THE LAW OF THE ITERATED LOGARITHM FOR TAIL SUMS

by

SANTOSH GHIMIRE

M.Sc., Tribhuvan University, Nepal, 2001
M.S., Kansas State University, USA, 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
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Abstract

The main purpose of this thesis is to derive the law of the iterated logarithm for tail sums in various contexts in analysis. The various contexts are sums of Rademacher functions, general dyadic martingales, independent random variables and lacunary trigonometric series. We name the law of the iterated logarithm for tail sums as tail law of the iterated logarithm. We first establish the tail law of the iterated logarithm for sums of Rademacher functions and obtain both upper and lower bound in it. Sum of Rademacher functions is a nicely behaved dyadic martingale. With the ideas from the Rademacher case, we then establish the tail law of the iterated logarithm for general dyadic martingales. We obtain both upper and lower bound in the case of martingales. A lower bound is obtained for the law of the iterated logarithm for tail sums of bounded symmetric independent random variables. Lacunary trigonometric series exhibit many of the properties of partial sums of independent random variables. So we finally obtain a lower bound for the tail law of the iterated logarithm for lacunary trigonometric series introduced by Salem and Zygmund.

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Approved by:

Major Professor
Prof. Charles N. Moore

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Dedication

I would like to dedicate this to:

My father (late) **Madhusudan Upadhayaya**

My mother **Jyanti Devi Ghimire**

My daughter **Subi Ghimire.**

Chapter 1

Introduction

This chapter begins with some useful definitions and notation which will be repeatedly used in later chapters. We state some useful results and then discuss the origin of law of the iterated logarithm.

1.1 Dyadic martingales.

Before we define dyadic martingales, we discuss the meaning of the word 'martingale'. Originally martingale meant a strategy for betting in which you double your bet every time you lose. Let us consider a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails. The strategy is that the gambler doubles his bet every time he loses and continues the process, so that the first win would recover all previous losses plus win a profit equal to the original stake. This process of betting can be represented by a sequence of functions which is an example of dyadic martingale. Since a gambler with infinite wealth will, almost surely, eventually flip heads, the martingale betting strategy can be seen as a sure thing. Of course, no gambler in fact possesses infinite wealth, and the exponential growth of the bets will eventually bankrupt those who choose to use the martingale strategy.

A dyadic interval of the unit cube $[0,1)$ is of the form $Q_{nj} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ for $n, j \in \mathbb{Z}$. We reserve the symbol Q_n to denote a generic dyadic interval of length $\frac{1}{2^n}$. Let $\mathfrak{F}_n$ denote the $\sigma-$algebra generated by the dyadic intervals of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ on $[0, 1)$ and let $E(f_{n+1}|\mathfrak{F}_\mathfrak{n})$ denote the conditional expectation of f_{n+1} on $\mathfrak{F}_n$ which is defined as,

$$E(f_{n+1}|\mathfrak{F}_\mathfrak{n})(x) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy, \quad x \in Q_n.$$

Definition 1 (Dyadic martingales). *A dyadic martingale is a sequence of integrable functions $\{f_n\}_{n=0}^{\infty}$, with $f_n : [0, 1) \to \mathbb{R}$ such that for every n, f_n is $\mathfrak{F}_n-$measurable and $E(f_{n+1}|\mathfrak{F}_\mathfrak{n}) = f_n$ for all $n \geq 0$. The sequence $\{f_n\}_{n=0}^{\infty}$ is called a dyadic submartingale (resp. supermartingale) if we replace $=$ by $\geq$ (resp. $\leq$) in the expectation condition.*

Here $\mathfrak{F}_0 = \{[0, 1), \phi\}$, $\mathfrak{F}_1 = \{[0, 1), \phi, [0, 1/2), [1/2, 1)\}$ and so on. Hence f_n is measurable with respect to $\mathfrak{F}_n$ means for all $a \in \mathbb{R}$, the set $\{x : f_n(x) > a\}$ belongs to $\mathfrak{F}_n$. Consequently, the function f_n is constant on each of the n^{th} generation dyadic intervals Q_n. The expectation condition tells us that the f_n is the average of f_{n+1} on Q_n. Moreover, the existence of the conditional expectation can be justified by Radon-Nikodym Theorem.

If we think of $\{f_n\}_{n=0}^{\infty}$ as the "fortune of gambler" at the instant n of a game, then the first condition simply says the trivial fact that the result of the game totally determines the state of fortune at any instant. The second condition expresses that the game is 'fair' in the sense that the expected fortune after any trial must be same as that of the fortune before the trial.

1.2 Notation

For a dyadic martingale we have the following standard associated functions.

(i) Maximal function,

$$f_m^* = \sup_{1\leq k\leq m} |f_k|, \quad f^* = \sup_{1\leq k<\infty} |f_k|.$$

(ii) Martingale difference sequence, $\{d_k\}_1^{\infty}$, where $d_k(x) = f_k(x) - f_{k-1}(x)$.

(iii) Martingale square function or Quadratic characteristics,

$$S_n^2 f(x) = (S_n f(x))^2 = \sum_{k=1}^{n} d_k^2(x).$$

$$S^2 f(x) = (Sf(x))^2 = \sum_{k=1}^{\infty} d_k^2(x).$$

(iv) Martingale tail square function,

$$S_n'^2 f(x) = (S_n' f(x))^2 = \sum_{k=n+1}^{\infty} d_k^2(x).$$

We note that $\int_0^1 d_n(x)dx = 0$. For this let Q_{nj} be a dyadic interval of length $\frac{1}{2^n}$. Then we have,

$$\begin{aligned}\int_0^1 d_n(x)dx &= \sum_{j=1}^{2^n-1} \int_{Q_{nj}} d_n(x)dx \\ &= \sum_{j=1}^{2^n-1} \int_{Q_{nj}} [f_n(x) - f_{n-1}(x)]dx\end{aligned}$$

Using the fact that f_{n-1} is constant on Q_{nj}, we have

$$\begin{aligned}\int_0^1 d_n(x)dx &= \sum_{j=1}^{2^n-1} \left[\int_{Q_{nj}} f_n(x)dx - f_{n-1}(x)|Q_{nj}| \right] \\ &= \sum_{j=1}^{2^n-1} \left[|Q_{nj}| \frac{1}{|Q_{nj}|} \int_{Q_{nj}} f_n(x)dx - f_{n-1}(x)|Q_{nj}| \right] \\ &= \sum_{j=1}^{2^n-1} [f_{n-1}(x)|Q_{nj}| - f_{n-1}(x)|Q_{nj}|] \\ &= 0.\end{aligned}$$

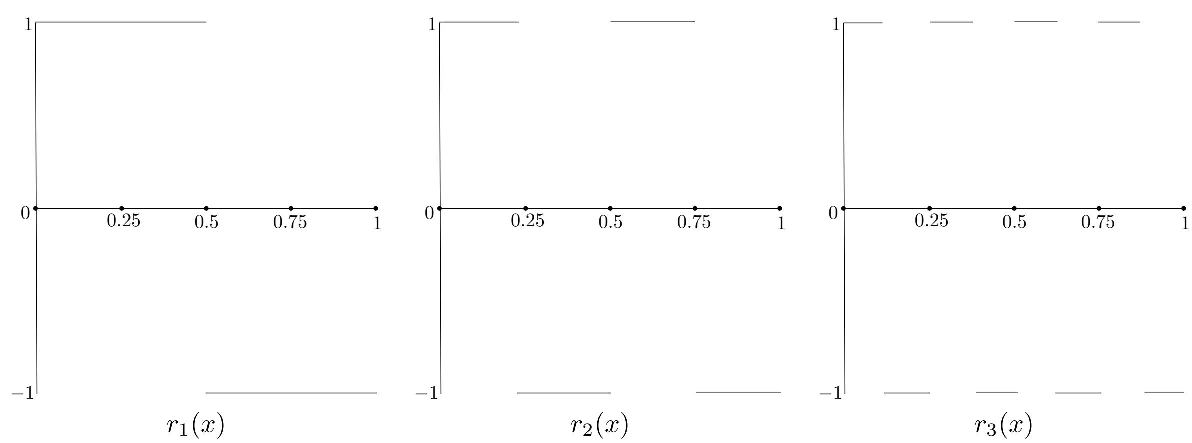
Also we have, $f_n(x) = \sum_{k=1}^{n} d_k(x) + f_0$. The martingale square function is a local version of variance and can also be understood as a discrete counterpart of the area function in Harmonic Analysis. They play an important role in the study of asymptotic behavior of dyadic martingales. We will see that the asymptotic behavior of a dyadic martingale is governed by the size of its quadratic variation. From the definition, we note that for any $x, y \in Q_n$, we have $S_n^2 f(x) = S_n^2 f(y)$. But the martingale tail square function, $S_n'^2 f(x)$ may not be equal to $S_n'^2 f(y)$.

1.3 Examples

Here we give some examples of dyadic martingales.

Example 1. The functions $\{r_k\}_{k=1}^{\infty}$ defined on $[0,1)$ by $r_k(x) = \text{sgn}(\sin 2^k \pi x)$ where sgn is given by $\text{sgn}(x) = 1$ for $x \geq 0$ and $\text{sgn}(x) = -1$ for $x < 0$ are called the Rademacher functions.

Figure 1.1: *Rademacher functions*



Here r_k alternates +1 and -1 on the dyadic intervals of the generation k as shown in Figure 1.1. Moreover, $r_k's$ are independent, identically distributed random variables with zero mean and variance one. Define $f_n = \sum_{k=1}^{n} a_k r_k$ where $\{a_k\}$ is a sequence of real numbers. Then $\{f_n\}$ is a dyadic martingale.

Example 2. Let $f \in L^1[0,1)$ and Q_n be a dyadic interval of length $\frac{1}{2^n}$ on $[0,1)$. Define $f_n(x) = \frac{1}{|Q_n|} \int_{Q_n} f(x)dx$, $x \in Q_n$. Then $\{f_n\}_{n=1}^{\infty}$ is a dyadic martingale on $[0,1)$.

1.4 Useful results

In this section, we state some useful results which will be frequently used in later chapters.

Lemma 2. *If $\{E_n\}$ is a sequence of sets on a $\sigma-$algebra $\mathfrak{F}$ with the property that $E_n \subset E_{n+1}$ for all n and $E = \bigcup_{n=1}^{\infty} E_n$, then $\lim_{n\to\infty} |E_n| = |E|$.*

Proof: Let us define $\{A_n\}$ as follows:

$$A_1 = E_1 \quad \text{and} \quad A_n = E_n - E_{n-1} \quad \text{for} \quad n = 2, 3, \cdots .$$

Clearly $A_n \in \mathfrak{F}$ and $A_i \cap A_j = \phi$ for all $i \neq j$. Moreover we have $E_n = A_1 \cup A_2 \cup \cdots \cup A_n$ and $E = \cup_{i=1}^{\infty} A_i$. Using the disjointness of $A_i's$ we have,

$$|E_n| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| \quad \text{and} \quad |E| = |\bigcup_{i=1}^{\infty} A_i| = \sum_{i=1}^{\infty} |A_i|.$$

Hence we have

$$\lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \sum_{i=1}^{\infty} |A_i| = \sum_{i=1}^{\infty} |A_i| = |E|.$$

Theorem 3. *For a dyadic martingale, we have*

$$\{x : f^*(x) < \infty\} \overset{a.s}{=} \{x : Sf(x) < \infty\} \overset{a.s.}{=} \{x : \lim f_n(x) \quad \textit{exists}\}$$

where $\overset{a.s.}{=}$ means that the sets are equal up to sets of measure zero.

Proof: For the proof, see [2].

Lemma 4 (Borel-Cantelli)**.** *If $\{A_n\}$ is a sequence of events and $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\{A_n \ \ i.o.\}) = 0$.*

Proof: We first note that,

$$\{A_n \ \ i.o.\} = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is the event which occur if and only if infinite number of events A_n occur.

$$\begin{aligned} P(\{A_n \ \ i.o.\}) &= P(\limsup_{n\to\infty} A_n) \\ &= P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) \\ &= \lim_{n\to\infty} P(\bigcup_{k=n}^{\infty} A_k) \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) \\ &= 0. \end{aligned}$$

Remark 1. The Borel-Cantelli Lemma can also be stated as:

"Let $\{E_k\}$ be a sequence of measurable sets in X, such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all $x \in X$ lie in at most finitely many of the sets E_k."

Lemma 5 (Borel-Cantelli, General version)**.** *If $\{A_n\}$ is a sequence of independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\{A_n \ \ i.o.\}) = 1$.*

Proof: We have,

$$\begin{aligned} 1 - P(\{A_n \ \ i.o.\}) &= P(\{A_n \ \ i.o.\}^c) \\ &= P(\{A_n \ \ i.o.\}) \\ &= P(\{\textstyle\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\}^c) \\ &= P(\textstyle\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c) \\ &= \lim_{n\to\infty} P(\textstyle\bigcap_{k=n}^{\infty} A_k^c). \end{aligned}$$

Clearly $\{A_k^c\}$ is a sequence of independent events as $\{A_k\}$ is independent. Then,

$$\begin{aligned} P(\textstyle\bigcap_{k=n}^{\infty} A_k^c) &= \lim_{N\to\infty} P(\textstyle\bigcap_{k=n}^{N} A_k^c) \\ &= \lim_{N\to\infty} \prod_{k=n}^{N} P(A_k^c) \\ &= \lim_{N\to\infty} \prod_{k=n}^{N} [1 - P(A_k)] \\ &\leq \lim_{N\to\infty} \prod_{k=n}^{N} \exp(-P(A_k)) \\ &= \lim_{N\to\infty} \exp\left(-\sum_{k=n}^{N} P(A_k)\right) \\ &= \exp\left(-\sum_{k=n}^{\infty} P(A_k)\right) \\ &= 0. \end{aligned}$$

Hence we have $1 - P(\{A_n \ \ i.o.\}) = 0$. Consequently, $P(\{A_n \ \ i.o.\}) = 1$.

Lemma 6 (Lévy's inequality)**.** *If $X_1, X_2, \cdots X_n$ be independent and symmetric random variables. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for all $\lambda > 0$,*

$$P\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right) \leq 2P(|S_n| \geq \lambda).$$

$$P\left(\max_{1\leq k\leq n} |X_k| \geq \lambda\right) \leq 2P(|S_n| \geq \lambda).$$

Proof: Define

$$A_k = \left\{x : \max_{1\leq j<k} S_j(x) < \lambda \leq S_k(x)\right\}$$

for $1 \leq k \leq n$. This means k is the smallest index for which $S_k(x) \geq \lambda$. Then using the fact that $X_1, X_2, \cdots, X_k$ independent from $X_{k+1}, X_{k+2}, \cdots, X_n$ we have,

$$\begin{aligned}
P\left(\{x : S_n(x) \geq \lambda\}\right) &= \sum_{k=1}^{n} P\left(A_k \cap \{x : S_n(x) \geq \lambda\}\right) \\
&\geq \sum_{k=1}^{n} P\left(A_k \cap \{x : S_n(x) \geq S_k(x)\}\right) \\
&= \sum_{k=1}^{n} P(A_k)P(S_n - S_k \geq 0) \\
&\geq \sum_{k=1}^{n} P(A_k)/2 \quad \text{using symmetry} \\
&= \frac{1}{2}P\left(\max_{1\leq k\leq n} S_k \geq \lambda\right).
\end{aligned}$$

Thus,

$$P\left(S_n \geq \lambda\right) \geq \frac{1}{2}P\left(\max_{1\leq k\leq n} S_k \geq \lambda\right). \tag{0.1}$$

Similarly we get,

$$P\left(S_n \geq \lambda\right) \geq \frac{1}{2}P\left(\max_{1\leq k\leq n} -S_k \geq \lambda\right). \tag{0.2}$$

From (0.1) and (0.2) we have,

$$2P\left(S_n \geq \lambda\right) \geq P\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right).$$

This gives the first result.

Again we define,

$$A_k = \left\{x : \max_{1\leq j<k} |X_j(x)| < \lambda \leq |X_j(x)|\right\}.$$

for $1 \leq k \leq n$. Fix k. We let $S_n^0 = 2X_k - S_n$. Then on A_k we have,

$$2\lambda \leq 2|X_k| \leq |S_n| + |S_n^0|.$$

Moreover,

$$\begin{aligned} P(A_k) &\leq P(A_k \cap \{|S_n| \geq \lambda\}) + P(A_k \cap \{|S_n^0| \geq \lambda\}). \\ &= 2P(A_k \cap \{|S_n| \geq \lambda\}). \end{aligned}$$

Then summing over k we have,

$$\begin{aligned} P(A) &= 2P(A \cap \{|S_n| \geq \lambda\}) \\ &\leq 2P(|S_n| \geq \lambda). \end{aligned}$$

Thus,

$$P\left(\max_{1\leq k\leq n} |X_k| \geq \lambda\right) \leq 2P(|S_n| \geq \lambda).$$

Lemma 7. *For any* λ

$$\frac{\lambda}{1+\lambda^2} e^{\frac{-\lambda^2}{2}} \leq \int_\lambda^\infty e^{\frac{-u^2}{2}} du \leq \frac{1}{\lambda} e^{\frac{-\lambda^2}{2}}.$$

Proof: For the proof, see [6].

Theorem 8 (Central limit theorem)**.** *Let $X_1, X_2, \cdots X_n$ be a sequence of independent identically distributed random variables with finite mean μ and variance σ^2. Define $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$. Then $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ converges to a standard normal distribution.*

Proof: For the proof, see page 236 of [3].

Theorem 9 (Hoeffding, 1963)**.** *Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with zero mean and bounded ranges: $a_i \leq Y_i \leq b_i, 1 \leq Y_i \leq n$. Then for each $\eta > 0$,*

$$P\left(\left|\sum_{i=1}^n Y_i\right| \geq \eta\right) \leq 2\exp\left(\frac{-2\eta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

Proof: For the proof, see [9].

Theorem 10. *Let $\{(X_n, \mathfrak{F}_n)\}$ be a submartingale and let ϕ be an increasing convex function defined on $\mathbb{R}$. If $\phi(x)$ is integrable for every n, then $\{(\phi(X_n), \mathfrak{F}_n)\}$ is also a submartingale.*

Proof: For the proof, see [6].

Lemma 11. *If X_i are independent random variables with the property $E(X_i) = 0$, then $S_n = \sum_{i=1}^{n} X_i$ is a martingale and S_n^2 is a submartingale.*

Proof: For the proof, see [6].

Theorem 12 (Doob's Maximal Inequality)**.** *If (X_n, β_n) is a submartingale, then for any $M > 0$,*

$$P\left(\max_{1\leq k\leq M} X_k \geq M\right) \leq \frac{1}{M}E(X_n^+) = \frac{1}{M}E(\max(X_n, 0))$$

Proof: For the proof, see [6].

1.5 Origin of law of the iterated logarithm.

Before we discuss the origin of law of the iterated logarithm, we first give the definition of normal numbers.

Definition 13 (Normal numbers)**.** Let us suppose that N takes values in $[0, 1)$ and consider its decimal and dyadic expansion as

$$N = \textstyle\sum_{n=1}^{\infty} \frac{X_n}{10^n}; \quad X_n \in \{0, 1, 2, \cdots, 9\} \quad N = \sum_{n=1}^{\infty} \frac{X_n}{2^n}; \quad X_n \in \{0, 1\}.$$

Now for a fixed $k, 0 \leq k \leq 9$, let $\omega_k^{(n)}(N)$ denotes the number of digits among the first $n-$ digits of N that are equal to k. Then $\frac{\omega_k^{(n)}(N)}{n}$ is the relative frequency of the digit k in the first n places and thus the limit $\lim\limits_{n\to\infty} \dfrac{\omega_k^{(n)}(N)}{n}$ is the frequency of k in N. Then the number N is called the normal to the base 10 if and only if this limit exists for each k and is equal to $\frac{1}{10}$. Similarly, the number N is called the normal to the base 2 if and only it the limit exists and is equal to $\frac{1}{2}$.

The first law of the iterated logarithm (LIL), introduced in probability theory, had its origin in attempts to perfect Borel's theorem on normal numbers. Precisely, the first

LIL was introduced to obtain the exact rate of convergence in the Borel's theorem. Many mathematicians obtained the different rates of convergence, but Khintchine was the one who obtained the exact rate of convergence. In order to describe Khintchine's result, we state a simple form of Borel's theorem on normal numbers.

Theorem 14 (Borel)**.** *If $N_n(t)$ denote the number of occurrences of the digit 1 in the first $n-$places of the binary expansion of a number $t \in [0,1)$, then $\lim_{n\to\infty} \frac{N_n(t)}{n} = \frac{1}{2}$ for a.e. t in Lebesgue measure.*

So by Borel's theorem we can conclude that a.e. $t \in [0,1)$ is a normal number. Here $n/2$ is the expected number of ones and the theorem gives the limit of the relative frequency of number of ones. But what can be said about the deviation $N_n(t) - n/2$? In order to answer this, we consider a special case as follows.

Suppose that X_n takes values ± 1 with probabilities $\frac{1}{2}$ (coin tossing model). We consider the unit interval with Lebesgue measure as a probability space. Then we can write $X_n(t) = 2b_n(t) - 1$, where b_n is the n^{th} digit in the binary expansion of $t \in [0,1)$. Let $S_n = \sum_{i=1}^n X_i$. Under this context the following results were obtained.

- Hausdorff (1913) obtained $|S_n| = O(n^{\frac{1}{2}+\varepsilon})$ a.e. for any $\varepsilon > 0$.
- Hardy and Littlewood (1914) obtained $|S_n| = O(\sqrt{n \log n})$ a.e.
- Khintchine (1923) obtained $|S_n| = O(\sqrt{n \log \log n})$ a.e.

In 1924, Khintchine obtained the definite answer to the size of the deviation in Borel's theorem and his result is given by,

Theorem 15 (Khintchine)**.** *If $N_n(t)$ denote the number of occurrences of the digit 1 in the first $n-$places of the binary expansion of a number $t \in [0,1)$, then for a.e. t, we have*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n \log \log n}} = 1 \tag{0.3}$$

This result is popularly known as Khintchine's law of the iterated logarithm (LIL). We note that

$$S_n(t) = \sum_{i=1}^{n} X_i(t) = \sum_{i=1}^{n} 2b_i(t) - \sum_{i=1}^{n} 1 = 2N_n(t) - n \tag{0.4}$$

Then using (0.4) in (0.3) we have,

$$\limsup_{n\to\infty} \frac{2N_n(t) - n}{\sqrt{2n \log\log n}} = 1$$

$$\limsup_{n\to\infty} \frac{N_n(t) - \frac{n}{2}}{\sqrt{\frac{1}{2} n \log\log n}} = 1$$

So Khintchine's LIL provides the size of the deviation in terms of expected mean and the deviation is of order $\sqrt{\frac{1}{2} n \log\log n}$. Because of the factor $\log\log n$ (iteration of log) in the deviation, Khintchine's law is popularly known as law of the iterated logarithm. Borel's theorem immediately follows from the Khintchine theorem. For this, we have

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n \log\log n}} = 1 \qquad \text{a.e.}$$

This gives us

$$\left| \frac{S_n(t)}{\sqrt{2n \log\log n}} \right| < 1 \quad \text{i.e} \quad \left| \frac{2b_n(t) - n}{\sqrt{2n \log\log n}} \right| < 1.$$

Again

$$\left| \frac{2N_n(t) - n}{\sqrt{2n \log\log n}} \right| < 1 \quad \text{i.e} \quad |2N_n(t) - n| < \sqrt{2n \log\log n}.$$

Hence we have

$$\left| \frac{N_n(t)}{n} - \frac{1}{2} \right| < \sqrt{\frac{\log\log n}{2n}}.$$

Then taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \left| \frac{N_n(t)}{n} - \frac{1}{2} \right| = 0, \quad \text{so} \quad \lim_{n\to\infty} \frac{N_n(t)}{n} = \frac{1}{2}.$$

We note that results of Hausdorff and Hardy-Littlewood also imply the conclusion of Borel's theorem. For this we note that for all $n \geq 1$, $\log n \leq n$ so that $\log\log n \leq \log n$. Consequently we have,

$$\sqrt{n \log\log n} \leq \sqrt{n \log n} \quad \text{i.e.} \quad \frac{|S_n|}{\sqrt{n \log\log n}} \leq \frac{|S_n|}{\sqrt{n \log n}}.$$

Thus,

$$|S_n| = O(\sqrt{n \log \log n}) \Longrightarrow |S_n| = O(\sqrt{n \log n}).$$

Khintchine's result on the rate of convergence is the first law of the iterated logarithm in the theory of probability. A few years later, the result of Khintchine was generalized by Kolmogorov to a wide class of sequences of independent random variables. We now state Kolmogorov's celebrated law of the iterated logarithm.

Theorem 16 (Kolmogorov, 1929). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with zero mean and variance one. Suppose that $|X_n|^2 \le \dfrac{\varepsilon_n n}{\log \log n}$ for some constants $\varepsilon_n \to 0$. Then for almost every ω,*

$$\limsup_{n\to\infty} \frac{S_n(\omega)}{\sqrt{2n \log \log n}} = 1$$

where $S_n = \sum_{i=1}^{n} X_i$.

We remark that in the above theorem, the mean of S_n is zero and $\sqrt{n}$ is the standard deviation of S_n. So Kolmogorov's LIL provides the size of oscillation of partial sum of independent random variable from its expected mean and the size is approximated in terms of standard deviation.

Next, we apply Kolmogorov's LIL to random walks to estimate the size of the walk in the long run. Consider the Rademacher functions, $\{r_k\}_{k=1}^{\infty}$. Set

$$\begin{aligned} f_1(x) &= r_1(x) \\ f_2(x) &= r_1(x) + r_2(x) \\ &\vdots \\ f_n(x) &= r_1(x) + r_2(x) + \ldots + r_n(x) \\ &\vdots \end{aligned}$$

Here $\{f_n(x)\}$ defines a random walk. In this random walk, we move 1 unit to the right if $r_i(x) = 1$ and to the left if $r_i(x) = -1$. Clearly, $\{f_n\}$ satisfies all the assumptions of

Kolmogorov's theorem. So by Kolmogorov's LIL, we have,

$$\limsup_{n\to\infty} \frac{f_n(x)}{\sqrt{2n\log\log n}} \leq 1.$$

For $\varepsilon > 0$, this gives us $|f_n(x)| \leq (1+\varepsilon)\sqrt{2n\log\log n}$ for n large. Here, the worst bound for the function $f_n(x)$ is n, i.e, $|f_n(x)| \leq n$. Thus, Kolmogorov's LIL gives the sharper asymptotic estimate, $|f_n(x)| \leq (1+\varepsilon)\sqrt{2n\log\log n}$. For sufficiently large n, the factor $\sqrt{2n\log\log n}$ is much smaller than n. This shows that in the long run the walker will fluctuate in between $-\sqrt{2n\log\log n}$ and $\sqrt{2n\log\log n}$.

Over the years people have made many efforts to obtain an analogue of Kolmogorov's LIL in various settings in analysis. Some of the existing settings are lacunary trigonometric series, martingales, harmonic functions to name just a few. But the first LIL in analysis was obtained in the setting of lacunary trigonometric series.

Definition 17 (Lacunary series)**.** *A real trigonometric series with the partial sums* $S_m(\theta) = \sum_{k=1}^{m}(a_k\cos n_k\theta + b_k\sin n_k\theta)$ *which has* $\frac{n_{k+1}}{n_k} > q > 1$ *is called* $q-$ *lacunary series.*

In the definition, the condition $\frac{n_{k+1}}{n_k} > q > 1$ is called gap condition which states that the sequence $\{n_k\}$ increases at least as rapidly as a geometric progression whose common ratio is bigger than 1. Lacunary series exhibit many of the properties of partial sums of independent random variables. In the modern probability theory, lacunary series are called 'weakly dependent' random variables. The law of the iterated logarithm in the setting of lacunary series was first given by Salem and Zygmund. This result of Salem and Zygmund is the first law of the iterated logarithm in analysis [1].

Theorem 18 (R. Salem and A. Zygmund, 1950)**.** *Suppose that* S_m *is a* $q-$ *lacunary series and the* n_k *are positive integers. Set* $B_m^2 = \frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)$ *and* $M_m = \max_{1\leq k\leq m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$*. Suppose also that* $B_m \longrightarrow \infty$ *as* $m \longrightarrow \infty$ *and* S_m *satisfies the Kolmogorov-type condition:* $M_m^2 \leq K_m \dfrac{B_m^2}{\log\log(e^e + B_m^2)}$ *for some sequence of numbers* $K_m \downarrow 0$*. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2\log\log B_m}} \leq 1$$

for almost every $\theta \in T$, unit circle.

Note that $\int_{-\pi}^{\pi} S_m(x)dx = 0$. This means that the mean of the partial sums is zero. Again,

$$\begin{aligned}
\sigma^2 = Var(S_m(x)) &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} S_m^2(x)dx - \left(\int_{-\pi}^{\pi} S_m(x)dx\right)^2\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_m^2(x)dx - 0 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\textstyle\sum_{k=1}^{m} a_k\cos(n_k x) + b_k\sin(n_k x)]^2 dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \textstyle\sum_{k=1}^{m}[a_k^2\cos^2(n_k x) + b_k^2\sin^2(n_k x)]dx \\
&= \frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)
\end{aligned}$$

Hence

$$\sigma = B_m = \sqrt{\frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)}.$$

This shows that B_m^2 is the variance of partial sums. So the theorem gives us the upper bound for the size of oscillation of partial sums from its expected mean and the order of the size depends on the size of standard deviation. Salem and Zygmund assumed n_k to be positive integers and they only obtained the upper bound. Erdös and Gál were the first to make progress towards the other inequality. They obtained the following result for a particular form of lacunary series given by the following theorem.

Theorem 19 (Erdös and Gál, 1955). *Suppose $S_m(\theta) = \sum_{k=1}^{m} \exp(in_k\theta)$ is a $q-$ lacunary series and n_k are integers. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{m\log\log m}} = 1$$

for almost every θ in the unit circle.

Later, M. Wiess gave the complete analogue of Kolmogorov's LIL in this setting. This result was the part of her Ph.D. thesis.

Theorem 20 (M.Weiss, 1959). *Suppose $S_m(\theta) = \sum_{k=1}^{m}(a_k\cos n_k\theta + b_k\sin n_k\theta)$ is a $q-$ lacunary series. Set $B_m = \left(\frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)\right)^{\frac{1}{2}}$ and $M_m = \max_{1\le k\le m}(|a_k|^2+|b_k|^2)^{\frac{1}{2}}$. Suppose* *also that* $B_m \to \infty$ *as* $m \to \infty$ *and* S_m *satisfies the Kolmogorov-type condition:* $M_m^2 \leq K_m \dfrac{B_m^2}{\log\log(e^e + B_m^2)}$ *for some sequence of numbers* $K_m \downarrow 0$*. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} = 1$$

for almost every θ *in the unit circle.*

There is another type of LIL in the case of independent random variables introduced by Kai Lal Chung.

Theorem 21 (Chung, 1948)**.** *Let* $\{X_n; n \geq 1\}$ *be a sequence of independent identically distributed random variables with common distribution* F *with zero mean and variance* σ^2*, and with finite third moment* $E(|X|^3) < \infty$*. Then* $\liminf_{n\to\infty} \sqrt{\dfrac{\log\log n}{n}} \max_{1\leq j\leq n} |S_j| = \dfrac{\sigma\pi}{\sqrt{8}}$ *with probability* 1.

Next, we discuss another law of the iterated logarithm introduced by Salem and Zygmund. In this LIL, they considered tail sums of the lacunary series instead of n^{th} partial sums.

Theorem 22 (R. Salem and A. Zygmund, 1950)**.** *Suppose a lacunary series* $\tilde{S}_N(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ *where* $c_k^2 = a_k^2 + b_k^2$ *satisfies* $\sum_{k=1}^{\infty} c_k^2 < \infty$*. Define* $\tilde{B}_N = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ *and* $\tilde{M}_N = \max_{k\geq N} |c_k|$*. Suppose that* $\tilde{B}_1 < \infty$ *and that* $\tilde{M}_N^2 \leq K_N \left(\dfrac{\tilde{B}_N^2}{\log\log \frac{1}{\tilde{B}_N}}\right)$ *for some sequence of numbers* $K_N \downarrow 0$ *as* $N \to \infty$*. Then*

$$\limsup_{N\to\infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \log\log \frac{1}{\tilde{B}_N}}} \leq 1$$

for almost every θ *in the unit circle.*

This result is popularly known as tail law of the iterated logarithm. We remark that the condition $\sum_{k=1}^{\infty} c_k^2 < \infty$ says that the given lacunary series converges a.e. and $\tilde{S}_N(\theta) = \displaystyle\sum_{k=1}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta) - \sum_{k=1}^{N-1}(a_k \cos n_k\theta + b_k \sin n_k\theta)$. This shows that the tail LIL gives the rate of convergence of partial sums of lacunary series to its limit function. Furthermore, the rate of convergence depends upon the standard deviation of the tail sums.

1.6 Organization of the thesis

The purpose of this thesis is to obtain an analogue of Salem-Zygmund's tail law of the iterated logarithm in various contexts in analysis. The various contexts are Rademacher functions, dyadic martingales, independent random variables, and lacunary trigonometric series. We first establish the tail law of the iterated logarithm for sums of Rademacher functions which are nicely behaved dyadic martingales. Employing the idea from the Rademacher functions, we then derive the tail law of the iterated logarithm for dyadic martingales and then obtain the tail law of the iterated logarithm for independent random variables and lacunary series. The thesis is organized as below.

