WEYL FILTRATION DIMENSION AND SUBMODULE STRUCTURES FOR B_2

by

MATTHEW BESWICK

M.A., Nottingham University, UK, 1999

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2009

Abstract

Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field of positive prime characteristic. Let $L(\lambda)$ and $\nabla(\lambda)$ be the simple and induced finite dimensional rational G-modules with p-singular dominant highest weight λ. In this thesis, the concept of Weyl filtration dimension of a finite dimensional rational G-module is studied for some highest weight modules with p-singular highest weights inside the p^2-alcove when G is of type B_2. In Chapter 4, intertwining morphisms, a diagonal G-module morphism and tilting modules are used to compute the Weyl filtration dimension of $L(\lambda)$ with λ p-singular and inside the p^2-alcove. It is shown that the Weyl filtration dimension of $L(\lambda)$ coincides with the Weyl filtration dimension of $\nabla(\lambda)$ for almost all (all but one of the 6 facet types) p-singular weights inside the p^2-alcove. In Chapter 5 we study some submodule structures of Weyl (and there translations), Vogan, and tilting modules with both p-regular and p-singular highest weights. Most results are for the p^2-alcove only although some concepts used are alcove independent.

WEYL FILTRATION DIMENSION AND SUBMODULE STRUCTURES FOR B_2

by

MATTHEW BESWICK

M.A., Nottingham University, UK, 1999

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2009

Approved by:

Major Professor
Zongzhu Lin

Abstract

Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field of positive prime characteristic. Let $L(\lambda)$ and $\nabla(\lambda)$ be the simple and induced finite dimensional rational G-modules with p-singular dominant highest weight λ. In this thesis, the concept of Weyl filtration dimension of a finite dimensional rational G-module is studied for some highest weight modules with p-singular highest weights inside the p^2-alcove when G is of type B_2. In Chapter 4, intertwining morphisms, a diagonal G-module morphism and tilting modules are used to compute the Weyl filtration dimension of $L(\lambda)$ with λ p-singular and inside the p^2-alcove. It is shown that the Weyl filtration dimension of $L(\lambda)$ coincides with the Weyl filtration dimension of $\nabla(\lambda)$ for almost all (all but one of the 6 facet types) p-singular weights inside the p^2-alcove. In Chapter 5 we study some submodule structures of Weyl (and there translations), Vogan, and tilting modules with both p-regular and p-singular highest weights. Most results are for the p^2-alcove only although some concepts used are alcove independent.

Table of Contents

List of Figures

Acknowledgments

I would like to thank my supervisor, Zongzhu Lin, for taking me on as his student, sharing his knowledge, and being understanding and patient with me. I would also like to thank my parents for pushing me in my education and supporting me all the way! I would also like to thank the referee for pointing out my math and English errors, and saving me from reproving some existing results. Comments and suggestions made by the referee on how to extend the result to all dominant weights for B_2 were also much appreciated.

Chapter 1

Preliminaries

1.1 Introduction

This thesis is concerned with the finite dimensional representation theory of algebraic groups over a field k with prime characteristic $p \neq 0$. This is a group with extra "topological" structure coming from algebraic geometry. Like any algebraic structure we would like to understand its representation theory. It is not my aim to give much of a review of the subject. For a good review of the subject up to 88' the reader is referred Lin [L1]. Assuming G is "nice enough," e.g., $G = SL_2(k)$ we want to answer questions like, what are all the simple representations of G? How many are there? Is the representation theory semisimple? What does the representation theory tell us about the algebraic groups? Can we parametrize all indecomposable modules? What happens when we tensor product modules together? What are all the projective, or injective modules? If I know all the representations of an algebraic group of this type, what additional information do I need, if any, to reconstruct the group? The theory is well understood if $char(k) = 0$, the theory is semisimple and we have a nice classification theory. However, for prime characteristic ground field much less is known. The theory is not semisimple in this case, and only a few of the theorems from the characteristic zero case carry over. For example take $G = SL_2(k)$. If $char(k) = 0$ then it is easy to show that the simple modules $H(n)$ correspond to homogeneous polynomials of integer degree n. However, if $char(k) = p \neq 0$ and we use the G-module map $H(n) \to H(pn)$ defined by

$f(x,y) \to [f(x,y)]^p$ we see that $H(n)$ maps to a proper submodule of $H(pn)$ so $H(pn)$ is no longer simple.

Let G be a connected, simply connected, semisimple algebraic group over an algebraically closed field of positive characteristic $p \neq 0$. Let B be a Borel subgroup of G and T a maximal torus. Let G/B be the corresponding flag variety. Let R be the root system of G with respect to T so that B corresponds to the negative roots. Let X denote the set of weights of T. The finite dimensional rational representations of such G are the finite dimensional rational G-modules, referred to as G-modules below. There is a special family of G-modules, the cohomology groups of a line bundle on the flag variety denoted $H^i(\lambda)$, $\lambda \in X$. In this thesis we will be mainly concerned with the *induced modules* $\nabla(\lambda) = H^0(\lambda)$ and there "duals" the *Weyl modules* $\Delta(\lambda)$. In the early 70's Kempf proved that if λ is a so called dominant weight the only non vanishing cohomology group is $\nabla(\lambda) = H^0(\lambda)$. Moreover, these induced modules are zero if λ is not dominant. This is called Kempfs vanishing theorem. The induced modules have simple socles $L(\lambda)$, and all simple modules are realized in this way.

There is a character formula for these cohomology modules([J, II 5.11]) and by Kempf's vanishing theorem the characters of the simple modules can be written in terms of the characters of these Weyl modules(or dually the induced modules),

$$\mathrm{ch}(L(\lambda)) = \sum_{\mu \in X^+} [\sum_i (-1)^i \dim \mathrm{Ext}^i_G(L(\lambda), \nabla(\mu))] \,\mathrm{ch}(\Delta(\mu)).$$

The *Weyl filtration dimension* of $L(\lambda)$ can be defined as

$$\max\{i \mid \mathrm{Ext}^i_G(L(\lambda), \nabla(\mu)) \neq 0, \ \mu \in X^+\}.$$

It is a conjecture of Lusztig [Lu] that the character of the simple modules is also given by an alternating sum over the affine Weyl group of certain Kazhdan-Lusztig polynomials evaluated at 1:

$$\mathrm{ch}(L(w \cdot \lambda)) = \Sigma_{w' \in W_p} (-1)^{d(w\cdot\lambda) - d(w'\cdot\lambda)} P_{w'\cdot w_0, w\cdot w_0}(1) \,\mathrm{ch}(w' \cdot \lambda).$$

It has been shown that the Lusztig conjecture is true for large primes, or in the restricted region known as the p^2-alcove for the low rank groups considered in this thesis. For the so called *p-regular* λ's the Weyl filtration dimension is the maximum degree of these Kazhdan-Lusztig polynomials. For general reductive algebraic groups Parker [P1] has shown that for p-regular weights the Weyl filtration dimension of $L(\lambda)$, denoted by $\mathrm{wfd}(L(\lambda))$, coincides with the $\mathrm{wfd}(\nabla(\lambda))$, and is equal to the number of hyperplanes between λ and the fundamental alcove. However, for the computations done so far in rank 2 in this thesis, and by Parker in [P3], the $\mathrm{wfd}(L(\lambda))$ for *p-singular* weights grows much slower and is zero for the Steinberg weights. For G of type B_2 Lin [L1] has shown that the radical filtration of the Weyl modules for p-singular weights inside the p^2-alcove can be computed from that of the p-regular weights by applying the translation functor. This was possible since the Lusztig conjecture holds in this region, beyond the p^2-alcove little seems to be known about the radical structure of the Weyl modules.

This thesis is structured as follows. Chapter one introduces the basic concepts required to understand the rest of the thesis. The reader can refer to Jantzen's book [J] if need be. Chapter 2 deals with the notion of Weyl filtration dimension. In chapter 3, the radical filtration of $\Delta(\lambda)$, homomorphisms between certain Weyl modules, and a diagonal map are used to compute $\mathrm{wfd}(L(\lambda))$ for p-singular highest weights inside the p^2-alcove for type B_2. We show that for λ in an arbitrary p-box with lower vertex λ_v, $\mathrm{wfd}(L(\lambda))$ is given by figure 4.24. The n in figure 4.24 is found by including the number of p-walls parallel to the shortest root between the fundamental alcove and λ_v, counting the wall containing λ_v. Next we show that for most p-singular weights inside the p^2-alcove for B_2 $\mathrm{wfd}(\nabla(\lambda)) = \mathrm{wfd}(L(\lambda))$. Also it is shown that $\mathrm{wfd}(\nabla(\lambda)) \leq \mathrm{wfd}(L(\lambda))$ for a certain family of weights using some results about structure of tilting modules and translates of induced modules. For SL_2 and A_2 the situation is much clearer and was proved in [P3]. We give SL_2 as an example. The last chapter is devoted to studying submodule structures of tilting, Vogan, and cohomology modules and translates of cohomology modules.

1.2 Representations of algebraic groups

For more details on the following definitions the reader is referred to [J]. Let G be a connected, simply connected, semisimple algebraic group over an algebraically closed field k of prime characteristic $p > 0$. Throughout this thesis we will assume $p \neq 2$.

1.2.1 Roots, weights, and the Weyl group

Let T be a maximal torus of G and denote its root system by R. Let B be the Borel subgroup of G containing T corresponding to the negative roots. R^+ is the set of positive roots and $S \subseteq R^+$ is the set of simple roots. $(X(T), \leq)$ is the group of characters of T and the usual partial order defined by

$$\lambda \geq \mu \iff \lambda - \mu = \Sigma_{\beta_i \in R^+} \beta_i.$$

Let W be the Weyl group acting on $X(T) = X$, generated by all s_α defined by, $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, with $\alpha \in S$, $\lambda \in X$. $\langle \ , \ \rangle$ is an inner product on $X \otimes \mathbb{R} = E$. The *dominant* weights X^+ are all the $\lambda \in X$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. We use the "dot" action of W on E given by

$$w \cdot (\lambda) = w(\lambda + \rho) - \rho,$$

for all $w \in W$, $\lambda \in E$, where ρ is half the sum of the positive roots. $W_p = W \rtimes p\mathbb{Z}R$ is the affine Weyl group generated by all $s_{\alpha,np}$ defined by

$$s_{\alpha,np} \cdot \lambda = s_\alpha \cdot \lambda + np\alpha,$$

with $\lambda \in E$, $\alpha \in R$ and $n \in \mathbf{Z}$. Here R is the set of roots and $\mathbb{Z}R$ is the root lattice. We denote the one dimensional B-module corresponding by k_λ.

1.2.2 Cohomology groups

For a given weight λ we can define the i-th cohomology group by

$$H^i(\lambda) \cong R^i \operatorname{Ind}_B^G(k_\lambda).$$

Here $\mathrm{Ind}_B^G : B \to G$ is the usual induction function. We can define the induced module to be

$$\nabla(\lambda) = \mathrm{Ind}_B^G(k_\lambda).$$

The Weyl module $\Delta(\lambda)$ is the "dual" of the induced module. This dual is not the linear dual but the involutory contravariant dual. If M is a G-module then we shall denote this dual as M^o. G_r is the r-th Frobenius kernel of the Frobenius map $F : G \to G$. Denote the r-th Frobenius twist of a G-module M by $M^{[r]}$. If M is a G-module such that G_r acts trivially on M, then there exists a G-module N such that $M = N^{[r]}$.

The *rth Steinberg module* is defined to be

$$St_r = \nabla((p^r - 1)\rho) \cong \Delta((p^r - 1)\rho) \cong L((p^r - 1)\rho).$$

A *hyperplane* or p*-wall* is a set

$$H_{\alpha,np} = \{\lambda \in E \mid \langle \lambda + \rho, \alpha^\vee \rangle = np\},\ \alpha \in R^+, n \in Z^+.$$

The connected components of the complement of the union of all the hyperplanes are called *alcoves*, the *fundamental alcove* is defined to be:

$$C = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p,\ \alpha \in R^+\},$$

and it's closure is

$$\bar{C} = \{\lambda \in X \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p,\ \alpha \in R^+\}.$$

Define $d(\lambda)$ to be the number of hyperplanes between λ and the fundamental alcove. More precisely, if λ is contained in an alcove, there exists a unique n_α with $\alpha \in R^+$ such that $n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p$, we can then write

$$d(\lambda) = \sum_{\alpha \in R^+} n_\alpha.$$

Let

$$X_r = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle \le p^r, \forall\, \alpha \in S\}.$$

Any translation of X_r by an element of $p^r X$ is called a p^r*-box*, and any $\lambda \in X$ can be written uniquely in the form

$$\lambda = \lambda_0 + p^r \lambda_1,$$

where $\lambda_0 \in X_r$ and $\lambda_1 \in X$. λ is contained in the p^r-box $X_r + p^r\lambda_1$ and we denote the upper and lower vertices of this p^r-box by

$$\lambda^v = p^r\lambda_1 + (p^r - 1)\rho,$$

and,

$$\lambda_v = p^r\lambda_1 - \rho,$$

respectively. A weight $\lambda \in X$ is p*-regular* if $\langle \lambda + \rho, \alpha^\vee \rangle \neq kp, k \in \mathbf{N}$, and unique $\alpha \in R^+$, otherwise λ is said to be p*-singular*. A weight satisfying $\langle \lambda + \rho, \alpha^\vee \rangle = mp^k$, where $\alpha \in R^+$, and $m \in \mathbb{Z}$ is said to be on a p^k*-wall*. The p^2*-alcove* is defined to be all $\lambda \in X^+$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \leq p^2$, $\alpha \in R^+$. A *facet* is defined to be any set of the form

$$F = \{\lambda \in E \mid \langle \lambda + \rho,\ \alpha^\vee \rangle = n_\alpha p,\ \forall \alpha \in R_0^+, (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p,\ \forall \alpha \in R_1^+\},$$

where the R_i^+'s are disjoint partitions of R^+, and $n_\alpha \in \mathbf{Z}$. The closure of F, $\bar{F}$ is obtained by replacing both $<$ signs by $\leq$. The *upper closure* is denoted by $\hat{F}$ and is obtained by replacing only the right most $<$ with $\leq$.

1.2.3 Strong Linkage

We can define a new order called *strong linkage* on X, denoted by $\uparrow$, as follows: $\mu \uparrow \lambda$ if and only if $\mu \leq \lambda$ and there is a sequence of simple reflections $s_i \in S$ for $i = 1, .., k$, such that

$$\mu \leq s_1 \cdot \mu \leq (s_1 s_2) \cdot \mu \leq \cdots \leq (s_k \cdots s_1) \cdot \mu = \lambda.$$

Consider the cohomology groups $H^i(\lambda)$. Andersen [A2] showed that if $L(\mu)$ is a composition factor then $\mu \uparrow \lambda$. This is called the *strong linkage principle.*

1.3 G-modules and Extensions

In this thesis we are concerned only with the finite dimensional representations of G. So we consider only the category of finite dimensional rational G-modules which we refer to from now on as G-modules. Let $\lambda \in X^+$, a *highest weight module* with highest weight λ is any quotient of $\Delta(\lambda)$. $\Delta(\lambda)$ is thus the universal highest weight module having λ as it's highest weight. We have the usual notion of composition series and the *composition factors* will be the simple highest weight modules. Given a G-module M the *radical* of M $\operatorname{Rad}(M)$ is the smallest submodule of M such that $M/\operatorname{Rad}(M)$ is semisimple. The *extension groups* between two G-modules M and N, $\operatorname{Ext}^i_G(M,N)$, are the groups of equivalence classes of exact sequences between M and N. Two exact sequences are equivalent if they are isomorphic in a certain sense. For more on the definitions of the extension groups the reader is referred to [M]. Throughout we shall make use of the contravariant functor $\operatorname{Ext}^\bullet_G(-,\nabla(\lambda))$. Given a short exact sequence of G-modules

$$0 \to M \to N \to Q \to 0,$$

we can apply this functor and get a long exact sequence in the usual way

$$0 \to \operatorname{Hom}(Q,\nabla(\lambda)) \to \operatorname{Hom}(N,\nabla(\lambda)) \to \operatorname{Hom}(M,\nabla(\lambda)) \to \operatorname{Ext}^1(Q,\nabla(\lambda))$$

$$\to \operatorname{Ext}^1(N,\nabla(\lambda)) \to \operatorname{Ext}^1(M,\nabla(\lambda)) \to \operatorname{Ext}^2(Q,\nabla(\lambda)) \cdots .$$

Here we have dropped the G in $\operatorname{Ext}^\bullet_G(-,\nabla(\lambda))$, and will continue this convention below.

