HIRZEBRUCH-RIEMANN-ROCH THEOREM FOR DIFFERENTIAL GRADED ALGEBRAS

by

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
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Abstract

Recall the classical Riemann-Roch theorem for curves: Given a smooth projective complex curve and two holomorphic vector bundles E, F on it, the Euler form

$$\chi(E, F) = \dim \operatorname{Hom}(E, F) - \dim \operatorname{Ext}^1(E, F)$$

can be computed in terms of the ranks and the degrees of the vector bundles. Remarkably, there are a number of similarly looking formulas in algebra. The simplest example is the Ringel formula in the theory of quivers. It expresses the Euler form of two finite-dimensional representations of a quiver algebra in terms of a certain pairing of their dimension vectors. The existence of Riemann-Roch type formulas in these two settings is a consequence of a deeper similarity in the structure of the corresponding *derived categories* - those of sheaves on curves and of modules over quiver algebras.

The thesis is devoted to a version of the Riemann-Roch formula for *abstract* derived categories. By the latter we understand the derived categories of *differential graded* (DG) categories. More specifically, we work with the categories of *perfect modules* over DG algebras. These are a simultaneous generalization of the derived categories of modules over associative algebras and the derived categories of schemes. Given an arbitrary DG algebra A, satisfying a certain finiteness condition, we define and explicitly describe a canonical pairing on its Hochschild homology. Then we give an explicit formula for the Euler character of an arbitrary perfect A-module; the character is an element of the Hochschild homology of A. In this setting, our noncommutative Riemann-Roch formula expresses the Euler characteristic of the Hom-complex between any two perfect A-modules in terms of the pairing of their Euler characters.

One of the main applications of our results is a theorem that the aforementioned pairing on the Hochschild homology is non-degenerate when the DG algebra satisfies a smoothness condition. This theorem implies a special case of the well-known noncommutative Hodge-to-de Rham degeneration conjecture. Another application is related to mathematical physics: We explicitly construct an open-closed topological field theory from an arbitrary Frobenius algebra and then, following ideas of physicists, interpret the noncommutative Riemann-Roch formula as a special case of the so-called topological Cardy condition.

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Dedication

To my father - my first math teacher

Chapter 1

Introduction

1.1 Outline

Recall the Riemann-Roch theorem for vector bundles over curves: Given a smooth projective complex curve C of genus g and two vector bundles E, F over C,

$$\chi(E,F) = deg(F)rk(E) - rk(F)deg(E) + (1-g)rk(F)rk(E),$$

where $\chi(E,F)$ is the Euler form

$$\chi(E,F) = \dim \operatorname{Hom}(E,F) - \dim \operatorname{Ext}^1(E,F)$$

and rk, deg stand for the rank and the degree of a vector bundle.

Remarkably, there are a number of similarly looking formulas in noncommutative algebra. For example, consider the quiver algebra $\mathbb{C}Q_l$ where Q_l is the quiver with two vertices, 1 and 2, and l arrows from 1 to 2. Given two finite-dimensional $\mathbb{C}Q_l$-modules M and N, Ringel's formula says [56]

$$\chi(M,N) = d_1(N)d_1(M) + d_2(N)d_2(M) - ld_2(N)d_1(M),$$

where

$$\chi(M,N) = \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}^1(M,N)$$

and $d_i(M) = \dim(e_iM)$ (here e_i is the idempotent in $\mathbb{C}Q_l$ corresponding to vertex i).

There are at least two explanations of the similarity of the above two formulas. The first explanation is that the abelian category $\mathsf{mod}(\mathbb{C}Q_l)$ of finite-dimensional $\mathbb{C}Q_l$-modules satisfies all the properties of abelian categories of coherent sheaves on curves [54]. Thus, the bounded derived categories of $\mathsf{mod}(\mathbb{C}Q_l)$ and $\mathsf{coh}(C)$ should have a very similar structure. The second explanation is that the derived category of $\mathsf{mod}(\mathbb{C}Q_l)$ is equivalent to the derived category of the noncommutative projective space $N\mathbf{P}^{l-1}$, introduced in [29], which *is* a noncommutative curve from the viewpoint of noncommutative algebraic geometry.

Both explanations refer to derived categories, and this is not surprising: the left-hand side of the Riemann-Roch formula contains the derived functor Ext^1 of Hom. This suggests that there might exist a version of the Riemann-Roch formula for some *abstract* derived categories, not related to any space or abelian category, which will describe the Euler form χ in terms of some pairing and some characteristic classes of objects. The present work is devoted to a result of this type. A brief description of the abstract Riemann-Roch formula will be presented in Section 1.3. But before presenting it, we should explain what we understand by abstract derived categories and describe the class of categories we will be working with.

The classical derived categories - those of sheaves on varieties or modules over algebras - are constructed in two steps. The input data are *differential graded (DG) categories of complexes* (of sheaves or modules, respectively). The first step consists in passing to the corresponding homotopy categories. The second step consists in formally inverting those morphisms in the homotopy categories that induce invertible maps at the level of cohomology. A similar procedure can be applied to an arbitrary DG category, and the outcome are the aforementioned abstract derived categories (see [6, 17, 32, 65]).

There is a very general algorithm for producing Riemann-Roch type formulas for arbitrary DG categories (see the next section). However, in order to obtain more specific results one has to work in some reasonable generality.

One way to extract an interesting class of DG categories is due to Keller, Bondal, and

Van den Bergh [5]. Namely, let A be a differential graded (DG) algebra, i.e. an associative algebra with a grading and a differential. There is a notion of a perfect A-module; these are analogs of bounded complexes of projective modules over usual algebras (their precise definition can be found in Section 2.2). They form a DG category, $\mathsf{Perf} A$. It turns out that the derived categories of such DG categories are a simultaneous generalization of the derived categories of modules over associative algebras and certain important derived categories associated to algebraic varieties. Namely, let X be a projective variety. One has a notion of a perfect complex on X; the simplest example is a bounded complex of vector bundles. Perfect complexes form a DG category, $\mathsf{Perf} X$. Keller, Bondal, and Van den Bergh have shown that for any X there exists a DG algebra A such that the derived category of $\mathsf{Perf} X$ is equivalent to the derived category of $\mathsf{Perf} A$. For example, the derived category of $\mathsf{Perf}\, \mathbf{P}^1$ is equivalent to the derived category of finite dimensional modules over the path algebra of the Kronecker quiver:

$$\bullet \rightrightarrows \bullet$$

Observe that such an A exists for affine varieties as well - just set A equal to the algebra of regular functions on such a variety. In fact, Keller, Bondal, and Van den Bergh have proven that this observation holds true for any "reasonable" space (a precise statement can be found in [5]).

A as above is not unique even in the affine case. But any such DG algebra encodes (co)homological invariants and some geometric properties of X. The simplest example is the computation of the Hodge cohomology of X in terms of $\mathsf{Perf} X$ in the case of a smooth projective variety:

$$\mathsf{HH}_n(A) = \oplus_i \mathrm{H}^{i-n}(\Omega^i_X), \tag{1.1.1}$$

where the left-hand side is the so-called n-th Hochschild homology group of A. Also, X is proper iff the total cohomology of A is finite-dimensional, and X is smooth iff A is a perfect A-bimodule [38]. We will take the latter two conditions as the definitions of properness and smoothness for abstract DG algebras. The first property is central to the present work, although we will discuss smooth[1] DG algebras as well (see Section 4.3).

All this suggests one to develop a formal geometric language that would treat abstract DG algebras as algebras of "functions" on noncommutative spaces. However, it should be done carefully as different DG algebras may give rise to the same "spaces" due to the non-uniqueness issue we mentioned above. A better approach is to associate noncommutative spaces to *equivalence classes* of DG algebras as follows. Following [65], we will say that two DG algebras A and B are Morita-equivalent if their perfect categories $\mathsf{Perf}A$ and $\mathsf{Perf}B$ are "essentially the same" (the precise term is 'quasi-equivalent'). In view of the above discussion, each variety gives rise to a fixed Morita-equivalence class. Therefore it is reasonable to think of an *arbitrary* Morita-equivalence class as representing some noncommutative scheme or, better yet, a *noncommutative DG-scheme*. Any DG algebra from the equivalence class should be viewed as "the" algebra of regular functions on this noncommutative DG-scheme, and $\mathsf{Perf}A$ plays the role of $\mathsf{Perf}X$. Of course, a real definition of noncommutative DG-schemes should also include a description of morphisms between them. We won't discuss it here referring the reader to more thorough treatments of the subject [17, 31, 63–66].

The above point of view agrees with the philosophy of *derived noncommutative algebraic geometry*. This subject was initiated in the beginning of 90's based on the previous extensive study of derived categories of coherent sheaves undertaken by the Moscow school (A. Beilinson, A. Bondal, M. Kapranov, D. Orlov, A. Rudakov et al). Later on, it was greatly enriched by new ideas and examples coming from M. Kontsevich's homological mirror symmetry program. Further important ideas and results in the field are due to A. Bondal and M. Van den Bergh, T. Bridgeland, V. Drinfeld, B. Keller, M. Kontsevich and Y. Soibelman, D. Orlov, R. Rouquier, B. Toen and others.

To conclude this overview, let us mention one potential application of derived noncommutative algebraic geometry. It is related to the homological mirror symmetry program. The program, we recall, asserts that the relationship, discovered by physicists, between

[1] A more conventional term is 'homologically smooth'.

holomorphic and symplectic invariants of two mirror Calabi-Yau manifolds should be a consequence of a relationship between the derived category of coherent sheaves of the first manifold and the so-called derived Fukaya category of the second one. The latter is defined in symplectic geometric terms, and there is not any underlying "symplectic" abelian category. Kontsevich indicated in [36] that it should be possible to extract the holomorphic and symplectic invariants from the derived categories of sheaves and the derived Fukaya categories of Calabi-Yau manifolds using the same algebraic algorithm. From this viewpoint, both categories are objects of the same nature, and one of the goals of derived noncommutative algebraic geometry is to provide a language and technique that will allow one to treat the holomorphic and the symplectic categories in a uniform way.

1.2 A categorical version of the Hirzebruch-Riemann-Roch theorem

Let us turn now to the subject of the thesis, the Hirzebruch-Riemann-Roch (HRR) theorem in the above noncommutative setting. We will start with very general (and oversimplified) categorical considerations.

Fix a ground field, k, and consider the tensor category of small k-linear DG categories, morphisms being DG functors. Fix also a *homology theory* on the latter category, i.e. a covariant tensor functor $\mathbf{H}$ to a tensor category of modules over a commutative ring[2] K, satisfying the following axioms:

(1) $\mathbf{H}$ respects quasi-equivalences.

(2) For any DG algebra A the canonical embedding $A \to \mathsf{Perf} A$ induces an isomorphism

$$\mathbf{H}(A) \simeq \mathbf{H}(\mathsf{Perf} A).$$

(3) $\mathbf{H}(k) = K$ (then, by (2), $\mathbf{H}(\mathsf{Perf} k) = K$).

[2] One can take $\mathbb{Z}$-graded, $\mathbb{Z}/2$-graded modules, modules that are complete in some topology etc.

Notice that (1) and (2) together imply that $\mathbf{H}$ descends to an invariant of noncommutative DG-schemes. Also, by the very definition of $\mathbf{H}$, there exists a functorial Künneth type isomorphism

$$\mathbf{H}(\mathcal{A}) \otimes_K \mathbf{H}(\mathcal{B}) \simeq \mathbf{H}(\mathcal{A} \otimes \mathcal{B}).$$

Let us add to this list one more condition:

(4) For any DG category $\mathcal{A}$ there is a functorial isomorphism

$$^\vee : \mathbf{H}(\mathcal{A}) \simeq \mathbf{H}(\mathcal{A}^{\mathsf{op}})$$

which is the identity in the case $\mathcal{A} = k$.

We will assume that the above isomorphisms satisfy all the natural properties and compatibility conditions one can imagine [3].

To describe what we understand by the abstract HRR theorem for noncommutative DG-schemes, we need to define the Chern character map with values in the homology theory $\mathbf{H}$. This is a function $\mathsf{Ch}^{\mathcal{A}}_{\mathbf{H}} : \mathcal{A} \to \mathbf{H}(\mathcal{A})$, one for each DG category $\mathcal{A}$, defined as follows. Take an object $N \in \mathcal{A}$ and consider the DG functor $T_N : k \to \mathcal{A}$ that sends the unique object of k to N. Then [8, 34]

$$\mathsf{Ch}^{\mathcal{A}}_{\mathbf{H}}(N) = \mathbf{H}(T_N)(1_K).$$

Clearly, the Chern character is functorial: For any two DG categories $\mathcal{A}, \mathcal{B}$ and any DG functor $F : \mathcal{A} \to \mathcal{B}$

$$\mathsf{Ch}^{\mathcal{B}}_{\mathbf{H}} \circ F = \mathbf{H}(F) \circ \mathsf{Ch}^{\mathcal{A}}_{\mathbf{H}}.$$

From now on, we will focus on proper DG categories, i.e. DG categories that correspond to proper noncommutative DG-schemes. Let $\mathcal{A}$ be a proper DG category. Consider the DG functor

$$\mathbf{Hom}_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A}^{\mathsf{op}} \to \mathsf{Perf}k, \quad N \otimes M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, N).$$

[3] The right definition of a homology theory should be formulated in terms of the category of noncommutative motives [37].

By (3), it induces a linear map $\mathbf{H}(\mathbf{Hom}_{\mathcal{A}}) : \mathbf{H}(\mathcal{A} \otimes \mathcal{A}^{\mathsf{op}}) \to K$. One can compose it with the Künneth isomorphism to get a K-bilinear pairing

$$\langle\, , \,\rangle_{\mathcal{A}} : \mathbf{H}(\mathcal{A}) \times \mathbf{H}(\mathcal{A}^{\mathsf{op}}) \to K.$$

Now we are ready to formulate the HRR theorem: For any proper DG category $\mathcal{A}$ and any two objects $N, M \in \mathcal{A}$

$$\mathsf{Ch}_{\mathbf{H}}^{\mathsf{Perf}k}(\mathrm{Hom}_{\mathcal{A}}(M,N)) = \langle\, \mathsf{Ch}_{\mathbf{H}}^{\mathcal{A}}(N)\, ,\, \mathsf{Ch}_{\mathbf{H}}^{\mathcal{A}}(M)^{\vee}\, \rangle_{\mathcal{A}}. \tag{1.2.1}$$

Indeed, it follows from the functoriality of the isomorphism $^{\vee}$ that

$$(\mathbf{H}(T_M)(1_K))^{\vee} = \mathbf{H}(T_{M^{\mathsf{op}}})(1_K)$$

where M^{op} stands for M viewed as an object of $\mathcal{A}^{\mathsf{op}}$. Then

$$\begin{aligned}\langle\, \mathsf{Ch}_{\mathbf{H}}^{\mathcal{A}}(N)\, ,\, \mathsf{Ch}_{\mathbf{H}}^{\mathcal{A}}(M)^{\vee}\, \rangle_{\mathcal{A}} &= \mathbf{H}(\mathbf{Hom}_{\mathcal{A}})\,(\mathbf{H}(T_N)(1_K) \otimes (\mathbf{H}(T_M)(1_K))^{\vee})\\ = \mathbf{H}(\mathbf{Hom}_{\mathcal{A}})\,(\mathbf{H}(T_N)(1_K) \otimes \mathbf{H}(T_{M^{\mathsf{op}}})(1_K)) &= \mathbf{H}(\mathbf{Hom}_{\mathcal{A}})\,(\mathbf{H}(T_{N\otimes M^{\mathsf{op}}})(1_K))\\ = \mathbf{H}(\mathbf{Hom}_{\mathcal{A}} \circ T_{N\otimes M^{\mathsf{op}}})(1_K) = \mathbf{H}(\mathrm{Hom}_{\mathcal{A}}(M,N))(1_K) &= \mathsf{Ch}_{\mathbf{H}}^{\mathsf{Perf}k}(\mathrm{Hom}_{\mathcal{A}}(M,N)).\end{aligned}$$

In this very general form, the HRR theorem is almost tautological. For it to be of any use, one needs to find a way to compute the right-hand side of (1.2.1) for a given proper noncommutative DG-scheme and any pair of perfect complexes on it. In this work, we solve this problem in the case $K = k$, $\mathbf{H} = \mathsf{HH}_{\bullet}$, where $\mathsf{HH}_{\bullet}$ stands for the Hochschild homology[4] (see Section 2.3 for the definition). This choice of the homology theory can be motivated as follows.

First of all, there is a classical character map from the Grothendieck group of a ring to its Hochschild homology - the so called Dennis trace map [41]. Its sheafified version appeared in [8] in connection with the index theorem for elliptic pairs [59, 60] (the definition of the Chern character given above mimics the one given in [8]).

[4]The most difficult axioms (1) and (2) in our "definition" of the homology theory were proved for $\mathsf{HH}_{\bullet}$ by B. Keller in [35].

In the algebraic geometric context, the relevance of the Hochschild homology to the HRR theorem can be explained as follows. There is a version of the HRR theorem for compact complex manifolds [47, 48], in which the Chern class of a coherent sheaf takes values in the Hodge cohomology $\oplus_i \mathrm{H}^i(\Omega^i_X)$ (see also [26]). A new proof of this result was obtained in [42, 43] using an algebraic-differential calculus (see also [11, 51]). This latter approach emphasizes the importance of viewing the Chern character as a map to the Hochschild homology $\mathsf{HH}_0(X)$ of the space X. The "usual" Chern character is then obtained via the Hochschild-Kostant-Rosenberg isomorphism $\mathsf{HH}_0(X) \cong \oplus_i \mathrm{H}^i(\Omega^i_X)$. This point of view was further developed in [10]. Namely, it was explained in [10] (see also [12]) how to obtain a categorical version of the HRR theorem, similar to the one above, starting from the cohomology theory

$$\text{smooth spaces} \to \text{graded vector spaces}, \quad X \mapsto \mathsf{HH}_\bullet(X)$$

("smooth spaces" are understood in a broad sense: these are usual schemes as well as various almost commutative ones such as orbifolds). Finally, the transition from X to its categorical incarnation, $\mathsf{Perf}\,X$, is based on the fact that $\mathsf{HH}_\bullet(X)$ is isomorphic to the Hochschild homology of the DG category $\mathsf{Perf}\,X$, which was proved in [34].

Before we move on to the description of the main results of the paper, we would like to mention a notational convention we are going to follow.

Following [8] (see also [34]), we will call the Chern character $\mathsf{Ch}_{\mathsf{HH}}$ with values in the Hochschild homology the *Euler* character and use the notation Eu.

1.3 Main results

Let us describe the main results of this work.

Fix a ground field k and a proper DG algebra A over k (as we mentioned earlier, the properness means $\sum_n \dim \mathrm{H}^n(A) < \infty$).

The first main result is the computation of the Euler class $\mathsf{eu}(L)$ of an arbitrary perfect

DG A-module L. Here $\mathsf{eu}(L)$ stands for the unique element in $\mathsf{HH}_0(A)$ that corresponds to $\mathsf{Eu}(L) \in \mathsf{HH}_0(\mathsf{Perf}\,A)$ under the canonical isomorphism $\mathsf{HH}_\bullet(A) \simeq \mathsf{HH}_\bullet(\mathsf{Perf}\,A)$ (see axiom (2) in Section 1.2). The following theorem is proved in Section 3.3.

Theorem 1. *Let $N = (\bigoplus_j A[r_j], d + \alpha)$ be a twisted DG A-module and L a homotopy direct summand of N which corresponds to a homotopy idempotent $\pi : N \to N$. Then*

$$\mathsf{eu}(L) = \sum_{l=0}^{\infty} (-1)^l \mathsf{str}(\pi[\underbrace{\alpha|\dots|\alpha}_{l}])$$

Roughly speaking, in this formula π and α are elements of a DG analog of the matrix algebra $\mathsf{Mat}(A)$, $\pi[\alpha|\dots|\alpha]$ is an element of the Hochschild chain complex of this DG matrix algebra, and str is an analog of the usual trace map $\mathsf{tr} : \mathsf{Mat}(A) \to A$ (see [24, 41]). Note that α is upper-triangular, so the series terminates.