In chapter 2, we derive some standard inequalities which will be used in later chapters and derive the martingale analogue of Kolmogorov's law of the iterated logarithm.

Chapter 3 begins with the derivation of the tail law of the iterated logarithm for sums of Rademacher functions. We then derive the tail LIL for dyadic martingales and construct an example of a dyadic martingale which does not follow the tail LIL. In this chapter, we only focus on the upper bound in the LIL for these functions.

In chapter 4, we obtain a lower bound in the tail LIL for sums of Rademacher functions. We also introduce the tail law of the iterated logarithm for sums of independent random variables and obtain a lower bound for it.

In chapter 5, we obtain a lower bound in the tail law of the iterated logarithm for dyadic martingales and finally in chapter 6, we obtain the lower bound in the tail law of the iterated logarithm for lacunary series introduced by Salem and Zygmund.

Chapter 2

Law of the iterated logarithm.

In this chapter, we first derive two useful martingale inequalities and then obtain an analogue of Kolmogorov's law of the iterated logarithm in the case of dyadic martingales. The martingale analogue of Kolmogorov's law of the iterated logarithm was first derived by W. Stout. Stout obtained the martingale analogue using a probabilistic approach. We will derive it using the harmonic analysis approach.

2.1 Martingales inequalities.

We first prove a Lemma, called Rubin's Lemma which will be used in our martingale inequalities. The proof of this lemma can also be found in [10], [11], and [4].

Lemma 23 (Rubin)**.** *For a dyadic martingale* $\{f_n\}, f_0 = 0$

$$\int_0^1 \exp\left(f_n(x) - \frac{1}{2}S_n^2 f(x)\right) dx \leq 1.$$

Proof: We claim that

$$g(n) = \int_0^1 \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right) dx$$

is a decreasing function of n. Let Q_{nj} be an arbitrary n^{th} generation dyadic interval. We have $\sum_{k=0}^{n} d_k(x) = f_n$ and f_n is constant on Q_{nj}. Using this we have,

$$\begin{aligned} g(n+1) &= \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp\left(\sum_{k=0}^{n+1} d_k(x) - \frac{1}{2}\sum_{k=0}^{n+1} d_k^2(x)\right) dx \\ &= \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right) \exp\left(d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)\right) dx \\ &= \sum_{j=0}^{2^n} \left[\exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right)\right]_{Q_{nj}} \int_{Q_{nj}} \exp\left(d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)\right) dx. \end{aligned}$$

Let $Q'_{(n+1)j}$ and $Q''_{(n+1)j}$ be the dyadic subintervals of Q_{nj}. Suppose d_{n+1} takes the value α on $Q'_{(n+1)j}$. Then by the expectation condition, d_{n+1} takes the value $-\alpha$ on $Q''_{(n+1)j}$. This gives,

$$\begin{aligned} \int_{Q_{nj}} \exp\left(d_{n+1}(x) - \frac{1}{2}d_{n+1}^2(x)\right) dx &= \int_{Q'_{(n+1)j}} \exp\left(\alpha - \frac{1}{2}\alpha^2\right) dx + \int_{Q''_{(n+1)j}} \exp\left(-\alpha - \frac{1}{2}\alpha^2\right) dx \\ &= \left[\exp\left(\alpha - \frac{1}{2}\alpha^2\right) + \exp\left(-\alpha - \frac{1}{2}\alpha^2\right)\right] \frac{1}{2^{n+1}} \\ &= 2\exp\left(-\frac{\alpha^2}{2}\right) \frac{e^{\alpha} + e^{-\alpha}}{2} \quad \frac{1}{2^{n+1}} \\ &= 2\exp\left(-\frac{\alpha^2}{2}\right) \cosh\alpha \quad \frac{1}{2^{n+1}}. \end{aligned}$$

Now using the elementary fact that $\cosh x \le e^{\frac{x^2}{2}}$, we have

$$\begin{aligned} g(n+1) &\le \sum_{j=0}^{2^n} \left[\exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right)\right]_{Q_{nj}} 2\exp\left(-\frac{\alpha^2}{2}\right) \exp\left(\frac{\alpha^2}{2}\right) \frac{1}{2^{n+1}} \\ &= \sum_{j=0}^{2^n} \left[\exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right)\right]_{Q_{nj}} |Q_{nj}| \\ &= \sum_{j=0}^{2^n} \int_{Q_{nj}} \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right) dx \\ &= g(n). \end{aligned}$$

Let Q_{11} and Q_{12} be the dyadic subintervals of Q_0. Assume that d_1 takes value θ on Q_{11} so that it takes value $-\theta$ on Q_{12}.

$$
\begin{aligned}
g(1) &= \int_0^1 \exp\left(d_1(x) - \frac{1}{2}d_1^2(x)\right) dx \\
&= \int_0^{\frac{1}{2}} \exp\left(\theta - \frac{1}{2}\theta^2\right) dx + \int_{\frac{1}{2}}^1 \exp\left(-\theta - \frac{1}{2}\theta^2\right) dx \\
&= \exp\left(\theta - \frac{1}{2}\theta^2\right)\frac{1}{2} + \exp\left(-\theta - \frac{1}{2}\theta^2\right)\frac{1}{2} \\
&= \exp\left(-\frac{1}{2}\theta^2\right)\frac{(e^{\theta} + e^{-\theta})}{2} \\
&= \exp\left(-\frac{1}{2}\theta^2\right)\cosh\theta \\
&\le \exp\left(-\frac{1}{2}\theta^2\right)\exp\left(\frac{1}{2}\theta^2\right) \\
&= 1.
\end{aligned}
$$

Since $g(n)$ is decreasing and $g(1) \le 1$ we conclude,

$$
\int_0^1 \exp\left(\sum_{k=0}^{n} d_k(x) - \frac{1}{2}\sum_{k=0}^{n} d_k^2(x)\right) dx \le 1.
$$

Hence,

$$
\int_0^1 \exp\left(f_n(x) - \frac{1}{2}S_n^2 f(x)\right) dx \le 1.
$$

This completes the proof of Rubin's lemma. ■

Note that if we rescale the sequence $\{f_n\}$ by λ, then the Lemma gives,

$$
\int_0^1 \exp\left(\lambda f_n(x) - \frac{1}{2}\lambda^2 S_n^2 f(x)\right) dx \le 1.
$$

This shows that the Rubin's lemma is an inhomogeneous type inequality.

Now we prove our first martingale inequality.

Lemma 24. *For a dyadic martingale $\{f_n\}$ and $\lambda > 0$ we have*

$$
\left|\left\{x \in [0,1) : \sup_{m\ge 1} |f_m(x)| > \lambda\right\}\right| \le 6\exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2}\right).
$$

Proof: Fix n. Let $\lambda > 0$, $\gamma > 0$. Then

$$f_{n-1}(x) = \frac{1}{|Q_{n-1}|} \int_{Q_{n-1}} f_n(y)dy, \quad x \in Q_{n-1}, \quad |Q_{n-1}| = \frac{1}{2^{n-1}}$$

$$f_{n-2}(x) = \frac{1}{|Q_{n-2}|} \int_{Q_{n-2}} f_n(y)dy, \quad x \in Q_{n-2}, \quad |Q_{n-2}| = \frac{1}{2^{n-2}}$$

and so on.

Hence for every $m \geq n$,

$$f_m(x) = \frac{1}{|Q_m|} \int_{Q_m} f_n(y)dy, \quad x \in Q_m, \quad |Q_m| = \frac{1}{2^m}.$$

Fix x. Then $\sup_{1 \leq m \leq n} |f_m(x)| \leq M|f_n|(x)$ where Mf_n is the Hardy-Littlewood maximal function of f_n. Then using Jensen's inequality we have,

$$\begin{aligned} \exp(\gamma|f_m(x)|) &= \exp\left(\gamma \left| \int_{Q_m} f_n(y) d\left(\frac{y}{|Q_m|}\right) \right| \right) \\ &\leq \frac{1}{|Q_m|} \int_{Q_m} \exp(\gamma|f_n(y)|)dy \\ &\leq M(e^{\gamma|f_m(x)|})(x). \end{aligned}$$

Then the Hardy-Littlewood maximal estimate gives,

$$\begin{aligned} \left| \left\{ x \in [0,1) : \sup_{1 \leq m \leq n} |f_m(x)| > \lambda \right\} \right| &= \left| \left\{ x \in [0,1) : \sup_{1 \leq m \leq n} e^{\gamma|f_m(x)|} > e^{\gamma\lambda} \right\} \right| \\ &\leq \left| \left\{ x \in [0,1) : M(e^{\gamma|f_m|})(x) > e^{\gamma\lambda} \right\} \right| \\ &\leq \frac{3}{e^{\gamma\lambda}} \int_0^1 \exp(\gamma|f_n(y)|)dy \\ &= \frac{3}{e^{\gamma\lambda}} \exp\left(\frac{\gamma^2}{2}||S_nf||_\infty^2\right) \int_0^1 \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}||S_nf||_\infty^2\right) dy \\ &\leq \frac{3}{e^{\gamma\lambda}} \exp\left(\frac{\gamma^2}{2}||S_nf||_\infty^2\right) \int_0^1 \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2f(y)\right) dy. \end{aligned}$$

Applying the Rubin's Lemma we have,

$$\begin{aligned}
&\int_0^1 \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2 f\right) dy \\
&= \int_{\{y: f_n(y)\geq 0\}} \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2 f(y)\right) dy + \int_{\{y: f_n(y)< 0\}} \exp\left(\gamma|f_n(y)| - \frac{\gamma^2}{2}S_n^2 f(y)\right) dy \\
&= \int_{\{y: f_n(y)\geq 0\}} \exp\left(\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2 f(y)\right) dy + \int_{\{y: f_n(y)< 0\}} \exp\left(-\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2 f(y)\right) dy \\
&\leq \int_0^1 \exp\left(\gamma f_n(y) - \frac{\gamma^2}{2}S_n^2 f(y)\right) dy + \int_0^1 \exp\left((-\gamma) f_n(y) - \frac{(-\gamma)^2}{2}S_n^2 f(y)\right) dy \\
&\leq 1+1 \\
&= 2.
\end{aligned}$$

Thus,

$$\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| \leq \frac{6}{e^{\gamma\lambda}} \exp\left(\frac{\gamma^2}{2}||S_n f||_\infty^2\right).$$

Choose $\gamma = \dfrac{\lambda}{||S_n f||_\infty^2}$. Then we have,

$$\begin{aligned}
\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| &\leq \frac{6}{\exp\left(\frac{\lambda^2}{||S_n f||_\infty^2}\right)} \exp\left(\frac{\lambda^2||S_n f||^2}{2||S_n f||_\infty^4}\right) \\
&\leq 6\exp\left(\frac{-\lambda^2}{2||S_n f||_\infty^2}\right).
\end{aligned}$$

For the dyadic martingale $\{f_n\}$,

$$S_n^2 f(x) = \sum_{k=1}^n d_k^2(x) \nearrow S^2 f(x) = \sum_{k=1}^\infty d_k^2(x).$$

This gives, $||S_n f||_\infty^2 \leq ||Sf||_\infty^2$. Consequently,

$$\frac{-1}{2||S_n f||_\infty^2} \leq \frac{-1}{2||Sf||_\infty^2}.$$

So we have,

$$\left|\left\{x \in [0,1) : \sup_{1\leq m\leq n} |f_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2}\right).$$

Define $E_n := \left\{ x \in [0,1) : \sup_{1\le m\le n} |f_m(x)| > \lambda \right\}$ and $E := \left\{ x \in [0,1) : \sup_{1\le m} |f_m(x)| > \lambda \right\}$. Clearly $E_n \subset E_{n+1}$ and $E = \cup_{k=1}^{\infty} E_k$. Then we have, $\lim_{n\to\infty} |E_n| = |E|$ (See Lemma 2 for the proof). Thus,

$$\begin{aligned}\left|\left\{ x \in [0,1) : \sup_{1\le m} |f_m(x)| > \lambda \right\}\right| &\le \lim_{n\to\infty} \left|\left\{ x \in [0,1) : \sup_{1\le m\le n} |f_m(x)| > \lambda \right\}\right| \\ &\le \lim_{n\to\infty} 6\exp\left(\frac{-\lambda^2}{2||S_n f||_\infty^2}\right) \\ &= 6\exp\left(\frac{-\lambda^2}{2||S_n f||_\infty^2}\right).\end{aligned}$$

So,

$$\left|\left\{ x \in [0,1) : \sup_{m\ge 1} |f_m(x)| > \lambda \right\}\right| \le 6\exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2}\right).$$

This completes the proof of the first martingale inequality. ■

Now using the above martingale inequality, we prove a martingale inequality for tail sums.

Lemma 25. *For a dyadic martingale $\{f_n\}$, with $\lambda > 0$ and, n fixed positive integer we have,*

$$\left|\left\{ x \in [0,1) : \sup_{m\ge n} |f(x) - f_m(x)| > \lambda \right\}\right| \le 12\exp\left(\frac{-\lambda^2}{8||S'_n f||_\infty^2}\right).$$

Proof: Fix n. Define a sequence $\{g_m\}$ as follows,

$$g_m(x) = \begin{cases} 0, & \text{if } m \le n; \\ f_m(x) - f_n(x), & \text{if } m > n. \end{cases}$$

We first show that $\{g_m\}$ is a dyadic martingale. Clearly for every m, g_m is measurable with respect to the sigma algebra $\mathfrak{F}_\mathfrak{m}$. Let $m > n$. Then using the fact that f_m is constant on the cube Q_m we have,

$$
\begin{aligned}
E(g_{m+1}|\mathfrak{F}_{\mathfrak{m}})(x) &= \frac{1}{|Q_m|}\int_{Q_m}[f_{m+1}(x) - f_n(x)]dx \\
&= \frac{1}{|Q_m|}\int_{Q_m} f_{m+1}(x)dx - \frac{1}{|Q_m|}\int_{Q_m} f_n(x)dx \\
&= \frac{1}{|Q_m|}\int_{Q_m} f_{m+1}(x)dx - f_n(x) \\
&= f_m(x) - f_n(x) \\
&= g_m(x).
\end{aligned}
$$

Thus we have $E(g_{m+1}|\mathfrak{F}_{\mathfrak{m}}) = g_m$. This shows that $\{g_m\}$ is a martingale. Then applying Lemma 24 for this martingale, we get

$$
\left|\left\{x \in [0,1) : \sup_{m\geq 1}|g_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2|||Sg||^2_\infty}\right).
$$

But, $g_m(x) = 0$ for $m \leq n$. Hence,

$$
\left|\left\{x \in [0,1) : \sup_{m\geq n}|g_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||Sg||^2_\infty}\right).
$$

Again,

$$
\begin{aligned}
S^2g(x) = \sum_{k=0}^{\infty} d_k^2(x) &= \sum_{k=0}^{\infty}[g_{k+1}(x) - g_k(x)]^2 \\
&= \sum_{k=n}^{\infty}[g_{k+1}(x) - g_k(x)]^2 \\
&= \sum_{k=n}^{\infty}[f_{k+1}(x) - f_n(x) - f_k(x) + f_n(x)]^2 \\
&= \sum_{k=n+1}^{\infty}[f_{k+1}(x) - f_k(x)]^2 \\
&= \sum_{k=n+1}^{\infty} d_k^2(x) \\
&= S_n'^2 f(x).
\end{aligned}
$$

This gives,

$$
\left|\left\{x \in [0,1) : \sup_{m\geq n}|g_m(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||S_n' f||^2_\infty}\right).
$$

i.e.

$$\left|\left\{x \in [0,1) : \sup_{m\geq n} |f_m(x) - f_n(x)| > \lambda\right\}\right| \leq 6\exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2}\right) \tag{0.1}$$

Clearly,

$$\{x : |f(x) - f_n(x)| > \lambda\} \subset \{x : \sup_{m\geq n} |f_m(x) - f_n(x)| > \lambda\}.$$

So we have,

$$|\{x : |f(x) - f_n(x)| > \lambda\}| \leq |\{x : \sup_{m\geq n} |f_m(x) - f_n(x)| > \lambda\}|.$$

Consequently,

$$|\{x : |f(x) - f_n(x)| > \lambda\}| \leq 6\exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2}\right). \tag{0.2}$$

By the triangle inequality we have,

$$\begin{aligned}\sup_{m\geq n} |f(x) - f_m(x)| &\leq \sup_{m\geq n}(|f(x) - f_n(x)| + |f_n(x) - f_m(x)|) \\ &= |f(x) - f_n(x)| + \sup_{m\geq n} |f_n(x) - f_m(x)|.\end{aligned}$$

This gives,

$$\left\{x : \sup_{m\geq n} |f(x) - f_m(x)| > \lambda\right\} \subset \left\{x : \sup_{m\geq n} |f(x) - f_n(x)| > \frac{\lambda}{2}\right\} \bigcup \left\{x : \sup_{m\geq n} |f_n(x) - f_m(x)| > \frac{\lambda}{2}\right\}.$$

Therefore,

$$\left|\left\{x : \sup_{m\geq n} |f(x) - f_m(x)| > \lambda\right\}\right| \leq \left|\left\{x : |f(x) - f_m(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : \sup_{m\geq n} |f_n(x) - f_m(x)| > \frac{\lambda}{2}\right\}\right|.$$

Then using (0.1) and (0.2) in the above inequality we get,

$$\begin{aligned}\left|\left\{x : \sup_{m\geq n} |f(x) - f_m(x)| > \lambda\right\}\right| &\leq 6\exp\left(\frac{-(\frac{\lambda}{2})^2}{2||S'_n f||_\infty^2}\right) + 6\exp\left(\frac{-(\frac{\lambda}{2})^2}{2||S'_n f||_\infty^2}\right) \\ &= 12\exp\left(\frac{-\lambda^2}{8||S'_n f||_\infty^2}\right).\end{aligned}$$

Thus,

$$\left|\left\{x : \sup_{m\geq n} |f(x) - f_m(x)| > \lambda\right\}\right| \leq 12\exp\left(\frac{-\lambda^2}{8||S'_n f||_\infty^2}\right).$$

This completes the proof of our second martingale inequality. ■

2.2 Law of the iterated logarithm for dyadic martingales.

Burkholder and Gundy proved (See Theorem 3)

$$\{x : Sf(x) < \infty\} \stackrel{a.s}{=} \{x : \lim f_n \quad \text{exists}\}$$

where $\stackrel{a.s}{=}$ means the sets are equal upto a set of measure zero. From this result, we observe that dyadic martingales $\{f_n\}$ behave asymptotically well on the set $\{x : Sf(x) < \infty\}$. But what can be said about the asymptotic behavior of dyadic martingales on the complement of the given set? Its behavior is quit pathological on the set $\{x : Sf(x) = \infty\}$; in particular it is unbounded a.e. on this set. But it is possible to obtain the size of growth of $|f_n|$ on the set $\{x : Sf(x) = \infty\}$? The rate of growth of $|f_n|$ on $\{x : Sf(x) = \infty\}$ is precisely given by the martingale analogue of Kolmogorov's law of the iterated logarithm. W. Stout proved the law of the iterated logarithm for martingales using a probabilistic approach. Here we derive the law of the iterated logarithm for dyadic martingales using a harmonic analysis approach.

Theorem 26. *If $\{f_n\}_{n=0}^{\infty}$ is a dyadic martingale on $[0,1)$ then,*

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} \leq 1$$

almost everywhere on the set where $\{f_n\}$ is unbounded.

Proof: Let $\theta > 1$ and $\delta > 0$. We note that for every $x \in [0,1)$, we have either $S_n f(x) > \theta^k$ for some n or $S_n f(x) \leq \theta^k$, for every n, and thus, $Sf(x) \leq \theta^k$. We define stopping time as;

$$\gamma_k(x) = \begin{cases} \min(n : S_{n+1}f(x) > \theta^k) \\ \infty, \quad if \quad Sf(x) \leq \theta^k \end{cases}$$

So by stopping time, γ_k is the smallest index such that $S_{\gamma_k+1}f(x) > \theta^k$. This means $S_{\gamma_k} f(x) \leq \theta^k$. Define,

$$\tilde{f}_n(x) = f_{n\wedge\gamma_k}(x) = \begin{cases} f_1(x), f_2(x), \ldots, f_{\gamma_k}(x), f_{\gamma_k}(x), \ldots, & \text{for } \gamma_k \neq \infty \\ f_1(x), f_2(x), f_3(x), \ldots, & \text{if } \gamma_k = \infty. \end{cases}$$

We first show that $S\tilde{f} \leq \theta^k$. So for $n < \gamma_k(x)$, we have $S\tilde{f}_n(x) = Sf_n(x) \leq Sf_{\gamma_k}(x) \leq \theta^k$. Again if $n \geq \gamma_k(x)$, then $S\tilde{f}_n(x) = Sf_{\gamma_k}(x) \leq \theta^k$. Thus, $\forall n$ $S\tilde{f}_n(x) \leq \theta^k$. Then, $\lim_{n\to\infty} S\tilde{f}_n(x) \leq \theta^k$. So we have $S\tilde{f} \leq \theta^k$.

Choose $\lambda = (1+\delta)\theta^k\sqrt{2\log\log\theta^k}$. Then using Lemma 24 for the dyadic martingale $\{\tilde{f}_n\}$ with the chosen λ, we get

$$\begin{aligned}
\left|\left\{x \in [0,1) : \sup_{n\geq 1} |\tilde{f}_n(x)| > (1+\delta)\theta^k\sqrt{2\log\log\theta^k}\right\}\right| &\leq 6\exp\left(\frac{-(1+\delta)^2\theta^{2k}2\log\log\theta^k}{2||Sf||_\infty^2}\right) \\
&\leq 6\exp\left(\frac{-(1+\delta)^2\theta^{2k}2\log\log\theta^k}{2\theta^{2k}}\right) \\
&= 6\exp\left(-(1+\delta)^2\theta^{2k}\log\log\theta^k\right) \\
&= 6\exp\left(\log(k\log\theta)^{-(1+\delta)^2}\right) \\
&= 6(k\log\theta)^{-(1+\delta)^2} \\
&= \frac{6}{(k\log\theta)^{(1+\delta)^2}}.
\end{aligned}$$

Summing over all k, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|\left\{x \in [0,1) : \sup_{n\geq 1} |\tilde{f}_n(x)| > (1+\delta)\theta^k\sqrt{2\log\log\theta^k}\right\}\right| &\leq \sum_{k=1}^{\infty}\frac{6}{(\log\theta)^{(1+\delta)^2}}\frac{1}{k^{(1+\delta)^2}} \\
&= \frac{6}{(\log\theta)^{(1+\delta)^2}}\sum_{k=1}^{\infty}\frac{1}{k^{(1+\delta)^2}} \\
&< \infty.
\end{aligned}$$

Then by Borel-Cantelli Lemma (Lemma 4) we have for a.e. x,

$$\sup_{n\geq 1} |\tilde{f}_n(x)| \leq (1+\delta)\theta^k\sqrt{2\log\log\theta^k}$$

for sufficiently large k, say, $k \geq M$, M depends on x. Thus for a.e. x, we have,

$$\sup_{n\geq 1} |f_{n\wedge\gamma_k}(x)(x)| \leq (1+\delta)\theta^k\sqrt{2\log\log\theta^k}$$

for sufficiently large $k \geq M$. We choose x such that $f_n(x)$ is unbounded. Then from the Theorem 3 we have,

$$\{x : Sf(x) < \infty\} \overset{a.e.}{=} \{x : f_n(x) \quad \text{converges}\}.$$

So we have $Sf(x) = \infty$. Then $\gamma_1(x) \le \gamma_2(x) \le \gamma_3(x) \le \dots$ i.e. for every i, $\gamma_i(x) < \infty$.

Let $n \ge \gamma_M$. Then choose k such that $\gamma_k(x) < n \le \gamma_{k+1}(x)$. Here, $\gamma_k(x) < n$ gives $\gamma_k(x) \le n-1$. Thus, $S_n f(x) = S_{n-1+1} f(x) > \theta^k$. Using this, we have

$$\begin{aligned}
|f_n(x)| &\le \sup_{1 \le m \le \gamma_{k+1}} |f_{m \wedge \gamma_{k+1}}(x)| \\
&\le \sup_{m \ge 1} |f_{m \wedge \gamma_{k+1}}(x)| \\
&\le (1+\delta)\theta^{k+1}\sqrt{2 \log \log \theta^{k+1}} \\
&= (1+\delta)\theta^k \theta \sqrt{2 \log(\log \theta^k + \log \theta)} \\
&< (1+\delta) S_n f(x) \theta \sqrt{2 \log(\log S_n f(x) + \log \theta)}.
\end{aligned}$$

So,

$$\limsup_{n \to \infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2 \log(\log S_n f(x))}} < (1+\delta)\theta \limsup_{n \to \infty} \sqrt{\frac{2 \log(\log S_n f(x) + \log \theta)}{2 \log(\log S_n f(x))}}.$$

We show,

$$\limsup_{n \to \infty} \sqrt{\frac{\log(\log S_n f(x) + \log \theta)}{\log(\log S_n f(x))}} = 1.$$

Let $X = \log(S_n f(x))$. Then

$$\begin{aligned}
\limsup_{n \to \infty} \sqrt{\frac{\log(\log S_n f(x) + \log \theta)}{\log(\log S_n f(x))}} &= \sqrt{\limsup_{X \to \infty} \frac{\log(X + \log \theta)}{\log X}} \\
&= 1.
\end{aligned}$$

Therefore for a.e. x,

$$\limsup_{n \to \infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2 \log \log S_n f(x)}} < (1+\delta)\theta.$$

Letting $\theta \searrow 1$ we get,

$$\limsup_{n \to \infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2 \log \log S_n f(x)}} \le 1+\delta.$$

This can be done for every $\delta > 0$. Hence we have for a.e. x,

$$\limsup_{n \to \infty} \frac{|f_n(x)|}{S_n f(x)\sqrt{2 \log \log S_n f(x)}} \le 1.$$

This completes the proof of the law of the iterated logarithm for dyadic martingales. ■

Chapter 3

The tail law of the iterated logarithm.

In this chapter, we first establish the tail law of the iterated logarithm for sums of Rademacher functions. Sums of Rademacher functions are nicely behaved dyadic martingales. We then derive the tail law of the iterated logarithm for dyadic martingales. Moreover, with the help of an example we will show that the tail law of the iterated logarithm is not true in general.

3.1 The tail law of the iterated logarithm for sums of Rademacher functions.

We first prove a lemma which will be used in the proof of the tail LIL for sums of Rademacher functions.

Lemma 27. *Let $f_n = \sum_{k=1}^{n} a_k r_k$, $f = \sum_{k=1}^{\infty} a_k r_k$ where $\{a_k\}$ is a sequence of numbers and $\{r_k\}$ is a sequence of Rademacher functions. Then for a fixed n and $\lambda > 0$ we have,*

$$\left|\left\{x : \sup_{m \geq n} |f(x) - f_m(x)| > \lambda\right\}\right| \leq 24 \exp\left(\frac{-\lambda^2}{2 \parallel S'_n f \parallel_\infty^2}\right).$$

Proof: Let $d_i = f_i - f_{i-1}$. Then $d_i = \sum_{k=1}^{i} a_k r_k - \sum_{k=1}^{i-1} a_k r_k = a_i r_i$. Here, each d_i has mean 0 and variance 1. Moreover, they are independent and symmetric random variables. So using Lévy's inequality (Lemma 6, Chapter 1), we have

$$\mathrm{P}\left(|\max_{1 \leq j \leq n} \textstyle\sum_{i=1}^{j} d_j| > \lambda\right) \leq 2\mathrm{P}\left(|\textstyle\sum_{k=1}^{n} d_k| > \lambda\right).$$

Let $N >> n$. Then we have,

$$\mathrm{P}\left(|\max_{0\le j\le N-n-1}\sum_{i=0}^{j} d_{N-i}| > \lambda\right) \le 2\mathrm{P}\left(|\sum_{i=0}^{N-n-1} d_{N-i}| > \lambda\right).$$

Thus,

$$\mathrm{P}(|\max d_N, d_N+d_{N-1},\cdots, d_N+d_{N-1}+\cdots+d_{n+1}| > \lambda) \le 2\mathrm{P}(|d_N+d_{N-1}+\cdots+d_{n+1}| > \lambda).$$

This gives,

$$\mathrm{P}(|\max f_N - f_{N-1}, f_N - f_{N-2}, \ldots, f_N - f_n| > \lambda) \le 2\mathrm{P}(|f_N - f_n| > \lambda)$$

i.e.

$$\left|\left\{x\in[0,1) : |\max_{N-1\ge m\ge n} f_N(x) - f_m(x)| > \lambda\right\}\right| \le 2\left|\{x : |f_N(x) - f_n(x)| > \lambda\}\right|.$$

Using the fact that $\sup_k |a_k| > \lambda \Rightarrow |\sup_k a_k| > \lambda$ or $|\sup_k(-a_k)| > \lambda$ we have,

$$\begin{aligned}&\left\{x\in[0,1) : \max_{N\ge m\ge n}|f_N(x) - f_m(x)| > \lambda\right\}\\ &= \left\{x\in[0,1) : \left|\max_{N\ge m\ge n} f_N(x) - f_m(x)\right| > \lambda\right\}\bigcup\left\{x\in[0,1) : \left|\max_{N\ge m\ge n} -f_N(x) + f_m(x)\right| > \lambda\right\}.\end{aligned}$$

Then,

$$\begin{aligned}&\left|\left\{x\in[0,1) : \max_{N\ge m\ge n}|f_N(x) - f_m(x)| > \lambda\right\}\right|\\ &\le \left|\left\{x\in[0,1) : \left|\max_{N\ge m\ge n} f_N(x) - f_m(x)\right| > \lambda\right\}\right| + \left|\left\{x\in[0,1) : \left|\max_{N\ge m\ge n}(-f_N)(x) - (-f_m)(x)\right| > \lambda\right\}\right|\\ &< |\{x : |f_N(x) - f_n(x)| > \lambda\}| + |\{x : |f_N(x) - f_n(x)| > \lambda\}|\\ &= 2\left|\{x : |f_N(x) - f_n(x)| > \lambda\}\right|.\end{aligned}$$

Thus,

$$\left|\left\{x\in[0,1) : \max_{N\ge m\ge n}|f_N(x) - f_m(x)| > \lambda\right\}\right| \le 2\left|\{x : |f_N(x) - f_n(x)| > \lambda\}\right|.$$

Now using equation (0.1) of Chapter 2 we have,

$$\left|\left\{x\in[0,1) : \sup_{m\ge n}|f_m(x) - f_n(x)| > \lambda\right\}\right| \le 6\exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2}\right).$$

Clearly,

$$|\{x : |f_N(x) - f_n(x)| > \lambda\}| \leq \left| \left\{ x \in [0,1) : \sup_{m \geq n} |f_m(x) - f_n(x)| > \lambda \right\} \right|.$$

Thus,

$$|\{x \in [0,1) : |f_N(x) - f_n(x)| > \lambda\}| \leq 6 \exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2} \right).$$

Therefore,

$$\left| \left\{ x \in [0,1) : \sup_{N \geq m \geq n} |f_N(x) - f_m(x)| > \lambda \right\} \right| \leq 12 \exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2} \right).$$

Let $E_N = \left\{ x \in [0,1) : \sup_{N \geq m \geq n} |f_N(x) - f_m(x)| > \lambda \right\}$ and $E = \bigcup_{k=1}^{\infty} E_k$. Clearly $E_N \subset E_{N+1}$. Then $|E| = \lim_{N \to \infty} |E_N|$ (See Lemma 2, Chapter 1 for the proof). Next we show,

$$\left\{ x \in [0,1) : \sup_{m \geq n} |f(x) - f_m(x)| > \lambda \right\} \subset E.$$

Let x be such that $\sup_{m \geq n} |f(x) - f_m(x)| > \lambda$. Then for sufficiently large N we have, $\sup_{N \geq m \geq n} |f_N(x) - f_m(x)| > \lambda$. This means $x \in E_N$ for sufficiently large N so that $x \in E$. Then,

$$\begin{aligned}
\left| \left\{ x \in [0,1) : \sup_{m \geq n} |f(x) - f_m(x)| > \lambda \right\} \right| &\leq |E| \\
&= \lim_{N \to \infty} |E_N| \\
&= \lim_{N \to \infty} \left| \left\{ x \in [0,1) : \sup_{N \geq m \geq n} |f_N(x) - f_m(x)| > \lambda \right\} \right| \\
&\leq 12 \lim_{N \to \infty} \exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2} \right) \\
&= 12 \exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2} \right).
\end{aligned}$$

Thus,

$$\left| \left\{ x \in [0,1) : \sup_{m \geq n} |f(x) - f_m(x)| > \lambda \right\} \right| \leq 12 \exp\left(\frac{-\lambda^2}{2||S'_n f||_\infty^2} \right).$$

This completes the proof of the Lemma. ■

Theorem 28 (Tail LIL for Rademacher functions)**.** *Let $\{r_k\}_{k=1}^{\infty}$ be the sequence of Rademacher functions and $\{a_n\}_{n=1}^{\infty}$ be a sequence with $\sum_{n=1}^{\infty} a_n^2 < \infty$. Set $f(t) = \sum_{k=1}^{\infty} a_k r_k(t)$, $f_n(t) = \sum_{k=1}^{n} a_k r_k(t)$, $S_n'^2 f(t) = \sum_{j=n+1}^{\infty} a_k^2$. Then,*

$$\limsup_{n\to\infty} \frac{|f(t) - f_n(t)|}{\sqrt{2S_n'^2 f(t) \log\log \frac{1}{S_n'^2 f(t)}}} \leq 1$$

for a.e. t.

