Void Fraction as a Function of Depth and Pressure Drops of Packed Beds of Porous Media Formed by Granular Materials

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Abstract. *A mathematical equation that describes the void fraction as a function of location, bulk density at the surface, bulk density at the location of interest in the bed, maximum bulk density, bin geometry, friction properties of the grain, and the ratio of lateral to vertical pressures of the grain in a deep packed bed of bulk solids was developed. Variation in the void fraction caused variation in the pressure drop per unit of bed depth. Since bulk density and void fraction are functionally related to each other, the void fraction was expressed as a mathematical function of bed depth, and the model was verified using fundamentals of mathematics. In order to develop the model, Ergun's equation was used and integrated over the bed depth, substituting a void fraction that varied with the bed depth. Model results were compared to Shedd's data. Agreement between Shedd's data and this model was good to a bed depth of 10 m. For bed depths greater than 10 m, results from this model gave much greater pressure drop values than were given by Shedd. The model showed that pressure drop per unit of bed depth increases exponentially with the bed depth.*

Keywords. *Granular material, Packed bed, Pressure depth, Void fraction.*

Defining porous media is extremely difficult, but we can somehow describe them (Scheidegger, 1960). We can say that they are solid bodies containing pores. But pores cannot be defined in exact terms. The description of porous media is rather intuitive. A body is porous if it contains voids, capillaries, or free space that is interconnected or non-interconnected. Accordingly, bulks of cereal grains, such as corn, wheat, rice, sorghum, etc., are examples of porous media. Flow of fluids through porous media encompasses a fluid and an ensemble of the porous medium and the interaction between the two. As a result, the flow characteristics of a fluid flowing through porous media will be different from the flow through an empty conduit.

Understanding and estimating pressure drops of fluids through packed beds of granular materials are extremely important for many engineering applications. In chemical engineering, pressure drops are important in reactor design, diffusion analysis, heat and mass transfer studies, power and energy calculations, and many other applications. In biological and food systems engineering, knowledge of pressure drops is important in aeration and drying studies and system design. In particle technology, pressure drops are used to characterize particles and find their aerodynamic and other properties. These are only a few of the applications of pressure drops in science and technology.

The subject of fluid dynamics in cereal grains is concerned with the void space of the porous medium as filled with the fluid. Therefore, applying the mechanics of fluids that fill the void spaces is necessary. The most important and fundamental law of flow through homogenous porous media is that of Darcy (1856). Darcy's law states that, macroscopically, the velocity of a fluid flowing through a porous medium is directly proportional to the pressure gradient acting on the fluid. Darcy's law holds only for the viscous range of the flow. For liquids at high velocities and for gases at very low and high velocities, Darcy's law becomes invalid (Scheidegger, 1960). It is rather restricted in its usefulness.

The first reported work of any importance in the area of fluid flow through packed materials is that of Zeisberg (1919). His conclusions were: (1) the free space naturally varied with the type of packing and with the manner of disposing it in the tower; (2) the resistance offered by a tower packing to gas flow was, of course, dependent on both the free space and the surface exposed, but just what effect each might have it was impossible to predict; and (3) the resistance was proportional to the square of the velocity of flow.

Blake (1922) applied the method of dimensional homogeneity to determine the resistance of packing to fluid flow. Kozeny (1927) advanced the so-called hydraulic radius theory, which treats the porous medium as a bundle of capillary tube of equal length. These tubes are not necessarily of circular cross-section. Carman (1937) studied fluid flow through granular beds to verify Kozeny's theory experimentally. The theory was applied for the viscous range of flow only. Ergun (1952) used a high-temperature oven coke, Eagle coke, and glass, lead, and copper spheres to develop a semi-empirical equation that was applicable for both viscous and turbulent flow.

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MODEL DEVELOPMENT AND VALIDATION

Void fraction is defined as the air space available in a unit volume of granular material. Let us assume a granular material of bulk density γ_z and true density γ_T and a pack void fraction of ε_z, where the subscript z signifies the dependence of the property on the bed depth (z, m). True density remains constant, while bulk density and the void fraction vary with the bed depth. The void fraction is expressed by equation 1:

$$\varepsilon_z = 1 - \frac{\gamma_z}{\gamma_T} \qquad (1)$$

where

ε_z = void fraction (dimensionless)
γ_z = bulk density (kg m^{-3})
γ_T = true density (kg m^{-3}).