1.4 Translation functors

We say λ and μ belong to the same *block* if $\operatorname{Ext}^1(L(\lambda),L(\mu)) \neq 0$. The category of G-modules can be divided up into subcategories each corresponding to a block. The following lemma tells us when we have the potential for an extension between two simple modules, and is useful when considering submodule structures of the Weyl modules:

Lemma 1.4.1. *Let $\mu, \lambda \in X^+$ such that $\mu \leq \lambda$. Then,*

$$\mathrm{Ext}^1(L(\lambda), L(\mu)) \cong \mathrm{Hom}(\mathrm{Rad}(\Delta(\lambda)), L(\mu)).$$

Let $\lambda \in X$, a block is equivalently defined as $W_p \cdot \lambda$. Given a G-module M $pr_\lambda M$ is the *projection* onto the block containing λ. $pr_\lambda M$ is the largest submodule of M such that all it's composition factors have highest weights in $W_p \cdot \lambda$. pr_λ is an exact functor. Let λ and $\mu \in \bar{C} \cap X^+$, and M be any G-module. Define the *translation functors* T_μ^λ via,

$$T_\mu^\lambda M = pr_\lambda(L(\nu) \otimes pr_\mu M).$$

Where ν is the the unique weight in $W(\lambda - \mu) \cap X^+$. The translation functors are exact and have the following properties.

$$T_\lambda^\mu L(w \cdot \lambda) \cong \begin{cases} L(w \cdot \mu) & \text{if } w \cdot \mu \in \hat{F}, \\ 0 & \text{otherwise.} \end{cases}$$

The translation functors take us from block to block.

The following lemma is from [J, II 7.11]

Lemma 1.4.2. *Let $\lambda, \mu \in \bar{C}$ and let F be the facet containing λ. If $\mu \in \bar{F}$, then for all $w \in W_p$ and $i \in N$ we have*

$$T_\lambda^\mu H^i(w \cdot \lambda)) \cong H^i(w \cdot \mu), \ \forall i \geq 0.$$

In particular if we have,

$$T_\lambda^\mu \nabla(w \cdot \lambda)) \cong \nabla(w \cdot \mu),$$

and,

$$T_\lambda^\mu \Delta(w \cdot \lambda)) \cong \Delta(w \cdot \mu).$$

For G of type B_2 if μ is in a p-wall in the upper closure of the alcove containing λ then we have,

$$T_\lambda^\mu T_\mu^\lambda L(\mu) \cong L(\mu) \oplus L(\mu),$$

$$T_\lambda^\mu T_\mu^\lambda \Delta(\mu) \cong \Delta(\mu) \oplus \Delta(\mu),$$

$$T_\lambda^\mu T_\mu^\lambda \nabla(\mu) \cong \nabla(\mu) \oplus \nabla(\mu).$$

Chapter 2

Weyl filtration dimension

2.1 Definitions and properties

A G-module M has a *Weyl filtration* if there exist a filtration

$$0 = M_0 \subset M_1 \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong \Delta(\mu_i)$ for some $\mu_i \in X(T)$. Similarly if $M_i/M_{i-1} \cong \nabla(\mu_i)$ then we say M has a *good filtration*. The definition and lemma that follow can be found in [FP].

Definition 2.1.1. Let M be a G-module. We say M has *Weyl filtration dimension* d, $\operatorname{wfd}(M) = d$, if M has a minimal resolution

$$0 \to M_d \to M_{d-1} \to \cdots \to M_1 \to M_0 \to M \to 0,$$

and each M_i has Weyl filtration, for all $i = 0, \ldots, d$.

Lemma 2.1.2. *For a G-module M,*

$$\begin{aligned}\operatorname{wfd}(M) &= \max\{i \mid \operatorname{Ext}^i(M, \nabla(\sigma)) \neq 0, \textit{ for some } \sigma \in X^+\} \\ &= \max\{i \mid \operatorname{Ext}^i(M, N) \neq 0, \textit{ for some } N \textit{ with good filtration}\}.\end{aligned}$$

For any G-module M, $0 \leq \operatorname{wfd}(M) < \infty$. For a proof of finiteness see [J, II 6.20] or [P1] lemma 2.9.

Remark 2.1.3. Let M be a G-module with $\mathrm{wfd}(M) = n$. For notational convenience, n shall be written as a superscript M^n. The following lemma will be used repeatedly in what follows.

Lemma 2.1.4. *If M^n, N^m and P^k are G-modules , then*

1. $\mathrm{wfd}(M \oplus N) = max\{n, m\}$.
2. *If* $0 \to L(\mu)^0 \to M^n \to L(\lambda)^1 \to 0$, *and M is not a Weyl module, then* $n = 1$.
3. $0 \to M^n \to N^0 \to P^0 \to 0 \Rightarrow n = 0$.
4. $0 \to M^n \to N^m \to P^k \to 0$ *and* $n \geq k \Rightarrow m = n$.
5. $0 \to M^n \to N^m \to P^k \to 0, n > m \Rightarrow k = n + 1$.
6. $0 \to M^n \to N^m \to P^k \to 0$ *and* $m > n \Rightarrow k = m$.
7. $0 \to M^n \to N^m \to P^k \to 0$ *and* $k \geq n + 2 \Rightarrow m = k$.
8. $0 \to M^n \to N^m \to P^k \to 0$ *and* $m < k \Rightarrow n = k - 1$.

Proof. (1) $\mathrm{Ext}^i_G(M \oplus N, \nabla(\sigma)) \cong \mathrm{Ext}^i_G(M, \nabla(\sigma)) \oplus \mathrm{Ext}^i_G(N, \nabla(\sigma))$, $\forall\, \sigma \in X^+$.

(2) Applying $\mathrm{Ext}^\bullet_G(-, \nabla(\delta))$, $\delta \in X^+$, shows that $n = 1$ or $n = 0$. M is not a Weyl module, so if $n = 0$ M must have a non-trivial Weyl filtration. However, $L(\mu) \subset M$ is the only possible non-trivial Weyl filtration but $M/L(\mu) \cong L(\lambda)^1$ and this is not a Weyl module, so n=1.

(3) Apply $\mathrm{Ext}^\bullet_G(-, \nabla(\delta))$, $\delta \in X^+$.

$$\ldots \to 0 \to \mathrm{Ext}^1(N, \nabla(\delta)) \to \mathrm{Ext}^1(M, \nabla(\delta)) \to \mathrm{Ext}^2(P, \nabla(\delta)) \to 0 \to \ldots$$

The fact that $\mathrm{Ext}^2(P, \nabla(\delta)) = \mathrm{Ext}^1(N, \nabla(\delta)) = 0$, and that everything else will be zero in the exact sequence gives the result.

(4) Apply $\mathrm{Ext}^\bullet_G(-, \nabla(\delta))$, $\delta \in X^+$.

$$\ldots \to \mathrm{Ext}^n(N, \nabla(\delta)) \to \mathrm{Ext}^n(M, \nabla(\delta)) \to \mathrm{Ext}^{n+1}(P, \nabla(\delta)) \to 0 \to \ldots$$

We have $\mathrm{Ext}^{n+1}(P, \nabla(\delta)) = 0$ for all $\delta \in X$, and there exists $\delta \in X^+$ such that $\mathrm{Ext}^n(M, \nabla(\delta)) \cong k$. Everything to the right of the zero in the exact sequence being zero along with the above implies the result.

(5) to (8) are proved similarly by application of $\mathrm{Ext}_G^\bullet(-, \nabla(\delta)),\ \delta \in X^+$. □

The following theorem is due to Parker [P1].

Theorem 2.1.5. *Let G be connected and reductive, if $\lambda \in X^+$ is p-regular, then*

$$\mathrm{wfd}(L(\lambda)) = \mathrm{wfd}(\nabla(\lambda)) = \mathrm{gfd}(\Delta(\lambda)) = d(\lambda).$$

This theorem does not hold for p-singular weights. For example take $G = SL_2$ and $\lambda = (p^r - 1)\rho$, where $r \in \mathbb{Z}^+$. We call these weights *Steinberg weights.* These weights correspond to the rth Steinberg modules St_r.

$$\mathrm{wfd}(St_r) = \mathrm{wfd}(L((p^r - 1)\rho)) = \mathrm{wfd}(\Delta((p^r - 1)\rho)) = 0.$$

The $\mathrm{wfd}(L(\lambda))$ with λ p-singular for SL_2 was computed by Parker in [P3]. We compute this case in the next chapter as an example. To find $\mathrm{wfd}(L(\lambda))$ with λ p-singular for SL_2 we shall use a proposition from [J]. First we state a lemma which follows from the spectral sequence [J, I 6.6(1)].

Lemma 2.1.6. *For every $\lambda \in X^+$ there exists a G-module N_λ with good filtration such that*

$$N_\lambda^{[r]} = \mathrm{Hom}_{G_r}(St_r, \nabla(\lambda)).$$

Proposition 2.1.7. *If M is any G-module, then* $\mathrm{wfd}(M) = \mathrm{wfd}(St_r \otimes M^{[r]})$.

Proof. By 2.1.6 and [J, I.6.6(1)], $\forall \lambda \in X^+$ there is a module N_λ with good filtration such that,

$$\mathrm{Ext}_G^i(M, N_\lambda) \cong \mathrm{Ext}_{G/G_r}^i(M^{[r]}, \mathrm{Hom}_{G_r}(St_r, \nabla(\lambda))) \cong \mathrm{Ext}_G^i(St_r \otimes M^{[r]}, \nabla(\lambda))$$

$$\Rightarrow \mathrm{wfd}(M^{[r]} \otimes St_r) \leq \mathrm{wfd}(M).$$

Using the identity $\mathrm{Hom}_{G_r}(St_r, N^{[r]} \otimes St_r) \cong \mathrm{Hom}_{G_r}(St_r, St_r) \otimes N^{[r]} \cong N^{[r]}$ and [J, I.6.6(1)],

$$\mathrm{Ext}^i_G(M, N) \cong \mathrm{Ext}^i_{G/G_r}(M^{[r]}, N^{[r]}) \cong \mathrm{Ext}^i_{G/G_r}(M^{[r]}, \mathrm{Hom}_{G_r}(St_r, N^{[r]})$$

$$\mathrm{Ext}^i_{G/G_r}(M^{[r]}, \mathrm{Hom}_{G_r}(St_r, N^{[r]} \otimes St_r)) \cong \mathrm{Ext}^i_G(M^{[r]} \otimes St_r, N^{[r]} \otimes St_r)$$

$$\Rightarrow \mathrm{Ext}^i_G(M, N) \cong \mathrm{Ext}^i_G(M^{[r]} \otimes St_r, N^{[r]} \otimes St_r).$$

So if N has good filtration,

$$\mathrm{gfd}(N^{[r]} \otimes St_r) \leq \mathrm{gfd}(N) = 0 \Rightarrow \mathrm{wfd}(M) \leq \mathrm{wfd}(M^{[r]} \otimes St_r).$$

□

2.2 Weyl filtration dimension of Vogan modules

Definition 2.2.1. let $\mu, \lambda \in X^+$ with λ contained inside an alcove and μ in the upper closure of the alcove containing λ. Consider $T^\lambda_\mu L(\mu)$. We call these modules *Vogan modules* and denote them as $U(\mu)$.

Andersen [A3] has shown that if the Lusztig conjecture is true then these modules have simple socle and head equal to $L(\lambda)$ and that there is a short exact sequence

$$L(\lambda) \to \mathrm{Rad}(U(\mu)) \to L(s \cdot \lambda) \oplus M.$$

Here s is a simple reflection such that $s \cdot \lambda \geq \lambda$ and M is a direct sum of simple modules which can be computed using the technique's below.

Proposition 2.2.2. *Let G be of type B_2 and let $\mu, \lambda \in X^+$ with λ contained inside an alcove with μ a non Steinberg weight in the upper closure of that alcove. Let $U(\mu)$ be the corresponding Vogan module. Then* $\mathrm{wfd}(U(\mu)) = \mathrm{wfd}(L(\mu))$.

Proof. $\mathrm{wfd}(L(\mu)) = \mathrm{wfd}(L(\mu) \oplus L(\mu)) = \mathrm{wfd}(T^\mu_\lambda T^\lambda_\mu L(\mu)) \leq \mathrm{wfd}(T^\lambda_\mu L(\mu)) \leq \mathrm{wfd}(L(\mu))$. □

2.3 Tilting modules

Modules with both good filtration and Weyl filtration are called *Tilting modules*. For every $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$ with highest weight λ. $T(\lambda)$ is uniquely determined by it's character. Every tilting module is a direct sum of these indecomposable tilting modules, and a direct sum of tilting modules is a tilting module. Tilting modules are self dual. Let $T(\mu)$ be the indecomposable tilting module with singular highest weight μ. Let λ be inside the alcove which has μ in it's closure. For G of type B_2 we have,

$$T_\lambda^\mu T_\mu^\lambda T(\mu) \cong T(\mu) \oplus T(\mu).$$

The indecomposable tilting modules form a basis for the grothendieck group. The first examples of tilting modules for B_2 are the simple modules with good filtration dimension equal to zero. Translations off the wall of such simple modules will be examples of Vogan modules that are tilting modules for B_2. Examples of these can be found in figure 5.6.

Chapter 3

SL_2(an example)

If $G = SL_2$ we can identify X^+ with $\mathbb{N}$. Let λ be a p-singular weight. λ has *p-adic decomposition*

$$\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_n p^n,$$

where $\lambda_0 = p-1$. If $\lambda_j = p-1, \forall\ j < i$, and $\lambda_i < p-1$ we can write,

$$\lambda = (p^i - 1) + \mu p^i,$$

where $\mu = \lambda_i + \cdots + \lambda_n p^{n-i}$ is p-regular. Now,

$$\mu = (p-a) + \lambda_{i+1} p + \cdots + \lambda_n p^{n-i} = (1 + \lambda_{i+1} + \cdots + \lambda_n p^{n-1-i})p - a,\ a > 1.$$

Using 2.1.5 $d(\mu) = \lambda_{i+1} + \cdots + \lambda_n p^{n-1-i}$, it can be shown that

$$(\lambda_{i+1} + \cdots + \lambda_n p^{n-i-1})p^{i+1} - 1 < \lambda = (1 + \lambda_{i+1} + \lambda_{i+2} p + \cdots + \lambda_n p^{n-i-1})p^{i+1} - b,$$

$b < p^{i+1} - 1$. It follows that $d(\mu) =$ the number of p^{i+1}-walls between λ and the fundamental alcove C.

Theorem 3.0.1. *For $G = SL_2$ and $\lambda = (p^i - 1) + \mu p^i$*

$$\mathrm{wfd}(L(\lambda)) = \mathrm{wfd}(\nabla(\lambda)) = \mathrm{wfd}(\nabla(\mu)) = d(\mu).$$

Proof. Since $\rho = 1$ for SL_2,

$$\nabla(\lambda) = \nabla((1+\mu)p^r - 1) \cong St_r \otimes \nabla(\mu)^{[r]} \Rightarrow \text{wfd}(\nabla(\lambda)) = \text{wfd}(St_r \otimes \nabla(\mu)^{[r]}).$$

Since μ is p-regular, 2.1.7 and 2.1.5 give,

$$\text{wfd}(\nabla(\lambda)) = \text{wfd}(\nabla(\mu)) = d(\mu).$$

To compute $\text{wfd}(L(\lambda))$, Let $\lambda_j \in X_1,\ j = 0, \cdots, r$, and write λ in it's p-adic decomposition

$$\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots + \lambda_r p^r,$$

where, $\lambda_0 = p - 1$. Using [J, II.3.16]

$$L(\lambda_0 + (\lambda_1 + \lambda_2 p + \cdots + \lambda_r p^{r-1})p) \cong L(\lambda_0) \otimes L(\lambda_1 + \lambda_2 p + \cdots + \lambda_r p^{r-1})^{[1]}$$

$$\Rightarrow L(\lambda) \cong St_1 \otimes L(\lambda_1 + \lambda_2 p + \cdots + \lambda_r p^{r-1})^{[1]}.$$

Now by 2.1.7,

$$\text{wfd}(L(\lambda)) = \text{wfd}(L(\lambda_1 + \lambda_2 p + \cdots \lambda_r p^{r-1})).$$

Since $\lambda_1 = p - 1$, repeat the above process until you reach $\lambda_i \neq p - 1$ to get,

$$\text{wfd}(L(\lambda)) = \text{wfd}(L(\lambda_i + \lambda_{i+1} p + \cdots \lambda_r p^{r-i})).$$

$\mu = \lambda_i + \lambda_{i+1} p + \cdots \lambda_r p^{r-i}$ p-regular implies,

$$\text{wfd}(L(\lambda)) = \text{wfd}(L(\mu)) = d(\mu).$$

□

Chapter 4

B_2

For the rest of the thesis we will assume $p \neq 2$. Consider G of type B_2, for weights inside the p^2-alcove not too close to the upper p^2-alcove, Lin [L1] has shown that the 'generic' (In this context we take generic to mean far enough away from the chamber walls) radical filtration of the Weyl modules with p-singular highest weights can be obtained by applying the translation functors to the 'generic' radical structure of the Weyl modules with p-regular highest weights. A cancellation principle (see [H2] for more details), stated below, can be used to find the radical filtration of the Weyl modules with dominant p-singular highest weights close to the chamber walls. In this case the radical filtration of $\Delta(\lambda)$, with p-singular highest weight, has at most 3 radical layers:

$$\mathrm{Rad}^3(\Delta(\lambda)) \subseteq \mathrm{Rad}^2(\Delta(\lambda)) \subseteq \mathrm{Rad}(\Delta(\lambda)) \subseteq \Delta(\lambda).$$

Figure 4.1 illustrates the radical filtrations of all $\Delta(\lambda)$ with generic p-singular highest weights, there are six 1-dimensional facets as indicated in figure 4.1. The zeros in these diagrams denote the highest weight of the corresponding Weyl module. The Weyl filtration dimension of the simples can be computed, thanks to the following simple fact: Consider the short exact sequence

$$0 \to \mathrm{Rad}(\Delta(\lambda)) \to \Delta(\lambda) \to L(\lambda) \to 0.$$

Assuming $\Delta(\lambda) \neq L(\lambda)$ and applying 2.1.4(5), we get

$$\mathrm{wfd}(L(\lambda)) = \mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))) + 1. \tag{4.1}$$

By using the cancellation principle and equation (4.1), all $\mathrm{wfd}(L(\lambda))$ in the first p-box can be computed. Consider figure 4.2, the numbers correspond to $\mathrm{wfd}(L(\lambda))$ with highest weights at those positions. The chamber walls are indicated by thick lines, and all Weyl modules are simple at the places indicated by zeros. The Weyl module with highest weight λ with $\mathrm{wfd}(L(\lambda)) = 1$ in the figure has two composition factors, it is not semisimple, and has simple top $L(\lambda)$.