Proof: We first show that $\{f_n\}_{n=1}^{\infty}$ is a dyadic martingale. For this we note that for $1 \leq i \leq n, a_i r_i$ is measurable with respect to $\mathfrak{F}_\mathfrak{n}$ and so is the sum $\sum_{i=1}^{n} a_i r_i = f_n$. Again for $|Q_n| = \frac{1}{2^n}$, we have

$$\begin{aligned}
E(f_{n+1}|\mathfrak{F}_\mathfrak{n}) &= \frac{1}{|Q_n|} \int_{Q_n} f_{n+1}(x)dx \\
&= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n+1} a_k r_k(x) dx \\
&= \frac{1}{|Q_n|} \int_{Q_n} \left[\sum_{k=1}^{n} a_k r_k(x) + a_{n+1} r_{n+1}(x) \right] dx \\
&= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} a_k r_k(x) dx + a_{n+1} \int_{Q_n} r_{n+1}(x) dx \\
&= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} a_k r_k(x) dx + 0 \\
&= \sum_{k=1}^{n} a_k r_k(x) \frac{1}{|Q_n|} \int_{Q_n} dx \\
&= \sum_{k=1}^{n} a_k r_k(x) \\
&= f_n.
\end{aligned}$$

Let $\theta > 1$. Define $n_1 \leq n_2 \leq \cdots, n_k \to \infty$ by

$$n_k = \min\left(n : \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k} \right).$$

Using the previous Lemma (Lemma 27) for a fixed m we have,

$$|\{t : \sup_{n \geq m} |f(t) - f_n(t)| > \lambda\}| \leq 12 \exp\left(\frac{-\lambda^2}{2 \parallel S_m' f \parallel_\infty^2} \right).$$

Then using the above estimate for n_k we have,

$$|\{t : \sup_{n \geq n_k} |f(t) - f_n(t)| > \lambda\}| \leq 12 \exp\left(\frac{-\lambda^2}{2 \parallel S'_{n_k} f \parallel_\infty^2}\right).$$

But

$$S'^2_{n_k} f(t) = \sum_{j=n_k+1}^{\infty} (f_j(t) - f_{j-1}(t))^2 = \sum_{j=n_k+1}^{\infty} (a_j r_j(t))^2 = \sum_{j=n_k+1}^{\infty} a_j^2.$$

So,

$$|\{t : \sup_{n \geq n_k} |f(t) - f_n(t)| > \lambda\}| \leq 12 \exp\left(\frac{-\lambda^2}{2 \sum_{j=n_k+1}^{\infty} a_j^2}\right).$$

We choose $\lambda = (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log\theta^k}$ where $\varepsilon > 0$. Then, from the above estimate, we have

$$\left|\left\{t : \sup_{n \geq n_k} |f(t) - f_n(t)| > (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log\theta^k}\right\}\right| \leq 12 \exp\left(\frac{-(1+\varepsilon)^2 \frac{2}{\theta^k} \log\log\theta^k}{2 \sum_{j=n_k+1}^{\infty} a_j^2}\right).$$

Using $\sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}$ we get,

$$\begin{aligned}
\left|\left\{t : \sup_{n \geq n_k} |f(t) - f_n(t)| > (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log\theta^k}\right\}\right| &\leq 12 \exp\left(\frac{-(1+\varepsilon)^2 \frac{1}{\theta^k} \log\log\theta^k}{\frac{1}{\theta^k}}\right) \\
&= 12 \exp\left(-(1+\varepsilon)^2 \log\log\theta^k\right) \\
&= 12 \exp\left(\log(k\log\theta)^{-(1+\varepsilon)^2}\right) \\
&= 12 (k\log\theta)^{-(1+\varepsilon)^2} \\
&= 12 \frac{1}{(\log\theta)^{(1+\varepsilon)^2}} \frac{1}{k^{(1+\varepsilon)^2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{k=1}^{\infty} \left|\left\{t : \sup_{n \geq n_k} |f(t) - f_n(t)| > (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log\theta^k}\right\}\right| &< \sum_{k=1}^{\infty} 12 \frac{1}{(\log\theta)^{(1+\varepsilon)^2}} \frac{1}{k^{(1+\varepsilon)^2}} \\
&= 12 \frac{1}{(\log\theta)^{(1+\varepsilon)^2}} \sum_{k=1}^{\infty} \frac{1}{k^{(1+\varepsilon)^2}} \\
&< \infty.
\end{aligned}$$

So by Borel-Cantelli Lemma (Lemma 4, Chapter 1) for a.e. t,

$$\sup_{n \geq n_k} |f(t) - f_n(t)| \leq (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log \theta^k} \tag{0.1}$$

for sufficiently large k, say $k \geq M$ such that M depends on t. Fix t. Choose $n \geq n_M$. Then $\exists$ $k \geq M$ such that $n_k \leq n < n_{k+1}$. Now by the definition of n_{k+1}, we have $S'^2_{n_{k+1}} f(t) < \frac{1}{\theta^{k+1}}$. But $n < n_{k+1}$ so that $S'^2_n f(t) \geq \frac{1}{\theta^{k+1}}$. Thus,

$$\sum_{j=n+1}^{\infty} a_j^2 \geq \frac{1}{\theta^{k+1}}.$$

Again $n_k \leq n$ implies

$$\sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^{k}}.$$

Thus,

$$\frac{1}{\theta^{k+1}} \leq \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^{k}}. \tag{0.2}$$

Then using (0.2) in (0.1), we have

$$\begin{aligned}
|f(t) - f_n(t)| &\leq \sup_{m \geq n_k} |f(t) - f_m(t)| \\
&\leq (1+\varepsilon)\sqrt{\frac{2}{\theta^k} \log\log \theta^k} \\
&= (1+\varepsilon)\sqrt{\theta}\sqrt{\frac{2}{\theta^{k+1}} \log\log \theta^k} \\
&< (1+\varepsilon)\sqrt{\theta}\sqrt{2 \sum_{j=n+1}^{\infty} a_j^2 \quad \log\log\left(\frac{1}{\sum_{j=n+1}^{\infty} a_j^2}\right)}.
\end{aligned}$$

Thus for a.e. t we have,

$$\limsup_{n\to\infty} \frac{|f(t) - f_n(t)|}{\sqrt{2 \sum_{j=n+1}^{\infty} a_j^2 \quad \log\log\left(\frac{1}{\sum_{j=n+1}^{\infty} a_j^2}\right)}} < (1+\varepsilon)\sqrt{\theta}.$$

Letting $\theta \searrow 1$, we get

$$\limsup_{n\to\infty} \frac{|f(t)-f_n(t)|}{\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \quad \log\log\left(\frac{1}{\sum_{j=n+1}^{\infty} a_j^2}\right)}} < (1+\varepsilon).$$

This is true $\forall \varepsilon > 0$. Hence for a.e. t,

$$\limsup_{n\to\infty} \frac{|f(t)-f_n(t)|}{\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \quad \log\log\left(\frac{1}{\sum_{j=n+1}^{\infty} a_j^2}\right)}} \leq 1.$$

This completes the proof of the tail law of the iterated logarithm for sums of Rademacher functions. ■

Remark 2. In the above theorem, we have $S^2 f(x) = \sum_{n=1}^{\infty} a_n^2 < \infty$. Then by Theorem 3, Chapter 1, $\lim f_n(x)$ exists. So the tail law of the iterated logarithm gives the rate of convergence of the sequence $\{f_n\}$ to its limit function f and rate of convergence depends on the tail sums of the square function.

3.2 The tail law of the iterated logarithm for dyadic martingales.

In this section, we employ the idea from the Rademacher case to obtain the tail law of the iterated logarithm for dyadic martingales. Moreover, we will later note that the tail law of the iterated logarithm in not true in general which will be justified by an example.

Theorem 29 (Tail LIL for dyadic martingales)**.** *Let $\{f_n\}_{n=0}^{\infty}$ be a dyadic martingale. Assume that there exists a constant $C < \infty$ such that $\left|\frac{S_n' f(x)}{S_n' f(y)}\right| \leq C \ \forall x, y \in I_{nj}$ for $n = 1, 2, 3, \cdots, j \in \{0, 1, 2, 3, \cdots, 2^n - 1\}$ where $I_{nj} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$. Then*

$$\limsup_{n\to\infty} \frac{|f(x)-f_n(x)|}{\sqrt{2S_n'^2 f(x) \log\log \frac{1}{S_n'^2 f(x)}}} \leq 2C$$

for a.e. x.

Proof: Let $\theta > 1$. Define functions $\gamma_1 \leq \gamma_2 \leq \cdots$ by

$\gamma_k(x) = \min \left\{ n : x \in I_{nj} \quad \text{for some} \quad j \in \{1, 2, 3, \ldots .2^n\} \quad \text{and} \quad \forall y \in I_{nj}, \quad S'_n f(y) < \frac{1}{\theta^k} \right\}.$

Now by Lemma 25, Chapter 2 for each I_{nj} we have,

$$\frac{1}{|I_{nj}|} \left| \left\{ y \in I_{nj} : \sup_{n \geq m} |f(y) - f_n(y)| > \lambda \right\} \right| \leq 12 \exp \left(\frac{-\lambda^2}{8 \parallel S'^2_m f|_{I_{nj}} \parallel_\infty} \right).$$

Now using the above estimate for $\gamma_k(y)$, we have,

$$\frac{1}{|I_{nj}|} |\{y \in I_{nj} : \sup_{n \geq \gamma_k(y)} |f(y) - f_n(y)| > \lambda\}| \leq 12 \exp \left(\frac{-\lambda^2}{8 \parallel S'^2_{\gamma_k(y)} f|_{I_{nj}} \parallel_\infty} \right). \tag{0.3}$$

Here,

$$\begin{aligned} S'_{\gamma_k(y)} f(y)|_{I_{nj}} &< \frac{1}{\theta^k} \\ \parallel S'_{\gamma_k(y)} f|_{I_{nj}} \parallel^2_\infty &\leq \frac{1}{\theta^{2k}}. \\ \frac{-\lambda^2}{\parallel S'_{\gamma_k(y)} f|_{I_{nj}} \parallel^2_\infty} &\leq -\lambda^2 \theta^{2k}. \end{aligned}$$

$$\text{So} \quad \exp \left(\frac{-\lambda^2}{8 \parallel S'_{\gamma_k(y)} f|_{I_{nj}} \parallel^2_\infty} \right) \leq \exp \left(\frac{-\lambda^2 \theta^{2k}}{8} \right). \tag{0.4}$$

Then using (0.4) in (0.3), we get

$$|\{y \in I_{nj} : \sup_{n \geq \gamma_k(y)} |f_n(y) - f(y)| > \lambda\}| \leq 12 |I_{nj}| \exp \left(\frac{-\lambda^2 \theta^{2k}}{8} \right).$$

Now summing over all such I_{nj} we get,

$$\left| \{y \in [0, 1) : \sup_{n \geq \gamma_k(y)} |f_n(y) - f(y)| > \lambda\} \right| \leq 12 \exp \left(\frac{-\lambda^2 \theta^{2k}}{8} \right). \tag{0.5}$$

Summing over all over all generations we have,

$$\sum_{k=1}^{\infty} \left| \left\{ y \in [0, 1) : \sup_{n \geq \gamma_k(y)} |f_n(y) - f(y)| > \lambda \right\} \right| \leq \sum_{k=1}^{\infty} 12 \exp \left(\frac{-\lambda^2 \theta^{2k}}{8} \right). \tag{0.6}$$

Choose $\lambda = (2 + \varepsilon) \sqrt{\frac{2}{\theta^{2k}} \log \log \theta^k}$ where $\varepsilon > 0$. Then using (0.6) for the chosen λ, we have

$$\sum_{k=1}^{\infty}\left|\left\{y\in[0,1):\sup_{n\geq\gamma_k(y)}|f_n(y)-f(y)|>(2+\varepsilon)\sqrt{\frac{2}{\theta^{2k}}\log\log\theta^k}\right\}\right|$$
$$\leq\sum_{k=1}^{\infty}12\exp\left(\frac{-(2+\varepsilon)^2\frac{2}{\theta^{2k}}\log\log\theta^k\theta^{2k}}{8}\right)$$
$$=12\sum_{k=1}^{\infty}\exp\left(\frac{-(2+\varepsilon)^2\log\log\theta^k}{4}\right)$$
$$=12\sum_{k=1}^{\infty}\exp\left(\log(k\log\theta)^{\frac{-(2+\varepsilon)^2}{4}}\right)$$
$$=12\sum_{k=1}^{\infty}(k\log\theta)^{\frac{-(2+\varepsilon)^2}{4}}$$
$$=\frac{12}{(\log\theta)^{\frac{(2+\varepsilon)^2}{4}}}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{(2+\varepsilon)^2}{4}}}$$
$$<\infty.$$

Thus,

$$\sum_{k=1}^{\infty}\left|\left\{y\in[0,1):\sup_{n\geq\gamma_k(y)}|f_n(y)-f(y)|>(2+\varepsilon)\sqrt{\frac{2}{\theta^{2k}}\log\log\theta^k}\right\}\right|<\infty.$$

Hence, by Borel-Cantelli Lemma (Lemma 4, Chapter 1) for a.e. y we have,

$$\sup_{n\geq\gamma_k(y)}|f_n(y)-f(y)|\leq(2+\varepsilon)\sqrt{\frac{2}{\theta^{2k}}\log\log\theta^k} \tag{0.7}$$

for sufficiently large k, say $k\geq M$, M depends on y. Fix y. Choose $n\geq n_M$, then $\exists\ j$ such that $y\in I_{nj}$ and $\exists\ k\geq M$ such that $\gamma_k(y)\leq n<\gamma_{k+1}(y)$. By the definition of γ_k we have, $S'_{\gamma_k(y)}f(y)<\frac{1}{\theta^k}$ and $\gamma_k(y)\leq n$. This gives,

$$S'_{\gamma_n(y)}f(y)<\frac{1}{\theta^k}. \tag{0.8}$$

Again by the definition of γ_{k+1}, $S'_{\gamma_{k+1}(y)}f(y)<\frac{1}{\theta^{k+1}}$ and $n<\gamma_{k+1}(y)$. So for some $y_0\in I_{n_j}$, $S'_nf(y_0)\geq\frac{1}{\theta^{k+1}}$. But $y,y_0\in I_{nj}$, so by assumption $\left|\frac{S'_nf(y_0)}{S'_nf(y)}\right|\leq C$. Thus,

$$CS'_nf(y)\geq S'_nf(y_0)\geq\frac{1}{\theta^{k+1}}.$$

i.e.

$$CS_n^{'} f(y) \geq \frac{1}{\theta^{k+1}}. \tag{0.9}$$

Combining (0.8) and (0.9) we have,

$$\frac{1}{C\theta^{k+1}} \leq S_n^{'} f(y) < \frac{1}{\theta^k}. \tag{0.10}$$

Using (0.10) in (0.7), we have

$$\begin{aligned}
|f_n(y) - f(y)| &\leq \sup_{m \geq \gamma_k(y)} |f_m(y) - f(y)| \\
&\leq (2+\varepsilon)\sqrt{\frac{2}{\theta^{2k}} \log\log \theta^k} \\
&= (2+\varepsilon) \quad \theta \quad \frac{\sqrt{2}}{\theta^{k+1}} \quad \sqrt{\log\log \theta^{2k}} \quad \sqrt{\frac{\log\log \theta^k}{\log\log \theta^{2k}}} \\
&\leq (2+\varepsilon) \quad \theta \quad C\sqrt{2}S_n^{'} f(y) \quad \sqrt{\log\log \frac{1}{S_n^{'2} f(y)}} \quad \sqrt{\frac{\log k + \log\log \theta}{\log 2k + \log\log \theta}}.
\end{aligned}$$

$$\frac{|f_n(y) - f(y)|}{\sqrt{2S_n^{'2} f(y) \log\log \frac{1}{S_n^{'2} f(y)}}} \leq (2+\varepsilon)\theta C \sqrt{\frac{\log k + \log\log \theta}{\log 2k + \log\log \theta}}.$$

Also we know that as $n \to \infty$ so does k.

$$\limsup_{n\to\infty} \frac{|f_n(y) - f(y)|}{\sqrt{2S_n^{'2} f(y) \log\log \frac{1}{S_n^{'2} f(y)}}} \leq (2+\varepsilon)\theta C \sqrt{\limsup_{k\to\infty} \frac{\log k + \log\log \theta}{\log 2k + \log\log \theta}}.$$

Here,

$$\limsup_{k\to\infty} \frac{\log k + \log\log \theta}{\log 2k + \log\log \theta} = 1.$$

So for a.e. y we have,

$$\limsup_{n\to\infty} \frac{|f_n(y) - f(y)|}{\sqrt{2S_n^{'2} f(y) \log\log \frac{1}{S_n^{'2} f(y)}}} < (2+\varepsilon)\theta C.$$

Letting $\theta \searrow 1$ we get,

$$\limsup_{n\to\infty} \frac{|f_n(y) - f(y)|}{\sqrt{2S_n'^2 f(y) \log\log \frac{1}{S_n'^2 f(y))}}} \leq C(2+\varepsilon).$$

This is true for $\forall \varepsilon > 0$. Thus for a.e. y we have,

$$\limsup_{n\to\infty} \frac{|f_n(y) - f(y)|}{\sqrt{2S_n'^2 f(y) \log\log \frac{1}{S_n'^2 f(y))}}} \leq 2C.$$

This proves the tail law of the iterated logarithm for dyadic martingales. ∎

Remark 3. From the assumption, we get $Sf(x) < \infty$ for a.e. x. This shows that the sequence $\{f_n(x)\}$ converges by Theorem 3. Thus the tail law of the iterated logarithm gives the rate of convergence of dyadic martingales $\{f_n\}$ to its limit function f. Moreover, the rate of convergence depends on the tail sums of martingale square function.

Corollary 30. *Let $\{f_n\}_{n=0}^{\infty}$ be a dyadic martingale. Fix $\theta > 1$ Define stopping times,*
$n_k(x) = \min\left\{n : x \in I_{n_j} \quad \textit{for some} \quad j \in \{1,2,3,....2^n\} \quad \textit{and} \quad \forall y \in I_{n_j}, \quad S_n' f(y) < \frac{1}{\theta^k}\right\}.$
Then for the sequence of stopping times $n_k(x)$,

$$\limsup_{k\to\infty} \frac{|f_{n_k(x)}(x) - f(x)|}{\sqrt{2S_{n_k(x)}'^2 f(x) \log\log\left[\frac{1}{S_{n_k(x)}'^2 f(x)}\right]}} < \sqrt{3}$$

for a.e. x.

Proof: We first prove the following estimate for $\lambda > 0, \eta > 0$,

$$\left|\left\{x \in [0,1) : |f(x) - f_n(x)| > \lambda, \quad S_n' f(x) < \eta\lambda\right\}\right| \leq 6\exp\left(\frac{-1}{2\eta^2}\right). \tag{0.11}$$

From equation (0.2) of Chapter 2 we have,

$$|\{x : |f(x) - f_n(x)| > \lambda\}| \leq 6\exp\left(\frac{-\lambda^2}{2||S_n' f||_\infty^2}\right).$$

Here $S_n^{'}f(x) < \eta\lambda$ gives $||S_n'f||_\infty^2 \leq \eta^2\lambda^2$. So, $\dfrac{-1}{||S_n'f||_\infty^2} \leq \dfrac{-1}{\eta^2\lambda^2}$. Then

$$\begin{aligned}\left|\left\{x \in [0,1) : |f(x) - f_n(x)| > \lambda, \quad S_n^{'}f(x) < \eta\lambda\right\}\right| &\leq 6\exp\left(\frac{-\lambda^2}{2||S_n'f||_\infty^2}\right)\\ &\leq 6\exp\left(\frac{-\lambda^2}{2\eta^2\lambda^2}\right)\\ &= 6\exp\left(\frac{-1}{2\eta^2}\right).\end{aligned}$$

Choose $\lambda = \dfrac{(1+\varepsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^l}$ and $\eta = \dfrac{\theta}{(1+\varepsilon)\sqrt{2\log\log\theta^{2l}}}$ where $\theta > 1$ and $\varepsilon > 0$. Then using (0.11) we have,

$$\begin{aligned}&\left|\left\{x \in [0,1) : |f(x) - f_n(x)| > \frac{(1+\varepsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^l}, \quad S_n^{'}f(x) < \frac{1}{\theta^{l-1}}\right\}\right|\\ &\leq 6\exp\left(\frac{-(1+\varepsilon)^2 2\log\log\theta^{2l}}{2\theta^2}\right)\\ &= 6\exp\left(\log(2l\log\theta)^{\dfrac{-(1+\varepsilon)^2}{\theta^2}}\right)\\ &= 6(2l\log\theta)^{\dfrac{-(1+\varepsilon)^2}{\theta^2}}\\ &= \frac{6}{(2l\log\theta)^{\frac{(1+\varepsilon)^2}{\theta^2}}}\\ &= \frac{6}{(2\log\theta)^{\frac{(1+\varepsilon)^2}{\theta^2}}}\left(\frac{1}{l}\right)^{\frac{(1+\varepsilon)^2}{\theta^2}}.\end{aligned}$$

Choose $\varepsilon = \sqrt{3}\theta - 1$. Then we have $\dfrac{(1+\varepsilon)^2}{\theta^2} = 3$. Thus,

$$\begin{aligned}&\left|\left\{x \in [0,1) : |f(x) - f_n(x)| > \frac{(1+\varepsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^l}, \quad S_n^{'}f(x) < \frac{1}{\theta^{l-1}}\right\}\right|\\ &\leq 6\left(\frac{1}{2\log\theta}\right)^3\frac{1}{l^3}\\ &= \frac{C}{l^3} \quad \text{(say)}.\end{aligned}$$

Let $g(x) = \sqrt{x\log\log\dfrac{1}{x}}$. Clearly $g(x)$ is an increasing function. So for $\dfrac{1}{\theta^{2l}} \leq S_n^{2'}f(x)$, we have

$$\sqrt{2S_n'^2 f(x)\log\log\frac{1}{S_n'^2 f(x)}} \geq \sqrt{2\frac{1}{\theta^{2l}}\log\log\theta^{2l}}. \tag{0.12}$$

Now using (0.12), we have

$$\begin{aligned}
&\left|\left\{x\in[0,1): |f(x)-f_n(x)| > (1+\varepsilon)\sqrt{2S_n'^2 f(x)\log\log\frac{1}{S_n'^2 f(x)}}\right\}\right| \\
&= \left|\bigcup_{l=k+1}^{\infty}\left\{x\in[0,1): |f(x)-f_n(x)| > (1+\varepsilon)\sqrt{2S_n'^2 f(x)\log\log\frac{1}{S_n'^2 f(x)}},\quad \frac{1}{\theta^l}\leq S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right| \\
&\leq \left|\bigcup_{l=k+1}^{\infty}\left\{x\in[0,1): |f(x)-f_n(x)| > (1+\varepsilon)\sqrt{2\frac{1}{\theta^{2l}}\log\log\theta^{2l}},\quad S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right| \\
&= \left|\bigcup_{l=k+1}^{\infty}\left\{x\in[0,1): |f(x)-f_n(x)| > \frac{(1+\varepsilon)}{\theta^l}\sqrt{2\log\log\theta^{2l}},\quad S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right| \\
&\leq \sum_{l=k+1}^{\infty}\left|\left\{x\in[0,1): |f(x)-f_n(x)| > \frac{(1+\varepsilon)}{\theta^l}\sqrt{2\log\log\theta^{2l}},\quad S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right| \\
&\leq \sum_{l=k+1}^{\infty}\frac{C}{l^3}.
\end{aligned}$$

Clearly,

$$\sum_{l=k+1}^{\infty}\frac{1}{l^3} \leq \int_k^{\infty}\frac{1}{x^3}dx = \left[\frac{-1}{2x^2}\right]_k^{\infty} = \frac{1}{k^2}.$$

Thus,

$$\left|\left\{x\in[0,1): |f(x)-f_n(x)| > (1+\varepsilon)\sqrt{2S_n'^2 f(x)\log\log\frac{1}{S_n'^2 f(x)}}\right\}\right| \leq \frac{C}{k^2}.$$

This can be done for every $n_k(x)$. So summing over all k we have,

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|\left\{x\in[0,1): |f(x)-f_{n_k(x)}(x)| > (1+\varepsilon)\sqrt{2S_{n_k(x)}'^2 f(x)\log\log\frac{1}{S_{n_k(x)}'^2 f(x)}}\right\}\right| &\leq \sum_{k=1}^{\infty}\frac{C}{k^2} \\
&= C\sum_{k=1}^{\infty}\frac{1}{k^2} \\
&< \infty.
\end{aligned}$$

So, by Borel-Cantelli (Lemma 4, Chapter 1) for a.e x, there exists M which depends on x such that for every $k \geq M$,

$$|f(x) - f_{n_k(x)}(x)| \leq (1+\varepsilon)\sqrt{2S'^2_{n_k(x)}f(x)\log\log\frac{1}{S'^2_{n_k(x)}f(x)}}.$$

But $\varepsilon = \sqrt{3}\theta - 1$. So,

$$\frac{|f(x) - f_{n_k(x)}(x)|}{\sqrt{2S'^2_{n_k(x)}f(x)\log\log\left[\frac{1}{S'_{n_k(x)}f(x)}\right]}} \leq \sqrt{3}\theta.$$

We note that as $n \to \infty$ so does k. Letting $\theta \searrow 1$ we get for a.e. x,

$$\limsup_{k\to\infty}\frac{|f(x) - f_{n_k(x)}(x)|}{\sqrt{2S'^2_{n_k(x)}f(x)\log\log\left[\frac{1}{S'_{n_k(x)}f(x)}\right]}} \leq \sqrt{3}.$$

■

Remark 4. This is true for every stopping time. But we can not estimate the behavior of the limsup in between any two stopping times as two consecutive stopping times might be significantly different.

Next, let f be an integrable function such that $f'(x)$ is continuous and $\forall x$, $0 < m \leq f'(x) \leq M$. Let us define

$$f_n(x) = \mathbb{E}(f|\mathfrak{F}_n)(x)$$

where $\mathfrak{F}_n$ is the $\sigma-$algebra generated by the dyadic intervals of length $\frac{1}{2^n}$ on $[0,1)$. Clearly the sequence $\{f_n\}$ defines a dyadic martingale. Then,

$$\begin{aligned}
d_n(x) &= f_n(x) - f_{n-1}(x) \\
&= \frac{1}{|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_{n-1}} f(y)dy \quad (Q_n \subset Q_{n-1}) \\
&= \frac{1}{|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{\widetilde{Q}_n} f(y)dy \quad (\widetilde{Q}_n \cup Q_n = Q_{n-1}) \\
&= \frac{1}{|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_n} f(u+|Q_n|)du \quad (\text{using} \quad y = u + |Q_n|) \\
&= \frac{1}{|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{2|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_n} f(u+|Q_n|)du \\
&= \frac{1}{2|Q_n|}\int_{Q_n} f(y)dy - \frac{1}{|Q_{n-1}|}\int_{Q_n} f(u+|Q_n|)du \\
&= \frac{1}{2|Q_n|}\int_{Q_n} \left[f(y) - f(y+|Q_n|)\right] dy.
\end{aligned}$$

Now by mean value theorem there exists z such that $f(y) - f(y+|Q_n|) = f^{'}(z)|Q_n|$. But $|f^{'}(z)| \leq M$. Thus $|f(y) - f(y+|Q_n|)| \leq M|Q_n|$. Then we have,

$$|d_n(x)| \leq \frac{1}{2|Q_n|}\int_{Q_n} M|Q_n|dy = \frac{1}{2|Q_n|} M|Q_n||Q_n| = \frac{M}{2^{n+1}}. \tag{0.13}$$

Now using $f^{'}(x) \geq m$, we have

$$\begin{aligned}
d_n(x) &= \frac{1}{2|Q_n|}\int_{Q_n} \left[f(y) - f(y+|Q_n|)\right] dy \\
&= \frac{1}{2|Q_n|}\int_{Q_n} f^{'}(z)|Q_n|dy \\
&\geq \frac{1}{2}\int_{Q_n} mdy \\
&= \frac{m}{2^{n+1}}.
\end{aligned}$$

Hence we have,

$$|d_n(x)| \geq \frac{m}{2^{n+1}}. \tag{0.14}$$

Now

$$|f(x) - f_n(x)| = |\textstyle\sum_{k=n+1}^{\infty} d_k(x)| \leq \sum_{k=n+1}^{\infty} |d_k(x)|.$$

From (0.13) we have

$$|d_k(x)| \leq \frac{M}{2^{k+1}}.$$

Then we get

$$|f(x) - f_n(x)| \leq \sum_{k=n+1}^{\infty} \frac{M}{2^{k+1}} = \frac{M}{2^{n+1}}.$$

Again using $|d_k(x)| \geq \frac{m}{2^{k+1}}$, we have

$$S_n'^2 f(x) = \sum_{k=n+1}^{\infty} d_k^2(x) \geq \sum_{k=n+1}^{\infty} \frac{m^2}{2^{2k+2}} = \frac{m^2}{3.2^{2n+2}}. \tag{0.15}$$

Let us take $g(x) = \sqrt{x \log\log \frac{1}{x}}$. Clearly $g(x)$ is an increasing function. So we have,

$$g\left(S_n'^2 f(x)\right) \geq g\left(\frac{m^2}{3.2^{2n+2}}\right)$$
$$\sqrt{S_n'^2 f(x) \log\log \frac{1}{S_n'^2 f(x)}} \geq \sqrt{\frac{m^2}{3.2^{2n+2}} \log\log \frac{3.2^{2n+2}}{m^2}}.$$

So we have

$$\frac{1}{\sqrt{S_n'^2 f(x) \log\log \frac{1}{S_n'^2 f(x)}}} \leq \frac{1}{\sqrt{\frac{m^2}{3.2^{2n+2}} \log\log \frac{3.2^{2n+2}}{m^2}}}. \tag{0.16}$$

Hence using (0.16) we have,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{|f_n(y) - f(y)|}{\sqrt{2S_n'^2 f(y) \log\log \frac{1}{S_n'^2 f(y)}}} &\leq \limsup_{n\to\infty} \frac{\frac{M}{2^{n+1}}}{\sqrt{\frac{m^2}{3.2^{2n+2}} \log\log \frac{3.2^{2n+2}}{m^2}}} \\
&= \limsup_{n\to\infty} \frac{\sqrt{3}M}{\sqrt{m^2 \log\log \frac{3.2^{2n+2}}{m^2}}} \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This shows that there is no need to use our theorem to find the limit of the quotient for functions with continuous and bounded derivatives as it is trivial in this case. Clearly we do have functions $f(x) = \sum_{n=1}^{\infty} a_n r_n(x)$ for which limsup in law of the iterated logarithm is nontrivial where $\{r_n\}$ is sequence of Rademacher functions. We want to look for functions for $Lip(\alpha)$ type which is slightly more general than differentiable function.

Next, we show that limsup is nontrivial for functions $Lip(\alpha)$ for $\frac{1}{2} < \alpha < 1$. For this we consider $f(x) = \sum_{n=1}^{\infty} a_n \sin(2^n x)$. Clearly, the series satisfies the gap condition. So it is a lacunary series. If $\sum a_n^2 < \infty$, then it converges a.e. We choose a_n such that $f(x) \in Lip(\alpha)$. In order to choose a_n we recall a Theorem from [12].

Theorem 31. For the function $f(x)$ for which the Fourier series is lacunary to belong to the class $Lip(\alpha)$ $(0 < \alpha < 1)$ it is necessary and sufficient that its Fourier coefficients are of order $\mathcal{O}(n^{-\alpha})$.