To present our next result, we observe that the pairing

$$\mathsf{HH}_\bullet(\mathsf{Perf}\,A) \times \mathsf{HH}_\bullet((\mathsf{Perf}\,A)^{\mathsf{op}}) \to \mathsf{HH}_\bullet(\mathsf{Perf}\,k) \simeq k,$$

defined earlier in Section 1.2, induces a pairing

$$\mathsf{HH}_\bullet(\mathsf{Perf}\,A) \times \mathsf{HH}_\bullet(\mathsf{Perf}\,A^{\mathsf{op}}) \to k. \tag{1.3.1}$$

This is due to the existence of a canonical quasi-equivalence of DG categories (see (3.2.4)):

$$D : \mathsf{Perf}\,A^{\mathsf{op}} \to (\mathsf{Perf}\,A)^{\mathsf{op}}, \quad M \mapsto DM = \mathrm{Hom}_{\mathsf{Perf}\,A^{\mathsf{op}}}(M, A).$$

In fact, we "twist" the exposition in the main text (Section 3.1) and work exclusively with the pairing (1.3.1). The reason is that it can be defined very explicitly without referring to its categorical nature. Besides, it induces a pairing

$$\langle\, ,\, \rangle : \mathsf{HH}_\bullet(A) \times \mathsf{HH}_\bullet(A^{\mathsf{op}}) \to k \tag{1.3.2}$$

via the canonical isomorphisms $\mathsf{HH}_\bullet(A) \simeq \mathsf{HH}_\bullet(\mathsf{Perf}\,A)$, $\mathsf{HH}_\bullet(A^{\mathsf{op}}) \simeq \mathsf{HH}_\bullet(\mathsf{Perf}\,A^{\mathsf{op}})$. This latter pairing is described explicitly in our next theorem, which is obtained by combining results of Section 3.2 (see formulas (3.2.2), (3.2.3)) and Theorem 3.4.1.

Theorem 2. *Let a, b be two elements of $\mathsf{HH}_\bullet(A)$, $\mathsf{HH}_\bullet(A^{\mathsf{op}})$, respectively. Then*

$$\langle\, a, b\, \rangle = \int a \wedge b.$$

Here $\wedge : \mathsf{HH}_\bullet(A) \times \mathsf{HH}_\bullet(A^{\mathsf{op}}) \to \mathsf{HH}_\bullet(\mathrm{End}_k(A))$, $\int : \mathsf{HH}_\bullet(\mathrm{End}_k(A)) \to k$ are defined as follows:

(1) If $\sum_a a_0[a_1|\dots|a_l]$ (resp. $\sum_b b_0[b_1|\dots|b_m]$) is a cycle in the Hochschild chain complex of A (resp. A^{op}) representing the homology class a (resp. b) then

$$a \wedge b = \sum_{a,b} \mathsf{sh}\left(L(a_0)[L(a_1)|\dots|L(a_l)] \otimes R(b_0)[R(b_1)|\dots|R(b_m)] \right),$$

where $L(a_i)$ (resp. $R(b_j)$) stands for the operator in A of left (resp. right) multiplication with a_i (resp. b_j); sh is the well known shuffle-product (see Section 2.4).

(2) $\int$ is what we call the Feigin-Losev-Shoikhet trace [21, 52]. It is described explicitly in Theorem 3.4.1 (Section 3.4).

Furthermore, recall that there should exist a canonical isomorphism ${}^\vee : \mathsf{HH}_\bullet(A) \simeq \mathsf{HH}_\bullet(A^{\mathsf{op}})$ (see axiom (4) in Section 1.2). In fact, the isomorphism is easy to describe explicitly (see Section 3.2). By summarizing the above discussion, we get the following version of the noncommutative HRR theorem:

Theorem 3. *For any perfect DG A-modules N, M*

$$\chi(M, N)(:= \chi(\mathrm{Hom}_{\mathsf{Perf}\,A}(M, N))) = \int \mathsf{eu}(N) \wedge \mathsf{eu}(M)^\vee.$$

The only thing that needs to be explained here is where $\chi(\mathrm{Hom}_{\mathsf{Perf}\,A}(M, N))$ came from. According the categorical HRR theorem, described in the previous section, the left-hand of the above equality should equal $\mathsf{eu}(\mathrm{Hom}_{\mathsf{Perf}\,A}(M, N))$. However, the Euler class of a perfect DG k-module is nothing but its Euler characteristic. This is a consequence of the following "expected" fact, which we prove in Section 3.1: for any A the Euler character Eu descends to a character on the Grothendieck group of the triangulated category $\mathsf{Ho}(\mathsf{Perf}A)$, the homotopy category of $\mathsf{Perf}A$.

Note that the noncommutative HRR formula doesn't include any analog of the Todd class. The Todd class seems to emerge in the case when a noncommutative space, $\widehat{X}$, is "close" to a commutative one, X (for example, $\widehat{X}$ is a deformation quantization of X). In such cases various homology theories of $\widehat{X}$ can be identified with certain cohomology rings associated with X and the Todd class of X appears because of this identification. For some classes of noncommutative spaces one can try to define an analog of the Todd class "by hand" but, in general, a categorical definition of the Todd class doesn't seem to exist.

In the main text we do not refer to the categorical version of the HRR theorem to prove Theorem 3. Instead, we derive it from a more general statement (Theorem 3.1.4). Roughly speaking, this statement says the following: If A and B are two proper DG algebras and X is a perfect $A-B$-bimodule then the map $\mathsf{HH}_\bullet(\mathsf{Perf}A) \to \mathsf{HH}_\bullet(\mathsf{Perf}B)$, induced by the DG functor $-\otimes_A X : \mathsf{Perf}A \to \mathsf{Perf}B$, is given by a "convolution" with the Euler class of X. Later, in Section 4.3, we use this result again to prove the following

Theorem 4. *Let A be a proper homologically smooth DG algebra. Then the pairing $\langle\, ,\, \rangle$ is non-degenerate.*

It is this application that was the original motivation for the author to study the Euler classes in the DG setting [5]. It implies, in particular, the noncommutative Hodge-to-De Rham degeneration conjecture [38] for homologically smooth algebras whose non-zero graded components sit in non-positive degrees (see Section 4.3). Hopefully, the reader will accept all this as an excuse for twisting the exposition and not mentioning the categorical HRR in what follows.

[5] I am grateful to Y. Soibelman for suggesting to me to "write up" the proof of this statement.

In Chapter 4 we treat some "toy" examples of proper noncommutative DG-schemes and the HRR formulas for them. Namely, in Section 4.1 we discuss what we call directed algebras. Basically, these are some quiver algebras with relations but we find the quiver-free description more convenient when it comes to proving general facts about such algebras. Many commutative schemes, viewed as noncommutative ones, are described by directed algebras. Namely, this is so when the scheme possesses a strongly exceptional collection [4]. The HRR formula for such algebras (see (4.1.3)) is essentially Ringel's formula [56, Section 2.4]. Section 4.2 is about proper noncommutative DG-schemes "responsible" for orbifold singularities of the form $\mathbb{C}^n/G$, where G is a finite subgroup of $SL_n(\mathbb{C})$. Namely, we look at the noncommutative DG-scheme related to the derived category of complexes of G-equivariant coherent sheaves on $\mathbb{C}^n$ with supports at the origin. We conjecture that the underlying DG algebra is the cross-product $\Lambda^\bullet\mathbb{C}^n \rtimes \mathbb{C}[G]$ and we derive the HRR formula for some perfect modules over this algebra (see (4.2.1)).

The material of the last Chapter is related to the main subject in a somewhat indirect way. Nevertheless, it served to us as yet another important motivation at later stages of this research. It is related to a well-known interpretation of the classical HRR theorem in the framework of topological string theory. Namely, let X be a smooth projective Calabi-Yau variety, i.e. a smooth projective variety with a fixed nowhere-vanishing holomorphic top-degree form. The so-called $N = 2$ super-symmetric σ-model with target X gives rise to an open-closed topological field theory called the B-model associated with X [15, 40, 44]. An abstract open-closed topological field theory is characterized, in particular, by its category of boundary conditions (a.k.a. D-branes). In the case of the B-model, the boundary conditions are essentially the bounded complexes of holomorphic vector bundles on X. It has been observed by both physicists and mathematicians (see e.g. [10, 29, 49]) that for the B-model, one of the most non-trivial axioms of open-closed topological field theories - the (topological) Cardy condition - boils down to an identity which contains the HRR formula as a special case. Furthermore, as it has been shown in [15], the natural setting to study open-closed topological field theories is precisely the one of derived noncommutative geometry. More precisely, [15] establishes an equivalence between certain categories of open-closed topological field theories on one hand and noncommutative Calabi-Yau spaces, in the sense of derived noncommutative geometry, on the other. Thus, it is natural to expect that the HRR formula, obtained in the present work, should admit the interpretation as a "Cardy-like" condition in the open-closed topological field theories coming from noncommutative spaces. In Section 5.3, we explicitly describe the field theories associated with "0-dimensional" noncommutative Calabi-Yau spaces, i.e. Frobenius algebras[6]. In the subsequent Section, we explain how our noncommutative HRR theorem for Frobenius algebras is a special case of the Cardy condition in the corresponding field theories. The key observation is a relationship between two pairings on the Hochschild homology of a Frobenius algebra - the pairing that we obtain in this work (it exists on the Hochschild homology of an arbitrary proper DG algebra) and the one coming from the corresponding field theory (this exists on the Hochschild homology of noncommutative proper Calabi-Yau spaces by [15, 38]). It has been conjectured by Y. Soibelman and K. Costello that this relationship holds true in the case of an arbitrary noncommutative Calabi-Yau space. A precise statement of the conjecture can be found in the same Section. We note that the conjecture has been established recently in the "commutative" context, i.e. for B-models; see [50].

1.4 Other viewpoints on the noncommutative HRR theorem

In this section, we provide a very brief account of other Riemann-Roch type theorems in Noncommutative Geometry we are aware of.

Let us begin with the afore-mentioned preprint [10] which partially inspired the present work. The approach taken in [10] is based on an alternative description of the Hochschild

[6]For simplicity, we restrict ourselves to algebras satisfying certain regularity condition in order to be able to use the axiomatics of open-closed topological field theories given in [40, 44].

homology of a *smooth* proper space X in terms of the Serre functor $S_X : D^b(X) \to D^b(X)$. Namely,

$$\mathsf{HH}_\bullet(X) \simeq \mathrm{Ext}^\bullet_{\mathsf{Fun}}(S_X^{-1}, I_X),$$

where I_X is the identity endofunctor of $D^b(X)$ and the extensions are taken in a suitably defined triangulated category of endofunctors. In [61] we generalized the above isomorphism to the case of an arbitrary smooth proper noncommutative DG-scheme. However, proving that the above definition gives rise to a homology theory on the category of smooth proper noncommutative DG-schemes (in other words, lifting the above definition on the level of DG categories) will require some efforts [10, Appendix B]. Besides, the “traditional” definition of the Hochschild homology we use in this paper works for an arbitrary, not necessarily smooth scheme.

Other analogs of the Riemann-Roch theorem were obtained in [55], [45] in the framework of Noncommutative Algebraic Geometry [1], [57], [58], [16]. The exposition in [45] is closer to ours in that it emphasizes the importance of triangulated categories in connection with Riemann-Roch type results. Our approach and the above two approaches to the noncommutative Riemann-Roch theorem are not completely unrelated since many interesting noncommutative schemes give rise to noncommutative DG schemes [5, Section 4].

Last, but not least, various index theorems have been proved in frameworks of A. Connes' Noncommutative Geometry [14, 62] and Deformation Quantization [8, 20, 46, 67].

Chapter 2

Hochschild homology of DG algebras and DG categories

2.1 DG algebras, DG categories, and DG functors

Throughout the paper, we work over a fixed ground field k. All vector spaces, algebras, linear categories are defined over k.

To begin with, let us briefly recall some standard definitions/conventions concerning DG algebras and DG categories [31].

We consider unital DG algebras with no restrictions on the $\mathbb{Z}$-grading. If A is a DG algebra

$$A = \bigoplus_{n\in\mathbb{Z}} A^n, \quad d = d_A : A^n \to A^{n+1}$$

then A^{op} will stand for the opposite DG algebra, i.e. A^{op} coincides with A as a complex[1] of vector spaces and the product on A^{op} is given by

$$a' \otimes a'' \mapsto (-1)^{|a'||a''|} a'' a'$$

(here and further, $|n|$ denotes the degree of a homogeneous element n of a graded vector space). Similar conventions apply to DG categories.

If A is a DG algebra, $\mathsf{Mod}A$ will stand for the DG category of right DG A-modules. The

[1] Here and further, by a *complex* we understand a *cochain complex*.

objects of this category, we recall, are (cochain) complexes of vector spaces

$$M = \bigoplus_{m \in \mathbb{Z}} M^m, \quad d_M : M^m \to M^{m+1}$$

acted on by A from the right so that the action is compatible with the gradings and the differentials:

$$M^m \otimes A^n \to M^{m+n},$$

$$d_M(m \cdot a) = d_M(m) \cdot a + (-1)^{|m|} m \cdot d(a).$$

Given two DG A-modules L and M, the n-the graded component $\mathrm{Hom}^n_{\mathsf{Mod}A}(L, M)$ of the complex of morphisms consists of families $(f^p)_{p \in \mathbb{Z}}$ of maps

$$f^p : L^p \to M^{p+n}$$

respecting the A-actions. The differential on this morphism complex is given by

$$d(f^p) = (d_M \cdot f^p - (-1)^n f^{p+1} \cdot d_L).$$

The homotopy category of a DG category $\mathcal{A}$ will be denoted by $\mathsf{Ho}(\mathcal{A})$. Let us recall the definition of the standard triangulated structure on $\mathsf{Ho}(\mathsf{Mod}A)$. The shift functor is defined in the obvious way:

$$M[1]^m = M^{m+1}, \quad d_{M[1]} = -d_M.$$

The distinguished triangles are defined as follows. Let $p : L \to M$ be a degree 0 closed morphism: $d(p) = 0$. The cone $\mathsf{Cone}(p)$ of the morphism p is a DG A-module defined by

$$\mathsf{Cone}(p) = \left(\begin{array}{c} L[1] \\ \oplus \\ M \end{array}, \begin{pmatrix} d_{L[1]} & 0 \\ p & d_M \end{pmatrix} \right)$$

(the direct sum is taken in the category of *graded* A-modules). There are obvious degree 0 closed morphisms $q : M \to \mathsf{Cone}(p)$ and $r : \mathsf{Cone}(p) \to L[1]$. A triangle in $\mathsf{Ho}(\mathsf{Mod}A)$ is, by definition, a sequence $X \to Y \to Z \to X[1]$ isomorphic to a sequence of the form $L \xrightarrow{p} M \xrightarrow{q} \mathsf{Cone}(p) \xrightarrow{r} L[1]$.

Let N be a right DG A-module. By a homotopy direct summand of N we will understand a DG A-module L that satisfies the following property: there exist two degree 0 closed morphisms $f : N \to L$ and $g : L \to N$ such that $fg = 1_L$ in $\mathsf{Ho}(\mathsf{Mod}A)$. In this case, $\pi = gf : N \to N$ is easily seen to be a homotopy idempotent, i.e. $\pi^2 = \pi$ in $\mathsf{Ho}(\mathsf{Mod}A)$.

To conclude this section, let us mention one important notion that will be used later on. Fix two DG categories $\mathcal{A}$ and $\mathcal{B}$ and consider the DG category $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ of DG functors from $\mathcal{A}$ to $\mathcal{B}$ with the morphisms being families of natural transformations [31]. A degree 0 closed morphism $f \in \mathrm{Hom}_{\mathsf{Fun}(\mathcal{A},\mathcal{B})}(F, G)$ will be called a *weak homotopy equivalence* if for any $N \in \mathcal{A}$ the morphism $f(N) : F(N) \to G(N)$ is an isomorphism in $\mathsf{Ho}(\mathcal{B})$.

2.2 Perfect modules

Let A be a DG algebra. A can be viewed as a full DG subcategory of $\mathsf{Mod}A$ with a single object. The embedding $A \hookrightarrow \mathsf{Mod}A$ factors through a full subcategory $\mathsf{Perf}A \subset \mathsf{Mod}A$ of perfect A-modules. This subcategory is defined as follows (see [7]).

Let us say that a DG A-module N is *finitely generated free* if it is of the form $K \otimes A$ where K is a finite dimensional graded vector space (equivalently, it is a finite direct sum of shifts of A). We will say that $N \in \mathsf{Mod}A$ is *finitely generated semi-free* if it can be obtained from a finite set of finitely generated free A-modules (equivalently, a finite set of shifts of A) by successive taking the cones of degree 0 closed morphisms. Finally, a *perfect* DG A-module is a homotopy direct summand of a finitely generated semi-free DG A-module.

Note that this definition is slightly more general than the one given in [7]. The authors of [7] require perfect modules to be semi-free but we don't. For example, a complex of vector spaces is a perfect k-module in our sense iff it has finite dimensional total cohomology and it is perfect in the sense of [7] if, in addition, it is bounded above. The reason we prefer not to restrict ourselves to semi-free modules will be clear from Proposition 2.2.4 below. It suffices for us to stay within the class of homotopically projective modules: N is homotopically projective iff $\mathrm{Hom}_{\mathsf{Mod}A}(N, L)$ is acyclic whenever L is acyclic. Every finitely generated semi-free module N is known to be homotopically projective [17, Section 13]. It follows that every perfect module in our sense is homotopically projective as well.

The following result is well known (and is not hard to prove):

Proposition 2.2.1. *The DG category* $\mathsf{Perf}A$ *is closed under passing to homotopically equivalent modules, taking shifts and cones of degree 0 morphisms, and taking homotopy direct summands.*

Let us list some simple useful facts about DG functors between the categories of perfect modules.

Proposition 2.2.2. *Let* A, B *be DG algebras and* $F : \mathsf{Mod}A \to \mathsf{Mod}B$ *a DG functor. The DG functor* F *preserves the subcategories of perfect modules iff* $F(A) \in \mathsf{Perf}B$.

To prove this proposition, observe that F preserves homotopy direct summands and cones of degree 0 morphisms.

For two DG algebras A, B and a bimodule $X \in \mathsf{Mod}(A^{\mathsf{op}} \otimes B)$ let us denote by T_X the DG functor

$$- \otimes_A X : \mathsf{Mod}A \to \mathsf{Mod}B.$$

The following statement is a straightforward consequence of the last proposition:

Corollary 2.2.3. *Suppose a bimodule* $X \in \mathsf{Mod}(A^{\mathsf{op}} \otimes B)$ *is perfect as a DG* B*-module. Then* T_X *preserves perfect modules.*

Recall that a DG algebra A is called *proper* if $\sum_n \dim \mathrm{H}^n(A) < \infty$.

Proposition 2.2.4. *Let* A *be a proper DG algebra and* B *an arbitrary DG algebra. Then for any* $X \in \mathsf{Perf}(A^{\mathsf{op}} \otimes B)$ *the DG functor* T_X *preserves perfect modules.*

In view of the above corollary, it is enough to show that X is perfect as a DG B-module. Suppose that X is a homotopy direct summand of a finitely generated semi-free DG $A^{\mathsf{op}}\otimes B$-module Y and Y is obtained from $(A^{\mathsf{op}} \otimes B)[m_1], \ldots, (A^{\mathsf{op}} \otimes B)[m_l]$ by successive taking cones of degree 0 closed morphisms. As a B-module, $A^{\mathsf{op}} \otimes B$ is homotopically equivalent to the finitely generated free module $\mathrm{H}^\bullet(A) \otimes B$ (this is where we use the properness of A and the fact that we are working over a field!). Thus, as a B-module, Y is homotopically equivalent to a finitely generated semi-free module. Then X, as a B-module, is a homotopy direct summand of a module that is homotopy equivalent to a finitely generated semi-free module. This, together with Proposition 2.2.1, finishes the proof.

Let us recall one more result about perfect modules which we will need later on. The fact that perfect modules are homotopically projective implies the following result (cf. [2, Corollary 10.12.4.4]):

Proposition 2.2.5. *If P is a perfect right DG A-module then $P \otimes_A N$ is acyclic for every acyclic DG A^{op}-module N.*

2.3 Hochschild homology

We begin by recalling the definition of the Hochschild homology groups $\mathsf{HH}_p(A)$, $p \in \mathbb{Z}$, of a DG algebra A.