4.1 Notation for facets

We will focus on singular weights contained in the 1-dimensional facets indicated in figure 4.2. They will be indicated by $\downarrow$, $\uparrow$, $\backslash$, $/$, , and $\rightarrow$, which should be clear from their shapes in the pattern. In order to keep track of all the computations the following notation for the position of a weight is used: Let α and β be the shortest and longest simple roots respectively. A weight will be written as $(\rightarrow, i, j)$ to indicate that it has facet type $\rightarrow$, and satisfies the two conditions,

$$(i-1)p < \langle \lambda + \rho, \alpha^\vee \rangle \leq ip,$$

$$(j-1)p < \langle \lambda + \rho, \beta^\vee \rangle \leq jp.$$

Note that $(ip-1)\omega_\alpha + (jp-1)\omega_\beta$ is the upper vertex of the p-box containing the facets. For example the highest weight λ in figure 4.3 below is at $(\leftarrow, 3, 1)$. The labels in figure 4.1 show the notation for each facet.

4.2 Cancellation principle

To compute the radical filtration of $\Delta(\lambda)$ close to the chamber walls for λ dominant, use figure 4.1 along with the following *cancellation principle:* If a composition factor is on a chamber wall or outside the dominant chamber, it is not included in the radical filtration. If the upper vertex of the p-box containing a weight outside the dominant chamber is uniquely

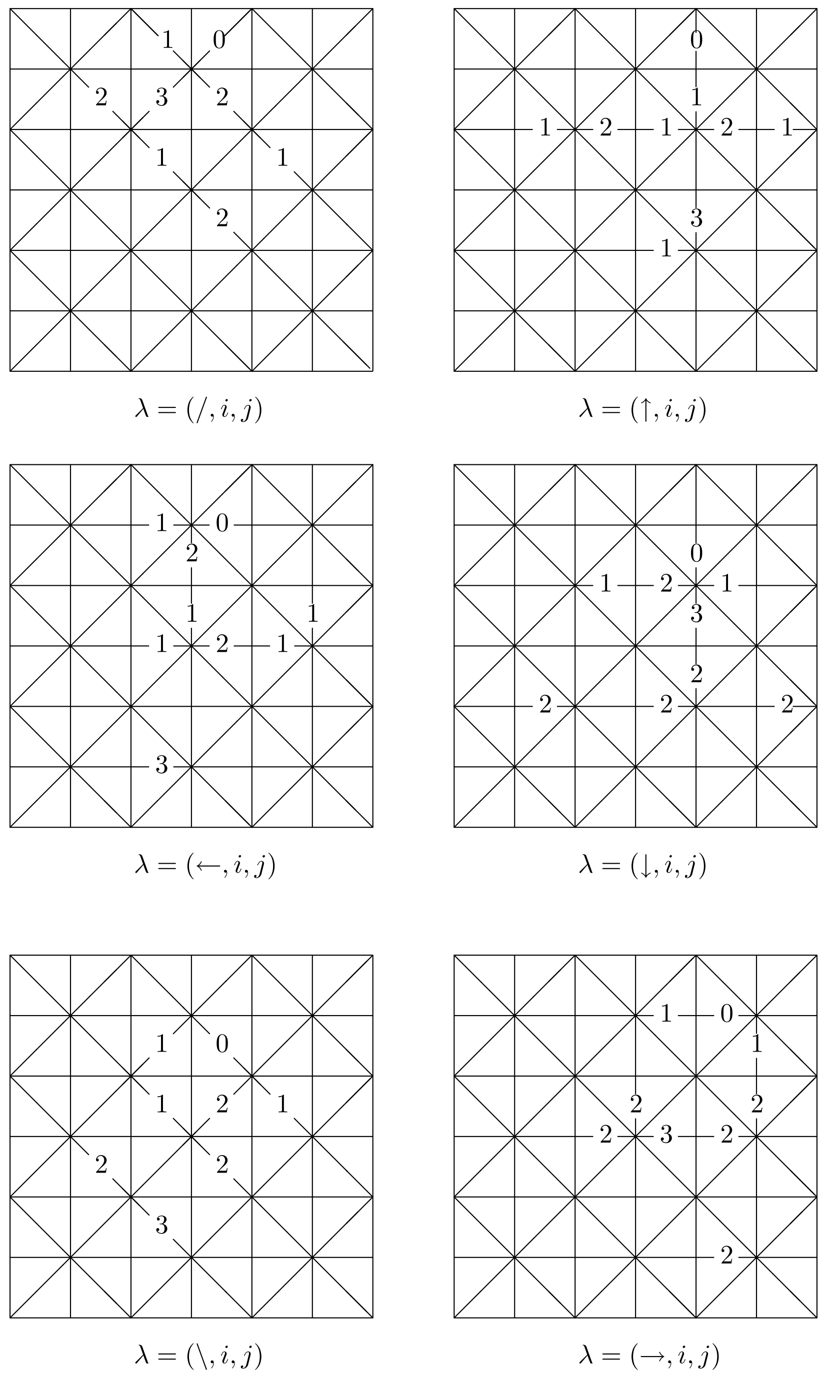


Figure 4.1: *Generic radical filtration of* $\Delta(\lambda)$ *for the 6 facet types.*

reflected in, across a chamber wall, and the weight lands on a dominant weight in the generic radical filtration pattern, then cancel that dominant weight from the radical filtration. For an illustration of this compare figure 4.5 to figure 4.1, the x's indicate where a weight is canceled by such a reflection.

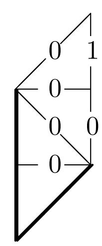


Figure 4.2: wfd$(L(\lambda))$ *in first* p*-box.*

4.3 Calculation methods used

Three different methods of calculation are needed to compute wfd$(L(\lambda))$:

Method 1: We use embeddings of Weyl modules, or quotients of Weyl modules, which have highest weights related by the affine Weyl group in the direction of a simple root.

Method 2: In some cases the embeddings of method 1 are not available. When we use the cancellation principle to reflect weights into the dominant chamber we can have a break down in the construction after Lemma 4.4.1(see below). However, in some cases the existence of a non-zero map between Weyl modules suffices.

Method 3: For some weights neither of the above methods work and we are forced to use the diagonal map $\Delta : M \to M \oplus M$ defined by $x \mapsto (x, x)$, with M being any G-module.

We now discuss the different methods separately and give an example of each.

4.4 Embeddings (*method 1*)

In some cases $\mathrm{wfd}(L(\lambda))$ can be found by using embeddings of Weyl modules, or quotients of Weyl modules, with highest weights related by elements of W_p in the direction of simple roots. It turns out that the radical filtrations of the generic Weyl modules (figure 4.1) can be determined from the induced modules Z_1 of subgroup schemes G_1T. We use some lemmas from [L1] and [L3] and induce up from the infinitesimal level, first some notations. For B_2 the Weyl group is generated by the reflections $\{s_\alpha, s_\beta\}$, where α and β are the simple roots, from now on fix a simple root α. Define the following action: Let δ be a dominant weight such that $np < \langle \delta + \rho, \alpha^\vee \rangle \leq (n+1)p$, then

$$(s_\alpha^*) \cdot \delta = s_\alpha \cdot \delta + (n+1)p\alpha.$$

Basically, (s_α^*) is the reflection in the closest upper α-hyperplane. Denote j-th powers by $(s_\alpha^*)^j$, and set $\delta_j^\alpha = (s_\alpha^*)^j \cdot \delta$. Let P denote the minimal parabolic containing B and the root subgroup, U_α. Denote the Frobenius kernel of P by P_1, and set

$$\mathrm{Ind}_B^{P_1B}(\lambda) = Z_1^\alpha(\lambda)$$

and

$$\mathrm{Ind}_{P_1B}^{G_1B} Z_1^\alpha(\lambda) = Z_1(\lambda).$$

Denote the the P_1B-socle of $Z_1^\alpha(\lambda)$ by $L_1^\alpha(\lambda)$ and the G_1B-composition factors. of $Z_1(\lambda)$ by $L_1(\gamma)$. assume $0 \leq np < \langle \lambda + \rho, \alpha^\vee \rangle < (n+1)p$, and let μ be a weight such that $0 < \langle \mu + \rho, \alpha^\vee \rangle < p$ and $\lambda = \mu_n^\alpha$. The reader can find proof of the following lemma in [L1, 2.1, 2.3]. We will also assume that λ in the lemma below is not on an α-hyperplane.

Lemma 4.4.1. *There is an exact sequence of G_1B-modules*

$$0 \to \operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\lambda) \to Z_1(\mu_n^{\alpha}) \xrightarrow{\psi_n} Z_1(\mu_{n-1}^{\alpha}) \to \cdots \to Z_1(\mu_2^{\alpha}) \xrightarrow{\psi_2} Z_1(\mu_1^{\alpha}) \to \operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\mu) \to 0.$$

4.5 Chain of embeddings

Divide up the sequence in the lemma into short exact sequences. For example if $n = 2$, $\lambda = \mu_2^{\alpha}$, and there are two short exact sequences,

$$0 \to \operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\lambda) \to Z_1(\lambda) \xrightarrow{\psi_2} \operatorname{Im}(\psi_2) \to 0$$

$$0 \to \operatorname{Im}(\psi_2) \to Z_1(\mu_1^{\alpha}) \to \operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\mu) \to 0.$$

Now apply the right derived functor $R^i \operatorname{Ind}_{G_1B}^{G}$ to these sequences to get long exact sequences. Suppose there are no cancellations coming from reflecting (uniquely) an upper vertex of a weight outside the dominant chamber and canceling. Using [L3] lemma 5.2, we have that every composition factor $L_1(\sigma)$ of $Z_1(\lambda)$ satisfies

$$R^1 \operatorname{Ind}_{G_1B}^{G}(L_1(\sigma)) = 0.$$

In particular

$$R^1 \operatorname{Ind}_{G_1B}^{G} L_1(\mu_1^{\alpha}) = R^1 \operatorname{Ind}_{G_1B}^{G}(\operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\mu_1^{\alpha})) = 0.$$

Since $\operatorname{Im}(\psi_2)$ is a submodule of $Z_1(\lambda)$, $R^1 \operatorname{Ind}_{G_1B}^{G}(\operatorname{Im}\psi_2) = 0$. This reduces the long exact sequences to the short exact sequences,

$$0 \to \operatorname{Ind}_{G_1B}^{G} \operatorname{Ind}_{P_1B}^{G_1B} L_1^{\alpha}(\lambda) \to \operatorname{Ind}_{G_1B}^{G} Z_1(\lambda) \to \operatorname{Ind}_{G_1B}^{G}(\operatorname{Im}(\psi_2)) \to 0,$$

$$0 \to \operatorname{Ind}_{G_1B}^{G} \operatorname{Im}(\psi_2) \to \operatorname{Ind}_{G_1B}^{G} Z_1(\mu_1^{\alpha}) \to \operatorname{Ind}_{G_1B}^{G}(\operatorname{Ind}_{P_1B}^{G_1B}(L_1^{\alpha}(\mu))) \to 0.$$

The following two facts are from infinitesimal theory:

i) If $\delta \in X^+$ satisfies $0 < \langle \delta + \rho, \alpha^{\vee} \rangle < p$, then $\operatorname{Ind}_{G_1B}^{G}(\operatorname{Ind}_{P_1B}^{G_1B}(L_1^{\alpha}(\delta))) \cong \nabla(\delta)$.

ii) For $\lambda \in X$. $R^i \operatorname{Ind}_{G_1B}^{G}(Z_1(\lambda)) \cong H^i(\lambda)$, $\forall i \geq 0$.

To prove ii), we first note that $\operatorname{Ind}_B^{G_1B}$ is an exact functor. Transitivity of induction and the Grothendieck spectral sequence gives the result.

Using i) and ii) the short exact sequences in the example become,

$$0 \to N_\lambda \to \nabla(\lambda) \to M_\lambda \to 0$$

$$0 \to M_\lambda \to \nabla(\mu_1^\alpha) \to \nabla(\mu) \to 0$$

Where $N_\lambda \cong \operatorname{Ind}_{G_1B}^{G} \operatorname{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda)$ and $M_\lambda \cong \operatorname{Ind}_{G_1B}^{G}(\operatorname{Im}(\psi_2))$.
Dualizing,

$$0 \to M_\lambda^o \to \Delta(\lambda) \to N_\lambda^o \to 0$$

$$0 \to \Delta(\mu) \to \Delta(\mu_1^\alpha) \to M_\lambda^o \to 0.$$

So, $M_\lambda^o \cong \Delta(\mu_1^\alpha)/\Delta(\mu) \hookrightarrow \Delta(\lambda)$. From these short exact sequences we can apply lemma 2.1.4 to get $\operatorname{wfd}(M_\lambda^o) = 1$, remembering remark 2.1.3 we write $(M_\lambda^o)^1$.

Notation: From now on the last module that embeds, for a given chain of embeddings, shall be denoted by M_λ^α, where α indicates the simple root and λ is the highest weight of the corresponding Weyl module, similarly we denote the final quotient $\Delta(\lambda)/M_\lambda^\alpha$ by N_λ^α. In the example above $M_\lambda^o = M_\lambda^\alpha$. For the purpose of illustration, in the figures below, the simple modules $L(\lambda)$ with $\operatorname{wfd}(L(\lambda)) = n$ will be written as λ^n. Put another way, the superscript of the highest weight denotes the Weyl filtration dimension of the simple module with that highest weight.

Remark 4.5.1. Note that $\operatorname{wfd}(M_\lambda^\alpha) \leq n-1$ for a chain of length n. Applying Lemma 2.1.4(8) gives $\operatorname{wfd}(M_\lambda^\alpha) = n-1$. Increasing the chain length by 1 increases $\operatorname{wfd}(M_\lambda^\alpha)$ by 1.

Example 4.5.1. Figure 4.3 illustrates the radical filtration for the Weyl module with highest weight located at $(\leftarrow, 3, 1)$. This figure is attained by applying the cancellation principle to figure 4.1. Note μ is shown only to illustrate that the chain of embeddings to be used will start from $\Delta(\mu)$, $L(\mu)$ is not a composition factor of $\Delta(\lambda)$. Figure 4.4 indicates some of the submodule structure of $\Delta(\lambda)$, the Greek letters indicate simple modules with the respective highest weights, and different rows indicate the radical structure. A line between two composition factors means that the subquotient corresponding to those two simples is not semisimple, and so the top composition factor can not drop down to the next radical layer. Using 4.4.1 we get two short exact sequences,

$$\Delta(\mu) \to \Delta(\omega_1) \to (M_\lambda^\alpha)^1,$$

$$(M_\lambda^\alpha)^1 \to \mathrm{Rad}(\Delta(\lambda)) \to L(\omega_2^1) \oplus L(\omega_3^1).$$

Finally, applying 2.1.4(1) and 2.1.4(4) gives $\mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))) = 1$, (4.1) implies $\mathrm{wfd}(L(\lambda)) = 2$.

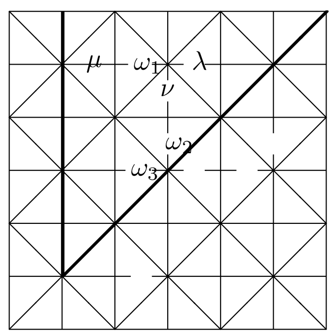


Figure 4.3: $\Delta(\lambda)$ *for example 4.5.1.*

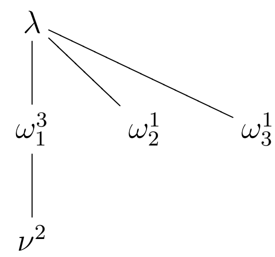


Figure 4.4: *Submodule structure of* $\Delta(\lambda)$ *for 4.5.1.*

4.6 No embeddings (*method 2*)

In the above example we made use of embeddings of Weyl modules. Unfortunately, If there is a cancellation coming from a reflection, for the first map in any chain, then $R^1(\mathrm{Ind}_{G_1B}^G(L(\mu_1^\alpha)))$ is not necessarily zero. So we can not use the construction above. However, even though we do not have an embedding, for weights 'close' to each other, there is a non zero map $f : \Delta(\mu) \to \Delta(\mu_1)$. The following definition and lemma are from [J, II 6.22] and [J, II 6.24(2)] respectively.

Definition 4.6.1. Let $\lambda, \mu \in X^+$ with $\mu \uparrow \lambda$. We say that μ is *close* to λ if and only if:
(1) There is no $\alpha \in R^+$ with $\mu \uparrow \lambda - p\alpha$.
(2) If $\mu \uparrow \lambda' \uparrow \lambda$ for some $\lambda' \in X$, then $\lambda' \in X^+$.

