ALGEBRAIC DEFORMATION OF A MONOIDAL CATEGORY

by

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B.Sc., Tribhuvan University, Nepal, 1988
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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Abstract

This dissertation begins the development of the deformation theorem of monoidal categories which accounts for the function that all arrow-valued operations, composition, the arrow part of the monoidal product, and structural natural transformation are deformed.

The first chapter is review of algebra deformation theory. It includes the Hochschild complex of an algebra, Gerstenhaber's deformation theory of rings and algebras, Yetter's deformation theory of a monoidal category, Gerstenhaber and Schack's bialgebra deformation theory and Markl and Shnider's deformation theory for Drinfel'd algebras.

The second chapter examines deformations of a small k-linear monoidal category. It examines deformations beginning with a naive computational approach to discover that as in Markl and Shnider's theory for Drinfel'd algebras, deformations of monoidal categories are governed by the cohomology of a multicomplex. The standard results concerning first order deformations are established. Obstructions are shown to be cocycles in the special case of strict monoidal categories when one of composition or tensor or the associator is left undeformed.

At the end there is a brief conclusion with conjectures.

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Dedication

To my parents, mother Ratnamaya, late father Karnabahadur and to my son, Saurav

Chapter 1

Introduction

A mathematical object may admit many additional structures of a given type. For instance, an even dimensional manifold may admit many complex structures; a vector space may admit many algebraic structures; a category may admit many monoidal structures.

Classifying these structures up to some relevant notion of equivalence, analytic equivalence, isomorphism, or monoidal equivalence in the examples given is often a difficult problem.

Deformation theory or properly, infinitesimal deformation theory contributes to the solution by classifying structures arbitrarily close to a given structure in the sense of lying in formal infinitesimal neighborhoods of various orders.

The original deformation theory of Froelicher-Nijenhuis-Kodaira-Spencer dealt with analytic structures on manifolds. An analogous theory due to Gerstenhaber[1],[2] dealt with deformation of associative rings and algebras. In both theories, the first order deformations are classified by a certain cohomology group, while obstructions to extending a deformation to higher order are cohomology classes of one higher cohomological dimension. Later, Gerstenhaber's deformation theory for associative algebra was extended to Lie algebra by Nijenhuis and Richardson. Gerstenhaber observed that the first Hochschild[3] cohomology group $H^1(A, A)$ of an algebra, A, classifies all the infinitesimal deformations of automorphisms of

A. Similarly the second Hochschild cohomology group $H^2(A,A)$ of the algebra A classifies deformations (of the multiplication) of the algebra A, in particular, if $H^2(A,A)=0$, A admits no deformations and is termed 'rigid'. Fox[4] provided some examples of deformation of associative algebra with graphs and obstruction computations. Gerstenhaber's deformation theory of algebra was extended to associative, coassociative bialgebras by Gerstenhaber and Schack[5][1992]. They showed that Hochschild complex of an associative algebra and the dual to Hochschild complex for a coassociative coalgebra, the Cartier complex[6][1956], are compatible in a way which allows construction of a double complex (the Hochschild complex in one direction and the Cartier on the other direction). Gerstenhaber and Schack's deformation theory for bialgebras was extended to Drinfel'd algebras by Markl and Shnider[7][1996]. The extension, of compatibility condition between the Hoschschild and Cartier differentials failed and the deformations were governed not by a double complex but by a multicomplex. This work is an extension of Gerstenhaber and Schack's compatibility condition of two complexes to the Hochschild complex[8] for $k-$linear category and Yetter's complex[9] for a monoidal category.

We review the main parts of algebraic deformation theory historically with some proofs to explain the expected form of our own results.

1.1 The Hochschild Homology and Cohomology of an Associative Unital Algebra

Let k be a field and A be a k-algebra, M be an A-A bimodule. We obtain a simplicial k-module $M\otimes A^{\otimes *}$ with $[n]\rightarrow M\otimes A^{\otimes n}$ $(M\otimes A^{\otimes 0}=M)$ by defining

$$\partial_i(m\otimes a_1\otimes ...\otimes a_n)=\begin{cases} ma_1\otimes ...\otimes a_n & if \quad i=0 \\ m\otimes ...\otimes a_ia_{i+1}\otimes ...\otimes a_n & if \quad 0<i<n \\ a_nm\otimes a_1\otimes ...\otimes a_{n-1} & if \quad i=n \end{cases}$$

$$\sigma_i(m\otimes a_1\otimes ...\otimes a_n)=m\otimes a_1\otimes ...\otimes a_i\otimes 1\otimes a_{i+1}\otimes ...\otimes a_n$$

for all $a_i \in A, m \in M$ where both ∂_i and σ_i are multilinear. So the relations are well-defined and the required identities are readily verified for the simplicial complex. That is,

$$0 \leftarrow M \overset{\partial_0 - \partial_1 = d_1}{\leftarrow} M \otimes A \overset{d_2}{\leftarrow} M \otimes A^{\otimes 2} \overset{d_3}{\leftarrow} M \otimes A^{\otimes 3} \overset{d_4}{\leftarrow} \cdots$$

is a simplicial complex. Similarly $[n] \to Hom_k(A^{\otimes n}, M)$ with

$$(\partial^i \phi)(a_0, a_1, ..., a_n) = \begin{cases} a_0\phi(a_1, ..., a_n) & if \quad i = 0 \\ \phi(a_0, ..., a_i a_{i+1}, ..., a_n) & if \quad 0 < i < n \\ \phi(a_0, a_1, ..., a_{n-1}) & if \quad i = n \end{cases}$$

$$(\sigma^i \phi)(a_1, ..., a_n) = \phi(a_1, ..., a_i, 1, a_{i+1}, ..., a_n).$$

Where $d_n = \sum_{i=1}^{n} (-1)^i \partial_i$ and $d^n = \sum_{i=1}^{n} (-1)^i \partial^i$.
The n^{th} homology of the former complex is

$$H_n(A, M) = \pi_n(M \otimes A^{\otimes *}) = H_n C(M \otimes A^{\otimes *})$$

and is called the Hochschild homology of algebra A with coefficients in M and the n^{th} cohomology of later cocomplex is denoted by

$$H^n(A, M) = \pi^n Hom_k(A^{\otimes *}, M) = H^n C(Hom_k(A^{\otimes *}, M)),$$

and is called the Hochschild cohomology of algebra A with coefficients in M. In the deformation theory of algebras, one considers the Hochschild cohomology with coefficients in A itself.

1.2 Deformation of Rings and Algebras

Definition 1.2.1. *Let A be an associative algebra over a field k and V be the underlying vector space of A. Let $k[\varepsilon]/\langle \varepsilon^n \rangle = R$ and $V_R = V \otimes_k R$. Then any bilinear function $f : V \times V \to V$ can be extended to $\hat{f} : V_R \times V_R \to V_R$. Let $\hat{f} = \sum_i f^{(i)} \varepsilon^i$, $f^{(0)} = f$.*

In particular, if f is a multiplication, we can write $f(a,b) = a * b \overset{simply}{=} ab$, $\hat{f} = \hat{*}$ and $a\hat{*}b = ab + \mu^{(1)}(a,b)\varepsilon + \mu^{(2)}(a,b)\varepsilon^2 + \cdots$. If $*$ is associative, we want $\hat{*}$ also to be an associative. Thus we need to have

$$(a\hat{*}b)\hat{*}c = a\hat{*}(b\hat{*}c)$$

which implies

$$\sum_{\substack{i+j=n \\ 0\leq i,j}} [\mu^{(i)}(\mu^{(j)}(a,b),c) - \mu^{(i)}(a,\mu^{(j)}(b,c))] = 0 \text{ for } n \geq 0 \tag{1.1}$$

For $n = 0$, it is just the associativity of old multiplication of algebra. If $n = 1$, it is

$$a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}(a,b)c = 0,$$

which is precisely the condition that $\mu^{(1)}$ be the Hochschild 2-cocycle of the complex on A with coefficient in A. That is, an infinitesimal of deformation of multiplication of an algebra A is a Hochschild 2-cocycle. If A was commutative and we require commutativity of the deformed multiplication, then we have another relation
$a\hat{*}b = b\hat{*}a$ implies $\mu^{(i)}(a,b) = \mu^{(i)}(b,a) \quad \forall i$.
A finite deformation of multiplication is of 1-parameter family and the infinitesimal of it is in $Z^2(A,A)$. But for any $f \in Z^2(A,A)$, it is not necessary that f is to be an infinitesimal of 1-parameter family. If it is such, then Gerstenhaber termed this as integrable. This implies that the existence of infinite sequence of relations which may be interpreted as the vanishing of "obstructions" to the integration of f.

Definition 1.2.2. *(The Trivial Deformation): A deformation $F_t : V_R \times V_R \to V_R$ of an associative algebra, V is trivial if $F_t(a,b) = ab + \sum_i \mu^{(i)}(a,b)\epsilon^i$ and $\mu^{(i)} = 0 \quad \forall i > 0$.*

Note: This definition implies that the trivial deformation is simply extension by scalars.

Definition (Equivalent Deformations): Two deformations $V_R, \bar{V}_R$ of an associative algebra V are called equivalent if there is an R-algebra homomorphism $\hat{f} : V_R \to \bar{V}_R$ such that $\hat{f}$ reduces modulo ϵ to an identity map on V.
As we are concerned with deformations up to equivalence, we call any deformation trivial if it is equivalent to the trivial deformation.

Note that if $\mu^{(1)}, \mu^{(2)}, ..., \mu^{(n-1)} = 0$ then $\delta(\mu^{(n)}) = 0$ i.e. $\mu^{(n)} \in Z^2(A, A)$. Furthermore, if $\mu^{(n)} \in B^2(A, A)$ then there exists an $\phi_n \in C^1(A, A)$ such that $\mu^{(n)} = \delta(\phi_n)$. Then setting $\Phi_\epsilon = a + \phi_n(a)\epsilon$ then we have

$$\Phi_\epsilon^{-1}(\Phi_\epsilon(a).\Phi_\epsilon(b)) = ab + \mu^{(n+1)}(a, b)\epsilon^{n+1} + \mu^{(n+2)}(a, b)\epsilon^{n+2} + \cdots$$

then $\mu^{(n+1)} \in Z^2(A, A)$.

For the first order deformation $a\hat{*}b = ab + \mu^{(1)}(a, b)\epsilon$ to extend to a second order deformation $a\hat{*}b = ab + \mu^{(1)}(a, b)\epsilon + \mu^{(2)}(a, b)\epsilon^2$ we must have,

$$\mu^{(1)}(\mu^{(1)}(a, b), c) - \mu^{(1)}(a, \mu^{(1)}(b, c)) = d_H(\mu^{(2)})(a, b, c).$$

That is, if $\mu^{(2)}$ exists, it most cobound the left quantity above. Whether such a $\mu^{(2)}$ exists or not, Gerstenhaber proved

Proposition 1.2.3 (Gerstenhaber)**.** *The quantity* $\mu^{(1)}(\mu^{(1)}(a,b),c) - \mu^{(1)}(a,\mu^{(1)}(b,c)) = \omega^{(1)}(a,b,c)$ *is a 3-cocycle.*

Proof. $\delta(\omega^{(1)})(a,b,c,d)$

$= a\omega^{(1)}(b,c,d) - \omega^{(1)}(ab,c,d) + \omega^{(1)}(a,bc,d) - \omega^{(1)}(a,b,cd) + \omega^{(1)}(a,b,c)d$

$= a\mu^{(1)}(\mu^{(1)}(b,c),d) - a\mu^{(1)}(b,\mu^{(1)}(c,d)) - \mu^{(1)}(\mu^{(1)}(ab,c),d) + \mu^{(1)}(ab,\mu^{(1)}(c,d))$
$+ \mu^{(1)}(\mu^{(1)}(a,bc),d) - \mu^{(1)}(a,\mu^{(1)}(bc,d)) - \mu^{(1)}(\mu^{(1)}(a,b),cd) + \mu^{(1)}(a,\mu^{(1)}(b,cd))$
$+ \mu^{(1)}(\mu^{(1)}(a,b),c)d - \mu^{(1)}(a,\mu^{(1)}(b,c))d$

$= a\mu^{(1)}(\mu^{(1)}(b,c),d)$
$- a\mu^{(1)}(b,\mu^{(1)}(c,d)) + \mu^{(1)}(ab,\mu^{(1)}(c,d)) - \mu^{(1)}(\mu^{(1)}(ab,c),d) + \mu^{(1)}(\mu^{(1)}(a,bc),d)$
$- \mu^{(1)}(a,\mu^{(1)}(bc,d)) + \mu^{(1)}(a,\mu^{(1)}(b,cd)) - \mu^{(1)}(\mu^{(1)}(a,b),cd) + \mu^{(1)}(\mu^{(1)}(a,b),c)d$
$- \mu^{(1)}(a,\mu^{(1)}(b,c))d$

$\overset{Linearity}{=} a\mu^{(1)}(\mu^{(1)}(b,c),d)$
$- \mu^{(1)}(a,b)\mu^{(1)}(c,d) + \mu^{(1)}(a,b\mu^{(1)}(c,d)) - \mu^{(1)}(a\mu^{(1)}(b,c),d) + \mu^{(1)}(\mu^{(1)}(a,b)c,d)$
$- \mu^{(1)}(a,b\mu^{(1)}(c,d)) + \mu^{(1)}(a,\mu^{(1)}(b,c)d) + \mu^{(1)}(a,b)\mu^{(1)}(c,d) - \mu^{(1)}(\mu^{(1)}(a,b)c,d)$
$- \mu^{(1)}(a,\mu^{(1)}(b,c))d$

$= 0$

□

The quantity $\omega^{(1)}$ obstructs extension of a first order deformation to the second order deformation. If this is zero as a cohomology class, then we can extend the deformation to second order as

$$a\hat{*}b = a * b + \mu^{(1)}(a,b)\varepsilon + \mu^{(2)}(a,b)\varepsilon^2$$

otherwise we can not. The general obstruction is

$$\sum_{\substack{i+j=n+1\\0\leq i,j}} [\mu^{(i)}(\mu^{(j)}(a,b),c) - \mu^{(i)}(a,\mu^{(j)}(b,c))] = \omega^{(n)}(a,b,c).$$

Theorem 1.2.4. *The obstruction $\omega^{(n)}$ is 3-cocycle for all n.*

Proof. We have

$$-d(\mu^{(k)})(a,b,c) = \omega^{(k-1)}(a,b,c) = \sum_{\substack{i+j=k\\0\leq i,j\leq k-1}} [\mu^{(i)}(\mu^{(j)}(a,b),c) - \mu^{(i)}(a,\mu^{(j)}(b,c))], k \leq n$$

Then

$d(\omega^{(n)})(a,b,c,d) = a\omega^{(n)}(b,c,d) - \omega^{(n)}(ab,c,d) + \omega^{(n)}(a,bc,d) - \omega^{(n)}(a,b,cd) + \omega^{(n)}(a,b,c)d$

$= \sum_{\substack{i+j=n+1\\0\leq i,j\leq n}} [a\mu^{(i)}(\mu^{(j)}(b,c),d) - a\mu^{(i)}(b,\mu^{(j)}(c,d)) - \mu^{(i)}(\mu^{(j)}(ab,c),d) + \mu^{(i)}(ab,\mu^{(j)}(c,d))$
$+ \mu^{(i)}(\mu^{(j)}(a,bc),d) - \mu^{(i)}(a,\mu^{(j)}(bc,d)) - \mu^{(i)}(\mu^{(j)}(a,b),cd) + \mu^{(i)}(a,\mu^{(j)}(b,cd))$
$+ \mu^{(i)}(\mu^{(j)}(a,b),c)d - \mu^{(i)}(a,\mu^{(j)}(b,c))d]$

$= \sum_{\substack{i+j=n+1\\0\leq i,j\leq n}} [a\mu^{(i)}(\mu^{(j)}(b,c),d) - a\mu^{(i)}(b,\mu^{(j)}(c,d)) - \mu^{(i)}(\mu^{(j)}(ab,c) - \mu^{(j)}(a,bc),d)$
$+ \mu^{(i)}(ab,\mu^{(j)}(c,d)) - \mu^{(i)}(a,\mu^{(j)}(bc,d) - \mu^{(j)}(b,cd)) - \mu^{(i)}(\mu^{(j)}(a,b),cd)$
$+ \mu^{(i)}(\mu^{(j)}(a,b),c)d - \mu^{(i)}(a,\mu^{(j)}(b,c))d]$

$= \sum_{\substack{i+j=n+1\\0\leq i,j\leq n}} [a\mu^{(i)}(\mu^{(j)}(b,c),d) - a\mu^{(i)}(b,\mu^{(j)}(c,d))$
$- \mu^{(i)}(a\mu^{(j)}(b,c) - \mu^{(j)}(a,b)c + \sum_{\substack{p+q=j\\0\leq p,q<j}} [\mu^{(p)}(\mu^{(q)}(a,b),c) - \mu^{(p)}(a,\mu^{(q)}(b,c))],d)$
$- \mu^{(i)}(a,b\mu^{(j)}(c,d) - \mu^{(j)}(b,c)d + \sum_{\substack{p+q=j\\0\leq p,q<j}} [\mu^{(p)}(\mu^{(q)}(b,c),d) - \mu^{(p)}(b,\mu^{(q)}(c,d))])$
$+ \mu^{(i)}(ab,\mu^{(j)}(c,d)) - \mu^{(i)}(\mu^{(j)}(a,b),cd) + \mu^{(i)}(\mu^{(j)}(a,b),c)d - \mu^{(i)}(a,\mu^{(j)}(b,c))d]$

$= \sum_{\substack{i+j=n+1\\0\leq i,j\leq n}} [a\mu^{(i)}(\mu^{(j)}(b,c),d) - \mu^{(i)}(a\mu^{(j)}(b,c),d) + \mu^{(i)}(a,\mu^{(j)}(b,c)d) - \mu^{(i)}(a,\mu^{(j)}(b,c))d$

$$
\begin{aligned}
&- a\mu^{(i)}(b,\mu^{(j)}(c,d)) + \mu^{(i)}(ab,\mu^{(j)}(c,d)) - \mu^{(i)}(a,b\mu^{(j)}(c,d)) + \mu^{(i)}(a,b)\mu^{(j)}(c,d)\\
&- \mu^{(i)}(a,b)\mu^{(j)}(c,d) + \mu^{(i)}(\mu^{(j)}(a,b)c,d) - \mu^{(i)}(\mu^{(j)}(a,b),cd) + \mu^{(i)}(\mu^{(j)}(a,b),c)d\\
&- \mu^{(i)}(\textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(a,b),c) - \mu^{(p)}(a,\mu^{(q)}(b,c))],d)\\
&- \mu^{(i)}(a,\textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(b,c),d) - \mu^{(p)}(b,\mu^{(q)}(c,d))])]
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\substack{i+j=n+1\\0\le i,j\le n}}[-\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(a,\mu^{(j)}(b,c)),d) - \mu^{(p)}(a,\mu^{(q)}(\mu^{(j)}(b,c),d))]\\
&+ \textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(a,b),\mu^{(j)}(c,d)) - \mu^{(p)}(a,\mu^{(q)}(b,\mu^{(j)}(c,d)))]\\
&+ \textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(\mu^{(j)}(a,b),c),d) - \mu^{(p)}(\mu^{(j)}(a,b),\mu^{(q)}(c,d))]\\
&- \mu^{(i)}(\textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(a,b),c) - \mu^{(p)}(a,\mu^{(q)}(b,c))],d)\\
&- \mu^{(i)}(a,\textstyle\sum_{\substack{p+q=j\\0\le p,q<j}}[\mu^{(p)}(\mu^{(q)}(b,c),d) - \mu^{(p)}(b,\mu^{(q)}(c,d))])]
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\substack{i+j=n+1\\0\le i,j\le n}}\sum_{\substack{p+q=j\\0\le p,q<j}}[-\{\mu^{(p)}(\mu^{(q)}(a,\mu^{(j)}(b,c)),d) - \mu^{(p)}(a,\mu^{(q)}(\mu^{(j)}(b,c),d))\}\\
&+ \{\mu^{(p)}(\mu^{(q)}(a,b),\mu^{(j)}(c,d)) - \mu^{(p)}(a,\mu^{(q)}(b,\mu^{(j)}(c,d)))\}\\
&+ \{\mu^{(p)}(\mu^{(q)}(\mu^{(j)}(a,b),c),d) - \mu^{(p)}(\mu^{(j)}(a,b),\mu^{(q)}(c,d))\}\\
&- \mu^{(i)}(\mu^{(p)}(\mu^{(q)}(a,b),c) - \mu^{(p)}(a,\mu^{(q)}(b,c)),d)\\
&- \mu^{(i)}(a,\mu^{(p)}(\mu^{(q)}(b,c),d) - \mu^{(p)}(b,\mu^{(q)}(c,d)))]
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\substack{i+j+k=n+1\\0<i,j,k\le n}}[-\mu^{(i)}(\mu^{(j)}(a,\mu^{(k)}(b,c)),d) + \mu^{(i)}(a,\mu^{(j)}(\mu^{(k)}(b,c),d))\\
&+ \mu^{(i)}(\mu^{(j)}(a,b),\mu^{(k)}(c,d)) - \mu^{(i)}(a,\mu^{(j)}(b,\mu^{(k)}(c,d))) + \mu^{(i)}(\mu^{(j)}(\mu^{(k)}(a,b),c),d)\\
&- \mu^{(i)}(\mu^{(j)}(a,b),\mu^{(k)}(c,d)) - \mu^{(i)}(\mu^{(j)}(\mu^{(k)}(a,b),c) - \mu^{(j)}(a,\mu^{(k)}(b,c)),d)\\
&- \mu^{(i)}(a,\mu^{(j)}(\mu^{(k)}(b,c),d) - \mu^{(j)}(b,\mu^{(k)}(c,d)))]
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\substack{i+j+k=n+1\\0<i,j,k\le n}}[\{\mu^{(i)}(\mu^{(j)}(a,\mu^{(k)}(b,c)),d) - \mu^{(i)}(\mu^{(j)}(a,b),\mu^{(k)}(c,d))\}\\
&- \{\mu^{(i)}(\mu^{(j)}(a,\mu^{(k)}(b,c)),d) - \mu^{(i)}(\mu^{(j)}(a,b),\mu^{(k)}(c,d))\}]
\end{aligned}
$$

$$= 0$$

□

Theorem 1.2.5. *(Gerstenhaber) Let μ_t be a 1-parameter family of deformations of an algebra A. Then μ_t is equivalent to a family of $\mu^{(i)}$'s*

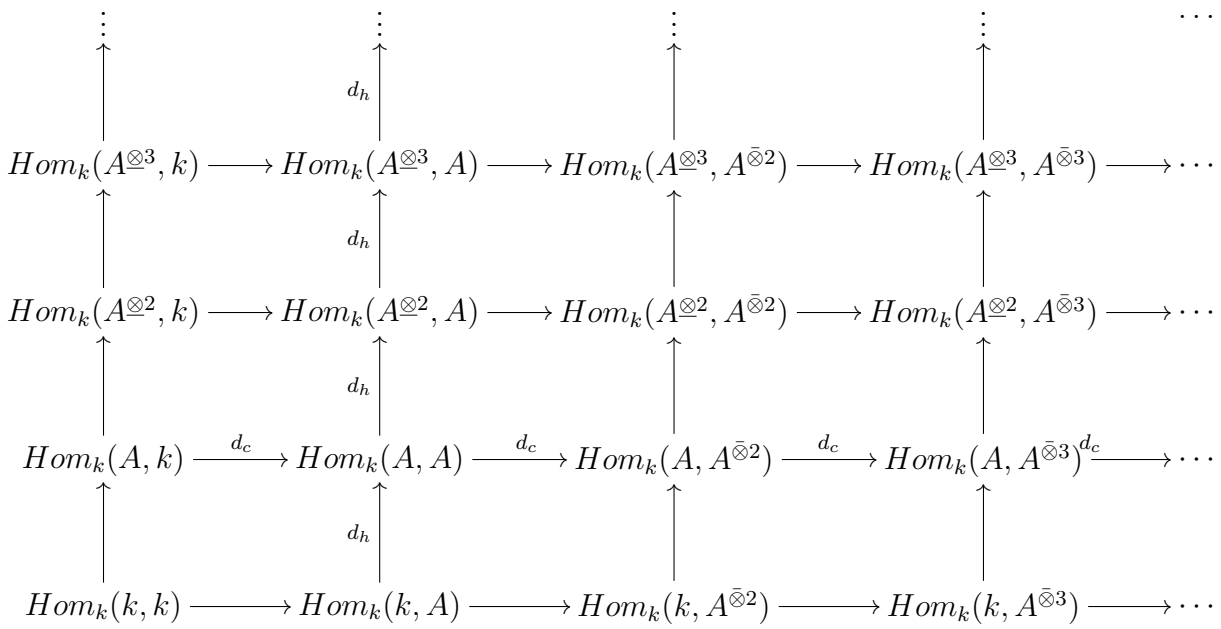
$$g_\epsilon(a,b) = ab + \mu^{(n+1)}(a,b)\epsilon^{n+1} + \mu^{(n+2)}(a,b)\epsilon^{n+2} + \cdots$$

where the first non-vanishing cochain if $\mu^{(n)} \in Z^2(A,A)$ and not cohomologous to zero.

Corollary 1.2.1. *If $H^2(A,A) = 0$ then A is rigid.*

1.3 Gerstenhaber and Schack's Double Complex

Building on results of Gerstenhaber, Gerstenhaber and Schack[1992] showed that the Hochschild complex of an associative algebra and the Cartier complex of coassociative coalgebra are compatible in the sense that



is a double complex where the underline of a $\otimes$ indicates the imposition of the obvious induced module structure and the overline indicates the imposition of the obvious induced comodule structure.

1.4 The Hochschild Cohomology of k-linear Categories

There is a long-known a folk-theorem[10] that generalizes Gerstenhaber's results from k-algebras to small k-linear categories.

Following Yetter, we make,

Definition 1.4.1. *If $\mathcal{C}$ and $\mathcal{D}$ are $k-$linear categories, the Hochschild complex[11] of parallel functor $F, G : \mathcal{C} \to \mathcal{D}$ has cochain groups given by*

$$X^n(F,G) := \prod_{x_0,x_1,\ldots,x_n} Hom_k(\mathcal{C}(x_0,x_1) \otimes \ldots \otimes \mathcal{C}(x_{n-1},x_n), \mathcal{D}(F(x_0),G(x_n)))$$

with the coboundary,
$d_Y(\psi)(f_0 \otimes \ldots \otimes f_n) = F(f_0) \otimes \psi(f_1 \otimes \ldots \otimes f_n) + \sum_{i=1}^{n} \psi(f_1 \otimes \ldots \otimes f_{i-1}f_i \otimes \ldots \otimes f_n)$
$+ (-1)^{n+1}\psi(f_0 \otimes \ldots \otimes f_{n-1}) \otimes G(f_n)$.

Then $d_Y^2 = 0$.

The Folk theorem then can be stated precisely as:

Theorem 1.4.2 (Folk Theorem)**.** *For any small k-linear category $H^2(Id_{\mathcal{C}}, Id_{\mathcal{C}})$ classifies deformation of (the composition) of $\mathcal{C}$ up to equivalence. Moreover obstructions to extensions of deformations to higher order, given by formulas formally identical to those in Gerstenhaber are 3-cocycles in $C^\bullet(\mathcal{C}) = X^\bullet(Id_{\mathcal{C}}, Id_{\mathcal{C}})$.*

1.5 Drinfel'd Algebra Deformations

Definition 1.5.1. *(Drinfel'd Algebra) An algebra $A = (V, ., \Delta, \phi)$ where $(V, ., \Delta)$ is an associative not necessarily coassociative, unital and counital k-algebra, ϕ is an invertible element of $V^{\otimes 3}$, and the usual coassociativity property is replaced by quasi-coassociativity:*

$$(\mathbf{1} \otimes \triangle)\triangle.\phi = \phi.(\triangle \otimes \mathbf{1})\triangle \tag{1.2}$$

where the '.' is used to indicate the multiplication of A and the induced multiplication on $V^{\otimes 3}$. Moreover, ϕ must satisfy

$$(\mathbf{1^2} \otimes \triangle)(\phi).(\triangle \otimes \mathbf{1^2})(\phi) = (\mathbf{1} \otimes \phi).(\mathbf{1} \otimes \triangle \otimes \mathbf{1})(\phi).(\phi \otimes \mathbf{1}) \tag{1.3}$$

$1 \in V$ the unit element and $\mathbf{1}$, the identity map on V, and if $\epsilon : V \to k$, k the base field, is the counit of coalgebra $(V, \triangle)$ then $(\epsilon \otimes \mathbf{1})\triangle = (\mathbf{1} \otimes \epsilon)\triangle = \mathbf{1}$.

Note that the bialgebra is a Drinfel'd algebra with $\phi = 1$.

Markl and Shnider[1996] extended the Gerstenhaber and Schack[1992] bialgebra deformation bicomplex to a Drinfel'd algebra deformation multicomplex. Because of the non-associativity nature of Drinfel'd algebra, the vertical and horizontal differential do not cancel each other. This adds significant complication of computation to the interactions among parts of the deformations and is not easy to handle by hand. The interactions of can not be encoded in a bicomplex. Markl and Shnider had to introduce additional differentials and used the term 'homotopy differentials'. They are not exactly a differential, but are differential up to homotopy.