Let us use the notation sa to denote an element $a \in A$ viewed as an element of the "suspension" $sA = A[1]$. Thus, $|sa| = |a| - 1$. Let $\mathsf{C}_\bullet(A) = A \otimes T(A[1]) = \bigoplus_{n=0}^{\infty} A \otimes A[1]^{\otimes n}$ equipped with the induced grading. We will denote elements of $A \otimes A[1]^{\otimes n}$ by a_0, if $n = 0$, and $a_0[a_1|a_2|\ldots|a_n]$ otherwise (i.e. $a_0[a_1|a_2|\ldots|a_n] = a_0 \otimes sa_1 \otimes sa_2 \otimes \ldots \otimes sa_n$). $\mathsf{C}_\bullet(A)$ is equipped with the degree 1 differential $b = b_0 + b_1$, where b_0 and b_1 are two anti-commuting degree 1 differentials given by

$$b_0(a_0) = da_0, \quad b_1(a_0) = 0, \tag{2.3.1}$$

and

$$b_0(a_0[a_1|a_2|\dots|a_n]) = da_0[a_1|a_2|\dots|a_n] - \sum_{i=1}^{n}(-1)^{\eta_{i-1}}a_0[a_1|a_2|\dots|da_i|\dots|a_n],$$

$$b_1(a_0[a_1|a_2|\dots|a_n]) = (-1)^{|a_0|}a_0a_1[a_2|\dots|a_n] + \sum_{i=1}^{n-1}(-1)^{\eta_i}a_0[a_1|a_2|\dots|a_ia_{i+1}|\dots|a_n] \\ -(-1)^{\eta_{n-1}(|a_n|+1)}a_na_0[a_1|a_2|\dots|a_{n-1}]$$

for $n \neq 0$. Here $\eta_i = |a_0| + |sa_1| + \dots + |sa_i|$. $\mathsf{C}_\bullet(A)$ is called the Hochschild chain[2] complex of A. Then

$$\mathsf{HH}_p(A) = \mathrm{H}^p(\mathsf{C}_\bullet(A)).$$

Let $\mathcal{A}$ be a (small) DG category. Its Hochschild chain complex is defined as follows. Fix a non-negative integer n. We will denote the set of sequences $\{X_0, X_1, \dots, X_n\}$ of objects of $\mathcal{A}$ by $\mathcal{A}^{n+1}$ (the objects in the sequence are not required to be different). Fix an element $\mathbb{X} = \{X_0, X_1, \dots, X_n\} \in \mathcal{A}^{n+1}$ and denote by $\mathsf{C}(\mathcal{A}, \mathbb{X})$ the graded vector space $\mathrm{Hom}_{\mathcal{A}}(X_n, X_0) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{n-1}, X_n)[1] \otimes \dots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1)[1]$. Equip the space

$$\mathsf{C}_\bullet(\mathcal{A}) = \bigoplus_{n\geq 0}\bigoplus_{\mathbb{X}\in\mathcal{A}^{n+1}} \mathsf{C}(\mathcal{A}, \mathbb{X})$$

with the differential $b = b_0 + b_1$ where b_0 and b_1 are given by formulas analogous to (2.3.1),(2.3.2). The complex $\mathsf{C}_\bullet(\mathcal{A})$ is the Hochschild chain complex of the DG category $\mathcal{A}$ and its cohomology

$$\mathsf{HH}_p(\mathcal{A}) = \mathrm{H}^p(\mathsf{C}_\bullet(\mathcal{A}))$$

is the Hochschild homology of $\mathcal{A}$.

Obviously, any DG functor $F : \mathcal{A} \to \mathcal{B}$ between two DG categories $\mathcal{A}$, $\mathcal{B}$ induces a morphism of complexes $\mathsf{C}(F) : \mathsf{C}_\bullet(\mathcal{A}) \to \mathsf{C}_\bullet(\mathcal{B})$ and, as a result, a linear map

$$\mathsf{HH}(F) : \mathsf{HH}_\bullet(\mathcal{A}) \to \mathsf{HH}_\bullet(\mathcal{B}).$$

[2]This classical terminology is slightly confusing since the Hochschild *chain* complex is a *cochain* complex in the sense of homological algebra.

Being applied to the embedding $A \to \mathsf{Perf}\, A$, the above construction yields a morphism of complexes $\mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(\mathsf{Perf}\, A)$. The following result was proved in [35] (see also [31]):

Theorem 2.3.1. *The morphism $\mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(\mathsf{Perf}\, A)$ is a quasi-isomorphism.*

Later on, we will need yet another result proved in [35] (see Section 3.4 of loc.cit.):

Theorem 2.3.2. *Let A and B be two DG algebras and $F, G : \mathsf{Perf}\, A \to \mathsf{Perf}\, B$ two DG functors. If there is a weak homotopy equivalence $F \to G$ then $\mathsf{HH}(F) = \mathsf{HH}(G)$.*

2.4 Künneth isomorphism

Let us recall the construction of the Künneth isomorphism

$$\bigoplus_n \mathsf{HH}_n(A) \otimes \mathsf{HH}_{N-n}(B) \simeq \mathsf{HH}_N(A \otimes B)$$

where A, B are two DG algebras. The formula below is borrowed from [41] (see also [67] where the differential graded case is discussed).

Let us fix a DG algebra A. The first ingredient of the construction is the shuffle product

$$\mathsf{sh} : \mathsf{C}_\bullet(A) \otimes \mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(A)$$

defined as follows. For two elements $a_0'[a_1'|a_2'|\ldots|a_n'], a_0''[a_1''|a_2''|\ldots|a_m''] \in \mathsf{C}_\bullet(A)$ the shuffle product is given by the formula:

$$\mathsf{sh}(a_0'[a_1'|a_2'|\ldots|a_n'] \otimes a_0''[a_1''|a_2''|\ldots|a_m'']) = (-1)^* \cdot a_0'a_0''\, \mathsf{sh}_{nm}[a_1'|\ldots|a_n'|a_1''|\ldots|a_m''] \quad (2.4.1)$$

Here $* = |a_0''|(|sa_1'| + \ldots + |sa_n'|)$ and

$$\mathsf{sh}_{nm}[x_1|\ldots|x_n|x_{n+1}|\ldots|x_{n+m}] = \sum_\sigma \pm[x_{\sigma^{-1}(1)}|\ldots|x_{\sigma^{-1}(n)}|x_{\sigma^{-1}(n+1)}|\ldots|x_{\sigma^{-1}(n+m)}]$$

where the sum is taken over all permutations that don't shuffle the first n and the last m elements and the sign in front of each summand is computed according to the following rule: for two homogeneous elements x, y, the transposition $[\dots|x|y|\dots] \to [\dots|y|x|\dots]$ contributes $(-1)^{|x||y|}$ to the sign.

Now let B be another DG algebra. Denote by ι^A, ι^B the natural embeddings of DG algebras

$$A \to A \otimes B, \quad B \to A \otimes B.$$

They induce morphisms of complexes:

$$\mathsf{C}(\iota^A) : \mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(A \otimes B), \quad \mathsf{C}(\iota^B) : \mathsf{C}_\bullet(B) \to \mathsf{C}_\bullet(A \otimes B).$$

Theorem 2.4.1. *The composition* K *of the maps*

$$\mathsf{C}_\bullet(A) \otimes \mathsf{C}_\bullet(B) \xrightarrow{\mathsf{C}(\iota^A)\otimes\mathsf{C}(\iota^B)} \mathsf{C}_\bullet(A \otimes B) \otimes \mathsf{C}_\bullet(A \otimes B) \xrightarrow{\mathsf{sh}} \mathsf{C}_\bullet(A \otimes B)$$

respects the differentials and induces a quasi-isomorphism of complexes.

The morphism $\mathsf{K} : \mathsf{C}_\bullet(A) \otimes \mathsf{C}_\bullet(B) \to \mathsf{C}_\bullet(A \otimes B)$ defined above admits a generalization to the case of DG categories. Namely, let $\mathcal{A}$ and $\mathcal{B}$ be two (small) DG categories. Fix a set $\{X_0, X_1, \dots, X_n\}$ of objects of $\mathcal{A}$ and a set $\{Y_0, Y_1, \dots, Y_m\}$ of objects of $\mathcal{B}$. For two elements

$$f_n[f_{n-1}|\dots|f_0] \in \mathrm{Hom}_{\mathcal{A}}(X_n, X_0) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{n-1}, X_n)[1] \otimes \dots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1)[1],$$

$$g_m[g_{m-1}|\dots|g_0] \in \mathrm{Hom}_{\mathcal{B}}(Y_m, Y_0) \otimes \mathrm{Hom}_{\mathcal{B}}(Y_{m-1}, Y_m)[1] \otimes \dots \otimes \mathrm{Hom}_{\mathcal{B}}(Y_0, Y_1)[1]$$

define $\mathsf{K}\left(f_n[f_{n-1}|f_{n-2}|\dots|f_0] \otimes g_m[g_{m-1}|g_{m-2}|\dots|g_0]\right)$ as

$$\pm(f_n \otimes g_m)\, \mathsf{sh}_{nm}[f_{n-1}|\dots|f_0|g_{m-1}|\dots|g_0],$$

where the sign is computed as before and sh_{nm} is defined by the formula

$$[f_{n-1} \otimes 1_{Y_m}|\dots|f_0 \otimes 1_{Y_m}|1_{X_0} \otimes g_{m-1}|\dots|1_{X_0} \otimes g_0]+$$

$$+(-1)^{|sf_0||sg_{m-1}|}[f_{n-1} \otimes 1_{Y_m}|\dots|1_{X_1} \otimes g_{m-1}|f_0 \otimes 1_{Y_{m-1}}|\dots|1_{X_0} \otimes g_0] + \dots$$

Other terms in this sum are obtained from the first one by shuffling the f-terms with the g-terms according to the following rule:

$$[\dots|f_k \otimes 1_{Y_{l+1}}|1_{X_k} \otimes g_l|\dots] \to (-1)^{|sf_k||sg_l|}[\dots|1_{X_{k+1}} \otimes g_l|f_k \otimes 1_{Y_l}|\dots]$$

Let A and B be two DG algebras. We have the obvious embedding of DG categories

$$\mathsf{Perf}\,A \otimes \mathsf{Perf}\,B \to \mathsf{Perf}(A \otimes B),$$

which induces a morphism of complexes

$$\mathsf{C}_\bullet(\mathsf{Perf}\,A \otimes \mathsf{Perf}\,B) \to \mathsf{C}_\bullet(\mathsf{Perf}(A \otimes B)).$$

Let us denote the composition

$$\mathsf{C}_\bullet(\mathsf{Perf}\,A) \otimes \mathsf{C}_\bullet(\mathsf{Perf}\,B) \xrightarrow{\mathsf{K}} \mathsf{C}_\bullet(\mathsf{Perf}\,A \otimes \mathsf{Perf}\,B) \to \mathsf{C}_\bullet(\mathsf{Perf}(A \otimes B)) \tag{2.4.2}$$

by the same letter K. As an immediate corollary of Theorems 2.3.1 and 2.4.1, we get the following result:

Proposition 2.4.2. *The map* $\mathsf{K} : \mathsf{C}_\bullet(\mathsf{Perf}\,A) \otimes \mathsf{C}_\bullet(\mathsf{Perf}\,B) \to \mathsf{C}_\bullet(\mathsf{Perf}(A \otimes B))$ *is a quasi-isomorphism.*

Indeed, we have the commutative diagram

$$\begin{array}{ccc}
\mathsf{C}_\bullet(\mathsf{Perf}\,A) \otimes \mathsf{C}_\bullet(\mathsf{Perf}\,B) & \longrightarrow & \mathsf{C}_\bullet(\mathsf{Perf}(A \otimes B)) \\
\uparrow & & \uparrow \\
\mathsf{C}_\bullet(A) \otimes \mathsf{C}_\bullet(B) & \longrightarrow & \mathsf{C}_\bullet(A \otimes B)
\end{array}$$

in which the vertical arrows and the arrow on the bottom are quasi-isomorphisms.

Finally, we will formulate two more results about the Künneth map (2.4.2). Both results follow directly from the definition of K.

Proposition 2.4.3. *Let A, B, and C be three DG algebras. The diagram*

$$\begin{array}{ccc} \mathsf{C}_\bullet(\mathsf{Perf}A)\otimes\mathsf{C}_\bullet(\mathsf{Perf}B)\otimes\mathsf{C}_\bullet(\mathsf{Perf}C) & \xrightarrow{\mathsf{K}\otimes 1} & \mathsf{C}_\bullet(\mathsf{Perf}(A\otimes B))\otimes\mathsf{C}_\bullet(\mathsf{Perf}C) \\ \downarrow{\scriptstyle 1\otimes\mathsf{K}} & & \downarrow{\scriptstyle \mathsf{K}} \\ \mathsf{C}_\bullet(\mathsf{Perf}A)\otimes\mathsf{C}_\bullet(\mathsf{Perf}(B\otimes C)) & \xrightarrow{\mathsf{K}} & \mathsf{C}_\bullet(\mathsf{Perf}(A\otimes B\otimes C)) \end{array}$$

commutes. In other words, K *is associative.*

Proposition 2.4.4. *Let A, B, C, D be DG algebras. Let $X \in \mathsf{Mod}(A^{\mathsf{op}}\otimes C)$ and $Y \in \mathsf{Mod}(B^{\mathsf{op}}\otimes D)$ be bimodules satisfying the conditions of Corollary 2.2.3. Then the diagram*

$$\begin{array}{ccc} \mathsf{C}_\bullet(\mathsf{Perf}A)\otimes\mathsf{C}_\bullet(\mathsf{Perf}B) & \xrightarrow{\mathsf{K}} & \mathsf{C}_\bullet(\mathsf{Perf}(A\otimes B)) \\ \downarrow{\scriptstyle \mathsf{C}(T_X)\otimes\mathsf{C}(T_Y)} & & \downarrow{\scriptstyle \mathsf{C}(T_{X\otimes_k Y})} \\ \mathsf{C}_\bullet(\mathsf{Perf}C)\otimes\mathsf{C}_\bullet(\mathsf{Perf}D) & \xrightarrow{\mathsf{K}} & \mathsf{C}_\bullet(\mathsf{Perf}(C\otimes D)) \end{array}$$

commutes.

Chapter 3

Hirzebruch-Riemann-Roch theorem

3.1 Euler character

Let A be a DG algebra and N a perfect right DG A-module. Consider the DG functor $T_N = - \otimes_k N : \mathsf{Perf}k \to \mathsf{Perf}A$. The Euler class $\mathsf{Eu}(N) \in \mathsf{HH}_0(\mathsf{Perf}A)$ is defined by the formula (cf. [8],[35])

$$\mathsf{Eu}(N) = \mathsf{HH}(T_N)(1_k).$$

In other words, $\mathsf{Eu}(N)$ is the class of the identity morphism 1_N in $\mathsf{HH}_0(\mathsf{Perf}A)$.

Let us list some basic properties of the Euler character map.

The following statement follows from Theorem 2.3.2:

Proposition 3.1.1. *If $N, M \in \mathsf{Perf}A$ are homotopically equivalent then $\mathsf{Eu}(N) = \mathsf{Eu}(M)$. In other words, Eu descends to objects of $\mathsf{Ho}(\mathsf{Perf}A)$.*

The following result means that the Euler class descends to the Grothendieck group of the triangulated category $\mathsf{Ho}(\mathsf{Perf}A)$.

Proposition 3.1.2. *For any $N \in \mathsf{Perf}A$ one has $\mathsf{Eu}(N[1]) = -\mathsf{Eu}(N)$ and for any triangle $L \xrightarrow{p} M \xrightarrow{q} N \xrightarrow{r} L[1]$ in $\mathsf{Ho}(\mathsf{Perf}A)$ one has*

$$\mathsf{Eu}(M) = \mathsf{Eu}(L) + \mathsf{Eu}(N). \tag{3.1.1}$$

Let us prove the first part. We have to show that $1_N + 1_{N[1]}$ is homologous to 0 in $\mathsf{C}_\bullet(\mathsf{Perf}\, A)$. Denote by $1_{N,N[1]}$ (resp. $1_{N[1],N}$) the identity endomorphism of N viewed as a morphism from N to $N[1]$ (resp. from $N[1]$ to N). Then

$$b(1_{N,N[1]}[1_{N[1],N}]) = b_1(1_{N,N[1]}[1_{N[1],N}]) =$$
$$= -(1_{N,N[1]}1_{N[1],N} + 1_{N[1],N}1_{N,N[1]}) = -(1_{N[1]} + 1_N)$$

Let us prove the second part. By Proposition 3.1.1, it suffices to prove (3.1.1) for $N = \mathsf{Cone}(p)$. Consider the following morphisms:

$$j_1 = \begin{pmatrix} 1_{L[1]} \\ 0 \end{pmatrix} : L[1] \to \mathsf{Cone}(p), \quad q_1 = \begin{pmatrix} 1_{L[1]} & 0 \end{pmatrix} : \mathsf{Cone}(p) \to L[1],$$
$$j_2 = \begin{pmatrix} 0 \\ 1_M \end{pmatrix} : M \to \mathsf{Cone}(p), \quad q_2 = \begin{pmatrix} 0 & 1_M \end{pmatrix} : \mathsf{Cone}(p) \to M.$$

It is easy to see that

$$d(j_1) = j_2 \cdot p, \quad d(q_1) = 0, \quad d(j_2) = 0, \quad d(q_2) = -p \cdot q_1.$$

(In these formulas, p is viewed as a degree 1 morphism from $L[1]$ to M.) The following computation finishes the proof:

$$1_{\mathsf{Cone}(p)} - 1_{L[1]} - 1_M = j_1q_1 + j_2q_2 - q_1j_1 - q_2j_2 = [j_1, q_1] + [j_2, q_2]$$
$$= b(j_1[q_1] + j_2[q_2]) - b_0(j_1[q_1] + j_2[q_2]) = b(j_1[q_1] + j_2[q_2]) - (d(j_1)[q_1] - j_2[d(q_2)])$$
$$= b(j_1[q_1] + j_2[q_2]) - (j_2p[q_1] + j_2[pq_1]) = b(j_1[q_1] + j_2[q_2] - j_2[p|q_1]).$$

To formulate the main result of this section, we need a pairing

$$\mathsf{HH}_n(\mathsf{Perf}\, A) \times \mathsf{HH}_{-n}(\mathsf{Perf}\, A^{\mathsf{op}}) \to k, \quad n \in \mathbb{Z},$$

where A is a proper DG algebra. Here is the definition.

Let us equip A with a left DG $A \otimes A^{\mathsf{op}}$-module structure as follows:

$$(a' \otimes a'')a = (-1)^{|a''||a|}a'aa''.$$

We will denote the resulting A-bimodule by Δ.

Consider the DG functor:

$$T_\Delta : \mathsf{Mod}(A \otimes A^{\mathsf{op}}) \to \mathsf{Mod}k, \quad N \mapsto N \otimes_{A \otimes A^{\mathsf{op}}} A$$

The following proposition is an immediate consequence of Corollary 2.2.3.

Proposition 3.1.3. *If A is proper then T_Δ induces a DG functor $\mathsf{Perf}(A \otimes A^{\mathsf{op}}) \to \mathsf{Perf}k$.*

We can use this to define a pairing

$$\langle\, , \,\rangle : \mathsf{HH}_n(\mathsf{Perf}A) \times \mathsf{HH}_{-n}(\mathsf{Perf}A^{\mathsf{op}}) \to k, \quad n \in \mathbb{Z} \tag{3.1.2}$$

via the composition of morphisms of complexes

$$\mathsf{C}_\bullet(\mathsf{Perf}A) \otimes \mathsf{C}_\bullet(\mathsf{Perf}A^{\mathsf{op}}) \xrightarrow{\mathsf{K}} \mathsf{C}_\bullet(\mathsf{Perf}(A \otimes A^{\mathsf{op}})) \xrightarrow{\mathsf{C}(T_\Delta)} C_\bullet(\mathsf{Perf}k)$$

and the fact that $\mathsf{HH}_n(\mathsf{Perf}k) \simeq \mathsf{HH}_n(k)$ is k, if $n = 0$, and 0 otherwise.