Lemma 4.6.2. *Let* $\lambda \in X_+$. *Suppose* $\mu \in X$ *is maximal for* $\mu \uparrow \lambda$ *and* $\mu \neq \lambda$. *If* $\mu \in X^+$ *and if* $\mu \neq \lambda - p\,\alpha$ *for all* $\alpha \in R^+$, *then there is a non zero map* $f : \Delta(\mu) \to \Delta(\lambda)$.

For more detailed information on these maps between Weyl modules the reader can refer to [K].

Example 4.6.1. Consider figure 4.5, the x's show where cancellation has occurred by reflecting the upper vertex of an outside weight into the dominant chamber across the chamber wall. Figure 4.6 shows some of the structure of the Weyl modules $\Delta(\lambda)$ and $\Delta(\mu)$, note that $\omega_3 = (\rightarrow, 1, 1)$ in this figure. Since μ is close to λ we have a non-zero map $f : \Delta(\mu) \to \Delta(\lambda)$. $\mathrm{Im}(f)$ is a submodule of $\mathrm{Rad}(\Delta(\lambda))$ and $L(\mu)$ must be in the image, so by definition of radical, we have

$$\mathrm{Im}(f) = \mathrm{Rad}(\Delta(\lambda)).$$

Using the short exact sequence,

$$L(\omega_3^0) \to \Delta(\mu) \to \mathrm{Im}(f),$$

and 2.1.4 we compute

$$\mathrm{wfd}(\mathrm{Im}(f)) = \mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))) = 1,$$

using (4.1) we have that $\mathrm{wfd}(L(\lambda)) = 2$.

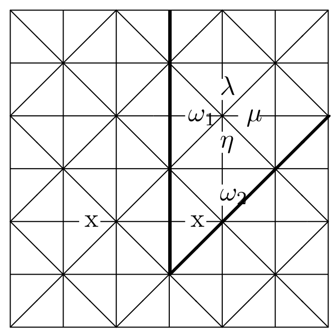


Figure 4.5: $\Delta(\lambda)$ *for example 4.6.1.*

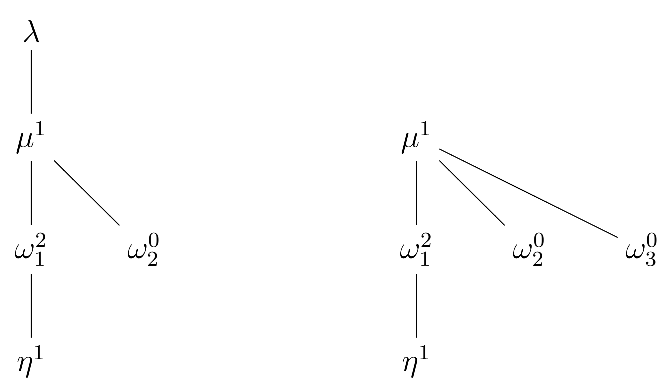


Figure 4.6: $\Delta(\lambda)$ *and* $\Delta(\mu)$ *for 4.6.1.*

4.7 A diagonal map (*method 3*)

Even though we have the above embeddings and maps, for some Weyl module structures this is still not enough to compute the Weyl filtrations of their simple heads. Consider the diagonal map $f : L(\alpha) \mapsto L(\alpha) \oplus L(\alpha)$ defined by $x \mapsto (x, x)$. This is a G-module homomorphism, and will be used in the following way: Let S be a G-module with simple head $L(\alpha)$ and radical $L(\epsilon) \oplus L(\gamma)$. Let V be a G-module with $\operatorname{Rad}(V) = L(\alpha)$ and $V/\operatorname{Rad}(V) \cong L(\omega_1) \oplus L(\omega_2)$. Suppose we have two submodules N_{ω_1} and N_{ω_2} of V such that

$$N_{\omega_1}/\operatorname{Rad}(N_{\omega_1}) \cong L(\omega_1)$$

and,

$$N_{\omega_2}/\operatorname{Rad}(N_{\omega_2}) \cong L(\omega_2).$$

Suppose there are non-split short exact sequences,

$$0 \to L(\epsilon) \to \operatorname{Rad}(N_{\omega_1}) \to L(\alpha) \to 0,$$

$$0 \to L(\gamma) \to \operatorname{Rad}(N_{\omega_2}) \to L(\alpha) \to 0.$$

Figure 4.7: *S and V for 4.7.1*

Figure 4.8: $\operatorname{Rad}(N_{\omega_1})$ *and* $\operatorname{Rad}(N_{\omega_2})$ *for 4.7.1*

Figure 4.9: *M and Q_Δ for 4.7.1*

Lemma 4.7.1. *There is a short exact sequence*

$$0 \to S \to N_{\omega_1} \oplus N_{\omega_2} \to V \to 0,$$

where $S, N_{\omega_1}, N_{\omega_2}$, and V are the modules defined above.

Proof. There is a short exact sequence

$$0 \to L(\gamma) \oplus L(\epsilon) \xrightarrow{i} N_{\omega_1} \oplus N_{\omega_2} \xrightarrow{\phi} M \to 0,$$

where M is as shown in figure 4.9. The diagonal map $\Delta : L(\alpha) \to L(\alpha) \oplus L(\alpha)$ can be composed with the embedding $i : L(\alpha) \oplus L(\alpha) \to M$. Denote the composition by $\Delta^* = i \circ \Delta$. There is a short exact sequence,

$$0 \to L(\alpha) \xrightarrow{\Delta^*} M \xrightarrow{f} Q_\Delta \to 0.$$

Where Q_Δ is shown in figure 4.9. We have $Ker(f\phi) \cong S$. □

The above proof becomes more obvious if we think in terms of cases: there are only finitely many modules $Ker(f\phi)$ can actually be.

Example 4.7.1. Assume that λ is in the p^2-alcove. Figure 4.10 shows the radical filtration of the Weyl module at $(\leftarrow, 2, 2)$. Figure 4.11 shows the radical of the Weyl module at $(\leftarrow, 2, 2)$. Note that figure 8 shows submodule structure gained from method 1. Since we have no cancellations coming from a reflection, lemma 4.4.1 implies we will have an embedding,

$$M_\lambda^\alpha = \Delta(\omega_1^4) \hookrightarrow \mathrm{Rad}(\Delta(\lambda)).$$

Consider the quotient

$$N = \mathrm{Rad}(\Delta(\lambda))/\Delta(\omega_1^4)$$

as shown in figure 4.11. If any of the ω_i^2, $i = 2,3,4,5$ drops down to the next radical layer, then 2.1.4(4) implies that the Weyl filtration dimension of the quotient is 2 and so $\operatorname{wfd}(L(\lambda)) = 3$. If none drop down then we can use 4.7.1 to get a short exact sequence

$$S \to \Delta(\omega_2) \oplus \Delta(\omega_3) \to V,$$

where S and V are the modules shown in figure 4.7. The submodule structure of $\Delta(\alpha_2)$ is shown in figure 4.12, here $\gamma = (\downarrow, 1, 1)$ and $\epsilon = (\uparrow, 1, 1)$. The fact that S is a subquotient of $\Delta(\alpha_2)$ enables us to show that $\operatorname{wfd}(S) = 1$. We can now use 2.1.4 to show $\operatorname{wfd}(V) = 2$, so we have

$$V^2 \to \operatorname{Rad}(\Delta(\lambda))/\Delta(\omega_1^4) \to L(\omega_4^2) \oplus L(\omega_5^2),$$

2.1.4 implies

$$\operatorname{wfd}(\operatorname{Rad}(\Delta(\lambda))/\Delta(\omega_1^4)) = 2,$$

therefore $\operatorname{wfd}(L(\lambda)) = 3$.

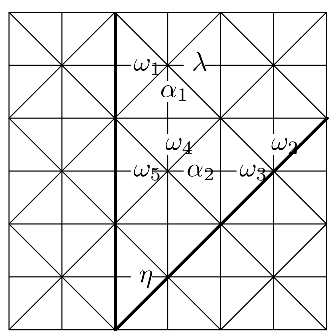


Figure 4.10: $\Delta(\lambda)$ *for example 4.7.1*

$$
\begin{array}{cccc}
\omega_2^2 & \omega_3^2 & \omega_4^2 & \omega_5^2 \\
 & & & \\
\alpha_2^1 & & & \\
 & & & \\
\eta^0 & & &
\end{array}
$$

Figure 4.11: *Submodule structure of N for 4.7.1*

$$
\begin{tikzcd}
\alpha_2^1 \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\omega_5^2 \arrow[d, dash] & \gamma^0 & \eta^0 \\
\epsilon^0 & &
\end{tikzcd}
$$

Figure 4.12: *Submodule structure of $\Delta(\alpha_2)$ for 4.7.1*

4.8 Main result

Let $\lambda \in X^+$ be a p-singular weight in a p-box with lower vertex λ_v. Let n denote the number of walls parallel to α (the short simple root) between zero and λ_v, including the wall containing λ_v. We shall show that for a given p-box, $\mathrm{wfd}(L(\lambda))$ is given by figure 4.24.

The proof of the main result is a proof by induction. The proof will split into two main parts. The first part proves that the conjectured formula (figure11) holds for all weights close to the chamber walls. The Weyl modules with such highest weights have less composition factors than the generic case, via cancellation. The second part proves the formula holds for Weyl modules with no cancellations of any kind, we call such modules *stabilized.*

Remark. The induction for the diagonal hyperplanes in figure 4.1 only depends on weights on diagonal hyperplanes to the left of and below it, so one can do the induction for these facet types separately. Similarly, the weights on horizontal and vertical hyperplanes don't depend on the diagonal ones. The only difference in the proof of each case is that the diagonal case does not need the the diagonal map discussed in 4.7.

4.9 Induction on weights close to the chamber walls

First we must prove that the formula holds for all weights close to the dominant chamber walls. We use two inductions, one along each chamber wall. This is done until the Weyl module structure stabilizes.

Lemma 4.9.1. *Theorem 4.10.1 holds for* $L(\lambda)$ *with* $\lambda = (\backslash, k, 1)$ *in the* p^2*-alcove.*

Proof. Figure 5.12 shows that $\mathrm{wfd}(L(\lambda)) = 0$ when $k = 1$. The Weyl module for $k = 2$ has only one radical layer consisting of the Weyl modules, $\Delta(\omega_1)$ and $\Delta(\omega_1)$, and equation (4.1) implies $\mathrm{wfd}(L(\lambda)) = 1$. Figures 4.13 and 4.14 show the chain of embeddings, the radical structure, and the submodule structure of $\Delta(\lambda)$ with $k = 3$. We have a short exact sequence

$$0 \to M_\lambda^\alpha \to \mathrm{Rad}(\Delta(\lambda)) \to L(\omega_2)^1 \to 0,$$

where $\mathrm{wfd}(M_\lambda^\alpha) = 1$. Lemma 2.1.4(4) implies $\mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))) = 1$, equation (4.1) implies $\mathrm{wfd}(L(\lambda)) = 2$. In general figures 4.15 and 4.16 show the two possible horizontal chains of embeddings and we have a short exact sequence

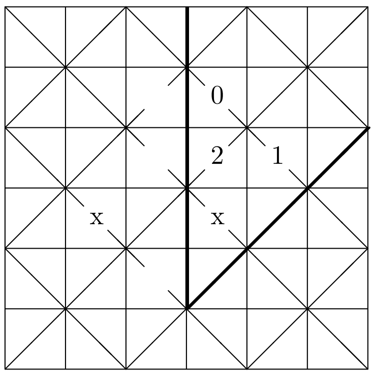
$$(M_\lambda^\alpha)^{k-1} \to \mathrm{Rad}(\Delta(\lambda)) \to L(\omega_2)^{k-1}.$$

Lemma 2.1.4(4) implies $\mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))) = k-1$. Hence equation (4.1) gives $\mathrm{wfd}(L(\lambda)) = k$. □

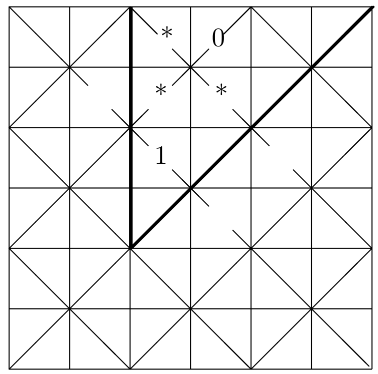
Lemma 4.9.2. *Theorem 4.10.1 holds for* $L(\lambda)$ *with* $\lambda = (\downarrow, 1, k)$ *in the* p^2*-alcove.*

Proof. Induction on k for $(\downarrow, 1, k)$. The formula holds for $k = 1$ and $k = 2$ (example 4.6.1). Figures 4.17 and 4.18 show the Weyl module structure of $\Delta(\lambda)$ and $\Delta(\omega)$ respectively. Figure 4.19 shows the corresponding positions of the weights. By [J, II 6.42] there is a non-zero map $f : \Delta(\omega) \to \Delta(\lambda)$. The fact that $\mathrm{Im}(f)$ is a submodule of $\Delta(\lambda)$ and has $L(\omega)$ as a composition factor we have $\mathrm{Im}(f) = \mathrm{Rad}(\Delta(\lambda)$. There is a short exact sequence,

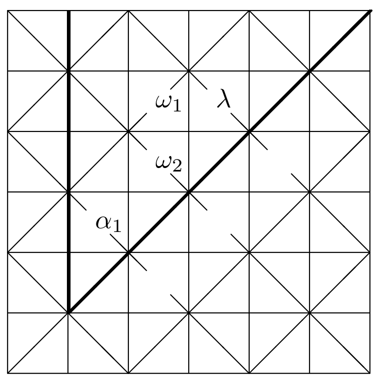
$$0 \to (M_\omega^\beta)^t \to \ker(f) \to L(\alpha_5)^{k-2} \to 0.$$



$(\leftarrow, 1, 2)$ and $n = 2$.



$(/, 2, 1)$ and $n = 1$.



$(\backslash, 3, 1)$ and $n = 2$.

Figure 4.13: *Chain of embeddings and radical structure for $k = 3$ in lemma 4.9*

$$\begin{tikzcd} \lambda \arrow[d, dash] \arrow[dr, dash] & \\ \omega_1^1 \arrow[d, dash] & \omega_2^1 \\ \alpha_1^0 & \end{tikzcd}$$

Figure 4.14: $\Delta(\lambda)$ *for* $(\backslash, 3, 1)$

For large k, $t \leq k-4$, lemma 2.1.4(7) then implies $\text{wfd}(\ker(f)) = k-2$, and so $\text{wfd}(\text{Im}(f)) = \text{wfd}(\text{Rad}(\Delta(\lambda))) = k-1$. Equation (4.1) gives $\text{wfd}(L(\lambda)) = k$. $\square$

The wfd of the simples for the other facet types close to the chamber walls can be computed similarly, with repeated use of (4.1), and 2.1.4 although the method may change. The results are summarized in the table below. The tables in figures 4.20, 4.21 and 4.22 summarize how 4.10.1 is verified for the simple modules with the indicated highest weights. We let $i, j = 0, 1, 2 \cdots$, hcl= horizontal chain length, dcl=diagonal chain length, and dm=diagonal map.

4.10 Induction on stabilized weights

With the formula proved for weights close to the chamber wall, we can proceed with the induction for stabilized weights. First note that for Weyl modules with highest weights far enough away from the chamber walls the structure of M_λ^α, $\alpha \in S$ stabilizes. This structure is independent of where the chain of embeddings began. For example, for a stabilized weight far away from the chamber walls of type $\leftarrow$ or $\rightarrow$, there are two possible chains of embeddings. The first begins with an embedding from $(\leftarrow, 1, j)$ to $(\rightarrow, 2, j)$, the second starts from the embedding from $(\rightarrow, 1, j)$ to $(\leftarrow, 2, j)$. It can be shown that for the purposes of the induction M_λ^α has the same form for both chains. The formula is proved by induction for each facet type: $(\uparrow, i, j), (\downarrow, i, j), (\rightarrow, i, j), (\leftarrow, i, j), (/, i, j)$, and $(\backslash, i, j)$.