We choose $a_n = \frac{1}{k^\alpha}$, so that $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x)$. Let us construct martingales from the given function as follows.

$$f_n(x) = \mathbf{E}(f|\mathfrak{F}_n) = \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x) dx.$$

So we have,

$$\begin{aligned} f_n(x) &= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x) dx \\ &= \frac{1}{|Q_n|} \int_{Q_n} \left[\sum_{k=1}^{n} \frac{1}{k^\alpha} \sin(2^k \pi x) + \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x) \right] dx \\ &= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} \frac{1}{k^\alpha} \sin(2^k \pi x) dx + 0 \\ &= \frac{1}{|Q_n|} \sum_{k=1}^{n} \int_{Q_n} \frac{1}{k^\alpha} \sin(2^k \pi x) dx. \end{aligned}$$

By mean value theorem there exits c such that,

$$\sin(2^k \pi x) - \sin(2^k \pi y) = 2^k \pi (x - y) \cos(2^k \pi c). \tag{0.17}$$

Hence using (0.17) we have,

$$
\begin{aligned}
f(x) - f_n(x) - \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x) &= f(x) - f_n(x) - \left(f(x) - \sum_{k=1}^{n} \frac{1}{k^\alpha} \sin(2^k \pi x) \right) \\
&= \sum_{k=1}^{n} \frac{1}{k^\alpha} \sin(2^k \pi x) - \frac{1}{|Q_n|} \sum_{k=1}^{n} \int_{Q_n} \frac{1}{k^\alpha} \sin(2^k \pi x) dx \\
&= \frac{1}{|Q_n|} \sum_{k=1}^{n} \int_{Q_n} \frac{1}{k^\alpha} \sin(2^k \pi x) dy - \frac{1}{|Q_n|} \sum_{k=1}^{n} \int_{Q_n} \frac{1}{k^\alpha} \sin(2^k \pi y) dy \\
&= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} \frac{1}{k^\alpha} [\sin(2^k \pi x) - \sin(2^k \pi y)] dy \\
&= \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} \frac{1}{k^\alpha} 2^k \pi (x-y) \cos(2^k \pi c) dy \\
&\le \frac{1}{|Q_n|} \int_{Q_n} \sum_{k=1}^{n} \frac{1}{k^\alpha} 2^k \pi (x-y) dy.
\end{aligned}
$$

Here

$$
\int_{Q_n} (x-y) dy = \int_{\{y : y > x\}} |x-y| dy \le \int_x^{x+|Q_n|} (x-y) dy = \left[\frac{1}{2}(x-y)^2 \right]_x^{x+|Q_n|} = \frac{1}{2}|Q_n|^2
$$

Hence,

$$
\begin{aligned}
f(x) - f_n(x) - \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x) &\le \frac{1}{|Q_n|} \sum_{k=1}^{n} \frac{1}{k^\alpha} 2^k \pi |Q_n|^2 \\
&= \sum_{k=1}^{n} \frac{1}{k^\alpha} 2^k \pi \frac{1}{2^n} \\
&= \frac{\pi}{2^n} \sum_{k=1}^{n} \frac{2^k}{k^\alpha} \\
&= \frac{\pi}{2^n} \left(\sum_{k=1}^{l} \frac{2^k}{k^\alpha} + \sum_{k=l+1}^{n} \frac{2^k}{k^\alpha} \right).
\end{aligned}
$$

We choose l such that $2^{l+1} = \frac{2^n}{n^\alpha}$ so that $\log_2 2^{l+1} = \log_2 2^n n^\alpha$. This shows that

$$
l = n + 1 - \alpha \log_2 n.
$$

Then using $\alpha \log_2 n \leq \frac{n}{2}$ for large n, we have

$$
\begin{aligned}
\frac{1}{2^n} \sum_{k=1}^{n} \frac{2^k}{k^\alpha} &= \frac{1}{2^n} \left(\sum_{k=1}^{l} \frac{2^k}{k^\alpha} + \sum_{k=l+1}^{n} \frac{2^k}{k^\alpha} \right) \\
&= \frac{1}{2^n} \left(2^{l+1} + \sum_{k=l+1}^{n} \frac{2^k}{k^\alpha} \right) \\
&\leq \frac{1}{2^n} \left(2^{n-\alpha \log_2 n} + \frac{1}{(n+1)^\alpha} \sum_{k=l+1}^{n} 2^k \right) \\
&\leq \frac{1}{n^\alpha} + \frac{1}{2^n} \frac{2^{n+1}}{(l+1)^\alpha} \\
&\leq \frac{1}{n^\alpha} + \frac{2}{(l+1)^\alpha} \\
&\leq \frac{1}{n^\alpha} + \frac{2}{(n-\alpha \log_2 n)^\alpha} \\
&\leq \frac{1}{n^\alpha} + \frac{2}{(\frac{n}{2})^\alpha} \\
&= \frac{1+2^{\alpha+1}}{n^\alpha} \\
&= \frac{C_\alpha}{n^\alpha} \quad \text{(say)}.
\end{aligned}
$$

We have,

$$
\limsup_{N\to\infty} \frac{\sum_{k=N}^{\infty} \frac{1}{k^\alpha} \sin(2^k \pi x)}{\sqrt{\sum_{k=N}^{\infty} \frac{1}{k^{2\alpha}} \log\log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N}^{\infty} \frac{1}{k^{2\alpha}}}}}} = 1.
$$

Next we show

$$
|d_m(x)| \leq \frac{C_\alpha}{m^\alpha}
$$

for some constant C.

$$
\begin{aligned}
&|d_m(x)| \\
&= |f_m(x) - f_{m-1}(x)| \\
&= \frac{1}{2|Q_m|}\int_{Q_m}\sum_{k=1}^{\infty}\frac{1}{k^\alpha}\left[\sin(2^k\pi x) - \sin(2^k\pi(x+|Q_m|))\right]dx \\
&= \left|\frac{1}{2|Q_m|}\int_{Q_m}\left[\sum_{k=1}^{m}\frac{1}{k^\alpha}\sin(2^k\pi x) - \sin(2^k\pi(x+|Q_m|)) + \sum_{k=m+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x) - \sin(2^k\pi(x+|Q_m|))\right]dx\right| \\
&= \left|\frac{1}{2|Q_m|}\int_{Q_m}\left[\sum_{k=1}^{m}\frac{1}{k^\alpha}\sin(2^k\pi x) - \sin(2^k\pi(x+|Q_m|))\right]dx\right| \\
&\le \frac{1}{2|Q_m|}\int_{Q_m}\sum_{k=1}^{m}\frac{1}{k^\alpha}(x+|Q_m|-x)2^k\pi|\cos(2^k\pi c)|dx \quad \text{(Using MVT)} \\
&\le \frac{1}{2|Q_m|}\int_{Q_m}\sum_{k=1}^{m}\frac{1}{k^\alpha}|Q_m|2^k\pi dx \\
&= \frac{1}{2}\sum_{k=1}^{m}\frac{2^k}{k^\alpha}\frac{\pi}{2^m} \\
&\le \frac{(1+2^{\alpha+1})}{m^\alpha} \\
&= \frac{C_\alpha}{m^\alpha}.
\end{aligned}
$$

So

$$\sum_{k=N+1}^{\infty} d_k^2(x) \le \sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}.$$

Define $g(x) = \sqrt{x\log\log\frac{1}{\sqrt{x}}}$. Clearly $g(x)$ is an increasing function. So

$$g\left(\sum_{k=N+1}^{\infty} d_k^2(x)\right) \le g\left(\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}\right).$$

This gives,

$$\limsup_{N\to\infty}\frac{1}{\sqrt{\sum_{k=N+1}^{\infty} d_k^2(x)\log\log\frac{1}{\sqrt{\sum_{k=N}^{\infty} d_k^2(x)}}}} \ge \limsup_{N\to\infty}\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}.$$

Then,

$$\limsup_{N\to\infty} \frac{f(x)-f_N(x)}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} d_k^2(x)\log\log\frac{1}{\sqrt{\sum_{k=N}^{\infty} d_k^2(x)}}}} \geq \limsup_{N\to\infty} \frac{f(x)-f_N(x)}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}$$

$$= \limsup_{N\to\infty} \frac{f(x)-f_N(x)-\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)+\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}$$

$$\geq \limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)-|f(x)-f_N(x)-\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)|}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}$$

$$\geq \limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)-\frac{C_\alpha}{N^\alpha}}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}$$

$$\geq \limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^\alpha}\sin(2^k\pi x)}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}} - \limsup_{N\to\infty} \frac{\frac{C_\alpha}{N^\alpha}}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}}.$$

But

$$\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}} = \sqrt{\frac{1}{(2\alpha-1)N^{2\alpha-1}}} \approx \frac{\sqrt{N}}{N^\alpha}.$$

This gives,

$$\limsup_{N\to\infty} \frac{\frac{C_\alpha}{N^\alpha}}{\sqrt{\displaystyle\sum_{k=N+1}^{\infty} \frac{C_\alpha^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_\alpha^2}{k^{2\alpha}}}}}} = \limsup_{N\to\infty}\frac{C_\alpha}{N^\alpha}\frac{N^\alpha}{\sqrt{N}\sqrt{\log\log(\frac{N^\alpha}{N})}}$$

$$= \limsup_{N\to\infty} C_\alpha \frac{1}{\sqrt{N}\sqrt{\log\log(\frac{N^\alpha}{N})}}$$

$$= 0.$$

Thus we have,

$$\limsup_{N\to\infty} \frac{f(x)-f_N(x)}{\sqrt{\sum_{k=N+1}^{\infty} d_k^2(x)\log\log\frac{1}{\sqrt{\sum_{k=N}^{\infty} d_k^2(x)}}}} \geq \limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^{\alpha}}\sin(2^k\pi x)}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_{\alpha}^2}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_{\alpha}^2}{k^{2\alpha}}}}}}$$

$$= \frac{1}{C_{\alpha}}\limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^{\alpha}}\sin(2^k\pi x)}{\sqrt{\sum_{k=N+1}^{\infty}\frac{1}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_{\alpha}^2}{k^{2\alpha}}}}}}.$$

But

$$\limsup_{N\to\infty} \frac{\sqrt{\sum_{k=N+1}^{\infty}\frac{1}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_{\alpha}^2}{k^{2\alpha}}}}}}{\sqrt{\sum_{k=N+1}^{\infty}\frac{1}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{1}{k^{2\alpha}}}}}} = 1. \tag{0.18}$$

Using (0.18) together with the assumption, we get

$$\limsup_{N\to\infty} \frac{\sum_{k=N+1}^{\infty}\frac{1}{k^{\alpha}}\sin(2^k\pi x)}{\sqrt{\sum_{k=N+1}^{\infty}\frac{1}{k^{2\alpha}}\log\log\frac{1}{\sqrt{\sum_{k=N+1}^{\infty}\frac{C_{\alpha}^2}{k^{2\alpha}}}}}} = 1.$$

Hence we have,

$$\limsup_{N\to\infty} \frac{f(x)-f_N(x)}{\sqrt{\sum_{k=N+1}^{\infty} d_k^2(x)\log\log\frac{1}{\sqrt{\sum_{k=N}^{\infty} d_k^2(x)}}}} \geq \frac{1}{C_{\alpha}}.$$

This shows that there are functions in $Lip(\alpha)$ with $\frac{1}{2} < \alpha < 1$ for which law of the iterated logarithm is nontrivial. We note that LIL is trivial for functions with continuous and bounded derivatives. We showed that there are functions in $Lip(\alpha)$ with $\frac{1}{2} < \alpha < 1$ doe which it is nontrivial. Note that $f(x)$ is differentiable a.e. for $f \in Lip(\alpha)$ $(\alpha = 1)$. This means the gap is very narrow.

Next, we show that the tail law of the iterated logarithm is not true in general with help of an example.

3.3 Tail LIL for dyadic martingales is not true in general.

Consider the Rademacher functions $\{r_i\}$ on $[0,1)$. Let us define partial sum as $S_n = \sum_{k=1}^{n} r_i$. Clearly $r_i's$ are independent, identically distributed random variables with mean $\mu = 0$ and variance $\sigma^2 = 1$. Then using central limit theorem (Theorem 8), $\dfrac{S_n - n\mu}{\sqrt{n\sigma^2}}$ converges in distribution to $N(0,1)$. This means that the distribution function of $\dfrac{S_n - n\mu}{\sqrt{n\sigma^2}}$ converges to the distribution function of $N(0,1)$. Thus we get,

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq \lambda\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{\frac{-u^2}{2}} du.$$

Then,

$$\left|\left\{t : \frac{S_n(t) - n.0}{\sqrt{n}.1} \leq \lambda\right\}\right| \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{\frac{-u^2}{2}} du.$$

This gives,

$$\left|\left\{t : \frac{S_n(t) - n.0}{\sqrt{n}.1} > \lambda\right\}\right| \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{\frac{-u^2}{2}} du.$$

So,

$$\left|\left\{t : \frac{r_1(t) + r_2(t) + r_3(t) + \ldots + r_n(t)}{\sqrt{n}} > \lambda\right\}\right| \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{\frac{-u^2}{2}} du.$$

Let $0 < \alpha < \frac{1}{e}$ be fixed. With this α, we get

$$\left|\left\{t : \frac{\alpha r_1(t) + \alpha r_2(t) + \alpha r_3(t) + \ldots + \alpha r_n(t)}{\alpha\sqrt{n}} > \lambda\right\}\right| \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{\frac{-u^2}{2}} du.$$

Let $\lambda = \gamma\sqrt{\log\log\left(\frac{1}{\alpha}\right)}$ where γ is some large number. From Lemma 7, we have

$$\frac{\lambda}{1+\lambda^2} e^{\frac{-\lambda^2}{2}} \leq \int_{\lambda}^{\infty} e^{\frac{-u^2}{2}} du. \tag{0.19}$$

Choose α_1 and γ_1. Using the inequality (0.19) for the given λ and we choose n_1 large enough that,

$$\left|\left\{t : \frac{\alpha_1 r_1(t) + \alpha_1 r_2(t) + \alpha_1 r_3(t) + \ldots + \alpha_1 r_{n_1}(t)}{\sqrt{n_1}} > \alpha_1\lambda\right\}\right| > \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\lambda}{1+\lambda^2} e^{\frac{-\lambda^2}{2}}.$$

$$\left|\left\{t : \frac{\alpha_1 r_1(t) + \alpha_1 r_2(t) + \alpha_1 r_3(t) + \ldots + \alpha_1 r_{n_1}(t)}{\sqrt{n_1}} > \alpha_1\gamma_1\sqrt{\log\log\left(\frac{1}{\alpha_1}\right)}\right\}\right|$$

$$> \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\gamma_1\sqrt{\log\log\left(\frac{1}{\alpha_1}\right)}}{1+\gamma_1^2\log\log\left(\frac{1}{\alpha_1}\right)}\exp\left(\frac{-\gamma_1^2\log\log\left(\frac{1}{\alpha_1}\right)}{2}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\gamma_1\sqrt{\log\log\left(\frac{1}{\alpha_1}\right)}}{1+\gamma_1^2\log\log\left(\frac{1}{\alpha_1}\right)}\left(\frac{1}{\log\left(\frac{1}{\alpha_1}\right)}\right)^{\frac{\gamma_1^2}{2}}.$$

Define a martingale,

$$f_0(t) = 0, \quad f_1(t) = \frac{\alpha_1 r_1(t)}{\sqrt{n_1}}, \quad f_2(t) = \frac{\alpha_1 r_1(t)}{\sqrt{n_1}} + \frac{\alpha_1 r_2(t)}{\sqrt{n_1}}$$

$$f_{n_1}(t) = \frac{\alpha_1 r_1(t)}{\sqrt{n_1}} + \frac{\alpha_1 r_2(t)}{\sqrt{n_1}} + \ldots + \frac{\alpha_1 r_{n_1}(t)}{\sqrt{n_1}}.$$

In this case, then

$$d_1 = \frac{\alpha}{\sqrt{n}}r_1(t), \quad d_2 = \frac{\alpha}{\sqrt{n}}r_2(t), \ldots, d_n = \frac{\alpha}{\sqrt{n}}r_n(t).$$

$S_{n_1}^2 f(t) = d_1^2 + d_2^2 + d_3^2 + \cdots + d_{n_1}^2 = \alpha_1^2$. Hence,

$$\left|\left\{t : \frac{|f_{n_1}(t) - f_0(t)|}{S_{n_1}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_1}f(t)}\right]}} > \gamma\right\}\right| > C(\alpha_1,\gamma_1)$$

where

$$C(\alpha_1,\gamma_1) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\gamma_1\sqrt{\log\log\left(\frac{1}{\alpha_1}\right)}}{1+\gamma_1^2\log\log\left(\frac{1}{\alpha_1}\right)}\left(\frac{1}{\log\left(\frac{1}{\alpha_1}\right)}\right)^{\frac{\gamma_1^2}{2}}.$$

Note that $C(\alpha,\gamma)$ is small as $\alpha \to 0, \gamma \to \infty$. So at least on the fraction $C(\alpha_1,\gamma_1)$ of the n_1^{th} generation dyadic intervals, we have

$$\frac{|f_{n_1}(t) - f_0(t)|}{S_{n_1}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_1}f(t)}\right]}} > \gamma_1. \tag{0.20}$$

Those intervals on which (0.20) holds we leave as it is and continue the process on the others. So $\exists\ n_2$ such that

$$\frac{|f_{n_2}(t) - f_{n_1}(t)|}{S_{n_2,n_1}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_2,n_1}f(t)}\right]}} > \gamma_1$$

where $S^2_{n_2,n_1}f(t) = \sum_{j=n_1+1}^{n_2} d_j^2(t)$. Note we may take $n_2 = 2n_1$. In other words $n_2 - n_1 = n_1 - 0$. Hence $\exists\ n_1 < n_2 < n_3 \ldots < n_l = M_1$ (say) and E_1 such that $|E_1^c| < \frac{1}{101^2}$ with the property that $\forall\ t \in E\ \exists\ n_{k+1}, n_k$ with $n_1 \leq n_k < n_{k+1} \leq M_1$ such that

$$\frac{|f_{n_{k+1}}(t) - f_{n_k}(t)|}{S_{n_{k+1},n_k}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{k+1},n_k}f(t)}\right]}} > \gamma_1$$

where

$$S^2_{n_{k+1},n_k}f(t) = \sum_{j=n_k+1}^{n_{k+1}} d_j^2(t) = S^2_{n_2,n_1}f(t) = S^2_{n_1}f(t) = \alpha_1^2.$$

Hence in this generation we have $\gamma_1, \alpha_1, M_1, E_1$. We choose $\gamma_2 > \gamma_1$ and $\alpha_2 < \alpha_1$. Then we consider M_1^{th} generation dyadic intervals and repeat the process. Then we get $M_1 < n_{l+1}$ such that

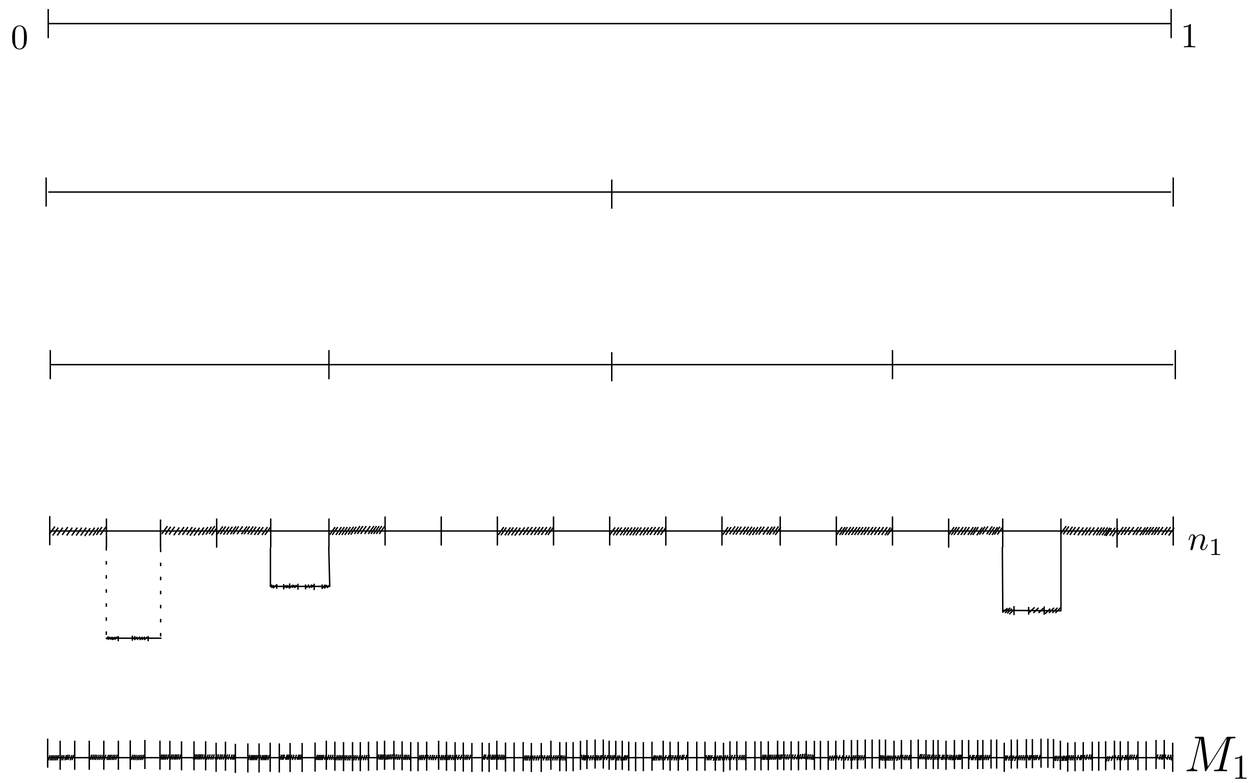
$$\frac{|f_{n_{l+1}}(t) - f_{n_l}(t)|}{S_{n_{l+1},n_l}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{n+1},n_n}f(t)}\right]}} > \gamma_2 \tag{0.21}$$

where $S^2_{n_{n+1},n_n}f = \alpha_2^2$ and this is true on the fraction $C(\alpha_2, \gamma_2)$ of each of the $2^{n_{l+1}}$ intervals of n_{l+1}^{th} generation. We leave those intervals where (0.21) holds and continue the process on the others. Then $\exists\ n_{l+2}$ such that

$$\frac{|f_{n_{l+2}}(t) - f_{n_{l+1}}(t)|}{S_{n_{l+2},n_{l+1}}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{l+2},n_{l+1}}f(t)}\right]}} > \gamma_2.$$

Here $n_{l+2} - n_{l+1} = n_{l+1} - n_l$.

Figure 3.1: *Construction of Martingales*



Then we get $M_1 < n_{l+1} < n_{l+2} \ldots < n_p := M_2$ and E_2 such that $|E_2^c| < \dfrac{1}{102^2}$ satisfying the property that $\forall\ t \in E_2\ \exists\ n_{j+1}, n_j$ such that $M_1 \leq n_j < n_{j+1} \leq M_2$ with

$$\frac{|f_{n_{j+1}}(t) - f_{n_j}(t)|}{S_{n_{j+1},n_j}f(t)\sqrt{\log\log\left[\dfrac{1}{S_{n_{j+1},n_j}f(t)}\right]}} > \gamma_2$$

where $S^2_{n_{j+1},n_j}f(t) = \alpha_2^2$ and $S^2_{n_{l+1},n_l}f = S^2_{n_{l+2},n_{l+1}}f$. Hence, in this generation we have $\gamma_2, \alpha_2, M_2, E_2$. We continue the process in this way considering $\alpha_q \to 0$ and $\gamma_q \to \infty$ which gives us,

$$M_1 < M_2 < M_3 \ldots < M_q < \ldots$$

and sets E_q with $|E_q^c| < \frac{1}{(100+q)^2}$ such that $\forall\ t \in E_q\ \exists\ n_s$ with $M_{q-1} \leq n_s < n_{s+1} \leq M_q$ and

$$\frac{|f_{n_{s+1}}(t) - f_{n_s}(t)|}{S_{n_{s+1},n_s}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{s+1},n_s}f(t)}\right]}} > \gamma_q.$$

Because of the stopping rule, note that $f_n(t) = f_{M_q}(t)\ \forall\ n$ with $n_{s+1} \leq n \leq M_q$. Also $S_{n_{s+1},n_s} = \alpha_q$. Here $\sum_{k=1}^{\infty} |E_k^c| < \infty$. Hence by Borel-Cantelli for a.e. t, eventually $t \in E_k$ for sufficiently large k. Consider a t at which $t \in E_k, k \geq L$, say. So for a given $k \geq L\ \exists\ n_q, n_{q+1}$ such that $M_{k-1} \leq n_q < n_{q+1} \leq M_k$ we have

$$\frac{|f_{n_{q+1}}(t) - f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > \gamma_k.$$

Now,

$$S'_{n_q}f(t)^2 = \sum_{j=n_q+1}^{\infty} d_j^2(t) = \sum_{j=n_q+1}^{n_{q+1}} d_j^2(t) + \sum_{j=n_{q+1}+1}^{M_k} d_j^2(t) + \sum_{M_k+1}^{M_{k+1}} d_j^2(t) + \ldots$$

But $f_n(t) = f_{M_k}(t)\ \forall\ n$ with $n_{q+1} \leq n \leq M_q$ so $\sum_{j=n_{q+1}+1}^{M_{k+1}} d_j^2(t) = 0$.

Then

$$S'_{n_q}f(t)^2 \leq \alpha_k^2 + b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \ldots$$

where b_k is a number such that

$$[1 - C(\alpha_k, \gamma_k)]^{b_k+1} = \frac{1}{(100+k)^2}.$$

$$(b_k + 1)\log[1 - C(\alpha_k, \gamma_k)] = -2\log(100 + k).$$

Now $\log(1 - x) \approx -x$. So, it suffices to consider

$$(b_k + 1)(-C(\alpha_k, \gamma_k)) < -2\log(100 + k)$$

i.e.

$$b_k > \frac{2\log(100 + k)}{C(\alpha_k, \gamma_k)} - 1.$$

Thus

$$b_k > \frac{2\log(100+k)}{\left[\dfrac{1}{2}\dfrac{1}{\sqrt{2\pi}}\dfrac{\gamma_k\sqrt{\log\log(\frac{1}{\alpha_k})}}{1+\gamma_k^2\log\log(\frac{1}{\alpha_k})}\left(\dfrac{1}{\log(\frac{1}{\alpha_k})}\right)^{\dfrac{\gamma_k^2}{2}}\right]} - 1.$$

So we need to choose two sequences α_k, γ_k satisfying the following properties:

(a) $\alpha_k \to 0, \gamma_k \to \infty$

(b) If $b_k = \dfrac{4\sqrt{2\pi}\log(100+k)}{\left[\dfrac{\gamma_k\sqrt{\log\log(\frac{1}{\alpha_k})}}{1+\gamma_k^2\log\log(\frac{1}{\alpha_k})}\left(\dfrac{1}{\log(\frac{1}{\alpha_k})}\right)^{\dfrac{\gamma_k^2}{2}}\right]}$

then, $\quad b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \ldots < \dfrac{1}{2}\alpha_k^2.$

We choose $\alpha_k = \dfrac{1}{e^{e^k}}$, $\gamma = \log k$. Clearly, the first property is satisfied. Here $\log\log(\frac{1}{\alpha_k}) = k$, $\log(\frac{1}{\alpha_k}) = e^k$. So,

$$b_k = \frac{4\sqrt{2\pi}(1+k(\log k)^2)\exp\left(\dfrac{k(\log k)^2}{2}\right)}{\sqrt{k}}\frac{\log(k+100)}{\log(k)}.$$

We claim that $b_{k+l} \leq 56\sqrt{2\pi}\exp{(k+l)^2}\ \forall\ k \geq 1, \forall\ l \geq 1$.

We first note that

$$\frac{\log(k+l+100)}{\log(k+l)} \leq 7.$$

Then we have,

$$\begin{aligned}
b_{k+l} &= 4\sqrt{2\pi}\frac{1+(k+l)(\log(k+l))^2}{\sqrt{k+l}}\exp\left[\frac{(k+l)(\log(k+l))^2}{2}\right]\frac{\log(k+l+100)}{\log(k+l)} \\
&\leq 4\sqrt{2\pi}\frac{2(k+l)\log(k+l)^2}{\sqrt{k+l}}\exp\left[\frac{(k+l)^2}{2}\right].7 \\
&= 56\sqrt{2\pi}\sqrt{k+l}\log(k+l)^2\exp\left[\frac{(k+l)^2}{2}\right] \\
&\leq 56\sqrt{2\pi}\exp\left[\frac{(k+l)^2}{2}\right]\exp\left[\frac{(k+l)^2}{2}\right] \\
&= 56\sqrt{2\pi}\exp(k+l)^2.
\end{aligned}$$

So $b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \dots$

$$\leq 56\sqrt{2\pi}\left[e^{(k+1)^2}\left(\frac{1}{e^{2e^k}}\right)^{e} + e^{(k+2)^2}\left(\frac{1}{e^{2e^k}}\right)^{e^2} + \dots\right]$$

$$= 56\sqrt{2\pi}\left[e^{(k+1)^2}\left(\frac{1}{e^{2e^k}}\right)^{e-1}\left(\frac{1}{e^{2e^k}}\right)^{1} + e^{(k+2)^2}\left(\frac{1}{e^{2e^k}}\right)^{e^2-2}\left(\frac{1}{e^{2e^k}}\right)^{2} + \dots\right].$$

Now we claim that $\dfrac{\exp[(k+l)^2]}{\exp[2e^k(e^l-l)]} \leq \dfrac{1}{168\sqrt{2\pi}}$ for $l = 1, 2, 3\dots$ and for $k \geq 2$. It suffices to prove that

$$\log(168\sqrt{2\pi}) + (k+l)^2 \leq 2(e^l - l)e^k.$$

Note that for $k \geq 2$ and for $l \geq 1$,

$$\log(168\sqrt{2\pi}) + (k+l)^2 \leq 2(k+l)^2 \leq e^{k+l} \leq 2e^k(e^l - l).$$

Hence,

$$b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \dots \leq \frac{56\sqrt{2\pi}}{168\sqrt{2\pi}}\left[\frac{1}{e^{2e^k}} + \left(\frac{1}{e^{2e^k}}\right)^2 + \dots\right] = \frac{1}{3}\frac{\frac{1}{e^{2e^k}}}{1 - \frac{1}{e^{2e^k}}}.$$

We have $\dfrac{1}{e^{2e^k}-1} \leq \dfrac{3}{2}\dfrac{1}{e^{2e^k}}$. Then

$$b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \dots \leq \frac{1}{2}\frac{1}{e^{2e^k}} = \frac{1}{2}\alpha_k^2.$$

$$S'_{n_q}f(t)^2 = \alpha_k^2 + b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \dots \leq \alpha_k^2 + \frac{1}{2}\alpha_k^2 = \frac{3}{2}\alpha_k^2.$$

Thus,

$$\alpha_k^2 < S'_{n_q}f(t)^2 < \frac{3}{2}\alpha_k^2.$$

Consequently, $S'_{n_q}f(t) \approx S_{n_{q+1},n_q}f(t)$. Here

$$\frac{|f_{n_{q+1}}(t) - f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > \gamma_k.$$

We need

$$\frac{|f(t)-f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > \gamma_k.$$

But

$$\frac{|f_{n_{q+1}}(t)-f(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} + \frac{|f(t)-f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > \gamma_k.$$

We now claim that

$$\frac{|f_{n_{q+1}}(t)-f(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} < (.1)\gamma_k$$

so that

$$\frac{|f(t)-f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > (.9)\gamma_k.$$

Here we note that $f_{n_{q+1}}(t) = f_{M_k}(t)$. We again use the estimate,

$$|\{t : |f(t) - f_{n_{q+1}}(t)| > \lambda\}| \le 6\exp\left(\frac{-\lambda^2}{2 \parallel S'_{n_{q+1}}f \parallel_\infty^2}\right).$$

Taking $\lambda = 0.1\alpha_k\gamma_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}$,

$$\left|\left\{t : |f(t)-f_{n_{q+1}}(t)| > 0.1\alpha_k\gamma_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}\right\}\right| \le 6\exp\left(\frac{-0.01\alpha_k^2\gamma_k^2\log\log\left[\frac{1}{\alpha_k}\right]}{2\parallel S'_{n_{q+1}}f\parallel_\infty^2}\right)$$
$$= 6\exp\left(\frac{-0.01\alpha_k^2\gamma_k^2\log\log\left[\frac{1}{\alpha_k}\right]}{2(b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \ldots)}\right).$$

But

$$b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \ldots \leq \frac{1}{2}\alpha_k^2$$
$$\frac{-1}{2(b_{k+1}\alpha_{k+1}^2 + b_{k+2}\alpha_{k+2}^2 + \ldots)} \leq \frac{-1}{\alpha_k^2}.$$

$$\left|\left\{t : |f(t) - f_{n_{q+1}}(t)| > 0.1\alpha_k\gamma_k\sqrt{\log\log\frac{1}{\alpha_k}}\right\}\right| \leq 6\exp\left(\frac{-0.01\alpha_k^2\gamma_k^2\log\log\left[\frac{1}{\alpha_k}\right]}{\alpha_k^2}\right)$$
$$= \frac{6}{e^{.01k(\log k)^2}}.$$

$$\sum_{k=1}^{\infty}|\{t : |f(t) - f_{n_{q+1}}(t)\}| > 0.1\alpha_k\gamma_k\sqrt{\log\log\frac{1}{\alpha_k}}\}| \leq \sum_{k=1}^{\infty}\frac{6}{e^{.01k(\log k)^2}}$$
$$\leq 6 + 600\sum_{k=2}^{\infty}\frac{1}{k(\log k)^2}.$$

Since $\int_2^{\infty}\frac{1}{x(\log x)^2}dx$ is convergent, $\sum_{k=2}^{\infty}\frac{1}{k(\log k)^2} < \infty$. Hence,

$$\sum_{k=1}^{\infty}\left|\left\{t : |f(t) - f_{n_{q+1}}(t)| > 0.1\alpha_k\gamma_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}\right\}\right| < \infty.$$

So, by Borel-Cantelli for a.e. t with sufficiently large k,

$$|f(t) - f_{n_{q+1}}(t)| \leq 0.1\alpha_k\gamma_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}$$
$$\frac{|f(t) - f_{n_{q+1}}(t)|}{0.1\alpha_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}} \leq 0.1\gamma_k.$$

This gives

$$\frac{|f(t) - f_{n_q}(t)|}{\alpha_k\sqrt{\log\log\left[\frac{1}{\alpha_k}\right]}} > 0.9\gamma_k.$$
$$\frac{|f(t) - f_{n_q}(t)|}{S_{n_{q+1},n_q}f(t)\sqrt{\log\log\left[\frac{1}{S_{n_{q+1},n_q}f(t)}\right]}} > 0.9\gamma_k.$$

But $S'_{n_q}f(t) \approx S_{n_{q+1},n_q}f(t)$.

$$\frac{|f(t) - f_{n_q}(t)|}{S'_{n_q} f(t) \sqrt{\log\log\left[\frac{1}{S'_{n_q} f(t)}\right]}} > 0.9\gamma_k.$$

Since it is true for a subsequence $\{n_q\}$, we have

$$\limsup_{n\to\infty} \frac{|f(t) - f_n(t)|}{S'_n f(t) \sqrt{\log\log\left[\frac{1}{S'_n f(t)}\right]}} > 0.9\gamma_k.$$

Note that

$$\limsup_{n\to\infty} \frac{\log\log\left[\frac{1}{S'^2_n f(t)}\right]}{\log\log\left[\frac{1}{S'_n f(t)}\right]} = \limsup_{n\to\infty} \frac{\log 2 + \log\log\left[\frac{1}{S'_n f(t)}\right]}{\log\log\left[\frac{1}{S'_n f(t)}\right]} = 1.$$

Thus we get,

$$\limsup_{n\to\infty} \frac{|f(t) - f_n(t)|}{S'_n f(t) \sqrt{\log\log\left[\frac{1}{S'^2_n f(t)}\right]}} > 0.9\gamma_k.$$

Here γ_k is a large number. This shows that tail law of the iterated logarithm is not true in general for general dyadic martingales.