Before we formulate the main result of this section, let us introduce the following notation. For a bimodule $X \in \mathsf{Perf}(A^{\mathsf{op}} \otimes B)$ we will denote by $\mathsf{Eu}'(X)$ the element

$$\mathsf{K}^{-1}(\mathsf{Eu}(X)) \in \bigoplus_n \mathsf{HH}_{-n}(\mathsf{Perf}A^{\mathsf{op}}) \otimes \mathsf{HH}_n(\mathsf{Perf}B),$$

where K is the Künneth isomorphism.

Theorem 3.1.4. *Let A be a proper DG algebra, B an arbitrary DG algebra, and X any object of $\mathsf{Perf}(A^{\mathsf{op}} \otimes B)$. If $y \in \mathsf{HH}_\bullet(\mathsf{Perf}A)$ then $\mathsf{HH}(T_X)(y) = \langle\, y\, , \mathsf{Eu}'(X)\, \rangle$. That is, if*

$$\mathsf{Eu}'(X) = \sum_n x'_{-n} \otimes x''_n \in \bigoplus_n \mathsf{HH}_{-n}(\mathsf{Perf}A^{\mathsf{op}}) \otimes \mathsf{HH}_n(\mathsf{Perf}B),$$

then $\mathsf{HH}(T_X)(y) = \sum_n \langle\, y\, , x'_{-n}\, \rangle \cdot x''_n$.

To prove this, observe that T_X can be described as a composition of the following DG functors

$$\mathsf{Perf}A \xrightarrow{-\otimes_k X} \mathsf{Perf}(A \otimes A^{\mathsf{op}} \otimes B) \xrightarrow{T_{\Delta \otimes_k B}} \mathsf{Perf}B$$

Thus, $\mathsf{HH}(T_X) = \mathsf{HH}(T_{\Delta\otimes_k B}) \circ \mathsf{HH}(- \otimes_k X)$. It follows from the definition of the Künneth isomorphism K that the diagram

$$\begin{array}{ccc} \mathsf{HH}_\bullet(\mathsf{Perf}\,A) & \xrightarrow{\mathsf{HH}(-\otimes_k X)} & \mathsf{HH}_\bullet(\mathsf{Perf}(A\otimes A^{\mathsf{op}}\otimes B)) \\ {\scriptstyle 1\otimes\mathsf{Eu}(X)}\downarrow & \nearrow{\scriptstyle \mathsf{K}} & \\ \mathsf{HH}_\bullet(\mathsf{Perf}\,A)\otimes \mathsf{HH}_0(\mathsf{Perf}(A^{\mathsf{op}}\otimes B)) & & \end{array}$$

commutes. By conjugating with $1\otimes\mathsf{K}$, we get the following commutative diagram:

$$\begin{array}{ccc} \mathsf{HH}_\bullet(\mathsf{Perf}\,A) & \xrightarrow{\mathsf{HH}(-\otimes_k X)} & \mathsf{HH}_\bullet(\mathsf{Perf}(A\otimes A^{\mathsf{op}}\otimes B)) \\ {\scriptstyle 1\otimes\mathsf{Eu}'(X)}\downarrow & \nearrow{\scriptstyle \mathsf{K}\circ(1\otimes\mathsf{K})} & \\ \mathsf{HH}_\bullet(\mathsf{Perf}\,A)\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,A^{\mathsf{op}})\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,B) & & \end{array}$$

Furthermore, by Proposition 2.4.4 the diagram

$$\begin{array}{ccc} \mathsf{HH}_\bullet(\mathsf{Perf}(A\otimes A^{\mathsf{op}}\otimes B)) & \xrightarrow{\mathsf{HH}(T_{\Delta\otimes_k B})} & \mathsf{HH}_\bullet(\mathsf{Perf}(k\otimes B)) \simeq \mathsf{HH}_\bullet(\mathsf{Perf}\,B) \\ {\scriptstyle \mathsf{K}^{-1}}\downarrow & & \uparrow{\scriptstyle \mathsf{K}} \\ \mathsf{HH}_\bullet(\mathsf{Perf}(A\otimes A^{\mathsf{op}}))\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,B) & \xrightarrow{\mathsf{HH}(T_\Delta)\otimes 1} & \mathsf{HH}_\bullet(\mathsf{Perf}\,k)\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,B) \end{array}$$

commutes. Conjugating with $\mathsf{K}\otimes 1$ gives us the following commutative diagram:

$$\begin{array}{ccc} \mathsf{HH}_\bullet(\mathsf{Perf}(A\otimes A^{\mathsf{op}}\otimes B)) & \xrightarrow{\mathsf{HH}(T_{\Delta\otimes_k B})} & \mathsf{HH}_\bullet(\mathsf{Perf}(k\otimes B)) \simeq \mathsf{HH}_\bullet(\mathsf{Perf}\,B) \\ {\scriptstyle (\mathsf{K}^{-1}\otimes 1)\mathsf{K}^{-1}}\downarrow & & \uparrow{\scriptstyle \mathsf{K}} \\ \mathsf{HH}_\bullet(\mathsf{Perf}\,A)\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,A^{\mathsf{op}})\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,B) & \xrightarrow{(\mathsf{HH}(T_\Delta)\mathsf{K})\otimes 1} & \mathsf{HH}_\bullet(\mathsf{Perf}\,k)\otimes \mathsf{HH}_\bullet(\mathsf{Perf}\,B) \end{array}$$

By concatenating the top arrows of the former and the latter diagrams, we get the following result:

$$\mathsf{HH}(T_{\Delta\otimes_k B})\circ\mathsf{HH}(-\otimes_k X) = \mathsf{K}\circ((\mathsf{HH}(T_\Delta)\mathsf{K})\otimes 1)\circ(\mathsf{K}^{-1}\otimes 1)\circ\mathsf{K}^{-1}\circ\mathsf{K}\circ(1\otimes\mathsf{K})\circ(1\otimes\mathsf{Eu}'(X)).$$

By associativity of the Künneth isomorphism (Proposition 2.4.3), the latter product is nothing but $\mathsf{K}\circ((\mathsf{HH}(T_\Delta)\mathsf{K})\otimes 1)\circ(1\otimes\mathsf{Eu}'(X))$ which finishes the proof.

Theorem 3.1.4 generates several corollaries. The first one, the Hirzebruch-Riemann-Roch type formula, will be formulated and proved in the next section. Another corollary, which concerns homologically smooth DG algebras, will be described in Section 4.3.

3.2 Hirzebruch-Riemann-Roch theorem

Essentially, the Hirzebruch-Riemann-Roch theorem is the following result:

Theorem 3.2.1. *Let A be a proper DG algebra. Then, for any $N \in \mathsf{Perf}\, A$, $M \in \mathsf{Perf}\, A^{\mathsf{op}}$,*

$$\sum_n (-1)^n \dim \mathrm{H}^n(N \otimes_A M) = \langle \mathsf{Eu}(N), \mathsf{Eu}(M) \rangle. \tag{3.2.1}$$

This theorem is an easy corollary of the results of the previous section. Indeed, consider the DG functors:

$$T_N = - \otimes_k N : \mathsf{Perf}\, k \to \mathsf{Perf}\, A, \quad T_M = - \otimes_A M : \mathsf{Perf}\, A \to \mathsf{Perf}\, k,$$

$$T_{N \otimes_A M} = - \otimes_k (N \otimes_A M) : \mathsf{Perf}\, k \to \mathsf{Perf}\, k.$$

Clearly, $T_{N \otimes_A M} = T_M T_N$ and so we get

$$\mathsf{Eu}(N \otimes_A M) = \mathsf{HH}(T_{N \otimes_A M})(1_k) = \mathsf{HH}(T_M T_N)(1_k)$$

$$= \mathsf{HH}(T_M)(\mathsf{HH}(T_N)(1_k)) = \mathsf{HH}(T_M)(\mathsf{Eu}(N)) = \langle \mathsf{Eu}(N), \mathsf{Eu}(M) \rangle$$

where the last equality holds by Theorem 3.1.4. What remains is to observe that, for a perfect DG k-module X,

$$\mathsf{Eu}(X) = \sum_n (-1)^n \dim \mathrm{H}^n(X).$$

This latter statement is a corollary of Propositions 3.1.1 and 3.1.2, along with the fact that X is homotopy equivalent to $\mathrm{H}^\bullet(X)$.

Let us explain how one can compute the right-hand side of (3.2.1).

First of all, observe that, by Theorem 2.3.1, the pairing (3.1.2) induces a pairing on $\mathsf{HH}_\bullet(A) \times \mathsf{HH}_\bullet(A^{\mathsf{op}})$. Let us fix two cycles

$$\sum a_0[a_1|\dots|a_l] \in \mathsf{C}_\bullet(A), \quad \sum b_0[b_1|\dots|b_m] \in \mathsf{C}_\bullet(A^{\mathsf{op}})$$

($\sum$ indicates that these are sums of several terms) and denote by a (resp. b) the corresponding elements in $\mathsf{HH}_\bullet(A)$ (resp. $\mathsf{HH}_\bullet(A^{\mathsf{op}})$). Let us describe $\langle\, a, b\,\rangle$ more explicitly.

Consider the composition of DG functors

$$A \otimes A^{\mathsf{op}} \to \mathsf{Perf}(A \otimes A^{\mathsf{op}}) \xrightarrow{T_\Delta} \mathsf{Perf}\, k,$$

where $A \otimes A^{\mathsf{op}}$ is viewed as a DG category with one object. Clearly, the unique object of $A \otimes A^{\mathsf{op}}$ gets mapped under this composition to $A \in \mathsf{Perf}\, k$ and an element $x \otimes y \in A \otimes A^{\mathsf{op}}$, viewed as a morphism in the DG category $A \otimes A^{\mathsf{op}}$, gets mapped to the operator $L(x)R(y) \in \mathrm{End}_k(A)$, where

$$L(x) : c \mapsto xc, \quad R(y) : c \mapsto (-1)^{|c||y|} cy$$

are the operators of left multiplication with x resp. right multiplication with y.

Since the operators of left multiplication commute with operators of right multiplication, we can define a product

$$a \wedge b = \sum_{a,b} \pm L(a_0)R(b_0)\mathsf{sh}_{lm}[L(a_1)| \dots |L(a_l)|R(b_1)| \dots |R(b_m)] \tag{3.2.2}$$

on $\mathsf{HH}_\bullet(A) \times \mathsf{HH}_\bullet(A^{\mathsf{op}})$ with values in $\mathsf{HH}_\bullet(\mathrm{End}_k(A))$ (the formula for $\pm$ and the definition of sh_{lm} are the same as in (2.4.1)). Then

$$\langle\, a, b \,\rangle = \int a \wedge b \tag{3.2.3}$$

where $\int$ is defined as follows. Let X be a perfect DG k-module. Then we have an embedding of DG categories[1] $\mathrm{End}_k(X) \to \mathsf{Perf}\, k$ which sends the unique object of the first category to X, viewed as an object of $\mathsf{Perf}\, k$. Then $\int$ is the map from $\mathsf{HH}_\bullet(\mathrm{End}_k(X))$ to $\mathsf{HH}_\bullet(\mathsf{Perf}\, k) \simeq k$ induced by this embedding.

Furthermore, let us use the notation $\mathsf{eu}(N)$ to denote the element in $\mathsf{HH}_0(A)$ corresponding to $\mathsf{Eu}(N)$ under the isomorphism $\mathsf{HH}_0(A) \to \mathsf{HH}_0(\mathsf{Perf}\, A)$. We are ready to rewrite the right-hand side of (3.2.1):

$$\langle \mathsf{Eu}(N), \mathsf{Eu}(M) \rangle = \int \mathsf{eu}(N) \wedge \mathsf{eu}(M).$$

[1]For a complex X of vector spaces $\mathrm{End}_k(X)$ will stand either for the DG algebra $\oplus_n \mathrm{End}_k^n(X)$, where $\mathrm{End}_k^n(X)$ is the subspace of degree n linear maps, or for the corresponding DG category with one object.

It turns out that there are very explicit formulas for $\int$ and eu which will be derived in the next section.

To conclude this section, we will rewrite the Hirzebruch-Riemann-Roch formula in a more conventional form. Namely, we will use (3.2.1) to derive a formula that expresses the Euler form

$$\chi(M,N) = \sum_n (-1)^n \dim \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Perf}\,A)}(M, N[n])$$

in terms of the Euler classes of M and N, where M and N are two perfect DG A-modules.

Consider the following (contravariant) DG functor

$$D : \mathsf{Mod}A \to \mathsf{Mod}A^{\mathsf{op}}, \quad M \mapsto DM = \mathrm{Hom}_{\mathsf{Mod}A}(M, A). \tag{3.2.4}$$

It is not hard to show that this DG functor preserves perfect modules. Moreover, its square is isomorphic to the identity endofunctor of $\mathsf{Perf}\,A$ and, thus, D is a quasi-equivalence of the DG categories $(\mathsf{Perf}\,A)^{\mathsf{op}}$ and $\mathsf{Perf}\,A^{\mathsf{op}}$. The crucial property of this functor is the following fact: for any perfect DG A-modules there is a natural quasi-isomorphism of complexes

$$N \otimes_A DM \cong \mathrm{Hom}_{\mathsf{Perf}\,A}(M, N).$$

Thus, the formula (3.2.1) can be written as follows: for any $N, M \in \mathsf{Perf}\,A$

$$\chi(M,N) = \langle \mathsf{Eu}(N), \mathsf{Eu}(DM) \rangle = \int \mathsf{eu}(N) \wedge \mathsf{eu}(DM). \tag{3.2.5}$$

Finally, we notice that $\mathsf{eu}(DM)$ can be expressed in terms of $\mathsf{eu}(M)$. More precisely

Proposition 3.2.2. *For any DG algebra, the formula*

$$(a_0[a_1|a_2|\ldots|a_n])^\vee = (-1)^{n+\sum_{1\le i<j\le n}|sa_i||sa_j|} a_0[a_n|a_{n-1}|\ldots|a_1]. \tag{3.2.6}$$

defines an isomorphism $^\vee : \mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(A^{\mathsf{op}})$. *One has*

$$\mathsf{eu}(DM) = \mathsf{eu}(M)^\vee.$$

Let us prove it. Clearly, the morphism (3.2.6) is invertible. We have to show that it commutes with the differentials. It is obvious that $^\vee$ respects the first differential b_0 as its definition doesn't involve multiplication. Let us show by a direct computation that $^\vee$ commutes with the second differential b_1. Let us denote the multiplication in A^{op} by $*$. To simplify computations, we will also use the notations $\xi_i = |a_0| + |sa_n| + |sa_{n-1}| + \ldots + |sa_{i+1}|$ and $f(a_1, a_2, \ldots, a_n) = \sum_{1 \le i < j \le n} |sa_i||sa_j|$. One has:

$$
\begin{aligned}
b_1((a_0[a_1|a_2|\ldots|a_n])^\vee) &= (-1)^{n+f(a_1,a_2,\ldots,a_n)} b_1(a_0[a_n|a_{n-1}|\ldots|a_1]) \\
&= (-1)^{n+f(a_1,a_2,\ldots,a_n)}((-1)^{|a_0|} a_0 * a_n[a_{n-1}|\ldots|a_1] \\
&\quad + \sum_{i=1}^{n-1} (-1)^{\xi_i} a_0[a_n|a_{n-1}|\ldots|a_{i+1} * a_i|\ldots|a_1] \\
&\quad - (-1)^{\xi_1(|a_1|+1)} a_1 * a_0[a_n|a_{n-1}|\ldots|a_2]) \\
&= (-1)^{n+f(a_1,a_2,\ldots,a_n)}((-1)^{|a_0|+|a_0||a_n|} a_n a_0[a_{n-1}|\ldots|a_1] \\
&\quad + \sum_{i=1}^{n-1} (-1)^{\xi_i + |a_{i+1}||a_i|} a_0[a_n|a_{n-1}|\ldots|a_i a_{i+1}|\ldots|a_1] \\
&\quad - (-1)^{\xi_1(|a_1|+1)+|a_1||a_0|} a_0 a_1[a_n|a_{n-1}|\ldots|a_2])
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(b_1(a_0[a_1|a_2|\ldots|a_n]))^\vee &= (-1)^{|a_0|}(a_0 a_1[a_2|\ldots|a_n])^\vee \\
&\quad + \sum_{i=1}^{n-1} (-1)^{\eta_i}(a_0[a_1|a_2|\ldots|a_i a_{i+1}|\ldots|a_n])^\vee \\
&\quad - (-1)^{\eta_{n-1}(|a_n|+1)}(a_n a_0[a_1|a_2|\ldots|a_{n-1}])^\vee \\
&= (-1)^{|a_0|}(-1)^{n-1+f(a_2,\ldots,a_n)} a_0 a_1[a_n|\ldots|a_2] \\
+ \sum_{i=1}^{n-1} (-1)^{\eta_i}(-1)^{n-1+f(a_1,a_2,\ldots,a_i a_{i+1},\ldots,a_n)} & a_0[a_n|a_{n-1}|\ldots|a_i a_{i+1}|\ldots|a_1] \\
- (-1)^{\eta_{n-1}(|a_n|+1)}(-1)^{n-1+f(a_1,a_2,\ldots,a_{n-1})} & a_n a_0[a_{n-1}|a_{n-2}|\ldots|a_1]
\end{aligned}
$$

What remains is to compare the signs, i.e. to show that

$$
(-1)^{f(a_1,a_2,\ldots,a_n)}(-1)^{|a_0|+|a_0||a_n|} = (-1)^{\eta_{n-1}(|a_n|+1)}(-1)^{f(a_1,a_2,\ldots,a_{n-1})},
$$

$$
(-1)^{f(a_1,a_2,\ldots,a_n)}(-1)^{\xi_i+|a_{i+1}||a_i|} = -(-1)^{\eta_i}(-1)^{f(a_1,a_2,\ldots,a_i a_{i+1},\ldots,a_n)},
$$

$$(-1)^{f(a_1,a_2,\dots,a_n)}(-1)^{\xi_1(|a_1|+1)+|a_1||a_0|} = (-1)^{|a_0|}(-1)^{f(a_2,\dots,a_n)},$$

which is an easy computation.

To prove the second statement, observe that one can generalize the above formulas to the case of an arbitrary DG category to get a quasi-isomorphism ${}^\vee : \mathsf{C}_\bullet(\mathcal{A}) \to \mathsf{C}_\bullet(\mathcal{A}^{\mathsf{op}})$. In the case $\mathcal{A} = \mathsf{Perf}A$ one can compose it with $\mathsf{C}(D) : \mathsf{C}_\bullet((\mathsf{Perf}A)^{\mathsf{op}}) \to \mathsf{C}_\bullet(\mathsf{Perf}A^{\mathsf{op}})$ to get an isomorphism ${}^\vee : \mathsf{C}_\bullet(\mathsf{Perf}A) \to \mathsf{C}_\bullet(\mathsf{Perf}A^{\mathsf{op}})$. It is immediate that $\mathsf{Eu}(DM) = \mathsf{Eu}(M)^\vee$. It is also true, but is less obvious, that $\mathsf{eu}(DM) = \mathsf{eu}(M)^\vee$. This latter observation follows from the fact that the two quasi-isomorphisms ${}^\vee : \mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(A^{\mathsf{op}})$ and ${}^\vee : \mathsf{C}_\bullet(\mathcal{A}) \to \mathsf{C}_\bullet(\mathcal{A}^{\mathsf{op}})$ agree under the embeddings $A \to \mathsf{Perf}A$ and $A^{\mathsf{op}} \to \mathsf{Perf}A^{\mathsf{op}}$. The proposition is proved.

Here is the noncommutative Hirzebruch-Riemann-Roch formula in its ultimate form:

$$\chi(M,N) = \langle \mathsf{Eu}(N), \mathsf{Eu}(M)^\vee \rangle = \int \mathsf{eu}(N) \wedge \mathsf{eu}(M)^\vee. \tag{3.2.7}$$

3.3 Computing Euler classes

The aim of this section is to explain how to compute the Euler class $\mathsf{eu}(N) \in \mathsf{HH}_0(A)$ of a perfect DG A-module.