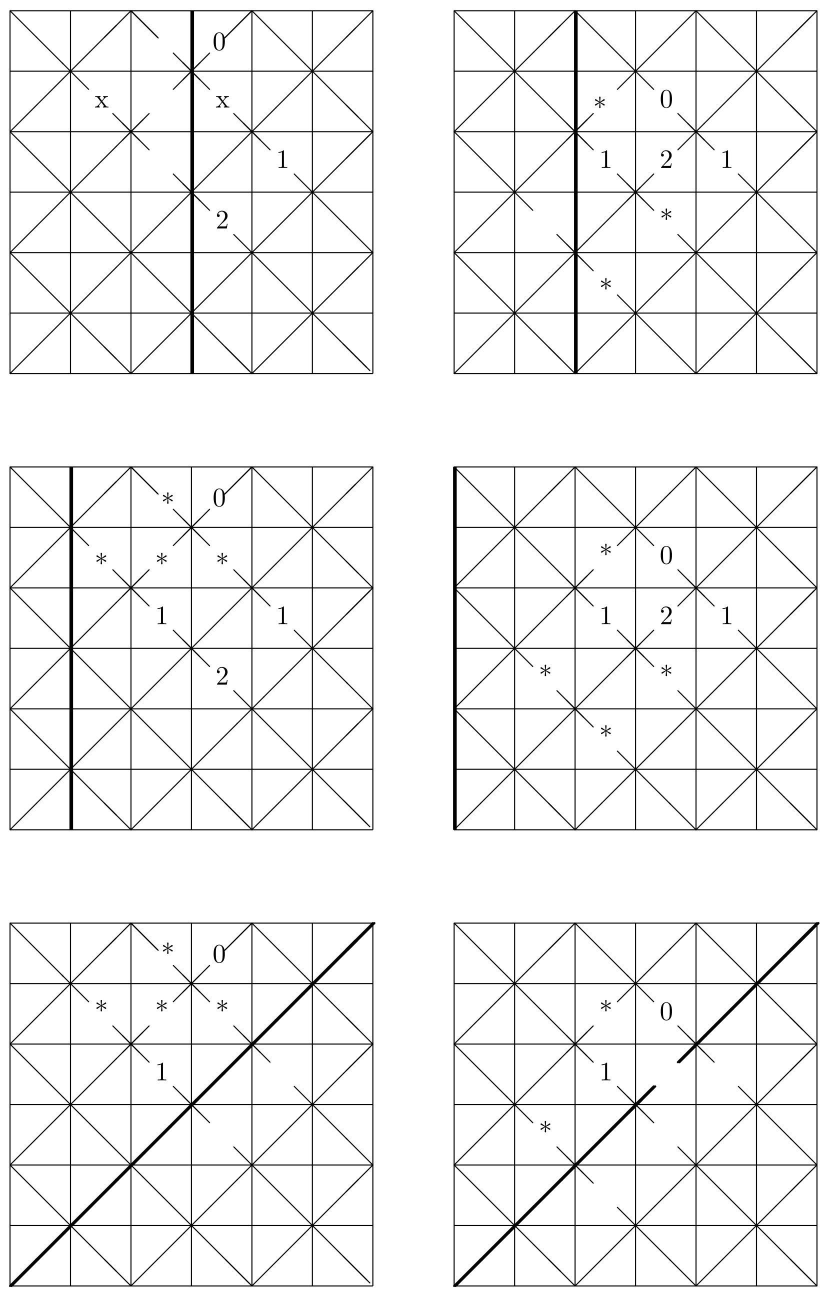


Figure 4.15: *First horizontal chain of embeddings for lemma 4.9*

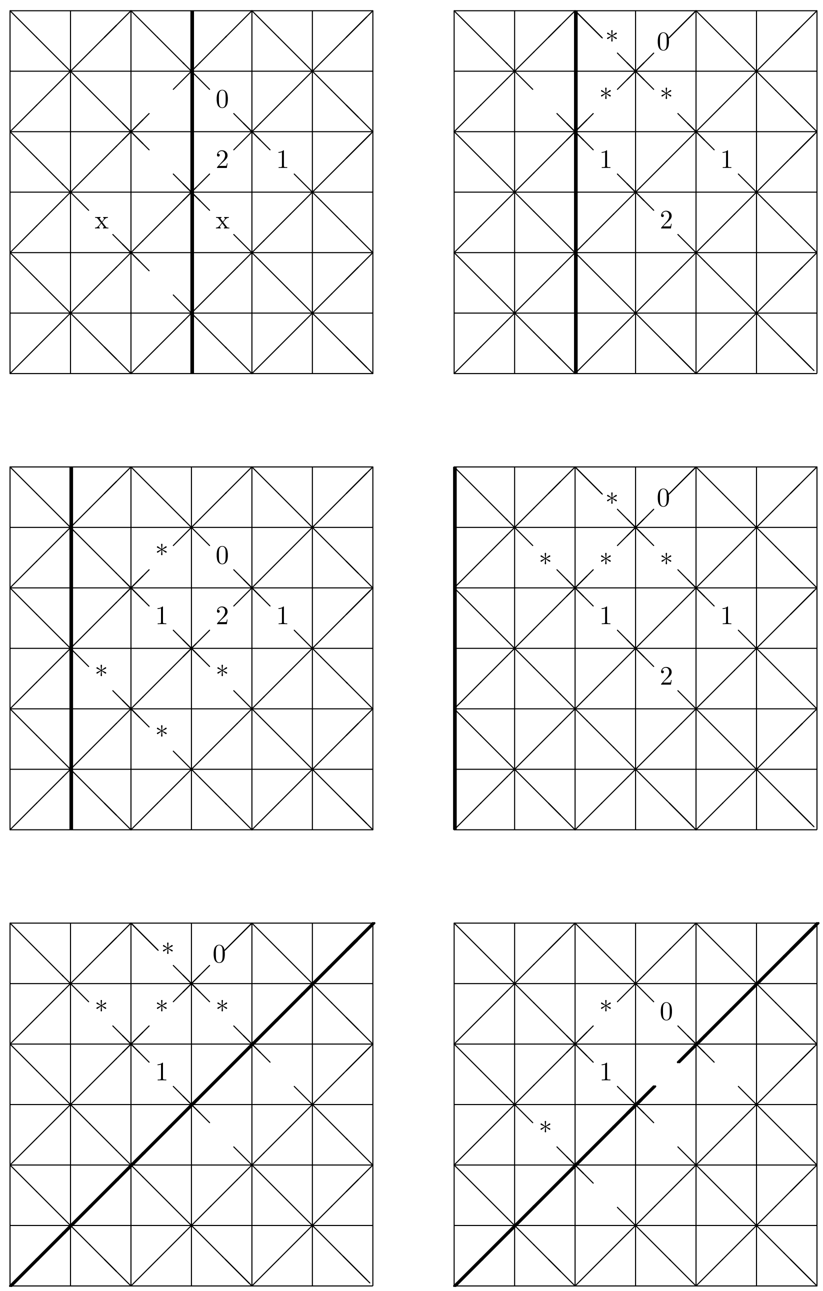


Figure 4.16: *Second horizontal chain of embeddings for lemma 4.9*

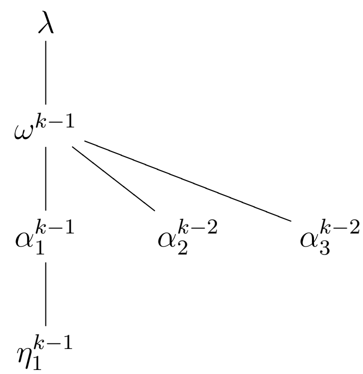


Figure 4.17: $\Delta(\lambda)$ *for* $(\downarrow, 1, j)$

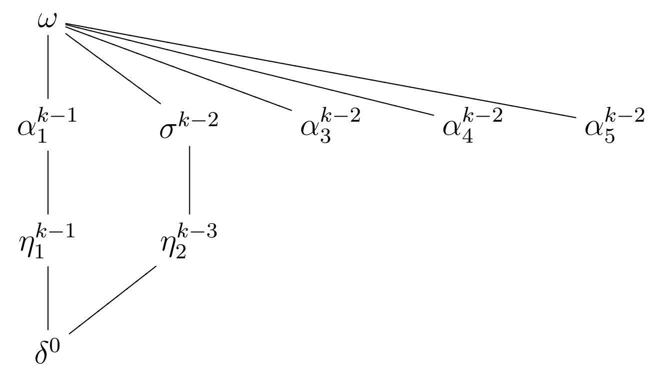


Figure 4.18: $\Delta(\omega)$ *for 4.9.2*

Case 1 : $(\leftarrow, \mathbf{i}, \mathbf{j})$:

The first, most important type we shall deal with is $(\leftarrow, i, j)$, for this weight stabilization occurs at $(\leftarrow, 2, 2)$, proved in 4.7.1. For $(\leftarrow, 3, 2)$, the Weyl module structure is the same as that of $(\leftarrow, 2, 2)$, the same method used for $(\leftarrow, 2, 2)$ works. Along the same horizontal wall the wfd's of the composition factors of a Weyl module will stay the same for same facet type. However, as we move along a horizontal wall from in the negative alpha direction the length of the horizontal chains will decrease. Therefore, wfd(M_λ^α) will decrease. Figure 16 shows the Weyl module structure of $(\leftarrow, i, j)$, we have the following short exact sequence,

$$0 \to V \to \mathrm{Rad}(\Delta(\lambda))/M_\lambda^\alpha \to (L(\omega_4^{k-1}) \oplus L(\omega_5^{k-1}))^{k-1} \to 0.$$

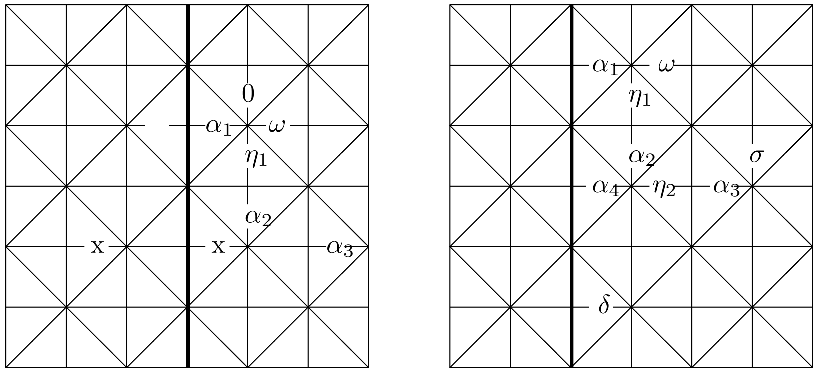


Figure 4.19: *Radical filtration for* $\Delta(\downarrow, 1, k)$ *and* $\Delta(\leftarrow, 2, k-1)$

Lemma 4.7.1 gives,

$$0 \to E^{k-2} \to ((N^{\beta}_{\omega_2})^{k-4} \oplus (N^{\beta}_{\omega_3})^{k-4})^{k-4} \to V \to 0.$$

There exists a short exact sequence

$$0 \to X^{k-3} \to \Delta(\alpha_2) \to E \to 0.$$

2.1.4 implies $\mathrm{wfd}(E) = k-2$, where $k \geq 2$.

Applying 2.1.4, $\mathrm{wfd}(V) = k-1 \implies \mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))/M^{\alpha}_{\lambda}) = k-1$. Observing that $\mathrm{wfd}(M^{\alpha}_{\lambda}) < k-3$, the result follows from 2.1.4.

Case 2 : $(\downarrow, \mathbf{i}, \mathbf{j})$:

There is a short exact sequence

$$0 \to (M^{\beta}_{\lambda})^{t} \to \mathrm{Rad}(\Delta(\lambda)) \to Q \to 0.$$

$\Delta(\leftarrow, i-1, j)$ implies $\mathrm{wfd}(Q) = k-1$. As chain length increases by one, k increases by two so $t \leq k-1$, and the result follows.

Facet type	Method	Induction using
$(\backslash, 1, j)$	2	dcl, $\Delta(\backslash, 2, j-1)$
$(\backslash, 2, j)$	1	dcl
$(\backslash, 3, j)$	1	dcl, $\Delta(\backslash, 3, j-1)$
$(\backslash, i, 1)$	1	hcl
$(\backslash, i, 2)$	1	hcl, $\Delta(\backslash, i, 1)$
$(/, 1, j)$	2	dcl, $\Delta(\backslash, 2, j-1)$
$(/, 2, j)$	1	hcl, $\Delta(/, 2, j-1)$
$(/, 3, j)$	1	hcl, $\Delta(/, 3, j-1)$
$(/, i, 1)$	1	hcl
$(/, i, 2)$	1	hcl, $\Delta(\backslash, i+1, 1)$
$(/, i, 3)$	1	hcl, $\Delta(\backslash, i+1, 2)$

Figure 4.20: *Table 1*

Facet type	Method	Induction using
$(\leftarrow, 1, j)$	1	dcl $\Delta(\downarrow, 2, j-1)$
$(\leftarrow, 2, j)$	1	dm $\Delta(\leftarrow, 2, j-1)$
$(\downarrow, 1, j)$	1,2	hcl, $\Delta(\leftarrow, 2, j-1)$
$(\downarrow, 2, j)$	1	hcl
$(\downarrow, 3, j)$	1	dcl $\Delta(\leftarrow, 2, j)$
$(\rightarrow, 1, j)$	2	dcl, $\Delta(\backslash, 2, j-1)$
$(\rightarrow, 2, j)$	1	hcl
$(\rightarrow, 3, j)$	1	hcl
$(\uparrow, 1, j)$	2	dcl, $\Delta(\leftarrow, 2, j-1)$
$(\uparrow, 2, j)$	1	hcl
$(\uparrow, 3, j)$	3	dcl

Figure 4.21: *Table 2*

Facet type	Method	Induction using
$(\leftarrow, i, 1)$	1	hcl
$(\leftarrow, i, 2)$	1,3	dm, $\Delta(\downarrow, i, 1)$
$(\downarrow, i, 1)$	2	$\Delta(\leftarrow, i-1, 1)$
$(\downarrow, i, 2)$	1	dcl,$\Delta(\leftarrow, i-1, 2)$
$(\downarrow, i, 3)$	1	dcl,$\Delta(\leftarrow, i-1, 3)$
$(\rightarrow, i, 1)$	2	$\Delta(\leftarrow, i-1, 1)$
$(\rightarrow, i, 2)$	1	hcl
$(\rightarrow, i, 3)$	1	hcl
$(\uparrow, i, 1)$	3	dm, $\Delta(\leftarrow, i-1, 1)$
$(\uparrow, i, 2)$	1,3	dm, $\Delta(\leftarrow, i-1, 2)$

Figure 4.22: *Table 3*

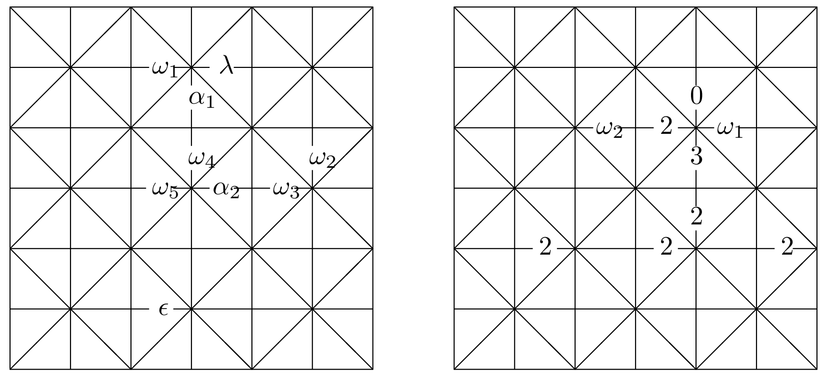


Figure 4.23: *Radical filtrations for* $\Delta(\leftarrow, i, j)$ *and* $\Delta(\downarrow, i, j)$

Case 3 : $(\rightarrow, \mathbf{i}, \mathbf{j})$:

Using the short exact sequences,

$$0 \rightarrow (M^{\alpha}_{\omega_1})^t \rightarrow \mathrm{Rad}(\Delta(\lambda)) \rightarrow Q \rightarrow 0$$

$$0 \rightarrow (L(\alpha_5))^{k-4} \rightarrow Q \rightarrow (L(\omega_2))^{k-1} \rightarrow 0.$$

Since $t < k-2$ it follows that $\mathrm{wfd}(Q) = k-1 = \mathrm{wfd}(\mathrm{Rad}(\Delta(\lambda))$.

Case 4 : $(\uparrow, \mathbf{i}, \mathbf{j})$:

$$0 \rightarrow (M^{\beta}_{\lambda})^t \rightarrow \mathrm{Rad}(\Delta(\lambda)) \rightarrow Q \rightarrow 0$$

This one is similar to $(\leftarrow, i, j)$. $\Delta(\leftarrow, i-1, j)$ and Lemma 4.7.1 gives the result.

Case 5 : $(\backslash, \mathbf{i}, \mathbf{j})$:

Consider the exact sequence $0 \rightarrow (M^{\alpha}_{\lambda})^t \rightarrow \mathrm{Rad}(\Delta(\lambda)) \rightarrow Q \rightarrow 0$. If there is any non split extension between simples in consecutive radical layers of Q $\mathrm{wfd}(Q) = k-1$. Since $t < k-2$ we are done. So assume there are no such non-split extensions. Consider $\Delta(\backslash, i+1, j-1)$.

We can use M_α^β and $\Delta(\alpha = (\backslash, i, j-1))$ to show $\mathrm{wfd}(N) = k-1$, where N is defined by

$$0 \to N \to Q \to L(\omega_3^{k-1}) \to 0$$

so 2.1.4 gives $\mathrm{wfd}(Q) = k-1$ and we are done.

Case 6 : $(/, \mathbf{i}, \mathbf{j})$:
Similar to $(\backslash, i, j)$.