Chapter 4

Lower bound in the tail law of the iterated logarithm.

In this chapter, we estimate a lower bound for the tail law of the iterated logarithm for sums of Rademacher functions. We also estimate a lower bound for the tail law of the iterated logarithm for independent bounded random variables.

4.1 Lower bound in the tail LIL for sums of Rademacher functions.

Theorem 32. *Let* $\{r_k\}_{k=1}^{\infty}$ *be the sequence of Rademacher functions and* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of numbers with* $\sum_{k=1}^{\infty} a_k^2 < \infty$*. Suppose that* $\{a_k\}_{k=1}^{\infty}$ *satisfies the property,*

$$\forall \varepsilon > 0, \quad \exists N : \forall n \geq N, \quad |a_n| < \varepsilon \sqrt{\textstyle\sum_{j=n}^{\infty} a_j^2}, \quad i.e., \quad a_n^2 < \varepsilon^2 \textstyle\sum_{j=n}^{\infty} a_j^2.$$

Then for a.e. ω *we have,*

$$\limsup_{n\to\infty} \frac{|\sum_{j=n}^{\infty} a_j r_j(\omega)|}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} a_j^2}}} \geq 1.$$

Proof: Let $\varepsilon > 0$, and assume that $\varepsilon << 1$. Let θ be sufficiently large. Choose $\delta > 0$ so that $(1+\delta)(1-\varepsilon^2) > 1$. . Define $n_1 \le n_2 \le \cdots, n_k \to \infty$ by

$$n_k = \min\left(n : \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k}\right).$$

We first claim that $\displaystyle\sum_{j=n_k+1}^{\infty} a_j^2 \approx \frac{1}{\theta^k}$ for n_k large.

We have,

$$\sum_{j=n_k}^{\infty} a_j^2 = a_{n_k}^2 + \sum_{j=n_k+1}^{\infty} a_j^2.$$

Then if n_k is sufficiently large, $a_{n_k}^2 < \varepsilon^2 \displaystyle\sum_{j=n_k}^{\infty} a_j^2$, and we get

$$\sum_{j=n_k}^{\infty} a_j^2 < \varepsilon^2 \sum_{j=n_k}^{\infty} a_j^2 + \sum_{j=n_k+1}^{\infty} a_j^2.$$

Then,

$$(1-\varepsilon^2)\sum_{j=n_k}^{\infty} a_j^2 < \sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}. \tag{0.1}$$

By definition of n_k,

$$(1-\varepsilon^2)\frac{1}{\theta^k} \le (1-\varepsilon^2)\sum_{j=n_k}^{\infty} a_j^2. \tag{0.2}$$

So from (0.1) and (0.2) we get,

$$(1-\varepsilon^2)\frac{1}{\theta^k} < \sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}.$$

Next, we show

$$\left|\left\{\omega \in [0,1) : \sup_{n \ge n_{k+1}} \left|\sum_{j=n+1}^{\infty} a_j r_j(w)\right| > \sqrt{\frac{2(1+\delta)(1-\varepsilon^2)}{\theta} \sum_{j=n_k+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right|$$
$$< \frac{24}{(k\log\theta)^{(1+\delta)(1-\varepsilon^2)}}.$$

Let us write $S_n'(\omega) = \displaystyle\sum_{j=n+1}^{\infty} a_j r_j(\omega)$ and $s_{n_k}'^2 = \displaystyle\sum_{j=n_k+1}^{\infty} a_j^2$.

Then

$$(1-\varepsilon^2)\theta \leq \frac{s'^2_{n_k}}{s'^2_{n_{k+1}}} \leq \frac{\theta}{1-\varepsilon^2}. \tag{0.3}$$

Now using (0.3) we have,

$$\begin{aligned}
&\left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} |S'_n(w)| > \sqrt{\frac{2(1+\delta)}{\theta} s'^2_{n_k} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&= \left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \frac{|S'_n(w)|}{s'_{n_{k+1}}} > \sqrt{\frac{2(1+\delta)}{\theta} \frac{s'^2_{n_k}}{s'^2_{n_{k+1}}} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&\leq \left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \frac{|S'_n(w)|}{s'_{n_{k+1}}} > \sqrt{\frac{2(1+\delta)}{\theta} \theta(1-\varepsilon^2) \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&= \left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \frac{|S'_n(w)|}{s'_{n_{k+1}}} > \sqrt{2(1+\delta)(1-\varepsilon^2) \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&= \left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} |S'_n(w)| > \sqrt{2(1+\delta)(1-\varepsilon^2) s'^2_{n_{k+1}} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&= \left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \left|\sum_{j=1}^{\infty} a_j r_j(w) - \sum_{j=1}^{n} a_j r_j(w)\right| > \sqrt{2(1+\delta)(1-\varepsilon^2) s'^2_{n_{k+1}} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right|.
\end{aligned}$$

Using Lemma 27, Chapter 3 for the sums of Rademacher functions we have,

$$\begin{aligned}
&\left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \left|\sum_{j=1}^{\infty} a_j r_j(w) - \sum_{j=1}^{n} a_j r_j(w)\right| > \sqrt{2(1+\delta)(1-\varepsilon^2) s'^2_{n_{k+1}} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&\leq 24 \exp\left(\frac{-2(1+\delta)(1-\varepsilon^2) s'^2_{n_{k+1}} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}{2 s'^2_{n_{k+1}}}\right) \\
&= 24 \exp\left(\log\left(\log\left(\frac{1}{s'^2_{n_k}}\right)\right)^{-(1+\delta)(1-\varepsilon^2)}\right) \\
&= 24 \left(\log\left(\frac{1}{s'^2_{n_k}}\right)\right)^{-(1+\delta)(1-\varepsilon^2)} \\
&\leq 24 \left(\frac{1}{\log \theta^k}\right)^{(1+\delta)(1-\varepsilon^2)}.
\end{aligned}$$

So,

$$\left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} \left|\sum_{j=1}^{\infty} a_j r_j(w) - \sum_{j=1}^{n} a_j r_j(\omega)\right| > \sqrt{2(1+\delta)(1-\varepsilon^2) s_{n_{k+1}}'^2 \log\log\left(\frac{1}{s_{n_k}'^2}\right)}\right\}\right|$$

$$< 24\left(\frac{1}{k\log\theta}\right)^{(1+\delta)(1-\varepsilon^2)}.$$

Thus,

$$\left|\left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} |S_n^{'}(\omega)| > \sqrt{\frac{2(1+\delta)(1-\varepsilon^2)}{\theta} s_{n_k}'^2 \log\log\left(\frac{1}{s_{n_k}'^2}\right)}\right\}\right| < 24\left(\frac{1}{k\log\theta}\right)^{(1+\delta)(1-\varepsilon^2)}.$$

Define

$$B := \left\{\omega \in [0,1) : \sup_{n \geq n_{k+1}} |S_n^{'}(\omega)| > \sqrt{\frac{2(1+\delta)(1-\varepsilon^2)}{\theta} s_{n_k}'^2 \log\log\left(\frac{1}{s_{n_k}'^2}\right)}\right\}.$$

Hence $|B| < \dfrac{24}{(\log\theta)^{(1+\delta)(1-\varepsilon^2)}} \dfrac{1}{k^{(1+\delta)(1-\varepsilon^2)}}$.

We recall a Theorem on exponential bounds from [12] page 119.

Theorem 33. *Let $\{X_k\}$ be a sequence of independent random variables with mean zero and variance σ_k^2 and $S_n = \sum_{k=1}^{n} X_k$, $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$. Let $c = \max_{k \leq n}\left|\frac{X_k}{s_n}\right|$ and $\eta > 0$. Then, given $\gamma > 0$, if $c = c(\gamma)$ is sufficiently small and $\eta = \eta(\gamma)$ is sufficiently large, then*

$$P\left(\frac{S_n}{s_n} > \eta\right) > \exp\left(-\frac{\eta^2}{2}(1+\gamma)\right).$$

Set $S_n = \sum_{k=m}^{n} a_k r_k$ so that $s_n^2 = \sum_{k=m}^{n} a_k^2$. Fix $\gamma > 0$ and choose $c(\gamma)$ as in the above Theorem. Suppose n_k is sufficiently large so that $\forall\ l \geq n_k + 1$, we have by assumption

$$|a_l| \leq \frac{c_\gamma}{2}\sqrt{\sum_{j=l}^{\infty} a_j^2} \leq \frac{c_\gamma}{2}\sqrt{\sum_{j=n_k+1}^{\infty} a_j^2}. \tag{0.4}$$

Again we choose n large enough that

$$\sqrt{\sum_{j=n_k+1}^{\infty} a_j^2} \leq 2\sqrt{\sum_{j=n_k+1}^{n} a_j^2}. \tag{0.5}$$

From (0.4) and (0.5) we have,

$$|a_l| \leq \frac{c_\gamma}{2}\sqrt{\sum_{j=n_k+1}^{\infty} a_j^2} \leq \frac{c_\gamma}{2} 2\sqrt{\sum_{j=n_k+1}^{n} a_j^2}.$$

i.e.,

$$\frac{|a_l|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} \leq c_\gamma.$$

Then

$$\max_{n_k+1\leq l\leq n} \frac{|a_l|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} \leq c_\gamma.$$

Using above Theorem (Theorem 33) we have,

$$\left|\left\{\omega \in [0,1) : \frac{|\sum_{j=n_k+1}^{n} a_j r_j(\omega)|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} > \lambda\right\}\right| > \exp\left(-\frac{\lambda^2}{2}(1+\gamma)\right).$$

Choose $\lambda = \sqrt{\frac{2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}$ where $\delta > 0$. Note that for sufficiently large n_k, λ is large enough as required by the Theorem. Then for this λ, we have

$$\begin{aligned}
&\left|\left\{\omega \in [0,1) : \frac{|\sum_{j=n_k+1}^{n} a_j r_j(\omega)|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} > \sqrt{\frac{2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| \\
&> \exp\left(\frac{-2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{s'^2_{n_k}}\right)\frac{(1+\gamma)}{2}\right) \\
&= \exp\left(-(1-\delta/2)\log\log\left(\frac{1}{s'^2_{n_k}}\right)\right) \\
&= \exp\left(\log\log\left(\frac{1}{s'^2_{n_k}}\right)^{-(1-\delta/2)}\right) \\
&= \log\left(\frac{1}{s'^2_{n_k}}\right)^{-(1-\delta/2)} \\
&\geq \left(\frac{1}{\log\left(\frac{\theta^k}{1-\varepsilon^2}\right)}\right)^{1-\frac{\delta}{2}} \\
&= \frac{1}{(k\log\theta + \log(1-\varepsilon^2))^{1-\frac{\delta}{2}}}.
\end{aligned}$$

Therefore for large k we have,

$$\left|\left\{\omega \in [0,1) : \frac{\left|\sum_{j=n_k+1}^{n} a_j r_j(\omega)\right|}{\sqrt{2\sum_{j=n_k+1}^{n} a_j^2 \frac{(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{s_{n_k}'^2}\right)}} > 1\right\}\right| > \frac{1}{2}\frac{1}{(k\log\theta)^{1-\frac{\delta}{2}}}.$$

i.e.,

$$\left|\left\{\omega \in [0,1) : \frac{\left|\sum_{j=n_k+1}^{\infty} a_j r_j(\omega) - \sum_{j=n+1}^{\infty} a_j r_j(w)\right|}{\sqrt{2\sum_{j=n_k+1}^{n} a_j^2 \log\log\left(\frac{1}{s_{n_k}'^2}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}}\right\}\right| > \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Thus,

$$\left|\left\{\omega \in [0,1) : \frac{|S_{n_k}'(w) - S_n'(\omega)|}{\sqrt{2\sum_{j=n_k+1}^{n} a_j^2 \log\log\left(\frac{1}{s_{n_k}'^2}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}}\right\}\right| > \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Using (0.1) for $n \geq n_{k+1}$, we have

$$\begin{aligned}
\sum_{j=n_k+1}^{n} a_j^2 &= \sum_{j=n_k+1}^{\infty} a_j^2 - \sum_{j=n+1}^{\infty} a_j^2 \\
&\geq (1-\varepsilon^2)\sum_{j=n_k}^{\infty} a_j^2 - \frac{1}{\theta^{k+1}} \\
&\geq (1-\varepsilon^2)\frac{1}{\theta^k} - \frac{1}{\theta^{k+1}} \\
&= \frac{1}{\theta^k}(1-\varepsilon^2-\frac{1}{\theta}) \\
&\geq s_{n_k}'^2\left(1-\varepsilon^2-\frac{1}{\theta}\right).
\end{aligned}$$

Thus, $$\sum_{j=n_k+1}^{n} a_j^2 \geq s_{n_k}'^2\left(1-\varepsilon^2-\frac{1}{\theta}\right).$$

Then,

$$\left|\left\{\omega \in [0,1) : \frac{|S'_{n_k}(w) - S'_n(\omega)|}{\sqrt{2s'^2_{n_k}(1-\varepsilon^2-\frac{1}{\theta})\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}}\right\}\right| \geq \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

$$\left|\left\{\omega \in [0,1) : \frac{|S'_{n_k}(w) - S'_n(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)}\right\}\right| \geq \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Define

$$S := \left\{\omega \in [0,1) : \frac{|S'_{n_k}(w) - S'_n(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)}\right\}.$$

Then $|S| \geq \dfrac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}$.

Again define,

$$G := \left\{\omega \in [0,1) : \frac{|S'_{n_{k+1}}(w) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}.$$

Next we show that $S \cap B^c \subset G$. Let $\omega \in S \cap B^c$. Then for all $n \geq n_{k+1}$ we have by triangle inequality,

$$\sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} < \frac{|S'_{n_k}(\omega)-S'_n(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}}$$

$$\leq \frac{|S'_{n_{k+1}}(\omega)-S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} + \frac{|S'_{n_{k+1}}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} + \frac{|S'_n(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}}$$

$$\leq \frac{|S'_{n_{k+1}}(\omega)-S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} + \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} + \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}$$

$$= \frac{|S'_{n_{k+1}}(\omega)-S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} + 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Thus,

$$\sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} < \frac{|S'_{n_{k+1}}(\omega)-S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}}.$$

This proves $S \cap B^c \subset G$. Then $|S-B| \leq |G|$ and so $|S|-|B| \leq |G|$.

So,

$$|G| \geq \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}} - \frac{24}{(k\log\theta)^{(1+\delta)(1-\varepsilon^2)}}.$$

For large k,

$$72(k\log\theta)^{1-\frac{\delta}{2}} \leq (k\log\theta)^{(1+\delta)(1-\varepsilon^2)} \quad \text{i.e.} \quad \frac{1}{3(k\log\theta)^{1-\frac{\delta}{2}}} \geq \frac{24}{(k\log\theta)^{(1+\delta)(1-\varepsilon^2)}}.$$

This gives

$$|G| \geq \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}} - \frac{1}{3(k\log\theta)^{1-\frac{\delta}{2}}} = \frac{1}{6(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Now summing over all k we have,

$$\sum_{k=1}^{\infty}\left|\left\{\omega : \frac{|S'_{n_{k+1}}(w) - S'_{n_k}(\omega)|}{\sqrt{s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}\right|$$
$$\geq \sum_{k=1}^{\infty}\frac{1}{6(k\log\theta)^{1-\frac{\delta}{2}}}$$
$$= \frac{1}{6\log\theta)^{1-\frac{\delta}{2}}}\sum_{k=1}^{\infty}\frac{1}{k^{1-\frac{\delta}{2}}}$$
$$= \infty.$$

So,

$$\sum_{k=1}^{\infty}\left|\left\{\omega : \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}\right| = \infty.$$

We note that $S'_{n_1}(\omega) - S'_{n_2}(\omega) = \sum_{j=n_1+1}^{n_2} a_j r_j(\omega)$, $S'_{n_2}(\omega) - S'_{n_3}(\omega) = \sum_{j=n_2+1}^{n_3} a_j r_j(\omega)$ and so on. Thus $\{S'_{n_k}(\omega) - S'_{n_{k+1}}(\omega)\}$ is a sequence of independent random variables. By Borel-Cantelli Lemma for a.e. ω, there is an infinite sequence $n_1 < n_2 < n_3 < \cdots$ such that,

$$\frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Again, by triangle inequality we have,

$$\frac{|S'_{n_{k+1}}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} + \frac{|S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k}\log\log\left(\frac{1}{s'^2_{n_k}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}. \tag{0.6}$$

We have

$$|B| = \left|\left\{\omega : \sup_{n\geq n_{k+1}}|S'_n(\omega)| > \sqrt{\frac{2(1+\delta)}{\theta}s'^2_{n_k}(1-\varepsilon^2)\log\log\left(\frac{1}{s'^2_{n_k}}\right)}\right\}\right| < \frac{24}{(\log\theta)^{1+\delta}}\frac{1}{k^{(1+\delta)(1-\varepsilon^2)}}.$$

So

$$\sum_{k=1}^{\infty}\left|\left\{\omega : \sup_{n\geq n_{k+1}} |S_n^{'}(\omega)| > \sqrt{\frac{2(1+\delta)}{\theta} s_{n_k}^{'2}(1-\varepsilon^2)\log\log\left(\frac{1}{s_{n_k}^{'2}}\right)}\right\}\right| < \sum_{k=1}^{\infty}\frac{24}{(\log\theta)^{(1+\delta)(1-\varepsilon^2)}}\frac{1}{k^{(1+\delta)(1-\varepsilon^2)}}$$

$$= \frac{24}{(\log\theta)^{(1+\delta)(1-\varepsilon^2)}}\sum_{k=1}^{\infty}\frac{1}{k^{(1+\delta)(1-\varepsilon^2)}}$$

$$< \infty.$$

Then by Borel-Cantelli Lemma for a.e. ω,

$$\sup_{n\geq n_{k+1}} |S_n^{'}(\omega)| \leq \sqrt{\frac{2(1+\delta)}{\theta} s_{n_k}^{'2}(1-\varepsilon^2)\log\log\left(\frac{1}{s_{n_k}^{'2}}\right)} \tag{0.7}$$

for sufficiently large $k \geq M$, say.

Thus from (0.34) and (0.7), for a.e. ω there exists an infinite sequence $n_1 < n_2 < n_3 < \cdots$ such that,

$$\frac{|S_{n_k}^{'}(\omega)|}{\sqrt{2s_{n_k}^{'2}\log\log\left(\frac{1}{s_{n_k}^{'2}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 3\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Since this is true for a subsequence $\{n_k\}_{k=1}^{\infty}$, we have,

$$\frac{|S_n^{'}(\omega)|}{\sqrt{2s_n^{'2}\log\log\left(\frac{1}{s_n^{'2}}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2-\frac{1}{\theta}\right)} - 3\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Letting $\theta \nearrow \infty, \varepsilon \searrow 0, \delta \searrow 0, \gamma \searrow 0$, we get,

$$\frac{|S_n^{'}(\omega)|}{\sqrt{2s_n^{'2}\log\log\left(\frac{1}{s_n^{'2}}\right)}} \geq 1.$$

$$\limsup_{n\to\infty}\frac{|S_n^{'}(\omega)|}{\sqrt{2s_n^{'2}\log\log\left(\frac{1}{s_n^{'2}}\right)}} \geq 1.$$

Then for a.e. ω we have,

$$\limsup_{n\to\infty} \frac{|\sum_{j=n+1}^{\infty} a_j r_j(\omega)|}{\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \log\log\left(\frac{1}{\sum_{j=n+1}^{\infty} a_j^2}\right)}} \geq 1.$$

Thus for a.e. ω we have,

$$\limsup_{n\to\infty} \frac{|\sum_{j=n}^{\infty} a_j r_j(\omega)|}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log\left(\frac{1}{\sum_{j=n}^{\infty} a_j^2}\right)}} \geq 1.$$

This completes the proof of our result. ■

Example. Let us consider a sequence $\{a_n\}$ given by $a_n = \frac{1}{n^2}$. Clearly $\sum_{k=1}^{\infty} a_k^2 < \infty$. Now we show that the sequence $\{a_k\}$ satisfies the hypothesis of our Theorem. We need to show,

$$\forall \varepsilon > 0, \quad \exists N : \quad \forall n \geq N, \quad a_n^2 < \varepsilon^2 \sum_{j=n}^{\infty} a_j^2.$$

For

$$\sum_{j=n}^{\infty} a_j^2 = \sum_{j=n}^{\infty} \frac{1}{j^4} \approx \int_n^{\infty} x^{-4} dx = \frac{1}{3n^3}$$

So

$$\frac{1}{n^4} < \varepsilon^2 \frac{1}{3n^3} \quad \text{i.e.} \quad \frac{1}{n} < \frac{\varepsilon^2}{3} \quad \text{which is true.}$$

Thus, the sequence $\{a_k\}$ where $a_k = \frac{1}{k^2}$ satisfies the hypothesis of our Theorem. Hence we have,

$$\limsup_{n\to\infty} \frac{|S_n'(w)|}{\sqrt{2s_n'^2 \log\log \frac{1}{s_n'^2}}} \geq 1.$$

Example. Let us consider a sequence of numbers $\{a_k\}$ given by $a_k = \frac{1}{2^k}$. For this sequence,

$$|S_n'(w)| \leq \sum_{k=n+1}^{\infty} |a_k r_k| \leq \sum_{j=n}^{\infty} \frac{1}{k^2} = \frac{1}{2^n}$$

and

$$s_n'^2 = \sum_{j=n+1}^{\infty} a_j^2 = \sum_{j=n+1}^{\infty} \frac{1}{4^j} = \frac{1}{3.2^{2n}} \approx \frac{1}{2^{2n}}.$$

Then

$$\begin{aligned}
\limsup_{n\to\infty} \frac{|S_n'(w)|}{\sqrt{2s_n'^2 \log\log \frac{1}{s_n'^2}}} &\approx \limsup_{n\to\infty} \frac{|S_n'(w)|}{\sqrt{2\frac{1}{2^{2n}} \log\log\left(\frac{1}{\frac{1}{2^{2n}}}\right)}} \\
&\leq \limsup_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n}\sqrt{2\log\log 2^{2n}}} \\
&= \limsup_{n\to\infty} \frac{1}{\sqrt{2(\log 2n + \log\log 2)}} \\
&= 0.
\end{aligned}$$

This shows that the given sequence does not satisfy the conclusion of our Theorem. Here we show that the given sequence does not satisfy our assumption either. For

$$a_n^2 = \frac{1}{2^{2n}} < \varepsilon^2 \sum_{j=n}^{\infty} a_j^2 = \varepsilon^2 \frac{3}{2.2^{2n}} \quad \text{i.e.} \quad 1 < \frac{3\varepsilon^2}{2}$$

which is not true. ■

4.2 The tail law of the iterated logarithm for independent random variables.

In this section, we prove the tail law of the iterated logarithm for symmetric, bounded, independent, and identically distributed random variables. There is much literature for the law of the iterated logarithm for independent random variables, however, the method we use here is different.

First we prove some Lemmas which will be used in our main Theorem.

Lemma 34. *If $\{X_n; n \geq 1\}$ are symmetric, bounded, independent, and identically distributed random variables, $-1 \leq X_i \leq 1$, $Y_i = a_i X_i$, $-a_i \leq Y_i \leq a_i$ with $E(X_n) = 0$, $E(X_n^2) = 1$, and $\{a_n; n \geq 1\}$ are real constants, then $\forall\ \eta > 0$, $\forall\ \lambda > 0$*

$$P\left(\left\{\omega : \sup_{m\geq 1} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}\right) \leq C \exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=1}^{\infty} a_k^2}\right).$$

Proof: Using the fact $\sup_k |a_k| > \lambda \Rightarrow \sup_k a_k > \lambda$ or $\sup_k(-a_k) > \lambda$, we have,

$$\left\{\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\} = \left\{\omega : \sup_{1\leq m\leq n} \sum_{i=1}^m Y_i(\omega) > \lambda\right\} \bigcup \left\{\omega : \sup_{1\leq m\leq n} -\sum_{i=1}^m Y_i(\omega) > \lambda\right\}.$$

So for any $\gamma > 0$,

$$\begin{aligned}
&P\left(\left\{\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(w)| > \lambda\right\}\right) \\
&\leq P\left(\left\{\omega : \sup_{1\leq m\leq n} \textstyle\sum_{i=1}^m Y_i(\omega) > \lambda\right\}\right) + P\left(\left\{\omega : \sup_{1\leq m\leq n} -\textstyle\sum_{i=1}^m Y_i(\omega) > \lambda\right\}\right) \\
&= P\left(\left\{\omega : \sup_{1\leq m\leq n} \exp\left(\textstyle\sum_{i=1}^m \gamma Y_i(\omega)\right) > e^{\gamma\lambda}\right\}\right) + P\left(\left\{\omega : \sup_{1\leq m\leq n} \exp\left(-\textstyle\sum_{i=1}^m \gamma Y_i(\omega)\right) > e^{\gamma\lambda}\right\}\right).
\end{aligned}$$

Now by Lemma 11, Chapter 1, $\sum_{i=1}^m Y_i(w)$ is a martingale. Also, $\exp(\gamma x)$ is convex and increasing. Then using Lemma 10, Chapter 1, $\exp\left(\gamma \sum_{i=1}^m Y_i(\omega)\right), \exp\left(-\gamma \sum_{i=1}^m Y_i(\omega)\right)$ are submartingales. Using Doob's maximal inequality (Theorem 12, Chapter 1), we have

$$\begin{aligned}
&P\left(\left\{\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}\right) \\
&\leq \frac{1}{e^{\gamma\lambda}} \int_\Omega \exp\left(\gamma |\textstyle\sum_{i=1}^n Y_i(w)|\right) dP + \frac{1}{e^{\gamma\lambda}} \int_\Omega \exp\left(\gamma |\textstyle\sum_{i=1}^n Y_i(w)|\right) dP \\
&= \frac{2}{e^{\gamma\lambda}} \int_\Omega \exp\left(\gamma |\textstyle\sum_{i=1}^n Y_i(w)|\right) dP.
\end{aligned}$$

Hence we have,

$$P\left(\left\{\omega : \sup_{1\le m\le n} |\textstyle\sum_{i=1}^{m} Y_i(\omega)| > \lambda\right\}\right) \le \frac{2}{e^{\gamma\lambda}} \int_\Omega \exp\left(\gamma|\textstyle\sum_{i=1}^{n} Y_i(\omega)|\right) dP. \tag{0.8}$$

Using Hoeffding's Theorem (Theorem 9, Chapter 1) we have,

$$P\left(\{\omega : |\textstyle\sum_{i=1}^{n} Y_i| \ge \lambda\}\right) \le 2\exp\left(\frac{-\lambda^2}{2\sum_{i=1}^{n} a_i^2}\right). \tag{0.9}$$

Next, we prove $\forall \eta > 0$ and $\Omega = [0,1)$

$$\int_\Omega \exp\left(\gamma|\textstyle\sum_{i=1}^{n} Y_i(\omega)|\right) dP \le 2\sqrt{2\pi} M \exp\left((1/2+\eta))\gamma^2 \textstyle\sum_{i=1}^{n} a_i^2\right).$$

We have a elementary result which follows from the Fubini's theorem,

$$\int_\Omega e^f\, dP = \int_{-\infty}^{\infty} e^\lambda P(\{f > \lambda\}) d\lambda. \tag{0.10}$$

Using (0.9) and (0.10), we have

$$\begin{aligned}
\int_\Omega \exp\left(\gamma|\textstyle\sum_{i=1}^{n} Y_i(\omega)|\right) dP &= \int_{-\infty}^{\infty} e^\lambda P\left(\{\omega : \gamma|\textstyle\sum_{i=1}^{n} Y_i(\omega| > \lambda\}\right) d\lambda \\
&= \int_{-\infty}^{\infty} e^\lambda P\left(\{\omega : |\textstyle\sum_{i=1}^{n} Y_i(\omega)| > \frac{\lambda}{\gamma}\}\right) d\lambda \\
&\le \int_{-\infty}^{\infty} e^\lambda \quad 2\exp\left(\frac{-\lambda^2}{2\gamma^2\sum_{i=1}^{n} a_i^2}\right) d\lambda \\
&= 2\int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\gamma^2\sum_{i=1}^{n} a_i^2}\left(\lambda^2 - 2\gamma^2\textstyle\sum_{i=1}^{n} a_i^2\lambda\right)\right) d\lambda \\
&= 2\int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\gamma^2\sum_{i=1}^{n} a_i^2}\left[\lambda^2 - 2\gamma^2\textstyle\sum_{i=1}^{n} a_i^2\lambda \pm (\gamma^2\textstyle\sum_{i=1}^{n} a_i^2\lambda)^2\right]\right) d\lambda \\
&= 2\exp\left(\frac{\gamma^2\sum_{i=1}^{n} a_i^2}{2}\right)\int_{-\infty}^{\infty} \exp\left(\frac{-\left[\lambda - \gamma^2\sum_{i=1}^{n} a_i^2\right]^2}{2\gamma^2\sum_{i=1}^{n} a_i^2}\right) d\lambda.
\end{aligned}$$

Substitute $u = \dfrac{\lambda - \gamma^2\sum_{i=1}^{n} a_i^2}{\gamma\sqrt{\sum_{i=1}^{n} a_i^2}}$ so that, $\gamma\sqrt{\sum_{i=1}^{n} a_i^2}du = d\lambda$. Then,

$$\begin{aligned}
\int_\Omega \exp\left(\gamma|\textstyle\sum_{i=1}^{n} Y_i(\omega)|\right) dP &= 2\exp\left(\frac{\gamma^2\sum_{i=1}^{n} a_i^2}{2}\right)\int_{-\infty}^{\infty} \exp\left(\frac{-u^2}{2}\right)\gamma\sqrt{\textstyle\sum_{i=1}^{n} a_i^2}du \\
&= 2\gamma\sqrt{\textstyle\sum_{i=1}^{n} a_i^2} \quad \exp\left(\frac{\gamma^2\sum_{i=1}^{n} a_i^2}{2}\right)\int_{-\infty}^{\infty} \exp\left(\frac{-u^2}{2}\right) du \\
&= 2\sqrt{2\pi}\gamma \quad \sqrt{\textstyle\sum_{i=1}^{n} a_i^2} \quad \exp\left(\frac{\gamma^2\sum_{i=1}^{n} a_i^2}{2}\right).
\end{aligned}$$

So we have,

$$\int_\Omega \exp\left(\gamma\,|\textstyle\sum_{i=1}^n Y_i(\omega)|\right) dP = 2\sqrt{2\pi} \quad \gamma\sqrt{\textstyle\sum_{i=1}^n a_i^2} \quad \exp\left(\frac{\gamma^2 \sum_{i=1}^n a_i^2}{2}\right).$$

This is not exactly what we want but it is the best we can get. Note that for given $\eta > 0$ there exists $M = M(\eta) > 0$ such that for all $X > 0$, $X\exp(\frac{1}{2}X^2) \leq M\exp\left([\frac{1}{2}+\eta]X^2\right)$. Then using this in above estimate we get,

$$\int_\Omega \exp\left(\gamma\,|\textstyle\sum_{i=1}^n Y_i(\omega)|\right) dP = 2\sqrt{2\pi}M(\eta)\exp\left([1/2+\eta]\gamma^2 \textstyle\sum_{i=1}^n a_i^2\right). \tag{0.11}$$

Choose $\gamma = \dfrac{\lambda}{\sum_{i=1}^n a_i^2}$. Then with this γ, we have from (0.8) and (0.11),

$$\begin{aligned}
P\left(\left\{\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}\right) &\leq \frac{C}{e^{\gamma\lambda}}\int_\Omega \exp\left(\gamma|\textstyle\sum_{i=1}^n Y_i(w)|\right) dP \\
&\leq \frac{2\sqrt{2\pi}C}{e^{\gamma\lambda}} M(\eta)\exp\left((1/2+\eta)\gamma^2\textstyle\sum_{i=1}^n a_i^2\right) \\
&= 2\sqrt{2\pi}CM(\eta)\exp\left((1/2+\eta-1)\frac{\lambda^2}{\sum_{i=1}^n a_i^2}\right) \\
&\leq 2\sqrt{2\pi}CM(\eta)\exp\left((-1/2+\eta)\frac{\lambda^2}{\sum_{i=1}^\infty a_i^2}\right).
\end{aligned}$$

Thus,

$$P\left(\left\{\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^\infty a_i^2}\right).$$

Set $E_n := \left\{\omega : \sup_{1\leq m\leq n} |\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}$. Clearly $E_n \subset E_{n+1}$. Define $E = \bigcup_{n=1}^\infty E_n$. Then we have, $\lim_{n\to\infty} |E_n| = |E|$. Hence,

$$\begin{aligned}
P\left(\left\{\omega : \sup_{m\geq 1} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right\}\right) &= \lim_{n\to\infty} |E_n| \\
&= \lim_{n\to\infty} P\left(\omega : \sup_{1\leq m\leq n} |\textstyle\sum_{i=1}^m Y_i(\omega)| > \lambda\right) \\
&\leq \lim_{n\to\infty} CM(\eta)\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^\infty a_i^2}\right) \\
&= CM(\eta)\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^\infty a_i^2}\right).
\end{aligned}$$

Thus,

$$P\left(\left\{\omega : \sup_{m\geq 1} |\textstyle\sum_{i=1}^{m} Y_i(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^{\infty} a_i^2}\right).$$

Now by the choice of η, we have

$$P\left(\left\{\omega : \sup_{m\geq 1} |\textstyle\sum_{i=1}^{m} Y_i(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{i=1}^{\infty} a_i^2}\right).$$

This completes the proof of our Lemma. ■

Next, we prove an exponential estimate for the tail sums of random variables.

