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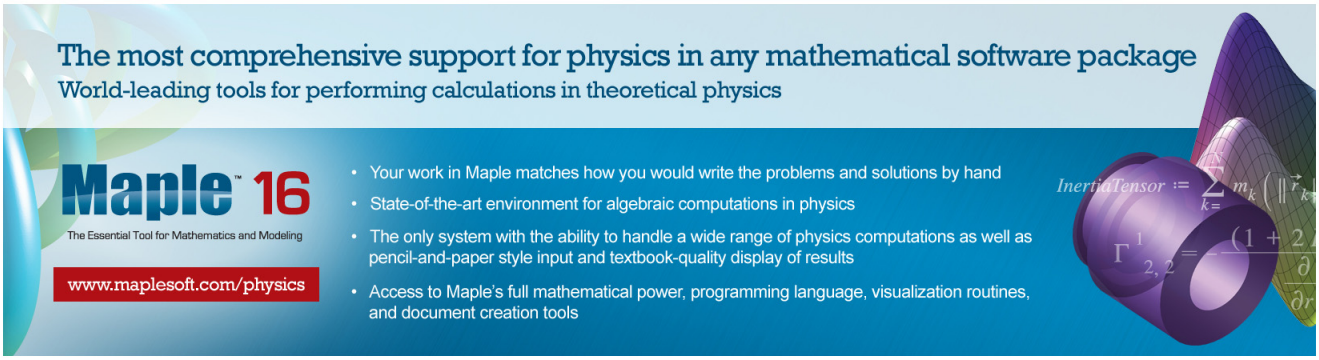

Inverse problem for an inhomogeneous Schrödinger equation

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Let $(l-k^2)u=-u''+q(x)u-k^2u=\delta(x)$, $x\in\mathbb{R}$, $\partial u/\partial|x|-iku\to 0$, $|x|\to\infty$. Assume that the potential $q(x)$ is real valued and compactly supported: $q(x)=\overline{q(x)}$, $q(x)=0$ for $|x|\geqslant 1$, $\int_{-1}^{1}|q|dx<\infty$, and that $q(x)$ produces no bound states. Let $u(-1,k)$ and $u(1,k)$, $\forall k>0$ be the data. It is shown that under the above assumptions these data determine $q(x)$ uniquely. © *1999 American Institute of Physics.* [S0022-2488(99)02108-8]

I. INTRODUCTION

For several decades, the following inverse problems of practical interest are open. Let

$$\nabla^2u+k^2u+k^2v(x)u=-\delta(x),\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

u satisfies the radiation condition at infinity, and $v(x)$ is a compactly supported piecewise-smooth function, $\operatorname{supp} v\subset\mathbb{R}^3_-:=\{x:x_3<0\}$.

The data are the values $u(x_1,x_2,0,k)$ for all $\hat{x}:=(x_1,x_2)\in\mathbb{R}^2$ and $k>0$.

(IP1) The inverse problem is the following.

Given the data, find $v(x)$.

Uniqueness of the solution to this problem is not proved. IP1 is not overdetermined: the data is a function of three variables, and $v(x)$ is also.

A similar inverse problem can be formulated: Let

$$\nabla^2u+k^2u-q(x)u=0,\quad \text{in } \mathbb{R}^3, \tag{1.2}$$

$$u=e^{ik\alpha\cdot x}+A(\alpha',\alpha,k)\frac{e^{ikr}}{r}+o\left(\frac{1}{r}\right),\quad r:=|x|\to\infty,\quad \alpha'=\frac{x}{r}, \tag{1.3}$$

where $\alpha\in S^2$ is a given unit vector, $q(x)$ is a real-valued piecewise-smooth function, $\operatorname{supp} q(x)\subset B_a:=\{x:|x|\leqslant a\}$, and S^2 is the unit sphere.

(IP2) *Given* $A(\alpha',\alpha_0,k)$ *for all* $\alpha'\in S^2$, *all* $k>0$ *and a fixed* $\alpha=\alpha_0\in S^2$, *find* $q(x)$.

The uniqueness of the solution to (IP2) is not proved.

The third problem is the following.

Let

$$\nabla^2u+k^2u-q(x)u=-\delta(x),\quad \text{in } \mathbb{R}^3; \tag{1.4}$$

u satisfies the radiation condition, and $q(x)$ is the same as in (IP2).