Definition 1.5.2. *A multicomplex $C^{(\bullet,\bullet)}$ is a bigraded complex with the differentials given by*

$$d_j : C^{(p,q)} \to C^{(p+j,q-j+1)}; q \geq j \geq 0$$

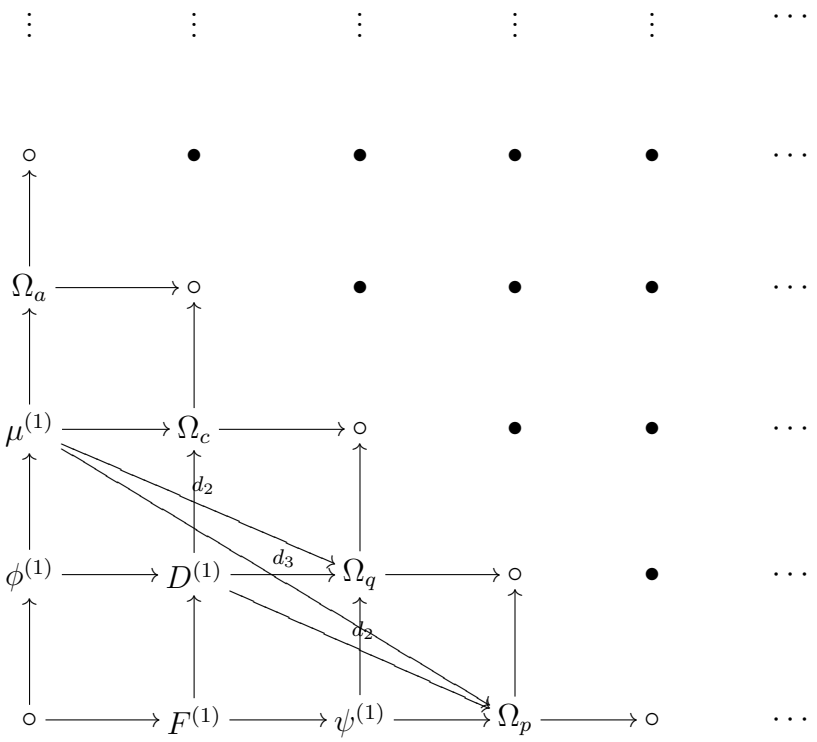
such that if

$$d = \begin{pmatrix} d_0 & 0 & 0 & \cdots & o \\ d_1 & d_0 & 0 & \cdots & o \\ d_2 & d_1 & d_0 & \cdots & o \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{n-1} & d_{n-2} & d_{n-3} & \cdots & d_0 \\ d_n & d_{n-1} & d_{n-2} & \cdots & d_1 \end{pmatrix} : \oplus_{i=1}^{n} C^{(n-i,i)} \to \oplus_{i=1}^{n+1} C^{(n-i,i)},$$

such that $d^2 = 0$.

Markl and Shnider used a geometrical approach and found that Stasheff polytopes provide descriptions of the complicated differentials using grouping objects by parenthesis [like

$(\bullet\bullet)\bullet \xrightarrow{\phi} \bullet(\bullet\bullet)]$ and ordering. As in Gerstenhaber and Schack, the vertical columns are the Hochschild complex but in the horizontal direction, the analog of the Cartier complex turns out to have differential which are not square zero. In case of Gerstenhaber and Schack, the first row was just trivial complex but this case it is not. The infinitesimal deformation of automorphisms lies at (1,0) and (1,1). Similarly, the infinitesimal deformation of multiplication and comultiplication lies at (1,2) and at (2,1) respectively. The infinitesimal deformation of ϕ lies at (3,0). The obstructions of associativity, compatibility, quasi-associativity, and pentagon lie at (1,3), (2,2), (3,1) and (4,0). The diagram looks like,



Such d_2 and d_3 are the homotopy differentials.

1.6 Yetter's Cohomology of a Monoidal Category

Following Mac Lane[12] we make,

Definition 1.6.1. *A* **graph** *(also called a "diagram scheme") is a set O of objects, a set A of arrows, and two functions $A \underset{cod}{\overset{dom}{\rightrightarrows}} O$. In this graph, the set of composable pairs of arrows is the set*

$$A \times_O A = \{\langle g, f\rangle | g, f \in A \text{ and } domg = codf\},$$

called the "product over O".

A **small category** *is a graph with two additional functions*

$$O \overset{id}{\rightarrow} A, \quad A \times_O A \overset{\circ}{\rightarrow} A$$

$$c \mapsto id_c, \quad \langle g, f\rangle \mapsto g \circ f$$

called the identity and composition, such that

$$dom(id\ a) = a = cod(id\ a), \quad dom(g \circ f) = domf, \quad cod(g \circ f) = codg$$

for all objects $a \in O$ and all composable pairs of arrows $\langle g, f\rangle \in A \times_O A$ and such that the associativity and unit axioms hold.

The expression $f : X \rightarrow Y$ is shorthand for the assertion that f is an arrow in a category, made clear in context, whose source is the object X and whose target is the object Y.

We let

$$\mathcal{C}(X, Y) = \{f : X \rightarrow Y | f \in Arr(\mathcal{C})\} = Hom_{\mathcal{C}}(X, Y)$$

and call these sets "hom-sets".

Definition 1.6.2. *For any unital commutative ring R, an R-linear category is a category whose hom-sets are each equipped with an R-modules structure and in which composition is R-bilinear.*

Definition 1.6.3. *$R-mod$ is an R-linear category, where R is a unital commutative rings, is a R-linear category with objects are R-modules and arrows are R-module homomorphisms. $k-V.S.$ for k is a field, the category with objects, k-vector spaces, and arrows are k-linear transformations between vector spaces,is k-linear category.*

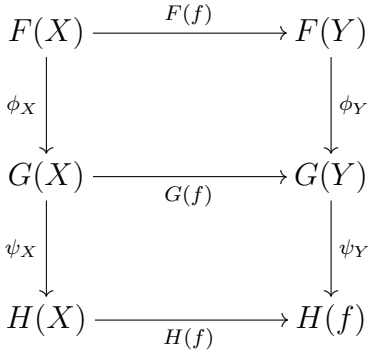
Definition 1.6.4. *(Functor) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an assignment to every object X of $\mathcal{C}$ of an object $F(X)$ of $\mathcal{D}$ such that for $X, Y, Z \in Ob(\mathcal{C})$, and to every arrow f of $\mathcal{C}$ an arrow $F(f)$ of $\mathcal{D}$, $X \xrightarrow{f} Y$ implies $F(X) \overset{F(f)}{\to} F(Y)$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ implies $F(X) \overset{F(f)}{\to} F(Y) \overset{F(g)}{\to} F(Z)$ and that $F(fg) = F(f)F(g)$.*

Definition 1.6.5. *(Natural Transformation) Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors from category $\mathcal{C}$ to category $\mathcal{D}$. Then a natural transformation ϕ from F to G is denoted by $\phi : F \Rightarrow G$ and is an assignment to each $X \in Ob(\mathcal{C})$ of an arrow $\phi_X : F(X) \to G(X)$ in $Arr(\mathcal{D})$ satisfying $\phi_X G(f) = F(f)\phi_Y$ for all $f : X \to Y$ in $\mathcal{C}$. That is,*

$$\begin{array}{ccc} F(X) & \xrightarrow{\quad F(f) \quad} & F(Y) \\ \Big\downarrow{\scriptstyle \phi_X} & & \Big\downarrow{\scriptstyle \phi_Y} \\ G(X) & \xrightarrow[\quad G(f) \quad]{} & G(Y) \end{array}$$

commutes.

If $F, G, H : \mathcal{C} \to \mathcal{D}$ are functors and $\phi : F \Rightarrow G, \quad \psi : G \Rightarrow H$ are natural transformations, then



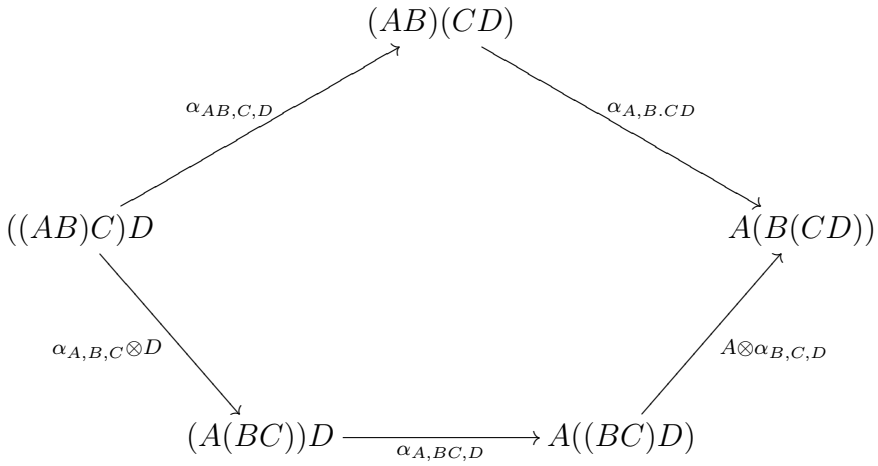
for any $f : X \to Y$ in $\mathcal{C}$. So $\phi\psi : F \Rightarrow G$ is a natural transformation.

A natural transformation ϕ such that each ϕ_x is an isomorphism is called a "natural isomorphism".

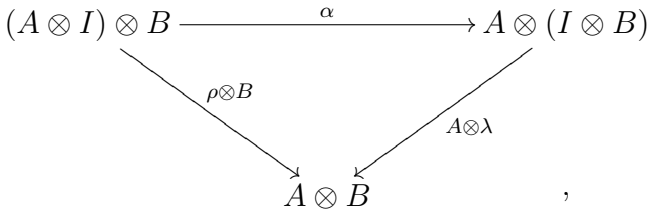
Definition 1.6.6. *(Equivalence of two categories) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\phi : FG \Rightarrow Id_{\mathcal{C}}$ and $\psi : GF \Rightarrow Id_{\mathcal{D}}$ then we say such a quadruple (F, G, ϕ, ψ) is called an equivalence of categories between categories $\mathcal{C}$ and $\mathcal{D}$.*

Definition 1.6.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. Then the product $\mathcal{C} \times \mathcal{D}$ is the category with objects $Ob(\mathcal{C} \times \mathcal{D}) = Ob(\mathcal{C}) \times Ob(\mathcal{D})$ and arrows $Arr(\mathcal{C} \times \mathcal{D}) = Arr(\mathcal{C}) \times Arr(\mathcal{D})$ and all operations (source, target, identity arrow, and composition) given coordinatewise by those of $\mathcal{C}$ and $\mathcal{D}$.*

Definition 1.6.8. *(Monoidal Category) A category $\mathcal{C}$ is a monoidal category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object I, and natural isomorphisms $\alpha : \otimes(\otimes \times \mathbf{1}_{\mathcal{C}}) \Rightarrow \otimes(\mathbf{1}_{\mathcal{C}} \times \otimes), \rho : \otimes I \Rightarrow \mathbf{1}_{\mathcal{C}}, \lambda : \mathbf{I}\otimes \Rightarrow \mathbf{1}_{\mathcal{C}}$ such that all instances of diagrams of the form*



and



,

commute. The commutativity of these diagrams are called the coherence conditions for a monoidal category $\mathcal{C}$*. In the case of* R*-linear categories we require that the maps induced on hom-sets by* $\otimes$ *be* R*-bilinear.*

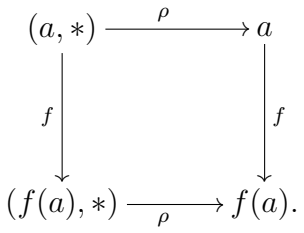
Example 1.6.9. $(\mathbf{Sets}, \times, *, \alpha, \rho, \lambda)$ *where* $\times$*=cartesian product*

$\{^*\}$*=singleton set*

$\alpha : ((a,b),c) \mapsto (a,(b,c))$

$\rho : (a,*) \mapsto a$ *and* $\lambda : (*,a) \mapsto a$

and the naturality for ρ is



Example 1.6.10. $(\mathbf{sets}, \bigsqcup, \phi, \alpha, \lambda, \rho)$[9]*, where ϕ is the empty set, is monoidal category.*

Definition 1.6.11. [9] *A tangle is an embedding $T : X \to I^3$ where $I = [0, 1]$ of a 1-manifold with boundary into the rectangular solid I^3 satisfying*

$$T(\partial X) = T(X) \bigcap \partial I^3 = T(X) \bigcap (I^2 \times \{0, 1\}).$$

Example 1.6.12. $(\mathbf{Tang}, \otimes, I, \alpha, \lambda, \rho)$*, where I is the empty tangle, and $\otimes$ is defined as: for two tangles T_1, T_2, $T_1 \otimes T_2$ has as underlying 1- manifold the disjoint union of the underlying 1-manifold of T_1 and T_2. $T_1 \otimes T_2$ is then the mapping of this 1-manifold the composition of $T_1 \coprod T_2$ with the map $\gamma : I^3 \coprod I^3 \to I^3$ given by*
$(x, y, z) \mapsto (x/2, y, z)$ for elements of the first summand, and
$(x, y, z) \mapsto ((x + 1)/2, y, z)$ for elements of the second summand.

Example 1.6.13. *Let k be a field. Then $(H - mod, \otimes = \otimes_k, k, \alpha, \lambda, \rho)$ where H is a bialgebra. This is a monoidal category.*

Example 1.6.14. *Let k be a field. Then $(R - mod, \otimes = \otimes_R, R, \alpha, \lambda, \rho)$ where R is a unital commutative algebra. This is a R-linear monoidal category.*

Example 1.6.15. *Let k be a field. Then $(k - v.s., \otimes = \otimes_k, k, \alpha, \lambda, \rho)$. This category is a k-linear monoidal category.*

Definition 1.6.16. *(Fusion Category)*[13] *By a fusion category $\mathcal{C}$ over a field k is a k-linear semisimple rigid tensor category with finitely many simple objects and finite dimensional*

hom-sets, and the hom-set $\mathcal{C}(I,I)$, which is necessarily a k-algebra under composition, is isomorphic to k.

Example 1.6.17. *Any fusion category is a k-linear monoidal category.*

Definition 1.6.18 (Strong Monoidal Functor)**.** *: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from monoidal category $\mathcal{C}$ to monoidal category $\mathcal{D}$ equipped with a natural isomorphism*
$\tilde{F} : F(.) \otimes F(.) \to F(. \otimes .)$ and an isomorphism $F_I : I \to F(I)$
satisfying

$$\begin{array}{ccc}
F((A\otimes B)\otimes C) & \xrightarrow{F(\alpha)} & F(A\otimes(B\otimes)) \\
\uparrow{\scriptstyle \tilde{F}^{A\otimes B,C}} & & \uparrow{\scriptstyle \tilde{F}_{A,B\otimes C}} \\
F(A\otimes B)\otimes F(C) & & F(A)\otimes F(B\otimes C) \\
\uparrow{\scriptstyle \tilde{F}^{A,B}\otimes F(C)} & & \uparrow{\scriptstyle F(A)\otimes\tilde{F}_{B,C}} \\
(F(A)\otimes F(B))\otimes F(C) & \xrightarrow{\beta} & F(A)\otimes(F(B)\otimes F(C))
\end{array}$$

$$\begin{array}{ccc}
F(A) & \xleftarrow{F(\rho_A)} & F(A\otimes I) \\
\uparrow{\scriptstyle \rho_{F(A)}} & & \uparrow{\scriptstyle \tilde{F}_{A,I}} \\
F(A)\otimes I & \xleftarrow[F(A)\otimes F_I]{} & F(A)\otimes F(I)
\end{array}
\qquad\qquad
\begin{array}{ccc}
F(A) & \xleftarrow{F(\lambda_A)} & F(I\otimes A) \\
\uparrow{\scriptstyle \lambda_{F(A)}} & & \uparrow{\scriptstyle \tilde{F}_{I,A}} \\
I\otimes F(A) & \xleftarrow[F_I\otimes F(A)]{} & F(I)\otimes F(A)
\end{array}$$

is called strong monoidal functor.

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is called *strict* if $\tilde{F}$ is the identity transformation and F_I is an identity arrow.

Definition 1.6.19. *(Monoidal natural transformation): A monoidal natural transformation is a natural transformation $\phi : F \Rightarrow G$ between two monoidal functors which satisfies*

$$\tilde{F}_{A,B}(\phi_{A\otimes B}) = \phi_A \otimes \phi_B(\tilde{G}_{A,B})$$

and $F_I = G_I(\phi_I)$.

Example 1.6.20. *For any k-bialgebra A, the functor from A-mod to k-vector space is a strict monoidal functor.*

Example 1.6.21. *The free group functor*

$$F : (Sets, \bigsqcup, \lambda, \rho, \alpha) \to (Grps, *, \lambda, \rho, \alpha)$$

equipped with structure maps induced by inclusions of generators and universal property of free groups is a strong monoidal functor.

Example 1.6.22. *For a Drinfel'd algebra R, the forgetful functor $F : R-bimod \to k-v.s.$ is a strong monoidal functor but not strict monoidal functor, unless R is a bialgebra, in which case it is strict.*

Definition 1.6.23. *(Monoidal Equivalence) A monoidal equivalence between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence of categories in which the functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are equipped with the structure of monoidal functors, and the natural isomorphisms $\phi : FG \Rightarrow Id_{\mathcal{C}}$ and $\psi : GF \Rightarrow Id_{\mathcal{D}}$ are both monoidal natural transformations. If there exists a monoidal equivalence between $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent.*

Definition 1.6.24. *A formal diagram in the theory of monoidal categories is a diagram in the free monoidal category on S for some set S.*

Theorem 1.6.25. *(Mac Lane's Coherence Theorem 1^{st} version)*[12]*: Every formal diagram in the theory of monoidal categories commutes. Consequently, any diagram which is the image of formal diagram under a (strict Monoidal) functor commutes.*
(2^{nd} version): Every monoidal category is monoidally equivalent to a strict monoidal category.

The following notion due to Yetter embodies Mac Lane's coherence theorem in a natural, useful way,

Definition 1.6.26. *A prolongation of an arrow f in a monoidal category is a map obtained by iterated monoidal product of f with identity maps of various objects.*

Definition 1.6.27. *(Padded composition): Given a monoidal category $\mathcal{C}$ and a sequence of maps $f_1, f_2, ..., f_n \in \mathcal{C}$ such that source of f_{i+1} is isomorphic to the target of f_i by a compositions of prolongations of structure maps, we denote by $\lceil f_1, f_2, ...f_n \rceil$,*
the composition $a_0 f_1 a_1 f_2 a_2 ... a_{n-1} f_n a_n$ where a_i are composition of prolongations of structure maps and

1. *source of a_0 is reduced and completely left parenthesized*
2. *The target of a_n is reduced and completely right parenthesized*
3. *Composite is well-defined.*

Padded composition has the following properties:

1. $\lceil f_1, ..., f_n \rceil = \lceil f_1, ..., f_i \rceil \lceil f_{i+1}, ..., f_n \rceil$
2. $\lceil f_1, ..., g \otimes I, ..., f_n \rceil = \lceil f_1, ..., g, ..., f_n \rceil = \lceil f_1, ..., I \otimes g, ..., f_n \rceil$

3. $\lceil f_1, ..., f_n \rceil = \lceil f_1, ..., f_{i-1}, \lceil f_i, f_{i+1}, ..., f_l \rceil, ..., f_n \rceil$

4. $\lceil f_1, ..., g \otimes h, ..., f_n \rceil = \lceil f_1, ..., \lceil g \rceil \otimes h, ..., f_n \rceil = \lceil f_1, ..., g \otimes \lceil h \rceil, ..., f_n \rceil$

1.6.1 Deformation of Monoidal Structure Maps

Let $\mathcal{C}$ be a k-linear category and A be a k-algebra. We can form a category $\mathcal{C} \otimes_k A = \hat{\mathcal{C}}$ by extension of scalars

$Ob(\hat{\mathcal{C}}) = Ob(\mathcal{C}) \quad Hom_{\hat{\mathcal{C}}}(X, Y) = Hom_{\mathcal{C}}(X, Y) \otimes_k A$

that is, if $f \in Hom_{\hat{\mathcal{C}}}(X, Y)$ and $g \in Hom_{\hat{\mathcal{C}}}(X, Z)$ then

$$\sum_i f_i \otimes a_i \hat{\circ} \sum_j g_j \otimes b_j = \sum_{i,j} f_i \circ_{\mathcal{C}} g_j a_i b_j.$$

If A is a local ring with m as its maximal ideal, then we can extend the composition on $\mathcal{C} \otimes A$, to a composition on $\mathcal{C}\hat{\otimes}A$, the category whose hom-sets are the m-adic completions of those of $\mathcal{C} \otimes A$, by continuity. For $A = k[\epsilon]/ < \epsilon^{n+1} >$, we denote $\mathcal{C} \otimes_k A = \mathcal{C}^{(n)}$ and if $A = k[|\epsilon|]$, $\mathcal{C}\hat{\otimes}A$ is denoted by $\mathcal{C}^{(\infty)}$.

Definition 1.6.28. *Given a k-linear monoidal category $\mathcal{C}$, an n^{th} order deformation of the structure maps of the category is a monoidal category structure on $\mathcal{C}^{(n)}$ whose structural functors are the extensions of those of $\mathcal{C}$ by bilinearity and whose structural natural transformations reduce modulo ϵ to those of $\mathcal{C}$.*

Definition 1.6.29. *Given a k-linear monoidal category $\mathcal{C}$, a formal deformation of structure maps of the category is a monoidal structure on $\mathcal{C}^{(\infty)}$, whose structural functors are the extensions of those of $\mathcal{C}$ by bilinearity and continuity, and whose structural natural transformations modulo ϵ is those of $\mathcal{C}$.*

Definition 1.6.30. *(The Trivial Deformation) The deformation of a monoidal category $\mathcal{C}$ whose structural natural transformations are the images of those in $\mathcal{C}$ under extension of*

scalars. It is denoted by $\mathcal{C}^{(n)}_{triv}$*(resp.* $\mathcal{C}^{(\infty)}_{triv}$*).*

Definition 1.6.31. *Two deformations of a monoidal category are equivalent if there is a monoidal equivalence between them whose structural natural isomorphism reduce to the identity modulo* ϵ*. As we are concerned with deformations up to equivalence, we say a deformation is trivial if it is equivalent to the trivial deformation.*

1.6.2 Deformation complexes of monoidal structure maps

Definition 1.6.32. *The deformation complex of the structure maps of a monoidal category* $(\mathcal{C}, \otimes, \alpha)$ *is the cochain complex* $(X^\bullet(\mathcal{C}), \delta)$ *where*
$X^n(\mathcal{C}) = Nat({}^n\otimes, \otimes^n)$ *and*
$d_Y(\phi)_{A_0,\ldots,A_n} = \lceil A_0 \otimes \phi_{A_1,\ldots,A_n} \rceil + \sum_i (-1)^i \lceil \phi_{A_0,\ldots,A_{i-1}\otimes A_i,\ldots,A_n} \rceil + \lceil (-1)^{n+1} \phi_{A_0,\ldots,A_{n-1}\otimes A_n} \rceil$.

Theorem 1.6.1 (Yetter)**.** *The first order deformations of a monoidal category* $\mathcal{C}$ *are classified up to equivalence by* $H^3(\mathcal{C})$*.*

Now consider the n^{th} order deformation of associator α which is given by
$\tilde{\alpha} = \sum_{i=0}^{n} \alpha^{(i)} \epsilon^i \quad \alpha^{(0)} = \alpha.$
Then the obstruction to extending to the $n+1^{st}$ order deformation is

$$\omega^{(n)}_{ABCD} = \sum_{\substack{i+j=n+1 \\ 0<i,j}} \alpha^{(i)}_{A\otimes BCD} \alpha^{(j)}_{ABC\otimes D} - \sum_{\substack{i+j=n+1 \\ 0<i,j}} [\alpha^{(i)}_{ABC} \otimes D] \alpha^{(j)}_{AB\otimes CD} [A \otimes \alpha^{(k)}_{BCD}]$$

$$= \sum_{\substack{i+j=n+1 \\ 0<i,j}} \partial_1\alpha^{(i)} \partial_3\alpha^{(j)} - \sum_{\substack{i+j=n+1 \\ 0<i,j}} \partial_0\alpha^{(i)} \partial_2\alpha^{(j)} \partial_4\alpha^{(k)}$$

where $\partial_0\alpha = A \otimes \alpha_{BDC}; \partial_1\alpha = \alpha_{A\otimes BCD}; \partial_2\alpha = \alpha_{AB\otimes CD}; \partial_3\alpha = \alpha_{ABC\otimes D}; \partial_4\alpha = \alpha_{ABC} \otimes D$ that is the operator ∂_i means the objects are pre-tensored at i^{th} place. Using this shorthand notation we can easily see that $\sum_{i=0}^{4} (-1)^i \partial_i \alpha^{(1)} = 0$, i.e. the first order deformation of

associator is a 3-cocycle.

Theorem 1.6.2 (Yetter)**.** *For all n, the n^{th} order obstruction is a 4-cocycle. Thus an $(n-1)^{th}$ order deformation of a monoidal structure extends to n^{th} order deformation if and only if the cohomology class $[\omega^{(n)}] \in H^4(\mathcal{C})$ vanishes.*

Chapter 2

Deformation of Monoidal Categories

Definition 2.0.33. *(Deformations of a monoidal category) Let $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ be a k-linear monoidal category and R be a unital commutative local k-algebra with maximal ideal m. The deformation of a monoidal category $\mathcal{C}$ is an R-linear category $\hat{\mathcal{C}}$ whose objects are those of $\mathcal{C}$ and whose hom-sets are given by $\hat{\mathcal{C}}(X,Y) = \mathcal{C}(X,Y)\hat{\otimes}R$, where $\hat{\otimes}$ denotes the m-adic completion of the $\otimes$ over k whose composition, arrow-part of $\otimes$ and structural natural isomorphisms reduce modulo m to those of $\mathcal{C}$. We denote by '$\hat{x}$', the deformed structure corresponding to x.*

Note: Through out we use the convention that'.' denotes the monoidal operation when it occurs in the argument of any arrow-value operation, e.g. $a.f \otimes k$ denote $(a \otimes f) \otimes k$ when we are considering the monoidal product of a and f as the first argument of $\otimes$.
We consider one parameter deformations, that is the case of $R = k[\epsilon]/\langle \epsilon^{n+1} \rangle; n \in \mathbf{N}$ or $R = k[|\epsilon|]$. As in the review of deformation theory in Chapter 1, we call a deformation of the first sort an n^{th}-order deformation, and of the latter sort a formal deformation.

Definition 2.0.34. *Two deformations $\hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ of $\mathcal{C}$ are equivalent if there is an R-linear monoidal equivalence between them, the arrow part of whose functors and natural isomorphisms reduce modulo m to identities.*

Proposition 2.0.35. *If $\hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are n^{th} order or formal deformations of $\mathcal{C}$ and $(F, \tilde{F}, F_1)$ is a monoidal functor from $\hat{\mathcal{C}}$ to $\tilde{\mathcal{C}}$ such that $F, \tilde{F}$ and F_1 reduce to ϵ to identity functor,*

identity natural transformation and identity arrow respectively, then $\hat{\mathcal{C}}$ *and* $\tilde{\mathcal{C}}$ *are equivalent.*

Sketch of proof. $F, \tilde{F}, F_1$ each have inverses by the same calculations that shows that formal power series with leading coefficients 1 have inverses. □

Proposition 2.0.36. *An* n^{th} *order deformation (resp. formal deformation) of a monoidal category consists of*

1. *Composition* $\circ$*:* $f \hat{\circ} g = \sum_i \mu^{(i)}(f,g)\varepsilon^i$*, where* $\mu^{(0)}(f,g) = f \circ g = fg(simply)$.
2. *Tensor* $\otimes$*:* $f \hat{\otimes} k = \sum_i \tau^{(i)}(f,k)\varepsilon^i$*, where* $\tau^{(0)}(f,k) = f \otimes g$.
3. *Associator* α*:* $\hat{\alpha} = \sum_i \alpha^{(i)}\varepsilon^i$*, where* $\alpha^{(0)} = \alpha$.

when the sums are bounded by n *(resp. run from* 0 *to infinity).*

Here f and g are composable arrows but f and k are not necessarily composable ones and $i = 1, 2, ...$ The supper script '(0)' means the base structure.

Proof. It suffices to observe that by linearity (resp. linearity and completeness), it is enough to specify composition and tensor on the arrows of $\mathcal{C}$. □

By a result of Yetter[11], any deformation is equivalent to one which preserves identity arrows;

Lemma 2.0.3. *If the identity arrow of* $\mathcal{C}$ *are identity arrows in* $\hat{\mathcal{C}}$*, then*

$$\mu^{(i)}(1, f) = \mu^{(i)}(f, 1) = 0$$

for all $i > 0$.

Proof. Since $f = 1 \circ f$, $f = f + \mu^{(1)}(1,f)\epsilon + \mu^{(2)}(1,f)\epsilon^2 + ...$ This implies that

$\mu^{(i)}(1, f) = 0, \quad i > 0.$ □

Lemma 2.0.4. $\tau^{(i)}(1,1) = 0$ *for all* $i > 0$.

Proof. The first order compatibility condition for deformations is given by
$\tau^{(1)}(a,f)b \otimes g - \tau(1)(ab,fg) + a \otimes f\tau^{(1)}(b,g)$
$= \mu^{(1)}(a,b) \otimes fg - \mu^{(1)}(a \otimes f, b \otimes g) + ab \otimes \mu^{(1)}(f,g).$
This gives $\tau^{(1)}(1,1) = 0$ if we take $a = 1 = f$. Taking second order condition and with the same choices of a and f, we get $\tau^{(2)}(1,1) = 0$. Similarly, we can see for the other orders. □

Consider the case $i = 1$, *i.e.* $\epsilon^2 = 0$. That is, first order deformations. Calculation of the four conditions that must be satisfied for the deformation to be a monoidal category, and the three condition which must be satisfied for two deformations to be equivalent reveal a natural cohomological structure.

1. Associativity of composition. That is,
   $(a\hat{\circ}b)\hat{\circ}c = a\hat{\circ}(b\hat{\circ}c)$ where a, b, c are composable arrows. So, the first order condition is,

$$\mu^{(1)}(a,b)c + \mu^{(1)}(ab,c) = \mu^{(1)}(a,bc) + a\mu^{(1)}(b,c).$$

$$\Rightarrow d_H(\mu^{(1)})(a,b,c) = 0$$

   which means, $\mu^{(1)}$ is a Hochschild 2-cocycle on the complex $\mathcal{C}^\bullet$ where

$$\mathcal{C}^n = \prod_{x_0,\ldots,x_n \in Ob(\mathcal{C})} Hom_k(\mathcal{C}(x_0,x_1) \times \ldots \times \mathcal{C}(x_{n-1},x_n), \mathcal{C}(x_0,x_n)).$$

2. Middle four interchange. That is,
   $A \otimes B \xrightarrow{a\otimes f} U \otimes V \xrightarrow[ab\otimes fg]{b\otimes g} X \otimes Y.$

which is,

$$(a \otimes f) \circ (b \otimes g) = (a \circ b) \otimes (f \circ g) : A \otimes B \to X \otimes Y. \tag{2.1}$$

Then, we need

$$(a \hat{\circ} b) \hat{\otimes} (f \hat{\circ} g) = (a \hat{\otimes} f) \hat{\circ} (b \hat{\otimes} g)$$

which we call "compatibility" of tensor and composition. The first order terms give us,

$$\mu^{(1)}(a,b) \otimes fg + \tau^{(1)}(ab, fg) + ab \otimes \mu^{(1)}(f,g) = \tau^{(1)}(a,g)b \otimes g + \mu^{(1)}(a \otimes f, b \otimes g) + a \otimes f\tau^{(1)}(b,g)$$

which can be written as

$$a \otimes f\tau^{(1)}(b,g) - \tau^{(1)}(ab, fg) + \tau^{(1)}(a,g)b \otimes g = ab \otimes \mu^{(1)}(f,g) - \mu^{(1)}(a \otimes f, b \otimes g) + \mu^{(1)}(a,b) \otimes fg$$

$$\Rightarrow d_H(\tau^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = d_Y(\mu^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix}.$$

With this relation, we see an interaction between $\tau^{(1)}$ and $\mu^{(1)}$, suggesting a double complex or perhaps a multi-complex, and that the arguments are those of the second Hochschild cochain group for $\mathcal{C} \boxtimes \mathcal{C}$ where $\boxtimes$ is Deligne product. Recall that the Deligne product of two (small) k-linear categories $\mathcal{A}$ and $\mathcal{B}$ is the k-linear category with objects $Ob(\mathcal{A}) \times Ob(\mathcal{B})$ and hom-sets given by $Hom((A,B),(A',B')) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B')$, with the obvious notions of composition and identity arrow.