Beds of granular materials have a tendency to pack, especially if the beds are deep and wide. Haque (2010) discovered that the bulk density of granular materials increases exponentially at the same rate as the vertical loadings exerted by the materials. He developed the following bulk density function:

$$\gamma_z = \gamma_0 \left[1 + \theta\left(1 - e^{-\alpha z}\right)\right] \qquad (2)$$

where

γ_0 = reference bulk density (kg m^{-3}), e.g., bulk density of the bulk solid at the bed surface ($z = 0$), i.e., the laboratory measurement of bulk density
γ_z = bulk density (kg m^{-3}) of the bulk solid at bed depth z (m)
γ_m = maximum bulk density (kg m^{-3}); $\gamma_m = \gamma_0[1 + \theta]$ when z is a large value
θ = $\dfrac{(\gamma_m - \gamma_0)}{\gamma_0}$, a constant (dimensionless), known as the compaction factor, that is dependent on the properties of the bulk solid, the storage container, and other external factors such as filling method, vibration, etc.
α = $\dfrac{\gamma_0 k \mu}{\gamma_m R_h}$, a constant (1/m) that is dependent on the properties of the bulk solid (including minimum and maximum bulk density), the storage container, and other external factors such as filling method, vibration, etc.
k = ratio of lateral to vertical pressure of the bulk solid in the bed (dimensionless)
μ = friction factor of the bulk material (dimensionless)
R_h = hydraulic radius of the storage container (m).

Replacing the bulk density in equation 1 with the value given in equation 2 yields equation 3:

$$\varepsilon_z = 1 - \frac{\gamma_0}{\gamma_T}\left[1 + \theta\left(1 - e^{-\alpha z}\right)\right] \qquad (3)$$

$$\varepsilon_z = \left(\frac{\gamma_T - \gamma_0}{\gamma_T}\right) - \frac{\gamma_0}{\gamma_T}\theta\left(1 - e^{-\alpha z}\right) \qquad (4)$$

$$\varepsilon_z = \left(\frac{\gamma_T - \gamma_0}{\gamma_T}\right) - \frac{(\gamma_m - \gamma_0)}{\gamma_T}\left(1 - e^{-\alpha z}\right) \qquad (5)$$

It is evident from equation 5 that the void fraction has a maximum and minimum value. The maximum value occurs at the surface of the bed, i.e., $z = 0$, and the minimum value occurs at a large depth. Let ε_m be the maximum value of the void fraction, and then the equation 5 can be rewritten as equation 6:

$$\varepsilon_z = \varepsilon_m\left[1 - \tau\left(1 - e^{-\alpha z}\right)\right] \qquad (6)$$

where

$\varepsilon_m = \dfrac{(\gamma_T - \gamma_0)}{\gamma_T}$, (dimensionless)

$\tau = \dfrac{(\gamma_m - \gamma_0)}{(\gamma_T - \gamma_0)}$, (dimensionless)

$\alpha = \dfrac{\gamma_0 k \mu}{\gamma_m R_h}$, (1/m).

Ergun's (1952) pressure drop, ΔP in N m^{-2} (Pa) for depth of bed L (m) is given by the following equation:

$$\frac{\Delta P}{L} = \frac{150\mu_0(1-\varepsilon)^2 V}{\varphi^2 d_p^2 \varepsilon^3} + \frac{1.75\rho(1-\varepsilon)V^2}{\varphi d_p \varepsilon^3} \qquad (7)$$

where

μ_0 = viscosity of fluid flowing through the bed (N-s m^{-2}, Pa-s, or kg m^{-1} s^{-1})
ρ = density of fluid flowing through the bed (kg m^{-3})
V = superficial or nominal fluid velocity based on the whole cross-section of flow (m s^{-1})
ε = void fraction (dimensionless constant)
φ = shape factor of the granular material (dimensionless)
d_p = equivalent particle diameter (m).

Rearranging equation 7 yields:

$$\frac{\Delta P}{L} = \frac{150\mu_0 V}{\varphi^2 d_p^2}\left[\varepsilon^{-3} - 2\varepsilon^{-2} + \varepsilon^{-1}\right]$$

$$+ \frac{1.75\rho V^2}{\varphi d_p}\left[\varepsilon^{-3} - \varepsilon^{-2}\right] \qquad (8)$$