Lemma 35. *If* $\{X_n; n \geq 1\}$ *are symmetric, bounded, independent, and identically distributed random variables,* $-1 \leq X_i \leq 1$, $Y_i = a_i X_i$, $-a_i \leq Y_i \leq a_i$ *with* $E(X_n) = 0$, $E(X_n^2) = 1$, *and* $\{a_n; n \geq 1\}$ *are real constants, then* $\forall\ \eta > 0$, $\forall\ \lambda > 0$,

$$P\left(\left\{\omega : \sup_{m\geq n} |\textstyle\sum_{k=1}^{\infty} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\right\}\right) \leq 12\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).$$

Proof: Fix n. Define,

$$b_k = \begin{cases} 0, & \text{if } k \leq n; \\ a_k, & \text{if } k > n. \end{cases}$$

Using the previous Lemma for $\{\sum_{k=1}^{m} b_k X_k\}$, we get

$$P\left(\left\{\omega : \sup_{m\geq 1} |\textstyle\sum_{k=1}^{m} b_k X_k(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=1}^{\infty} b_k^2}\right).$$

$$P\left(\left\{\omega : \sup_{m\geq n} |\textstyle\sum_{k=n+1}^{m} a_k X_k(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).$$

Thus,

$$P\left(\left\{\omega : \sup_{m\geq n} |\textstyle\sum_{k=1}^{m} a_k X_k(\omega) - \sum_{k=1}^{n} a_k X_k(\omega)| > \lambda\right\}\right) \leq CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right). \tag{0.12}$$

Let $N >> n$ where n is fixed. Then using Levy's inequality (Lemma 6, Chapter 1) we get,

$$P\left(\left\{\omega : \max_{n\leq j\leq N-1} |\textstyle\sum_{i=0}^{j-n} Y_{N-i}(\omega)| > \lambda\right\}\right) \leq 2P\left(\left\{\omega : |\textstyle\sum_{k=0}^{N-n-1} Y_{N-k}(\omega)| > \lambda\right\}\right).$$

$$P\left(\left\{\omega : \max_{n\leq m\leq N-1} |\textstyle\sum_{i=1}^{N} Y_i(\omega) - \sum_{i=1}^{m} Y_i(\omega)| > \lambda\right\}\right) \leq 2P\left(\left\{\omega : |\textstyle\sum_{i=1}^{N} Y_k(\omega) - \sum_{i=1}^{n} Y_k(\omega)| > \lambda\right\}\right).$$

So,

$$P\left(\left\{\omega : \max_{n\le m\le N} |\textstyle\sum_{i=1}^{N} Y_i(\omega) - \sum_{i=1}^{m} Y_i(\omega)| > \lambda\right\}\right) \le 2P\left(\left\{\omega : |\textstyle\sum_{i=1}^{N} Y_k(\omega) - \sum_{i=1}^{n} Y_k(\omega)| > \lambda\right\}\right). \tag{0.13}$$

Since $N >> n$, we have from (0.12),

$$P\left(\left\{\omega : |\textstyle\sum_{k=1}^{N} Y_k(\omega) - \sum_{k=1}^{n} Y_k(\omega)| > \lambda\right\}\right) \le CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right). \tag{0.14}$$

Hence from (0.13) and (0.14) we get,

$$P\left(\left\{\omega : \sup_{N\ge m\ge n} |\textstyle\sum_{k=1}^{N} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\right\}\right) \le CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).$$

Let $E_N := \{\omega : \sup_{N\ge m\ge n} |\sum_{k=1}^{N} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\}$ and $E := \bigcup_{k=1}^{\infty} E_k$. Then by Lemma 2 Chapter 1, we have $\lim_{N\to\infty} |E_N| = |E|$.

For sufficiently large N,

$$\sup_{m\ge n} |\textstyle\sum_{k=1}^{\infty} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda \Longrightarrow \sup_{N\ge m\ge n} \left|\sum_{k=1}^{N} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)\right| > \lambda.$$

This shows that $x \in E_N$ for sufficiently large N i.e. $x \in E$. Thus,

$$\begin{aligned}
P\left(\left\{\omega : \sup_{m\ge n} |\textstyle\sum_{k=1}^{\infty} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\right\}\right) &\le |E| \\
&\le \lim_{N\to\infty} |E_N| \\
&\le \lim_{N\to\infty} P\left(\omega : \sup_{N\ge m\ge n} |\sum_{k=1}^{N} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\right) \\
&\le \lim_{N\to\infty} CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right) \\
&\le CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).
\end{aligned}$$

Hence,

$$P\left(\left\{\omega : \sup_{m\ge n} |\textstyle\sum_{k=1}^{\infty} Y_k(\omega) - \sum_{k=1}^{m} Y_k(\omega)| > \lambda\right\}\right) \le CM(\eta)\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).$$

This completes the proof of the Lemma. ■

We now prove our main result.

Theorem 36. *If $\{X_n; n \geq 1\}$ are symmetric, bounded, independent, and identically distributed random variables, $-1 \leq X_i \leq 1$, $Y_i = a_i X_i$, $-a_i \leq Y_i \leq a_i$ with $E(X_n) = 0$, $E(X_n^2) = 1$, and $\{a_n; n \geq 1\}$ are real constants satisfying*

$$\sum_{j=1}^{\infty} a_j^2 < \infty \quad \text{and} \quad \forall \varepsilon > 0, \quad \exists N : \forall n \geq N, \quad |a_n| < \varepsilon \sqrt{\textstyle\sum_{j=n}^{\infty} a_j^2}, \quad \text{i.e.,} \quad a_n^2 < \varepsilon^2 \textstyle\sum_{j=n}^{\infty} a_j^2,$$

then for a.e. ω,

$$\limsup_{n \to \infty} \frac{\sum_{j=n}^{\infty} a_j X_j(\omega)}{\sqrt{2 \sum_{j=n}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} a_j^2}}} \geq 1.$$

Proof: Let $\varepsilon > 0$, $\eta > 0$ and assume that $\varepsilon << 1$, $\eta << 1$. Let θ be sufficiently large. Choose $\delta > 0$ so that $(1+\delta)(1-\varepsilon^2)(1-\eta) > 1$. Define $n_1 \leq n_2 \leq \cdots, n_k \to \infty$ by

$$n_k = \min\left(n : \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k}\right).$$

We first claim that $\displaystyle\sum_{j=n_k+1}^{\infty} a_j^2 \approx \frac{1}{\theta^k}$ for n_k large.

We have,

$$\sum_{j=n_k}^{\infty} a_j^2 = a_{n_k}^2 + \sum_{j=n_k+1}^{\infty} a_j^2.$$

Then if n_k is sufficiently large, $a_{n_k}^2 < \varepsilon^2 \displaystyle\sum_{j=n_k}^{\infty} a_j^2$, and we get

$$\sum_{j=n_k}^{\infty} a_j^2 < \varepsilon^2 \sum_{j=n_k}^{\infty} a_j^2 + \sum_{j=n_k+1}^{\infty} a_j^2.$$

Then,

$$(1-\varepsilon^2) \sum_{j=n_k}^{\infty} a_j^2 < \sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}. \tag{0.15}$$

By definition of n_k,

$$(1-\varepsilon^2)\frac{1}{\theta^k} \leq (1-\varepsilon^2) \sum_{j=n_k}^{\infty} a_j^2. \tag{0.16}$$

So from (0.15) and (0.16) we get,

$$(1-\varepsilon^2)\frac{1}{\theta^k} < \sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}.$$

This gives,

$$(1-\varepsilon^2)\theta \le \frac{\sum_{j=n_k+1}^{\infty} a_j^2}{\sum_{j=n_{k+1}+1}^{\infty} a_j^2} \le \frac{\theta}{1-\varepsilon^2}. \tag{0.17}$$

Using (0.17) we have,

$$P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \left|\sum_{i=n+1}^{\infty} a_i X_i(\omega)\right| > \sqrt{\frac{4(1+\delta)}{\theta} \sum_{j=n_k+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right)$$

$$= P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \frac{\left|\sum_{i=n+1}^{\infty} a_i X_i(\omega)\right|}{\sqrt{\sum_{j=n_{k+1}+1}^{\infty} a_j^2}} > \sqrt{\frac{4(1+\delta)}{\theta} \frac{\sum_{j=n_k+1}^{\infty} a_j^2}{\sum_{j=n_{k+1}+1}^{\infty} a_j^2} \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right)$$

$$\le P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \frac{\left|\sum_{i=n+1}^{\infty} a_i X_i(\omega)\right|}{\sqrt{\sum_{j=n_{k+1}+1}^{\infty} a_j^2}} > \sqrt{\frac{4(1+\delta)}{\theta}\theta(1-\varepsilon^2) \log\log \frac{1}{\sum_{j=n_{k+1}+1}^{\infty} a_j^2}}\right\}\right)$$

$$= P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \left|\sum_{i=n+1}^{\infty} a_i X_i(\omega)\right| > \sqrt{4(1+\delta)(1-\varepsilon^2) \sum_{j=n_{k+1}+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right)$$

$$= P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \left|\sum_{j=1}^{\infty} a_j X_j(\omega) - \sum_{j=1}^{n} a_j X_j(\omega)\right| > \sqrt{4(1+\delta)(1-\varepsilon^2) \sum_{j=n_{k+1}+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right).$$

Now using Lemma 34 Chapter 4, we get

$$P\left(\left\{\omega : \sup_{n\ge n_{k+1}} \left|\sum_{j=1}^{\infty} a_j X_j(\omega) - \sum_{j=1}^{n} a_j X_j(\omega)\right| > \sqrt{4(1+\delta)(1-\varepsilon^2) \sum_{j=n_{k+1}+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}\right\}\right)$$

$$\le 12CM(\eta) \exp\left(\frac{(-1+\eta)4(1+\delta)(1-\varepsilon^2)\sum_{j=n_{k+1}+1}^{\infty} a_j^2 \quad \log\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)}{2\sum_{j=n_{k+1}+1}^{\infty} a_j^2}\right)$$

$$= 12CM(\eta) \exp\left(\log\left(\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)^{2(-1+\eta)(1+\delta)(1-\varepsilon^2)}\right)$$

$$= 12CM(\eta)\left(\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)^{2(-1+\eta)(1+\delta)(1-\varepsilon^2)}$$

$$\le 12CM(\eta)\left(\log \theta^k\right)^{2(-1+\eta)(1+\delta)(1-\varepsilon^2)}.$$

Thus,

$$P\left(\left\{\omega : \sup_{n\geq n_{k+1}}\left|\sum_{j=1}^{\infty}a_jX_j(\omega)-\sum_{j=1}^{n}a_jX_j(\omega)\right| > \sqrt{4(1+\delta)(1-\varepsilon^2)\sum_{j=n_{k+1}+1}^{\infty}a_j^2\log\log\frac{1}{\sum_{j=n_k+1}^{\infty}a_j^2}}\right\}\right)$$

$$\leq 12CM(\eta)\left(\frac{1}{\log\theta^k}\right)^{2(1-\eta)(1+\delta)(1-\varepsilon^2)}$$

$$= 12CM(\eta)\frac{1}{k^{2(1-\eta)(1+\delta)(1-\varepsilon^2)}}\frac{1}{(\log\theta)^{2(1-\eta)(1+\delta)(1-\varepsilon^2)}}.$$

Summing over all k, we have

$$\sum_{k=1}^{\infty}P\left(\left\{\omega : \sup_{n\geq n_{k+1}}\left|\sum_{j=1}^{\infty}a_jX_j(\omega)-\sum_{j=1}^{n}a_jX_j(\omega)\right| > \sqrt{\frac{4(1+\delta)(1-\varepsilon^2)}{\theta}\sum_{j=n_k+1}^{\infty}a_j^2\log\log\frac{1}{\sum_{j=n_k+1}^{\infty}a_j^2}}\right\}\right)$$

$$\leq 12CM(\eta)\frac{1}{(\log\theta)^{2(1-\eta)(1+\delta)(1-\varepsilon^2)}}\sum_{k=1}^{\infty}\frac{1}{k^{2(1-\eta)(1+\delta)(1-\varepsilon^2)}}$$

$$< \infty.$$

By Borel-Cantelli Lemma for a.e. ω we have,

$$\sup_{n\geq n_{k+1}}\left|\sum_{j=1}^{\infty}a_jX_j(\omega)-\sum_{j=1}^{n}a_jX_j(\omega)\right| \leq \sqrt{\frac{4(1+\delta)(1-\varepsilon^2)}{\theta}\sum_{j=n_k+1}^{\infty}a_j^2\log\log\frac{1}{\sum_{j=n_k+1}^{\infty}a_j^2}}$$

for sufficiently large k, say $k\geq M$.

Hence for a.e ω,

$$\sup_{n\geq n_{k+1}}\frac{\left|\sum_{j=1}^{\infty}a_jX_j(\omega)-\sum_{j=1}^{n}a_jX_j(\omega)\right|}{\sqrt{2\sum_{j=n_k+1}^{\infty}a_j^2\log\log\frac{1}{\sum_{j=n_k+1}^{\infty}a_j^2}}} \leq \sqrt{\frac{2(1+\delta)(1-\varepsilon^2)}{\theta}} \tag{0.18}$$

for sufficiently large k, say $k\geq M$.

We now state a Theorem on exponential bounds (For proof see [12] page 119).

Theorem 37. *Let $\{X_k\}$ be a sequence of independent random variables with mean zero and variance σ_k^2 and $S_n=\sum_{k=1}^{n}X_k$, $s_n^2=\sum_{k=1}^{n}\sigma_k^2$. Let $c=\max_{k\leq n}\left|\frac{X_k}{s_n}\right|$ and $\eta>0$. Then, given $\gamma>0$, if $c=c(\gamma)$ is sufficiently small and $\lambda=\lambda(\gamma)$ is sufficiently large, then*

$$P\left(\frac{S_n}{s_n}>\lambda\right) > \exp\left(-\frac{\lambda^2}{2}(1+\gamma)\right).$$

Set $S_n = \sum_{k=m}^{n} a_k X_k$ so that $s_n^2 = \sum_{k=m}^{n} a_k^2$.

Fix $\gamma > 0$. Choose $c(\gamma)$ as in the above Theorem. Suppose n_k is sufficiently large so that $\forall l \geq n_k + 1$, we have by assumption

$$|a_l| \leq \frac{c_\gamma}{2} \sqrt{\sum_{j=l}^{\infty} a_j^2} \leq \frac{c_\gamma}{2} \sqrt{\sum_{j=n_k+1}^{\infty} a_j^2}. \tag{0.19}$$

Again we choose n large enough so that

$$\sqrt{\sum_{j=n_k+1}^{\infty} a_j^2} \leq 2 \sqrt{\sum_{j=n_k+1}^{n} a_j^2}. \tag{0.20}$$

From (0.19) and (38) we have,

$$|a_l| \leq \frac{c_\gamma}{2} \sqrt{\sum_{j=n_k+1}^{\infty} a_j^2} \leq \frac{c_\gamma}{2} 2 \sqrt{\sum_{j=n_k+1}^{n} a_j^2}.$$

i.e.,

$$\frac{|a_l|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} \leq c_\gamma.$$

Then

$$\max_{n_k+1 \leq l \leq n} \frac{|a_l|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} \leq c_\gamma.$$

Using the above Theorem we have,

$$P\left(\left\{\omega \in [0,1) : \frac{|\sum_{j=n_k+1}^{n} a_j X_j(\omega)|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} > \lambda\right\}\right) > \exp\left(-\frac{\lambda^2}{2}(1+\gamma)\right).$$

Choose $\lambda = \sqrt{\frac{2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)}$ where $\delta > 0$. Then for sufficiently large n_k,

λ is large as required by the above Theorem.

$$P\left(\left\{\omega \in [0,1) : \frac{|\sum_{j=n_k+1}^{n} a_j X_j(\omega)|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} > \sqrt{\frac{2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)}\right\}\right)$$

$$> \exp\left(\frac{-2(1-\delta/2)}{(1+\gamma)} \log\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)\frac{(1+\gamma)}{2}\right)$$

$$= \exp\left(-(1-\delta/2)\log\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)\right)$$

$$= \exp\left(\log\left(\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)\right)^{-(1-\delta/2)}\right)$$

$$= \left(\log\left(\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)\right)^{-(1-\delta/2)}$$

$$\geq \left(\log\left(\frac{\theta^k}{1-\varepsilon^2}\right)\right)^{-(1-\delta/2)}$$

$$= \frac{1}{(k\log\theta - \log(1-\varepsilon^2))^{1-\frac{\delta}{2}}}.$$

Therefore for large k we have,

$$P\left(\left\{\omega \in [0,1) : \frac{|\sum_{j=n_k+1}^{n} a_j X_j(\omega)|}{\sqrt{\sum_{j=n_k+1}^{n} a_j^2}} > \sqrt{\frac{2(1-\delta/2)}{(1+\gamma)} \log\log\theta^k}\right\}\right) > \frac{1}{2}\frac{1}{(k\log\theta)^{1-\frac{\delta}{2}}}.$$

So,

$$P\left(\left\{\omega : \frac{\left|\sum_{j=n_k+1}^{\infty} a_j X_j(\omega) - \sum_{j=n+1}^{\infty} a_j X_j(w)\right|}{\sqrt{2\sum_{j=n_k+1}^{n} a_j^2 \log\log\frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}}\right\}\right) > \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}. \tag{0.21}$$

Using (0.16) for $n \geq n_{k+1}$, we have

$$\begin{aligned}
\sum_{j=n_k+1}^{n} a_j^2 &= \sum_{j=n_k+1}^{\infty} a_j^2 - \sum_{j=n+1}^{\infty} a_j^2 \\
&\geq (1-\varepsilon^2) \sum_{j=n_k}^{\infty} a_j^2 - \frac{1}{\theta^{k+1}} \\
&\geq (1-\varepsilon^2) \frac{1}{\theta^k} - \frac{1}{\theta^{k+1}} \\
&= \frac{1}{\theta^k}(1-\varepsilon^2 - \frac{1}{\theta}) \\
&\geq \sum_{j=n_k+1}^{\infty} a_j^2 (1-\varepsilon^2 - \frac{1}{\theta}).
\end{aligned}$$

Thus,

$$\sum_{j=n_k+1}^{n} a_j^2 \geq \sum_{j=n_k+1}^{\infty} a_j^2 (1-\varepsilon^2 - \frac{1}{\theta}). \tag{0.22}$$

Then using (0.22) in (0.21) we have,

$$P\left(\left\{\omega : \frac{|\sum_{j=n_k+1}^{\infty} a_j X_j(\omega) - \sum_{j=n+1}^{\infty} a_j X_j(w)|}{\sqrt{\left(2\sum_{j=n_k+1}^{\infty} a_j^2 (1-\varepsilon^2 - \frac{1}{\theta}) \log\log \frac{1}{\sum_{j=n_k+1}^{n+1} a_j^2}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}}\right\}\right) > \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

$$P\left(\left\{\omega : \frac{|\sum_{j=n_k+1}^{\infty} a_j X_j(\omega) - \sum_{j=n+1}^{\infty} a_j X_j(\omega)|}{\sqrt{\left(2\sum_{j=n_k+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n_k+1}^{\infty} a_j^2}\right)}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2 - \frac{1}{\theta}\right)}\right\}\right) > \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Let us write $S'_{n_k}(\omega) = \sum_{j=n_k+1}^{\infty} a_j X_j(\omega)$ and $s'^2_{n_k} = \sum_{j=n_k+1}^{\infty} a_j^2$. We now claim:

$$P\left(\left\{w : \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}\right) >$$
$$P\left(\left\{\omega : \frac{|S'_{n_k}(\omega) - S'_{n}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1-\varepsilon^2 - \frac{1}{\theta}\right)}\right\}\right).$$

Using (0.18) for $n \geq n_{k+1}$ we have

$$\begin{aligned}\frac{|S'_{n_k}(\omega) - S'_n(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} &= \frac{|S'_{n_k}(\omega) - S'_{n_{k+1}}(w) + S'_{n_{k+1}}(\omega) - S'_n(w)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} \\ &\leq \frac{|S'_n(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} + \frac{|S'_{n_{k+1}}(v)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} + \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} \\ &< \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} + \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} + \frac{|S'_{n_{k+1}}(w) - S'_{n_k}(w)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}}.\end{aligned}$$

But we have,

$$\frac{|S'_{n_k}(\omega) - S'_n(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)},$$

so that

$$\sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} < \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} + \sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}} + \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}}.$$

$$\therefore \qquad \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

This proves our claim. Consequently we get,

$$P\left(\left\{\omega : \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}\right) \geq \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

Then,

$$\sum_{k=1}^{\infty} P\left(\omega : \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right) \geq \sum_{k=1}^{\infty} \frac{1}{2(k\log\theta)^{1-\frac{\delta}{2}}}.$$

So,

$$\sum_{k=1}^{\infty} P\left(\left\{\omega : \frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}\right\}\right) = \infty.$$

Here $S'_{n_2}(\omega) - S'_{n_1}(\omega) = \sum_{j=n_1+1}^{n_2} a_j X_j(\omega)$, $S'_{n_3}(\omega) - S'_{n_2}(\omega) = \sum_{j=n_2+1}^{n_3} a_j r_j(\omega)$ and so on. Note that $X_j's$ are independent random variables. Thus $\{S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)\}$ is a sequence of independent random variables. By Borel-Cantelli Lemma for a.e. ω, there is an infinite sequence $n_1 < n_2 < \cdots$ such that,

$$\frac{|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Using, $|S'_{n_{k+1}}(\omega) - S'_{n_k}(\omega)| < |S'_{n_{k+1}}(\omega)| + |S'_{n_k}(\omega)|$, we get

$$\frac{|S'_{n_{k+1}}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} + \frac{|S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Using (0.18) we get,

$$\sqrt{\frac{1+\delta}{\theta}} + \frac{|S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 2\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Thus we have,

$$\frac{|S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} > \sqrt{\frac{(1-\delta/2)}{(1+\gamma)}\left(1 - \varepsilon^2 - \frac{1}{\theta}\right)} - 3\sqrt{\frac{(1+\delta)(1-\varepsilon^2)}{\theta}}.$$

Letting $\theta \nearrow \infty$, $\varepsilon \searrow 0$, $\delta \searrow 0$, and $\gamma \searrow 1$ we have,

$$\frac{|S'_{n_k}(\omega)|}{\sqrt{2s'^2_{n_k} \log\log \frac{1}{s'^2_{n_k}}}} \geq 1.$$

Since this is true for a subsequence $\{n_k\}_{k=1}^{\infty}$, we have,

$$\limsup_{n\to\infty} \frac{|S'_n(\omega)|}{\sqrt{2s'^2_n \log\log \frac{1}{s'^2_n}}} \geq 1.$$

Thus we have for a.e. ω,

$$\limsup_{n\to\infty} \frac{\sum_{j=n}^{\infty} a_j X_j(\omega)}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} a_j^2}}} \geq 1.$$

This proves the tail law of the iterated logarithm for independent random variables. ■

Next, we state an iterated logarithm for weighted averages by Chow and Teicher (See [5])

Theorem 38. *If $\{X_n : n \geq 1\}$ are independent identically distributed random variables with $E(X_n) = 0, E(X_n^2) = 1$ and $\{a_n : n \geq 1\}$ are real constants satisfying*

(i) $\dfrac{a_n^2}{\sum_{j=n}^{\infty} a_j^2} \leq \dfrac{c}{n}$ *for some* $c \in (0, \infty)$

(ii) $\sum_{n=1}^{\infty} a_n^2 < \infty$,

then

$$P\left(\limsup_{n\to\infty} \frac{\sum_{j=n}^{\infty} a_j X_j}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log \dfrac{1}{\sum_{j=n}^{\infty} a_j^2}}} = 1\right) = 1.$$

Corollary 39. *In Theorem 38 admissible values for a_n are $a_n = \pm n^{\beta}$, $\beta < -\frac{1}{2}$; $a_n = \pm n^{\beta_1}(\log n)^{\beta_2}$, $\beta < -\frac{1}{2}$, or, $\beta_1 = -\frac{1}{2} > \beta_2$ etc.*

Our condition is weaker than given by the above theorem. So we find sequences which satisfy our condition but not their condition.

Here $a_n^2 < \varepsilon^2 \sum_{j=n}^{\infty} a_j^2$ is equivalent to find $f(x)$ such that

$$\begin{aligned}
f(x) &\le \varepsilon(x) \int_x^\infty f(t)dt \\
\frac{f(x)}{\int_x^\infty f(t)dt} &\le \varepsilon(x) \\
\frac{d}{dx}\left(\log \int_x^\infty f(t)dt\right) &\ge -\varepsilon(x) \\
\log \int_x^\infty f(t)dt &\ge -\int \varepsilon(x)dx. \\
\therefore \qquad \int_x^\infty f(t)dt &\ge e^{-\int \varepsilon(x)dx}.
\end{aligned}$$

Let $f(y) = \frac{1}{\sqrt{y}} e^{-\sqrt{y}}$. So,

$$\sum_{j=n}^{\infty} a_j^2 = \sum_{j=n}^{\infty} \frac{1}{\sqrt{j}} e^{-\sqrt{j}} \approx \int_n^\infty \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx = 2\int_{\sqrt{n}}^\infty e^{-u} du \approx e^{-\sqrt{n}}$$

$$\therefore \quad a_n^2 = \frac{1}{\sqrt{n}} e^{-\sqrt{n}} \le \varepsilon e^{-\sqrt{n}} = \varepsilon \sum_{j=n}^{\infty} a_j^2.$$

Thus

$$a_n^2 \le \varepsilon \sum_{j=n}^{\infty} a_j^2.$$

So we need to choose $\{a_n\}$ such that

(i) $a_n = \sqrt{\varepsilon(n) \exp\left(\int_1^n \varepsilon(y)dy\right)}$.

(ii) a_n goes to zero slower than $\frac{1}{n}$.

(ii) $\int_1^\infty \varepsilon(y)dy = \infty$.

Under these $\{a_n\}$ satisfies $a_n^2 \le \varepsilon \sum_{j=n}^{\infty} a_j^2$ but not $a_n^2 \le \frac{c\varepsilon}{n} \sum_{j=n}^{\infty} a_j^2$.

For this,

$$\begin{aligned}
\sum_{j=n}^{\infty} a_j^2 &= \sum_{j=n}^{\infty} \left[\varepsilon(j) \exp\left(-\int_1^j \varepsilon(y)dy \right) \right] \\
&\geq \int_n^{\infty} \varepsilon(x) \exp\left(-\int_1^x \varepsilon(y)dy \right) dx \\
&= \int_{\int_1^n \varepsilon(y)dy}^{\infty} e^{-u} du \qquad \text{(Using substitution} \quad -\int_1^x \varepsilon(y)dy = u \quad \text{and} \quad \frac{du}{dx} = \varepsilon(x)) \\
&= \exp\left(-\int_1^n \varepsilon(y)dy \right).
\end{aligned}$$

Thus,

$$\sum_{j=n}^{\infty} a_j^2 \geq \exp\left(-\int_1^n \varepsilon(y)dy \right). \tag{0.23}$$

But $a_n^2 = \varepsilon \exp\left(-\int_1^n \varepsilon(y)dy \right)$. Then using (0.23), we have $a_n^2 \leq \varepsilon \sum_{j=n}^{\infty} a_j^2$. On the other hand,

$$\begin{aligned}
\sum_{j=n}^{\infty} a_j^2 &= \sum_{j=n}^{\infty} \varepsilon(j) \exp\left(-\int_1^j \varepsilon(y)dy \right) \\
&\leq \int_{n-1}^{\infty} \varepsilon(x) \exp\left(-\int_1^x \varepsilon(y)dy \right) \\
&= \int_{\int_1^{n-1} \varepsilon(y)dy}^{\infty} e^{-u} du \\
&= \exp\left(-\int_1^{n-1} \varepsilon(y)dy \right).
\end{aligned}$$

So we have

$$\sum_{j=n}^{\infty} a_j^2 \leq \exp\left(-\int_1^{n-1} \varepsilon(y)dy \right). \tag{0.24}$$

Then using (0.24) we have,

$$\begin{aligned}
a_n^2 &= \varepsilon(n)\exp\left(-\int_1^n \varepsilon(y)dy\right)\\
&= \varepsilon(n)\exp\left(-\int_1^{n-1} \varepsilon(y)dy - \int_{n-1}^n \varepsilon(y)dy\right)\\
&\geq \varepsilon(n)\sum_{j=n}^{\infty} a_j^2 \exp\left(-\int_1^x \varepsilon(y)dy\right)\\
&\geq \varepsilon(n)\sum_{j=n}^{\infty} a_j^2 \frac{1}{2} \quad \text{for large} \quad n.
\end{aligned}$$

Hence,

$$a_j^2 \geq \frac{1}{2}\varepsilon(n)\sum_{j=n}^{\infty} a_j^2.$$

This implies, $a_j^2 \leq \frac{1}{2}\varepsilon(n)\sum_{j=n}^{\infty} a_j^2$ is not possible. We give some examples:

(i) $\varepsilon(n) = \frac{1}{n^\alpha}; \quad 0 < \alpha < 1$

(ii) $\varepsilon(n) = \frac{\log n}{n}$

Both of the above examples satisfy the properties:

(i) $\varepsilon(n) \to 0$ slower than $\frac{1}{n}$

(ii) $\int_1^\infty \varepsilon(y)dy = \infty$

For $\varepsilon(n) = \frac{\log n}{n}$, $a_n = \sqrt{\frac{\log n}{n}\exp\left(-\int_3^n \frac{\log y}{y}\right)dy} = \sqrt{\frac{\log n}{n}\exp\left(-\frac{(\log n)^2}{2}\right)}$ where the term $\exp\left(-\frac{(\log n)^2}{2}\right)$ goes to zero faster than $\frac{\log n}{n}$. Also $h(y) = \varepsilon(y)\exp\left(-\int_1^y \varepsilon(x)dx\right)$ is decreasing. For,

$$\begin{aligned}
h^{'}(y) &= \varepsilon^{'}(y)\exp\left(-\int_1^y \varepsilon(x)dx\right) + \varepsilon(y)\exp\left(-\int_1^y \varepsilon(x)dx\right)(-\varepsilon(y))\\
&= \varepsilon^{'}(y)\exp\left(-\int_1^y \varepsilon(x)dx\right) + (\varepsilon(y))^2\exp\left(-\int_1^y \varepsilon(x)dx\right)\\
&\leq 0 \qquad (\because \varepsilon^{'}(y) < 0, \varepsilon(y) \searrow 0).
\end{aligned}$$

4.3 Lower bound in the tail law of the iterated logarithm for dyadic martingales.

In this section, we obtain a lower bound for the tail law of the iterated logarithm for dyadic martingales.

Theorem 40. Let $\{f_n\}_{n=0}^{\infty}$ be a dyadic martingale. Assume that

(i) there exists $L < \infty$: $\dfrac{S_n'^2 f(x)}{S_n'^2 f(y)} < L, \forall x, y \in I_{nj}; n = 1, 2, 3 \cdots; j \in \{1, 2, \dots, 2^n - 1\}$

(ii) $\dfrac{d_n^2(x)}{S_{n-1}'^2 f(x)} = \dfrac{[f_n(x) - f_{n-1}(x)]^2}{\sum_{k=n}^{\infty}[f_n(x) - f_{n-1}(x)]^2} < \varepsilon$ for some $\varepsilon > 0$ i.e. jumps are not large and

(iii) $\sup\limits_{0 \le i \le n} \dfrac{d_i(x)}{E(\sum_{k=1}^{n} d_k^2(x))} \xrightarrow{\text{uniformly}} 0.$

Then,

$$\limsup_{n\to\infty} \frac{|f(x) - f_n(x)|}{\sqrt{2S_n'^2 f(x) \log\log \dfrac{1}{S_n'^2 f(x)}}} \ge \frac{1}{\sqrt{2L}}$$

for a.e. x.