The definition of a finitely generated semi-free module we gave in Section 2.2 is convenient for proving theorems but it is not explicit enough for the purposes of this section. A more explicit description was given in [6] and we will begin by recalling it.

Let A be a DG algebra. Let $\mathsf{Free}A$ be the DG subcategory in $\mathsf{Perf}A$ whose objects are finitely generated free DG A-modules, i.e. direct sums of modules of the form

$$A[r] = k[r] \otimes A, \quad r \in \mathbb{Z}.$$

Clearly,

$$\mathrm{Hom}_{\mathsf{Free}A}(A[r], A[s]) = \mathrm{Hom}_{\mathsf{Perf}A}(A[r], A[s]) \simeq A[s-r].$$

The differential on the morphism spaces of the DG category $\mathsf{Free}A$, as well as on the free modules themselves, will be denoted by d_{Free}.

The alternative description of finitely generated semi-free modules is based on the notion of twisted A-module. These are objects of a larger DG subcategory $\mathsf{Tw}A \supset \mathsf{Free}A$ in $\mathsf{Perf}\,A$. Namely, a twisted A-module is a right DG A-module of the form

$$(\bigoplus_{j=1}^{n} A[r_j], d_{\mathsf{Free}} + \alpha)$$

where $\alpha = (\alpha_{ij})$ is a strictly upper triangular $n \times n$-matrix of morphisms

$$\alpha_{ij} \in \mathrm{Hom}^1_{\mathsf{Free}A}(A[r_j], A[r_i])$$

satisfying the Maurer-Cartan equation

$$d_{\mathsf{Free}}(\alpha) + \alpha \cdot \alpha = 0.$$

Clearly, the differential d_{Tw} on $\mathrm{Hom}_{\mathsf{Perf}\,A}((\bigoplus_{j=1}^{n} A[r_j], d_{\mathsf{Free}} + \alpha), (\bigoplus_{i=1}^{m} A[s_i], d_{\mathsf{Free}} + \beta))$ is given by the formula

$$d_{\mathsf{Tw}}(f) = d_{\mathsf{Free}}(f) + \beta \cdot f - (-1)^{|f|} f \cdot \alpha.$$

It is not hard to show that any finitely generated semi-free module is isomorphic to a twisted A-module.

The main result of this section is a formula for the Euler class of a homotopy direct summand of a twisted A-module. Its formulation involves a (super-)trace map[2] str which we will describe now.

Let N be a DG A-module which is isomorphic to $\bigoplus_{j=1}^{n} A[r_j]$ as a graded A-module. Fix m homogeneous endomorphisms of N:

$$A', A'', \ldots, A^{(m)} \in \mathrm{End}_{\mathsf{Perf}\,A}(N).$$

Thus, each $A^{(k)}$ is an $n \times n$-matrix $(e(r_i, r_j) \otimes a^{(k)}_{ij})$ of morphisms

$$e(r_i, r_j) \otimes a^{(k)}_{ij} \in \mathrm{Hom}_{\mathsf{Perf}\,A}(A[r_j], A[r_i]),$$

[2] In the case of an associative algebra, this map coincides with the well-known trace map from Section 1.2 of [41].

where $a_{ij}^{(k)} \in A$ and $e(r_i, r_j) \in \mathrm{Hom}_{\mathsf{Perf}A}(A[r_j], A[r_i])$ is the morphism that sends the generator of $A[r_j]$ to the generator of $A[r_i]$. The endomorphisms give rise to an element $A'[A''|\ldots|A^{(m)}]$ of the Hochschild chain complex of the DG category $\mathsf{Perf}A$. Let us define $\mathsf{str}(A'[A''|\ldots|A^{(m)}]) \in \mathsf{C}_\bullet(A)$ by the formula

$$\mathsf{str}(A'[A''|\ldots|A^{(m)}]) = \sum_{j=1}^{n} \sum_{i_1,i_2,\ldots,i_{m-1}} (-1)^* \cdot a'_{ji_1}[a''_{i_1i_2}|\ldots|a^{(m)}_{i_{m-1}j}],$$

where $* = r_{i_1} + (r_{i_1} - r_j)|a'_{ji_1}| + (r_{i_2} - r_j)|sa''_{i_1i_2}| + \ldots + (r_{i_{m-1}} - r_j)|sa^{(m-1)}_{i_{m-2}i_{m-1}}|$.

Theorem 3.3.1. *Let $N_\alpha = (\bigoplus_{j=1}^{n} A[r_j], d_{\mathsf{Free}} + \alpha)$ and L be a homotopy direct summand of N_α corresponding to a homotopy idempotent $\pi : N_\alpha \to N_\alpha$. Then*

$$\mathsf{eu}(L) = \sum_{l=0}^{n-1} (-1)^l \mathsf{str}(\pi[\underbrace{\alpha|\ldots|\alpha}_{l}])$$

Let us prove the theorem.

Lemma 3.3.2. *In the above notation, $\mathsf{Eu}(L) = \pi$.*

We have to show that $1_L \in \mathrm{End}^0_{\mathsf{Perf}A}(L)$ and $\pi \in \mathrm{End}^0_{\mathsf{Perf}A}(N_\alpha)$ define the same element of $\mathsf{HH}_0(\mathsf{Perf}A)$. Let us fix some degree 0 closed morphisms $f : N_\alpha \to L$ and $g : L \to N_\alpha$ such that

$$fg = 1_L + [d_L, H_L], \quad gf = \pi + [d_{N_\alpha}, H_{N_\alpha}]$$

(see Section 2.1). Then

$$1_L - \pi = b(f[g] + H_{N_\alpha} - H_L).$$

The lemma is proved.

Let $N_\alpha = (\bigoplus_{j=1}^{n} A[r_j], d_{\mathsf{Free}} + \alpha)$ and π be as before. Let us introduce some new notations. We will write N_0 to denote the free DG A-module $(\bigoplus_{j=1}^{n} A[r_j], d_{\mathsf{Free}})$. For an endomorphism $f \in \mathrm{End}_{\mathsf{Perf}A}(N_\alpha)$, $\widetilde{f}$ (resp. $\overrightarrow{f}$, $\overleftarrow{f}$) will stand for f viewed as an element of $\mathrm{End}_{\mathsf{Perf}A}(N_0)$ (resp. $\mathrm{Hom}_{\mathsf{Perf}A}(N_\alpha, N_0)$, $\mathrm{Hom}_{\mathsf{Perf}A}(N_0, N_\alpha)$). For a morphism g we will write g_{ij} (resp. g_{i*},

g_{*j}) for the $n \times n$-matrix, viewed as a morphism between the same modules, whose ij-th entry (resp. i-th row, j-th column) coincides with that of g and other entries (resp. rows, columns) are 0.

The following lemma is a straightforward consequence of the definition of str:

Lemma 3.3.3. *One has*

$$\sum_{l=0}^{n-1}(-1)^l \mathsf{str}(\pi[\underbrace{\alpha|\dots|\alpha}_{l}]) = \sum_{l=0}^{n-1}\sum_{i_0,i_1,\dots,i_l}(-1)^l \mathsf{str}(\widetilde{\pi}_{i_0 i_1}[\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}])$$

in $\mathsf{HH}_0(\mathsf{Perf} A)$.

The next lemma is less straightforward:

Lemma 3.3.4. *One has*

$$\pi = \sum_{l=0}^{n-1}\sum_{i_0,i_1,\dots,i_l}(-1)^l \widetilde{\pi}_{i_0 i_1}[\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}]$$

in $\mathsf{HH}_0(\mathsf{Perf} A)$.

To prove this, pick a large N and apply the differential b to the element

$$\sum_{l=0}^{N}\sum_{i_0,i_1,\dots,i_l}(-1)^l \overrightarrow{\pi}_{i_0 *}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}].$$

Let us begin by computing the b_0-component:

$$\begin{aligned} b_0(\overrightarrow{\pi}_{i_0 *}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}]) &= d_{\mathsf{Tw}}(\overrightarrow{\pi}_{i_0 *})[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}] \\ &\quad - \overrightarrow{\pi}_{i_0 *}[d_{\mathsf{Tw}}(\overleftarrow{1}_{*i_1})|\widetilde{\alpha}_{i_1 i_2}|\dots|\widetilde{\alpha}_{i_l i_0}] \\ &\quad + \sum_{m=1}^{l}\overrightarrow{\pi}_{i_0 *}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1 i_2}|\dots|d_{\mathsf{Free}}(\widetilde{\alpha}_{i_m i_{m+1}})|\dots|\widetilde{\alpha}_{i_l i_0}]. \end{aligned}$$

Recall that π is closed, i.e. $d_{\mathsf{Free}}(\pi) + \alpha\pi - \pi\alpha = 0$. Therefore

$$\begin{aligned} d_{\mathsf{Tw}}(\overrightarrow{\pi}_{i_0 *}) = d_{\mathsf{Free}}(\overrightarrow{\pi}_{i_0 *}) - \overrightarrow{\pi}_{i_0 *}\alpha &= (\overrightarrow{\pi}\alpha)_{i_0 *} - (\widetilde{\alpha}\overrightarrow{\pi})_{i_0 *} - \overrightarrow{\pi}_{i_0 *}\alpha \\ &= -(\widetilde{\alpha}\overrightarrow{\pi})_{i_0 *} = -\sum_{k=1}^{n}\widetilde{\alpha}_{i_0 k}\overrightarrow{\pi}_{k*}. \end{aligned}$$

Furthermore,

$$d_{\mathsf{Tw}}(\overleftarrow{1}_{*i_1}) = \alpha\overleftarrow{1}_{*i_1} = \overleftarrow{\alpha}_{*i_1}, \quad d_{\mathsf{Free}}(\widetilde{\alpha}_{i_m i_{m+1}}) = -\sum_{k=1}^{n} \widetilde{\alpha}_{i_m k}\widetilde{\alpha}_{k i_{m+1}}.$$

Thus,

$$\begin{aligned} b_0(\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}]) = -\sum_{k=1}^{n} \widetilde{\alpha}_{i_0k}\overrightarrow{\pi}_{k*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}] \\ -\overrightarrow{\pi}_{i_0*}[\overleftarrow{\alpha}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}] \\ -\sum_{m=1}^{l}\sum_{k=1}^{n} \overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_mk}\widetilde{\alpha}_{ki_{m+1}}|\dots|\widetilde{\alpha}_{i_li_0}]. \end{aligned}$$

Let us compute now the b_1-component. Clearly, $b_1(\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_0}]) = \widetilde{\pi}_{i_0i_0} - \pi_{i_0i_0}$ and for $l \geq 1$

$$\begin{aligned} b_1(\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}]) = \widetilde{\pi}_{i_0i_1}[\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}] - \overrightarrow{\pi}_{i_0*}[\overleftarrow{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}] \\ -\sum_{m=1}^{l-1} \overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_mi_{m+1}}\widetilde{\alpha}_{i_{m+1}i_{m+2}}|\dots|\widetilde{\alpha}_{i_li_0}] \\ -\widetilde{\alpha}_{i_li_0}\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_{l-1}i_l}]. \end{aligned}$$

To finish the proof of Lemma 3.3.4, one needs to add the results of the above two computations, take the sum over l and $i_0, i_1, \dots, i_l$, and observe that the right-hand side of the formula for $b_0(\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\widetilde{\alpha}_{i_1i_2}|\dots|\widetilde{\alpha}_{i_li_0}])$ vanishes for l large enough since α is upper-triangular.

Now Theorem 3.3.1 follows from the above three lemmas and the following proposition:

Proposition 3.3.5. *Let N_0 be a free A-module. For any closed element $x \in \mathsf{C}_\bullet(\mathrm{End}_{\mathsf{Perf}A}(N_0))$ the image of the element $\mathsf{str}(x)$ under the natural map $\mathsf{C}_\bullet(A) \to \mathsf{C}_\bullet(\mathsf{Perf}A)$ is closed. Moreover, in this case x and $\mathsf{str}(x)$ define the same class in $\mathsf{HH}_\bullet(\mathsf{Perf}A)$.*

The idea of the proof resembles the one used in the proof of [41, Theorem 1.2.4]. Namely, suppose $N_0 = \bigoplus_{j=1}^{n} A[r_j]$. It is enough to show that there exists a degree -1 map

$$h : \mathsf{C}_\bullet(\mathrm{End}_{\mathsf{Perf}A}(N_0)) \to \mathsf{C}_\bullet(\mathsf{Perf}A)$$

such that for any $x \in \mathsf{C}_\bullet(\mathrm{End}_{\mathsf{Perf}A}(N_0))$

$$x - \mathsf{str}(x) = bh(x) + hb(x)$$

where b is the Hochschild differential[3]. Let us construct such an h.

To begin with, consider the following morphisms in $\mathsf{Perf}A$:

$$C_i : A \to N_0, \quad a \mapsto (-1)^{r_i} a \in A[r_i] \subset N_0 \quad (i = 1, \ldots, n)$$

$$R_i : N_0 \to A, \quad R_i|_{A[r_i]}(a) = a \in A, \quad R_i|_{A[r_j]} = 0 \quad i \neq j \quad (i = 1, \ldots, n)$$

Clearly, $\deg C_i = -r_i$, $\deg R_i = r_i$, and

$$\sum_{i=1}^{n} (-1)^{r_i} C_i R_i = 1_{N_0} \tag{3.3.1}$$

Elements of $\mathsf{C}_\bullet(\mathrm{End}_{\mathsf{Perf}A}(N_0))$ have the form $\sum A_0[A_1|\ldots|A_m]$ where the sum runs over tensors of various lengths and each A_k is an $n \times n$-matrix. We can (and will) assume that all the matrices A_k are homogeneous.

Define degree -1 maps $h_k : \mathsf{C}_\bullet(\mathrm{End}_{\mathsf{Perf}A}(N_0)) \to \mathsf{C}_\bullet(\mathsf{Perf}A)$ as follows:

$h_k(A_0[A_1|\ldots|A_m]) =$

$$= (-1)^{\xi_k} \sum A_0 C_{i_0}[R_{i_0} A_1 C_{i_1} | R_{i_1} A_2 C_{i_3} | \ldots | R_{i_{k-1}} A_k C_{i_k} | R_{i_k} | A_{k+1} | \ldots | A_m]$$

if $k \leq m$, and $h_k(\sum A_0[A_1|\ldots|A_m]) = 0$ otherwise. In the above formula, $\xi_k = |A_0| + |A_1| + \ldots + |A_k|$; the sum in the right-hand side is taken over all possible tuples $(i_1, i_2, \ldots, i_k)$. In what follows, we will omit the summation sign.

The sought-for homotopy h is the alternating sum of the h_k's:

$$h = \sum_{k=0}^{\infty} (-1)^k h_k.$$

In order to prove it, let us compute the products $h_k b_0$, $b_0 h_k$, $h_k b_1$, and $b_1 h_k$ where b_0 and b_1 are the components of the Hochschild differential: $b = b_0 + b_1$. We will begin by looking at b_0.

[3] In the left-hand side, x and $\mathsf{str}(x)$ are viewed as elements of $\mathsf{C}_\bullet(\mathsf{Perf}A)$.

On the one hand,

$$h_k(b_0(A_0[A_1|\dots|A_m])) = h_k(d(A_0)[A_1|\dots|A_m]) - \sum_{j=1}^{m}(-1)^{\eta_{j-1}}h_k(A_0[A_1|\dots|d(A_j)|\dots|A_m])$$

$$= (-1)^{\xi_k+1}d(A_0)C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k+1}\sum_{j=1}^{k}(-1)^{\eta_{j-1}}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}d(A_j)C_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=k+1}^{m}(-1)^{\eta_{j-1}}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|d(A_j)|\dots|A_m]$$

where $\eta_j = |A_0| + |sA_1| + \dots + |sA_j|$. On the other hand, if we set

$$\eta'_j = \begin{cases} |A_0C_{i_0}| + |sR_{i_0}A_1C_{i_1}| + \dots + |sR_{i_{j-1}}A_jC_{i_j}| = \eta_j + |C_{i_j}| & j \le k \\ |A_0C_{i_0}| + |sR_{i_0}A_1C_{i_1}| + \dots + |sR_{i_k}| + \dots + |sA_{j-1}| = \eta_{j-1} + 1 & j > k \end{cases}$$

then

$$b_0(h_k(A_0[A_1|\dots|A_m])) = (-1)^{\xi_k}d(A_0C_{i_0})[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=1}^{k}(-1)^{\eta'_{j-1}}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|d(R_{i_{j-1}}A_jC_{i_j})|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}(-1)^{\eta'_k}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|d(R_{i_k})|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=k+1}^{m}(-1)^{\eta'_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|d(A_j)|\dots|A_m]$$

$$= (-1)^{\xi_k}d(A_0)C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=1}^{k}(-1)^{\eta'_{j-1}+|R_{i_{j-1}}|}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}d(A_j)C_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=k+1}^{m}(-1)^{\eta'_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|d(A_j)|\dots|A_m]$$

$$= (-1)^{\xi_k}d(A_0)C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=1}^{k}(-1)^{\eta_{j-1}}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}d(A_j)C_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$-(-1)^{\xi_k}\sum_{j=k+1}^{m}(-1)^{\eta_{j-1}+1}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|d(A_j)|\dots|A_m]$$

Thus, we see that $h_kb_0=-b_0h_k$ for all k's, and as a result, $hb_0=-b_0h$.

Let us look now at the compositions of h_k with b_1. On the one hand,

$$h_k(b_1(A_0[A_1|\dots|A_m]))=(-1)^{|A_0|}h_k(A_0A_1[A_2|\dots|A_m])$$

$$+\sum_{j=1}^{m-1}(-1)^{\eta_j}h_k(A_0[A_1|\dots|A_jA_{j+1}|\dots|A_m])-(-1)^{\eta_{m-1}(|A_m|+1)}h_k(A_mA_0[A_1|\dots|A_{m-1}])$$

$$=(-1)^{\xi_{k+1}+|A_0|}A_0A_1C_{i_0}[R_{i_0}A_2C_{i_1}|R_{i_1}A_3C_{i_3}|\dots|R_{i_{k-1}}A_{k+1}C_{i_k}|R_{i_k}|A_{k+2}|\dots|A_m]$$

$$+\sum_{j=1}^{k}(-1)^{\xi_{k+1}+\eta_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jA_{j+1}C_{i_j}|\dots|R_{i_{k-1}}A_{k+1}C_{i_k}|R_{i_k}|A_{k+2}|\dots|A_m]$$

$$+\sum_{j=k+1}^{m-1}(-1)^{\xi_k+\eta_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_jA_{j+1}|\dots|A_m]$$

$$-(-1)^{\xi_k+|A_m|+\eta_{m-1}(|A_m|+1)}A_mA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_{m-1}]$$

On the other hand,

$$b_1(h_k(A_0[A_1|\dots|A_m]))=(-1)^{\xi_k}b_1(A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m])$$

$$=(-1)^{\xi_k+|A_0C_{i_0}|}A_0C_{i_0}R_{i_0}A_1C_{i_1}[R_{i_1}A_2C_{i_3}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+\sum_{j=1}^{k-1}(-1)^{\xi_k+\eta'_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}R_{i_j}A_{j+1}C_{i_{j+1}}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+(-1)^{\xi_k+\eta'_k}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+(-1)^{\xi_k+\eta'_k+|sR_{i_k}|}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}A_{k+1}|\dots|A_m]$$

$$+\sum_{j=k+1}^{m-1}(-1)^{\xi_k+\eta'_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_jA_{j+1}|\dots|A_m]$$

$$-(-1)^{\xi_k+\eta'_{m-1}(|A_m|+1)}A_mA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_{m-1}]$$

By taking into account the definition of η', ξ, and also formula (3.3.1), the latter expression can be written as follows:

$$(-1)^{\xi_k+|A_0|}A_0A_1C_{i_1}[R_{i_1}A_2C_{i_3}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+\sum_{j=1}^{k-1}(-1)^{\xi_k+\eta_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jA_{j+1}C_{i_{j+1}}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+(-1)^kA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}|\dots|R_{i_{k-1}}A_k|A_{k+1}|\dots|A_m]$$

$$-(-1)^kA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}A_{k+1}|\dots|A_m]$$

$$-\sum_{j=k+1}^{m-1}(-1)^{\xi_k+\eta_j+1}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_jA_{j+1}|\dots|A_m]$$

$$+(-1)^{\xi_k+|A_m|+\eta_{m-1}(|A_m|+1)}A_mA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_{m-1}]$$

In the above computations, we implicitly assumed that k and m were "generic" (i.e. k is less than m and greater than 1). A similar computation leads to the following result which takes into account special cases as well: set

$$\Upsilon_k=(-1)^{\xi_k+|A_0|}A_0A_1C_{i_1}[R_{i_1}A_2C_{i_3}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

$$+\sum_{j=1}^{k-1}(-1)^{\xi_k+\eta_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jA_{j+1}C_{i_{j+1}}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_m]$$

if $0<k<m+1$ and $\Upsilon_k=0$ otherwise;

$$\Phi_0=A_0[A_1|\dots|A_m],$$

$$\Phi_k=A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{j-1}}A_jC_{i_j}|\dots|R_{i_{k-1}}A_k|A_{k+1}|\dots|A_m]$$

if $0<k<m+1$ and $\Phi_k=0$ otherwise;

$$\Psi_k=\sum_{j=k+1}^{m-1}(-1)^{\xi_k+\eta_j}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_jA_{j+1}|\dots|A_m]$$

$$-(-1)^{\xi_k+|A_m|+\eta_{m-1}(|A_m|+1)}A_mA_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{k-1}}A_kC_{i_k}|R_{i_k}|A_{k+1}|\dots|A_{m-1}]$$

if $0\le k<m$,

$$\Psi_m=(-1)^{\xi_m+(\eta_m+|C_{i_m}|)(|R_{i_m}|+1)}R_{i_m}A_0C_{i_0}[R_{i_0}A_1C_{i_1}|\dots|R_{i_{m-1}}A_mC_{i_m}]$$

and $\Psi_k=0$ otherwise. Then

$$h_kb_1(A_0[A_1|\dots|A_m])=\Upsilon_{k+1}+\Psi_k$$

if $k < m$ and 0 otherwise;

$$b_1 h_k(A_0[A_1|\dots|A_m]) = \Upsilon_k + (-1)^k(\Phi_k - \Phi_{k+1}) - \Psi_k$$

when $k \leq m$ and 0 otherwise.