Now replacing ε, a constant in equations 7 and 8, with ε_z, a variable given by equation 1, and writing equation 8 in differential form and integrating between 0 to z yields:

$$\Delta P\int_0^z dp = C_1\left[\int_0^z \varepsilon_z^{-3} dz - 2\int_0^z \varepsilon_z^{-2} dz + \int_0^z \varepsilon_z^{-1} dz\right]$$

$$+ C_2\left[\int_0^z \varepsilon_z^{-3} dz - \int_0^z \varepsilon_z^{-2} dz\right]$$

where $C_1 = \dfrac{150\mu_0 V}{\varphi^2 d_p^2}$ and $C_2 = \dfrac{1.75\rho V^2}{\varphi d_p}$. Therefore:

$$\Delta P =$$
$$(C_1 + C_2)\int_0^z \varepsilon_z^{-3} dz - (2C_1 + C_2)\int_0^z \varepsilon_z^{-2} dz + C_1\int_0^z \varepsilon_z^{-1} dz \quad (9)$$

Solving for the three integrals on the right side of equation 9 proceeds as follows. For the first integral:

$$\int_0^z \varepsilon_z^{-3} dz = \varepsilon_m^{-3} \int_0^z \left[1 - \tau\left(1 - e^{-\alpha z}\right)\right]^{-3} dz$$

Substituting $x = 1 - \tau(1 - e^{-\alpha z})$ when $z = 0$, $x = 1 = a$, and $z = z$, then $x = 1 - \tau(1 - e^{-\alpha z}) = b$.

$dz = \dfrac{dx}{\alpha(1 - \tau - x)}$, the first integral becomes:

$$\int_a^b \frac{x^{-3}}{\alpha(1 - \tau - x)} dx$$

Now, substituting $du = x^{-3} dx$ and $v = \dfrac{1}{(1 - \tau - x)}$, the solution of the first integral on the right side of equation 9 is:

$$\frac{1}{4\varepsilon_m^3 \alpha \tau} \left\{ \frac{150\mu_0 V}{\varphi^2 d_p^2} + \frac{1.75\rho V^2}{\varphi d_p} \right\} \left[\frac{\left\{1 - \tau\left(1 - e^{-\alpha z}\right)^{-2}\right\}}{e^{-\alpha z}} - 1 \right] \quad (10)$$

Likewise, the second integral on the right side of equation 9 is:

$$\frac{-1}{2\varepsilon_m^2 \alpha \tau} \left\{ \frac{300\mu_0 V}{\varphi^2 d_p^2} + \frac{1.75\rho V^2}{\varphi d_p} \right\} \left[\frac{\left\{1 - \tau\left(1 - e^{-\alpha z}\right)^{-1}\right\}}{e^{-\alpha z}} - 1 \right] \quad (11)$$

The solution of the third integral on the right side of equation 9 is:

$$\frac{-1}{2\varepsilon_m \alpha \tau} \left\{ \frac{150\mu_0 V}{\varphi^2 d_p^2} \right\} \left[\frac{\ln\left\{1 - \tau\left(1 - e^{-\alpha z}\right)\right\}}{e^{-\alpha z}} \right] \quad (12)$$

Adding the solutions of all three integrals on the right side of equation 9 (i.e., eqs. 10, 11, and 12) yields:

$\Delta P =$

$$\frac{1}{4\varepsilon_m^3 \alpha \tau} \left\{ \frac{150\mu_0 V}{\varphi^2 d_p^2} + \frac{1.75\rho V^2}{\varphi d_p} \right\} \left[\frac{\left\{1 - \tau\left(1 - e^{-\alpha z}\right)^{-2}\right\}}{e^{-\alpha z}} - 1 \right]$$

$$- \frac{1}{2\varepsilon_m^2 \alpha \tau} \left\{ \frac{300\mu_0 V}{\varphi^2 d_p^2} + \frac{1.75\rho V^2}{\varphi d_p} \right\} \left[\frac{\left\{1 - \tau\left(1 - e^{-\alpha z}\right)^{-1}\right\}}{e^{-\alpha z}} - 1 \right]$$

$$- \frac{1}{2\varepsilon_m \alpha \tau} \left\{ \frac{150\mu_0 V}{\varphi^2 d_p^2} \right\} \left[\frac{\ln\left\{1 - \tau\left(1 - e^{-\alpha z}\right)\right\}}{e^{-\alpha z}} \right] \quad (13)$$

This equation could be used to calculate pressure drops from $z = 0$ to any depth z for deep beds of compactable granular materials using relevant properties of the granular material, the geometry of the storage container, and fluid properties. The equation combines both viscous and turbulent components of the flow.