Here the d_Y looks like Yetter's differential, but with arrows, rather than objects as arguments.

3. Naturality of α.

   We have,

$$\alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C). \tag{2.2}$$

   If $(A \otimes B) \otimes C \overset{(a\otimes f)\otimes k}{\to} (X \otimes Y) \otimes Z$, then we have the naturality square

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{ABC}} & A \otimes (B \otimes C) \\ {\scriptstyle (a\otimes f)\otimes k}\downarrow & & \downarrow{\scriptstyle a\otimes(f\otimes k)} \\ (X \otimes Y) \otimes Z & \xrightarrow[\alpha_{XYZ}]{} & X \otimes (Y \otimes Z) \end{array}$$

   that is,

$$\alpha_{ABC}[a \otimes (f \otimes k)] = [(a \otimes f) \otimes k]\alpha_{XYZ}.$$

   If the composition is not deformed, this gives

$$\begin{aligned} &\alpha^{(1)}_{ABC}[a \otimes (f \otimes k)] + \alpha_{ABC}\tau^{(1)}(a, f \otimes k) + \alpha_{ABC}[a \otimes \tau^{(1)}(f \otimes k)] \\ &= \tau^{(1)}(a \otimes f, k)\alpha_{XYZ} + [\tau^{(1)}(a, f) \otimes k]\alpha_{XYZ} + [(a \otimes f) \otimes k]\alpha^{(1)}_{XYZ} \end{aligned}$$

   which gives,

$$\begin{aligned} &\{\alpha^{(1)}_{ABC}[a \otimes (f \otimes k)] - [(a \otimes f) \otimes k]\alpha^{(1)}_{XYZ}\} \\ &+ \alpha_{ABC}[a \otimes \tau^{(1)}(f \otimes k)] - [\tau^{(1)}(a, f) \otimes k]\alpha_{XYZ} \\ &+ \alpha_{ABC}[a \otimes \tau^{(1)}(f \otimes k)] - [\tau^{(1)}(a, f) \otimes k]\alpha_{XYZ} = 0. \end{aligned}$$

This can be written as

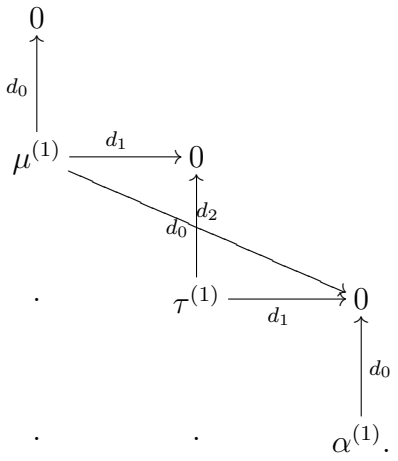
$$[d_H(\alpha^{(1)}) + d_Y(\tau^{(1)})](a, f, k) := [d_0(\alpha^{(1)}) + d_1(\tau^{(1)})](a, f, k) = 0$$

This follows the pattern of previous result and we can guess with the Hochschild's complexes on one direction and on the other with the Yetter's complexes. Assume the Hochschild's complex of composable arrows in the vertical direction and the Yetter's complex on the horizontal direction, at this point also Hochschild's coboundary and Yetter's coboundary cancel each other which is required condition for double complex. However, deforming the composition also, we get

$$\begin{aligned}
&\{\alpha^{(1)}_{ABC}[a \otimes (f \otimes k)] - [(a \otimes f) \otimes k]\alpha^{(1)}_{XYZ}\} + \alpha_{ABC}[a \otimes \tau^{(1)}(f \otimes k)] \\
&- [\tau^{(1)}(a, f) \otimes k]\alpha_{XYZ} + \alpha_{ABC}[a \otimes \tau^{(1)}(f \otimes k)] - [\tau^{(1)}(a, f) \otimes k]\alpha_{XYZ} \\
&+ \mu^{(1)}(\alpha_{ABC}, a \otimes (f \otimes k)) - \mu^{(1)}((a \otimes f) \otimes k, \alpha_{XYZ}) = 0.
\end{aligned}$$

$$\Rightarrow [d_0(\alpha^{(1)} + d_1(\tau^{(1)})](a, f, k) + d_2(\mu^{(1)})(a, f, k) = 0,$$

where $d_2(\mu^{(1)})$ inserts an instance of α as one of the arguments in both possible ways. This shows, if we deform the all of the associator, tensor, and composition, the deformation are not governed by a double complex. It is more than a double complex. We can expect it will be a multi-complex. According to the input arguments of $\mu^{(1)}, \tau^{(1)}$ and $\alpha^{(1)}$, and if we use the number of composable arrows involved as y-coordinate with 0 as the composable arrows when objects are the arguments, and number of tensorands (objects or arrows) as x-coordinate, we can think their position are at (1,2),(2,1) and (3,0) respectively. The relations we observed above can be pictorially represented as,



4. The pentagon identity.

   The condition

$$[\alpha_{ABC} \otimes D]\alpha_{AB.CD}[A \otimes \alpha_{BCD}] = \alpha_{A.BCD}\alpha_{ABC.D}, \tag{2.3}$$

   is called the pentagon identity. Again the dot '.' is used denote that objects or arrows are tensored before being used as arguments. Deforming only associator, we get

$$\lceil \alpha^{(1)}_{ABC} \otimes D\rceil - \lceil \alpha^{(1)}_{A.BCD}\rceil + \lceil \alpha^{(1)}_{AB.CD}\rceil - \lceil \alpha^{(1)}_{ABC.D}\rceil + \lceil A \otimes \alpha^{(1)}_{BCD}\rceil$$
$$\overset{Yetter}{=} \sum_{i=0}^{4}(-1)^i \partial_i \alpha^{(1)} = 0.$$

   Which is Yetter's cocycle condition of the Yetter's complex [at (4,0)]. Similarly, if we deform the tensor and associator, we get

$$\sum_{i=0}^{4}(-1)^i \partial_i \alpha^{(1)} + \tau^{(1)}(\alpha_{ABC}, D) + \tau^{(1)}(A, \alpha_{BCD}) = 0.$$

   Deforming composition also, we have

$$d_1(\alpha^{(1)}) + \tau^{(1)}(\alpha_{ABC}, D)\alpha_{AB.CD}[A \otimes \alpha_{BCD}] + [\alpha_{ABC} \otimes D]\alpha_{AB.CD}\tau^{(1)}(A, \alpha_{BCD})$$
$$+ \mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD})[A \otimes \alpha_{BCD}] + \mu^{(1)}([\alpha_{ABC} \otimes D]\alpha_{AB.CD}, A \otimes \alpha_{BCD})$$
$$- \mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D}) = 0.$$

Under Yetter's notation of padded composition but suppressing $\lceil\rceil$ marks, it is read as

$$d_1(\alpha^{(1)}) + \tau^{(1)}(\alpha_{ABC}, D) + \tau^{(1)}(A, \alpha_{BCD}) + \mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD})$$
$$+ \mu^{(1)}([\alpha_{ABC} \otimes D]\alpha_{AB.CD}, A \otimes \alpha_{BCD}) - \mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D}) = 0$$

which, then

$$[d_1(\alpha^{(1)}) + d_2(\tau^{(1)}) + d_3(\mu^{(1)})](A, B, C, D) = 0.$$

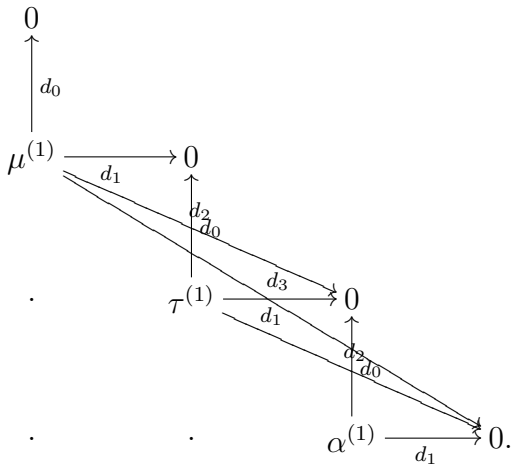
where

$$d_2(\tau^{(1)})(A, B, C, D) = \lceil\tau^{(1)}(\alpha_{ABC}, D)\rceil + \lceil\tau^{(1)}(A, \alpha_{BCD})\rceil$$

and

$$d_3(\mu^{(1)})(A, B, C, D) = \lceil\mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD})\rceil$$
$$+ \lceil\mu^{(1)}([\alpha_{ABC} \otimes D]\alpha_{AB.CD}, A \otimes \alpha_{BCD})\rceil - \lceil\mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D})\rceil.$$

This fills the another component for the multi-complex



Theorem 2.0.37. *$\mu^{(1)}, \tau^{(1)}, \alpha^{(1)}$ give a first order deformation if and only if they collectively are a cocycle in the total complex of a multicomplex with low order differentials are given by d_0 is the Hochschild differential, d_1 is the Yetter differential and*

(a) $d_2(\mu^{(1)})(a, f, k) = \mu^{(1)}(\alpha_{ABC}, a \otimes f.k) - \mu^{(1)}(a.f \otimes k, \alpha_{XYZ})$

(b) $d_3(\mu^{(1)})(A, B, C, D) = \lceil \mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD}) \rceil$
$+ \lceil \mu^{(1)}([\alpha_{ABC} \otimes D]\alpha_{AB.CD}, A \otimes \alpha_{BCD}) \rceil - \mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D}).$

(c) $d_2(\tau^{(1)})(A, B, C, D) = \lceil \tau^{(1)}(\alpha_{ABC}, D) \rceil + \lceil \tau^{(1)}(A, \alpha_{BCD}) \rceil.$

Let us now consider the conditions needed for the first order deformations to be equivalent:

5. Composition preservation
   $F(f \circ g) = F(f) \circ F(g)$ implies
   $F^{(1)}(fg) + \mu^{(1)}(f, g) = F^{(1)}(f)f + \nu^{(1)}(f, g) + fF^{(1)}(g)$

$\Rightarrow d_H(F^{(1)})(f;g) = [\nu^{(1)} - \mu^{(1)}](f;g).$

6. Natural Transformation

$$\tilde{F} : F(- \otimes -) \to F(-) \otimes F(-) \tag{2.4}$$

where $F : \mathcal{C} \to \mathcal{D}$ is a functor from monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{D}$. Then the naturality square

$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{\tilde{F}_{A,B}} & F(A) \otimes F(B) \\ {\scriptstyle F(f\otimes k)}\downarrow & & \downarrow{\scriptstyle F(f)\otimes F(k)} \\ F(X \otimes Y) & \xrightarrow[\tilde{F}_{X,Y}]{} & F(X) \otimes F(Y) \end{array}$$

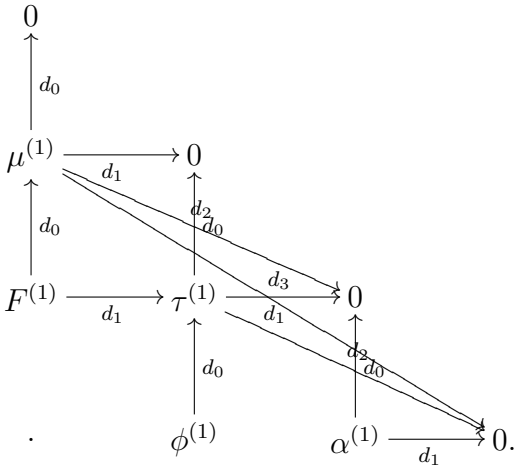
gives that $\tilde{F}_{A,B}F(f) \otimes F(k) = F(f \otimes k)\tilde{F}_{X,Y}$ which implies that, if
$\hat{\tilde{F}} = Id_{\mathcal{C}} + \phi^{(1)}\epsilon$, and $\hat{F} = Id_{\mathcal{C}} + F^{(1)}\epsilon$

$$\phi^{(1)}_{A,B} f \otimes k + F^{(1)}(f) \otimes k + f \otimes F^{(1)}(k) + \rho^{(1)}(f,k) = F^{(1)}(f \otimes k) + f \otimes k\phi^{(1)}_{X,Y} + \tau^{(1)}(f,k)$$

which is equivalent to say

$$[d_1(F^{(1)}) + d_0(\phi^{(1)})](f \otimes k) = \tau^{(1)}(f,k) - \rho^{(1)}(f,k).$$

Then the above picture is extended to,



7. Hexagon condition

$$\begin{array}{ccc} F((A\otimes B)\otimes C) & \xrightarrow{F(\alpha)} & F(A\otimes(B\otimes C)) \\ \uparrow{\scriptstyle \hat{F}_{A\otimes B,C}} & & \uparrow{\scriptstyle \hat{F}_{A,B\otimes C}} \\ F(A\otimes B)\otimes F(C) & & F(A)\otimes F(B\otimes C) \\ \uparrow{\scriptstyle \tilde{F}_{A,B}\otimes F(C)} & & \uparrow{\scriptstyle F(A)\otimes\tilde{F}_{B,C}} \\ [F(A)\otimes F(B)]\otimes F(C) & \xrightarrow[\alpha_{F(A),F(B),F(C)}]{} & F(A)\otimes[F(B)\otimes F(C)] \end{array}$$

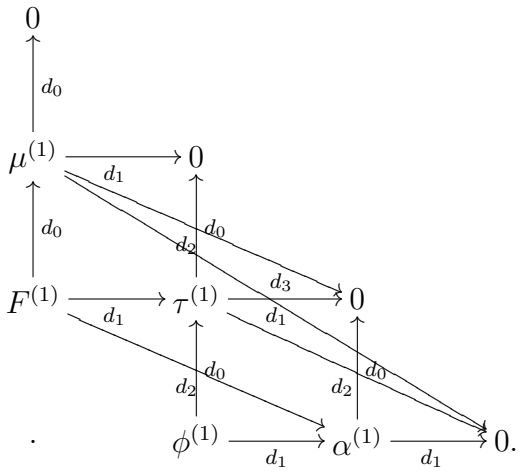
That is, $\tilde{F}_{A\otimes B,C}[\tilde{F}_{A,B}\otimes F(C)]\alpha_{F(A),F(B),F(C)} = F(\alpha)\tilde{F}_{A,B\otimes C}[F(A)\otimes\tilde{F}_{B,C}$ which implies that, if the deformation of F is $\hat{F} = Id_{\mathcal{C}} + F^{(1)}\epsilon$ and let the deformation of $\tilde{F}$ is given by $\hat{\tilde{F}} = Id_{\mathcal{C}} + \phi^{(1)}\epsilon$ with $\epsilon^2 = 0$, then,
$\lceil\phi^{(1)}_{A\otimes B,C}\rceil + \lceil\phi^{(1)}_{A,B}\otimes C\rceil + \lceil\alpha^{(1)}_{ABC}\rceil = \lceil F^{(1)}(\alpha)\rceil + \lceil a^{(1)}_{ABC}\rceil + \lceil\phi^{(1)}_{A,B\otimes C}\rceil + \lceil A\otimes\phi^{(1)}_{B,C}\rceil$.

That is,

$$[d_1(\phi^{(1)}) + d_2(F^{(1)})](A,B,C) = \lceil\alpha^{(1)}_{ABC}\rceil - \lceil a^{(1)}_{ABC}\rceil.$$

This says that if we don't deform the monoidal functor F, then the difference of two first order deformations of associator is cobounded by the Yetter coboundary of the first order deformation of monoidal natural transformation. But if the functor F is also deformed, the quantity $d_2(F^{(1)})$ is involved. The quantity $d_2(F^{(1)})$ is not necessarily zero. So, in this case, the Yetter complex is no more a chain complex because there is interaction of deformation of lax functor.

The pictorial representation of the multicomplex up to this point looks like



Also, we have collected those instances of differentials which are different than those of Hochschild and Yetter which arise consider first order deformations up to equivalence.

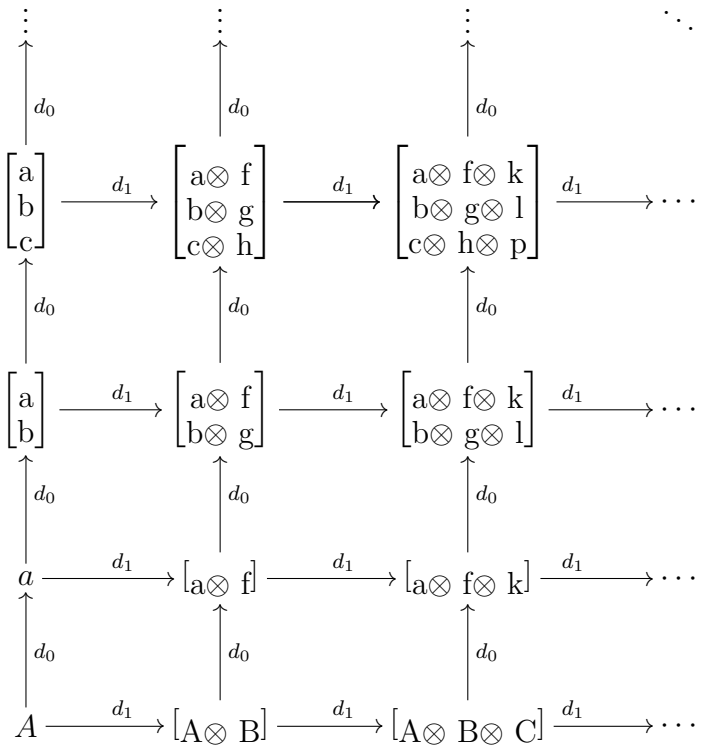
1. $d_2(F^{(1)})(A,B,C) = F^{(1)}(\alpha_{A,B,C})$
2. $d_2(\tau^{(1)})(A,B,C,D) = \tau^{(1)}(A,\alpha_{B,C,D}) - \tau^{(1)}(\alpha_{A,B,C},D)$
3. $d_2(\mu^{(1)})(a,f,k) = \mu^{(1)}(a.f\otimes k,\alpha_{X,Y,Z}) - \mu^{(1)}(\alpha_{A,B,C},a\otimes f.k)$, and
4. $d_3(\mu^{(1)})(A,B,C,D) = \lceil\mu^{(1)}(\alpha_{A,B,C}\otimes D,\alpha_{A,B.C,D})\rceil$
   $+\lceil\mu^{(1)}(\alpha_{A,B,C}\otimes D\alpha_{A,B.C,D},A\otimes\alpha_{B,C,d})\rceil - \lceil\mu^{(1)}(\alpha_{A.B,C,D}\otimes C,\alpha_{A,B,C.D})\rceil$

Thus, we have,

Theorem 2.0.38. *The first order deformations of monoidal category are equivalent if and only if the triples* $(\mu^{(1)}, \tau^{(1)}, \alpha^{(1)})$ *and* $(\nu^{(1)}, \rho^{(1)}, a^{(1)})$ *differ by coboundary of a pair* $(F^{(1)}, \phi^{(1)})$*, that is, if and only if they are cohomologous in the total complex of the multi-complex.*

That is, the first cohomology of the total complex classifies the deformation of k-linear monoidal category up to equivalence.

We have also determined the inputs for each cochain group in the lower left corner of the multi-complex. Omitting higher differentials, the multi-complex looks like



with other differentials running right and down and having total degree one. This is the

expected multi-complex.

Hence, at (p,q) of the multi-complex, if we let ${}^{p}\otimes : \mathcal{C}^{\boxtimes p} \to \mathcal{C}$ ($\otimes^{p} : \mathcal{C}^{\boxtimes p} \to \mathcal{C}$) be the functor, giving left (resp. right) parenthesize a tensor product,

$$C^{(p,q)} := \prod_{x_i \in Ob(\mathcal{C}^{\boxtimes p}); i=0,\ldots,q} Hom_k(\mathcal{C}^{\boxtimes p}(x_0,x_1)\otimes\mathcal{C}^{\boxtimes p}(x_1,x_2)\otimes\cdots\otimes\mathcal{C}^{\boxtimes p}(x_{q-1},x_q), \mathcal{C}({}^{p}\otimes x_0, \otimes^{p} x_q))$$

where $\boxtimes$ is the Deligne product. The p^{th} column of the multicomplex is the normalized Hochschild complex of parallel functors ${}^{p}\otimes$ and $\otimes^{p}$ in the sense of Definition 1.4.1.
This is a foundation of multi-complex. The results above give us instances for the formulas of the differentials in case of first order deformation of monoidal structures. In general,

Conjecture 2.0.39. *There exists a multicomplex $(C^{(\bullet,\bullet)})$ with the underlying graded module of the previous paragraph and differentials,*

$$d_j^{p,q} : C^{p,q} \to C^{p+j,q-j+1}; q \geq 0, p \geq 1, q \geq j \geq 0,$$

such that $d_0 = d_H, d_1 = d_Y$ and $d_j; j \geq 2$ as in the pictures above, is the sum (with appropriate signs) of all possible insertion of $j-1$ associators or $j-1$ prolongations of the associator in the argument of argument of d_j.

2.0.3 Obstructions to deformation of a k-linear monoidal category

We now consider the conditions on higher order terms in deformations. As expected, these give rise to a sequence of obstructions which must be cobounded (i.e. vanish in cohomology) for a higher order deformations to exist. We are concerned with three structures, composition '$\circ$', tensor '$\otimes$', and associator 'α'. Properly speaking, an obstruction is a cochain in the total complex of the multicomplex, with direct summands at (0,4), (1,3), (2,2) and (4,0). Each of these direct summands has an interpretation as obstructing one of the properties

of a monoidal category, associativity of composition, middle-four-interchange, naturality of the associator, and the pentagon conditions, respectively. By abuse of language, we refer to each of the summands as "an obstruction". Similarly the condition that the obstruction be a cocycle can be decomposed into five conditions, which we refer to as "being a cocycle at (a,b), for various bi-indices"

2.0.4 Obstruction to deformation of the associativity of composition

Consider the deformation of composition. Whether or not other structures are deformed, the new composition must be associative. That is,

$$(a\hat{\circ}b)\hat{\circ}c = a\hat{\circ}(b\hat{\circ}c).$$

So then, collecting the degree n terms on both sides and equating coefficients, we have, for each n,

$$\sum_{\substack{i+j=n \\ 0\leq i,j\leq n}} \mu^{(i)}(\mu^{(j)}(a,b),c) = \sum_{\substack{i+j=n \\ 0\leq i,j\leq n}} \mu^{(i)}(a,\mu^{(j)}(b,c)). \tag{2.5}$$

For $n = 0$, the relation is just associativity of the original composition. For $n = 1$,

$$a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}(a,b)c = d_H(\mu^{(1)})(a,b,c) = 0$$

which is the Hochschild 2-cocycle condition of infinitesimal, $\mu^{(1)}$, the infinitesimal deformation of the composition. Now for $n = 2$,

$$a\mu^{(2)}(b,c) - \mu^{(2)}(ab,c) + \mu^{(2)}(a,bc) - \mu^{(2)}(a,b)c = \mu^{(1)}(a,\mu^{(1)}(b,c)) - \mu^{(1)}(\mu^{(1)}(a,b),c),$$

that is,

$$d_H(\mu^{(2)})(a,b,c) = \mu^{(1)}(a,\mu^{(1)}(b,c)) - \mu^{(1)}(\mu^{(1)}(a,b),c).$$

If the right hand side is zero, $\mu^{(2)}$ is also a Hochschild 2-cocycle. In this case we can extend our deformation of composition to the second order. It's making it sound like the obstruction has to be zero as a cochain for there to be a higher order deformation, not zero in cohomology. This expression is called that primary obstruction of deformation of composition whether it is zero or not. We denote it by $\omega_c^{(1)}(a,b,c)$. In general, the n^{th} order obstruction of composition is denoted by

$$\omega_c^{(n)}(a,b,c) := \sum_{\substack{i+j=n+1 \\ 0\leq i,j\leq n}} [\mu^{(i)}(\mu^{(j)}(a,b),c) - \mu^{(i)}(a,\mu^{(j)}(b,c))]. \tag{2.6}$$

This obstruction lies at (1,3). If the obstructions are coboundary for $k \leq n$, we want to extend our deformation to $(n+1)^{th}$ order. Gerstenhaber proved that the obstructions of deformation of composition are cocycle. So, in our multi-complex case, on the Hochschild direction, the obstructions of composition are cocycles by Gerstenhaber.

2.0.5 Obstruction to deformation of compatibility between composition and tensor

The compatibility of composition and tensor, often called middle-four interchange, is:

$$(a \circ b) \otimes (f \circ g) = (a \otimes f) \circ (b \otimes g).$$

A similar argument to that given above shows that, the n^{th} obstruction to compatibility is given by, for all $n \geq 1$,

$$\omega_{cp}^{(n)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} := \sum_{\substack{i+j+k=n+1 \\ 0\leq i,j,k\leq n}} [\tau^{(i)}(\mu^{(j)}(a,b),\mu^{(k)}(f,g)) - \mu^{(i)}(\tau^{(j)}(a,f),\tau^{(k)}(b,g))]. \tag{2.7}$$

This obstruction lies at (2,2). For obstructions to be cocycles in our multi-complex, we need to show that the Hochschild coboundary of obstruction of compatibility and Yetter's coboundary of obstruction of composition should cancel each other.

2.0.6 Obstruction to deformation of naturality of associator

We have, the naturality condition of associator, α is

$$\alpha_{A,B,C}a \otimes (f \otimes k) = (a \otimes f) \otimes k\alpha_{X,Y,Z},$$

which we write,

$$\alpha_{ABC}a \otimes f.k = a.f \otimes k\alpha_{XYZ},$$

where as always the '.' is for tensor when the result is an argument in another operation and the composition operation '$\circ$' is dropped.

The n^{th} order obstruction of this identity is given by, for all $n \geq 1$,
$\omega_N^{(n)}(a, f, k)$

$$:= \sum_{\substack{i+j+k+l=n+1 \\ 0\leq i,j,k,l\leq n}} [\mu^{(i)}(\alpha_{ABC}^{(j)}, \tau^{(k)}(a, \tau^{(l)}(f, k))) - \mu^{(i)}(\tau^{(j)}(\tau^{(k)}(a, f), k), \alpha_{XYZ}^{(l)})]. \tag{2.8}$$

This obstruction lies at (3,1). To see that the obstructions are cocycles, we need to see that the d_0 coboundary of $\omega_N^{(n)}$ cancels the sum of the d_1 coboundary of $\omega_{cp}^{(n)}$ and d_2 coboudary of $\omega_c^{(n)}$.

2.0.7 Obstruction to deformation of pentagon condition

We have the pentagon condition as above. The n^{th} order obstruction of the structures of a k-linear category due to this identity is given by, for all $n \geq 1$,

$\omega_P^{(n)}(A,B,C,D)$

$$:= \sum_{\substack{i+j+k+l+r+s+t=n+1 \\ 0\leq i,j,k,l,r,s,t\leq n}} \mu^{(i)}(\mu^{(j)}(\tau^{(k)}(\alpha_{ABC}^{(l)},D),\alpha_{AB.CD}^{(r)}),\tau^{(s)}(A,\alpha_{BCD}^{(t)}))$$

$$- \sum_{\substack{i+j+k=n+1 \\ 0\leq i,j,k\leq n}} \mu^{(i)}(\alpha_{A.BCD}^{(j)},\alpha_{ABC.D}^{(k)}). \tag{2.9}$$

This obstruction lies at (4,0). To see the obstructions are cocycle, we need to show that d_0 coboundary of $\omega_P^{(n)}$cancels with the sum of d_1 coboundary of $\omega_N^{(n)}$, d_2 coboundary of $\omega_{cp}^{(n)}$ and d_3 coboundary of $\omega_c^{(n)}$ and d_1 coboundary of $\omega_P^{(n)}$ cancel with the sum of d_2 coboundary of $\omega_N^{(n)}$, d_3 coboundary of $\omega_{cp}^{(n)}$ and d_4 coboundary of $\omega_c^{(n)}$ with appropriate signs, analogous to the Drinfel'd algebra case considered by Markl and Shnider.

2.1 Obstructions are Cocycles

The calculations of the coboundary of obstructions, with current techniques, is much too complicated in the general case. We prove the standard result that obstructions are cocycles only in special cases.

Since every monoidal category is monoidally equivalent to strict monoidal category, throughout the following, we will consider only deformation of strict monoidal category. That is, the undeformed structure maps of monoidal category considered all identities.

2.1.1 Deforming one Structure at a Time

1. *Deforming composition only*

Theorem 2.1.1. *All obstructions are cocycles when only composition is deformed.*

Sketch of proof. WE give an explicit computational proof only on the case of the primary obstruction.

