On the other hand,

$$\tilde{d}^{r-1} \tilde{h}_0^r = \begin{pmatrix} d_{0a}^{r-1} h_0^r & 0 \\ \\ \Psi^r h_0^r & 0 \end{pmatrix},$$

and

$$\tilde{h}_0^{r+1} \tilde{d}^r = \begin{pmatrix} h_0^{r+1} d_{0a}^r & 0 \\ \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\tilde{d}^{r-1} \tilde{h}_0^r + \tilde{h}_0^{r+1} \tilde{d}^r = \begin{pmatrix} d_{0a}^{r-1} h_0^r + h_0^{r+1} d_{0a}^r & 0 \\ \\ \Psi^r h_0^r & 0 \end{pmatrix} = I - \tilde{F}_0^r \tilde{G}_0^r.$$

(b) It is similar to the part (a).