Proof: Let $0 < \delta < 1$ and θ be sufficiently large. Define $n_1 \le n_2 \le \cdots$ by,

$$n_k(x) = \min\left\{n : x \in I_{nj} \quad \text{for some} \quad j \in \{1, 2, \dots, 2^n\} \quad \text{and} \quad \forall y \in I_{nj} \quad S_n'^2 f(y) < \frac{1}{\theta^k}\right\}.$$

Again for a given $x \in [0, 1]$, $x \in I_{nj}$ where I_{nj} is a dyadic cube of side length $\dfrac{1}{2^{n_k}}$. Now, $\forall y \in I_{nj}$, $S_{n_k}'^2 f(y) < \dfrac{1}{\theta^k}$. But $S_{n_k-1}'^2 f(z) \ge \dfrac{1}{\theta^k}$ for some $z \in I_{nj}'$ where I_{nj}' is a parent dyadic cube of I_{nj} with length $|I_{nj}'| = \dfrac{1}{2^{n_k-1}}$. For $y \in I_{nj}$, we have $y \in I_{nj}'$. So $z, y \in I_{nj}'$. Then using assumption, we have

$$S_{n_k-1}'^2 f(z) < L S_{n_k-1}'^2 f(y).$$

Thus,

$$\frac{1}{L} S_{n_k-1}'^2 f(z) < S_{n_k-1}'^2 f(y). \tag{0.25}$$

We have, $S'^2_{n_k-1}f(y) = d^2_{n_k}(y) + S'^2_{n_k}f(y)$ so that $S'^2_{n_k}f(y) = -d^2_{n_k}(y) + S'^2_{n_k-1}f(y)$. Then using $\dfrac{d^2_{n_k}(y)}{S'^2_{n_k-1}f(y)} < \varepsilon$, we have

$$S'^2_{n_k}f(y) > -\varepsilon S'^2_{n_k-1}f(y) + S'^2_{n_k-1}f(y) = (1-\varepsilon)S'^2_{n_k-1}f(y).$$

Without loss of generality, take $\varepsilon = 1/2$. Then we have,

$$S'^2_{n_k}f(y) > (1-\varepsilon)S'^2_{n_k-1}f(y) = \frac{1}{2}S'^2_{n_k-1}f(y). \tag{0.26}$$

Then from (0.25) and (0.26) we have,

$$\frac{1}{\theta^k} > S'^2_{n_k}f(y) > \frac{1}{2}S'^2_{n_k-1}f(y) > \frac{1}{2L}S'^2_{n_k-1}f(z) > \frac{1}{2L}\frac{1}{\theta^k}.$$

Thus,

$$\frac{1}{2L}\frac{1}{\theta^k} < S'^2_{n_k}f(y) < \frac{1}{\theta^k}. \tag{0.27}$$

We first prove for a.e. x,

$$\limsup_{n\to\infty} \frac{|\sum_{j=n+1}^{\infty} d_j(x)|}{\sqrt{2S'^2_{n_k}f(x)\log\log\dfrac{1}{S'^2_{n_k}f(x)}}} \le 4\sqrt{\frac{(1+\delta)L}{\theta}}$$

for sufficiently large $k \ge M$. Let $\lambda > 0$ and I_{nj} be dyadic interval of generation n. Using Lemma 25, Chapter 2 we have,

$$\frac{1}{|I_{nj}|}\left|\left\{x \in I_{nj} : \sup_{n\ge n_{k+1}} |\textstyle\sum_{j=n+1}^{\infty} d_j(x)| > \lambda\right\}\right| \le 12\exp\left(\frac{-\lambda^2}{8||S'_{n_{k+1}}|_{I_{nj}}||^2_\infty}\right). \tag{0.28}$$

Here,

$$\begin{aligned}
S'^2_{n_{k+1}}f(y)|_{I_{n_j}} &< \frac{1}{\theta^{k+1}} \\
\| S'^2_{n_{k+1}}f|_{I_{n_j}} \|^2_\infty &\le \frac{1}{\theta^{k+1}}. \\
\frac{-\lambda^2}{\| S'_{n_{k+1}}f|_{I_{n_j}} \|^2_\infty} &\le -\lambda^2\theta^{k+1}.
\end{aligned}$$

Thus,

$$\exp\left(\frac{-\lambda^2}{8\| S'_{n_{k+1}}f|_{I_{n_j}} \|^2_\infty}\right) \le \exp\left(\frac{-\lambda^2\theta^{k+1}}{8}\right). \tag{0.29}$$

Then using (0.29) in (0.28), we get

$$\left|\left\{x \in I_{nj} : \sup_{n \geq n_{k+1}} |\textstyle\sum_{j=n+1}^{\infty} d_j(x)| > \lambda\right\}\right| \leq 12|I_{n_j}| \exp\left(\frac{-\lambda^2\theta^{k+1}}{8}\right).$$

Now summing over all such I_{nj}, we get

$$\left|\left\{x \in [0,1) : \sup_{n \geq n_{k+1}} |\textstyle\sum_{j=n+1}^{\infty} d_j(x)| > \lambda\right\}\right| \leq 12 \exp\left(\frac{-\lambda^2\theta^{k+1}}{8}\right). \tag{0.30}$$

Then using (0.30) we have,

$$\begin{aligned}
&\left|\left\{x \in [0,1] : \sup_{n \geq n_{k+1}} |\sum_{j=n+1}^{\infty} d_j(x)| > \sqrt{\frac{16L(1+\delta)}{\theta} S'^2_{n_k} f(x) \log\log \frac{1}{S'^2_{n_k} f(x)}}\right\}\right| \\
&\leq \left|\left\{x \in [0,1] : \sup_{n \geq n_{k+1}} |\sum_{j=n+1}^{\infty} d_j(x)| > \sqrt{\frac{16L(1+\delta)}{\theta} \frac{1}{2L\theta^k} \log\log \theta^k}\right\}\right| \\
&= \left|\left\{x \in [0,1] : \sup_{n \geq n_{k+1}} |\sum_{j=n+1}^{\infty} d_j(x)| > \sqrt{\frac{8(1+\delta)}{\theta^{k+1}} \log\log \theta^k}\right\}\right| \\
&\leq 12 \exp\left(-\frac{\frac{8(1+\delta)}{\theta^{k+1}}\theta^{k+1} \log\log \theta^k}{8}\right) \\
&= 12 \exp\left(\log(k\log\theta)^{-(1+\delta)}\right) \\
&= 12 \left(\frac{1}{k\log\theta}\right)^{1+\delta} \\
&= \frac{12}{(k\log\theta)^{1+\delta}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left|\left\{x \in [0,1] : \sup_{n \geq n_{k+1}} |\sum_{j=n+1}^{\infty} d_j(x)| > \sqrt{\frac{16L(1+\delta)}{\theta} S'^2_{n_k} f(x) \log\log \frac{1}{S'^2_{n_k} f(x)}}\right\}\right| \\
&< \sum_{k=1}^{\infty} \frac{12}{(k\log\theta)^{1+\delta}} \\
&< \infty.
\end{aligned}$$

So by Borel-Cantelli Lemma, for a.e. x,

$$\sup_{n \geq n_{k+1}} |\sum_{j=n+1}^{\infty} d_j(x)| \leq \sqrt{\frac{16L(1+\delta)}{\theta} S'^2_{n_k} f(x) \log\log \frac{1}{S'^2_{n_k} f(x)}}$$

for sufficiently large k, say $k \geq M$. Thus, for a.e. x,

$$\sup_{n \geq n_{k+1}} \frac{|\sum_{j=n+1}^{\infty} d_j(x)|}{\sqrt{2S_{n_k}'^2 f(x) \log\log \frac{1}{S_{n_k}'^2 f(x)}}} \leq 4\sqrt{\frac{(1+\delta)L}{\theta}} \tag{0.31}$$

for sufficiently large $k \geq M$.

Next, we prove a Lemma which is a version of the strong-form of the Borel-Cantelli Lemma.

Lemma 41. Let F_k be a collection of dyadic cubes whose union is $[0,1]$ and F_{k+1} is a refinement of F_k. Suppose that the maximum length of the elements of F_k tends to zero. Suppose $\mathcal{E}_k \subset F_k$ has the property:

$$\forall Q \in F_k, \qquad \left| Q \cap \bigcup_{J \in \mathcal{E}_{k+1}} J \right| > |Q| \frac{1}{\sqrt{k}}.$$

Set $E_k = \bigcup_{J \in \mathcal{E}_k} J$. Then for a.e. x, $x \in E_k$ i.o.

Proof: We show, $|\{x : x \in E_k \quad \text{i.o.}\}| = 1$. So,

$$\begin{aligned} 1 - |\{x : x \in E_k \quad \text{i.o.}\}| &= |\{x : x \in E_k \quad \text{i.o.}\}^c| \\ &= |\{x : x \in \cap_{N=1}^{\infty} \cup_{k=N}^{\infty} E_k\}^c| \\ &= |\{x : x \in \cup_{N=1}^{\infty} \cap_{k=N}^{\infty} E_k^c\}|. \end{aligned}$$

We note that,

$$\bigcup_{k=1}^{\infty} E_k^c \subset \bigcup_{k=2}^{\infty} E_k^c \subset \ldots \Longrightarrow \bigcup_{N=1}^{\infty} \left(\bigcap_{k=N}^{\infty} E_k^c \right) = \lim_{N \to \infty} \bigcap_{k=N}^{\infty} E_k^c.$$

So it suffices to show that,

$$1 - |\{x : x \in E_k \quad \text{i.o.}\}| = \lim_{N \to \infty} |\{x : x \in \bigcap_{k=N}^{\infty} E_k^c\}| = 0.$$

We estimate:

$$
\begin{aligned}
|\{x : x \in \bigcap_{k=N}^{l} E_k^c\}| &= |E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_l^c| \\
&= \left| \left(\bigcup_{Q \in (E_1^c \cap E_2^c \cap E_3^c \cap \dots E_{l-1}^c)} Q \right) \cap E_l^c \right| \\
&\le \sum_{Q \in (E_1^c \cap E_2^c \cap E_3^c \cap \dots E_{l-1}^c)} |Q \cap E_l^c|.
\end{aligned}
$$

We have $|Q \cap E_l| > \frac{1}{\sqrt{l-1}}|Q|$. Then,

$$
\begin{aligned}
|E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_l^c| &\le \sum_{Q \in (E_1^c \cap E_2^c \cap E_3^c \cap \dots E_{l-1}^c)} |Q \cap E_l^c| \\
&\le \sum_{Q \in (E_1^c \cap E_2^c \cap E_3^c \cap \dots E_{l-1}^c)} \left(1 - \frac{1}{\sqrt{l-1}}\right) |Q| \\
&= \left(1 - \frac{1}{\sqrt{l-1}}\right) \sum_{Q \in (E_1^c \cap E_2^c \cap E_3^c \cap \dots E_{l-1}^c)} |Q| \\
&= (1 - \frac{1}{\sqrt{l-1}})|E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_{l-1}^c| \\
&\le \left(1 - \frac{1}{\sqrt{l-1}}\right) \left(1 - \frac{1}{\sqrt{l-2}}\right) |E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_{l-2}^c| \\
&\vdots \\
&\le \left(1 - \frac{1}{\sqrt{l-1}}\right) \left(1 - \frac{1}{\sqrt{l-2}}\right) \dots \left(1 - \frac{1}{\sqrt{N+1}}\right) |E_N^c \cap E_{N+1}^c|.
\end{aligned}
$$

Again,

$$
\begin{aligned}
|E_N^c \cap E_{N+1}^c| &= |\bigcup_{Q \in E_N^c} (Q \cap E_{N+1}^c)| \\
&\le \sum_{Q \in E_N^c} |Q \cap E_{N+1}^c| \\
&\le \sum_{Q \in E_N^c} \left(1 - \frac{1}{\sqrt{N}}\right) |Q| \\
&\le \left(1 - \frac{1}{\sqrt{N}}\right).
\end{aligned}
$$

Then,

$$|E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_l^c| \leq \left(1 - \frac{1}{\sqrt{l-1}}\right)\left(1 - \frac{1}{\sqrt{l-2}}\right)\dots\left(1 - \frac{1}{\sqrt{N}}\right) = \prod_{j=N}^{l}\left(1 - \frac{1}{\sqrt{j}}\right).$$

So,

$$|E_N^c \cap E_{N+1}^c \cap E_3^c \cap \dots E_l^c| \leq \prod_{j=N}^{l}\left(1 - \frac{1}{\sqrt{j}}\right). \tag{0.32}$$

Then using (0.32) and the fact that $\sum a_k$ diverges if and only if $\prod_{j=1}^{\infty}(1 - a_j) = 0$, we get

$$\begin{aligned}
\lim_{N\to\infty} |\{x : x \in \bigcap_{k=N}^{\infty} E_k^c\}| &= \lim_{N\to\infty}\lim_{l\to\infty} |\{x : x \in \bigcap_{k=N}^{l} E_k^c\}| \\
&= \lim_{N\to\infty}\left(\lim_{l\to\infty}\prod_{j=N}^{l}\left(1 - \frac{1}{\sqrt{j}}\right)\right) \\
&= 0.
\end{aligned}$$

Hence, we have,

$$|\{x : x \in E_k \quad \text{i.o.}\}| = 1 \quad \text{i.e.} \quad x \in E_k \quad \text{i.o.}.$$

We first state a Central Limit Theorem for martingales from [8].

Theorem 42. *Let $\{S_{ni}, \mathfrak{F}_{ni}, -\infty < i < \infty\}$ be a zero-mean, square-integrable martingale array with differences $X_{ni} = S_{ni} - S_{n.i-1}$, and let η^2 be an a.s. finite random variable. Suppose that $\sup_{n,i} < \infty$ holds and $S_{n,-\infty} = 0$ a.s. If*

(i) $\sup_i X_{ni} \xrightarrow{P} 0$

(ii) $\sum_i X_{ni}^2 \xrightarrow{P} \eta^2$

(iii) $E\left(\sup_i X_{ni}^2\right)$ *is bounded in n and*

(iv) for all n, i $\mathfrak{F}_{ni} \subseteq \mathfrak{F}_{n+1,i}$,

then $S_{n\infty} \xrightarrow{d} Z$ where the random variable Z has characteristic function $E(\exp -\frac{1}{2}\eta^2 t^2)$.

If $\eta^2 > 0$ *a.s., then*

$$\frac{S_{n\infty}}{U_{n\infty}} \xrightarrow{d} N(0,1) \quad \textit{where} \quad U^2_{n\infty} = \sum_{i=-\infty}^{\infty} X^2_{ni}.$$

Next consider the following martingale array:

i=	$\cdots -2$	-1	0	1	2	3	4	5	$6\cdots$
n=1	$\cdots\frac{S_0}{s_1}$	$\frac{S_0}{s_1}$	$\frac{S_0}{s_1}$	$\frac{S_1}{s_1}$	$\frac{S_1}{s_1}$	$\frac{S_1}{s_1}$	$\frac{S_1}{s_1}$	$\frac{S_1}{s_1}$	$\frac{S_1}{s_1}\cdots$
n=2	$\cdots\frac{S_0}{s_2}$	$\frac{S_0}{s_2}$	$\frac{S_0}{s_2}$	$\frac{S_1}{s_2}$	$\frac{S_2}{s_2}$	$\frac{S_2}{s_2}$	$\frac{S_2}{s_2}$	$\frac{S_2}{s_2}$	$\frac{S_2}{s_2}\cdots$
n=3	$\cdots\frac{S_0}{s_3}$	$\frac{S_0}{s_3}$	$\frac{S_0}{s_3}$	$\frac{S_1}{s_3}$	$\frac{S_2}{s_3}$	$\frac{S_3}{s_3}$	$\frac{S_3}{s_3}$	$\frac{S_3}{s_3}$	$\frac{S_3}{s_3}\cdots$
n=4	$\cdots\frac{S_0}{s_4}$	$\frac{S_0}{s_4}$	$\frac{S_0}{s_4}$	$\frac{S_1}{s_4}$	$\frac{S_2}{s_4}$	$\frac{S_3}{s_4}$	$\frac{S_4}{s_4}$	$\frac{S_4}{s_4}$	$\frac{S_4}{s_4}\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For the above martingale array the Theorem 42 becomes:

Theorem 43. *Let* $\{S_n, \mathfrak{F}_n, n \geq 1\}$ *be a zero-mean, square integrable martingale. Let* $s_n^2 = E(\sum_{k=1}^n d_k^2(x))$. *Let* η^2 *be an a.s. finite random variable. Suppose* $\sup_{n, 0\le i\le n} E\left(\frac{S_i^2}{s_n^2}\right) = \sup_{n,0\le i\le n} \frac{1}{s_n^2} E\left(S_i^2\right) < \infty$. *If*

(i) $\sup_{0\le i\le n} \left|\frac{S_i}{s_n} - \frac{S_{i-1}}{s_n}\right| = \sup_{0\le i\le n} \frac{|d_i|}{s_n} \xrightarrow{P} 0$

(ii) $E\left[\sup_{0\le i\le n}\left(\frac{S_i}{s_n} - \frac{S_{i-1}}{s_n}\right)^2\right] = E\left(\sup_{0\le i\le n}\frac{d_i^2}{s_n^2}\right) = \frac{1}{s_n^2}E\left(\sup_{0\le i\le n} d_i^2\right)$ *is bounded in n*

(iii) $\sum_{i=0}^{n}\left(\frac{S_i}{s_n} - \frac{S_{i-1}}{s_n}\right)^2 = \frac{1}{s_n^2}\sum_{i=0}^{n} d_i^2 \xrightarrow{P} \eta^2$ *and*

(iv) $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$, *then*

$$S_{n,\infty} = \sum_{i=-\infty}^{\infty} X_{ni} = \sum_{i=-\infty}^{\infty}\left(\frac{S_i}{s_n} - \frac{S_{i-1}}{s_n}\right) = \frac{S_n}{s_n} - \frac{S_0}{s_n} = \frac{S_n - S_0}{s_n} \xrightarrow{d} Z,$$

where Z is a function with distribution $\exp\left(-\dfrac{\eta^2 t^2}{2}\right)$. *If $\eta^2 > 0$ a.s. then,*

$$\frac{S_n - S_0}{\sqrt{\sum_{i=1}^{n} d_i^2}} \xrightarrow{d} N(0,1).$$

Here condition (i) is satisfied by our assumption. For (iii), we note that $Sf(x) < \infty$ and $S_n'^2 f(x) < LS_n'^2 f(y)$. This gives us $S^2 f(x) < LS^2 f(y), \forall x, y \in I_{0j}$. Thus

$$\sum_{k=1}^{n} d_k^2(x) < L \sum_{k=1}^{n} d_k^2(y).$$

Without loss of generality assume $d_k(y)$ exits for some y, say, $y = \frac{1}{2}$. So

$$\sum_{k=1}^{\infty} d_k^2(x) < L \sum_{k=1}^{\infty} d_k^2\left(\frac{1}{2}\right).$$

Hence

$$\forall n \quad \sum_{i=1}^{n} d_i^2(x) < L \sum_{i=1}^{\infty} d_i^2\left(\frac{1}{2}\right).$$

By orthogonality we have,

$$s_n^2 = E(f_n^2) = E[(\textstyle\sum_{k=1}^{n} d_k(x))^2] = E[\sum_{k=1}^{n} d_k^2(x)].$$

Then

$$s_n^2 = E[\textstyle\sum_{k=1}^{n} d_k^2(x)] \leq E\left[L\left(\sum_{k=1}^{\infty} d_k^2(\frac{1}{2})\right)\right] = L\sum_{k=1}^{\infty} d_k^2(\frac{1}{2}) < \infty.$$

Therefore sequence $\left\{\dfrac{\sum_{k=1}^{n} d_k^2(x)}{E(\sum_{k=1}^{n} d_k^2(x))}\right\}_n = \left\{\dfrac{\sum_{k=1}^{n} d_k^2(x)}{s_n^2}\right\}_n$ is an increasing and bounded sequence. Hence by the Monotone Convergence Theorem it converges. So we have,

$$\frac{\sum_{k=1}^{n} d_k^2(x)}{E(\sum_{k=1}^{n} d_k^2(x))} \to \frac{Sf(x)}{s} > 0.$$

Hence condition (iii) is satisfied. Moreover condition (ii) is satisfied by our assumption (iii). Next we consider Q, a dyadic cube of side length $\frac{1}{2^{n_k}}$. Then employing Theorem 43 on the cube Q we have,

$$\frac{\sum_{j=1}^{n} d_j(x)}{\sqrt{\sum_{j=1}^{n} d_j^2(x)}} \xrightarrow{d} N(0,1).$$

Hence we have,

$$\frac{1}{|Q|}\left|\left\{x\in Q:\frac{\sum_{j=1}^{n}d_j(x)}{\sqrt{\sum_{j=1}^{n}d_j^2(x)}}>\lambda\right\}\right|\longrightarrow\frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\infty}e^{-\frac{t^2}{2}}dt.$$

Choose $\lambda=\sqrt{2(1-\frac{\delta}{2})\log\log\theta^k}$. For the chosen λ we choose n so large that $n\geq n_{k+1}$ such that

$$\frac{1}{|Q|}\left|\left\{x\in Q:\frac{\sum_{j=n_k+1}^{n}d_j(x)}{\left[\sum_{j=n_k+1}^{n}d_j^2(x)\right]^{\frac{1}{2}}}>\lambda\right\}\right|\geq\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\infty}e^{-\frac{t^2}{2}}dt.$$

Using the inequality $\frac{\lambda}{1+\lambda^2}e^{\frac{-\lambda^2}{2}}\leq\int_{\lambda}^{\infty}e^{\frac{-u^2}{2}}du$, we have

$$\frac{1}{|Q|}\left|\left\{x\in Q:\sum_{j=1}^{n}d_j(x)>\sqrt{2\left(1-\frac{\delta}{2}\right)\sum_{j=n_k+1}^{n}d_j^2(x)\log\log\theta^k}\right\}\right|\geq\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\lambda}{1+\lambda^2}e^{-\frac{\lambda^2}{2}}.$$

Again using $\frac{\lambda}{1+\lambda^2}\geq\frac{1}{2\lambda}$ we have,

$$\begin{aligned}
&\frac{1}{|Q|}\left|\left\{x\in Q:\sum_{j=1}^{n}d_j(x)>\sqrt{2\left(1-\frac{\delta}{2}\right)\sum_{j=n_k+1}^{n}d_j^2(x)\log\log\theta^k}\right\}\right|\quad\geq\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{2\lambda}e^{-\frac{\lambda^2}{2}}\\
&=\frac{1}{4\sqrt{2\pi}}\frac{1}{\sqrt{2(1-\frac{\delta}{2})\log\log\theta^k}}\exp\left(-\frac{2(1-\frac{\delta}{2})\log\log\theta^k}{2}\right)\\
&=\frac{1}{8\sqrt{\pi(1-\frac{\delta}{2})}}\frac{1}{\sqrt{\log(k\log\theta)}}\frac{1}{(k\log\theta)^{1-\frac{\delta}{2}}}\\
&=\frac{1}{8\sqrt{\pi(1-\frac{\delta}{2})}}\frac{1}{(k\log\theta)^{\frac{\delta}{4}}}\frac{1}{(k\log\theta)^{1-\frac{\delta}{2}}}\\
&=\frac{1}{8\sqrt{\pi(1-\frac{\delta}{2})}}\frac{1}{(k\log\theta)^{1-\frac{\delta}{4}}}\\
&=\frac{A}{(k\log\theta)^{1-\frac{\delta}{4}}}\quad\text{(say)}.
\end{aligned}$$

Now,

$$\begin{aligned}\sum_{j=n_k+1}^{n} d_j^2(x) &= \sum_{j=n_k+1}^{\infty} d_j^2(x) - \sum_{j=n+1}^{\infty} d_j^2(x)\\ &= S_{n_k}'^2 f(x) - S_n'^2 f(x)\\ &> \frac{1}{2L}\frac{1}{\theta^k} - \frac{1}{\theta^{k+1}} \qquad (n \geq n_{k+1})\\ &= \left(\frac{1}{2L} - \frac{1}{\theta}\right)\frac{1}{\theta^k}.\end{aligned}$$

Thus,

$$\sum_{j=n_k+1}^{n} d_j^2(x) > \left(\frac{1}{2L} - \frac{1}{\theta}\right)\frac{1}{\theta^k}. \tag{0.33}$$

Using (0.33), we have

$$\frac{1}{|Q|}\left|\left\{x \in Q : \sum_{j=n_k+1}^{n} d_j(x) > \sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)\frac{1}{\theta^k}\log\log\theta^k}\right\}\right| \geq \frac{A}{(k\log\theta+\alpha)^{1-\frac{\delta}{4}}}.$$

Again,

$$\begin{aligned}\left|\sum_{j=n_k+1}^{n} d_j(x)\right| &= \left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n+1}^{\infty} d_j(x)\right|\\ &\leq \left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right| + \left|\sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right| + \left|\sum_{j=n+1}^{\infty} d_j(x)\right|.\end{aligned}$$

We now prove,

$$\begin{aligned}&\frac{1}{|Q|}\left|\left\{x \in Q : \frac{\sum_{j=n_k+1}^{n} d_j(x)}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} > \sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)}\right\}\right|\\ &\leq \frac{1}{|Q|}\left|\left\{x \in Q : \frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} > \sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 2\sqrt{\frac{16L(1+\delta)}{\theta}}\right\}\right|.\end{aligned}$$

Here,

$$\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} \le \frac{\left|\sum_{j=n_k+1}^{n} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}$$

$$\le \frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right| + \left|\sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right| + \left|\sum_{j=n+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}$$

$$\le \frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} + \frac{\left|\sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} + \frac{\left|\sum_{j=n+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}$$

$$\le \frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} + 2\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

Hence,

$$\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 2\sqrt{\frac{16L(1+\delta)}{\theta}} < \frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}.$$

Then we have,

$$\frac{1}{|Q|}\left|\left\{x\in Q:\frac{\left|\sum_{j=n_k+1}^{\infty}d_j(x)-\sum_{j=n_{k+1}+1}^{\infty}d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}>\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)}-2\sqrt{\frac{16L(1+\delta)}{\theta}}\right\}\right|$$

$$=\frac{1}{|Q|}\left|\left\{x\in Q:\frac{\left|\sum_{j=n_{k+1}+1}^{n}d_j(x)\right|}{2\sqrt{\frac{1}{\theta^k}\log\log\theta^k}}>\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)}-2\sqrt{\frac{16L(1+\delta)}{\theta}}\right\}\right|$$

$$\geq\frac{A}{(k\log\theta+\alpha)^{1-\frac{\delta}{4}}}$$

$$\approx\frac{A}{\sqrt{k}}.$$

Hence,

$$\left|\left\{x\in Q:\frac{\left|\sum_{j=n_k+1}^{\infty}d_j(x)-\sum_{j=n_{k+1}+1}^{\infty}d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}>\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)}-2\sqrt{\frac{16L(1+\delta)}{\theta}}\right\}\right|\geq\frac{A|Q|}{\sqrt{k}}. \tag{0.34}$$

We note that $\sum_{j=n_k+1}^{n_{k+1}}d_j(x)$ is constant on each cube J of side length $\frac{1}{2^{n_{k+1}}}$. So there exists a sub collection $\mathcal{E}_{k+1}$ of cubes J such that $\forall x\in J, J\in\mathcal{E}_{k+1}$,

$$\frac{\left|\sum_{j=n_k+1}^{\infty}d_j(x)-\sum_{j=n_{k+1}+1}^{\infty}d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}}>\sqrt{2\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)}-2\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

Moreover, $\left|Q\bigcap\bigcup_{J\in\mathcal{E}_{k+1}}J\right|>\frac{A|Q|}{\sqrt{k}}$.

Hence by the Lemma 48, there is an infinite sequence $n_1 < n_2 < \cdots$ such that,

$$\frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2\frac{1}{\theta^k}\log\log\theta^k}} > \sqrt{\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 2\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

Then we have,

$$\frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2S_{n_k}'^2 f(x)\log\log\left(\frac{1}{2LS_{n_k}'^2 f(x)}\right)}} > \sqrt{\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 2\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

We note that,

$$\lim_{S_{n_k}'^2 f(x)\to 0} \frac{\log\log\frac{1}{2LS_{n_k}'^2 f(x)}}{\log\log\frac{1}{S_{n_k}'^2 f(x)}} = \lim_{S_{n_k}'^2 f(x)\to 0} \frac{\log\left(\log\frac{1}{S_{n_k}'^2 f(x)} + \log\frac{1}{2L}\right)}{\log\log\frac{1}{S_{n_k}'^2 f(x)}} = 1.$$

Thus if n_k is sufficiently large,

$$\frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x) - \sum_{j=n_{k+1}+1}^{\infty} d_j(x)\right|}{\sqrt{2S_{n_k}'^2 f(x)\log\log\left(\frac{1}{S_{n_k}'^2 f(x)}\right)}} > \sqrt{\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 3\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

We have from (0.31) for a.e. x

$$\sup_{n\geq n_{k+1}} \frac{|\sum_{j=n+1}^{\infty} d_j(x)|}{\sqrt{2S_{n_k}'^2 f(x)\log\log\frac{1}{S_{n_k}'^2 f(x)}}} \leq 4\sqrt{\frac{(1+\delta)L}{\theta}}$$

for sufficiently large $k \geq M$.

So there exists K such that for an infinite number of $n_k \geq K$, we get,

$$\therefore \qquad \sup_{n\geq n_{k+1}} \frac{|\sum_{j=n_k+1}^{\infty} d_j(x)|}{\sqrt{2S_{n_k}'^2 f(x)\log\log\left(\frac{1}{S_{n_k}'^2 f(x)}\right)}} \geq \sqrt{\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 3\sqrt{\frac{16L(1+\delta)}{\theta}}$$

So we have,

$$\frac{|\sum_{j=n_k+1}^{\infty} d_j(x)|}{\sqrt{2S_{n_k}'^2 f(x) \log\log\left(\frac{1}{S_{n_k}'^2 f(x)}\right)}} \geq \sqrt{\left(1-\frac{\delta}{2}\right)\left(\frac{1}{2L}-\frac{1}{\theta}\right)} - 3\sqrt{\frac{16L(1+\delta)}{\theta}}.$$

Let $\theta \nearrow \infty$ and $\delta \searrow 0$, then we have

$$\frac{\left|\sum_{j=n_k+1}^{\infty} d_j(x)\right|}{\sqrt{2S_{n_k}'^2 f(x) \log\log\left(\frac{1}{S_{n_k}'^2 f(x)}\right)}} \geq \frac{1}{\sqrt{2L}}.$$

This is true for subsequence $\{n_k\}$ so we have,

$$\limsup_{n\longrightarrow\infty} \frac{|f(x)-f_n(x)|}{\sqrt{2S_n'^2 f(x) \log\log\left(\frac{1}{S_n'^2 f(x)}\right)}} \geq \frac{1}{\sqrt{2L}}$$

for a.e. x. ■

Chapter 5

The tail law of the iterated logarithm for lacunary series.

In this chapter, we derive a lower bound in the law of the iterated logarithm for lacunary trigonometric series introduced by Salem and Zygmund.

5.1 Lower bound in the tail law of the iterated logarithm for lacunary series.

Salem and Zygmund obtained the following law of the iterated logarithm for tail sums of lacunary series.

Theorem 44 (R. Salem and A. Zygmund, 1950)**.** *Suppose a lacunary series with tail sums* $\tilde{S}_N(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ *where* $c_k^2 = a_k^2 + b_k^2$ *satisfies* $\sum_{k=1}^{\infty} c_k^2 < \infty$*. Define* $\tilde{B}_N = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ *and* $\tilde{M}_N = \max\limits_{k\geq N} |c_k|$*. Suppose that* $\tilde{B}_1 < \infty$ *and that* $\tilde{M}_N^2 \leq K_N \left(\frac{\tilde{B}_N^2}{\log\log \frac{1}{\tilde{B}_N}}\right)$ *for some sequence of numbers* $K_N \downarrow 0$ *as* $N \to \infty$*. Then*

$$\limsup_{N\to\infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \log\log \frac{1}{\tilde{B}_N}}} \leq 1$$

for almost every θ *in the unit circle.*

We prove a lower bound in the above tail law of the iterated logarithm. In the proof, we will need following Theorem from [7].

Theorem 45. *Suppose that n_k satisfies $\dfrac{n_k}{n_{k+1}} \geq q > 1$ and λ_{Nk} satisfies the following conditions:*

$$\sum_{k=1}^{\infty} \lambda_{Nk}^2 = 1, \quad \Lambda_N = \max_k |\lambda_{Nk}| = o(1) \quad as \quad N \to \infty.$$

Set $F_N(y) = P\left(\sum_{k=1}^{\infty} \lambda_{Nk}\omega_{n_k} < y\right)$ where $\omega_{n_k} = \sqrt{2}\cos(2\pi n_k x)$. Then the distributions $F_N(y)$ converge towards the normal law; moreover,

$$\sup_{-\infty<y<\infty} |F_N(y) - \Phi(y)| \leq C(q)\Lambda_N^{1/4},$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt$ and the constant $C(q)$ depends only on q.