The data are the values $u(x,k)|_{|x|=a}$.

(IP3) *Given the data* $u(x,k)|_{|x|=a}$ *for all* $k>0$ *and all* x *on the sphere* $S_a:=\{x:|x|=a\}$, *find* $q(x)$.

Uniqueness of the solution to (IP3) is not proved.

An overview of inverse problems and references one can find in Refs. 1–3.

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Our purpose in this paper is to study the one-dimensional analog of (IP3) and to prove for this analog a uniqueness theorem. The one-dimensional analog of (IP3) corresponds to a plasma equation in a layer.

Let

$$lu-k^2u:=-u''+q(x)u-k^2u=\delta(x), \quad x\in\mathbb{R}^1, \tag{1.5}$$

$$\frac{\partial u}{\partial|x|}-iku\to 0, \quad |x|\to\infty. \tag{1.6}$$

Assume that $q(x)$ is a real-valued function,

$$q(x)=0, \quad \text{for } |x|>1, \quad q\in L^\infty[-1,1]. \tag{1.7}$$

Suppose that the data,

$$\{u(-1,k),u(1,k)\}, \quad \forall k>0, \tag{1.8}$$

are given.

The inverse problem analogous to (IP3) is the following.

(IP) Given the data (1.8), find $q(x)$.

This problem, as well as (IP1)–(IP3), is of practical interest. One can think about finding the properties of an inhomogeneous slab (the governing equation is a plasma equation) from the boundary measurements of the field, generated by a point source inside the slab.

In the literature there are many results concerning various inverse problems for the homogeneous version of Eq. (1.5), but it seems that no results concerning (IP) are known.

Assume that the self-adjoint operator $l=-d^2/dx^2+q(x)$ in $L^2(\mathbb{R})$ has no negative eigenvalues [this is the case when $q(x)\geqslant 0$, for example]. The operator l is the closure in $L^2(\mathbb{R})$ of the symmetric operator l_0 defined on $C_0^\infty(\mathbb{R}^1)$ by the formula $l_0u=-u''+q(x)u$. Our result is the following.

Theorem 1: *Under the above assumptions IP has, at most, one solution.*

II. PROOF OF THEOREM 1

The solution to (1.5)–(1.6) is

$$u=\begin{cases}\dfrac{g(k)}{[f,g]}f(x,k), & x>0,\\[2ex] \dfrac{f(k)}{[f,g]}g(x,k), & x<0.\end{cases} \tag{2.1}$$

Here $f(x,k)$ and $g(x,k)$ solve homogeneous version of Eq. (1.5) and have the following asymptotics:

$$f(x,k)\sim e^{ikx}, \quad x\to+\infty, \quad g(x,k)\sim e^{-ikx}, \quad x\to-\infty, \tag{2.2}$$

$$f(k):=f(0,k), \quad g(k):=g(0,k), \tag{2.3}$$

$$[f,g]:=fg'-f'g=-2ika(k), \tag{2.4}$$

where the prime denotes differentiation with respect to the x variable, and $a(k)$ is defined by the equation

$$f(x,k)=b(k)g(x,k)+a(k)g(x,-k). \tag{2.5}$$

It is known (see, for example, Ref. 4) that

$$g(x,k)=-b(-k)f(x,k)+a(k)f(x,-k), \tag{2.6}$$

$$a(-k)=\overline{a(k)}, \quad b(-k)=\overline{b(k)}, \quad |a(k)|^2=1+|b(k)|^2, \quad k\in\mathbb{R}, \tag{2.7}$$

$$a(k)=1+O\left(\frac{1}{k}\right), \quad k\to\infty, \quad k\in\mathbb{C}_+; \quad b(k)=O\left(\frac{1}{k}\right), \quad |k|\to\infty, \quad k\in\mathbb{R}, \tag{2.7'}$$

$$[f(x,k),g(x,-k)]=2ikb(k), \quad [f(x,k),g(x,k)]=-2ika(k), \tag{2.8}$$

$a(k)$ is analytic in $\mathbb{C}_+$, $b(k)$, in general, does not admit analytic continuation from $\mathbb{R}$, but if $q(x)$ is compactly supported, then $a(k)$ and $b(k)$ are analytic functions of $k\in\mathbb{C}\backslash 0$.