We have,

(a) $\omega_c^{(1)}(a,b,c) = \mu^{(1)}(\mu^{(1)}(a,b),c) - \mu^{(1)}(a,\mu^{(1)}(b,c))$

(b) $\omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} = \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)$

(c) $\omega_N^{(1)}(a,f,k) = 0$

(d) $\omega_P^{(1)}(A,B,C,D) = \mu^{(1)}(\mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD}), A \otimes \alpha_{BCD})$

In this case, at $(4,0)$, $d_0(\omega_c^{(1)})(a,b,c,d) = 0$ by Gerstenhaber[2]. For calculation at $(2,3)$, the Yetter coboundary of obstruction of composition is

$$d_1(\omega_c^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix}$$
$$= abc \otimes \omega_c^{(1)} \begin{pmatrix} f \\ g \\ h \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} a \otimes f \\ b \otimes g \\ c \otimes h \end{pmatrix} + \omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes fgh$$

and the Hochschild coboundary

$$d_1(\omega_{cp}^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = a \otimes f\omega_{cp}^{(1)} \begin{pmatrix} b & g \\ c & h \end{pmatrix} - \omega_{cp}^{(1)} \begin{pmatrix} ab & fg \\ c & h \end{pmatrix} + \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ bc & gh \end{pmatrix}$$
$$- \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} c \otimes h,$$

which means

$$d_1(\omega_c^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = abc \otimes [\mu^{(1)}(\mu^{(1)}(f,g),h) - \mu^{(1)}(f,\mu^{(1)}(g,h))]$$
$$- \mu^{(1)}(\mu^{(1)}(a \otimes f, b \otimes g), c \otimes h) + \mu^{(1)}(a \otimes f, \mu^{(1)}(b \otimes g, c \otimes h))$$
$$+ [\mu^{(1)}(\mu^{(1)}(a,b),c) + \mu^{(1)}(a,\mu^{(1)}(b,c))] \otimes fgh$$

and

$$d_0(\omega_{cp}^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix}$$
$$= a \otimes f\mu^{(1)}(b,c) \otimes \mu^{(1)}(f,g) - \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h)$$
$$+ \mu^{(1)}(a,bc) \otimes \mu^{(1)}(f,gh) - \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)c \otimes h.$$

Since, the first order condition is that, $\mu^{(1)}$ is a Hochschild cocycle, the compatibility condition gives

$$\mu^{(1)}(a \otimes f, b \otimes g) = \mu^{(1)}(a,b) \otimes fg + ab \otimes \mu^{(1)}(f,g),$$

$$d_0(\omega_{cp}^{(1)}) = abc \otimes [\mu^{(1)}(\mu^{(1)}(f,g),h) - \mu^{(1)}(f,\mu^{(1)}(g,h))]$$
$$- \mu^{(1)}(\mu^{(1)}(a,b) \otimes fg + ab \otimes \mu^{(1)}(f,g), c \otimes h)$$
$$+ \mu^{(1)}(a \otimes f, \mu^{(1)}(b,c) \otimes gh + bc \otimes \mu^{(1)}(g,h))$$
$$+ [\mu^{(1)}(\mu^{(1)}(a,b),c) + \mu^{(1)}(a,\mu^{(1)}(b,c))] \otimes fgh$$

$$= abc \otimes \mu^{(1)}(\mu^{(1)}(f,g),h) - abc \otimes \mu^{(1)}(f,\mu^{(1)}(g,h))$$
$$- \mu^{(1)}(\mu^{(1)}(a,b),c) \otimes fgh - \mu^{(1)}(a,b)c \otimes \mu^{(1)}(fg,h)$$
$$- \mu^{(1)}(ab,c) \otimes \mu^{(1)}(f,g)h - abc \otimes \mu^{(1)}(\mu^{(1)}(f,g),h)$$
$$+ \mu^{(1)}(a,\mu^{(1)}(b,c)) \otimes fgh + a\mu^{(1)}(b,c) \otimes \mu^{(1)}(f,gh)$$
$$+ \mu^{(1)}(a,bc) \otimes f\mu^{(1)}(g,h) + abc \otimes \mu^{(1)}(f,\mu^{(1)}(g,h))$$
$$+ \mu^{(1)}(\mu^{(1)}(a,b),c) \otimes fgh + \mu^{(1)}(a,\mu^{(1)}(b,c)) \otimes fgh.$$

So,
$$d_0(\omega_{cp}^{(1)}) + d_1(\omega_c)$$

$$= -\mu^{(1)}(ab,c) \otimes \mu^{(1)}(f,g)h - \mu^{(1)}((a,b)c \otimes \mu^{(1)}(fg,h)$$
$$+ \mu^{(1)}(a,bc) \otimes f\mu^{(1)}(g,h) + a\mu^{(1)}(b,c) \otimes \mu^{(1)}(f,gh)$$

$$+ a\mu^{(1)}(b,c) \otimes f\mu^{(1)}(g,h) - \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h)$$
$$+ \mu^{(1)}(a,bc) \otimes \mu^{(1)}(f,gh) - \mu^{(1)}(a,b)c \otimes \mu^{(1)}(f,g)h$$

$$= -\mu^{(1)}(ab,c) \otimes \mu^{(1)}(f,g)h - \mu^{(1)}((a,b)c \otimes \mu^{(1)}(fg,h)$$
$$- \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h) - \mu^{(1)}(a,b)c \otimes \mu^{(1)}(f,g)h$$
$$+ \mu^{(1)}(a,bc) \otimes [f\mu^{(1)}(g,h) + \mu^{(1)}(f,gh)]$$
$$+ a\mu^{(1)}(b,c) \otimes [\mu^{(1)}(f,gh) + f\mu^{(1)}(g,h)]$$

$$= -\mu^{(1)}(ab,c) \otimes \mu^{(1)}(f,g)h - \mu^{(1)}((a,b)c \otimes \mu^{(1)}(fg,h)$$
$$- \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h) - \mu^{(1)}(a,b)c \otimes \mu^{(1)}(f,g)h$$
$$+ \mu^{(1)}(a,bc) \otimes [\mu^{(1)}(fg,h) + \mu^{(1)}(f,g)h]$$
$$+ a\mu^{(1)}(b,c) \otimes [\mu^{(1)}(f,g)h + \mu^{(1)}(fg,h)]$$

$$= [a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}((a,b)c]$$
$$\otimes [\mu^{(1)}(f,g)h + \mu^{(1)}(fg,h)] = 0.$$

At $(3,2)$, the coboundaries of the obstructions are

$$d_1(\omega_{cp}^{(1)}) \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix}$$
$$= ab \otimes \omega_{cp}^{(1)} \begin{pmatrix} f & k \\ g & l \end{pmatrix} - \omega_{cp}^{(1)} \begin{pmatrix} a \otimes f & k \\ b \otimes g & l \end{pmatrix} + \omega_{cp}^{(1)} \begin{pmatrix} a & f \otimes k \\ b & g \otimes l \end{pmatrix}$$
$$- \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} \otimes kl$$
$$=ab \otimes \mu^{(1)}(f,g) \otimes \mu^{(1)}(k,l) - \mu^{(1)}(a \otimes f, b \otimes g)\mu^{(1)}(k,l)$$
$$+ \mu^{(1)}(a,b) \otimes \mu^{(1)}(f \otimes k, g \otimes l) - \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g) \otimes kl$$
$$+ \mu^{(1)}(a,b) \otimes fg \otimes \mu^{(1)}(k,l) = 0$$

Since $\omega_N^{(1)}(a,f,k) = 0$; $\omega_P^{(1)}(A,B,C,D) = 0$, in the strict case by the identity preserving lemmas, we have,

$$d_2(\omega_{cp})\begin{pmatrix} a & f & k & m \end{pmatrix} = \omega_{cp}^{(1)}\begin{pmatrix} \alpha_{ABC} & 1_D \\ a \otimes f.k & m \end{pmatrix} + \omega_{cp}^{(1)}\begin{pmatrix} 1_A & \alpha_{BCD} \\ a & f \otimes k.m \end{pmatrix}$$
$$-\omega_{cp}^{(1)}\begin{pmatrix} a & f \otimes k.m \\ 1_X & \alpha_{YZW} \end{pmatrix} - \omega_{cp}^{(1)}\begin{pmatrix} a & f \otimes k.m \\ \alpha_{XYZ} & 1_W \end{pmatrix} = 0$$

$$d_3(\omega_{cp}^{(1)})(A,B,C,D) = \omega_{cp}^{(1)}\begin{pmatrix} \alpha_{ABC} \otimes D & E \\ \alpha_{AB.CD} & E \end{pmatrix} - \omega_{cp}^{(1)}\begin{pmatrix} \alpha_{AB.CD} & E \\ A \otimes \alpha_{BCD} & E \end{pmatrix}$$
$$+ \omega_{cp}^{(1)}\begin{pmatrix} A & \alpha_{BCD} \otimes E \\ A & \alpha_{BC.DE} \end{pmatrix} + \omega_{cp}^{(1)}\begin{pmatrix} A & \alpha_{BC.DE} \\ A & B \otimes \alpha_{CDE} \end{pmatrix} = 0$$

and similarly $d_4(\omega_c^{(1)}) = 0 = d_3(\omega_c^{(1)})$. Hence, in this case, the obstructions are cocycles.

□

2. *Deforming tensor only*

Theorem 2.1.2. *All obstructions are cocycles when only tensor is deformed.*

Sketch of proof. Again we give a computational proof only in the case of the primary obstruction. We have,

$$a\hat{\otimes}f = a \otimes f + \tau^{(1)}(a,f)\epsilon.$$

Then

(a) $d_0(\tau^{(1)})\,(a\ fb\ g)$
$= a \otimes f\tau^{(1)}(f,k) - \tau^{(1)}(ab,fg)$

$$+ \tau^{(1)}(a, f)b \otimes g = 0$$

(b) $d_1(\tau^{(1)})\,(a\ f\ k) = a \otimes \tau^{(1)}(f, k) - \tau^{(1)}(a \otimes f, k)$
$+ \tau^{(1)}(a, f \otimes k) - \tau^{(1)}(a, f) \otimes k = 0$(Under padding)

and

(c) $\omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

(d) $\omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = \tau^{(1)}(a, f)\tau^{(1)}(b, g)$

(e) $\omega_N^{(1)} \begin{pmatrix} a & f & k \end{pmatrix} = \tau^{(1)}(\tau^{(1)}(a, f), k) - \tau^{(1)}(a, \tau^{(1)}(f, k))$

(f) $\omega_p^{(1)} \begin{pmatrix} A & B & C & D \end{pmatrix} = \lceil \tau^{(1)}(\alpha_{ABC}, D)\tau^{(1)}(A, \alpha_{BCD}\rceil = 0.$

At (1,4), result follows trivially in this case. At (2,3),

$$d_1(\omega_c^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = abc \otimes \omega_c^{(1)} \begin{pmatrix} f \\ g \\ h \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} a \otimes f \\ b \otimes g \\ c \otimes h \end{pmatrix}$$
$$+ \omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes fgh = 0$$

$$d_0(\omega_{cp}^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = a \otimes f\omega_{cp}^{(1)} \begin{pmatrix} b & g \\ c & h \end{pmatrix} - \omega_{cp}^{(1)} \begin{pmatrix} ab & fg \\ c & h \end{pmatrix}$$
$$+ \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ bc & gh \end{pmatrix} \otimes fgh - \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} c \otimes h$$
$$= a \otimes f\tau^{(1)}(b,g)\tau^{(1)}(c,h) - \tau^{(1)}(ab,fg)\tau^{(1)}(c,h)$$
$$+ \tau^{(1)}(a,f)\tau^{(1)}(bc,gh) - \tau^{(1)}(a,f)\tau^{(1)}(b,g)c \otimes h$$
$$= [a \otimes f\tau^{(1)}(b,g) - \tau^{(1)}(ab,fg)]\tau^{(1)}(c,h)$$
$$+ \tau^{(1)}(a,f)[\tau^{(1)}(bc,gh) - \tau^{(1)}(b,g)c \otimes h]$$
$$= -\tau^{(1)}(a,f)b \otimes g\tau^{(1)}(c,h) + \tau^{(1)}(a,f)b \otimes g\tau^{(1)}(c,h) = 0$$

At(3,2),

$$d_2(\omega_c^{(1)}) \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix} = \omega_c^{(1)} \begin{pmatrix} a.f \otimes k \\ b.g \otimes l \\ \alpha_{XYZ} \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} a.f \otimes k \\ \alpha_{UVW} \\ b \otimes g.l \end{pmatrix}$$
$$+ \omega_c^{(1)} \begin{pmatrix} \alpha_{ABC} \\ a \otimes f.k \\ b \otimes g.l \end{pmatrix} = 0$$

$$d_0(\omega_N^{(1)}) \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix}$$
$$= [\text{a b c }]_L\omega_N^{(1)} \begin{pmatrix} a & f & k \end{pmatrix} - \omega_N^{(1)} \begin{pmatrix} ab & fg & kl \end{pmatrix} + \omega_N^{(1)} \begin{pmatrix} a & f & k \end{pmatrix} [\text{f g h}]_R$$

$$= [\text{a b c }]_L[\tau^{(1)}(\tau^{(1)}(b,g),l) - \tau^{(1)}(b,\tau^{(1)}(g,l))]$$
$$+ \tau^{(1)}(ab,\tau^{(1)}(fg,kl)) + \tau^{(1)}(\tau^{(1)}(a,f),k)[\text{f g h}]_R - \tau^{(1)}(a,\tau^{(1)}(f,k))[\text{f g h}]_R$$

$$= [\text{a b c }]_L\tau^{(1)}(\tau^{(1)}(b,g),l) - [\text{a b c }]_L\tau^{(1)}(b,\tau^{(1)}(g,l))$$
$$- \tau^{(1)}(\tau^{(1)}(a,f)b \otimes g,kl) - \tau^{(1)}(a \otimes f\tau^{(1)}(b,g),kl)$$
$$+ \tau^{(1)}(ab,\tau^{(1)}(f,k)g \otimes l) + \tau^{(1)}(ab,f \otimes k\tau^{(1)}(g,l))$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f),k)[\text{f g h}]_R - \tau^{(1)}(a,\tau^{(1)}(f,k))[\text{f g h}]_R$$

$$= [\text{a b c }]_L\tau^{(1)}(\tau^{(1)}(b,g),l) - [\text{a b c }]_L\tau^{(1)}(b,\tau^{(1)}(g,l))$$
$$- \tau^{(1)}(\tau^{(1)}(a,f),k)b.g \otimes l - \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b \otimes g,l)$$

$$- \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b, g) \otimes l - a.f \otimes k\tau^{(1)}(\tau^{(1)}(b, g), l)$$
$$+ a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g \otimes l) + \tau^{(1)}(a, \tau^{(1)}(f, k))b \otimes g.l$$
$$+ \tau^{(1)}(a, f \otimes k)b \otimes \tau^{(1)}(g, l) + a \otimes f.k\tau^{(1)}(b, \tau^{(1)}(g, l))$$
$$+ \tau^{(1)}(\tau^{(1)}(a, f), k)[\text{f g h}]_R - \tau^{(1)}(a, \tau^{(1)}(f, k))[\text{f g h}]_R$$

$$= -\tau^{(1)}(a \otimes f, k)\tau^{(1)}(b, g) \otimes l + a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g \otimes l)$$
$$+ \tau^{(1)}(a, f \otimes k)b \otimes \tau^{(1)}(g, l) - \tau^{(1)}(a, f) \otimes k\tau^{(1)}(b \otimes g, l)$$

$$d_1(\omega_{cp}^{(1)}) \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix} = ab \otimes \omega_{cp}^{(1)} \begin{pmatrix} f & k \\ g & l \end{pmatrix}$$
$$- \omega_{cp}^{(1)} \begin{pmatrix} a \otimes f & k \\ b \otimes g & l \end{pmatrix} + \omega_{cp}^{(1)} \begin{pmatrix} a & f \otimes k \\ b & g \otimes l \end{pmatrix}$$
$$- \omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} \otimes kl$$

$$= ab \otimes \tau^{(1)}(f, k)\tau^{(1)}(g, l) - \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b \otimes g, l)$$
$$+ \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b, g \otimes l) - \tau^{(1)}(a, f)\tau^{(1)}(b, g) \otimes kl$$

$$= ab \otimes \tau^{(1)}(f, k)\tau^{(1)}(g, l) - [a \otimes \tau^{(1)}(f, k) + \tau^{(1)}(a, f \otimes k)$$
$$- \tau^{(1)}(a.f) \otimes k][b \otimes \tau^{(1)}(g, l) + \tau^{(1)}(b, g \otimes l) - \tau^{(1)}(b, g) \otimes l]$$
$$+ \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b, g \otimes l) - \tau^{(1)}(a, f)\tau^{(1)}(b, g) \otimes kl$$

$$= -a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g \otimes l) + a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g) \otimes l$$
$$- \tau^{(1)}(a, f) \otimes k)b \otimes \tau^{(1)}(g, l) + \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b, g) \otimes l$$
$$+ \tau^{(1)}(a, f) \otimes kb \otimes \tau^{(1)}(g, l) + \tau^{(1)}(a, f) \otimes k\tau^{(1)}(b, g \otimes l)$$
$$- \tau^{(1)}(a, f) \otimes k\tau^{(1)}(b, g) \otimes l - \tau^{(1)}(a, f) \otimes k\tau^{(1)}(b, g) \otimes l$$

Thus combining, we get,

$$[d_0(\omega_N^{(1)}) + d_1(\omega_{cp}^{(1)})] \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix}$$

$$= -a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b, g \otimes l) + a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b,g) \otimes l$$
$$- \tau^{(1)}(a,f) \otimes k)b \otimes \tau^{(1)}(g,l) + \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b,g) \otimes l$$
$$+ \tau^{(1)}(a,f) \otimes kb \otimes \tau^{(1)}(g,l) + \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b, g \otimes l)$$
$$- \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b,g) \otimes l - \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b,g) \otimes l$$
$$- \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b,g) \otimes l + a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b, g \otimes l)$$
$$+ \tau^{(1)}(a, f \otimes k)b \otimes \tau^{(1)}(g,l) - \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b \otimes g, l)$$

$$= a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b,g) \otimes l + \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b,g) \otimes l$$
$$+ \tau^{(1)}(a,f) \otimes kb \otimes \tau^{(1)}(g,l) + \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b, g \otimes l)$$
$$- \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b,g) \otimes l - \tau^{(1)}(a,f)\tau^{(1)}(b,g) \otimes kl$$
$$- \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b,g) \otimes l - \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b \otimes g, l)$$

$$= a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b,g) \otimes l + \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b,g) \otimes l$$
$$+ \tau^{(1)}(a,f) \otimes kb \otimes \tau^{(1)}(g,l) + \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b, g \otimes l)$$
$$- a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b,g) \otimes l - \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b,g) \otimes l$$
$$- \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b, g \otimes l) - \tau^{(1)}(a,f) \otimes kb \otimes \tau^{(1)}(g,l) = 0$$

At (4,1),

$$d_0(\omega_P^{(1)})(a \quad f \quad k \quad m) = 0$$

$$d_1(\omega_N^{(1)})(a \quad f \quad k \quad m)$$

$$= a \otimes \omega_N^{(1)}(f \quad k \quad m) - \omega_N^{(1)}(a \otimes f \quad k \quad m)$$
$$+ \omega_N^{(1)}(a \quad f \otimes k \quad m) - \omega_N^{(1)}(a \quad f \quad k \otimes m)$$
$$+ \omega_N^{(1)}(a \quad f \quad k) \otimes m$$

$$= a \otimes \tau^{(1)}(\tau^{(1)}(f,k),m) - a \otimes \tau^{(1)}(f,\tau^{(1)}(k,m))$$
$$- \tau^{(1)}(\tau^{(1)}(a \otimes f,k),m) + \tau^{(1)}(a \otimes f,\tau^{(1)}(k,m))$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f \otimes k),m) - \tau^{(1)}(a,\tau^{(1)}(f \otimes k,m))$$
$$- \tau^{(1)}(\tau^{(1)}(a,f),k \otimes m) + \tau^{(1)}(a,\tau^{(1)}(f,k \otimes m))$$
$$+ \tau^{(1)}(\tau^{(1)}(a.f)k) \otimes m - \tau^{(1)}(a,\tau^{(1)}(f,k)) \otimes m$$

$$= \tau^{(1)}(a \otimes \tau^{(1)}(f,k),m) - \tau^{(1)}(a,\tau^{(1)}(f,k) \otimes m)$$
$$+ \tau^{(1)}(a,\tau^{(1)}(f,k) \otimes m - \tau^{(1)}(a \otimes f,\tau^{(1)}(k,m)$$
$$+ \tau^{(1)}(a,f \otimes \tau^{(1)}(k,m)) - \tau^{(1)}(a,f) \otimes \tau^{(1)}(k,m)$$
$$- \tau^{(1)}(\tau^{(1)}(a \otimes f,k),m) + \tau^{(1)}(a \otimes f,\tau^{(1)}(k,m))$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f \otimes k),m) - \tau^{(1)}(a,\tau^{(1)}(f \otimes k,m))$$
$$- \tau^{(1)}(\tau^{(1)}(a,f)k \otimes m) + \tau^{(1)}(a,\tau^{(1)}(f,k \otimes m))$$
$$+ \tau^{(1)}(a,f) \otimes \tau^{(1)}(k,m) - \tau^{(1)}(\tau^{(1)}(a,f) \otimes k,m)$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f),k \otimes m) - \tau^{(1)}(a,\tau^{(1)}(f,k)) \otimes m$$

$$= \tau^{(1)}(a \otimes \tau^{(1)}(f,k),m) - \tau^{(1)}(\tau^{(1)}(a \otimes f,k),m)$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f \otimes k),m) - \tau^{(1)}(\tau^{(1)}(a,f) \otimes k,m)$$
$$+ \tau^{(1)}(a,f \otimes \tau^{(1)}(k,m) + \tau^{(1)}(a,\tau^{(1)}(f \otimes k,m))$$
$$+ \tau^{(1)}(a,\tau^{(1)}(f,k \otimes m)) - \tau^{(1)}(a,\tau^{(1)}(f,k) \otimes m)$$
$$- \tau^{(1)}(a \otimes f,\tau^{(1)}(k,m) + \tau^{(1)}(a \otimes f,\tau^{(1)}(k,m)$$
$$- \tau^{(1)}(\tau^{(1)}(a,f),k \otimes m) + \tau^{(1)}(\tau^{(1)}(a,f),k \otimes m)$$

$$= 0,$$

$$d_2(\omega_{cp}^{(1)})(a \quad f \quad k \quad m) = \omega_{cp}^{(1)} \begin{pmatrix} a.f \otimes k & m \\ \alpha_{XYZ} & 1_W \end{pmatrix} + \omega_{cp}^{(1)} \begin{pmatrix} a & f.k \otimes m \\ 1_X & \alpha_{YZW} \end{pmatrix}$$

$$- \omega_{cp}^{(1)} \begin{pmatrix} \alpha_{ABC} & 1_D \\ a \otimes f.k & m \end{pmatrix} - \omega_{cp}^{(1)} \begin{pmatrix} 1_A & \alpha_{BCD} \\ a & f \otimes k.m \end{pmatrix}$$

$$= 0,$$

$$d_3(\omega_c^{(1)})(a \quad f \quad k \quad m) = \omega_c^{(1)} \begin{pmatrix} ((a.f).k).m \\ \alpha_{XYZ} \otimes 1_W \\ \alpha_{XY.ZW} \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} \\ \alpha_{ABC} \otimes 1_D \\ (a.(f.k)).m \\ \alpha_{XY.ZW} \end{pmatrix}$$
$$+ \omega_{cp}^{(1)} \begin{pmatrix} \alpha_{ABC} \otimes 1_D \\ \alpha_{AB.CD} \\ a.(f.(k.m)) \end{pmatrix} + \omega_c^{(1)} \begin{pmatrix} (a.(f.k)).m \\ \alpha_{XY.ZW} \\ 1_X \otimes \alpha_{YZW} \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} \alpha_{AB.CD} \\ a.((f.k).m) \\ 1_X \otimes \alpha_{YZW} \end{pmatrix}$$
$$+ \omega_c^{(1)} \begin{pmatrix} \alpha_{AB.CD} \\ 1_A \otimes \alpha_{BCD} \\ a.(f.(k.m)) \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} ((a.f).k).m \\ \alpha_{X.YZW} \\ \alpha_{XYZ.W} \end{pmatrix} + \omega_c^{(1)} \begin{pmatrix} \alpha_{A.BCD} \\ (a.f).(k.m) \\ \alpha_{XYZ.W} \end{pmatrix}$$
$$- \omega_c^{(1)} \begin{pmatrix} \alpha_{A.BCD} \\ \alpha_{ABC.D} \\ a.(f.(k.m)) \end{pmatrix} = 0,$$

and similarly, at (5,0), the coboundary of obstruction vanishes because inputs are associators and objects only. So the result.

□

3. Deforming the associator only.

Theorem 2.1.3. *The obstructions are cocycles when only associator is deformed.*

Sketch of proof. In this case all the obstructions are zero except the obstruction, $\omega_P^{(1)}$, the obstruction due to pentagon identity. Also, the Yetter coboundary of this non-zero obstruction is zero by a result of Yetter[9] as we are using the identity preserving deformation, $d_2(\omega_N^{(1)}$ vanishes. Since the coboundaries of $\alpha^{(1)}$ are zero in both the directions, it is a natural transformation and satisfies the pentagon identity. This implies that $d_0(\omega_P^{(1)}) = 0$. □

2.1.2 Deforming two Structures at a Time

Theorem 2.1.4. *All primary obstructions are cocycles when one of composition, tensor and associator is undeformed.*

Proof. 1. Tensor and the associator only are deformed (i.e. $\tau^{(1)}, \alpha^{(1)} \neq 0$ and $\mu^{(1)} = 0$)

Then the first order conditions are:

(a) $\mu^{(1)} = 0 \Rightarrow \omega_c^{(1)} = 0.$

(b) $d_0(\tau^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = \tau^{(1)}(a, f)b \otimes g - \tau^{(1)}(ab, fg)$
$+ a \otimes f\tau^{(1)}(b, g) = 0.$

(c) $d_1(\tau^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = \alpha_{ABC} a \otimes \tau^{(1)}(f, k) - \tau^{(1)}(a \otimes f, k)\alpha_{XYZ}$
$+ \alpha_{ABC}\tau^{(1)}(a, f \otimes k) - \tau^{(1)}(a, f) \otimes k\alpha_{XYZ}$
$= \lceil a \otimes \tau^{(1)}(f, k) \rceil - \lceil \tau^{(1)}(a \otimes f, k) \rceil + \lceil \tau^{(1)}(a, f \otimes k)$
$- \lceil \tau^{(1)}(a, f) \otimes k \rceil$
$= -d_0(\alpha^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = -\alpha^{(1)}_{ABC} a \otimes f.k + a.f \otimes k\alpha^{(1)}_{XYZ}$

(d) $d_2(\tau^{(1)}) \begin{pmatrix} A & B & C & D \end{pmatrix} = 0$

(e) $d_1(\alpha^{(1)}) \begin{pmatrix} A & B & C & D \end{pmatrix} = \lceil \alpha^{(1)}_{ABC} \otimes D \rceil - \lceil \alpha^{(1)}_{A.BCD} \rceil$
$+ \lceil \alpha^{(1)}_{AB.CD} \rceil - \lceil \alpha^{(1)}_{ABC.D} \rceil + \lceil A \otimes \alpha^{(1)}_{BCD} \rceil$
$\overset{Yetter}{=} \sum_{i=0}^{4} (-1)^i \partial_i \alpha^{(1)} = 0$

and the obstructions are:

(a) $\omega^{(1)}_{cp} \begin{pmatrix} a & f \\ b & g \end{pmatrix} = \tau^{(1)}(a, f)\tau^{(1)}(b, g)$

(b) $\omega_N^{(1)}\begin{pmatrix} a & f & k \end{pmatrix} = \alpha_{ABC}^{(1)}\tau^{(1)}(a, f\otimes k) + \alpha_{ABC}^{(1)}a\otimes\tau^{(1)}(f,k)$
$+ \alpha_{ABC}\tau^{(1)}(a,\tau^{(1)}(f,k)) - \tau^{(1)}(\tau^{(1)}(a,f),k)\alpha_{XYZ}$
$- \tau^{(1)}(a\otimes f,k)\alpha_{XYZ}^{(1)} - \tau^{(1)}(a,f)\otimes k\alpha_{XYZ}^{(1)}$

(c) $\omega_P^{(1)}\begin{pmatrix} a & f & k \end{pmatrix} = \lceil[\alpha_{ABC}^{(1)}\otimes D]\alpha_{AB.CD}^{(1)}\rceil$
$+ \lceil[\alpha_{ABC}^{(1)}\otimes D][A\otimes\alpha_{BCD}^{(1)}]\rceil + \lceil\alpha_{AB.CD}^{(1)}A\otimes\alpha_{BCD}^{(1)}\rceil$
$+ \lceil\tau^{(1)}(\alpha_{ABC}^{(1)},D)\rceil + \lceil\tau^{(1)}(A,\alpha_{BCD}^{(1)})\rceil - \alpha_{A.BCD}^{(1)}\alpha_{ABC.D}^{(1)}.$

Then we can easily see that, at (2,3)
$d_0(\omega_{cp}^{(1)})\begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = 0 = d_1(\omega_c^{(1)})\begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix}.$

At (3,2), we have,

$d_1(\omega_{cp}^{(1)})\begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix} = ab\otimes\omega_{cp}^{(1)}\begin{pmatrix} f & k \\ g & l \end{pmatrix} - \omega_{cp}^{(1)}\begin{pmatrix} a\otimes f & k \\ b\otimes g & l \end{pmatrix}$
$+ \omega_{cp}^{(1)}\begin{pmatrix} a & f\otimes k \\ b & g\otimes l \end{pmatrix} - \omega_{cp}^{(1)}\begin{pmatrix} a & f\otimes k \\ b & g \end{pmatrix}\otimes kl$
$= \alpha_{ABC}ab\otimes\tau^{(1)}(f,k)\tau^{(1)}(g,l) - \tau^{(1)}(a\otimes f,k)\tau^{(1)}(b\otimes g,l)\alpha_{XYZ}$
$+ \alpha_{ABC}\tau^{(1)}(a,f\otimes k)\tau^{(1)}(b,g\otimes l) - \tau^{(1)}(a,f)\tau^{(1)}(b,g)\otimes kl\alpha_{XYZ}$
$= \lceil ab\otimes\tau^{(1)}(f,k)\tau^{(1)}(g,l)\rceil - \lceil\tau^{(1)}(a\otimes f,k)\tau^{(1)}(b\otimes g,l)\rceil$
$+ \lceil\tau^{(1)}(a,f\otimes k)\tau^{(1)}(b,g\otimes l)\rceil - \lceil\tau^{(1)}(a,f)\tau^{(1)}(b,g)\otimes kl\rceil$

$d_0(\omega_N^{(1)})\begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix} = a.f\otimes k\omega_N^{(1)}\begin{pmatrix} b & g & l \end{pmatrix} - \omega_N^{(1)}\begin{pmatrix} ab & fg & kl \end{pmatrix}$
$+ \omega_N^{(1)}\begin{pmatrix} a & f & k \end{pmatrix} b\otimes g.l$