Our main result is:

Theorem 46. *Let $S_m(x) = \displaystyle\sum_{k=1}^{m} a_k \cos(2\pi n_k x)$ be a partial sum of a lacunary series where $\dfrac{n_{k+1}}{n_k} \geq q > 1$ and $\displaystyle\sum_{k=1}^{\infty} a_k^2 < \infty$. Assume that $\max_{k\geq N} a_k^2 = o\left(\dfrac{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}{\log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}}}\right)$. Then for a.e. x,*

$$\limsup_{n\longrightarrow\infty} \frac{|\displaystyle\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)|}{\sqrt{2 \quad \frac{1}{2}\displaystyle\sum_{k=n}^{\infty} a_k^2 \quad \log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=n}^{\infty} a_k^2}}}} \geq 1.$$

Proof: Let us write $f(x) = \sum_{k=1}^{\infty} a_k \cos(2\pi n_k x)$. We first prove: There exists constants A and C so that for every $\eta > 0$,

$$|\{x \in [0,1) : |f(x)| > \eta\}| \leq A \exp\left(\frac{-C\eta^2}{\sum_{k=1}^{\infty} a_k^2}\right).$$

This is proved by using the following Theorem from [12].

Theorem 47. *Consider the series $g(x) = \sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x)$ with $\alpha^2 = \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$ where $\frac{n_{k+1}}{n_k} \geq q > 1$. If $\alpha \leq 1$, then $\int_0^{2\pi} \exp(Cg^2)dx \leq A$, provided $C \leq C_0(q)$, with A an absolute constant.*

Then for any $\lambda > 0$ this gives,

$$e^{C\lambda^2}|\{x : |g(x)| > \lambda\}| \leq \int_{\{x:|g(x)|>\lambda\}} \exp(Cg^2)dx \leq \int_0^{2\pi} \exp(Cg^2)dx \leq A.$$

Thus,

$$|\{x : \quad |g(x)| > \lambda\}| \leq Ae^{-C\lambda^2}. \tag{0.1}$$

We have $f(x) = \sum_{k=1}^{\infty} a_k \cos(2\pi n_k x)$. Define $g(x) = \frac{1}{\sqrt{\sum_{k=1}^{\infty} a_k^2}} f(x)$ and $\lambda = \frac{1}{\sqrt{\sum_{k=1}^{\infty} a_k^2}}\eta$. Then $\alpha = 1$. Then using (0.1) we have,

$$\left|\left\{x \in [0,1) : \frac{f(x)}{\sqrt{\sum_{k=1}^{\infty} a_k^2}} > \frac{\eta}{\sqrt{\sum_{k=1}^{\infty} a_k^2}}\right\}\right| \leq A\exp\left(\frac{-C\eta^2}{\sum_{k=1}^{\infty} a_k^2}\right).$$

Thus,

$$|\{x \in [0,1) : f(x) > \eta\}| \leq A\exp\left(\frac{-C\eta^2}{\sum_{k=1}^{\infty} a_k^2}\right). \tag{0.2}$$

Now let M be a fixed large number. Define $N_1 \leq N_2 \leq \cdots$ by

$$N_l = \text{smallest}\left(N : \frac{1}{2}\sum_{k=N+1}^{\infty} a_k^2 < \frac{1}{M^l}\right).$$

Let $\varepsilon > 0$, assume $\varepsilon << 1$. Choose $\delta > 0$ so that $(1+\delta)(1-\varepsilon^2) > 1$. Finally choose μ so that

$$\frac{\mu\log(8\sqrt{\pi})}{1+\varepsilon} - \frac{1}{2}\frac{\mu\log\mu}{1+\varepsilon} + \frac{1}{1+\varepsilon} < 1.$$

Next, we claim that for l sufficiently large,

$$(1-\varepsilon^2)\frac{1}{M^l} < \frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 < \frac{1}{M^l}. \tag{0.3}$$

We have,

$$\sum_{k=N_l}^{\infty} a_k^2 = a_{N_l}^2 + \sum_{k=N_l+1}^{\infty} a_k^2.$$

For large l, $a_{N_l}^2 < \varepsilon^2 \sum_{k=N_l}^{\infty} a_k^2$, so that

$$\frac{1}{2}\sum_{j=N_l}^{\infty} a_k^2 < \frac{1}{2}\varepsilon^2 \sum_{j=N_l}^{\infty} a_k^2 + \frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2.$$

By definition of N_l we have,

$$\frac{1}{2}(1-\varepsilon^2)\sum_{k=N_l}^{\infty} a_l^2 < \frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 < \frac{1}{M^l}. \tag{0.4}$$

Again, by the definition of N_l we have

$$(1-\varepsilon^2)\frac{1}{M^l} \le (1-\varepsilon^2)\frac{1}{2}\sum_{k=N_l}^{\infty} a_k^2. \tag{0.5}$$

So from (0.4) and (0.5) we get,

$$(1-\varepsilon^2)\frac{1}{M^l} < \frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 < \frac{1}{M^l}.$$

This gives us

$$(1-\varepsilon^2)M < \frac{\sum_{k=N_l+1}^{\infty} a_k^2}{\sum_{k=N_{l+1}+1}^{\infty} a_k^2} < \frac{M}{1-\varepsilon^2}. \tag{0.6}$$

From (0.2) we have,

$$\left|\left\{x \in [0,1) : \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x) > \eta\right\}\right| \le A \exp\left(\frac{-C\eta^2}{\sum_{k=N_{l+1}+1}^{\infty} a_k^2}\right).$$

Choose

$$\eta = \sqrt{\frac{1+\delta}{CM}\sum_{k=N_l+1}^{\infty} a_k^2 \log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}}.$$

From (0.3) we have,

$$\sqrt{\frac{M^l}{(1-\varepsilon^2)}} > \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_{l+1}+1}^{\infty} a_k^2}} > \sqrt{M^l}.$$

Then we have,

$$-\log\log\left(\sqrt{\frac{M^l}{(1-\varepsilon^2)}}\right) < -\log\log\left(\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_{l+1}+1}^{\infty} a_k^2}}\right) < -\log\log\left(\sqrt{M^l}\right). \tag{0.7}$$

Then using (0.6) and (0.7) we have,

$$\left|\left\{x\in[0,1):\sum_{k=N_{l+1}+1}^{\infty}a_k\cos(2\pi n_k x)>\sqrt{\frac{(1+\delta)}{CM}\sum_{k=N_l+1}^{\infty}a_k^2\log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty}a_k^2}}}\right\}\right|$$

$$\leq A\exp\left(\frac{-C\frac{(1+\delta)}{CM}\sum_{k=N_l+1}^{\infty}a_k^2\log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty}a_k^2}}}{\sum_{k=N_{l+1}+1}^{\infty}a_k^2}\right)$$

$$\leq A\exp\left(\frac{-(1+\delta)(1-\varepsilon^2)M\log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty}a_k^2}}}{M}\right)$$

$$\leq A\exp\left(-(1+\delta)(1-\varepsilon^2)\log\log M^{\frac{l}{2}}\right)$$

$$=A\left(\frac{1}{2}\log M^l\right)^{-(1+\delta)(1-\varepsilon^2)}$$

$$=A\frac{1}{\left(\frac{1}{2}\log M\right)^{(1+\delta)(1-\varepsilon^2)}}\frac{1}{l^{(1+\delta)(1-\varepsilon^2)}}.$$

So we have,

$$\left|\left\{x\in[0,1):\sum_{k=N_{l+1}+1}^{\infty}a_k\cos\left(2\pi n_k x\right)>\sqrt{\frac{(1+\delta)}{CM}\sum_{k=N_l+1}^{\infty}a_k^2\log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty}a_k^2}}}\right\}\right|$$

$$\leq A\frac{1}{\left(\frac{1}{2}\log M\right)^{(1+\delta)(1-\varepsilon^2)}}\frac{1}{l^{(1+\delta)(1-\varepsilon^2)}}.$$

Consequently,

$$\sum_{l=1}^{\infty}\left|\left\{x\in[0,1):\sum_{k=N_{l+1}+1}^{\infty}a_k\cos\left(2\pi n_k x\right)>\sqrt{\frac{(1+\delta)}{CM}\sum_{k=N_l+1}^{\infty}a_k^2\log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty}a_k^2}}}\right\}\right|$$

$$\leq\sum_{l=1}^{\infty}A\frac{1}{\left(\frac{1}{2}\log M\right)^{(1+\delta)(1-\varepsilon^2)}}\frac{1}{l^{(1+\delta)(1-\varepsilon^2)}}$$

$$=A\frac{1}{\left(\frac{1}{2}\log M\right)^{(1+\delta)(1-\varepsilon^2)}}\sum_{l=1}^{\infty}\frac{1}{l^{(1+\delta)(1-\varepsilon^2)}}$$

$$<\infty.$$

So by Borel-Cantelli Lemma for a.e. x, we have,

$$\frac{\sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\sum_{k=N_l+1}^{\infty} a_k^2 \log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}}} \leq \sqrt{\frac{1+\delta}{CM}} \tag{0.8}$$

for sufficiently large l (Here l depends on x).

Again,

$$\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2 = \frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 - \frac{1}{2}\sum_{k=N_{l+1}+1}^{\infty} a_k^2 > (1-\varepsilon^2)\frac{1}{M^l} - \frac{1}{M^{l+1}} = \frac{1}{M^l}\left(1-\varepsilon^2-\frac{1}{M}\right).$$

By assumption for all sufficiently large N,

$$\max_{k\geq N} a_k^2 \leq \frac{\varepsilon}{2}\left(\frac{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}{\log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}}}\right).$$

Then, if l is large enough,

$$\begin{aligned}
\max_{k\geq N_l+1} a_k^2 &\leq \frac{\varepsilon}{2}\left(\frac{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}{\log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}}\right) \\
&\leq \frac{\varepsilon}{2}\frac{\frac{1}{M^l}}{\log\log\sqrt{M^l}} \\
&= \frac{\varepsilon/2}{(1-\varepsilon^2-\frac{1}{M})}\frac{\frac{1}{M^l}(1-\varepsilon^2-\frac{1}{M})}{\log l}\frac{\log l}{\log l+\log\log\sqrt{M}} \\
&< \frac{\varepsilon/2}{(1-\varepsilon^2-\frac{1}{M})}\frac{1}{\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.
\end{aligned}$$

We may assume that ε is small enough and M large enough so that $1-\varepsilon^2-\frac{1}{M} > \frac{1}{2}$. Then

$$\max_{k\geq N_l+1} a_k^2 < \frac{\varepsilon}{\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{0.9}$$

Thus ,

$$\max_{k\geq N_l+1} \frac{a_k}{\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}} \leq \sqrt{\frac{\varepsilon}{\log l}}.$$

Recall, we choose $\mu > 0$ so that

$$\frac{\mu \log(8\sqrt{\pi})}{1+\varepsilon} - \frac{1}{2}\frac{\mu \log \mu}{1+\varepsilon} + \frac{1}{1+\varepsilon} < 1.$$

Suppose l is large so that $\mu \log l >> 1$. Next, we define a sequence of numbers $l_1, l_2 \cdots l_{[]}$ where $[]$ is the greatest integer contained in $\dfrac{\mu \log l}{1+\varepsilon}$ as follows,

let l_1 be the first time such that,

$$\frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1} a_k^2 \geq \frac{1}{\mu \log l}\frac{1}{2}\sum_{N_l+1}^{N_{l+1}} a_k^2$$

so that

$$\frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 < \frac{1}{\mu \log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

Likewise, let l_2 be the first time such that,

$$\frac{1}{2}\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2 \geq \frac{1}{\mu \log l}\frac{1}{2}\sum_{N_l+1}^{N_{l+1}} a_k^2$$

so that

$$\frac{1}{2}\sum_{k=N_l+l_1+1}^{N_l+l_2-1} a_k^2 < \frac{1}{\mu \log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

Similarly we define $l_3 \cdots l_{[]}$. Using (0.9) we have,

$$\begin{aligned}
\sum_{k=N_l+1}^{N_l+l_1} a_k^2 &= \sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 + a_{N_l+l_1}^2 \\
&< \sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 + \varepsilon \frac{1}{\mu \log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2 \\
&< \frac{1}{\mu \log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2 + \varepsilon \frac{1}{\mu \log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2 \\
&= (1+\varepsilon)\frac{1}{\mu \log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.
\end{aligned}$$

So we have,

$$\frac{1}{\mu \log l}\frac{1}{2}\sum_{N_l+1}^{N_{l+1}} a_k^2 \leq \frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1} a_k^2 < (1+\varepsilon)\frac{1}{\mu \log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

As above we can show that $\forall j = 1, 2, \cdots, []$ with $l_0 = 0$,

$$\frac{1}{\mu \log l} \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2 \leq \frac{1}{2} \sum_{k=N_l+l_{j-1}+1}^{N_l+l_j} a_k^2 < (1+\varepsilon) \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{0.10}$$

Using (0.10), we have,

$$\begin{aligned}
&\frac{1}{2} \sum_{N_l+1}^{N_l+l_1} a_k^2 + \frac{1}{2} \sum_{N_l+l_1+1}^{N_l+l_2} a_k^2 + \cdots + \frac{1}{2} \sum_{N_l+l_{[]-1}+1}^{N_l+l_{[]}} a_k^2 \\
&< (1+\varepsilon) \frac{1}{\mu \log l} \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2 + (1+\varepsilon) \frac{1}{\mu \log l} \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2 + \cdots + (1+\varepsilon) \frac{1}{\mu \log l} \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2 \\
&= (1+\varepsilon) \frac{1}{\mu \log l} \left[\frac{\mu \log l}{1+\varepsilon} \right] \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2 \\
&\leq \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2.
\end{aligned}$$

Thus,

$$\frac{1}{2} \sum_{N_l+1}^{N_l+l_{[]}} a_k^2 < \frac{1}{2} \sum_{N_l+1}^{N_{l+1}} a_k^2.$$

This shows that $N_l + l_{[]}$ does not exceed N_{l+1}. Consider a dyadic cube Q such that $|Q| = \frac{1}{2^L}$ where L is the number such that $2^L \leq n_{N_l} < 2^{L+1}$. Then using the Theorem 45 we have,

$$\left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} \frac{a_k}{\sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}} \sqrt{2} \cos(2\pi n_k x) > \lambda \right\} \right| \geq \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty e^{\frac{-t^2}{2}} dt - c(q) \left(\frac{\varepsilon}{\log l} \right)^{\frac{1}{8}}.$$

Choose $\lambda = \sqrt{\frac{2}{\mu}}$. We can assume l large enough (l depends only on γ, q, and μ) such that,

$$\left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} \frac{a_k}{\sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}} \sqrt{2} \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}} \right\} \right| \geq \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2}{\mu}}}^\infty e^{\frac{-t^2}{2}} dt.$$

We use the inequality

$$\frac{\lambda}{1+\lambda^2}e^{\frac{-\lambda^2}{2}} \leq \int_\lambda^\infty e^{\frac{-u^2}{2}}du \leq \frac{1}{\lambda}e^{\frac{-\lambda^2}{2}},$$

to obtain,

$$\begin{aligned}\left|\left\{x \in Q : \sum_{k=N_l+1}^{N_l+l_1} \frac{a_k}{\sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}}\sqrt{2}\cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}}\right\}\right| &\geq \frac{1}{2\sqrt{2\pi}}\frac{\sqrt{\frac{2}{\mu}}}{1+\frac{2}{\mu}}e^{-\frac{1}{\mu}} \\ &\geq \frac{1}{2\sqrt{2\pi}}\frac{\sqrt{\mu}}{2\sqrt{2}}e^{-\frac{1}{\mu}} \\ &= \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}.\end{aligned}$$

Thus we have,

$$\left|\left\{x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k\cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}}\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1} a_k^2}\right\}\right| \geq \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}.$$

Let,

$$G(Q) = \left\{x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k\cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}}\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1} a_k^2}\right\}.$$

Let $h(x) = \sum_{k=N_l+1}^{N_l+l_1} a_k\cos(2\pi n_k x)$. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}|h^{'}(x)| &\leq \sum_{k=N_l+1}^{N_l+l_1} a_k \quad 2\pi \quad n_k \\ &\leq 2\pi\sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}\sqrt{\sum_{k=N_l+1}^{N_l+l_1} n_k^2}.\end{aligned}$$

Let L_1 be the number such that $2^{L_1} \leq n_{N_l+l_1} < 2^{L_1+1}$. Then,

$$\begin{aligned}
\sum_{k=N_l+1}^{N_l+l_1} n_k^2 &\leq n_{N_l+l_1}^2 + n_{N_l+l_1-1}^2 + n_{N_l+l_1-2}^2 + \cdots \\
&\leq \left(2^{L_1+1}\right)^2 + \frac{1}{q^2}\left(2^{L_1+1}\right)^2 + \frac{1}{q^4}\left(2^{L_1+1}\right)^2 + \cdots \\
&= \left(2^{L_1+1}\right)^2 \left(1 + \frac{1}{q^2} + \frac{1}{q^4} + \cdots\right) \\
&= \left(2^{L_1+1}\right)^2 \frac{q^2}{q^2-1}.
\end{aligned}$$

So,

$$\sqrt{\sum_{k=N_l+1}^{N_l+l_1} n_k^2} \leq 2^{L_1+1} \frac{q}{\sqrt{q^2-1}}.$$

Thus,

$$|h^{'}(x)| \leq 2\pi \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \quad 2 \quad 2^{L_1} \frac{q}{\sqrt{q^2-1}} = B \quad 2^{L_1} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \quad \text{say.}$$

Let $x \in G(Q)$ and consider y such that $|x-y| < \dfrac{1}{2^{L_1}}$. Then,

$$\begin{aligned}
|h(y)| &\geq |h(x)| - |h(y) - h(x)| \\
&\geq |h(x)| - \sup |h^{'}(x)| \frac{1}{2^{L_1}} \\
&\geq |h(x)| - \frac{1}{2^{L_1}} \quad B \quad 2^{L_1} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \\
&= |h(x)| - B \quad \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.
\end{aligned}$$

This gives,

$$\begin{aligned}
|h(y)| &\geq \sqrt{\frac{2}{\mu}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2} - B \quad \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \\
&= \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{\sqrt{\mu}}\right) \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2}.
\end{aligned}$$

So there exists a subcollection $\{Q^{'}\}$ of Q with $|Q^{'}| = \frac{1}{2^{L_1}}$ such that $\forall x \in Q^{'}$ we have,

$$\sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{N_l+l_1} a_k^2}$$

$$\geq \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

and $\left|\bigcup_{Q^{'}\cap G(Q)} Q^{'}\right| > \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}|Q|.$

Again on each $Q^{'}$ we use the Theorem 45 and obtain:

$$|G(Q')| = \left|\left\{x \in Q' : \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}}\sqrt{\frac{1}{2}\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2}\right\}\right| \geq \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}.$$

Then as above there exists a subcollection $\{Q^{''}\}$ with $|Q^{''}| = \frac{1}{2^{L_2}}$ where L_2 is the number such that $2^{L_2} \leq n_{N_l+l_2} < 2^{L_2+1}$ and $\forall x \in Q^{''}$

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{2}\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2}$$

$$\geq \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

and $\left|\bigcup_{Q^{''}\cap G(Q^{'})} Q^{''}\right| > \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}|Q^{'}|.$

We continue this process. Then there exists a subcollection of cubes $\{I\}$ with $|I| = \frac{1}{2^{L_{[]}}}$ where $[]$ is the greatest integer contained in $\frac{\mu\log l}{1+\varepsilon}$ and $L_{[]}$ is the number satisfying $2^{L_{[]}} \leq n_{N_l+l_{[]}} < 2^{L_{[]}+1}$ and $\forall x \in I$,

$$\sum_{k=N_l+l_{[]-1}+1}^{N_l+l_{[]}} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{2}\sum_{k=N_l+l_{[]-1}+1}^{N_l+l_{[]}} a_k^2}$$
$$\geq \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Moreover, $\left|\bigcup_{I\cap G(\tilde{Q})} I\right| > \frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}|\tilde{Q}|$ where $\tilde{Q}$ is the previous generation cube. Now on each I, we use the Theorem 45 and obtain,

$$\left|\left\{x\in I : \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k\cos(2\pi n_k x) > B\sqrt{\frac{1}{2}\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}\right\}\right| \geq \frac{1}{\sqrt{2\pi}}\int_B^\infty e^{\frac{-B^2}{2}}dt - c(q)\left(\frac{\varepsilon}{\log l}\right)^{\frac{1}{8}}$$
$$\geq \frac{1}{\sqrt{2\pi}}\frac{B^2}{1+B^2}e^{-\frac{B^2}{2}} - c(q)\left(\frac{\varepsilon}{\log l}\right)^{\frac{1}{8}}.$$

Let

$$G(I) = \left\{x\in I : \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k\cos(2\pi n_k x) > B\sqrt{\frac{1}{2}\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}\right\}.$$

Recall that B is a constant given by $B = 4\sqrt{2}\pi\frac{q}{\sqrt{q^2-1}}$. So for sufficiently large l we have,

$$\left|\left\{x\in I : \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k\cos(2\pi n_k x) > B\sqrt{\frac{1}{2}\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}\right\}\right| \geq \frac{1}{\sqrt{2\pi}}\frac{B^2}{1+B^2}e^{-\frac{B^2}{2}}.$$

Next, we consider dyadic subcubes of I, say, J with side length $|J| = \frac{1}{\tilde{L}}$ where $\tilde{L}$ is the number such that, $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}+1}$.

Let $h(x) = \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k\cos(2\pi n_k x)$. Then by the Cauchy-Schwarz inequality,

$$|h'(x)| \leq \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k \quad 2\pi \quad n_k$$
$$\leq 2\pi\sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}\sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} n_k^2}.$$

Then,

$$\begin{aligned}\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} n_k^2 &\le n_{N_{l+1}}^2 + n_{N_{l+1}-1}^2 + n_{N_{l+1}-2}^2 + \cdots \\ &\le \left(2^{\tilde{L}+1}\right)^2 + \frac{1}{q^2}\left(2^{\tilde{L}+1}\right)^2 + \frac{1}{q^4}\left(2^{\tilde{L}+1}\right)^2 + \cdots \\ &= \left(2^{\tilde{L}+1}\right)^2 \left(1 + \frac{1}{q^2} + \frac{1}{q^4} + \cdots\right) \\ &= \left(2^{\tilde{L}+1}\right)^2 \frac{q^2}{q^2-1}.\end{aligned}$$

So,

$$\sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} n_k^2} \le 2^{\tilde{L}+1}\frac{q}{\sqrt{q^2-1}}.$$

This gives,

$$|h^{'}(x)| \le 2\pi\sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}\left(2^{\tilde{L}+1}\right)\frac{q}{\sqrt{q^2-1}} = B2^{\tilde{L}}\sqrt{\frac{1}{2}\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}.$$

Let $x \in G(I)$. Consider y such that $|x-y| < \dfrac{1}{2^{\tilde{L}}}$. Then

$$\begin{aligned}|h(y)| &\ge |h(x)| - |h(y)-h(x)| \\ &\ge |h(x)| - \sup|h^{'}(x)|\frac{1}{2^{\tilde{L}}} \\ &\ge |h(x)| - \frac{1}{2^{\tilde{L}}} \quad B \quad 2^{\tilde{L}}\sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2} \\ &= |h(x)| - B \quad \sqrt{\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k^2}.\end{aligned}$$

This shows that there exists a collection of subcubes $\{J\}$ of I with $|J| = \dfrac{1}{2^{\tilde{L}}}$ with the property that $\forall x \in J$, $\displaystyle\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > 0$. Moreover, $\displaystyle\left| I \bigcap \bigcup_{J\cap G(I)} J \right| > \frac{1}{2\sqrt{2\pi}}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}$.

Adding the estimates from all of the above generations, we have

$$\sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) + \cdots + \sum_{k=N_l+l_{[]}-1}^{N_l+l_{[]}} a_k \cos(2\pi n_k x) + \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k \cos(2\pi n_k x)$$

$$> \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2} + \cdots + \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2} + 0$$

$$= \left[\frac{\mu\log l}{1+\varepsilon}\right]\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Also we have,

$$\left|\left\{x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left[\frac{\mu\log l}{1+\varepsilon}\right]\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right|$$

$$\geq |Q|\left(\frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}\right)^{\left[\frac{\mu\log l}{1+\varepsilon}\right]}\frac{1}{2\sqrt{2\pi}}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}.$$

Moreover, there exists a subcollection $\{J\}$ of Q with $|J| = \frac{1}{2^{\tilde{L}}}$ and

$$|Q \cap \bigcup J| > |Q|\left(\frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}\right)^{\left[\frac{\mu\log l}{1+\varepsilon}\right]}\frac{1}{2\sqrt{2\pi}}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}.$$

Again,

$$\left|\left\{x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \frac{\mu \log l}{1+\varepsilon}\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu \log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right|$$

$$\geq |Q|\left(\frac{1}{8\sqrt{\pi}}\sqrt{\mu}e^{-\frac{1}{\mu}}\right)^{\frac{\mu\log l}{1+\varepsilon}} . \frac{1}{2\sqrt{2}\pi}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}$$

$$= |Q|\left(\frac{1}{8\sqrt{\pi}}\right)^{\frac{\mu\log l}{1+\varepsilon}} (\sqrt{\mu})^{\frac{\mu\log l}{1+\varepsilon}} \left(e^{-\frac{1}{\mu}}\right)^{\frac{\mu\log l}{1+\varepsilon}} . \frac{1}{2\sqrt{2}\pi}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}$$

$$= |Q|\frac{1}{l^{\left(\frac{\mu\log 8\sqrt{\pi}}{1+\varepsilon}\right)}}\frac{1}{l^{\left(\frac{\mu\log\mu}{2(1+\varepsilon)}\right)}}\frac{1}{l^{\left(\frac{1}{1+\varepsilon}\right)}} . \frac{1}{2\sqrt{2}\pi}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}$$

$$= \frac{|Q|}{l^{\left(\frac{\mu\log 8\sqrt{\pi}}{1+\varepsilon}-\frac{\mu\log\mu}{2(1+\varepsilon)}+\frac{1}{1+\varepsilon}\right)}} . \frac{1}{2\sqrt{2}\pi}\frac{B}{1+B^2}e^{-\frac{B^2}{2}}$$

$$\geq C\frac{|Q|}{l}.$$

Consequently, $|Q \cap \bigcup J| > C\frac{|Q|}{l}$. Moreover,

$$\left\{x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left[\frac{\mu\log l}{1+\varepsilon}\right]\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\} = \bigcup_J J.$$

We may assume that l is large enough so that

$$\left[\frac{\mu\log l}{1+\varepsilon}\right] / \left(\frac{\mu\log l}{1+\varepsilon}\right) > \frac{1}{1+\varepsilon}.$$

Then we have,

$$\left|\left\{x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^2}\right)\sqrt{\log l \quad \frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right|$$

$$\geq \left|\left\{x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left[\frac{\mu\log l}{1+\varepsilon}\right]\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}}\right)\sqrt{\frac{1}{\mu\log l}\frac{1}{2}\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right|$$

$$\geq C\frac{|Q|}{l}.$$

Next, we use the following Lemma:

Lemma 48. *Let F_k be a collection of dyadic cubes whose union is $[0,1]$ and F_{k+1} is a refinement of F_k. Suppose that the maximum length of the elements of F_k tends to zero. Suppose $\mathcal{E}_k \subset F_k$ has the property:*

$$\forall Q \in F_k, \qquad \left| Q \cap \bigcup_{J \in \mathcal{E}_{k+1}} J \right| > |Q| \frac{C}{k}$$

Set $E_k = \bigcup_{J \in \mathcal{E}_k} J$. Then for a.e. x, $x \in E_k$ i.o.

Using above Lemma, for a.e. x there exists an infinite sequence of numbers $N_1 < N_2 < \cdots$ such that, for l sufficiently large

$$\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2} - \sqrt{2\mu} B}{(1+\varepsilon)^2} \right) \sqrt{\log l \quad \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

So we have,

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\log l \quad \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}} > \left(\frac{\sqrt{2} - \sqrt{2\mu} B}{(1+\varepsilon)^2} \right).$$

Again,

$$\begin{aligned}
\frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2 &= \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 - \frac{1}{2} \sum_{k=N_{l+1}+1}^{\infty} a_k^2 \\
&\geq \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 - \frac{1}{M^{l+1}} \\
&\geq \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 - \frac{1}{M(1-\varepsilon^2)} \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \\
&= \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \left(1 - \frac{1}{M(1-\varepsilon^2)} \right).
\end{aligned}$$

This gives,

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\log l \quad \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \left(1 - \frac{1}{M(1-\varepsilon^2)} \right)}} > \left(\frac{\sqrt{2} - \sqrt{2\mu} B}{(1+\varepsilon)^2} \right). \tag{0.11}$$

Then,

$$\sqrt{\frac{1-\varepsilon^2}{M^l}} \leq \sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}$$

gives

$$\log M^{\frac{l}{2}} - \log\sqrt{1-\varepsilon^2} \geq \log\left(\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}\right). \tag{0.12}$$

But,

$$\lim_{l\to\infty} \frac{\log\left(l\log\sqrt{M} - \log\sqrt{1-\varepsilon^2}\right)}{\log l} = 1.$$

So for sufficiently large l,

$$\frac{\log\left(l\log\sqrt{M} - \log\sqrt{1-\varepsilon^2}\right)}{\log l} < 1+\varepsilon.$$

This gives,

$$\log l > \frac{\log\left(l\log\sqrt{M} - \log\sqrt{1-\varepsilon^2}\right)}{1+\varepsilon} > \frac{\log\log\dfrac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}}{1+\varepsilon}. \tag{0.13}$$

Then using (0.11) and (0.13) we have,

$$\frac{\displaystyle\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\log\log\left(\dfrac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}\right)\dfrac{1}{(1+\varepsilon)^{\frac{1}{2}}}\displaystyle\sum_{k=N_l+1}^{\infty} a_k^2\left(1-\frac{1}{M(1-\varepsilon^2)}\right)}} \geq \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^2}\right).$$

Then,

$$\frac{\displaystyle\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\dfrac{1}{2}\displaystyle\sum_{k=N_l+1}^{\infty} a_k^2 \log\log\left(\dfrac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}\right)}} \geq \frac{1}{\sqrt{(1+\varepsilon)^2}}\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{1+\varepsilon}\right)\sqrt{1-\frac{1}{M(1-\varepsilon^2)}}.$$

Consequently,

$$\frac{\sum_{k=N_l+1}^{\infty} a_k \cos(2\pi n_k x) - \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 \log\log\left(\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}\right)}} \geq \frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^{5/2}}\sqrt{1-\frac{1}{M(1-\varepsilon^2)}}.$$

But from (0.8) for a.e. x we have,

$$\frac{\sum_{k=N_{l+1}+1}^{\infty} a_k \cos(\pi n_k x)}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 \log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}}} \leq \sqrt{\frac{2(1+\delta)}{CM}}$$

for sufficiently large l (depending on x).

Hence we have,

$$\frac{\sum_{k=N_l+1}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2 \log\log\left(\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N_l+1}^{\infty} a_k^2}}\right)}} \geq \frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^{5/2}}\sqrt{1-\frac{1}{M(1-\varepsilon^2)}} - \sqrt{\frac{2(1+\delta)}{CM}}.$$

Since this is true for the subsequence $\{N_l\}$, we have for a. e. x,

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)|}{\sqrt{\frac{1}{2}\sum_{k=n}^{\infty} a_k^2 \quad \log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=n}^{\infty} a_k^2}}}} \geq \frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^{5/2}}\sqrt{1-\frac{1}{M(1-\varepsilon^2)}} - \sqrt{\frac{2(1+\delta)}{CM}}.$$

By the choice of μ we have as $\varepsilon \searrow 0$, then $\mu \searrow 0$. Finally letting $M \nearrow \infty$, $\varepsilon \searrow 0$, $\delta \searrow 0$, we get,

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)|}{\sqrt{2 \quad \frac{1}{2}\sum_{k=n}^{\infty} a_k^2 \quad \log\log\frac{1}{\sqrt{\frac{1}{2}\sum_{k=n}^{\infty} a_k^2}}}} \geq 1.$$

Conclusion

In 1950, Salem and Zygmund introduced a law of the iterated logarithm for tail sums of lacunary trigonometric series. We call the law of the iterated logarithm for tail sums as tail law of the iterated logarithm. They only obtained the upper bound in their law of the iterated logarithm. In this thesis, we mainly focused to obtain an analogue of Salem and Zygmund's tail law of the iterated logarithm in the various contexts in analysis. The various contexts are sums of Rademacher functions, dyadic martingale and independent random variables. We first established the tail law of the iterated logarithm for sums of Rademacher functions. Sum of Rademacher functions is a nicely behaved martingale. Then employing the ideas from the Rademacher case, we introduced the tail law of the iterated logarithm for dyadic martingale. We obtained both upper and lower bounds in the tail LIL of Rademacher functions and dyadic martingale. Next, we established the tail LIL for independent, symmetric, random variables where we obtained the lower bound. Finally, we obtained a lower bound in the Salem and Zygmund's law of the iterated logarithm for lacunary trigonometric series.

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