Theorem 4.10.1. *Let λ be a dominant, p-singular weight inside the p^2-alcove with unique decomposition $\lambda_0 + p\lambda_1$, where $\lambda_0 \in X_1$, and $\lambda_1 \in X$. If $n = \langle \lambda_1, \alpha^\vee \rangle + 2\langle \lambda_1, \beta^\vee \rangle$, $\lambda_1 \neq 0$, then $\mathrm{wfd}(L(\lambda))$ is given by figure 4.24.*

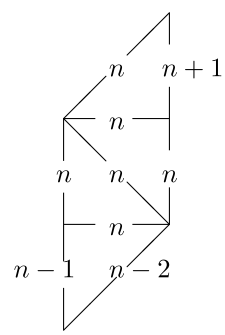


Figure 4.24: *Induction hypothesis for* $\mathrm{wfd}(L(\lambda))$ *in a* p*-box.*

4.11 Weyl filtration dimension of the induced modules for B_2

For all but the two facet types, $/$, and $\leftarrow$, we have $\mathrm{wfd}(L(\lambda)) \geq \mathrm{wfd}(L(\mu))$ for all composition factors $L(\mu)$ of $\nabla(\lambda)$, so 2.1.4(4) can be used to show that $\mathrm{wfd}(\nabla(\lambda)) = \mathrm{wfd}(L(\lambda))$ for most weight types inside p^2-alcove. First we deal with type $(\leftarrow, i, j)$. The following lemma is from [J, II 7.12.].

Lemma 4.11.1. *Let $\lambda \in X(T) \cap C$ and $\mu \in \bar{C}_{\mathbb{Z}}$. Suppose there is $s \in \Sigma$ with $\mathrm{stab}(\mu) = \{1, s\}$. Let $w \in W_p$ with $w \cdot \lambda \in X(T)_+$ and $w \cdot \lambda < ws \cdot \lambda$. Then there is a short exact sequence of G-modules*

$$0 \to \nabla(w \cdot \lambda) \to T_\mu^\lambda \nabla(w \cdot \mu) \to \nabla(ws \cdot \lambda) \to 0.$$

Theorem 4.11.2. *For weights of type $\nu = (\leftarrow, i, j)$ inside the p^2-alcove, we have*

$$\mathrm{wfd}(L(\nu)) = \mathrm{wfd}(\nabla(\nu)).$$

Proof. Let λ and $s \cdot \lambda$ be separated by a facet of type $\downarrow$, and ν be the dominant weight of type $(\leftarrow, i, j)$ in the upper closure of the alcove containing λ, as shown in figure 4.25. Translate the short exact sequence in the above lemma onto the horizontal wall above λ to obtain the short exact sequence,

$$0 \to \nabla(\nu) \to T_\lambda^\nu T_\mu^\lambda \nabla(\mu) \to \nabla(s \cdot \nu) \to 0.$$

Since $s \cdot \nu$ is of facet type $(\to, i, j)$ we have

$$\mathrm{wfd}(\nabla(s \cdot \nu)) = \mathrm{wfd}(L(s \cdot \nu)) = n,$$

where $n \in \mathbf{Z}^+$, and

$$\mathrm{wfd}(\nabla(\mu)) = \mathrm{wfd}(L(\mu)) = n - 1.$$

Since the translation functor is exact, so we have

$$\mathrm{wfd}(T_\lambda^\nu T_\mu^\lambda \nabla(\mu)) \leq \mathrm{wfd}(\nabla(\mu)) = n-1.$$

Hence, by 2.1.4(8),

$$\mathrm{wfd}(\nabla(\nu)) = n-1 = \mathrm{wfd}(L(\nu)).$$

$\square$

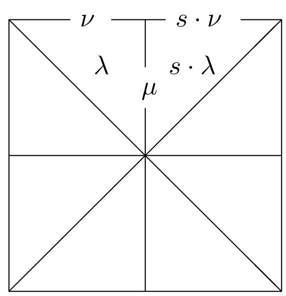


Figure 4.25: *Weights for theorem 4.11.2*

We wish to show that the equality $\mathrm{wfd}(L(\lambda)) = \mathrm{wfd}(\nabla(\lambda))$ holds for λ of type $(/, i, j)$. Unfortunately, we can only prove the equality for type $(/, 1, j)$ and type $(/, 2, 1)$ in the p^2-alcove. However, tilting module structures can be used to show that $\mathrm{wfd}(\nabla(\lambda)) \leq \mathrm{wfd}(L(\lambda))$ for λ of type $(/, 2, j)$ inside the p^2-alcove.

Firstly, for weights of type $(/, 1, j)$ inside the p^2-alcove we can use the fact that $\mathrm{wfd}(L(\lambda)) \geq \mathrm{wfd}(L(\mu))$ for all composition factors $L(\mu)$, and lemma 2.1.4 to prove the equality.

For type $(/, 2, j)$ we study the structures of translations of induced modules with specific singular highest weights. We shall also utilize the indecomposable tilting modules. In the late 90's Stroppel [St] computed the Weyl filtrations of the tilting modules in the B_2 case for the quantum group, which is parallel to the modular case inside the p^2-alcove.

We shall need the following lemma from [J, II 7.13].

Lemma 4.11.3. *Let $\lambda, \mu \in \bar{C}$ and $w \in W_p$ and $w \cdot \lambda \in X_+$. Then $T_\lambda^\mu \nabla(w \cdot \lambda)$ has a good filtration with factors $\nabla(ww_1 \cdot \mu)$, with $w_1 \in$ $Stab_{W_p}(\lambda)$, and*
$[T_\lambda^\mu \nabla(w \cdot \lambda) : \nabla(ww_1 \cdot \mu)] = 1$.

Consider the figures 5.26, 5.27 and 5.28 from chapter 5. The dots in these pictures represent the Weyl modules with highest weights in those positions in the Weyl/good filtration of the tilting module, with the obvious highest weight. By translating these pictures we can get the tilting modules with singular highest weights by removing the appropriate summands. Figures 4.27, 4.28, and 4.30 show the Weyl modules in the top part of the Weyl filtration of certain tilting modules. Note that there are more Weyl modules in these tilting filtration diagrams, but only those of concern are shown.

Remark 4.11.4. Any translate of an indecomposable tilting module with p-singular highest weight to a p-regular highest weight must also be an indecomposable tilting module. To see this we note that the translation off the wall must be tilting. If it is a direct sum of tilting modules then we can remove those summands with the lowest highest weights. Now we can translate back to the wall and we should get two copies of the original tilting module. Clearly, if we remove any tilting summands we will not get the two copies of the original module. Therefore, no summands could have been removed, so any translate of an indecomposable tilting module is also an indecomposable tilting module.

We will also need the following.

Lemma 4.11.5. *Let $T(\lambda)$ be the indecomposable tilting module of highest weight λ, with λ of type $(\backslash, 2, j)$ inside the p^2-alcove. If $\mu = (/, 1, j)$, then $\nabla(\mu)$ is not a quotient of $T(\lambda)$.*

Proof. Consider the radical filtration figures 4.1 for $\Delta(\lambda)$ and the translations (to the p-singular case) of the Weyl/good filtration figures 5.28 of tilting modules. Notice that $L(\mu)$

occurs as a composition factor in the second radical layer of $\Delta(\lambda)$, and $\Delta(\mu)$ is in the 'second layer' of the Weyl filtration of $T(\lambda)$. Now, $L(\mu)$ must occur as a composition factor in the radical layers above and below the radical layer containing λ in $T(\lambda)$. By the condition on the highest weights of the induced modules, the copy of $L(\mu)$ in the radical layer below λ must be the simple module in the induced module $\nabla(\mu)$ in the good filtration of $T(\lambda)$. Since this $L(\mu)$ has a non split extension with λ (by the self duality of $T(\lambda)$) we can not have $\nabla(\mu)$ being a quotient of $T(\lambda)$ as required. Note, translation of figure 5.28 (again to the p-singular weights) implies that this is the only copy of $\nabla(\mu)$ in the good filtration of $T(\lambda)$. □

Now we can use this lemma to prove the following:

Lemma 4.11.6. *Let μ be a p-singular weight of type $(\rightarrow, 1, j)$ or $(\leftarrow, 2, j)$ inside the p^2-alcove, and let $T_\mu^\lambda \nabla(\mu)$ be the translate off the wall containing μ. Denote the p-regular weights by λ and $s \cdot \lambda$ where $s \cdot \lambda < \lambda$. If we then translate using $T_\lambda^{\hat{\lambda}}$ where $\hat{\lambda}$ is in the upper closure of the alcove containing λ, and $\widehat{s \cdot \lambda}$ is in the lower closure of the alcove containing $s \cdot \lambda$, then $T_\lambda^{\hat{\lambda}} T_\mu^\lambda \nabla(\mu) = \nabla(\hat{\lambda}) \oplus \nabla(\widehat{s \cdot \lambda})$.*

Proof. Starting with 4.11.3 applied to $T(/, 2, j)$, inside the p^2-alcove, we see that the top part of the Weyl filtration of $T(/, 2, j)$ is as shown in figure 4.27. Consider the indecomposable tilting module with highest weight at $\epsilon = (\uparrow, 1, j)$. Now translate $T(\epsilon)$ off the wall and onto the upper closures as shown in figure 4.30. Note that the resulting translate is not a direct sum $T(/, 1, j) \oplus T(\backslash, 2, j)$ using figure 4.27. Clearly, if we remove the Weyl module with highest weight $(/, 1, j)$ from figure 4.30 we will not have sufficient Weyl modules for $T(/, 2, j)$ when we translate. Now, starting with $T(\rightarrow, 1, j)$ the top part of which is shown in figure 4.28 we translate off the wall and then onto the wall containing $T(\backslash, 1, j)$ we will get two Weyl modules with highest weight $(/, 1, j)$. We must remove one since the indecomposable tilting module only has one Weyl module in the filtration at this position by translation of figure 5.28. We cannot remove the factor coming from the translate of

$\Delta(\uparrow, 1, j)$ by lemma 4.11.5, therefore, we see that the lemma is satisfied for type $(\rightarrow, 1, j)$. To prove the theorem for type $(\rightarrow, 1, j)$ we begin with $T(\backslash, 2, j)$ and translate as shown in figure 4.30, the numbers indicate the number of copies of a given Wely module. Now we translate again as shown in figure 4.31. The last translate in figure 4.31 is a direct sum of tilting modules, one of which is $T(\hat{\lambda})$ where $\hat{\lambda} = (/, 2, j)$. So, we see that we can remove two copies of $T(\widehat{s \cdot \lambda})$ from the final translation. Considering $T(\backslash, 1, j)$ we see that we must have $T_{\lambda}^{\hat{\lambda}} T_{\mu}^{\lambda} \Delta(\mu) \cong \Delta(\hat{\lambda}) \oplus \Delta(\widehat{s \cdot \lambda})$, proving the last part. $\square$

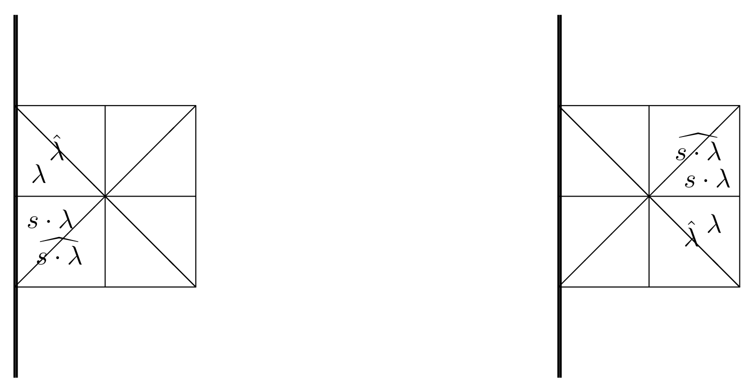


Figure 4.26: *Labeling for weights for lemma 4.11.6 and lemma 4.11.7.*

Lemma 4.11.7. *If* $\lambda = (/, 2, j)$ *is inside the* p^2*-alcove, then* $\operatorname{wfd}(\nabla(\lambda)) \leq \operatorname{wfd}(L(\lambda))$.

Proof. Consider figure 4.26. We have a short exact sequence

$$0 \rightarrow N \rightarrow T(\mu) \rightarrow \nabla(\mu) \rightarrow 0,$$

where $\mu = (\leftarrow, 2, j)$. By 4.11.6 we can translate the above sequence to get a short exact sequence

$$0 \to N' \to \oplus T_i \to \nabla(\lambda) \oplus \nabla(\nu) \to 0$$

where $\nu = (\backslash, 2, j)$. Now,

$$\mathrm{wfd}(N') \leq \mathrm{wfd}(N)$$

and Lemma 2.1.4 implies,

$$\mathrm{wfd}(\nabla(\lambda)) \leq \mathrm{wfd}(\nabla(\mu)) = \mathrm{wfd}(L(\lambda)).$$

□

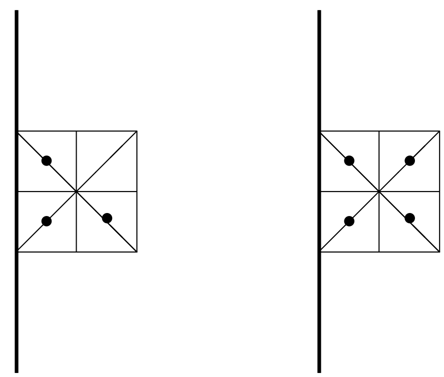


Figure 4.27: *Top part of Weyl filtration of $T(\backslash, 1, j+1)$ and $T(/, 2, j)$.*

Example 4.11.1. $\mathrm{wfd}(\nabla(\lambda)) = \mathrm{wfd}(L(\lambda))$ for $\lambda = (/, 2, 1)$. Consider the indecomposable tilting module $T(\lambda)$, in figure 4.32. We have a short exact sequence

$$0 \to N \to T(\mu) \to \nabla(\mu) \to 0$$

where $\mu = (\leftarrow, 2, 1)$. By 4.11.6 we can translate the above sequence to get a short exact sequence

$$0 \to N' \to \oplus T_i \to \nabla(\lambda) \oplus \nabla(\nu) \to 0$$

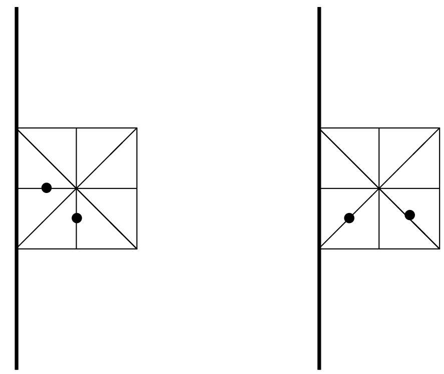


Figure 4.28: *Top part of Weyl filtration of $T(\rightarrow, 1, j)$ and $T(\backslash, 2, j)$.*

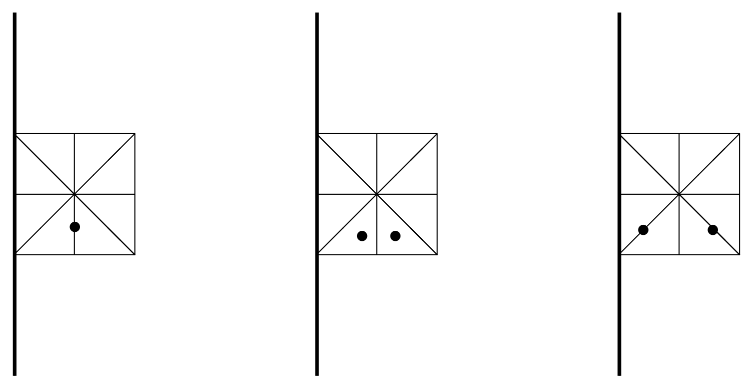


Figure 4.29: *Translation off the wall of top Weyl module of Weyl filtration of $T(\uparrow, 1, j)$.*

where $\nu = (\backslash, 2, 1)$. By 4.11.2 $\mathrm{wfd}(\nabla(\mu)) = 1$ and so lemma 2.1.4 implies $\mathrm{wfd}(N) = 0$. We then have $\mathrm{wfd}(N') \leq \mathrm{wfd}(N) = 0$, lemma 2.1.4 implies $\mathrm{wfd}(\nabla(\lambda)) \leq 1$. Since $\nabla(\lambda)$ is not a tilting module we have $\mathrm{wfd}(\nabla(\lambda)) = 1 = \mathrm{wfd}(L(\lambda))$ By 4.11.3 the indecomposable tilting module at $(/, 2, 1)$ is as shown in 4.32. We have

$$0 \rightarrow T(\backslash, 1, 2) \rightarrow T(\lambda)) \rightarrow \nabla(\lambda) \rightarrow 0.$$

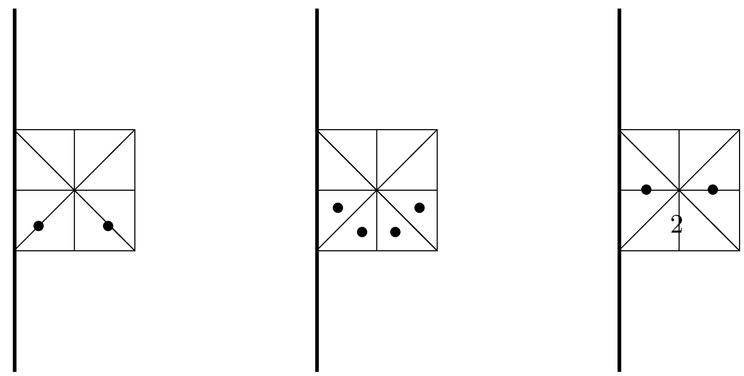


Figure 4.30: *Translation of* $T(\backslash, 2, j)$.