Now we are in position to complete the proof:

$$\sum_{k=0}^{\infty}(-1)^k(h_k b_1(A_0[A_1|\dots|A_m]) + b_1 h_k(A_0[A_1|\dots|A_m]))$$

$$= \sum_{k=0}^{m-1}(-1)^k(\Upsilon_{k+1} + \Psi_k) + \sum_{k=0}^{m}(-1)^k(\Upsilon_k + (-1)^k(\Phi_k - \Phi_{k+1}) - \Psi_k)$$

$$= \Phi_0 - (-1)^m \Psi_m$$

$$= A_0[A_1|\dots|A_m] - (-1)^{m+\xi_m+(\eta_m+|C_{i_m}|)(|R_{i_m}|+1)} R_{i_m} A_0 C_{i_0}[R_{i_0} A_1 C_{i_1}|\dots|R_{i_{m-1}} A_m C_{i_m}]$$

$$= A_0[A_1|\dots|A_m] - (-1)^{\eta_m r_{i_m}} R_{i_m} A_0 C_{i_0}[R_{i_0} A_1 C_{i_1}|\dots|R_{i_{m-1}} A_m C_{i_m}]$$

What remains is to observe that

$$\mathsf{str}(A_0[A_1|\dots|A_m]) = (-1)^{\eta_m r_{i_m}} R_{i_m} A_0 C_{i_0}[R_{i_0} A_1 C_{i_1}|\dots|R_{i_{m-1}} A_m C_{i_m}]$$

3.4 Computing the integral

In Section 3.2 we introduced an "integral"

$$\int : \mathsf{HH}_\bullet(\mathrm{End}_k(X)) \to \mathsf{HH}_\bullet(\mathsf{Perf}k) \simeq k$$

for any complex of vector spaces X with finite dimensional total cohomology. In this section we will present an explicit formula for this integral based on the results of [21] (see also [52]). This, together with (3.2.2), will give us an explicit formula for computing the pairing (3.2.3).

We will exclude the trivial case and assume that X has non-zero cohomology.

Let us fix a pair of degree 0 maps $p : X \to \mathrm{H}^\bullet(X)$ and $i : \mathrm{H}^\bullet(X) \to X$ that establish the homotopy equivalence between the complex X and its cohomology $\mathrm{H}^\bullet(X)$:

$$pi = 1_{\mathrm{H}^\bullet(X)}, \quad ip = 1_X - [d_X, H]$$

where $H : X \to X$ is a degree -1 map.

Here is an explicit formula for the integral:

Theorem 3.4.1. *The following map is a quasi-isomorphism:*

$$\phi : \mathsf{C}_\bullet(\mathrm{End}_k(X)) \to k, \quad T_1[T_2|\ldots|T_n] \mapsto \sum_{j=0}^{n-1} \mathsf{str}_{\mathrm{H}^\bullet(X)}(\mathcal{F}_n(\tau^j(T_1[T_2|\ldots|T_n]))),$$

where $\mathsf{str}_{\mathrm{H}^\bullet(X)}$ *is the ordinary super-trace,*

$$\tau(T_1[T_2|\ldots|T_n]) = (-1)^{|sT_n|(|sT_1|+\ldots+|sT_{n-1}|)} T_n[T_1|\ldots|T_{n-1}],$$

and $\mathcal{F}_n : \mathrm{End}_k(X)^{\otimes n} \to \mathrm{End}_k(\mathrm{H}^\bullet(X))$ *is given by*

$$\mathcal{F}_n(T_1[T_2|\ldots|T_n]) = pT_1HT_2H \cdot \ldots \cdot HT_n i.$$

Furthermore, the induced isomorphism $\mathsf{HH}_\bullet(\mathrm{End}_k(X)) \simeq k$ *coincides with* $\int$*.*

Let us sketch the idea of the proof. That ϕ is a morphism of complexes can be verified by a direct computation. Alternatively, this follows from Lemma 2.4 of [21] and the fact that the collection $\mathcal{F}_n$, $n = 1, 2, \ldots$, gives rise to an A_∞-morphism from the DG algebra $\mathrm{End}_k(X)$ to the DG algebra $\mathrm{End}_k(\mathrm{H}^\bullet(X))$. Moreover, the latter morphism is an A_∞-quasi-isomorphism, therefore $\mathsf{HH}_\bullet(\mathrm{End}_k(X)) \simeq \mathsf{HH}_\bullet(\mathrm{End}_k(\mathrm{H}^\bullet(X))) \simeq k$ which proves that ϕ is a quasi-isomorphism.

It remains to prove that the induced map $\mathsf{HH}_\bullet(\mathrm{End}_k(X)) \to k$ coincides with $\int$. Obviously, it suffices to fix a non-zero generator of $\mathsf{HH}_0(\mathrm{End}_k(X))$ and to show that the values of both functionals on this generator coincide. Let us start by describing a generator of $\mathsf{HH}_0(\mathrm{End}_k(X))$.

The endomorphism ip is an idempotent. Let us denote its image by $\mathrm{Harm}^\bullet(X)$. Clearly, $\mathrm{Harm}^\bullet(X)$ is a finite dimensional subspace of X isomorphic to $\mathrm{H}^\bullet(X)$. Fix n such that the component $\mathrm{Harm}^n(X)$ is non-zero and let π stand for the projection in $\mathrm{Harm}^\bullet(X)$ onto this component parallel to other graded components. Then the endomorphism $\Pi =$

$\pi i p \in \mathrm{End}^0_k(X)$ represents a non-zero element of $\mathsf{HH}_0(\mathrm{End}_k(X))$. It is immediate that $\phi(\Pi) = (-1)^n \dim \mathrm{H}^n(X)$.

On the other hand, Π and $p\pi i$ define the same element of $\mathsf{HH}_0(\mathsf{Perf}k)$ ($p\pi i$ is just for the projection in $\mathrm{H}^\bullet(X)$ onto the component $\mathrm{H}^n(X)$ parallel to other graded components). Indeed, $\Pi - p\pi i = \pi i p - p\pi i = b(\pi i[p])$. To finish the proof, observe that the element of $\mathsf{HH}_0(\mathsf{Perf}k)$, defined by $p\pi i$, coincides with the one, defined by $(-1)^n \dim \mathrm{H}^n(X) \cdot 1 \in \mathrm{End}_k(k)$.

Let us point out a couple of straightforward corollaries of Theorem 3.4.1 and formula (3.2.3).

Let A be a finite dimensional associative algebra. Then its Hochschild homology groups $\mathsf{HH}_\bullet(A)$ are concentrated in non-positive degrees. Therefore, among the pairings $\langle\, , \rangle : \mathsf{HH}_n(A) \times \mathsf{HH}_{-n}(A^{\mathsf{op}}) \to k$, only the one corresponding to $n = 0$ survives. In this case we have

Corollary 3.4.2. *For an associative algebra A, the pairing*

$$\langle\, , \rangle : A/[A, A] \times A^{\mathsf{op}}/[A^{\mathsf{op}}, A^{\mathsf{op}}] \to k$$

is given by

$$\langle a, b \rangle = \mathsf{tr}_A(L(a)R(b)).$$

(In the right-hand side, a and b stand for elements of A and A^{op}, respectively, and in the left-hand side a, b stand for the corresponding classes in the Hochschild homology.)

Let now A be a finite dimensional graded algebra. Since A is equipped with the trivial differential, we can set $H = 0$ in Theorem 3.4.1 and obtain

Corollary 3.4.3. *For a graded A, the pairing of two cycles*

$$a = a_0 + \sum a_0'[a_1'] + \sum a_0''[a_1''|a_2''] + \ldots \in \mathsf{C}_\bullet(A),$$

$$b = b_0 + \sum b_0'[b_1'] + \sum b_0''[b_1''|b_2''] + \ldots \in \mathsf{C}_\bullet(A)$$

is given by

$$\langle\, a,\, b \,\rangle = \mathsf{str}_A(L(a_0)R(b_0)).$$

Chapter 4

Applications

4.1 Directed algebras

In this section, we describe how the Hirzebruch-Riemann-Roch formula looks like for a special class of finite dimensional associative algebras.

Let $\mathcal{V}$ be a k-linear category with finite number of objects, say $\{v_s\}_{s\in S}$, and finite dimensional Hom-spaces. Suppose there is a bijection

$$f : \{1, 2, \ldots, n\} \to S$$

such that

$$\mathrm{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}) = \begin{cases} k & i = j \\ 0 & i > j \end{cases}. \tag{4.1.1}$$

Of course, f doesn't have to be unique. Let us denote the algebra of this category by $A(\mathcal{V})$:

$$A(\mathcal{V}) = \bigoplus_{s,t\in S} \mathrm{Hom}_{\mathcal{V}}(v_s, v_t).$$

We will call such algebras (as well as the underlying categories) *directed*.

Let us denote the abelian category of finite dimensional right $A(\mathcal{V})$-modules by $\mathsf{mod}A(\mathcal{V})$. The following simple result is very well known.

Proposition 4.1.1. *Any module $N \in \mathsf{mod}A(\mathcal{V})$ admits a projective resolution of finite length.*

Let us prove this. Fix a map f as above and denote $1_{v_{f(i)}}$ simply by 1_i. Denote also the projective modules $1_i\, A(\mathcal{V})$ by P_i. Clearly,

$$\dim \mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_i, P_j) = \dim \mathrm{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}).$$

Thus, by (4.1.1)

$$\dim \mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_i, P_j) = \begin{cases} k & i = j \\ 0 & i > j \end{cases}. \tag{4.1.2}$$

Fix $N \in \mathsf{mod} A(\mathcal{V})$. The canonical morphism

$$p : \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_i, N) \otimes_k P_i \to N$$

is surjective. The kernel of this morphism satisfies the property

$$\mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_n, \mathsf{Ker}\, p) = 0.$$

To see this, apply the functor $\mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_n, -)$ to the short exact sequence

$$0 \to \mathsf{Ker}\, p \to \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathsf{mod} A(\mathcal{V})}(P_i, N) \otimes_k P_i \to N \to 0$$

and use the property (4.1.2).

To finish the proof, apply the same argument to $\mathsf{Ker}\, p$ instead of N etc.

Observe that $\mathsf{HH}_0(A(\mathcal{V}))$ is spanned by the idempotents 1_{v_s}, $s \in S$ (or rather their classes in the quotient $A(\mathcal{V})/[A(\mathcal{V}), A(\mathcal{V})]$). In terms of these elements, the pairing $\langle\, ,\, \rangle$ on $\mathsf{HH}_0(A(\mathcal{V})) \times \mathsf{HH}_0(A(\mathcal{V})^{\mathsf{op}})$ is given by

$$\langle\, 1_t\, ,\, 1_s^{\vee}\, \rangle = \dim \mathrm{Hom}_{\mathcal{V}}(v_s, v_t).$$

Let us derive the Hirzebruch-Riemann-Roch formula for finite dimensional modules over directed algebras. It is well known and was obtained in [56, Section 2.4].

Let us keep the notations from the proof of Proposition 4.3.1. Set

$$d_{ij} := \dim \mathrm{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}).$$

Let $M, N \in \mathsf{mod}A(\mathcal{V})$. As we saw above, M and N admit finite length resolutions by direct sums of the projective modules P_i. Let us fix two such resolutions $P(M)$ and $P(N)$. We know that $\mathsf{eu}(P(M))$, $\mathsf{eu}(P(N))$ are linear combinations of 1_i's:

$$\mathsf{eu}(P(M)) = \sum_{i=1}^{n} a_i \cdot 1_i, \quad \mathsf{eu}(P(N)) = \sum_{i=1}^{n} b_i \cdot 1_i.$$

Since $1_j = \mathsf{eu}(P_j)$, we have

$$\begin{aligned}(\underline{\dim} M)_j := \mathrm{Hom}_{\mathsf{mod}A(\mathcal{V})}(P_j, M) = \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Mod}A(\mathcal{V}))}(P_j, P(M)) &= \langle\, \mathsf{eu}(P(M))\,,\, 1_j^\vee \,\rangle \\ &= \sum_{i=1}^{n} d_{ji} a_i\end{aligned}$$

and similarly $(\underline{\dim} N)_j = \sum_{i=1}^{n} d_{ji} b_i$. Therefore,

$$\begin{aligned}\sum_l (-1)^l \dim \mathrm{Ext}^l_{\mathsf{mod}A(\mathcal{V})}(M, N) = \chi(P(M), P(N)) &= \langle\, \mathsf{eu}(P(N))\,,\, \mathsf{eu}(P(M))^\vee \,\rangle \\ &= \sum_{i,j} b_i a_j d_{ji}.\end{aligned}$$

Since $a_j = \sum_k (d^{-1})_{jk} (\underline{\dim} M)_k, \quad b_i = \sum_k (d^{-1})_{il} (\underline{\dim} N)_l$, we get the following formula which is due to Ringel [56, Section 2.4]:

$$\sum_l (-1)^l \dim \mathrm{Ext}^l_{\mathsf{mod}A(\mathcal{V})}(M, N) = \sum_{i,j} (\underline{\dim} M)_i (d^{-1})_{ij} (\underline{\dim} N)_j. \tag{4.1.3}$$

4.2 Noncommutative DG-schemes arising from orbifold singularities

In this section, we will describe certain proper DG algebras[1] which arise from quotient singularities of the form $\mathbb{C}^n/G$, where G is a finite group.

Let $V = \mathbb{C}^n$ be a finite dimensional complex vector space and G a finite subgroup of $SL(V) \cong SL_n(\mathbb{C})$. Then G acts on the polynomial algebra $\mathbb{C}[V]$ via $(gf)(x) = f(g^{-1}x)$. The spectrum $X = V/G$ of the algebra $\mathbb{C}[V]^G$ of G-invariant polynomials is a singular affine

[1] All of them are DG algebras with the trivial differential.

variety. The central problem in the study of such singular varieties is to construct their "most economical" resolutions, which are called *crepant*: a resolution $\pi : Y \to X$ is crepant, if π preserves the canonical classes[2], i.e. $\pi^*(\omega_X) = \omega_Y$.

The derived Mckay correspondence [9, 28, 53] is a program around the following conjecture and various versions thereof:

For any crepant resolution $Y \to X$, the bounded derived category $D(Y)$ of coherent sheaves on Y is equivalent to the bounded derived category $D^G(V)$ of G-equivariant coherent sheaves on V.

In other words, all crepant resolutions of a fixed singularity are expected to be isomorphic as noncommutative DG-schemes. The conjecture is known to be true for finite subgroups of $SL(2)$ [28] and $SL(3)$ [9] (see also [3] for a result in higher dimensions).

Denote by $D_0^G(V)$ the subcategory in $D^G(V)$ of complexes supported at the origin $0 \in V$ and by $D_0(Y)$ the subcategory in $D(Y)$ of complexes supported at the exceptional fiber $\pi^{-1}(0)$ (in the latter formula 0 stands for the image of the origin of V under the canonical projection $V \to X$). Then the above equivalence of categories should induce an equivalence between $D_0(Y)$ and $D_0^G(V)$ [9].

The Ext groups between any two objects of $D_0^G(V)$ vanish in all but finitely many degrees and, thus, we are dealing with a proper noncommutative DG-scheme. This scheme is the main subject of the section.

Following [25, Section 6.2], consider the cross-product $\Lambda(V,G)$ of the exterior algebra ΛV and the group algebra of G. In other words, as a vector space $\Lambda(V,G)$ is the tensor product $\Lambda V \otimes \mathbb{C}[G]$. The product of two elements is given by

$$(v \otimes g)(w \otimes h) = (v \wedge g(w)) \otimes gh, \quad v, w \in \Lambda V,\ g, h \in G.$$

[2] A crepant resolutions of X, if exists, is a noncompact Calabi-Yau variety since the top-degree form on V is G-invariant and therefore the canonical sheaves of X and Y are trivial.

Equip $\Lambda(V,G)$ with the unique grading such that $\deg v = 1$ and $\deg g = 0$ for any $v \in V$ and $g \in G$. We will view $\Lambda(V,G)$ as a DG algebra with the trivial differential.

The following conjecture is motivated by [25]:

Conjecture. *There is an equivalence of triangulated categories*

$$D_0^G(V) \cong \mathsf{Ho}(\mathsf{Perf}\Lambda(V,G)).$$

Here is how the conjecture might be proved. The category $D_0^G(V)$ seems to be equivalent to the category $D^b(f.d.\,\mathbb{C}[V] \rtimes G)$, where $\mathbb{C}[V] \rtimes G$ is the cross-product of the polynomial algebra and the group algebra of G and $f.d.\,\mathbb{C}[V] \rtimes G$ is the abelian category of finite dimensional graded $\mathbb{C}[V] \rtimes G$-modules. Every such module admits a finite filtration by simple $\mathbb{C}[V] \rtimes G$-modules. The latter are the $\mathbb{C}[V] \rtimes G$-modules obtained from simple $\mathbb{C}[G]$-modules via "restriction of scalars"

$$\mathbb{C}[V] \rtimes G \to \mathbb{C}[G], \quad f(x) \otimes g \mapsto f(0)g, \quad f(x) \in \mathbb{C}[V],\ g \in G.$$

Let us denote the simple $\mathbb{C}[V] \rtimes G$-module, corresponding to an irreducible representations ρ of G, by S_ρ. Then, using the technique described in [30], we may conclude that $D^b(f.d.\,\mathbb{C}[V] \rtimes G)$ is equivalent to the category $\mathsf{Ho}(\mathsf{Perf}\mathcal{A})$ for some A_∞ algebra $\mathcal{A}$ with

$$\mathrm{H}^\bullet(\mathcal{A}) = \mathrm{Ext}^\bullet(\oplus_\rho S_\rho, \oplus_\rho S_\rho),$$

where the sum in the right-hand side is taken over irreducible representations of G. According to [25, Section 6.2], the algebra $\mathbb{C}[V] \rtimes G$ is quadratic and Koszul, and its Koszul dual is exactly $\Lambda(V,G)$. Then, by [30, Section 2.2], the A_∞ algebra $\mathcal{A}$ is formal. Finally, we expect that $\mathrm{Ext}^\bullet(\oplus_\rho S_\rho, \oplus_\rho S_\rho)$ is Morita equivalent to $\Lambda(V,G)$.