$= a.f\otimes k[\alpha_{UVW}^{(1)}\tau^{(1)}(b,g\otimes l) + \alpha_{UVW}^{(1)}b\otimes\tau^{(1)}(g,l)$
$+ \alpha_{UVW}\tau^{(1)}(b,\tau^{(1)}(g,l) - \tau^{(1)}(\tau^{(1)}(b,g),l)\alpha_{XYZ}$
$- \tau^{(1)}(b\otimes g,l)\alpha_{XYZ}^{(1)} - \tau^{(1)}(b,g)\otimes l\alpha_{XYZ}^{(1)}]$
$- [\alpha_{ABC}^{(1)}\tau^{(1)}(ab,fg\otimes kl) + \alpha_{ABC}^{(1)}ab\otimes\tau^{(1)}(fg,kl)$

$$+ \alpha_{ABC}\tau^{(1)}(ab, \tau^{(1)}(fg, kl) - \tau^{(1)}(\tau^{(1)}(ab, fg), kl)\alpha_{XYZ}$$
$$- \tau^{(1)}(ab \otimes fg, kl)\alpha^{(1)}_{XYZ} - \tau^{(1)}(ab, fg) \otimes kl\alpha^{(1)}_{XYZ}]$$
$$+ [\alpha^{(1)}_{ABC}\tau^{(1)}(a, f \otimes k) + \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f, k)$$
$$+ \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f, k) - \tau^{(1)}(\tau^{(1)}(a, f), k)\alpha_{UVW}$$
$$- \tau^{(1)}(a \otimes f, k)\alpha^{(1)}_{UVW} - \tau^{(1)}(a, f) \otimes k\alpha^{(1)}_{UVW}]b \otimes g.l.$$

Then,

$$[d_0(\omega^{(1)}_N) - d_1(\omega^{(1)}_{cp})] \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix}$$
$$= a.f \otimes k[\alpha^{(1)}_{UVW}\tau^{(1)}(b, g \otimes l) + \alpha^{(1)}_{UVW}b \otimes \tau^{(1)}(g, l)$$
$$+ \alpha_{UVW}\tau^{(1)}(b, \tau^{(1)}(g, l) - \tau^{(1)}(\tau^{(1)}(b, g), l)\alpha_{XYZ}$$
$$- \tau^{(1)}(b \otimes g, l)\alpha^{(1)}_{XYZ} - \tau^{(1)}(b, g) \otimes l\alpha^{(1)}_{XYZ}]$$
$$- [\alpha^{(1)}_{ABC}\tau^{(1)}(ab, fg \otimes kl) + \alpha^{(1)}_{ABC}ab \otimes \tau^{(1)}(fg, kl)$$
$$+ \alpha_{ABC}\tau^{(1)}(ab, \tau^{(1)}(fg, kl) - \tau^{(1)}(\tau^{(1)}(ab, fg), kl)\alpha_{XYZ}$$
$$- \tau^{(1)}(ab \otimes fg, kl)\alpha^{(1)}_{XYZ} - \tau^{(1)}(ab, fg) \otimes kl\alpha^{(1)}_{XYZ}]$$
$$+ [\alpha^{(1)}_{ABC}\tau^{(1)}(a, f \otimes k) + \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f, k)$$
$$+ \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f, k) - \tau^{(1)}(\tau^{(1)}(a, f), k)\alpha_{UVW}$$
$$- \tau^{(1)}(a \otimes f, k)\alpha^{(1)}_{UVW} - \tau^{(1)}(a, f) \otimes k\alpha^{(1)}_{UVW}]b \otimes g.l$$
$$- [\lceil ab \otimes \tau^{(1)}(f, k)\tau^{(1)}(g, l)\rceil - \lceil \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b \otimes g, l)\rceil$$
$$+ \lceil \tau^{(1)}(a, f \otimes k)\tau^{(1)}(b, g \otimes l)\rceil - \lceil \tau^{(1)}(a, f)\tau^{(1)}(b, g) \otimes kl\rceil]$$

$$= [a.f \otimes l\alpha^{(1)}_{UVW} - \alpha^{(1)}_{ABC}a \otimes f.k][\tau^{(1)}(b, g \otimes l) + b \otimes \tau^{(1)}(g, l)]$$
$$+ a.f \otimes l\alpha_{UVW}\tau^{(1)}(b, \tau^{(1)}(g, l)) - a.f \otimes l\tau^{(1)}(b, \tau^{(1)}(g, l))\alpha_{XYZ}$$
$$- a.f \otimes l\tau^{(1)}(b, g) \otimes l\alpha^{(1)}_{XYZ} - a.f \otimes l\tau^{(1)}(b \otimes g, l)\alpha^{(1)}_{XYZ}$$
$$- \alpha^{(1)}_{ABC}\tau^{(1)}(a, f \otimes k)b \otimes g.l - \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f, k)b \otimes g.l$$
$$- \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f, k)b \otimes g.l - \alpha_{ABC}a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g \otimes l)$$
$$- \alpha_{ABC}a \otimes f.k\tau^{(1)}(b, \tau^{(1)}(g, l)) - \alpha_{ABC}\tau^{(1)}(a, f \otimes k)b \otimes \tau^{(1)}(g, l)$$
$$+ \tau^{(1)}(\tau^{(1)}(a, f), k)b.g \otimes l\alpha_{XYZ} + \tau^{(1)}(a \otimes f, k)\tau^{(1)}(b, g) \otimes l\alpha_{XYZ}$$

$$+ a.f \otimes k\tau^{(1)}(\tau^{(1)}(b,g),l)\alpha_{XYZ} + \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b \otimes g,l)\alpha_{XYZ}$$
$$+ [\tau^{(1)}(a,f) \otimes k + \tau^{(1)}(a \otimes f,k)][b.g \otimes l\alpha^{(1)}_{XYZ} - \alpha^{(1)}_{UVW}b \otimes g.l]$$
$$+ a.f \otimes k\tau^{(1)}(b,g) \otimes l\alpha^{(1)}_{XYZ} + a.f \otimes k\tau^{(1)}(b \otimes g,l)\alpha^{(1)}_{XYZ}$$
$$+ \alpha^{(1)}_{ABC}\tau^{(1)}(a,f \otimes k)b \otimes g.l + \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f,k)b \otimes g.l$$
$$+ \alpha_{ABC}\tau^{(1)}(a,\tau^{(1)}(f,k))b \otimes g.l - \tau^{(1)}(\tau^{(1)}(a,f),k)\alpha_{UVW}b \otimes g.l$$
$$- [\alpha_{ABC}ab \otimes \tau^{(1)}(f,k)\tau^{(1)}(g,l) - \tau^{(1)}(a \otimes f,k)\tau^{(1)}(b \otimes g,l)\alpha_{XYZ}$$
$$+ \alpha_{ABC}\tau^{(1)}(a,f \otimes k)\tau^{(1)}(b,g \otimes l) - \tau^{(1)}(a,f)\tau^{(1)}(b,g) \otimes kl\alpha_{XYZ}].$$

Now using the first order conditions, the above expression,

$$= [\alpha_{ABC}a \otimes \tau^{(1)}(f,k) - \tau^{(1)}(a \otimes f,k)\alpha_{UVW} + \alpha_{ABC}\tau^{(1)}(a,f \otimes k)$$
$$- \tau^{(1)}(a,f) \otimes k\alpha_{UVW}][\tau^{(1)}(b,g \otimes l) + b \otimes \tau^{(1)}(g,l)]$$
$$+ a.f \otimes l\alpha_{UVW}\tau^{(1)}(b,\tau^{(1)}(g,l)) - a.f \otimes l\tau^{(1)}(b,\tau^{(1)}(g,l))\alpha_{XYZ}$$
$$- a.f \otimes l\tau^{(1)}(b,g) \otimes l\alpha^{(1)}_{XYZ} - a.f \otimes l\tau^{(1)}(b \otimes g,l)\alpha^{(1)}_{XYZ}$$
$$- \alpha^{(1)}_{ABC}\tau^{(1)}(a,f \otimes k)b \otimes g.l - \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f,k)b \otimes g.l$$
$$- \alpha_{ABC}\tau^{(1)}(a,\tau^{(1)}(f,k)b \otimes g.l - \alpha_{ABC}a \otimes \tau^{(1)}(f,k)\tau^{(1)}(b,g \otimes l)$$
$$- \alpha_{ABC}a \otimes f.k\tau^{(1)}(b,\tau^{(1)}(g,l)) - \alpha_{ABC}\tau^{(1)}(a,f \otimes k)b \otimes \tau^{(1)}(g,l)$$
$$+ \tau^{(1)}(\tau^{(1)}(a,f),k)b.g \otimes l\alpha_{XYZ} + \tau^{(1)}(a \otimes f,k)\tau^{(1)}(b,g) \otimes l\alpha_{XYZ}$$
$$+ a.f \otimes k\tau^{(1)}(\tau^{(1)}(b,g),l)\alpha_{XYZ} + \tau^{(1)}(a,f) \otimes k\tau^{(1)}(b \otimes g,l)\alpha_{XYZ}$$
$$+ [\tau^{(1)}(a,f) \otimes k + \tau^{(1)}(a \otimes f,k)][\alpha_{UVW}b \otimes \tau^{(1)}(g,l)$$
$$- \tau^{(1)}(b \otimes g,l)\alpha_{XYZ} + \alpha_{UVW}\tau^{(1)}(b,g \otimes l) - \tau^{(1)}(b,g) \otimes l\alpha_{XYZ}]$$
$$+ a.f \otimes k\tau^{(1)}(b,g) \otimes l\alpha^{(1)}_{XYZ} + a.f \otimes k\tau^{(1)}(b \otimes g,l)\alpha^{(1)}_{XYZ}$$
$$+ \alpha^{(1)}_{ABC}\tau^{(1)}(a,f \otimes k)b \otimes g.l + \alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f,k)b \otimes g.l$$
$$+ \alpha_{ABC}\tau^{(1)}(a,\tau^{(1)}(f,k))b \otimes g.l - \tau^{(1)}(\tau^{(1)}(a,f),k)\alpha_{UVW}b \otimes g.l$$
$$- [\alpha_{ABC}ab \otimes \tau^{(1)}(f,k)\tau^{(1)}(g,l) - \tau^{(1)}(a \otimes f,k)\tau^{(1)}(b \otimes g,l)\alpha_{XYZ}$$
$$+ \alpha_{ABC}\tau^{(1)}(a,f \otimes k)\tau^{(1)}(b,g \otimes l)$$
$$- \tau^{(1)}(a,f)\tau^{(1)}(b,g) \otimes kl\alpha_{XYZ}] = 0.$$

At (4,1), proof is in Appendix A.

At (5,0), all the coboundaries of obstructions are zero except for $d_1(\omega_P^{(1)})$. Now
$d_1(\omega_P^{(1)})(A,B,C,D,E) = A\otimes\omega_P^{(1)}(B,C,D,E) - \omega_P^{(1)}(A.B,C,D,E) + \omega_P^{(1)}(A,B.C,D,E)$
$- \omega_P^{(1)}(A,B,C.D,E) + \omega_P^{(1)}(A,B,C,D.E) - \omega_P^{(1)}(A,B,C,D)\otimes E$

$= A\otimes[\alpha^{(1)}_{BCD}\otimes 1_E\alpha^{(1)}_{BC.DE} + \alpha^{(1)}_{BCD}\otimes 1_E 1_B\otimes\alpha^{(1)}_{CDE} + \alpha^{(1)}_{BC.DE}1_B\otimes\alpha^{(1)}_{CDE}$
$- \alpha^{(1)}_{B.CDE}\alpha^{(1)}_{BCD.E} + \tau^{(1)}(\alpha^{(1)}_{BCD},1_E) + \tau^{(1)}(1_B,\alpha^{(1)}_{CDE})]$
$- [\alpha^{(1)}_{A.BCD}\otimes 1_E\alpha^{(1)}_{A.BC.DE} + \alpha^{(1)}_{A.BCD}\otimes 1_E 1_{A.B}\otimes\alpha^{(1)}_{CDE} + \alpha^{(1)}_{A.BC.DE}1_{A.B}\otimes\alpha^{(1)}_{CDE}$
$- \alpha^{(1)}_{(A.B).CDE}\alpha^{(1)}_{A.BCD.E} + \tau^{(1)}(\alpha^{(1)}_{A.BCD},1_E) + \tau^{(1)}(1_{A.B},\alpha^{(1)}_{CDE})]$
$+ [\alpha^{(1)}_{AB.CD}\otimes 1_E\alpha^{(1)}_{A(B.C).DE} + \alpha^{(1)}_{AB.CD}\otimes 1_E 1_A\otimes\alpha^{(1)}_{B.CDE} + \alpha^{(1)}_{A(B.C).DE}1_A\otimes\alpha^{(1)}_{B.CDE}$
$- \alpha^{(1)}_{A(B.C).DE}\alpha^{(1)}_{AB.CD.E} + \tau^{(1)}(\alpha^{(1)}_{AB.CD},1_E) + \tau^{(1)}(1_A,\alpha^{(1)}_{B.CDE})]$
$- [\alpha^{(1)}_{ABC.D}\otimes 1_E\alpha^{(1)}_{AB(C.D).E} + \alpha^{(1)}_{ABC.D}\otimes 1_E 1_A\otimes\alpha^{(1)}_{BC.DE} + \alpha^{(1)}_{AB(C.D).E}1_A\otimes\alpha^{(1)}_{BC.DE}$
$- \alpha^{(1)}_{AB(C.D).E}\alpha^{(1)}_{AB(C.D).E} + \tau^{(1)}(\alpha^{(1)}_{ABC.D},1_E) + \tau^{(1)}(1_A,\alpha^{(1)}_{BC.DE})]$
$+ [\alpha^{(1)}_{ABC}\otimes 1_{D.E}\alpha^{(1)}_{ABC.(D.E)} + \alpha^{(1)}_{ABC}\otimes 1_{D.E}1_A\otimes\alpha^{(1)}_{BCD.E} + \alpha^{(1)}_{ABC.(D.E)}1_A\otimes\alpha^{(1)}_{BCD.E}$
$- \alpha^{(1)}_{ABC.(D.E)}\alpha^{(1)}_{ABC.(D.E)} + \tau^{(1)}(\alpha^{(1)}_{ABC},1_{D.E}) + \tau^{(1)}(1_A,\alpha^{(1)}_{BCD.E})]$
$- [\alpha^{(1)}_{ABC}\otimes 1_D\alpha^{(1)}_{ABC.D} + \alpha^{(1)}_{ABC}\otimes 1_D 1_A\otimes\alpha^{(1)}_{BCD} + \alpha^{(1)}_{ABC.D}1_A\otimes\alpha^{(1)}_{BCD}$
$- \alpha^{(1)}_{ABC.D}\alpha^{(1)}_{ABC.D} + \tau^{(1)}(\alpha^{(1)}_{ABC},1_D) + \tau^{(1)}(1_A,\alpha^{(1)}_{BCD})]\otimes E.$

In this case, since the first order condition on the pentagon is exactly the first order condition of Yetter's case as we are assuming the identity preserving case, all the terms except those containing $\tau^{(1)}$ vanish by Yetter[9]. Using the pentagon identity inside the arguments of $\tau^{(1)}$ and then the first order condition from preservation of identity (lemma 2.0.4), we can see that the other terms also vanish.

2. Composition and tensor are deformed (i.e. $\mu^{(1)},\tau^{(1)}\neq 0$ and $\alpha^{(1)}=0$).
The first order conditions are

(a) $d_0(\mu^{(1)}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}(a,b)c = 0.$

(b) $d_1(\mu^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = ab\mu^{(1)}(f,g) - \mu^{(1)}(a \otimes f, b \otimes g) + \mu^{(1)}(a,b) \otimes fg.$

(c) $d_2(\mu^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = \mu^{(1)}(\alpha_{ABC}, a \otimes f.k) - \mu^{(1)}(a.f \otimes k, \alpha_{XYZ}) = 0.$

(d) $d_0(\tau^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = a \otimes f\tau^{(1)}(b,g) - \tau^{(1)}(ab,fg) + \tau^{(1)}(a,f)b \otimes g.$

$$d_0(\tau^{(1)}) = d_1(\mu^{(1)})$$

(e) $d_1(\tau^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = \alpha_{ABC}a \otimes \tau^{(1)}(f,k) - \tau^{(1)}(a.f,k)\alpha_{XYZ} + \alpha_{ABC}\tau^{(1)}(a,f.k)$
$-\tau^{(1)}(a,f) \otimes k\alpha_{XYZ} = 0$ since $\alpha^{(1)} = 0.$

(f) $d_3(\mu^{(1)}) \begin{pmatrix} A & B & C & D \end{pmatrix} = \mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD})$
$-\mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D}) + \mu^{(1)}(\alpha_{ABC} \otimes D\alpha_{AB.CD}, A \otimes \alpha_{BCD}) = 0$
since $\alpha^{(1)} = 0.$

And obstructions,

(a) $\omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu^{(1)}(\mu^{(1)}(a,b),c) - \mu^{(1)}(a,\mu^{(1)}(b,c))$

(b) $\omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} = \mu^{(1)}(\tau^{(1)}(a,f), b \otimes g) - \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g) + \mu^{(1)}(a \otimes f, \tau^{(1)}(b,g))$
$-\tau^{(1)}(\mu^{(1)}(a,b),fg) + \tau^{(1)}(a,f)\tau^{(1)}(b,g) - \tau^{(1)}(ab,\mu^{(1)}(f,g))$

(c) $\omega_N^{(1)} \begin{pmatrix} a & f & k \end{pmatrix} = \alpha_{ABC}\tau^{(1)}(\tau^{(1)}(a,f),k) - \tau^{(1)}(a,\tau^{(1)}(a,,k))\alpha_{XYZ}$

(d) $\omega_P^{(1)} \begin{pmatrix} A & B & C & D \end{pmatrix} = 0$

At (1,4), $d_0(\omega_c^{(1)}) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$ by Gerstenhaber.

At (2,3), $d_1(\omega_c^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = abc \otimes \omega_c^{(1)} \begin{pmatrix} f \\ g \\ h \end{pmatrix}$

$+\omega_c^{(1)} \begin{pmatrix} a \otimes f \\ b \otimes g \\ c \otimes h \end{pmatrix} - \omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes fgh$

$= abc \otimes \mu^{(1)}(\mu^{(1)}(f,g),h) - abc \otimes \mu^{(1)}(f,\mu^{(1)}(g,h)) - \mu^{(1)}(\mu^{(1)}(a.f,b.),c.h)$

$+ \mu^{(1)}(a.f,\mu^{(1)}(b.g,c.h)) + \mu^{(1)}(\mu^{(1)}(a,b),c) \otimes fgh - \mu^{(1)}(a,\mu^{(1)}(b,c)) \otimes fgh$

$d_0(\omega_{cp}^{(1)}) \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = a \otimes \begin{pmatrix} b & g \\ c & h \end{pmatrix} - \begin{pmatrix} ab & fg \\ c & h \end{pmatrix}$

$+ \begin{pmatrix} a & f \\ bc & gh \end{pmatrix} - \begin{pmatrix} a & f \\ b & g \end{pmatrix} c \otimes h$

$= a \otimes f\mu^{(1)}(\tau^{(1)}(b,g),c.h) + a \otimes f\mu^{(1)}(b.g,\tau^{(1)}(c,h)) + a \otimes f\tau^{(1)}(b,g)\tau^{(1)}(c,h)$

$- a \otimes f\tau^{(1)}(\mu^{(1)}(b,c),gh) - a \otimes f\tau^{(1)}(bc,\mu^{(1)}(g,h)) - a \otimes f\mu^{(1)}(b,v) \otimes \mu^{(1)}(g,h)$

$- \mu^{(1)}(\tau^{(1)}(ab,fg),c.h) - \mu^{(1)}(ab.fg,\tau^{(1)}(c,h)) - \tau^{(1)}(ab,fg)\tau^{(1)}(c,h)$

$+ \tau^{(1)}(\mu^{(1)}(ab,c),fgh) + \tau^{(1)}(abc,\mu^{(1)}(fg,h)) + \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h)$

$+ \mu^{(1)}(\tau^{(1)}(a,f)bc.gh) + \mu^{(1)}(a.f,\tau^{(1)}(bc,gh)) + \tau^{(1)}(a,f)\tau^{(1)}(bc,gh)$

$- \tau^{(1)}(\mu^{(1)}(a,bc),fgh) - \tau^{(1)}(abc,\mu^{(1)}(f,gh)) - \mu^{(1)}(a,bc) \otimes \mu^{(1)}(f,gh)$

$- \mu^{(1)}(\tau^{(1)}(a,f),b.g)c.h - \mu^{(1)}(a.f,\tau^{(1)}(b,g))c.h - \tau^{(1)}(a,f)\tau^{(1)}(b,g)c.h$

$+ \tau^{(1)}(\mu^{(1)}(a,b),fg)c.h + \tau^{(1)}(ab,\mu^{(1)}(f,g))c,h + \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)c.h$

$= a \otimes f\mu^{(1)}(\tau^{(1)}(b,g),c.h) + a \otimes f\mu^{(1)}(b.g,\tau^{(1)}(c,h)) + a \otimes f\tau^{(1)}(b,g)\tau^{(1)}(c,h)$

$$
\begin{aligned}
&- a \otimes f\tau^{(1)}(\mu^{(1)}(b,c),gh) - a \otimes f\tau^{(1)}(bc,\mu^{(1)}(g,h)) - a \otimes f\mu^{(1)}(b,v) \otimes \mu^{(1)}(g,h) \\
&- \mu^{(1)}(\tau^{(1)}(a,f)b.g,c.h) - \mu^{(1)}(\mu^{(1)}(a.f,b.g),c.h) - \mu^{(1)}(a.f,\tau^{(1)}(b,g),c.h) \\
&+ \mu^{(1)}(\mu^{(1)}(a,b).fg,c.h) + \mu^{(1)}(ab.\mu^{(1)}(f,g),c.h) - \mu^{(1)}(ab.fg,\tau^{(1)}(c,h)) \\
&- \tau^{(1)}(a,f)b.g\tau^{(1)}(c,h) - \mu^{(1)}(a.f,b.g)\tau^{(1)}(c,h) - a.f,\tau^{(1)}(b,g)\tau^{(1)}(c,h) \\
&+ \mu^{(1)}(a,b).fg\tau^{(1)}(c,h) + ab.\mu^{(1)}(f,g)\tau^{(1)}(c,h) + \tau^{(1)}(\mu^{(1)}(ab,c),fgh) \\
&+ \tau^{(1)}(abc,\mu^{(1)}(fg,h)) + \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h) + \mu^{(1)}(\tau^{(1)}(a,f),bc.gh) \\
&+ \mu^{(1)}(a.f,\tau^{(1)}(b,g)c.h) + \mu^{(1)}(a.f,\mu^{(1)}(b.g,c.h)) + \mu^{(1)}(a.f,b.g\tau^{(1)}(c,h)) \\
&- \mu^{(1)}(a.f,\mu^{(1)}(b,g) \otimes gh) - \mu^{(1)}(a.f,bc \otimes \mu^{(1)}(g,h)) + \tau^{(1)}(a,f)\tau^{(1)}(b,g)c.h \\
&+ \tau^{(1)}(a,f)\mu^{(1)}(b.g,c.h) + \tau^{(1)}(a,f)b.g\tau^{(1)}(c,h) + \tau^{(1)}(a,f)\mu^{(1)}(b,g) \otimes gh \\
&+ \tau^{(1)}(a,f)bc \otimes \mu^{(1)}(g,h) - \tau^{(1)}(\mu^{(1)}(a,bc),fgh) - \tau^{(1)}(abc,\mu^{(1)}(f,gh)) \\
&- \mu^{(1)}(a,bc) \otimes \mu^{(1)}(f,gh) - \mu^{(1)}(\tau^{(1)}(a,f),b.g)c.h - \mu^{(1)}(a.f,\tau^{(1)}(b,g))c.h \\
&- \tau^{(1)}(a,f)\tau^{(1)}(b,g)c.h + \tau^{(1)}(\mu^{(1)}(a,b),fg)c.h \\
&+ \tau^{(1)}(ab,\mu^{(1)}(f,g))c,h + \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)c.h
\end{aligned}
$$

$$
\begin{aligned}
&\overset{1^{st} \text{ order condition}}{=} a \otimes f\tau^{(1)}(b,g)\tau^{(1)}(c,h) - a \otimes f\tau^{(1)}(\mu^{(1)}(b,c),gh) - a \otimes f\tau^{(1)}(bc,\mu^{(1)}(g,h)) \\
&- a \otimes f\mu^{(1)}(b,v) \otimes \mu^{(1)}(g,h) - \mu^{(1)}(\mu^{(1)}(a.f,b.g),c.h) + \mu^{(1)}(\mu^{(1)}(a,b).fg,c.h) \\
&+ \mu^{(1)}(ab.\mu^{(1)}(f,g),c.h) + \mu^{(1)}(a,b).fg\tau^{(1)}(c,h) + ab.\mu^{(1)}(f,g)\tau^{(1)}(c,h) \\
&+ \tau^{(1)}(\mu^{(1)}(ab,c),fgh) + \tau^{(1)}(abc,\mu^{(1)}(fg,h)) + \mu^{(1)}(ab,c) \otimes \mu^{(1)}(fg,h) \\
&+ \mu^{(1)}(a.f,\mu^{(1)}(b.g,c.h)) - \mu^{(1)}(a.f,\mu^{(1)}(b,g) \otimes gh) - \mu^{(1)}(a.f,bc \otimes \mu^{(1)}(g,h)) \\
&+ \tau^{(1)}(a,f)\mu^{(1)}(b,g) \otimes gh + \tau^{(1)}(a,f)bc \otimes \mu^{(1)}(g,h) - \tau^{(1)}(\mu^{(1)}(a,bc),fgh) \\
&- \tau^{(1)}(abc,\mu^{(1)}(f,gh)) - \mu^{(1)}(a,bc) \otimes \mu^{(1)}(f,gh) + \tau^{(1)}(\mu^{(1)}(a,b),fg)c.h \\
&+ \tau^{(1)}(ab,\mu^{(1)}(f,g))c,h + \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)c.h
\end{aligned}
$$

Combining, we get

$$
\begin{aligned}
&[d_1(\omega_c^{(1)}) - d_0(\omega_{cp}^{(1)})] \begin{pmatrix} a & f \\ b & g \\ c & h \end{pmatrix} = abc \otimes \mu^{(1)}(\mu^{(1)}(f,g),h) - abc \otimes \mu^{(1)}(f,\mu^{(1)}(g,h)) \\
&+ \mu^{(1)}(\mu^{(1)}(a,b),c) \otimes fgh - \mu^{(1)}(a,\mu^{(1)}(b,c) \otimes fgh \\
&+ a.f\tau^{(1)}(b,c),gh) + a.f\tau^{(1)}(bc,\mu^{(1)}(g,h) + a.d\mu^{(1)}(b,c) \otimes \mu^{(1)}(g,h)
\end{aligned}
$$

$$- \mu^{(1)}(\mu^{(1)}(a,b).fg, c.h) - \mu^{(1)}(ab.\mu^{(1)}(f,g), c.h) - \mu^{(1)}(a,b).fg\tau^{(1)}(c,h)$$
$$- ab.\mu^{(1)}(f,g)\tau^{(1)}(c,h) - \tau^{(1)}(\mu^{(1)}(ab,c), fgh) - \tau^{(1)}(abc, \mu^{(1)}(fg,h))$$
$$- \mu^{(1)}(ab,c).\mu^{(1)}(fg,h) + \mu^{(1)}(a.f, \mu^{(1)}(c, b.gh) + \mu^{(1)}(a.f, bc.\mu^{(1)}(g,h))$$
$$+ \tau^{(1)}(a,f)\mu^{(1)}(b,c).gh + \tau^{(1)}(a.f)bc.\mu^{(1)}(g,g) + \tau^{(1)}(\mu^{(1)}(a,bc), fgh)$$
$$+ \tau^{(1)}(abc, \mu^{(1)}(f,gh)) + \mu^{(1)}(a,bc).\mu^{(1)}(f,gh)$$
$$- \tau^{(1)}(\mu^{(1)}(a,b), fg)c.h - \tau^{(1)}(ab, \mu^{(1)}(f,g)c.h - \mu^{(1)}(a,b).\mu^{(1)}(f,g)c \otimes h$$

Using the first condition on $\begin{pmatrix} \mu^{(1)}(a,b) & fg \\ c & h \end{pmatrix}$ and such hexagons, we find that the difference is

$$= a.f\mu^{(1)}(b,c).\mu^{(1)}(g,h) - \mu^{(1)}(a,b).\mu^{(1)}(f,g)c \otimes h$$
$$- \tau^{(1)}(\mu^{(1)}(ab,c), fgh) - \tau^{(1)}(abc, \mu^{(1)}(fg,h) - \mu^{(1)}(ab,c).\mu^{(1)}(fg,h)$$
$$+ \tau^{(1)}(\mu^{(1)}(a,bc), fgh) + \tau^{(1)}(abc, \mu^{(1)}(f,gh)) + \mu^{(1)}(a,bc).\mu^{(1)}(f,gh)$$
$$- \mu^{(1)}(a,b)c.\mu^{(1)}(fg,h) - \tau^{(1)}(\mu^{(1)}(a,b)c, fgh) - \mu^{(1)}(ab,c).\mu^{(1)}(f,g)h$$
$$- \tau^{(1)}(abc, \mu^{(1)}(f,g)h) + \mu^{(1)}(a,bc).f\mu^{(1)}(g,h) + \tau^{(1)}(abc, f\mu^{(1)}(g,h))$$
$$+ a\mu^{(1)}(b,c).\mu^{(1)}(f,gh) + \tau^{(1)}(a\mu^{(1)}(b,c), fgh) + \tau^{(1)}(a\mu^{(1)}(b,c), fgh)$$

$$= a\mu^{(1)}(b,c).f\mu^{(1)}(g,h) - \mu^{(1)}(a,b)c.\mu^{(1)}(f,g)h - \mu^{(1)}(ab,c).\mu^{(1)}(fg,h)$$
$$+ \mu^{(1)}(a,bc).\mu^{(1)}(f,gh) - \mu^{(1)}(a,b)c.\mu^{(1)}(fg,h) - \mu^{(1)}(ab,c).\mu^{(1)}(f,g)h$$
$$+ \mu^{(1)}(a,bc).f\mu^{(1)}(g,h) + a\mu^{(1)}(b,c).\mu^{(1)}(f,gh)$$

$$= a\mu^{(1)}(b,c).f\mu^{(1)}(g,h) + \mu^{(1)}(a,bc).\mu^{(1)}(f,gh)$$
$$+ a\mu^{(1)}(b,c).\mu^{(1)}(f,gh) + \mu^{(1)}(a,bc).f\mu^{(1)}(g,h)$$
$$- \mu^{(1)}(a,b)c.[\mu^{(1)}(f,g)h + \mu^{(1)}(fg,h)] - \mu^{(1)}(ab,c).[\mu^{(1)}(fg,h) + \mu^{(1)}(f,g)h]$$

$$= a\mu^{(1)}(b,c).f\mu^{(1)}(g,h) + \mu^{(1)}(a,bc).\mu^{(1)}(f,gh)$$
$$+ a\mu^{(1)}(b,c).\mu^{(1)}(f,gh) + \mu^{(1)}(a,bc).f\mu^{(1)}(g,h)$$
$$- \mu^{(1)}(a,b)c.[\mu^{(1)}(f,gh) + f\mu^{(1)}(g,h)] - \mu^{(1)}(ab,c).[f\mu^{(1)}(g,h) + \mu^{(1)}(f,gh)]$$

$= d_1(\mu^{(1)})(a,b,c)[f\mu^{(1)}(g,h) + \mu^{(1)}(f,gh)] = 0$

At (3,2), this is a special case of the proof in Appendix A.