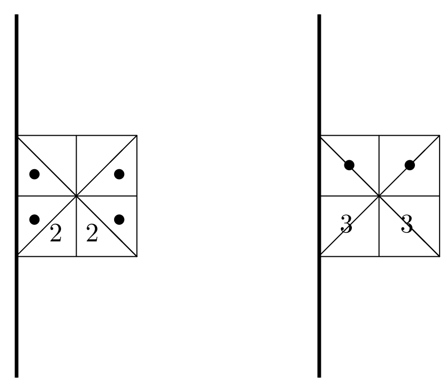


Figure 4.31: *Translation of figure 4.30.*

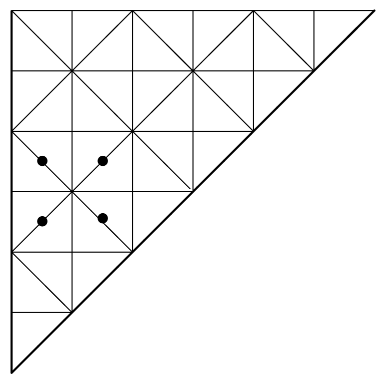


Figure 4.32: *Weyl filtration of* $T(/, 2, 1)$

Chapter 5

Submodule structures for B_2

In this chapter the submodule structures of the Weyl modules, Vogan modules, tilting modules, and translates of induced modules are studied. This is done in the p^2-alcove for singular and regular weights close to the chamber walls. The techniques by no means give all structures of all modules. I give an example of every technique I have found useful. Tilting modules begin to get complicated when we move away from one of the chamber walls but stay close the other.

5.1 Weyl modules

5.1.1 p-regular weights

In this section we compute some of the submodule structures of the Weyl modules and Vogan modules. We shall use the ordering on weights given in figure 5.1. It is known that for weights far enough away from the dominant chamber walls the induced modules will have simple socle and head. However, the following example illustrates the known fact that this no longer holds for weights close to the chamber walls.

Example 5.1.2. The first Weyl module we will consider is $\Delta(\lambda_{41})$. The radical structure has three layers and the whole structure is shown in 5.7 and is found by applying two wall crossing functors to the two short exact sequences,

$$0 \to L(\lambda_2) \to \Delta(\lambda_{31}) \to L(\lambda_{31}) \to 0,$$

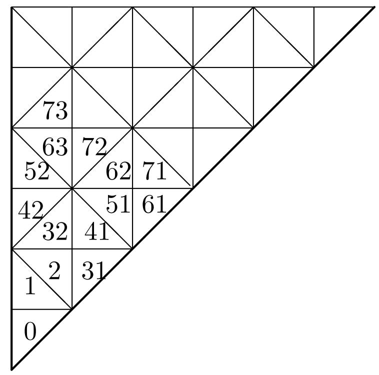


Figure 5.1: *Ordering on the regular weights for B_2.*

$$0 \to L(\lambda_2) \to \Delta(\lambda_{32}) \to L(\lambda_{32}) \to 0.$$

Applying wall crossing functors we get:

$$0 \to T_{\mu}^{\lambda_{41}} T_{\lambda_{31}}^{\mu} L(\lambda_2) \to T_{\mu}^{\lambda_{41}} T_{\lambda_{31}}^{\mu} \Delta(\lambda_{31}) \to T_{\mu}^{\lambda_{41}} T_{\lambda_{31}}^{\mu} L(\lambda_{31}) \to 0,$$

$$0 \to T_{\nu}^{\lambda_{41}} T_{\lambda_{32}}^{\nu} L(\lambda_2) \to T_{\nu}^{\lambda_{41}} T_{\lambda_{32}}^{\nu} \Delta(\lambda_{32}) \to T_{\nu}^{\lambda_{41}} T_{\lambda_{31}}^{\nu} L(\lambda_{32}) \to 0.$$

Where $\mu = (\rightarrow, 2, 1)$ and $\nu = (\uparrow, 1, 1)$. This tells us that we have exact sequences

$$0 \to U(\mu) \hookrightarrow T_{\mu}^{\lambda_{41}} T_{\lambda_{31}}^{\mu} \Delta(\lambda_{31}) \to U(\gamma)$$

$$0 \to U(\nu) \hookrightarrow T_{\nu}^{\lambda_{41}} T_{\lambda_{32}}^{\nu} \Delta(\lambda_{32}) \to U(\epsilon)$$

$\gamma = (\leftarrow, 1, 1)$ and $\epsilon = (\downarrow, 1, 1)$ In this case $U(\gamma)$ and $U(\epsilon)$ are tilting modules.

The submodule structures of the Vogan modules and some of the submodule structure of the translates of induced/Weyl modules with singular highest weights in the p^2-alcove can be computed using the technique in the above example.

5.1.3 p-singular weights

Example 5.1.4. The first Weyl module we will consider whose submodule structure can not be found by the chain of embeddings has highest weight at $\lambda = (\leftarrow, 2, 1)$. We begin with

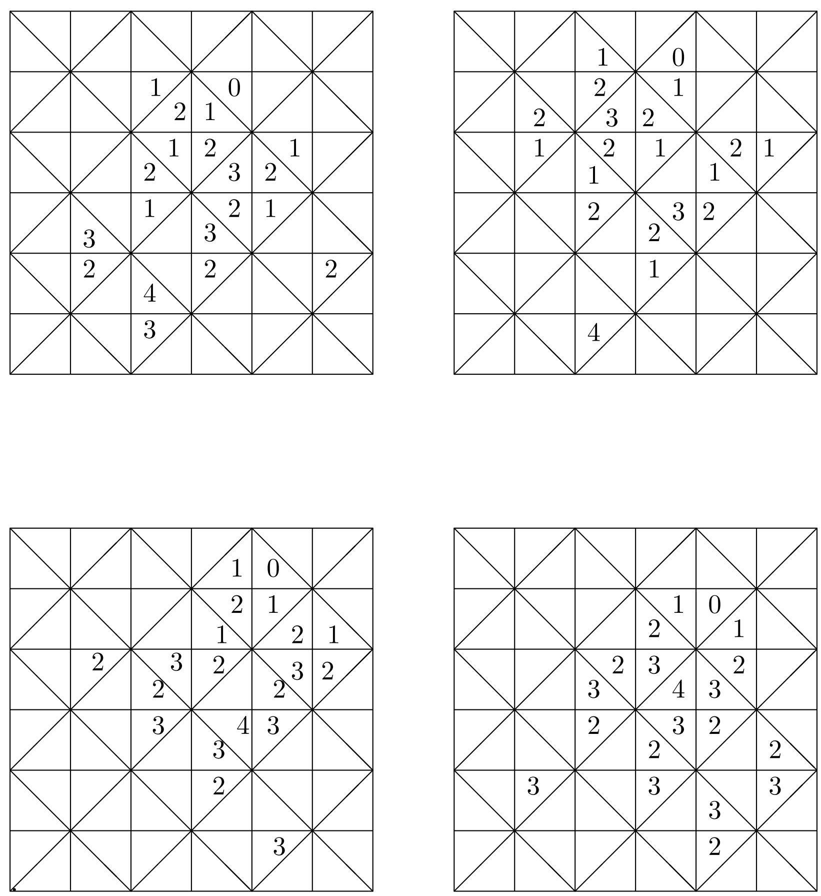


Figure 5.2: *Generic regular B_2 composition factors for $\nabla(\lambda)$.*

Figure 5.3: *$T_{\mu}^{\lambda_{31}}\Delta(\mu)$ and $T_{\nu}^{\lambda_{32}}\Delta(\nu)$ for example 5.1.2.*

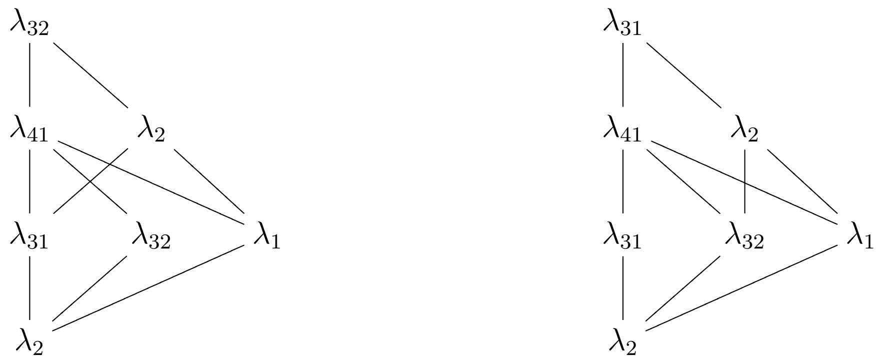


Figure 5.4: *Submodule structure of $T_{\mu}^{\lambda_{31}}\Delta(\mu)$ and $T_{\nu}^{\lambda_{32}}\Delta(\nu)$ for example 5.1.2.*

$\Delta(\mu)$ where $\mu = (\backslash, 2, 1)$. This Weyl module has semisimple radical $L(\mu_1) \oplus L(\mu_2)$, where $\mu_1 = (/, 1, 1)$ and $\mu_2 = (\backslash, 1, 1)$. We begin with the short exact sequence,

$$0 \to L(\mu_1) \oplus L(\mu_2) \to \Delta(\mu) \to L(\mu) \to 0.$$

Now, translate off the wall to get

$$0 \to U(\mu_1) \oplus U(\mu_2) \to T_{\mu}^{\lambda_{41}}\Delta(\mu) \to U(\mu) \to 0$$

Both $U(\mu_1)$ and $U(\mu_2)$ are indecomposable tilting modules and there structures are shown in figure 5.8. The structure of $T_{\mu}^{\lambda_{41}}\Delta\mu)$ due to the Vogan modules is as shown in figure 5.9. By identifying the images of the composition factors in $T_{\mu}^{\lambda_{41}}\Delta(\mu)$ with the composition factors in $T_{\lambda_{51}}^{\lambda}T_{\mu}^{\lambda_{41}}\Delta(\mu)$ we see that

$$T_{\lambda_{51}}^{\lambda}U(\mu_2) \cong L(\downarrow, 1, 1) \oplus L(\to, 1, 1) \hookrightarrow \Delta(\lambda).$$

Hence the structure is as shown in 5.14.

$$
\begin{tikzcd}[column sep=small]
& \lambda_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\lambda_{31} \arrow[dr, dash] & & \lambda_1 \arrow[dl, dash] \\
& \lambda_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
& \lambda_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\lambda_{32} \arrow[dr, dash] & & \lambda_1 \arrow[dl, dash] \\
& \lambda_2 &
\end{tikzcd}
$$

Figure 5.5: *$U(\downarrow, 1, 1)$ and $U(\leftarrow, 1, 1)$ for example 5.1.2.*

$$
\begin{tikzcd}
\lambda_{31} \arrow[d, dash] \\
\lambda_{41} \arrow[d, dash] \\
\lambda_{31}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\lambda_{32} \arrow[d, dash] \\
\lambda_{41} \arrow[d, dash] \\
\lambda_{32}
\end{tikzcd}
$$

Figure 5.6: *$U(\nu)$ and $U(\mu)$ for example 5.1.2.*

The above technique along with the chain of embeddings fails to give all the structures of all Weyl modules with singular highest weights inside the p^2-alcove. For example for the structure of $\Delta(\lambda)$ when $\lambda = (\uparrow, 1, 2)$. The known radical structure is as shown in fig 5.25. In the next section we will see how tilting modules can be used to find some of the submodule structure of such Weyl modules.

Now we give an example of a generic Weyl module whose structure can be completely determined by the chain of embeddings.

Example 5.1.5. When the highest weight of a Weyl module is "generic", meaning far enough away from the chamber walls, the Weyl module has simple head and simple socle. Let λ be generic of type $(\rightarrow, i, j)$. Note for the generic case the final embeddings are independent of i and j. By lemma 5.21 there are no extensions between $L(\omega_1)$ and $L(\alpha_5)$ and $L(\omega_2)$ and $L(\alpha_3)$. The total structure is as shown in 5.15.

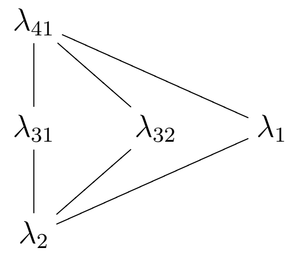


Figure 5.7: $\Delta(\lambda_{41})$ *for example 5.1.2.*

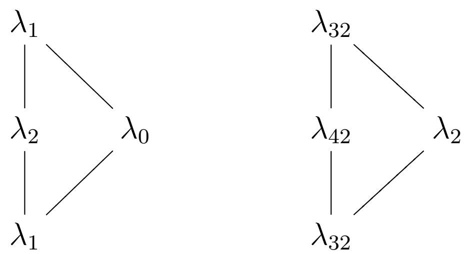


Figure 5.8: $U(\mu_2)$ *and* $U(\mu_1)$*.*

5.2 Indecomposable tilting modules

Finally, we consider some techniques to find the submodule structures of some of the simple indecomposable tilting modules with highest weight inside the p^2-alcove for B_2.

5.2.1 p-regular weights

For the regular weights the Vogan modules tell us a lot about the structure.

Example 5.2.2. The first indecomposable tilting module with regular highest weight that is not a Vogan module has highest weight λ_{41}. The structure is comprised of Vogan modules see 5.16

5.2.3 p-singular weights

Here we show that verification of the identity $\mathrm{wfd}(L(\lambda)) = \mathrm{wfd}(\nabla(\lambda))$ for $\lambda = (/, 2, 1)$ allows us to deduce some of the structure of $T(\lambda)$. For this same example one can use the Weyl filtration dimension to compute the submodule structure for $\nabla(\lambda)$.

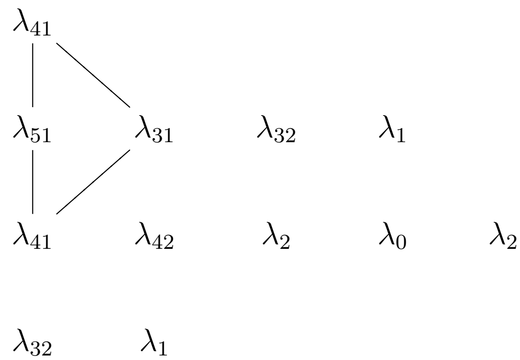


Figure 5.9: $T_{\mu}^{\lambda_{41}}\Delta(\mu)$.

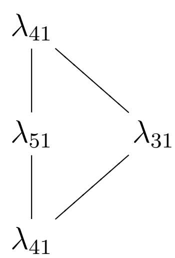


Figure 5.10: $U(\mu)$.

Example 5.2.4. Let $\mu_1 = (\backslash, 1, 2)$ and $\lambda = (/, 2, 1)$. Consider example 4.11.1. Consider figure 5.18 the structure illustrated in this figure is figured out using the Weyl filtration shown in figure 4.32 and the self duality of tilting modules. Note that there can be no extension between $L(\epsilon)$ and $L(\lambda)$ by lemma 1.4.1. However, there can be an extension between $L(\omega)$ and $L(\mu_2)$. From example 4.11.1 we know that $\mathrm{wfd}(\nabla(\lambda)) = 1$ and so we have a short exact sequence,

$$0 \to N^0 \to T(\lambda) \to \nabla(\lambda)^1 \to 0.$$

For some G-module N. It follows from 2.1.4(8) that $\mathrm{wfd}(N) = 0$, so N has Weyl filtration. Considering all possible Weyl filtrations we see that $\Delta(\omega)$, shown in figure 5.21 must be a factor and so there is in fact a non simple extension between $L(\omega)$ and $L(\mu_2)$. Therefore, $N^0 \cong T(\mu_1)$ and we have a short exact sequence

$$0 \to N^0 \to T(\lambda) \to \nabla(\lambda)^1 \to 0.$$

$$
\begin{tikzcd}
\Delta(\uparrow,1,1) \arrow[d, dash, dashed] \\
\Delta(\lambda)
\end{tikzcd}
$$

Figure 5.11: $T^{\lambda}_{\lambda_{51}} T^{\lambda_{41}}_{\mu} \Delta(\mu)$.

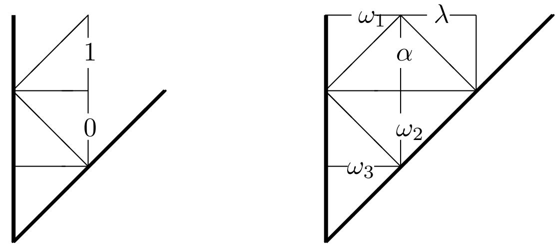


Figure 5.12: $\Delta(\leftarrow,2,1)$ *and* $\Delta(\uparrow,1,1)$ *for example 5.1.4.*

The total structure of $\nabla(\lambda)$ and $T(\lambda)$ are given by figures 5.19 and 5.24.

Example 5.2.5. For $\Delta(\lambda)$, where $\lambda = (\uparrow, 1, 2)$ the radical structure is as shown in figure 5.25. The chain of embeddings and Vogan modules give us no more information. The Weyl factors in the Weyl filtration of $T(\lambda)$ are as shown in figure 5.26. The Tilting module structure is shown in figure 5.24. Self duality means that there is a non simple extension between $L(\omega)$ and $L(\alpha_3)$. Lemma 1.4.1 means there can be non simple extensions between $L(\omega)$ and $L(\alpha_2)$ and $L(\omega)$ and $L(\alpha_1)$. So the structure is as shown in figure 5.25 and some structure still remains to be found.