RESULTS AND DISCUSSION

The Ergun equation (Ergun, 1952) has some limitations in its applicability for granular materials that are compactable, such as cereal grains, because the equation assumes that the void space remains constant in the entire bed. In addition, the equation does not include many important variables of granular products, such as the friction factor of the grain and the size of the storage container. Both the friction factor of the grain and the size of the storage container are known to contribute to the compaction of grain in beds, consequently causing a significant increase in the rate of pressure drop as the depth increases. Equation 13 has been developed to include many factors, including not only the fluid properties of the flow through the granular material, such as viscosity, density, and velocity, but also the properties of the granular material and the storage container's size and geometry. The properties of the granular material include particle size, shape, non-compacted and compacted bulk densities, true density, and frictional characteristics. Overbearing loads imparted by grain are also included in the equation.

Calculations using equation 13 have been performed to arrive at figures 1 through 4 for wheat, corn, soybeans, and sorghum, respectively. All calculations were based on ambient airflow through grain stored in a bin of 9.144 m (30 ft) diameter and 30.48 m (100 ft) height with a hydraulic radius of 2.286 m (7.5 ft). The coefficient of friction of the grain was 0.4, and the ratio of horizontal to vertical loads was 0.5. The viscosity, density, and superficial velocity of the air were assumed to be 0.00001821 kg m^{-1} s^{-1}, 1.205 kg m^{-3}, and 0.0431292 m s^{-1} (1/10 cfm per bu), respectively. Additional assumptions and statistics used to calculate the values for the four grains in figures 1 through 4 were gathered from various literature sources, including *ASABE Standards* (2010), Karimi et al. (2009), Mwithiga and Sifuna (2005), Tavakoli et al. (2009), Deshpande et al. (1993), Chang (1988), and Pearson and Brabec (2006) and are listed in table 1.

Shedd's data (Shedd, 1953) are almost universally used as a basis for determining pressure drops in various grains. The values calculated using equation 13 were compared with data presented by Shedd, which were based on a loose-fill packed bed. This comparison was done by estimating a pressure drop value from the plots presented in ASABE Standard D272.3 (*ASABE Standards*, 1996). Since Shedd's pressure drop was given per unit of bed depth, that value was multiplied by the depths in figures 1 through 4. The figures show that the pressure drop calculated using equation 13 conforms relatively well with Shedd's data if the bed depth is about 10 m or less, but the two methods differ drastically for deeper bins. Shedd's data were based on a loose-fill packed bed and as such considerably underestimate the actual pressure drops in storage bins. Design engineers use somewhat arbitrary correction factors (e.g., 1.5) to Shedd's data to conform to practical needs caused by compaction (Hall, 1957). The

Table 1. Bulk densities, particle diameters, and shape factors of four common grains.

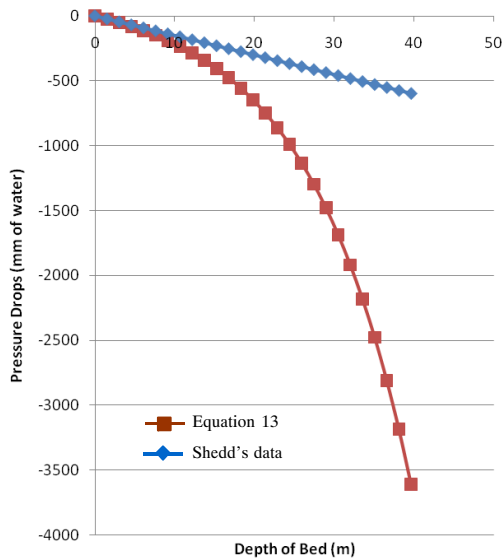|	Wheat	Corn	Soybeans	Sorghum
γ_0 (kg m^{-3})	772	721	772	721
γ_m (kg m^{-3})	856	801	840	830
γ_T (kg m^{-3})	1285	1186	1200	1318
d_p (m)	0.0060	0.00823	0.0066	0.00394
φ (unitless)	0.5793	0.6200	0.8102	0.7370

Figure 1. Pressure drops of wheat in a 9.144 m diameter × 30.48 m high (30 ft × 100 ft) grain bin using equation 13 vs. Shedd's data.
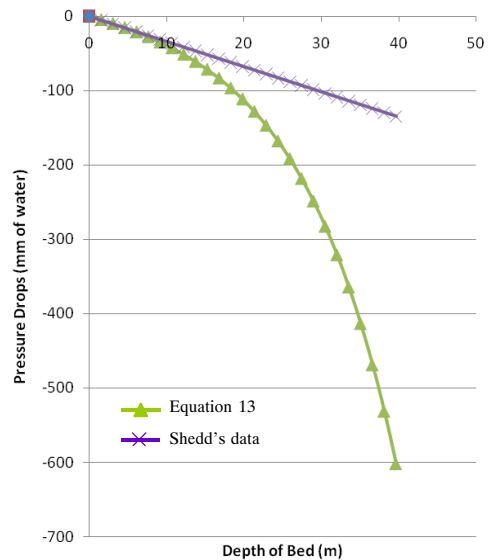

Figure 3. Pressure drops of soybeans in a 9.144 m diameter × 30.48 m high (30 ft × 100 ft) grain bin using equation 13 vs. Shedd's data.