At (4,1), we have

$\omega_P^{(1)} = 0$ and so $d_0(\omega_P^{(1)}) = 0$.

Also, $d_2(\omega_{cp}^{(1)}) = 0 = d_3(\omega_c^{(1)})$. To prove the result, we need to see $d_1(\omega_N^{(1)}) = 0$. For this,

$d_1(\omega_N^{(1)}) \begin{pmatrix} a & f & k & m \end{pmatrix}$

$= a.\omega_N^{(1)}(f,k,m) - \omega_N^{(1)}(a.f,k,m)$
$+ \omega_N^{(1)}(a,f.k,m) - \omega_N^{(1)}(a,f,k.m) + \omega_N^{(1)}(a,f,k).m$

$= a.\tau^{(1)}(\tau^{(1)}(\tau^{(1)}(f,k),m) - a.\tau^{(1)}(f,\tau^{(1)}(k,m) - \tau^{(1)}(\tau^{(1)}(a.f,k),m)$
$+ \tau^{(1)}(a.f,\tau^{(1)}(k,m)) + \tau^{(1)}(\tau^{(1)}(a,f.k),m) - \tau^{(1)}(a,\tau^{(1)}(f.k,m))$
$- \tau^{(1)}(\tau^{(1)}(a,f),k.m) + \tau^{(1)}(a,\tau^{(1)}(f,k.m)) + \tau^{(1)}(\tau^{(1)}(a,f),k).m$
$+ \tau^{(1)}(a,\tau^{(1)}(f,k)).m$

$= \tau^{(1)}(a.\tau^{(1)}(f,k),m) - \tau^{(1)}(a,\tau^{(1)}(f,k).m) + \tau^{(1)}(a,\tau^{(1)}(f,k)).m$
$- \tau^{(1)}(a.f,\tau^{(1)}(k,m)) + \tau^{(1)}(a,f.\tau^{(1)}(k,m)) - \tau^{(1)}(a,f).\tau^{(1)}(k,m)$
$- \tau^{(1)}(\tau^{(1)}(a.f,k),m) + \tau^{(1)}(a.f,\tau^{(1)}(k,m)) + \tau^{(1)}(\tau^{(1)}(a,f.k),m)$
$- \tau^{(1)}(a,\tau^{(1)}(f.k,m)) - \tau^{(1)}(\tau^{(1)}(a,f),k.m) + \tau^{(1)}(a,\tau^{(1)}(f,k.m))$
$+ \tau^{(1)}(a,f).\tau^{(1)}(k,m) - \tau^{(1)}(\tau^{(1)}(a,f).k,m) + \tau^{(1)}(\tau^{(1)}(a,f),k.m)$
$+ \tau^{(1)}(a,\tau^{(1)}(f,k)).m = 0$

At (5,0), all the coboundaries of all obstructions are 0s. Hence, the obstructions are cocycles in this case too.

3. Composition and the associator only are deformed (i.e. $\mu^{(1)}, \alpha^{(1)} \neq 0$ and $\tau^{(1)} = 0$). The first order conditions are

(a) $d_0(\mu^{(1)}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}(a,b)c = 0.$

(b) $d_1(\mu^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = ab\mu^{(1)}(f,g) - \mu^{(1)}(a \otimes f, b \otimes g) + \mu^{(1)}(a,b) \otimes fg = 0.$

(c) $d_2(\mu^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = \mu^{(1)}(\alpha_{ABC}, a \otimes f.k) - \mu^{(1)}(a.f \otimes k, \alpha_{XYZ}) = 0.$

(d) $d_0(\tau^{(1)}) \begin{pmatrix} a & f \\ b & g \end{pmatrix} = 0.$

$$d_0(\tau^{(1)}) = d_1(\mu^{(1)}) = 0$$

(e) $d_1(\tau^{(1)}) \begin{pmatrix} a & f & k \end{pmatrix} = 0.$

(f) $d_3(\mu^{(1)}) \begin{pmatrix} A & B & C & D \end{pmatrix} = \mu^{(1)}(\alpha_{ABC} \otimes D, \alpha_{AB.CD}) - \mu^{(1)}(\alpha_{A.BCD}, \alpha_{ABC.D})$
$+\mu^{(1)}(\alpha_{ABC} \otimes D\alpha_{AB.CD}, A \otimes \alpha_{BCD}) = 0$, since $\alpha^{(1)} = 0$.

And obstructions,

(a) $\omega_c^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu^{(1)}(\mu^{(1)}(a,b),c) - \mu^{(1)}(a,\mu^{(1)}(b,c))$

(b) $\omega_{cp}^{(1)} \begin{pmatrix} a & f \\ b & g \end{pmatrix} = \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g)$

(c) $\omega_N^{(1)}\left(\begin{array}{ccc} a & f & k \end{array}\right) = \mu^{(1)}(\alpha_{ABC}^{(1)}, a \otimes f.k) - \mu^{(1)}(a.f \otimes k, \alpha_{XYZ}^{(1)})$

(d) $\omega_P^{(1)}\left(\begin{array}{cccc} A & B & C & D \end{array}\right) = \alpha_{ABC}^{(1)} \otimes D\alpha_{AB.CD}^{(1)} + \alpha_{ABC}^{(1)} \otimes DA \otimes \alpha_{BCD}^{(1)}$
$+\alpha_{AB.CD}^{(1)} A \otimes \alpha_{BCD}^{(1)} - \alpha_{A.BCD}^{(1)} \alpha_{ABC.D}^{(1)}$
under Yetter's padded composition.

As before, at (1,4) is Gerstenhaber's proof, at (2,3) and (3,2) are the particular cases of previous composition and tensor case and the appendix respectively. At (4,1)
$d_3(\omega_c^{(1)}) = 0 = d_2(\omega_{cp}^{(1)})$
For the other differentials of obstruction, as $\alpha^{(1)}$ is natural in this case, vanishes independently.

At (5,0), all the coboundaries of obstructions are zero except the Yetter coboundary of $\omega_P^{(1)}$. Since, we are assuming the category is strict one, this is also cocycle by a result of Yetter[9].

Thus the result follows. □

2.2 A Geometrical Approach to Simplify Messy Calculations

We have seen that the direct calculation of the coboundaries of obstructions, even in special cases are not simple. So, to handle the general case, we will use geometrical encoding of the obstructions and cobounding conditions. For this we are going to construct polytope or cell decomposition of sphere such that edges of each face represent the deformable arrow-valued operations and with them the corresponding deformation terms of each order and each face represents both relations the deformation terms satisfy up to some order and instances of an obstruction entry of each order greater than one depending on how the operations around a face are assembled, or a difference of terms which must trivially be zero.

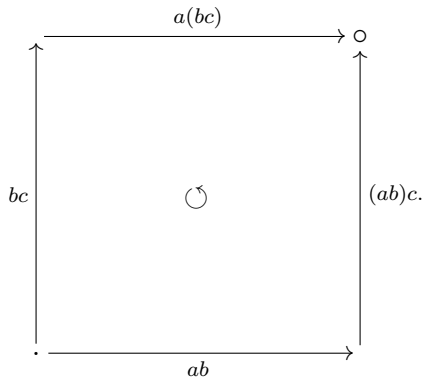
As an example, consider the first order obstruction of deformation of composition is given by

$$\omega^{(1)}(a,b,c) = \mu^{(1)}(\mu^{(1)}(a,b),c) - \mu^{(1)}(a,\mu^{(1)}(b,c)),$$

and the first order condition is given by

$$a\mu^{(1)}(b,c) - \mu^{(1)}(ab,c) + \mu^{(1)}(a,bc) - \mu^{(1)}(a,b)c = 0.$$

Those two conditions can be represented by the following square:

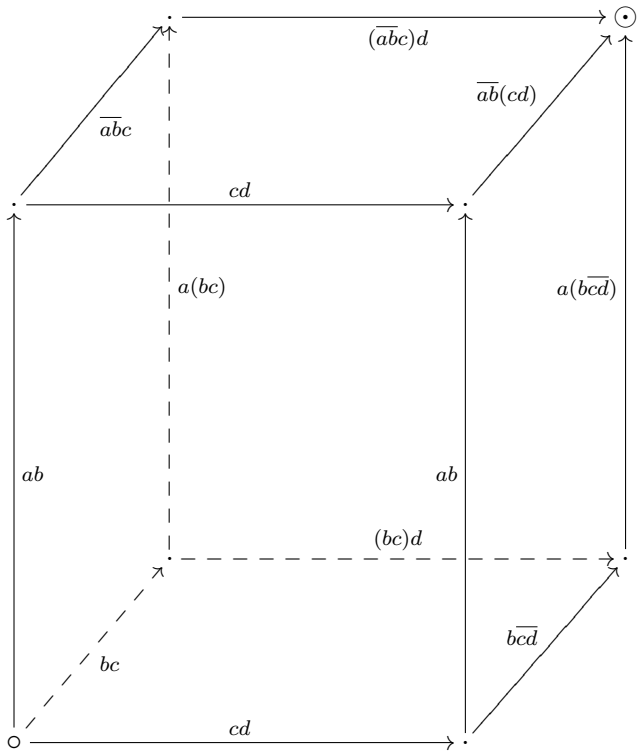


We can call it, square $[a,b,c]$. For the first order condition, we are taking one degree deformation from each edge and the orientation gives the sign of the terms on the expression. But, we need to remember that each term should have all the arrows used. So, the first side on the clockwise direction gives us the first term of the first order condition and the second gives us the third term. Similarly, the other sides give us the second and last term of the same.

For the first obstruction terms, we use the same sign rule, but spread a total degree of two over the edges of each oriented "half" of the boundary so that each edge gets at most one. Similarly, second order condition, we spread degree 2 in all possible ways keeping the sign rule in mind, which gives

$a\mu^{(2)}(b,c) + \mu^{(2)}(a,bc) + \mu^{(1)}(a,\mu^{(1)}(b,c)) = \mu^{(2)}(a,b) + \mu^{(2)}(ab,c) + \mu^{(1)}(\mu^{(1)}(a,b),c)$. Note that this is equivalent to the condition that the Hochschild coboundary of $\mu^{(1)}$ equals the first obstruction $\omega^{(1)}$. The second obstruction can be obtained as spreading the degree 3 in all possible ways such that each edge gets at most degree 2. That is,
$\omega_c^{(2)}(a,b,c) = \mu^{(2)}(\mu^{(1)}(a,b),c) + \mu^{(1)}(\mu^{(2)}(a,b),c) - \mu^{(2)}(a,\mu^{(1)}(b,c)) - \mu^{(1)}(a,\mu^{(2)}(b,c))$.
In general, spreading total degree n over all edges on each of the two halves of the boundary, so then each edge has degree less than n and taking the difference gives the obstruction to extend to degree n, which letting the edges have degrees up to and including n gives the n^{th} order condition that $\mu^{(n)}$ cobounds $\omega^{(n)}$.

Now, the Gerstenhaber cocycle condition of obstruction to deformation of associative composition can be calculated from a polytope, whose edges represent deformable structures, here the composition only,



To keep track of the calculations of the cocycle condition for the obstructions to deformation of composition from chapter 1, the lowest face is the square [b, c, d], the top face is the square [ab,c,d], the back face is the square [a,bc,d], the right face it the square [a,b,cd], the left one is the square [a, b, c] and the front face is a trivial square.

In the calculations in the proofs of Proposition 1.2.3 and Theorem 1.2.4, we used instances of the lower order conditions (that $\mu^{(1)}$ is a coboundary and that each $\mu^{(k)}$ cobounds the k^{th} obstruction) on each face to replace instances of the label on the common edge abc on the other face with expressions not involving the common label, thereby showing that the sum of the obstruction-type expressions on the square $[ab, c, d]$ and the square $[a, b, c]$ equals the obstruction-like expression corresponding to the hexagon obtained by gluing the squares on their common edge by spreading the total degree among the edges of each oriented side as described above for the squares. Similarly, we glued the squares $[b, c, d]$ and $[a, b, cd]$ through the common edge bcd. In the proof of Theorem 1.2.4 these steps occur where the nested summations first appear. The remaining sides (the trivial front, and $[a, bc, d]$) glue on even more trivially: the terms corresponding to one half of the boundary cancel some of those in the hexagons obtained by the previous gluing, and the result is an obstruction like expression corresponding to the hexagon with edges ab, cd, $ab(cd)$, bc, $(bc)d$, and $a(bcd)$, occurring twice with opposite signs.

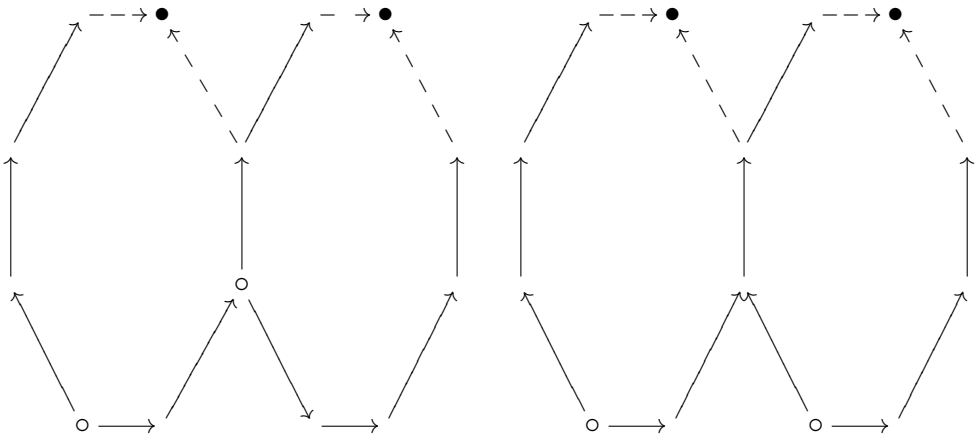
This suggests taking a geometrical approach to proving the obstructions are cocycles if we can get suitable cell decomposition of a sphere, with edges labeled by instances of arrow-valued operations for each of the bidegrees (1,2), (2,3), (3,2), (4,1) and (5,0).

Consider a polygon whose edges are labeled with instances of deformable arrow-valued operations which are either arrow (or object) variables or values of arrow-valued operations

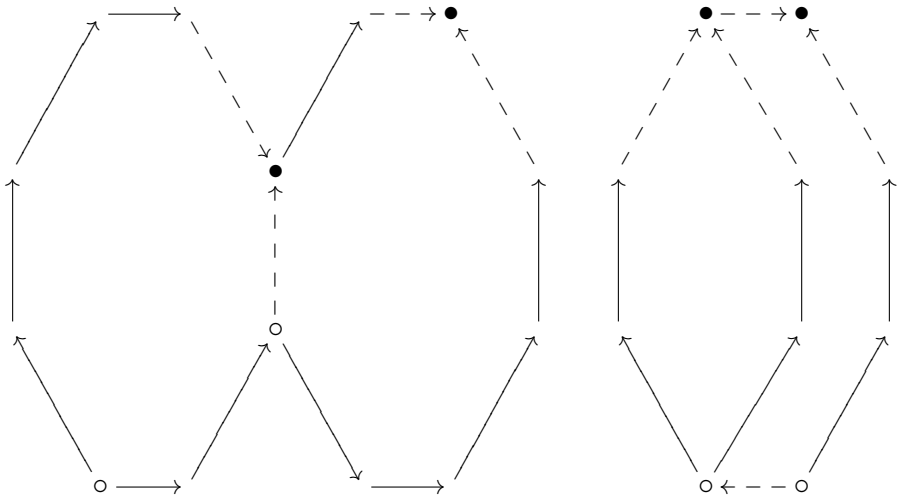
occurring earlier on a path from a global source to a global target. Each polygon encodes a equational condition at each order: the vanishing of the difference of the expressions resulting from spreading the order among the edges of the two directed paths in all possible way with the sign chosen by which path is reveal to give the boundary orientation. Each polygon encodes an obstruction-type expression at each order greater or equal to one: such that the difference of the expressions in which the total order could all be on a single edge (with other order 0).

Definition 2.2.1. *(Admissible and Non-admissible Polygon Combinations) Two polygons with edges representing arrow-valued operations with one source and one target are called admissible if when they are glued together on the edges which represent same instance of an arrow-valued operation, the boundary of resulting polygon has unique source and target.*

Example:



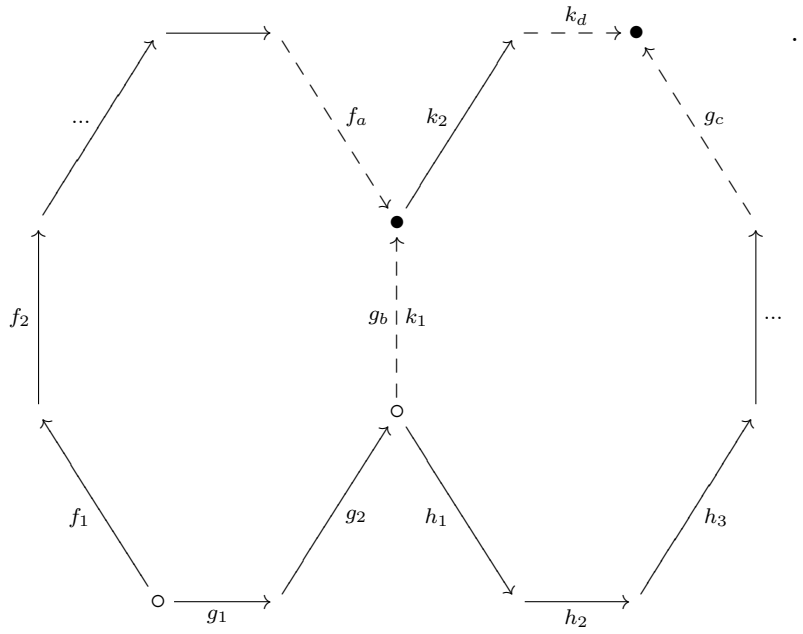
These combination of polygons are non-admissible because they have not unique source and targets for the boundary polygon. So, we see that if the sharing side(s) is(are) dissolved, the polygon will not have unique either source or target in this case. But the polygons,



are admissible because they have unique source and target if the shared side(s) is(are) dissolved.

Lemma 2.2.1. *If two polygons have sides representing arrow-valued deformable structures, the first order conditions described by both polygons hold and they are admissible, then the sum of the second order obstruction-type expressions given by the two polygons is equal to the obstruction-type expression given by the larger polygon obtained by deleting the shared edges.*

Proof. Consider two polygons each with one source and one target and are admissible. The sides of this polygon are represented as:



One side of the first polygon is denoted by arrows $f_i's$, $i = 1, 2, ..., a$ and on the other side is denoted by arrows $g_i's$, $i = 1, 2, ..., b$. Similarly, for the other polygon,$h_i's$, $i = 1, 2, ..., c$ and $k_i's$, $i = 1, 2, ..., d$ respectively.

If we are considering only first order deformation of arrows, i.e.

$\hat{f}_i = f_i + f_i^{(1)}\epsilon$ and $\hat{g}_j = g_j + g_j^{(1)}\epsilon$ where $i = 1, 2, ..., a$ and $i = 1, 2, ..., b$. Then the first order condition is given by,

$$\sum_{1 \leq i \leq a} f_i^{(1)} - \sum_{1 \leq j \leq b} g_j^{(1)} = 0.$$

The first obstruction of deformation on the boundary is given by,

$$\sum_{1 \leq i < j \leq a} f_i^{(1)} f_j^{(1)} - \sum_{1 \leq p < q \leq b} g_p^{(1)} g_q^{(1)}.$$

Now, gluing a polygonal face which satisfy the admissible condition in our sense. Assume that the side common to both polygons is g_b and the arrows on the g-side are given by h_p's, $p = 1, 2, ..., c$, and that on the f-side are given by k_q, $q = 1, 2, ..., d$, i.e. $g_b = k_1$ and $\hat{h}_p = h_p + h_p^{(1)}\epsilon$ and $\hat{k}_p = k_p + k_p^{(1)}\epsilon$ where $p = 1, 2, ..., c$ and $q = 1, 2, ..., d$. Then the first order condition on the second polygon is

$$\sum_{1\le r\le c} h_r^{(1)} - \sum_{1\le s\le d} k_s^{(1)} = 0,$$

and the first obstruction is

$$\sum_{1\le r<s\le c} h_r^{(1)} h_s^{(1)} - \sum_{1\le t<u\le d} k_t^{(1)} k_u^{(1)}.$$

Then, for the resulting polygon, we want to get the result from the condition from the individual polygon as;

Note that $k_1 = g_b$. So, then solving the value of k_1 from the first order condition of second polygon and substituting it to the first order condition from the first polygon, that is,

$$g_b^{(1)} = k_1 = \sum_{1\le r\le c} h_r^{(1)} - \sum_{1<s\le d} k_s^{(1)},$$

and so,

$$0 = \sum_{1\le i\le a} f_i^{(1)} - \sum_{1\le j\le b} g_j^{(1)} = \sum_{1\le i\le a} f_i^{(1)} - \sum_{1\le j<b} g_j^{(1)} - g_b^{(1)}$$

$$= \sum_{1\le i\le a} f_i^{(1)} - \sum_{1\le j<b} g_j^{(1)} - [\sum_{1\le r\le c} h_r^{(1)} - \sum_{1<s\le d} k_s^{(1)}]$$

$$= [\sum_{1\le i\le a} f_i^{(1)} + \sum_{1<s\le d} k_s^{(1)}] - [\sum_{1\le r\le c} h_r^{(1)} + \sum_{1<s\le d} k_s^{(1)}].$$

which is the first order condition on the boundary of glued polygon. Also,

$$\sum_{1\le i<j\le a} f_i^{(1)} f_j^{(1)} - \sum_{1\le p<q\le b} g_p^{(1)} g_q^{(1)} = \sum_{1\le i<j\le a} f_i^{(1)} f_j^{(1)} - \sum_{1\le p<q<b} g_p^{(1)} g_q^{(1)} - \sum_{1\le p<b} g_p^{(1)} g_b^{(1)}$$

$$= \sum_{1\leq i<j\leq a} f_i^{(1)} f_j^{(1)} - \sum_{1\leq p<q<b} g_p^{(1)} g_q^{(1)} - \sum_{1\leq p<b} g_p^{(1)} [\sum_{1\leq r\leq c} h_r^{(1)} - \sum_{1<s\leq d} k_s^{(1)}].$$

and similarly,

$$\sum_{1\leq r<s\leq c} h_r^{(1)} h_s^{(1)} - \sum_{1\leq t<u\leq d} k_t^{(1)} k_u^{(1)} = \sum_{1\leq r<s\leq c} h_r^{(1)} h_s^{(1)} - \sum_{1<u\leq d} k_1^{(1)} k_u^{(1)} - \sum_{1<t<u\leq d} k_t^{(1)} k_u^{(1)}$$

$$= \sum_{1\leq r<s\leq c} h_r^{(1)} h_s^{(1)} - \sum_{1<u\leq d} [\sum_{1\leq i\leq a} f_i^{(1)} - \sum_{1\leq p<b} g_p^{(1)}] k_u^{(1)} - \sum_{1<t<u\leq d} k_t^{(1)} k_u^{(1)}.$$

Combining both, the first obstructions, we get

$$\sum_{1\leq i<j\leq a} f_i^{(1)} f_j^{(1)} - \sum_{1\leq p<q\leq b} g_p^{(1)} g_q^{(1)} - \sum_{1\leq p<b} g_p^{(1)} [\sum_{1\leq r\leq c} h_r^{(1)} - \sum_{1<s\leq d} k_s^{(1)}]$$

$$- \sum_{1\leq r<s\leq c} h_r^{(1)} h_s^{(1)} + \sum_{1<u\leq d} [\sum_{1\leq i\leq a} f_i^{(1)} - \sum_{a\leq p<b} g_p^{(1)}] k_u^{(1)} + \sum_{1<t<u\leq d} k_t^{(1)} k_u^{(1)}$$

$$= [\sum_{1\leq i<j\leq a} f_i^{(1)} f_j^{(1)} + \sum_{1<u\leq d} \sum_{1\leq i\leq a} f_i^{(1)} k_u^{(1)} + \sum_{1<t<u\leq d} k_t^{(1)} k_u^{(1)}]$$

$$-[\sum_{1\leq p<q\leq b} g_p^{(1)} g_q^{(1)} + \sum_{1\leq p<b} \sum_{1\leq r\leq c} g_p^{(1)} h_r^{(1)} + \sum_{1\leq r<s\leq c} h_r^{(1)} h_s^{(1)}].$$

which is the required first obstruction on the deformation due to glued polygon. The contribution due to the shared edge after gluing cancel and we see the contribution of the

boundary of the resultant polygon only.

The proof in the second admissible case is similar. □

Further,

Lemma 2.2.2. *If two polygons have sides representing arrow-valued deformable structures, and the equational conditions represented by both polygons hold for all orders less than or equal to n, and the polygons are admissible, then the sum of the n^{th} order obstruction-type expressions given by the two polygons is equal to the n^{th} obstruction-type expression given by the larger polygon obtained by deleting the shared edges.*

Note: The notion $f \circ g$ is used to denote $g(f)$ to follow the diagrammatic order if an expression is too long to fit on the same line.

Proof. We have shown the case $n = 1$. For the general case, we have,

$$\hat{f} = \sum_{i=0}^{n} f^{(i)}\epsilon^i \text{ where } n \geq 1.$$

Then the lower order conditions on the faces for $l < n$ are given by,

$$\sum_{\phi_1+\phi_2+...+\phi_a=l} f_1^{(\phi_1)} f_2^{(\phi_2)} ... f_a^{(\phi_a)} - \sum_{\psi_1+\psi_2+...+\psi_b=l} g_1^{(\psi_1)} g_2^{(\psi_2)} ... g_b^{(\psi_b)} = 0$$

and

$$\sum_{\rho_1+\rho_2+...+\rho_c=l} h_1^{(\rho_1)} h_2^{(\rho_2)} ... h_a^{(\rho_c)} - \sum_{\nu_1+\nu_2+...+\nu_d=l} k_1^{(\nu_1)} k_2^{(\nu_2)} ... k_d^{(\psi_d)} = 0$$

that is,

$$g_b^{(l)} = k_1^{(l)} = \sum_{\phi_1+\phi_2+...+\phi_a=l} f_1^{(\phi_1)} f_2^{(\phi_2)} ... f_a^{(\phi_a)} - \sum_{\psi_1+\psi_2+...+\psi_{b-1}=l} g_1^{(\psi_1)} g_2^{(\psi_2)} ... g_{b-1}^{(\psi_{b-1})}$$

and

$$g_b^{(l)} = k_1^{(l)} = \sum_{\rho_1+\rho_2+\ldots+\rho_c=l} h_1^{(\rho_1)} h_2^{(\rho_2)} \ldots h_a^{(\rho_c)} - \sum_{\nu_2+\nu_3+\ldots+\nu_d=l} k_2^{(\nu_2)} k_3^{(\nu_3)} \ldots k_d^{(\psi_d)}.$$

The n^{th} order obstruction-type expressions are given by,

$$\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=n+1 \\ 0\leq\phi_i\leq n}} f_1^{(\phi_1)} f_2^{(\phi_2)} \ldots f_a^{(\phi_a)} - \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_b=n+1 \\ 0\leq\psi_i\leq n}} g_1^{(\psi_1)} g_2^{(\psi_2)} \ldots g_b^{(\psi_b)}$$

and

$$\sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=n+1 \\ 0\leq\rho_i\leq n}} h_1^{(\rho_1)} h_2^{(\rho_2)} \ldots h_a^{(\rho_c)} - \sum_{\substack{\nu_1+\nu_2+\ldots+\nu_d=n+1 \\ 0\leq\nu_i\leq n}} k_1^{(\nu_1)} k_2^{(\nu_2)} \ldots k_d^{(\nu_d)}$$

So, using above lower order conditions, in the n^{th} order obstruction-type expressions, we have

$$\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=n+1 \\ 0\leq\phi_i\leq n}} f_1^{(\phi_1)} f_2^{(\phi_2)} \ldots f_a^{(\phi_a)} - \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=n+1-\psi_b \\ 0\leq\psi_i\leq n}} g_1^{(\psi_1)} g_2^{(\psi_2)} \ldots g_{b-1}^{(\psi_{b-1})}$$
$$\circ\big[\sum_{\rho_1+\rho_2+\ldots+\rho_c=\psi_b} h_1^{(\rho_1)} h_2^{(\rho_2)} \ldots h_a^{(\rho_c)} - \sum_{\nu_2+\nu_3+\ldots+\nu_{d-1}=\psi_b} k_2^{(\nu_2)} k_3^{(\nu_3)} \ldots k_d^{(\psi_d)} \big].$$

and

$$\sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=n+1 \\ 0\leq\rho_i\leq n}} h_1^{(\rho_1)} h_2^{(\rho_2)} \ldots h_a^{(\rho_c)} - \sum_{\substack{\nu_2+\ldots+\nu_d=n+1-\nu_1 \\ 0\leq\nu_i\leq n}} k_1^{(\nu_1)} k_2^{(\nu_2)} \ldots k_{d-1}^{(\nu_{d-1})}$$
$$\circ\big[\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=\nu_1 \\ 0\leq\phi_i}} f_1^{(\phi_1)} f_2^{(\phi_2)} \ldots f_a^{(\phi_a)} - \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=\nu_1 \\ 0\leq\psi_i}} g_1^{(\psi_1)} g_2^{(\psi_2)} \ldots g_{b-1}^{(\psi_{b-1})} \big].$$