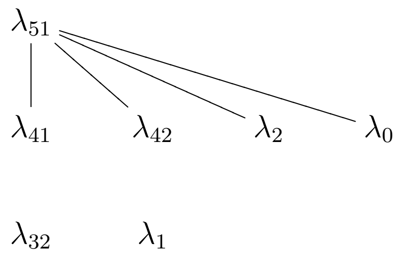


Figure 5.13: $\Delta(\lambda)$ *for 5.1.4.*

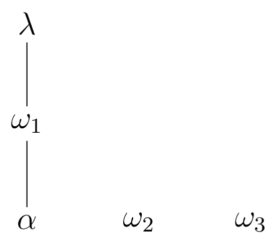


Figure 5.14: $\Delta(\lambda)$ *for 5.1.4.*

5.3 Closing remarks

The methods used throughout this thesis are very ad hoc and the computations get complicated very quickly, even in the p^2-alcove. The author has not pushed these techniques to their limits to try to find all of the structures of the Weyl modules. The Weyl filtration dimension can be seen to be a useful tool in understanding submodule structures, especially close to the fundamental alcove. For weights far away from the walls the Weyl modules will have simple head and simple socle but still it seems that all submodule structures can not be completely found in the B_2 case with the techniques of this thesis. The main question though has to be: what are the radical filtrations outside the fundamental alcove for B_2.

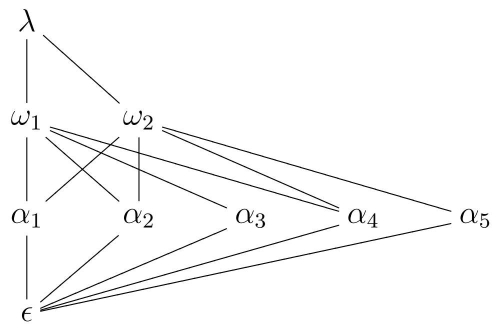


Figure 5.15: $\Delta(\rightarrow, i, j)$.

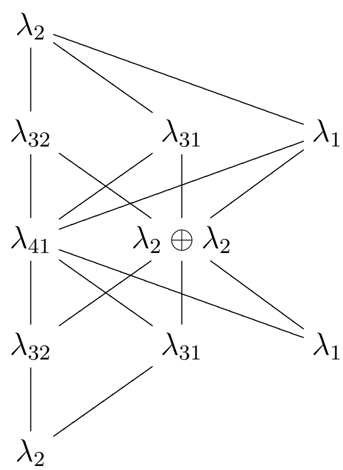


Figure 5.16: $T(\lambda_{41})$.

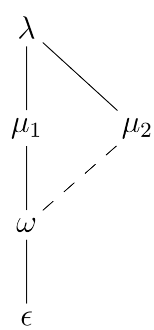


Figure 5.17: $\nabla(/, 2, 1)$ *from embeddings.*

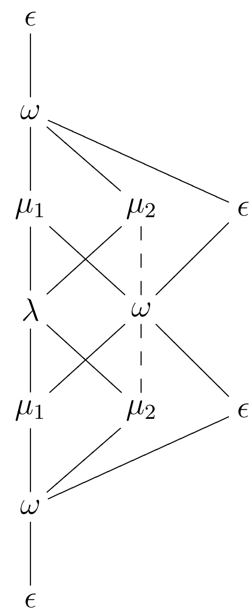


Figure 5.18: $T(\lambda = (/, 2, 1))$.

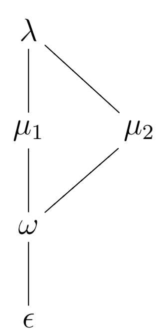


Figure 5.19: *Total* $\nabla(/, 2, 1)$ *structure.*

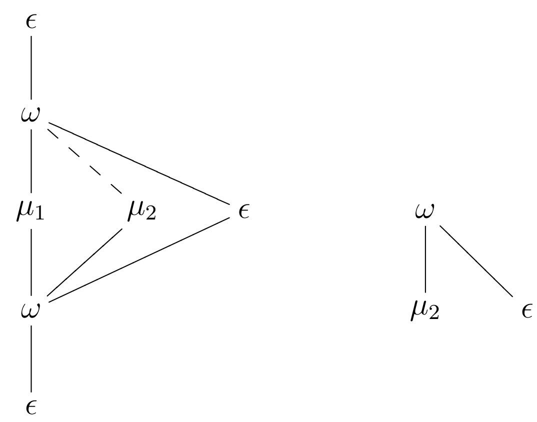


Figure 5.20: *N and $\Delta(\omega)$ for example 5.21.*

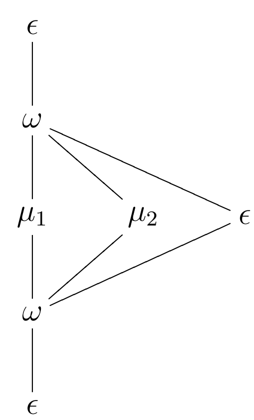


Figure 5.21: $T(\lambda = (\backslash, 1, 2))$.

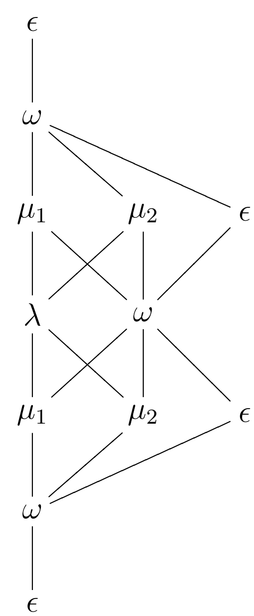


Figure 5.22: *Total* $T(\lambda = (/, 2, 1))$ *structure.*

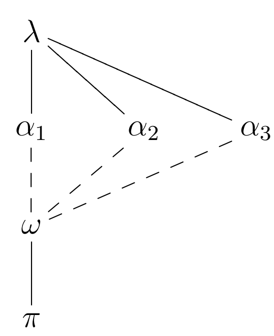


Figure 5.23: $\Delta(\lambda)$ *with* $\lambda = (\uparrow, 1, 2)$.

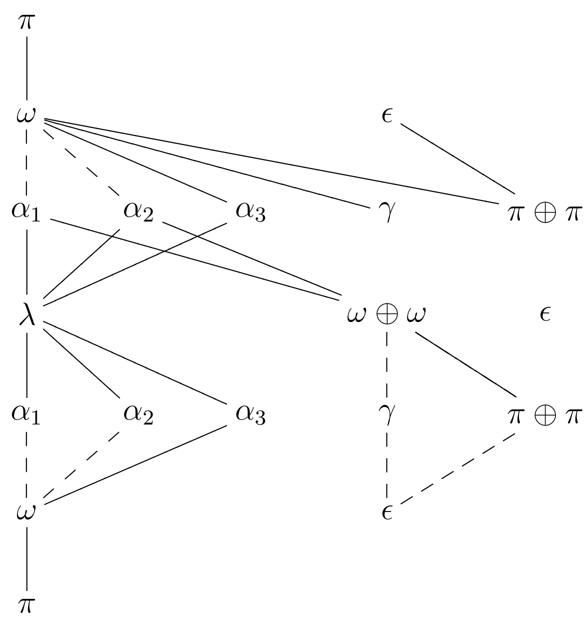


Figure 5.24: *Weyl filtration structure of* $T(\lambda = (\uparrow, 1, 2))$.

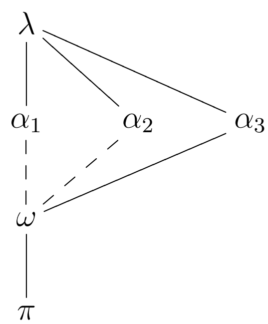


Figure 5.25: $\Delta(\lambda)$ *with* $\lambda = (\uparrow, 1, 2)$ *from* $T(\lambda)$.

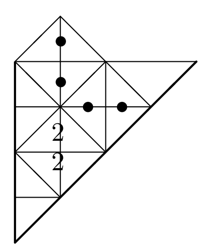


Figure 5.26: *Weyl filtration factors for* $T(\uparrow, 1, 2)$.

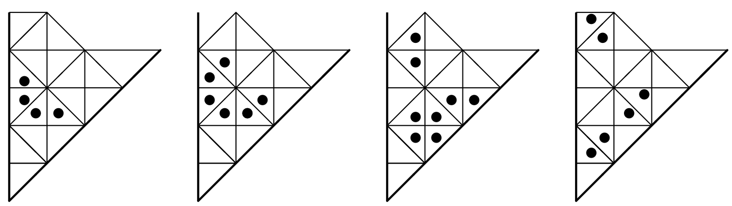


Figure 5.27: *Weyl filtration of tilting modules.*

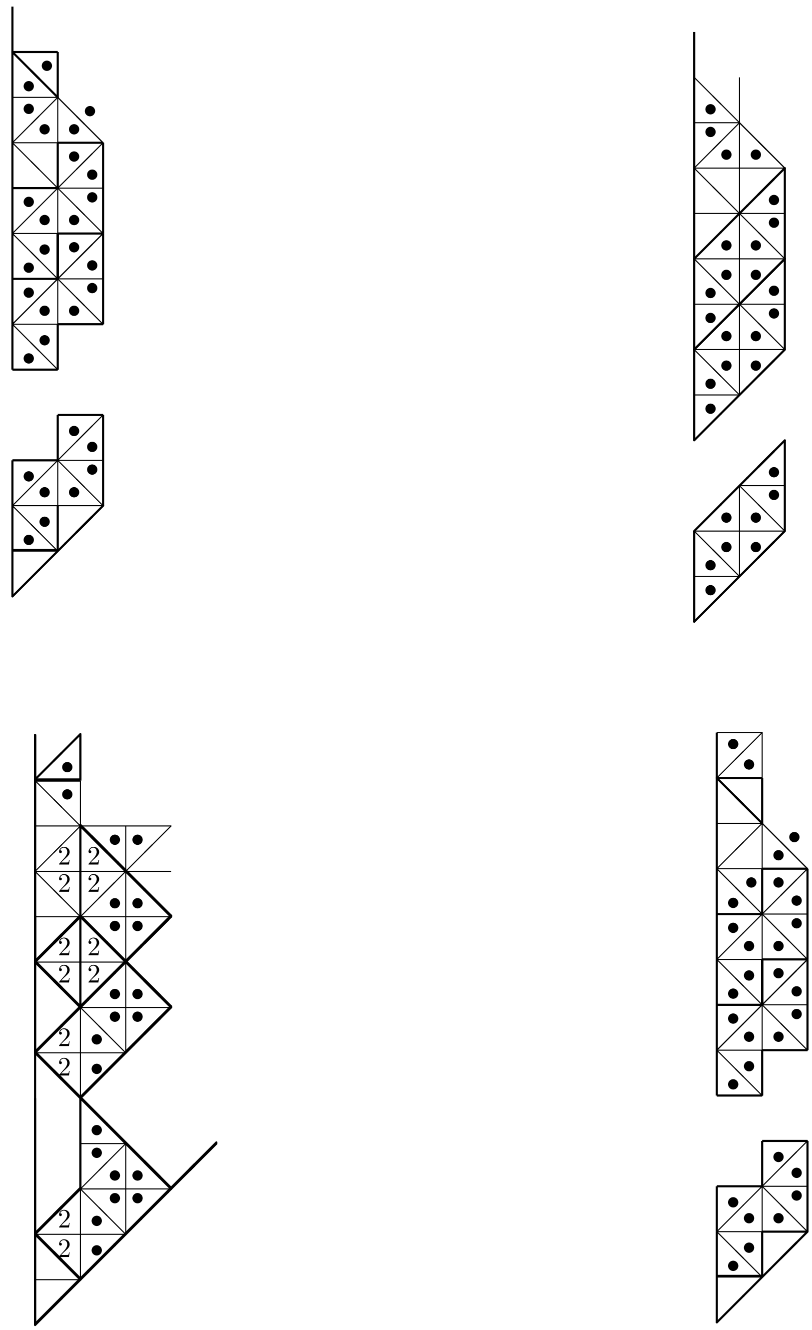


Figure 5.28: *Weyl filtration of tilting modules.*

Chapter 6

Quantum groups

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $U_q(\mathfrak{g})$, where $q^\ell = 1$ be the quantised universal enveloping algebra, constructed by Lusztig using divided powers.. We have all the usual objects similar to the Modular case. $L_q(\lambda)$ are the simple modules, $V_q(\lambda)$ is the Weyl module accompanied, again accompanied by it's dual. $W_l = W \ltimes l\mathbb{Z}R$. It is expected that the representation theory of G when restricted to the p^2-alcove is parallel to the representation theory of the corresponding quantum group at a p^{th} root of unity. In [So] an algorithm is conjectured for a good filtration of any indecomposable tilting module in the p^2-alcove for B_2 in the quantum group setting. Stroppel [St] uses this algorithm to find good filtrations for indecomposable tilting modules with highest weights in the p^2-alcove. The quantum versions of the results in this thesis are as follows.

Theorem 6.0.1. *For the quantum group U_q, theorem 4.10.1 holds for all singular weights.*

Theorem 6.0.2. *For the quantum group U_q we have* $\mathrm{wfd}(L_q(\lambda)) = \mathrm{wfd}(\nabla_q(\lambda))$ *for the weight types as shown in chapter 4.*

We also expect that all submodule structures considered inside p^2-alcove in the previous chapter will be the same as the quantum case.

6.0.1 Kazhdan Lusztig polynomials

As mentioned in the introduction, for regular weights, we can define the Weyl filtration as the maximum degree of the Kazhdan Lusztig polynomials:

$$\mathrm{wfd}(L(\lambda)) = \max\{deg(P_{\lambda,\mu}(q)) \,|\, \mu \uparrow \lambda\}.$$

The results of this thesis can then be used to define the degree of an analogous Kazhdan-Lusztig type polynomial for singular weights. This will be investigated in future work.

Bibliography

[A1] H.H. Andersen, On the structure of Weyl modules, *Math. Z.* **170** (1980), 1-14.

[A2] H.H. Andersen, The strong linkage principle, *J. reine angew Math.* **315** (1980), 53-59.

[A3] H.H. Andersen, An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups, *Adv. in Math.* **60** (1986), 125-153.

[CPS] E.Cline, B.Parshall, L.Scott, Finite dimensional algebras and highest weight categories, *J. reine angew Math.* **391** (1988), 85-99.

[DS] S.R. Doty and J.B. Sullivan, On the structure of higher cohomology modules of line bundles on G/B, *J.Algebra* **114** (1988), 286-332.

[D] S.Donkin, *The q-schur Algebra*, London Math. Soc. Lecture Note Series 253, Cambridge University Press, Cambridge (1998).

[EP] K. Erdmann and A.E. Parker, On the global and ∇-filtration dimension of quasi-hereditary algebras, *J. Pure and Applied Alg.* **194**(2004), 92-111.

[FP] E.M. Friedlander and B.J. Parshall, Cohomology of Lie algebras and algebraic groups, *Amer J. Math.* **108** (1986), 235-253.

[H1] J.E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, **21** Springer-Verlag, Berlin/Heidelberg/New York 1975.

[H2] J.E. Humphreys, Cohomology of G/B in characteristic p, *Adv. in Math.* **59** (1986), 170-183.

[J] J.C. Jantzen, *Representations of Algebraic Groups*, Pure Appl. Math. 131, Academic Press, Boston/Orlando 1987.

[Je] J.G. Jensen, PhD thesis *On Tilting modules for low rank groups*, Phd thesis, university of Aarhus 1997.

[K] M. Koppinen, Homomorphisms between neighboring Weyl modules, *Journal of Algebra.* **103** (1986), 302-319.

[L1] Z. Lin, Socle series of cohomology groups of line bundles on G/B, *J. Pure Appl. Algebra* **72** (1991), 275-294.

[L2] Z. Lin, *The structure of cohomology of line bundles on the flag varieties for some groups of rank 2*, PhD thesis, University of Massachusetts (1989).

[L3] Z. Lin, structure of cohomology of line bundles on G/B for semisimple groups, *Journal of Algebra* **134** (1990) 225-256.

[L4] Z. Lin, Comparison of Extensions of Modules for Algebraic Groups and Quantum Groups, *Proceedings of Symposia in Pure Mathematics*, **63** (1998) 355-367.

[Lu] G. Lusztig, Hecke Algebras and Jantzens generic Decomposition Patterns, *Adv. in Math* **37** (1980), 121-164.

[M] S. MacLane, *Homology*, Berlin, Springer, 1963.

[P1] A.E. Parker, On the good filtration dimension of Weyl modules for a linear algebraic group, *J. Reine Angew. Math* **562** (2003), 5-21.

[P2] A.E. Parker, *On the Global Dimension of Schur Algebras and Related Algebras,* Ph.D. thesis, University of London, 2001.

[P3] A.E. Parker, the global dimension of the Schur algebras for GL_2 and GL_3, *J. Algebra* **241** (2001), 340-378.

[So] W. Soergel, Kazhdan-Lusztig Polynomials and a combinatoric for Tilting Modules, *Representation Theory (an electronic Journal of the AMS)* Volume 1 (1997), 83-114.

[Sp] T.A. Springer, *Linear algebraic groups*, Progress in mathematics; 9, Birkhauser, Boston/Basel/Stuttgart 1980.

[St] C. Stroppel *Untersuchungen zu den parabolishen Kazhdan-Lusztig-Polynomen fur affine Weyl-Gruppen*, Masters thesis, Freiburg University (1997)