Whether the conjecture is true or not, it is clear that the algebraic triangulated categories of the form $\mathsf{Ho}(\mathsf{Perf}\Lambda(V,G))$ should play a role in the study of the quotient singularities.

Let us compute the pairing $\langle\, , \rangle$ on $\mathsf{HH}_0(\Lambda(V,G)) \times \mathsf{HH}_0(\Lambda(V,G)^{\mathsf{op}})$.

We start by noticing that, in general, the space $\mathsf{HH}_0(\Lambda(V,G))$ is infinite dimensional (this is already so in the simplest case $V=\mathbb{C}$, $G=\{1\}$). However, the pairing $\langle\,,\,\rangle$ vanishes on a subspace of finite codimension (this follows from Corollary 3.4.3). In fact, the pairing is determined by its restriction onto the finite dimensional subspace

$$\mathsf{HH}_0(\mathbb{C}[G]) \times \mathsf{HH}_0(\mathbb{C}[G]^{\mathsf{op}}) \subset \mathsf{HH}_0(\Lambda(V,G)) \times \mathsf{HH}_0(\Lambda(V,G)^{\mathsf{op}}).$$

(Here we are using the natural embedding $\mathbb{C}[G] \to \Lambda(V,G)$ which induces an embedding $\mathsf{HH}_0(\mathbb{C}[G]) \to \mathsf{HH}_0(\Lambda(V,G))$.) Furthermore, it is well known that $\mathsf{HH}_0(\mathbb{C}[G])$ is spanned by (the homology classes of) the characters of irreducible representations of G. Let us denote the character of an irreducible representation ρ by χ_ρ:

$$\chi_\rho = \sum_g \mathsf{tr}(\rho(g))g.$$

Using basic harmonic analysis on G, it is easy to show that the element $\pi_\rho = \frac{\dim\rho}{|G|}\chi_\rho$ is an idempotent in $\Lambda(V,G)$ (it is nothing but the Euler class of the DG $\Lambda(V,G)$-module $\pi_\rho \cdot \Lambda(V,G)$). Thus, we have to compute

$$\langle\, \pi_{\rho_1}, \pi_{\rho_2}^\vee \,\rangle = \mathsf{str}_{\Lambda(V,G)}(L(\pi_{\rho_1})R(\pi_{\rho_2}))$$

for two irreducible representations ρ_1, ρ_2.

Let W be the space of some representation of G. Then $W \otimes \mathbb{C}[G]$ carries a natural $\mathbb{C}[G]$-bimodule structure, defined as follows:

$$g(w \otimes h)k = g(w) \otimes ghk, \quad w \in W,\ g,h,k \in G.$$

In particular, the graded components $\Lambda^n(V,G) = \Lambda^n V \otimes \mathbb{C}[G]$ of the algebra $\Lambda(V,G)$ are $\mathbb{C}[G]$-bimodules and we have

$$\begin{aligned}\mathsf{str}_{\Lambda(V,G)}(L(\pi_{\rho_1})R(\pi_{\rho_2})) &= \sum_{n=0}^{\dim V} (-1)^n\, \mathsf{tr}_{\Lambda^n(V,G)}(L(\pi_{\rho_1})R(\pi_{\rho_2})) \\ &= \sum_{n=0}^{\dim V} (-1)^n\, \dim(\pi_{\rho_1}\Lambda^n(V,G)\pi_{\rho_2}).\end{aligned}$$

Therefore, we will start by computing $\dim(\pi_{\rho_1}(W \otimes \mathbb{C}[G])\pi_{\rho_2})$ for an arbitrary W.

Let us introduce a matrix d^W of non-negative integers by the following formula:

$$W \otimes \rho = \bigoplus_{\sigma} d^W_{\sigma\rho}\, \sigma,$$

where ρ and σ run through the set of irreducible representations of G. Let us denote the representation, dual to ρ, by ρ'. Then, as a $\mathbb{C}[G]$-bimodule

$$W \otimes \mathbb{C}[G] = \bigoplus_{\rho} (W \otimes \rho) \boxtimes \rho' = \bigoplus_{\rho,\sigma} d^W_{\sigma\rho}\, \sigma \boxtimes \rho'.$$

Thus,

$$\dim(\pi_{\rho_1}(W \otimes \mathbb{C}[G])\pi_{\rho_2}) = \dim \rho_1 \, \dim \rho_2 \, d^W_{\rho_1\rho_2},$$

which gives us the following formula for $\langle \pi_{\rho_1}, \pi^\vee_{\rho_2} \rangle$:

$$\langle \pi_{\rho_1}, \pi^\vee_{\rho_2} \rangle = \dim \rho_1 \, \dim \rho_2 \sum_{n=0}^{\dim V} (-1)^n \, d^{\Lambda^n V}_{\rho_1\rho_2}. \tag{4.2.1}$$

4.3 On noncommutative Hodge-to-de Rham degeneration

Recall [38] that a DG algebra is said to be homologically smooth if there is a perfect right DG $A^{\mathsf{op}} \otimes A$-module $P(A)$ together with a quasi-isomorphism $P(A) \to A$ of right DG $A^{\mathsf{op}} \otimes A$-modules.

To have an example at hand, observe that

Proposition 4.3.1. *Any directed algebra is homologically smooth.*

Indeed, it is clear that $A(\mathcal{V})^{\mathsf{op}} \otimes A(\mathcal{V}) \cong A(\mathcal{V}^{\mathsf{op}} \otimes \mathcal{V})$. Therefore, by Proposition 4.1.1, any finite dimensional $A(\mathcal{V})^{\mathsf{op}} \otimes A(\mathcal{V})$-module admits a finite projective resolution. What remains is to apply this to $A(\mathcal{V})$ and observe that any finite complex of projective bimodules over an associative algebra is a perfect DG bimodule in our sense.

The aim of this section is to prove that the pairing

$$\langle\, ,\, \rangle : \mathsf{HH}_n(\mathsf{Perf}\, A) \times \mathsf{HH}_{-n}(\mathsf{Perf}\, A^{\mathsf{op}}) \to k,$$

is non-degenerate for any proper homologically smooth DG algebra A. The proof is based on the observation that the pairing is inverse to the Euler class $\mathsf{Eu}(A)$ of the A-bimodule A. The author learned about this idea from [39].

Theorem 4.3.2. *Let A be a proper homologically smooth DG algebra. Then the pairing $\langle\, ,\, \rangle$ is non-degenerate.*

Indeed, fix a perfect resolution $P(A) \xrightarrow{p} A$ in the category of right DG $A^{\mathsf{op}} \otimes A$-modules. Then, for any right perfect DG A-module X, we have a morphism

$$1 \otimes p : X \otimes_A P(A) \to X \otimes_A A \simeq X.$$

By Proposition 2.2.5, $1 \otimes p$ is a quasi-isomorphism. On the other hand, by Proposition 2.2.4, both $X \otimes_A P(A)$ and X are perfect and, in particular, homotopically projective. It is well known that a quasi-isomorphism between two homotopically projective modules is actually a homotopy equivalence (see, for instance, the proof of Lemma 10.12.2.2 in [2]). Thus, $1 \otimes p : X \otimes_A P(A) \to X \otimes_A A \simeq X$ is a homotopy equivalence.

What we have just proved is that the quasi-isomorphism $P(A) \xrightarrow{p} A$ gives rise to a weak homotopy equivalence of the DG functors $T_{P(A)} \to \mathrm{Id}_{\mathsf{Perf}\, A}$ where Id stands for the identity endofunctor. Then, as a corollary of Theorem 2.3.2, we get the following result: the linear map $\mathsf{HH}(T_{P(A)}) : \mathsf{HH}_\bullet(\mathsf{Perf}\, A) \to \mathsf{HH}_\bullet(\mathsf{Perf}\, A)$ coincides with the identity map. On the other hand, by Theorem 3.1.4, the map $\mathsf{HH}(T_{P(A)})$ is given by the 'convolution' with $\mathsf{Eu}'(P(A))$, so the convolution with $\mathsf{Eu}'(P(A))$ is the identity map. This proves that the left kernel of the pairing is trivial, i.e. for any n we have an embedding

$$\mathsf{HH}_n(\mathsf{Perf}\, A) \to \mathsf{HH}_{-n}(\mathsf{Perf}\, A^{\mathsf{op}})^*.$$

One of the results of [61] says that the Hochschild homology of an arbitrary proper homologically smooth DG algebra is finite dimensional. Thus, to prove that the right kernel of the pairing is trivial, it is enough to show that $\dim \mathsf{HH}_n(\mathsf{Perf}\, A) = \dim \mathsf{HH}_{-n}(\mathsf{Perf}\, A^{\mathsf{op}})$. This can be done by replacing A by A^{op} in the above argument.

Let us point out one interesting corollary of this result[3]:

Corollary 4.3.3. *If A is a homologically smooth proper associative algebra then*

$$\mathsf{HH}_n(A) = \begin{cases} A/[A, A] & n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Indeed, the Hochschild homology of such an algebra is concentrated in non-positive degrees. Thus, by the non-degeneracy of the pairing, the Hochschild homology groups, sitting in negative degrees, have to vanish.

This corollary, together with Proposition 4.3.1, implies $\mathsf{HH}_n(A(\mathcal{V})) = 0$ for any directed algebra $A(\mathcal{V})$ and any $n \neq 0$. This result was obtained by a different method in [13].

Another application of the corollary is related to the so-called noncommutative Hodge-to-de Rham degeneration conjecture. Roughly speaking, the conjecture claims that the B-operator $B : \mathsf{HH}_\bullet(A) \to \mathsf{HH}_{\bullet-1}(A)$ (see [22, 67]) vanishes whenever A is proper and homologically smooth. It was formulated, in a stronger form, by M. Kontsevich and Y. Soibelman [38] and proved, in the partial case of DG algebras concentrated in non-negative degrees, by D. Kaledin [27]. The above corollary implies the conjecture in the case of DG algebras concentrated in non-positive degrees. Indeed, by their very definition the Hochschild homology groups of a DG algebra, graded by non-positive integers, may be non-trivial in non-positive degrees only whence the result.

[3] If k is perfect, this result also follows from Proposition 2.5 of [33] and Morita invariance of the Hochschild homology.

Chapter 5

The HRR theorem and topological field theories

5.1 Reminder on topological field theories

In this section, we recall the definition of and some basic facts about topological field theories (TFTs) and their open-closed versions. Our main sources are [15, 40, 44].

According to Atiyah, a TFT[1] is a rule that assigns a finite-dimensional vector space C to the circle S^1, the tensor power $C^{\otimes n}$ to the disjoint union of n circles, and linear maps between the tensor powers to (isomorphism classes of) oriented 2-dimensional cobordisms between the unions of circles. The axioms that this assignment is required to satisfy are most conveniently expressed by saying that it is a monoidal functor from the symmetric monoidal category $2Cob$, whose objects are closed oriented 1-manifolds and morphisms are 2-cobordisms, to the category of finite-dimensional vector spaces.

TFTs are well studied objects: it is a classical result that the image of the circle under a functor as above is a *commutative Frobenius algebra*, and conversely, any such algebra determines a TFT. A commutative Frobenius algebra, we recall, is a finite-dimensional unital commutative algebra C equipped with a functional $\theta_C : C \to k$ such that the pairing

$$C \otimes C \to k, \quad a \otimes b \mapsto \theta_C(ab)$$

is non-degenerate.

[1] Here and further, we consider only the 2-dimensional case.

There is an important refinement of the classical definition of TFT. The resulting notion is called an *open-closed* TFT. The “old” definition is a part (or the *closed sector*) of the new one. Let us outline what open-closed TFTs are.

In the closed case, we are dealing with a geometric category whose objects are unions of circles and morphisms are the usual 2-dimensional cobordisms, i.e. surfaces with boundary. Accordingly, the connected components of the boundary of each surface are of two types: “incoming” circles and “outgoing” circles. In the open-closed case, one is working with a more intricate geometric category. For starters, the objects of this category are unions of circles *and* segments. The morphisms are *2-cobordisms with corners*, or more precisely, 2-dimensional surfaces whose boundary circles may be of the following types:

- some of the circles are labeled as “incoming” or “outgoing”;
- the remaining circles contain disjoint unions of segments labeled as “open incoming” or “open outgoing”; the complement of these segments in each circle is labeled as “free” (it is possible for a boundary component to be completely free).

The composition of morphisms in this category is defined in the natural way: one stitches outgoing circles and open segments of the first surface to incoming circles and open segments of the second one. Some of the free boundaries of the two surfaces get stitched at the end points of open segments to form new free boundaries of the resulting surface.

From the point of view of applications in topological string theory, it is necessary to extend the definition of this geometric category as follows. One fixes a set Λ, whose elements are abstract versions of D-branes in string theory, and labels the connected free boundaries of surfaces with elements of this set. Then the stitching of surfaces is required to be compatible with the labeling. The previous definition corresponds essentially to the case of a unique D-brane.

Now we are in position to define open-closed TFTs: these are monoidal functors from the open-closed geometric category (with a fixed set Λ of D-branes, if necessary) to the monoidal category of finite-dimensional vector spaces.

Again, from the point of view of applications, it is sometimes important to work with open-closed TFTs valued in some other categories, i. e. in the categories of $\mathbb{Z}_2$-graded or $\mathbb{Z}$-graded finite-dimensional vector spaces. We will be working in the latter setting.

It is clear from the above definition that an open-closed TFT is comprised of the following data

- a $\mathbb{Z}$-graded space C that the TFT assigns to a circle (the closed sector)
- a collection of $\mathbb{Z}$-graded vector spaces $O_{\lambda\mu}$ for each pair of elements $\lambda, \mu \in \Lambda$ that the TFT assigns to an open segment whose end point are labeled λ and μ (the open sector)

and that these spaces have to satisfy a number of properties and compatibility conditions. The complete list of properties was written down independently by Lazaroiu and Moore-Segal [40, 44]. Here it is.

First of all, the space C carries a graded commutative Frobenius algebra structure. The definition is the same as in the non-graded case with the additional requirement that the trace be of degree 0.

Furthermore, the elements of Λ together with the spaces $O_{\lambda\mu}$ turn out to form a category. More precisely, there are "composition maps"

$$O_{\lambda\mu} \times O_{\mu\nu} \to O_{\lambda\nu}$$

and the "identity morphisms" $1_\lambda \in O_{\lambda\lambda}$ satisfying the usual properties. In addition to that, this category of D-branes possesses the *Calabi-Yau* property. It means that one has traces

$$\theta_\lambda : O_{\lambda\lambda} \to k \quad \forall \lambda \in \Lambda$$

which, together with the composition maps, induce non-degenerate symmetric pairings on the morphism spaces

$$O_{\lambda\mu} \times O_{\mu\lambda} \to O_{\lambda\lambda} \to k$$

Here "symmetric" means that for any homogeneous morphisms $f' \in O_{\lambda\mu}$, $f'' \in O_{\mu\lambda}$

$$\theta_\lambda(f'f'') = (-1)^{|f'||f''|}\theta_\mu(f''f')$$

Finally, the commutative Frobenius algebra C and the Calabi-Yau category of D-branes have to satisfy certain compatibility properties which can be formulated as follows. There exist unital graded algebra homomorphisms

$$\iota_\lambda : C \to O_{\lambda\lambda},$$

one for each λ, such that

$$\iota_\lambda(c)f = f\iota_\mu(c), \quad \forall c \in C, f \in O_{\lambda\mu} \tag{5.1.1}$$

In addition, if we introduce the adjoint maps

$$\iota^\lambda : O_{\lambda\lambda} \to C, \quad \theta_C(\iota^\lambda(f)c) = \theta_\lambda(f\iota_\lambda(c)) \tag{5.1.2}$$

then the so-called *Cardy condition* has to be satisfied: for all $f' \in O_{\lambda\lambda}$ and $f'' \in O_{\mu\mu}$

$$\mathsf{str}_{O_{\lambda\mu}}(L(f')R(f'')) = \theta_C(\iota^\lambda(f')\iota^\mu(f'')) \tag{5.1.3}$$

where $L(f')$ (resp. $R(f'')$) stands for the operator in $O_{\lambda\mu}$ of left (resp. right) multiplication with f' (resp. f'').

The aim of this chapter is to write out a class of examples of open-closed TFTs coming from *noncommutative 0-dimensional smooth proper Calabi-Yau spaces* and to show that the Cardy condition in these examples can be interpreted as a generalized noncommutative Hirzebruch-Riemann-Roch theorem.

5.2 Noncommutative Calabi-Yau spaces

The most general definition of a Calabi-Yau structure on noncommutative spaces was given in [38]. We will be interested in some simple examples of noncommutative Calabi-Yau spaces, and this section contains all the relevant definitions and facts. For simplicity, we will assume from now on that the characteristic of the ground field is 0.

To begin with, let us recall that by a *trace* on a DG algebra A one understands a (homogeneous) functional $\theta : A \to k$ such that

$$\theta(da) = 0, \quad \theta([a, b]) = 0, \quad a, b \in A.$$

Let A be a proper DG algebra. Suppose the algebra possesses a degree $-d$ trace θ satisfying the following condition: the induced degree $-d$ pairing

$$\mathrm{H}^{\bullet}(A) \times \mathrm{H}^{\bullet}(A) \to k, \quad (a, b) \mapsto \theta(ab)$$

is non-degenerate. Then the pair (A, θ) is called a d-dimensional (proper) Calabi-Yau DG algebra and the corresponding noncommutative proper DG scheme is said to be Calabi-Yau [38][2]. Sometimes, we will write A instead of (A, θ).

Observe that the algebra $\Lambda(V, G)$, which we studied in Section 4.2, carries a natural structure of a dim V-dimensional CY DG algebra. Namely, fix a non-zero element $\omega \in \Lambda^{\dim V} V$ and set [25]:

$$\tau_\omega(v \otimes g) = \begin{cases} 0 & v \in \Lambda^n V,\ n < \dim V \\ \delta_{1g} & v = \omega \end{cases}.$$

Before we proceed any further, we would like to mention that there exist noncommutative analogs of affine Calabi-Yau spaces. Their definition is different and is based on the notion of noncommutative dualizing complex [23, 38]. A good example of such an algebra is the cross-product $\mathbb{C}[V] \rtimes G$, mentioned in Section 4.2. We will be studying a very special class of noncommutative Calabi-Yau spaces which are Calabi-Yaus from either point of view.

From now on, all our Calabi-Yau algebras will be finite-dimensional non-graded. These are necessarily 0-dimensional. Clearly, such a Calabi-Yau algebra is nothing but a noncommutative Frobenius algebra[3], i.e. an associative finite-dimensional unital algebra A with

[2]Actually, the authors of [38] work with CY A_∞ algebras.

[3]More precisely, it is what is called "symmetric Frobenius algebra".

a non-degenerate symmetric trace θ. In addition, we will require our Frobenius algebras to be homologically smooth in the sense of Section 4.3. Here is the first basic property of homologically smooth Frobenius algebras:

Proposition 5.2.1. *Any homologically smooth Frobenius algebra A is separable, i.e. it is projective as an A-bimodule.*

Indeed, it is easy to see that the homological smoothness implies finiteness of the cohomological dimension of A [18]. It was shown in [19] that Frobenius algebras may have cohomological dimension 0 or ∞. Thus, any homologically smooth Frobenius algebra is of cohomological dimension 0. Finally, algebras have cohomological dimension 0 iff they are separable.

Separable algebras are the simplest homologically smooth algebras, and the above fact simplifies the study of homologically smooth Frobenius algebras drastically. For example, if A is a separable algebra then $\mathsf{Perf}A$ is quasi-equivalent to its much smaller subcategory. We will formulate a precise statement below.