Hence, combining two obstructions, we get,

$$\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=n+1 \\ 0\leq\phi_i\leq n}} f_1^{(\phi_1)} f_2^{(\phi_2)} \ldots f_a^{(\phi_a)} - \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=n+1-\psi_b \\ 0\leq\psi_i\leq n}} g_1^{(\psi_1)} g_2^{(\psi_2)} \ldots g_{b-1}^{(\psi_{b-1})}$$

$$\circ[\sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=\psi_b\\0\leq\rho_i}} h_1^{(\rho_1)}h_2^{(\rho_2)}\ldots h_a^{(\rho_c)} - \sum_{\substack{\nu_2+\nu_3+\ldots+\nu_d=\psi_b\\0\leq\nu_i}} k_2^{(\nu_2)}k_3^{(\nu_3)}\ldots k_d^{(\psi_d)}]$$

$$- \sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=n+1\\0\leq\rho_i\leq n}} h_1^{(\rho_1)}h_2^{(\rho_2)}\ldots h_a^{(\rho_c)} + \sum_{\substack{\nu_2+\ldots+\nu_d=n+1-\nu_1\\0\leq\nu_i\leq n}} k_1^{(\nu_1)}k_2^{(\nu_2)}\ldots k_{d-1}^{(\nu_{d-1})}$$

$$\circ[\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=\nu_1\\0\leq\phi_i}} f_1^{(\phi_1)}f_2^{(\phi_2)}\ldots f_a^{(\phi_a)} - \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=\nu_1\\0\leq\psi_i}} g_1^{(\psi_1)}g_2^{(\psi_2)}\ldots g_{b-1}^{(\psi_{b-1})}]$$

$$= [\sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=n+1\\0\leq\phi_i\leq n}} f_1^{(\phi_1)}f_2^{(\phi_2)}\ldots f_a^{(\phi_a)} + \sum_{\substack{\nu_2+\nu_3+\ldots+\nu_d=n+1\\0\leq\nu_i}} k_2^{(\nu_2)}k_3^{(\nu_3)}\ldots k_d^{(\psi_d)}$$

$$+ \sum_{\substack{\nu_2+\ldots+\nu_d=n+1-\nu_1\\0\leq\nu_i\leq n}} \sum_{\substack{\phi_1+\phi_2+\ldots+\phi_a=\nu_1\\0\leq\phi_i;\nu_1>0}} k_2^{(\nu_2)}k_3^{(\nu_3)}\ldots k_d^{(\nu_d)}f_1^{(\phi_1)}f_2^{(\phi_2)}\ldots f_a^{(\phi_a)}]$$

$$-[\sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=n+1\\0\leq\rho_i\leq n}} h_1^{(\rho_1)}h_2^{(\rho_2)}\ldots h_a^{(\rho_c)} + \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=n+1\\0\leq\psi_i}} g_1^{(\psi_1)}g_2^{(\psi_2)}\ldots g_{b-1}^{(\psi_{b-1})}$$

$$+ \sum_{\substack{\psi_1+\psi_2+\ldots+\psi_{b-1}=n+1-\psi_b\\0\leq\psi_i\leq n}} \sum_{\substack{\rho_1+\rho_2+\ldots+\rho_c=\psi_b\\0\leq\rho_i;\psi_b>0}} g_1^{(\psi_1)}g_2^{(\psi_2)}\ldots g_{b-1}^{(\psi_{b-1})}h_1^{(\rho_1)}h_2^{(\rho_2)}\ldots h_a^{(\rho_c)}]$$

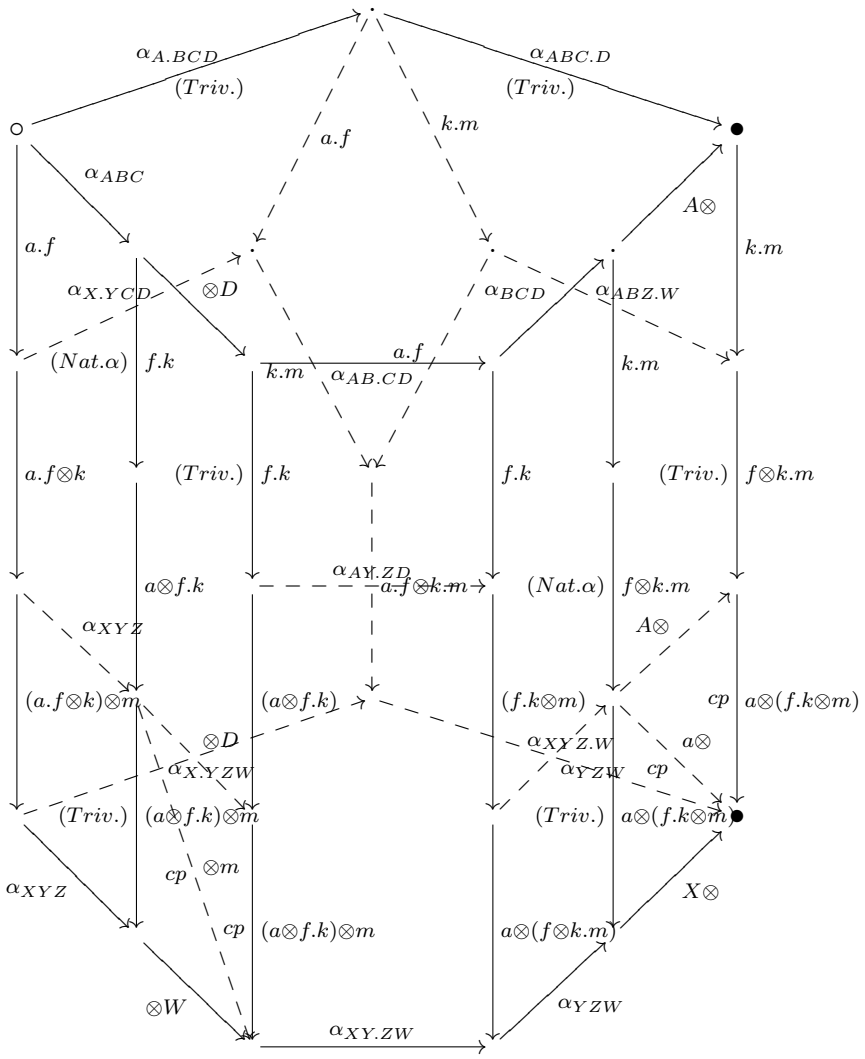
since, for $\psi_b \neq 0$ and $\nu_1 \neq 0$, the first mixed-terms of k and g cancel with the same from the second. This is the n^{th} order obstruction due to the boundary of the glued polygon in which, effect due to glued side vanishes.

□

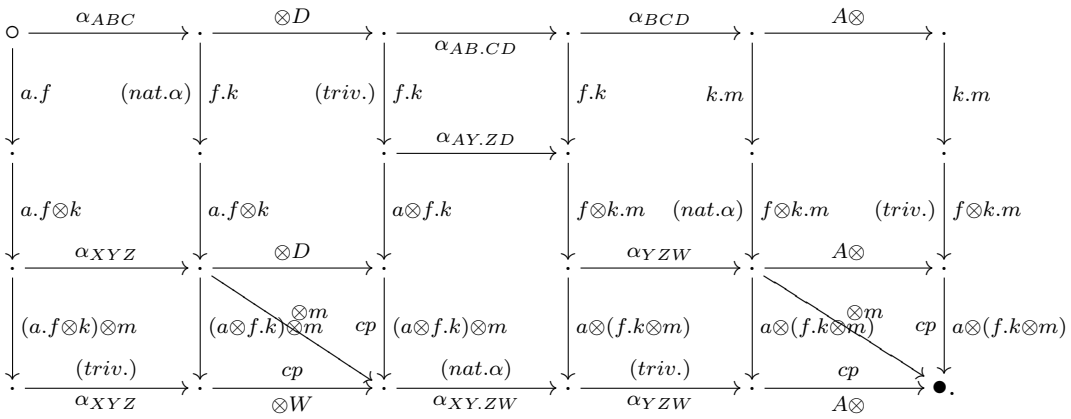
As an example of applying the lemmas, we have

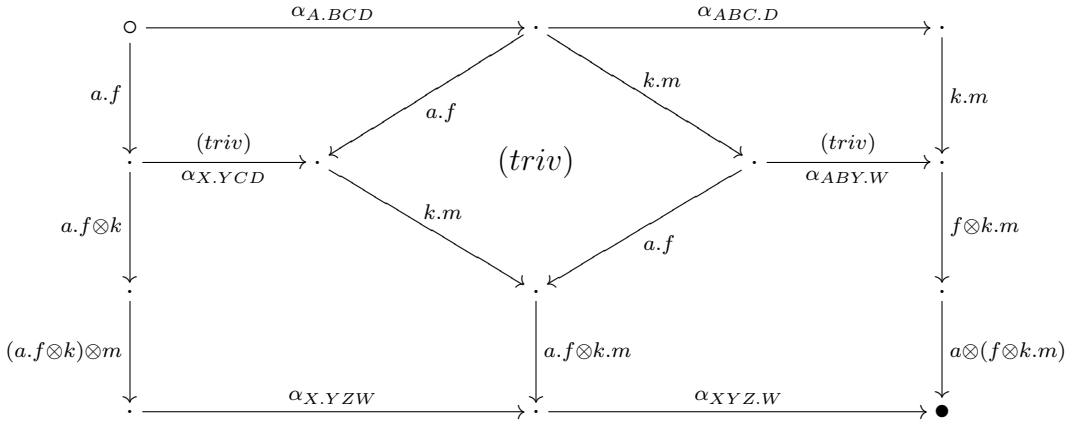
Proposition 2.2.3. *The obstructions satisfy the cocycle condition at (4,1) when tensor and associator are deformed.*

Sketch of proof. For this we have the cell decomposition of the sphere,



For simplicity, to understand the application of lemma here, let us consider first the front and back faces only. It is,





Now, the right-top most hexagon, is a trivial face, that is, all of the equational conditions and obstruction-type expressions named by it vanish trivially. The upper triangle on the bottom of it is middle four interchange and is assumed that it commutes up to order n. So, we can apply lemma to glue with and extend the boundary of face of hexagon to the figure adjoining this triangle because they are admissible polygons. The lower triangle is also the compatibility condition of tensor and composition (here the composition is not deformed), so it commutes up to order n and is admissible with the previously obtained figure. Applying the lemma again, we can extend the obstruction to the boundary to the octagon after gluing the last triangle we just considered. Now, consider the hexagon to the left of the resultant octagon, which is a naturality hexagon for the associator α, so

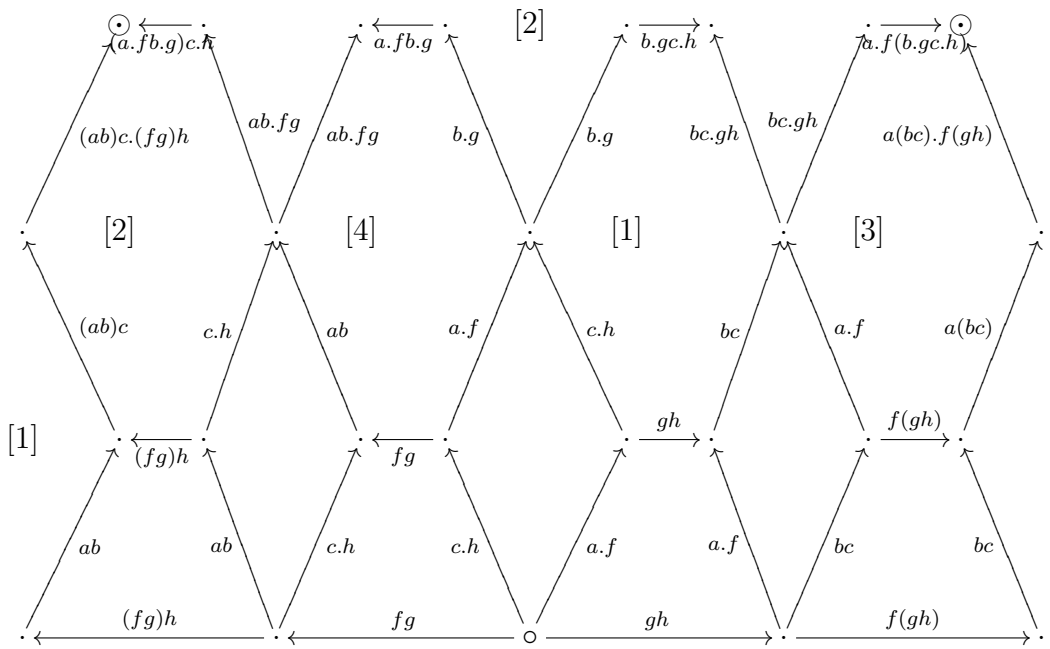
commutes up to n^{th} order. Applying the lemma, we extend the boundary of the polygon adjoining it. Then we can glue with the lower trivial square using lemma again and obtain the obstruction on the boundary of the resulting decagon. Next, we glue the top middle trivial square and then lower naturality hexagon with the help of lemma. Proceeding in the same way, for the polygons on the other two left columns, we can extend the obstruction of the deformation to the boundary of the front face. Following the same trick, on the back of the whole shell, the top three squares are trivial ones and the lower hexagons are naturality hexagons which commute up to order n and are admissible one after the other. Right top polygon is admissible with right bottom polygon. The result of those is admissible with the middle trivial diamond and the obtained polygon is admissible with the left top trivial quadrilateral. Then the obtained polygon is admissible with the lower left hexagon, So, we can apply lemma to get the obstruction around the boundary of resultant back face. Now, the top and bottom of the original cell decomposition of the sphere are the pentagon identity, which, by assumption, commutes up to order n. Both the faces are admissible with front as well as back polygon. Using lemma, we can glue one pentagon with front face and the other with the back. Hence we get the obstruction of the deformations at (4,1) can be obtained around the boundary of front and back faces with opposite signs, so adding together, cancel each other. Hence the cocycle condition at (4,1) holds.

□

Theorem 2.2.2. *In the case of a strict monoidal category and identity preserving deformations, the obstructions to the deformation of k-linear monoidal category are cocycles at (1,4), (2,3) and (3,2).*

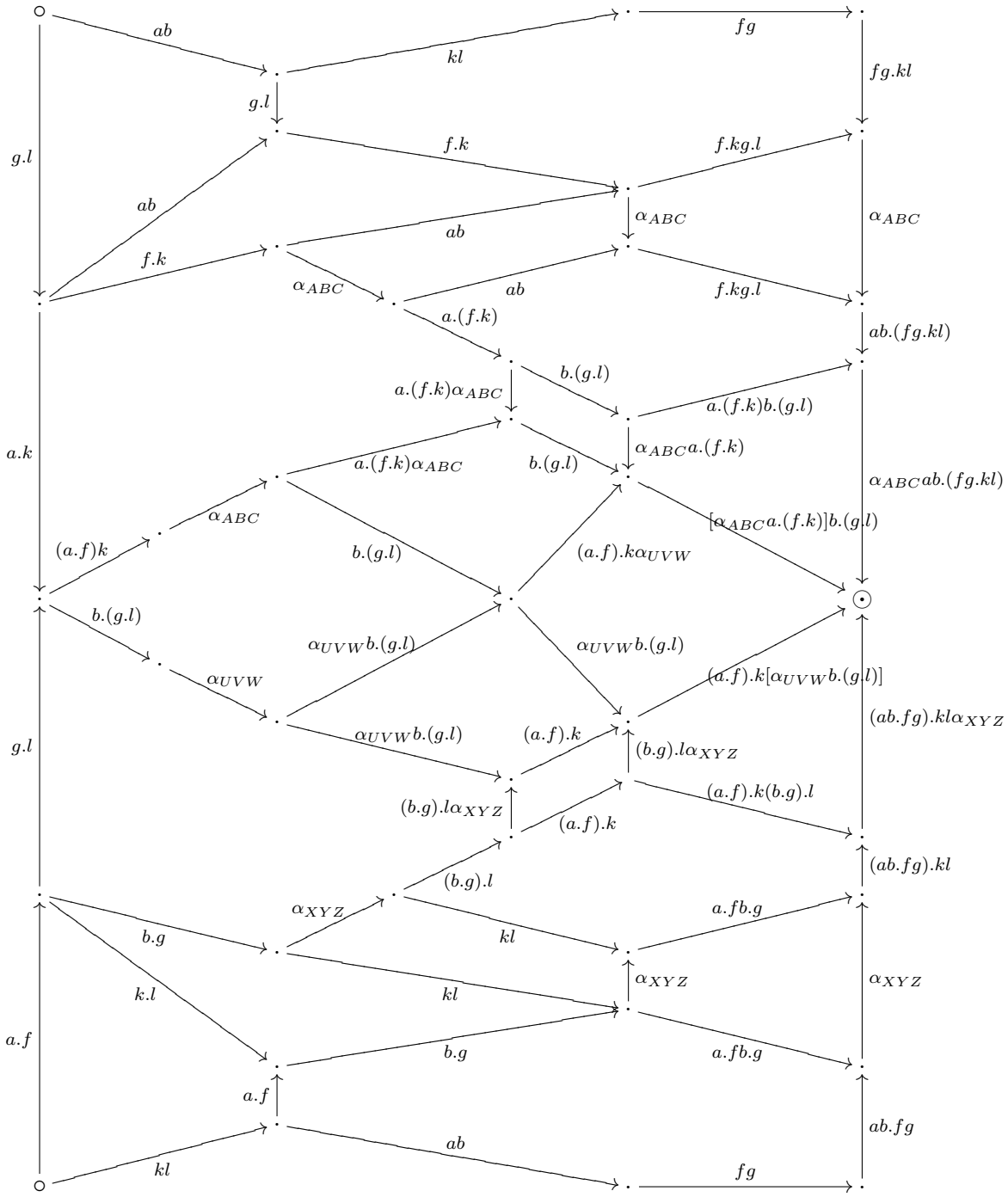
Conjecture 2.2.3. *Under the same hypothese all obstructions are cocylcles.*

Proof of the Theorem. At (1,4), this is just the folk-generalization of Gerstenhaber's result. At (2,3), the cell decomposition of sphere, with the edges of deformable structures, looks like,



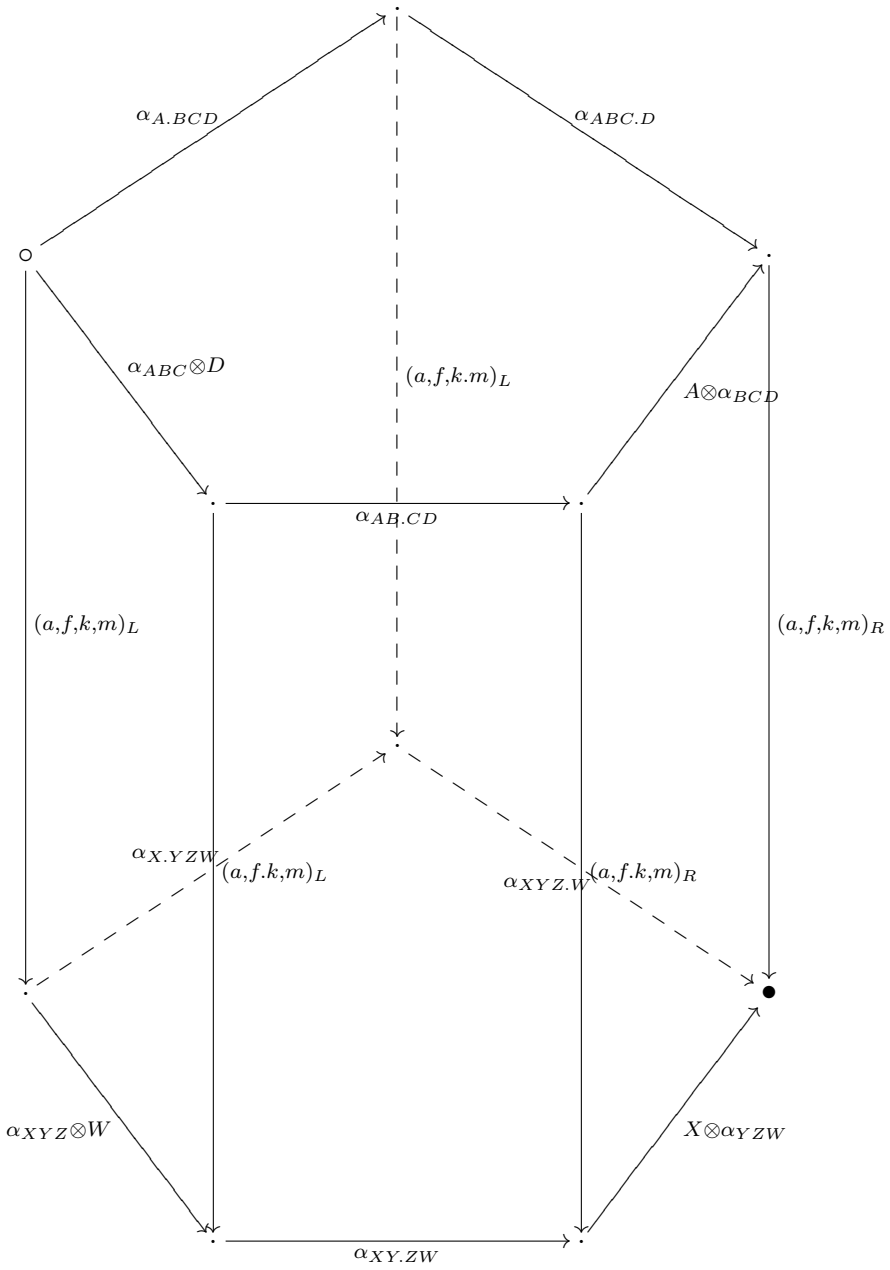
Hence, by use of lemma, they are cocycles.

At (3,2), the cell decomposition of sphere, with the edges of deformable structures, looks like,

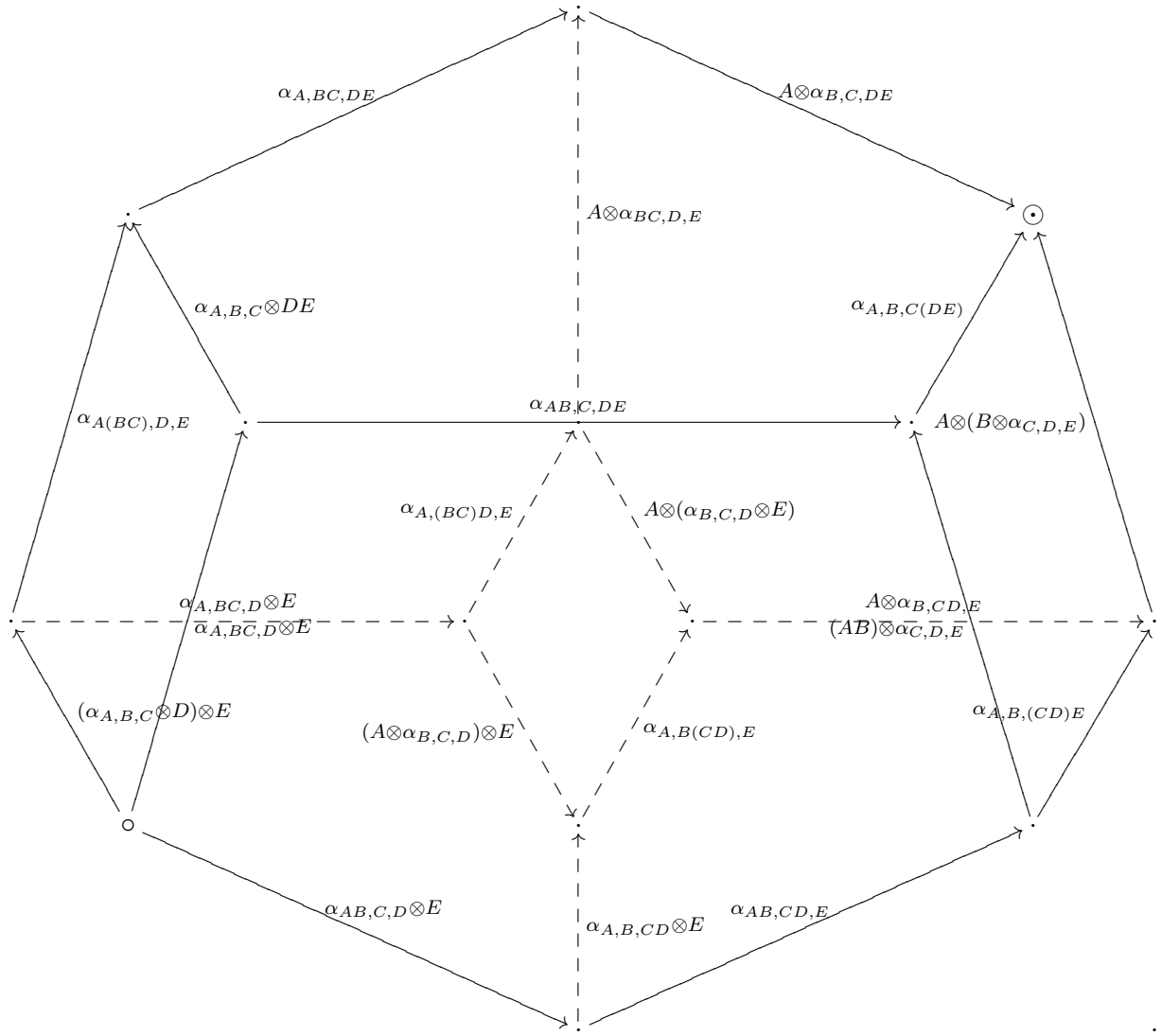


□

To prove the conjecture,at(4,1), the cell decomposition of sphere will be the blown up version of,



and at (5,0), it will be the blown up version of 'The Stasheff polytope' with edges the deformable structures on its edges



2.3 Conclusion

1. *The deformations of identity functor and natural transformation are classified up to equivalence by $H^2(\mathcal{C})$, that of composition, arrow part of tensor,and that of associator are classified up to equivalence by $H^3(\mathcal{C})$.*

2. *The obstructions to the deformation of a k-linear monoidal category are 4-cocycles in special cases.*

Conjecture 2.3.1. *There are higher homotopy differentials which have the structure of a multicomplex.*

Conjecture 2.3.2. *All obstructions are cocycles without additional hypothesis.*

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Appendix A

Appendix: Some hand calculations

For brevity we omit the padded composition marks throughout. Every term in the calculations which follow should be understood as a padded composition.

The cocycle condition of obstruction at (4,1) if tensor and associator only are deformed:

Proof. $d_0(\omega_p^{(1)})(a,f,k,m)$
$= [a,f,k,m]_L\omega_p^{(1)}(X,Y,Z,W) - \omega_p^{(1)}(A,B,C,D)[a,f,k,m]_R$

$= [a,f,k,m]_L[\alpha^{(1)}_{XYZ}\otimes 1_W\alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{XYZ}\otimes 1_W 1_X\otimes\alpha^{(1)}_{YZW}$
$+ \alpha^{(1)}_{XY.ZW}1_X\otimes\alpha^{(1)}_{YZW} - \alpha^{(1)}_{X.YZW}\alpha^{(1)}_{XYZ.W}$
$+ \tau^{(1)}(\alpha^{(1)}_{XYZ},1_W) + \tau^{(1)}(1_X,\alpha^{(1)}_{YZW})]$
$- [\alpha^{(1)}_{ABC}\otimes 1_D\alpha^{(1)}_{AB.CD} + \alpha^{(1)}_{ABC}\otimes 1_D 1_A\otimes\alpha^{(1)}_{BCD}$
$+ \alpha^{(1)}_{AB.CD}1_A\otimes\alpha^{(1)}_{BCD} - \alpha^{(1)}_{A.BCD}\alpha^{(1)}_{ABC.D}$
$+ \tau^{(1)}(\alpha^{(1)}_{ABC},1_D) + \tau^{(1)}(1_A,\alpha^{(1)}_{BCD})][a,f,k,m]_R$
$= \alpha^{(1)}_{ABC}\otimes 1_D[a,f.k,m]_L\alpha^{(1)}_{XY.ZW} + \tau^{(1)}(a,f.k)\otimes m\alpha^{(1)}_{XY.ZW}$
$+ a.\tau^{(1)}(f,k)\otimes m\alpha^{(1)}_{XY.ZW} - \tau^{(1)}(a.f,k)\otimes m\alpha^{(1)}_{XY.ZW}$
$- \tau^{(1)}(a,f).k\otimes m\alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC}\otimes 1_D[a,f.k,m]_L 1_X\otimes\alpha^{(1)}_{YZW}$
$+ \tau^{(1)}(a,f.k)\otimes m1_X\otimes\alpha^{(1)}_{YZW} + a.\tau^{(1)}(f,k)\otimes m1_X\otimes\alpha^{(1)}_{YZW}$
$- \tau^{(1)}(a.f,k)\otimes m1_X\otimes\alpha^{(1)}_{YZW} - \tau^{(1)}(a,f).k\otimes m1_X\otimes\alpha^{(1)}_{YZW}$