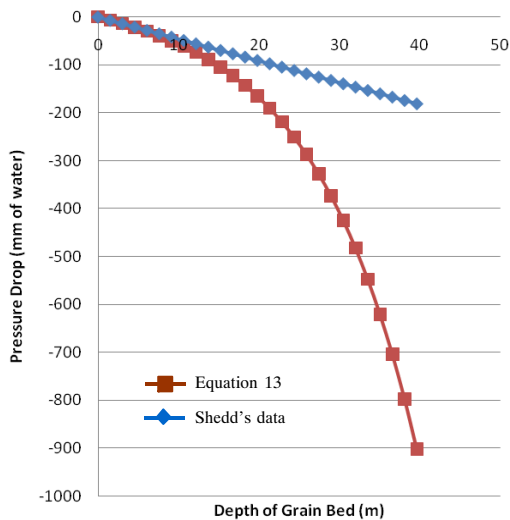

Figure 2. Pressure drops of corn in a 9.144 m diameter × 30.48 m high (30 ft × 100 ft) grain bin using equation 13 vs. Shedd's data.
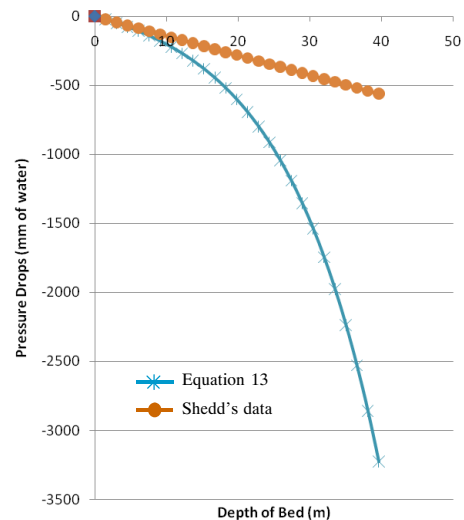

Figure 4. Pressure drops of sorghum in a 9.144 m diameter × 30.48 m high (30 ft × 100 ft) grain bin using equation 13 vs. Shedd's data.

Ergun equation is not applicable for deep beds of compactable granular material. The equation developed in this study provides a theoretical method of calculating pressure drops that represent practical situations more accurately.

It is evident from the figures that the pressure drop per unit of bed depth is lowest near the surface and generally agrees with the values given by Shedd. However, as the bed depth increases, the pressure drop per unit of depth also progressively increases, but the rate of increase decreases and eventually becomes constant. Apparently, the total pressure drop across the bed depends on the length of the bed: the deeper the bed, the greater the pressure drop per unit of bed depth. This is in drastic contrast with Shedd's data.

Since corn is of larger diameter than wheat and sorghum and compacts less than those grains, the difference in pressure drops between Shedd's data and the values calcu-

lated using equation 13 is also relatively less. As expected, wheat and sorghum display the greatest pressure drops per unit of bed depth along with the highest variation between Shedd's values and this study. Soybeans compact less because of their high shape factor value (almost spherical) and consequently have the lowest pressure drop and the lowest variation with respect to Shedd's data. Among the four grains, as with Shedd's data, wheat displayed the greatest pressure drops, followed by sorghum, corn, and soybeans.

This study resulted in a tool to more accurately estimate the pressure drops across any applicable bed depth rather than assuming that the pressure drop per unit of bed depth remains constant throughout the bed. The latter is not true because increases in grain compaction deep inside the bed increase the pressure drop as well.

Conclusion

A mathematical model has been developed for determining void fractions at different depths of granular materials in deep beds. The model follows an exponential function exactly with the same rate of change as bulk density. The void fraction has a maximum value at the surface of the bed and a minimum value at the bottom of the bed.

The equation shows that the pressure drop per unit of bed depth is not equal at all depths but rather increases as the depth increases. Comparison with Shedd's data reveals that for shallow beds (less than 10 m depth) the pressure drop calculations for both methods generally agree; however, for deep beds, Shedd's data grossly underestimate the values calculated with the model developed in this study. These two models will allow scientists and engineers to design systems and processes for flow, diffusion, and chemical reactor problems through packed beds.

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