Let $\mathsf{Proj}_{gr} A$ stand for the full subcategory in $\mathsf{Perf}A$ consisting of direct summands of free modules (here "free module" means a module from the subcategory $\mathsf{Free}A \subset \mathsf{Perf}A$; see Section 3.3). Thus, we can think of objects of $\mathsf{Proj}_{gr} A$ as pairs $P = (\bigoplus_{j=1}^{n} A[r_j], \pi)$ where π is an idempotent endomorphism of P. Morphisms from one such module, $P' = (\bigoplus_{j=1}^{n} A[r_j], \pi')$, to another one, $P'' = (\bigoplus_{j=1}^{m} A[s_j], \pi'')$, are $m \times n$-matrices f of elements from A (with shifted degrees) satisfying the condition

$$f\pi' = \pi'' f = f$$

We can treat the category $\mathsf{Proj}_{gr} A$ as a DG category even though the differential on the morphism spaces is 0. Thus, it makes sense to speak about its homotopy category, $\mathsf{Ho}(\mathsf{Proj}_{gr} A)$. The following fact is well known:

Proposition 5.2.2. $\mathsf{Ho}(\mathsf{Proj}_{gr}\,A)$ *is a triangulated category. Moreover, the inclusion* $\mathsf{Proj}_{gr}\,A \subset \mathsf{Perf}\,A$ *induces an equivalence of the triangulated categories* $\mathsf{Ho}(\mathsf{Proj}_{gr}\,A)$ *and* $\mathsf{Ho}(\mathsf{Perf}\,A)$.

To conclude this section, let us equip the category $\mathsf{Proj}_{gr}\,A$, for a separable Frobenius algebra (A, θ), with a natural Calabi-Yau structure (see the previous section). This will justify our use of the term "noncommutative Calabi-Yau space".

Fix a module $P \in \mathsf{Proj}_{gr}\,A$ and for any endomorphism $f \in \mathrm{Hom}_{\mathsf{Proj}_{gr}\,A}(P, P)$ set

$$\theta_P(f) = \theta(\mathsf{str}(f))$$

Then one can easily prove that

Proposition 5.2.3. *With the traces defined as above,* $\mathsf{Proj}_{gr}\,A$ *is a Calabi-Yau category.*

5.3 Open-closed TFTs from 0-dimensional noncommutative Calabi-Yau spaces

The aim of this section is to explicitly construct an open-closed TFTs associated with an arbitrary Frobenius separable algebra A. Apparently, all the formulas we present below can be derived from a much more general approach of K. Costello [15] but we are not going to do it here.

The open sector in our examples is described by the Calabi-Yau category $\mathsf{Proj}_{gr}\,A$ introduced in the previous section. More rigorously, we should work with the opposite category: for two modules $P', P'' \in \mathsf{Proj}_{gr}\,A$ we set

$$O_{P'P''} = \mathrm{Hom}_{\mathsf{Proj}_{gr}\,A}(P', P'')$$

and define the product maps

$$O_{P'P''} \times O_{P''P'''} \to O_{P'P'''}$$

by

$$(f', f'') \mapsto f' \star f'' = (-1)^{|f'||f''|} f'' f'$$

The closed sector is determined by a commutative Frobenius algebra structure on the Hochschild homology $\mathsf{HH}_\bullet(A)$. It is well known (and it is also a very special case of Corollary 4.3.3) that for a separable algebra A the Hochschild homology is concentrated in degree 0. So in this case, the commutative Frobenius algebra will be non-graded. Let us describe it explicitly.

Let us fix a separable Frobenius algebra (A, θ) and let $\sum_i \xi_i' \otimes \xi_i'' \in A \otimes A$ stand for the symmetric tensor inverse to the pairing defined by θ:

$$a = \sum_i \xi_i' \theta(a \xi_i''), \quad \forall a \in A.$$

Define a linear map $m : A \otimes A \to A$ by

$$m(a \otimes b) = \sum_i \xi_i' a \xi_i'' b.$$

Proposition 5.3.1. *The map m descends to a well-defined associative commutative product*

$$m : \mathsf{HH}_0(A) \otimes \mathsf{HH}_0(A) \to \mathsf{HH}_0(A).$$

In addition, $\xi = \sum_i \xi_i' \xi_i''$ is invertible in A and the inverse element is the unit of $\mathsf{HH}_0(A)$.

The proof of the first statement is straightforward once we notice that the tensor $\sum_i \xi_i' \otimes \xi_i''$ satisfies the following properties which, in their turn, follow directly from its definition:

$$\sum_i a\xi_i' \otimes \xi_i'' = \sum_i \xi_i' \otimes \xi_i'' a, \tag{5.3.1}$$

$$\sum_i \xi_i' a \otimes \xi_i'' = \sum_i \xi_i' \otimes a \xi_i''. \tag{5.3.2}$$

Let us prove for example that the map m descends to a well-defined map from $\mathsf{HH}_0(A) \otimes \mathsf{HH}_0(A)$ to $\mathsf{HH}_0(A)$. Indeed, it follows from (5.3.2) that m descends to a well-defined map from $\mathsf{HH}_0(A) \otimes A$ to A. Then

$$\sum_i \xi_i' a \xi_i'' (bc - cb) = \sum_i \xi_i' a \xi_i'' bc - \sum_i \xi_i' a \xi_i'' cb$$

$$\stackrel{(5.3.1)}{=} \sum_i b\xi_i' a\xi_i'' c - \sum_i \xi_i' a\xi_i'' cb \equiv 0 \textit{ modulo } [A, A]$$

which means that m descends further to a map from $\mathsf{HH}_0(A) \otimes \mathsf{HH}_0(A)$.

The associativity and the commutativity of the resulting product on $\mathsf{HH}_0(A)$ are proved in a similar manner.

Notice that by (5.3.1) the element ξ is in the center of A. Therefore, the invertibility of ξ would immediately imply that ξ^{-1} is the unit in $\mathsf{HH}_0(A)$. Thus, it remains to prove that ξ is invertible in A. To prove it, notice that for any $a \in A$

$$\mathsf{tr}_A(L(a)) = \theta(a\xi)$$

where, as previously, $L(a)$ stands for the operator of left multiplication with a. This follows from the fact that under the canonical isomorphism $\mathrm{End}_k(A) \cong A \otimes A^*$ the operator $L(a)$ corresponds to $\sum_i a\xi_i' \otimes \theta(\xi_i'' \cdot -)$. It is well known that for a separable algebra (over a field of characteristic 0) the pairing $(a, b) \mapsto \mathsf{tr}_A(L(a)L(b))$ is non-degenerate. Therefore, by the above equality, the operator $a \mapsto a\xi$ is invertible, i.e. ξ is invertible. Proposition 5.3.1 is proved completely.

To complete our description of the closed sector, we need to equip the commutative algebra $\mathsf{HH}_0(A)$ with a non-degenerate trace.

Proposition 5.3.2. *The trace θ descends to a non-degenerate trace $\theta_{\mathsf{HH}} : \mathsf{HH}_0(A) \to k$.*

To prove it, we will show that the pairing on $\mathsf{HH}_0(A)$ induced by this trace coincides with the pairing $\langle a, b\rangle = \mathsf{tr}_A(L(a)R(b))$ studied in the previous chapters. Then the non-degeneracy will follow from Theorem 4.3.2.

Observe that under the canonical isomorphism $\mathrm{End}_k(A) \cong A \otimes A^*$ the operators $L(a)$, $R(b)$ get mapped to the elements

$$\sum_i a\xi_i' \otimes \theta(\xi_i'' \cdot -), \quad \sum_j \xi_j' b \otimes \theta(\xi_j'' \cdot -),$$

respectively. Therefore,

$$\mathsf{tr}_A(L(a)R(b)) = \sum_{i,j} \theta(\xi''_j a\xi'_i)\theta(\xi''_i\xi'_j b) = \sum_i \theta(\xi''_i\xi'_j \sum_j \theta(\xi''_j a\xi'_i)b) = \sum_i \theta(\xi''_i a\xi'_i b)$$

which finishes the proof.

The remaining part of this section is devoted to constructing the "mixed" (i.e. open-closed) sector. That is, we need to introduce maps

$$\iota_P : \mathsf{HH}_0(A) \to \mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P,P), \quad \iota^P : \mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P,P) \to \mathsf{HH}_0(A), \quad P \in \mathsf{Proj}_{gr} A$$

satisfying all the properties from Section 5.1.

Let us fix a module $P = (\bigoplus_{j=1}^{n} A[r_j], \pi)$ and define the maps ι_P and ι^P by

$$\iota_P(a) = \pi \cdot D_n(\sum_i \xi'_i a \xi''_i), \quad \iota^P(f) = \mathsf{str}(f)$$

In the first formula, $D_n(c)$ stands for the diagonal $n \times n$ matrix whose diagonal entries are all equal to $c \in A$; we have to multiply the matrix by the idempotent π for otherwise it will not belong to $\mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P,P)$. In the second formula, the super-trace in the right-hand side is an element of A but we are looking at its image in $\mathsf{HH}_0(A)$. Observe that by (5.3.1) $\sum_i \xi'_i a \xi''_i$ is in the center of A. This will play a crucial role in what follows.

Proposition 5.3.3. *The maps ι_P and ι^P satisfy properties (5.1.1) and (5.1.2). Also, ι_P is a unital algebra homomorphism.*

To prove (5.1.1), fix two modules, $P' = (\bigoplus_{j=1}^{n} A[r_j], \pi')$ and $P'' = (\bigoplus_{j=1}^{m} A[s_j], \pi'')$, an $m \times n$-matrix f, defining a morphism from P' to P'', and an element $a \in \mathsf{HH}_0(A)$. Since $f\pi' = \pi'' f = f$ and $\sum_i \xi'_i a \xi''_i$ is central, one has

$$\iota_{P'}(a) \star f = f\iota_{P'}(a) = f\pi' D_n(\sum_i \xi'_i a\xi''_i) = fD_n(\sum_i \xi'_i a\xi''_i)$$

$$= D_m(\sum_i \xi'_i a\xi''_i)f = D_m(\sum_i \xi'_i a\xi''_i)\pi'' f = \pi'' D_m(\sum_i \xi'_i a\xi''_i)f$$

$$= \iota_{P''}(a) f = f \star \iota_{P''}(a).$$

Let us prove (5.1.2). Fix a module, $P = (\bigoplus_{j=1}^{n} A[r_j], \pi)$, an $n \times n$-matrix f, defining an endomorphism of P, and an element $a \in \mathsf{HH}_0(A)$. We have to show that

$$\theta(\sum_i \xi_i' \mathsf{str}(f) \xi_i'' a) = \theta(\mathsf{str}(f \star \pi D_n(\sum_i \xi_i' a \xi_i'')))$$

This immediately follows from the cyclic invariance of θ and the equality $\pi f = f$.

To prove that ι_P is a unital algebra homomorphism, observe that for any two elements $a, b \in \mathsf{HH}_0(A)$

$$\pi D_n(\sum_i \xi_i' a \xi_i'') \star \pi D_n(\sum_j \xi_j' b \xi_j'') = \pi D_n(\sum_j \xi_j' b \xi_j'') \pi D_n(\sum_i \xi_i' a \xi_i'')$$

$$= \pi^2 D_n(\sum_j \xi_j' b \xi_j'' \sum_i \xi_i' a \xi_i'') = \pi D_n(\sum_{i,j} \xi_i' a \xi_i'' \xi_j' b \xi_j'') \overset{by (5.3.1)}{=} \pi D_n(\sum_{i,j} \xi_j' \xi_i' a \xi_i'' b \xi_j'')$$

$$= \pi D_n(\sum_j \xi_j' (\sum_i \xi_i' a \xi_i'' b) \xi_j'')$$

That ι_P preserves the units is obvious.

What remains is to prove

Proposition 5.3.4. *The Cardy condition (5.1.3) is satisfied in our setting.*

Consider two modules, $P' = (\bigoplus_{j=1}^{n} A[r_j], \pi')$ and $P'' = (\bigoplus_{j=1}^{m} A[s_j], \pi'')$, and two endomorphisms, $f' \in \mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P', P')$ and $f'' \in \mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P'', P'')$. We need to prove that

$$\mathsf{str}_{O_{P'P''}}(L(f')R(f'')) = \theta(\sum_i \xi_i' \mathsf{str}(f') \xi_i'' \mathsf{str}(f'')) \tag{5.3.3}$$

Notice that the operators $L(f')$ and $R(f'')$ in the left-hand side are defined via the product $\star$. We would like to rewrite in terms of the product of morphisms (i.e. the matrix product):

$$\mathsf{str}_{\mathrm{Hom}_{\mathsf{Proj}_{gr} A}(P', P'')}(R(f')L(f'')) = \theta(\sum_i \xi_i' \mathsf{str}(f') \xi_i'' \mathsf{str}(f'')) \tag{5.3.4}$$

The first crucial observation we are going to use is that the right-hand side of (5.3.4) equals $\mathsf{tr}_A(L(\mathsf{str}(f''))R(\mathsf{str}(f')))$. We established this while proving Proposition 5.3.2. Thus, the Cardy condition is equivalent to

$$\mathsf{str}_{\mathrm{Hom}_{\mathsf{Proj}_{gr}A}(P',P'')}(R(f')L(f'')) = \mathsf{tr}_A(L(\mathsf{str}(f''))R(\mathsf{str}(f'))) \tag{5.3.5}$$

The second observation we will need is that the super-trace in the left-hand side can be computed over the larger space

$$\mathrm{Hom}_{\mathsf{Proj}_{gr}A}(\bigoplus_{j=1}^{n} A[r_j], \bigoplus_{j=1}^{m} A[s_j]) \supset \mathrm{Hom}_{\mathsf{Proj}_{gr}A}(P', P'')$$

This follows from the equalities $\pi' f' = f'$ and $f''\pi'' = f''$ which imply that the operator $R(f')L(f'')$ vanishes on the "orthogonal" complement of $\mathrm{Hom}_{\mathsf{Proj}_{gr}A}(P', P'')$ in the larger spaces. As a result, (5.3.5) should follow from its special case, namely, when $P' = \bigoplus_{j=1}^{n} A[r_j]$, $P'' = \bigoplus_{j=1}^{m} A[s_j]$, and f', f'' are arbitrary endomorphisms of P', P'', respectively. It remains to prove this special case.

Since both hand-sides of (5.3.5) are bilinear in f' and f'', it is enough to prove the formula in the case when both f' and f'' are matrices (of sizes $n \times n$ and $m \times m$, respectively) with a single non-zero component. It is also clear that both hand-sides of (5.3.5) vanish unless the non-zero components of f' and f'' are on the main diagonals. So let us assume that

$$f'|_{A[r_j]} = \begin{cases} a' & j = k \\ 0 & \text{otherwise} \end{cases} \qquad f''|_{A[s_j]} = \begin{cases} a'' & j = l \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathsf{str}_{\mathrm{Hom}_{\mathsf{Proj}_{gr}A}(P',P'')}(R(f')L(f'')) = \mathsf{str}_{\mathrm{Hom}_{\mathsf{Proj}_{gr}A}(A[r_k],A[s_l])}(R(a')L(a''))$$
$$= \mathsf{str}_{A[s_l - r_k]}(R(a')L(a'')) = (-1)^{s_l - r_k}\mathsf{tr}_A(R(a')L(a''))$$

On the other hand,

$$\mathsf{tr}_A(L(\mathsf{str}(f''))R(\mathsf{str}(f'))) = \mathsf{tr}_A(L((-1)^{s_l}a'')R((-1)^{r_k}a')) = (-1)^{s_l + r_k}\mathsf{tr}_A(L(a'')R(a'))$$

Proposition 5.3.4 is proved.

5.4 The Cardy condition vs. the HRR theorem

In this section, we will comment on the relationship between the Cardy condition in open-closed TFTs and the noncommutative HRR theorem obtained previously.

The results of the preceding section show that the Cardy condition in the open-closed TFT associated with a Frobenius algebra can be viewed as a generalization of the noncommutative HRR for modules over this algebra. Namely, the Cardy condition (5.3.3) is equivalent to (5.3.5), and the latter reduces to the noncommutative HRR formula in the special case $f' = 1_{P'}$, $f'' = 1_{P''}$. In view of this observation, it is natural to look for a similar field-theoretic interpretation of the general noncommutative HRR formula (for an arbitrary Calabi-Yau DG algebra). Below, we will describe the relevant class of topological field theories and formulate a conjecture which, we believe, should be the key step in understanding physicists' thesis "The Cardy condition is a generalized Riemann-Roch theorem".

The aforementioned class of field theories consists of the so-called open-closed *topological conformal field theories* (TCFTs). A detailed definition of these, along with a historical overview of the subject, can be found in [15]. The conjecture that we want to formulate is related to certain *closed* TCFTs, so let us briefly recall the definition of the closed TCFT sector.

Fix a non-negative integer d. A degree d closed TCFT is defined as follows. Let $\mathcal{M}(n,m) = \bigcup_{g\geq 0} \mathcal{M}_g(n,m)$ denote the moduli space of Riemann surfaces with n incoming and m outgoing boundaries (g denotes the genus). Fix a graded vector space $C_\bullet$. By definition, $C_\bullet$ carries a structure of degree d TCFT if one has a collection of linear maps

$$\mathrm{H}_\bullet(\mathcal{M}(n,m)) \otimes C_\bullet^{\otimes n} \to C_\bullet^{\otimes m}, \quad n \geq 1,\ m \geq 0 \tag{5.4.1}$$

(here $\mathrm{H}_\bullet$ in the left-hand side denotes the singular homology) satisfying the following conditions:

(1) the maps are compatible with the operation

$$\mathcal{M}(m,l) \times \mathcal{M}(n,m) \to \mathcal{M}(n,l)$$

of gluing two surfaces along the boundary components and the operation

$$\mathcal{M}(n,m)\times\mathcal{M}(p,q)\to\mathcal{M}(n+p,m+q)$$

of taking the disjoint union of surfaces;

(2) elements of $\mathrm{H}_\bullet(\mathcal{M}_g(n,m))$ act by operators of degree $d(2-2g-n-m)$.

Important examples of closed TCFTs come from proper Calabi-Yau DG algebras (see Section 5.2). Recall that a d-dimensional proper Calabi-Yau DG algebra is a pair (A,θ) where A is a proper DG algebra and θ is a degree $-d$ trace on A. One has [15, 38]:

For any d-dimensional Calabi-Yau DG algebra A the Hochschild homology $\mathsf{HH}_\bullet(A)$ carries a canonical structure of degree d closed TCFT.[4]

One immediate consequence of this result is that there is a natural degree 0 pairing – let us denote it by $\langle\ ,\ \rangle_\theta$ – on the Hochschild homology of a d-dimensional Calabi-Yau DG algebra given by the surface with two incoming boundary circles and no outgoing boundaries (it is a generator of $\mathrm{H}_0(\mathcal{M}_0(2,0))$). The following conjecture relates this pairing to the one constructed in the present work[5]:

Conjecture. *For any Calabi-Yau DG algebra A, the pairing $\langle\ ,\ \rangle_\theta$ coincides with the pairing (3.2.3), i.e. for any $a,b\in\mathsf{HH}_\bullet(A)$*

$$\langle\, a,b\,\rangle_\theta=\langle\, a,b^\vee\,\rangle, \tag{5.4.2}$$

where $^\vee$ is the isomorphism $\mathsf{HH}_\bullet(A)\to\mathsf{HH}_\bullet(A^{\mathsf{op}})$ defined by (3.2.6).

Notice that we already verified this conjecture in the case of Frobenius algebras while proving Proposition 5.3.2, and it was this statement that eventually allowed us to deduce the Cardy condition in that setting (compare the formulas (5.3.4) and (5.3.5)!). We believe

[4] In fact, a much stronger result is obtained in [15, 38], namely, that the action (5.4.1) exists on the level of complexes that compute the singular homology of the moduli spaces and the Hochschild homology of the algebra.

[5] This conjecture was suggested to the author by Y. Soibelman and K. Costello.

that the general case of the conjecture will play the same role in the setting of open-closed TCFTs associated with general Calabi-Yau algebras and categories.

Notice that the conjecture would generate another important result. Namely, together with Theorem 4.3.2, it would imply the following statement conjectured in [38, Section 11.6]:

Corollary. *For any homologically smooth Calabi-Yau DG algebra A, the pairing* $\langle\ ,\ \rangle_\theta$ *is non-degenerate.*

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