The functions

$$A_1(k):=\frac{g(k)f(1,k)}{-2ika(k)}, \quad A_2(k):=\frac{f(k)g(-1,k)}{-2ika(k)}, \tag{2.9}$$

are the data; they are known for all $k>0$. Therefore one can assume the functions

$$h_1(k):=\frac{g(k)}{a(k)}, \quad h_2(k):=\frac{f(k)}{a(k)}, \tag{2.10}$$

to be known for all $k>0$, because

$$f(1,k)=e^{ik}, \quad g(-1,k)=e^{ik}, \tag{2.11}$$

as follows from the assumption (1.7) and from (2.2).

From (2.10), (2.6), and (2.5), it follows that

$$a(k)h_1(k)=-b(-k)f(k)+a(k)f(-k)=-b(-k)h_2(k)a(k)+h_2(-k)a(-k)a(k), \tag{2.12}$$

$$a(k)h_2(k)=b(k)a(k)h_1(k)+a(k)h_1(-k)a(-k). \tag{2.13}$$

From (2.12) and (2.13), it follows that

$$-b(-k)h_2(k)+h_2(-k)a(-k)=h_1(k), \tag{2.14}$$

$$b(k)h_1(k)+a(-k)h_1(-k)=h_2(k). \tag{2.15}$$

Eliminating $b(-k)$ from (2.14) and (2.15), one gets

$$a(k)h_1(k)h_2(k)+a(-k)h_1(-k)h_2(-k)=h_1(k)h_1(-k)+h_2(-k)h_2(k), \tag{2.16}$$

or

$$a(k)=m(k)a(-k)+n(k), \quad k\in\mathbb{R}, \tag{2.17}$$

where

$$m(k):=-\frac{h_1(-k)h_2(-k)}{h_1(k)h_2(k)}, \quad n(k):=\frac{h_1(-k)}{h_2(k)}+\frac{h_2(-k)}{h_1(k)}. \tag{2.18}$$

Problem (2.17) is a Riemann problem (see Ref. 5 for the theory of this problem) for the pair $\{a(k),a(-k)\}$, the function $a(k)$ is analytic in $\mathbb{C}_+ := \{k: k \in \mathbb{C}, \operatorname{Im} k > 0\}$ and $a(-k)$ is analytic in $\mathbb{C}_-$. The functions $a(k)$ and $a(-k)$ tend to one as k tends to infinity in $\mathbb{C}_+$ and, respectively, in $\mathbb{C}_-$; see Eq. (2.7′).

The function $a(k)$ has finitely many simple zeros at the points ik_j, $1 \leqslant j \leqslant J$, $k_j > 0$, where $-k_j^2$ are the negative eigenvalues of the operator l defined by the differential expression $lu = -u'' + q(x)u$ in $L^2(\mathbb{R})$.

The zeros ik_j are the only zeros of $a(k)$ in the upper half-plane k.

Define

$$\operatorname{ind} a(k) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} d \ln a(k). \tag{2.19}$$

One has

$$\operatorname{ind} a = J, \tag{2.20}$$

where J is the number of negative eigenvalues of the operator l, and, using (2.10), (2.20) and (2.18), one gets

$$\operatorname{ind} m(k) = -2[\operatorname{ind} h_1(k) + \operatorname{ind} h_2(k)] = -2[\operatorname{ind} g(k) + \operatorname{ind} f(k) - 2J]. \tag{2.21}$$

Since l has no negative eigenvalues, it follows that $J = 0$.

In this case $\operatorname{ind} f(k) = \operatorname{ind} g(k) = 0$ (see Lemma 1 below), so $\operatorname{ind} m(k) = 0$, and $a(k)$ is uniquely recovered from the data as the solution of (2.17), which tends to one at infinity; see Eq. (2.7′). If $a(k)$ is found, then $b(k)$ is uniquely determined by Eq. (2.15), and so the reflection coefficient $r(k) := b(k)/a(k)$ is found. The reflection coefficient determines a compactly supported $q(x)$ uniquely (see Ref. 2).

To make this paper self-contained, let us outline a proof of the last claim using an argument different from the one given in Ref. 2.

If $q(x)$ is compactly supported, then the reflection coefficient $r(k) := b(k)/a(k)$ is meromorphic. Therefore, its values for all $k > 0$ determine uniquely $r(k)$ in the whole complex k plane as a meromorphic function. The poles of this function in the upper half-plane are the numbers ik_j, $j = 1,2,...,J$. They determine uniquely the numbers k_j, $1 \leqslant j \leqslant J$, which are a part of the standard scattering data $\{r(k), k_j, s_j, 1 \leqslant j \leqslant J\}$, where s_j are the norming constants.

Note that if $a(ik_j) = 0$ then $b(ik_j) \neq 0$: otherwise Eq. (2.5) would imply $f(x,ik_j) \equiv 0$, in contradiction to the first relation (2.2).

If $r(k)$ is meromorphic, then the norming constants can be calculated by the formula $s_j = -i[b(ik_j)/\dot{a}(ik_j)] = -i \operatorname{Res}_{k=ik_j} r(k)$, where the dot denotes differentiation with respect to k, and Res denotes the residue. So, for compactly supported potential, the values of $r(k)$ for all $k > 0$ determine uniquely the standard scattering data, that is, the reflection coefficient, the bound states $-k_j^2$, and the norming constants s_j, $1 \leqslant j \leqslant J$. These data determine the potential uniquely.

Theorem 1 is proved. □

Lemma 1: If $J = 0$ then $\operatorname{ind} f = \operatorname{ind} g = 0$.

Proof: We prove $\operatorname{ind} f = 0$. The proof of the equation $\operatorname{ind} g = 0$ is similar. Since $\operatorname{ind} f(k)$ equals the number of zeros of $f(k)$ in $\mathbb{C}_+$, we have to prove that $f(k)$ does not vanish in $\mathbb{C}_+$. If $f(z) = 0$, $z \in \mathbb{C}_+$, then $z = ik$, $k > 0$, and $-k^2$ is an eigenvalue of the operator l in $L^2(0,\infty)$, with the boundary condition $u(0) = 0$.

From the variational principle one can find the negative eigenvalues of the operator l in $L^2(\mathbb{R}_+)$ with the Dirichlet condition at $x = 0$ as consecutive minima of the quadratic functional. The minimal eigenvalue is

$$-k^2 = \inf \int_0^{\infty} [u'^2 + q(x)u^2] dx := \kappa_0, \quad u \in \mathring{H}^1(\mathbb{R}_+), \quad \|u\|_{L^2(\mathbb{R}_+)} = 1, \tag{2.22}$$

where $\mathring{H}^1(\mathbb{R}_+)$ is the Sobolev space of $H^1(\mathbb{R}_+)$ functions, satisfying the condition $u(0)=0$.

On the other hand, if $J=0$, then

$$0 \leqslant \inf \int_{-\infty}^{\infty} [u'^2 + q(x)u^2]dx := \kappa_1, \quad u \in H^1(\mathbb{R}), \quad \|u\|_{L^2(\mathbb{R})} = 1. \tag{2.23}$$

Since any element u of $\mathring{H}^1(\mathbb{R}_+)$ can be considered as an element of $H^1(\mathbb{R})$ if one extends u to the whole axis by setting $u=0$ for $x<0$, it follows from the variational definitions (2.22) and (2.23) that $\kappa_1 \leqslant \kappa_0$. Therefore, if $J=0$, then $\kappa_1 \geqslant 0$, and therefore $\kappa_0 \geqslant 0$. This means that operator l on $L^2(\mathbb{R}_+)$ with the Dirichlet condition at $x=0$ has no negative eigenvalues. This means that $f(k)$ does not have zeros in $\mathbb{C}_+$, if $J=0$. Thus $J=0$ implies $\operatorname{ind} f(k)=0$.

Lemma 1 is proved. □

Remark 2: The above argument shows that, in general,

$$\operatorname{ind} f \leqslant J \quad \text{and} \quad \operatorname{ind} g \leqslant J, \tag{2.24}$$

so that (2.21) implies

$$\operatorname{ind} m(k) \geqslant 0. \tag{2.25}$$

Therefore the Riemann problem (2.17) is always solvable.

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