$$
\begin{aligned}
&+ \alpha^{(1)}_{AB.CD}[a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ a \otimes \tau^{(1)}(f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a \otimes f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(a, f.k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} - \alpha^{(1)}_{A.BCD}[a.f, k, m]_L \alpha^{(1)}_{XYZ.W} \\
&- \tau^{(1)}(a.f, k \otimes m) \alpha^{(1)}_{XYZ.W} - a.f \otimes \tau^{(1)}(k, m) \alpha^{(1)}_{XYZ.W} \\
&+ \tau^{(1)}(a.f, k) \otimes m \alpha^{(1)}_{XYZ.W} + \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k) \otimes m - \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R \alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k) \otimes m] \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) + \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k).m] \\
&- \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) \\
&- \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k) \otimes m] - \alpha^{(1)}_{AB.CD}[a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) + \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) \\
&+ \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k).m] + \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f) \otimes k.m \\
&+ \alpha^{(1)}_{A.BCD} \tau^{(1)}(a \otimes f, \otimes k.m) + \alpha^{(1)}_{A.BCD}[a, f, k.m] \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f \otimes k.m) - \alpha^{(1)}_{A.BCD} a \otimes \tau^{(1)}(f, k.m) \\
&+ \tau^{(1)}(a.f \otimes k \alpha^{(1)}_{XYZ}, m) - \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ} \otimes 1_W \\
&+ \tau^{(1)}(a, f.k \otimes m \alpha^{(1)}_{YZW}) - \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(\alpha^{(1)}_{ABC} a \otimes f.k, m) + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) \\
&- \tau^{(1)}(a, \alpha^{(1)}_{BCD} f \otimes k.m) + 1_A \otimes \alpha^{(1)}_{BCD} \tau^{(1)}(a, f \otimes k.m)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_L \alpha^{(1)}_{XY.ZW} + \tau^{(1)}(a, f.k) \otimes m \alpha^{(1)}_{XY.ZW} \\
&+ a.\tau^{(1)}(f, k) \otimes m \alpha^{(1)}_{XY.ZW} - \tau^{(1)}(a.f, k) \otimes m \alpha^{(1)}_{XY.ZW} \\
&- \tau^{(1)}(a, f).k \otimes m \alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_L 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \tau^{(1)}(a, f.k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} + a.\tau^{(1)}(f, k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(a.f, k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a, f).k \otimes m 1_X \otimes \alpha^{(1)}_{YZW}
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha^{(1)}_{AB.CD}[a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ a \otimes \tau^{(1)}(f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a \otimes f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(a, f.k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} - \alpha^{(1)}_{A.BCD}[a.f, k, m]_L \alpha^{(1)}_{XYZ.W} \\
&- \tau^{(1)}(a.f, k \otimes m) \alpha^{(1)}_{XYZ.W} - a.f \otimes \tau^{(1)}(k, m) \alpha^{(1)}_{XYZ.W} \\
&+ \tau^{(1)}(a.f, k) \otimes m \alpha^{(1)}_{XYZ.W} + \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k) \otimes m - \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R \alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k) \otimes m] \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) + \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k).m] \\
&- \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) \\
&- \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k) \otimes m] - \alpha^{(1)}_{AB.CD}[a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) + \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) \\
&+ \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k).m] + \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f) \otimes k.m \\
&+ \alpha^{(1)}_{A.BCD} \tau^{(1)}(a \otimes f, \otimes k.m) + \alpha^{(1)}_{A.BCD}[a, f, k.m] \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f \otimes k.m) - \alpha^{(1)}_{A.BCD} a \otimes \tau^{(1)}(f, k.m) \\
&+ \tau^{(1)}(a.f \otimes k \alpha^{(1)}_{XYZ}, m) - \tau^{(1)}(\alpha^{(1)}_{ABC} a \otimes f.k, m) \\
&+ \tau^{(1)}(a, f.k \otimes m \alpha^{(1)}_{YZW}) - \tau^{(1)}(a, \alpha^{(1)}_{BCD} f \otimes k.m) \\
&- \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ} \otimes 1_W - \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) + 1_A \otimes \alpha^{(1)}_{BCD} \tau^{(1)}(a, f \otimes k.m)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_L \alpha^{(1)}_{XY.ZW} + \tau^{(1)}(a, f.k) \otimes m \alpha^{(1)}_{XY.ZW} \\
&+ a.\tau^{(1)}(f, k) \otimes m \alpha^{(1)}_{XY.ZW} - \tau^{(1)}(a.f, k) \otimes m \alpha^{(1)}_{XY.ZW} \\
&- \tau^{(1)}(a, f).k \otimes m \alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_L 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \tau^{(1)}(a, f.k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} + a.\tau^{(1)}(f, k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(a.f, k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a, f).k \otimes m 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \alpha^{(1)}_{AB.CD} [a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ a \otimes \tau^{(1)}(f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a \otimes f.k, m) 1_X \otimes \alpha^{(1)}_{YZW} \\
&- \tau^{(1)}(a, f.k) \otimes m 1_X \otimes \alpha^{(1)}_{YZW} - \alpha^{(1)}_{A.BCD} [a.f, k, m]_L \alpha^{(1)}_{XYZ.W} \\
&- \tau^{(1)}(a.f, k \otimes m) \alpha^{(1)}_{XYZ.W} - a.f \otimes \tau^{(1)}(k, m) \alpha^{(1)}_{XYZ.W} \\
&+ \tau^{(1)}(a.f, k) \otimes m \alpha^{(1)}_{XYZ.W} + \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k) \otimes m - \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R \alpha^{(1)}_{XY.ZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) \\
&- \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) - \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k) \otimes m] \\
&- \alpha^{(1)}_{ABC} \otimes 1_D [a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} + \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a, f.k \otimes m) \\
&+ \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k, m) + \alpha^{(1)}_{ABC} \otimes 1_D a \otimes [\tau^{(1)}(f, k).m] \\
&- \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) - \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) \\
&- \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k) \otimes m] - \alpha^{(1)}_{AB.CD} [a, f.k, m]_R 1_X \otimes \alpha^{(1)}_{YZW} \\
&+ \alpha^{(1)}_{AB.CD} \tau^{(1)}(a, f.k \otimes m) + \alpha^{(1)}_{AB.CD} a \otimes \tau^{(1)}(f.k, m) \\
&+ \alpha^{(1)}_{AB.CD} a \otimes [\tau^{(1)}(f, k).m] + \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f) \otimes k.m \\
&+ \alpha^{(1)}_{A.BCD} \tau^{(1)}(a \otimes f, \otimes k.m) + \alpha^{(1)}_{A.BCD} [a, f, k.m] \alpha^{(1)}_{XYZ.W} \\
&- \alpha^{(1)}_{A.BCD} \tau^{(1)}(a, f \otimes k.m) - \alpha^{(1)}_{A.BCD} a \otimes \tau^{(1)}(f, k.m) \\
&+ \tau^{(1)}(a.\tau^{(1)}(f, k), m) + \tau^{(1)}(\tau^{(1)}(a, f.k), m) \\
&- \tau^{(1)}(\tau^{(1)}(a, f).k, m) - \tau^{(1)}(\tau^{(1)}(a.f, k), m) \\
&+ \tau^{(1)}(a, \tau^{(1)}(f, k \otimes m)) + \tau^{(1)}(a, f \otimes \tau^{(1)}(k, m)) \\
&- \tau^{(1)}(a, \tau^{(1)}(f, k).m) - \tau^{(1)}(a, \tau^{(1)}(f.k, m)) \\
&- \tau^{(1)}(a.f \otimes k, m) \alpha^{(1)}_{XYZ} \otimes 1_W - \tau^{(1)}(a, f.k \otimes m) 1_X \otimes \alpha^{(1)}_{YZW}
\end{aligned}
$$

$$+ \alpha^{(1)}_{ABC} \otimes 1_D \tau^{(1)}(a \otimes f.k, m) + 1_A \otimes \alpha^{(1)}_{BCD} \tau^{(1)}(a, f \otimes k.m)$$

$$d_1(\omega_N^{(1)})(a,f,k,m) = a \otimes \omega_N^{(1)}(f,k,m) - \omega_N^{(1)}(a \otimes f,k,m)$$
$$+ \omega_N^{(1)}(a,f \otimes k,m) - \omega_N^{(1)}(a,f,k \otimes m) + \omega_N^{(1)}(a,f,k) \otimes m$$

$$= a \otimes [\alpha_{BCD}^{(1)} \tau^{(1)}(f,k.m) + \alpha_{BCD}^{(1)} f \otimes \tau^{(1)}(k,m) + \tau^{(1)}(f,\tau^{(1)}(k,m))$$
$$- \tau^{(1)}(f.k,m)\alpha_{YZW}^{(1)} - \tau^{(1)}(f,k).m\alpha_{YZW}^{(1)} - \tau^{(1)}(\tau^{(1)}(f,k),m)]$$
$$- [\alpha_{A.BCD}^{(1)} \tau^{(1)}(a.f,k \otimes m) + \alpha_{A.BCD}^{(1)} a.f \otimes \tau^{(1)}(k,m) + \tau^{(1)}(a.f,\tau^{(1)}(k,m))$$
$$- \tau^{(1)}(a.f \otimes k,m)\alpha_{X.YZW}^{(1)} - \tau^{(1)}(a.f,k) \otimes m\alpha_{X.YZW}^{(1)} - \tau^{(1)}(\tau^{(1)}(a.f,k),m)]$$
$$+ [\alpha_{AB.CD}^{(1)} \tau^{(1)}(a,f.k \otimes m) + \alpha_{AB.CD}^{(1)} a \otimes \tau^{(1)}(f.k,m) + \tau^{(1)}(a,\tau^{(1)}(f.k,m))$$
$$- \tau^{(1)}(a \otimes f.k,m)\alpha_{XY.ZW}^{(1)} - \tau^{(1)}(a,f.k) \otimes m\alpha_{XY.ZW}^{(1)} - \tau^{(1)}(\tau^{(1)}(a,f.k),m)]$$
$$- [\alpha_{ABC.D}^{(1)} \tau^{(1)}(a,f \otimes k.m) + \alpha_{ABC.D}^{(1)} a \otimes \tau^{(1)}(f,k.m) + \tau^{(1)}(a,\tau^{(1)}(f,k.m))$$
$$- \tau^{(1)}(a \otimes f,k.m)\alpha_{XYZ.W}^{(1)} - \tau^{(1)}(a,f) \otimes k.m\alpha_{XYZ.W}^{(1)} - \tau^{(1)}(\tau^{(1)}(a,f),k.m)]$$
$$+ [\alpha_{ABC}^{(1)} \tau^{(1)}(a,f.k) + \alpha_{ABC}^{(1)} a \otimes \tau^{(1)}(f,k) + \alpha_{ABC} \tau^{(1)}(a,\tau^{(1)}(f,k))$$
$$- \tau^{(1)}(a.f,k)\alpha_{XYZ}^{(1)} - \tau^{(1)}(a,f) \otimes k\alpha_{XYZ}^{(1)} - \tau^{(1)}(\tau^{(1)}(a,f),k)] \otimes m$$

Now

$$d_0 + d_1 = [a.\tau^{(1)}(f,k) \otimes m\alpha^{(1)}_{XY.ZW} - \tau^{(1)}(a.f,k) \otimes m\alpha^{(1)}_{XY.ZW}$$
$$- \tau^{(1)}(a,f).k \otimes m\alpha^{(1)}_{XY.ZW} + \tau^{(1)}(a,f.k) \otimes m1_X \otimes \alpha^{(1)}_{YZW}$$
$$+ a.\tau^{(1)}(f,k) \otimes m1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a.f,k) \otimes m1_X \otimes \alpha^{(1)}_{YZW}$$
$$- \tau^{(1)}(a,f).k \otimes m1_X \otimes \alpha^{(1)}_{YZW} + \tau^{(1)}(a,f.k \otimes m)1_X \otimes \alpha^{(1)}_{YZW}$$
$$- \tau^{(1)}(a \otimes f.k,m)1_X \otimes \alpha^{(1)}_{YZW} - \tau^{(1)}(a,f.k) \otimes m1_X \otimes \alpha^{(1)}_{YZW}$$
$$- a.f \otimes \tau^{(1)}(k,m)\alpha^{(1)}_{XYZ.W} + \tau^{(1)}(a.f,k) \otimes m\alpha^{(1)}_{XYZ.W}$$
$$+ \tau^{(1)}(a.f \otimes k,m)\alpha^{(1)}_{XYZ.W} - \alpha^{(1)}_{ABC} \otimes 1_D\tau^{(1)}(a \otimes f.k,m)$$
$$+ \alpha^{(1)}_{ABC} \otimes 1_D\tau^{(1)}(a,f.k \otimes m) + \alpha^{(1)}_{ABC} \otimes 1_D a \otimes \tau^{(1)}(f.k,m)$$
$$- \alpha^{(1)}_{AB.CD}a \otimes [\tau^{(1)}(f,k) \otimes m] + \alpha^{(1)}_{AB.CD}a \otimes \tau^{(1)}(f.k,m)$$
$$+ \alpha^{(1)}_{AB.CD}a \otimes [\tau^{(1)}(f,k).m] + \alpha^{(1)}_{A.BCD}\tau^{(1)}(a,f) \otimes k.m$$
$$- \alpha^{(1)}_{A.BCD}\tau^{(1)}(a,f \otimes k.m) - \alpha^{(1)}_{A.BCD}a \otimes \tau^{(1)}(f,k.m)$$
$$+ \tau^{(1)}(a.\tau^{(1)}(f,k),m) - \tau^{(1)}(\tau^{(1)}(a,f).k,m)$$
$$+ \tau^{(1)}(a,f \otimes \tau^{(1)}(k,m)) - \tau^{(1)}(a,\tau^{(1)}(f,k).m)$$
$$- \tau^{(1)}(a.f \otimes k,m)\alpha^{(1)}_{XYZ} \otimes 1_W - \tau^{(1)}(a,f.k \otimes m)1_X \otimes \alpha^{(1)}_{YZW}$$
$$+ \alpha^{(1)}_{ABC} \otimes 1_D\tau^{(1)}(a \otimes f.k,m) + 1_A \otimes \alpha^{(1)}_{BCD}\tau^{(1)}(a,f \otimes k.m)]$$
$$+ a \otimes \omega^{(1)}_N(f,k,m) - \omega^{(1)}_N(a \otimes f,k,m)$$
$$+ \omega^{(1)}_N(a,f \otimes k,m) - \omega^{(1)}_N(a,f,k \otimes m) + \omega^{(1)}_N(a,f,k) \otimes m$$
$$+ [a \otimes [\alpha^{(1)}_{BCD}\tau^{(1)}(f,k.m) + \alpha^{(1)}_{BCD}f \otimes \tau^{(1)}(k,m) + \tau^{(1)}(f,\tau^{(1)}(k,m))$$
$$- \tau^{(1)}(f,k).m\alpha^{(1)}_{YZW} - \tau^{(1)}(\tau^{(1)}(f,k),m)]$$
$$- [\alpha^{(1)}_{A.BCD}a.f \otimes \tau^{(1)}(k,m) + \tau^{(1)}(a.f,\tau^{(1)}(k,m))$$
$$- \tau^{(1)}(a.f \otimes k,m)\alpha^{(1)}_{X.YZW} - \tau^{(1)}(a.f,k) \otimes m\alpha^{(1)}_{X.YZW}]$$
$$+ [\alpha^{(1)}_{AB.CD}\tau^{(1)}(a,f.k \otimes m) - \tau^{(1)}(a \otimes f.k,m)\alpha^{(1)}_{XY.ZW}]$$
$$- [\alpha^{(1)}_{ABC.D}\tau^{(1)}(a,f \otimes k.m) - \tau^{(1)}(a,f) \otimes k.m\alpha^{(1)}_{XYZ.W} - \tau^{(1)}(\tau^{(1)}(a,f),k.m)]$$
$$+ [\alpha^{(1)}_{ABC}a \otimes \tau^{(1)}(f,k) + \alpha_{ABC}\tau^{(1)}(a,\tau^{(1)}(f,k))$$
$$- \tau^{(1)}(a.f,k)\alpha^{(1)}_{XYZ} - \tau^{(1)}(a,f) \otimes k\alpha^{(1)}_{XYZ} - \tau^{(1)}(\tau^{(1)}(a,f),k)] \otimes m]$$

Using the lower order coboundary conditions, as in the identity $\alpha_{A.BCD}[\tau^{(1)}(a,f).k]\otimes m = \tau^{(1)}(a,f)\otimes k.m\alpha_{X.YZW}$, we can rearrange groups of terms in which each group will have a factor like $\lceil \alpha^{(1)}_{ABC}\otimes 1_D - \alpha^{(1)}_{A.BCD} + \alpha^{(1)}_{AB.CD} - \alpha^{(1)}_{ABC.D} + 1_A \otimes \alpha^{(1)}_{BCD}\rceil$ or the same expression on the target pentagon. Hence the obstructions are cocycle at (4,1) in this case.

□

By-hand calculation for cocycle condition of obstructions at (3,2) if all structures are deformed (Note: The superscripts in square brackets preceding terms indicate groups of terms which are cancelled, either simply with like terms of opposite sign or by applying a lower order relation.):

Proof. $[d_0(\omega_N^{(1)}) - d_1(\omega_{cp}^{(1)}) - d_2(\omega_c^{(1)})] \begin{pmatrix} a & f & k \\ b & g & l \end{pmatrix} = a.f \otimes k\omega_N^{(1)}(b,g,l) - \omega_N^{(1)}(ab, fg, kl)$
$+ \omega_N^{(1)}(a,f,k)b \otimes g.l - [ab \otimes \omega_{cp}^{(1)}(f,g;k,l) - \omega_{cp}^{(1)}(a \otimes f, k; b \otimes g, l)$
$+ \omega_{cp}^{(1)}(a, f \otimes k; b, g \otimes l) - \omega_{cp}^{(1)}(a,f;b,g) \otimes kl] - [\omega_c^{(1)}(\alpha_{ABC}; a \otimes f.k; b \otimes g.l)$
$- \omega_c^{(1)}(a.f \otimes k; \alpha_{UVW}; b \otimes g.l) + \omega_c^{(1)}(a.f \otimes k; b.g \otimes l; \alpha_{XYZ})]$

$= [a.f \otimes k\mu^{(1)}(\alpha_{UVW}^{(1)}, b \otimes g.l) + a.f \otimes k\mu^{(1)}(\alpha_{UVW}, \tau^{(1)}(b, g.l)$
$+ a.f \otimes k\mu^{(1)}(\alpha_{UVW}, b \otimes g.l) + a.f \otimes k\alpha_{UVW}^{(1)}\tau^{(1)}(b, g.l)$
$+ a.f \otimes k\alpha_{UVW}^{(1)} b \otimes \tau^{(1)}g, l) + a.f \otimes k\alpha_{UVW}\tau^{(1)}(b, \tau^{(1)}(g,l))$
$- a.f \otimes k\tau^{(1)}(b, \tau^{(1)}(g,l))\alpha_{XYZ} - a.f \otimes k\mu^{(1)}(\tau^{(1)}(b.g, l), \alpha_X YZ)$
$- a.f \otimes k\mu^{(1)}(\tau^{(1)}(b,g) \otimes l, \alpha_{XYZ}) - a.f \otimes k\mu^{(1)}(b.g \otimes l, \alpha_{XYZ}^{(1)})$
$- a.f \otimes k\tau^{(1)}(b,g) \otimes l\alpha_{XYZ}^{(1)} - a.f \otimes k\tau^{(1)}(b.g, l)\alpha_{XYZ}^{(1)}]$
$- [\alpha_{ABC}^{(1)}\tau^{(1)}(ab, fg \otimes kl) + \mu^{(1)}(\alpha_{ABC}^{(1)}, ab \otimes fg.kl)$
$+ \alpha_{ABC}^{(1)} ab \otimes \tau^{(1)}(fg, kl) + \mu^{(1)}(\alpha_{ABC}, ab \otimes \tau^{(1)}(fg, kl))$
$+ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(ab, fg.kl)) + \alpha_{ABC}\tau^{(1)}(ab, \tau^{(1)}(fg, kl))$
$- \tau^{(1)}(\tau^{(1)}(ab, fg), kl)\alpha_{XYZ} - \mu^{(1)}(\tau^{(1)}(ab, fg) \otimes kl, \alpha_{XYZ})$
$- \mu^{(1)}(\tau^{(1)}(ab.fg, kl), \alpha_{XYZ}) - \mu^{(1)}(ab.fg \otimes kl, \alpha_{XYZ}^{(1)})$
$- \tau^{(1)}(ab, fg) \otimes \alpha_{XYZ}^{(1)} - \tau^{(1)}(ab.fg, kl)\alpha_{XYZ}^{(1)}]$
$+ [\mu^{(1)}(\alpha_{ABC}^{(1)}, a \otimes f.kb \otimes g.l) + \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k))b \otimes g.l$
$+ \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f,k))b \otimes g.l + \alpha_{ABC}^{(1)}\tau^{(1)}(a, f.k)b \otimes g.l$
$+ \alpha_{ABC}^{(1)} a \otimes \tau^{(1)}(f,k)b \otimes g.l + \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f,k))b \otimes g.l$
$- \tau^{(1)}(\tau^{(1)}(a,f),k)\alpha_{UVW} b \otimes g.l - \mu^{(1)}(\tau^{(1)}(a.f,k), \alpha_{UVW})b \otimes g.l$

$$
\begin{aligned}
&- \mu^{(1)}(\tau^{(1)}(a,f)\otimes k, \alpha_{UVW})b\otimes g.l - \mu^{(1)}(a.f\otimes k, \alpha^{(1)}_{UVW})b\otimes g.l \\
&- \tau^{(1)}(a, f\otimes k)\alpha^{(1)}_{UVW} b\otimes g.l - \tau^{(1)}(a.f,k)\alpha^{(1)}_{UVW} b\otimes g.l] \\
&- \alpha_{ABC} ab\otimes [\tau^{(1)}(f,k)\tau^{(1)}(g,l) + \mu^{(1)}(\tau^{(1)}(f,k), g\otimes l) + \mu^{(1)}(f.k, \tau^{(1)}(g,l)) \\
&- \mu^{(1)}(f,g)\otimes\mu^{(1)}(k,l) - \tau^{(1)}(\mu^{(1)}(f,g), kl) - \tau^{(1)}(fg, \mu^{(1)}(k,l))] \\
&+ [\tau^{(1)}(a.f,k)\tau^{(1)}(b,g,l) + \mu^{(1)}(\tau^{(1)}(a.f,k), b.g\otimes l) + \mu^{(1)}(a.f\otimes k, \tau^{(1)}(b.g,l) \\
&- \mu^{(1)}(a.f, b.g)\otimes\mu^{(1)}(k,l) - \tau^{(1)}(\mu^{(1)}(a.f,b.g), kl) \\
&- \tau^{(1)}(ab.fg, \mu^{(1)}(k,l))]\alpha_{XYZ} \\
&- \alpha_{ABC}[\tau^{(1)}(a, f.k)\tau^{(1)}(b,g.l) + \mu^{(1)}(\tau^{(1)}(a,f.k), b\otimes g.l) \\
&+ \mu^{(1)}(a\otimes f.k, \tau^{(1)}(b,g.l)) - \mu^{(1)}(a,b)\otimes\mu^{(1)}(f.k, g.l) \\
&- \tau^{(1)}(\mu^{(1)}(a,b), fg.kl) - \tau^{(1)}(ab, \mu^{(1)}(f.k, g.l))] \\
&+ [\tau^{(1)}(a,f)\tau^{(1)}(b,g) + \mu^{(1)}(\tau^{(1)}(a,f), b.g) \\
&+ \mu^{(1)}(a.f, \tau^{(1)}(b,g)) - \mu^{(1)}(a,b)\otimes\mu^{(1)}(f,g) \\
&- \tau^{(1)}(\mu^{(1)}(a,b), fg) - \tau^{(1)}(ab, \mu^{(1)}(f,g))]\otimes\alpha_{XYZ} \\
&- \mu^{(1)}(\mu^{(1)}(\alpha_{ABC}, a\otimes f.k), b\otimes g.l) + \mu^{(1)}(\alpha_{ABC}, \mu^{(1)}(a\otimes f.k, b\otimes g.l)) \\
&+ \mu^{(1)}(\mu^{(1)}(a.f\otimes k, \alpha_{UVW}, b\otimes g.l) - \mu^{(1)}(a.f\otimes k, \mu^{(1)}(\alpha_{UVW}, b\otimes g.l)) \\
&- \mu^{(1)}(\mu^{(1)}(a.f\otimes k, b.g\otimes l), \alpha_{XYZ}) - \mu^{(1)}(a.f\otimes k, \mu^{(1)}(b.g\otimes l, \alpha_{XYZ}))
\end{aligned}
$$

$$
\begin{aligned}
&=^{[13]} a.f\otimes k\mu^{(1)}(\alpha^{(1)}_{UVW}, b\otimes g.l) + a.f\otimes\mu^{(1)}(\alpha_{UVW}, \tau^{(1)}(b,g.l)) \\
&+ a.f\otimes k\mu^{(1)}(\alpha_{UVW}, b\otimes\tau^{(1)}(g,l)) + a.f\otimes k\alpha^{(1)}_{UVW}\tau^{(1)}(b,g.l) \\
&+ a.f\otimes k\alpha^{(1)}_{UVW} b\otimes\tau^{(1)}(g,l) + a.f\otimes k\alpha^{(1)}_{UVW}\tau^{(1)}(b,\tau^{(1)}(g,l)) \\
&- a.f\otimes k\tau^{(1)}(b,\tau^{(1)}(b,l))\alpha_{XYZ} - a.f\otimes k\mu^{(1)}(\tau^{(1)}(b,g.l), \alpha_{XYZ}) \\
&- a.f\otimes k\mu^{(1)}(\tau^{(1)}(b,g)\otimes l, \alpha_{XYZ}) -^{[11]} a.f\otimes k\mu^{(1)}(b.g\otimes l, \alpha_{XYZ}) \\
&-^{[5]} a.f\otimes k\mu^{(1)}(b,g)\otimes l, \alpha_{XYZ}) -^{[4]} a.f\otimes k\tau^{(1)}(b.g,l)\alpha_{XYZ}) \\
&-^{[7]} \alpha^{(1)}_{ABC}\tau^{(1)}(a,f.k)\tau^{(1)}(b,g.l) -^{[12]} \alpha^{(1)}_{ABC}\mu^{(1)}(a\otimes f.k, b\otimes g.l) \\
&- \alpha^{(1)}_{ABC} a\otimes f.k\tau^{(1)}(b,g.l) + \alpha^{(1)}_{ABC}\mu^{(1)}(a,b)\otimes fg.kl \\
&+^{[1]} \alpha^{(1)}_{ABC} ab\otimes\mu^{(1)}(f.k, g.l) -^{[12]} \mu^{(1)}(\alpha^{(1)}_{ABC}, a\otimes f.kb\otimes g.l) \\
&- \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a,f.k)b\otimes g.l) -^{[2]} \mu^{(1)}(\alpha_{ABC}, \mu^{(1)}(a\otimes f.k, b\otimes g.l))
\end{aligned}
$$

$$
\begin{aligned}
&- \mu^{(1)}(\alpha_{ABC}, a \otimes f.k\tau^{(1)}(b, g.l)) + \mu^{(1)}(\alpha_{ABC}, \mu^{(1)}(a, b) \otimes fg.kl) \\
&+^{[9]} \mu^{(1)}(\alpha_{ABC}, ab \otimes \mu^{(1)}(f.k, g.l)) - \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f, k)b \otimes g.l) \\
&-^{[9]} \mu^{(1)}(\alpha^{(1)}_{ABC}, ab \otimes \mu^{(1)}(f.k, g.l)) - \mu^{(1)}(\alpha_{ABC}, a \otimes f.kb \otimes \tau^{(1)}b, l) \\
&+ \mu^{(1)}(\alpha_{ABC}, ab \otimes \mu^{(1)}(f, g)kl) + \mu^{(1)}(\alpha_{ABC}, ab \otimes fg.\mu^{(1)}(k, l)) \\
&-^{[8]} \alpha^{(1)}_{ABC}ab \otimes \tau^{(1)}(f, k)g.l - \alpha^{(1)}_{ABC}ab \otimes (f.k\tau^{(1)}(g, l)) \\
&-^{[1]} \alpha^{(1)}_{ABC}ab \otimes \mu^{(1)}(f.k, g.l) - \alpha^{(1)}_{ABC}ab \otimes \mu^{(1)}(f, g).kl \\
&+ \alpha^{(1)}_{ABC}ab \otimes fg.\mu^{(1)}(k, l) - \alpha_{ABC}\tau^{(1)}(ab, \tau^{(1)}(f, k)g.l) \\
&- \alpha_{ABC}\tau^{(1)}(ab, \mu^{(1)}(f.k, g.l) - \alpha_{ABC}\tau^{(1)}(ab, f.k\tau^{(1)}(g, l)) \\
&+ \alpha_{ABC}\tau^{(1)}(ab, \mu^{(1)}(f, g).kl) + \alpha_{ABC}\tau^{(1)}(ab, fg.\mu^{(1)}(k, l)) \\
&+ \tau^{(1)}(\tau^{(1)}(a, f)b.g, kl)\alpha_{XYZ} + \tau^{(1)}(a.f\tau^{(1)}(b, g), kl)\alpha_{XYZ} \\
&+ \tau^{(1)}(\mu^{(1)}(a.f, b.g), kl)\alpha_{XYZ} - \tau^{(1)}(\mu^{(1)}(a, b).fg, kl)\alpha_{XYZ} \\
&- \tau^{(1)}(ab.\mu^{(1)}(f, g), kl)\alpha_{XYZ} + \mu^{(1)}(\tau^{(1)}(a, f)b.g \otimes kl, \alpha_{XYZ}) \\
&+ \mu^{(1)}(a.f\tau^{(1)}(b, g) \otimes kl, \alpha_{XYZ}) +^{[10]} \mu^{(1)}(\mu^{(1)}(a.f, b.g) \otimes kl, \alpha_{XYZ}) \\
&- \mu^{(1)}(\mu^{(1)}(a, b).fg \otimes kl, \alpha_{XYZ}) - \mu^{(1)}(ab.\mu^{(1)}(f, g) \otimes kl, \alpha_{XYZ}) \\
&+ \mu^{(1)}(\tau^{(1)}(a.f, k)b.g \otimes l, \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k\tau^{(1)}(b.g, l), \alpha_{XYZ}) \\
&+^{[3]} \mu^{(1)}(\mu^{(1)}(a.f \otimes k, b.g \otimes l), \alpha_{XYZ}) -^{[10]} \mu^{(1)}(\mu^{(1)}(a.f, b.g) \otimes kl, \alpha_{XYZ}) \\
&- \mu^{(1)}(ab, fg) \otimes \mu^{(1)}(k, l)\alpha_{XYZ} + \mu^{(1)}(a.f \otimes kb.g \otimes l, \alpha^{(1)}_{XYZ}) \\
&+ \tau^{(1)}(a, f)b.g \otimes kl\alpha^{(1)}_{XYZ} +^{[6]} \mu^{(1)}(a.f, b.g) \otimes kl\alpha^{(1)}_{XYZ} \\
&+^{[5]} a.f\tau^{(1)}(b, g) \otimes kl\alpha^{(1)}_{XYZ} - \mu^{(1)}(a, b).fg \otimes kl\alpha^{(1)}_{XYZ} \\
&- ab.\mu^{(1)}f, g) \otimes kl\alpha^{(1)}_{XYZ} + \tau^{(1)}(a.f, k)b.g \otimes l\alpha^{(1)}_{XYZ} \\
&+^{[4]} a.f \otimes k\tau^{(1)}(b.g, l)\alpha^{(1)}_{XYZ} +^{[11]} \mu(1)(a.f \otimes k, b.g \otimes l)\alpha^{(1)}_{XYZ} \\
&-^{[6]} \mu^{(1)}(a.f, b.g) \otimes kl\alpha^{(1)}_{XYZ} - ab.fg \otimes \mu^{(1)}(k, l)\alpha^{(1)}_{XYZ} \\
&+^{[12]} \mu^{(1)}(\alpha^{(1)}_{ABC}, a \otimes f.kb \otimes g.l) +^{[7]} \alpha^{(1)}_{ABC}\tau^{(1)}(a, f.k)b \otimes g.l \\
&+^{[8]} \alpha^{(1)}_{ABC}a \otimes f.k\tau^{(1)}(b, g.l) + \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k))b \otimes g.l \\
&+ \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f, k))b \otimes g.l + \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f, k))b \otimes g.l \\
&- \tau^{(1)}(\tau^{(1)}(a, f), k)\alpha_{UVW}b \otimes g.l - \mu^{(1)}(\tau^{(1)}(a, f) \otimes k, \alpha_{UVW})b \otimes g.l \\
&-^{[13]} \mu^{(1)}(a.f \otimes k, \alpha^{(1)}_{UVW})b \otimes g.l - \tau^{(1)}(a.f, k)\alpha^{(1)}_{UVW})b \otimes g.l
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha_{ABC} ab \otimes \tau^{(1)}(f,k)\tau^{(1)}(g,l) - \alpha_{ABC} ab \otimes \mu^{(1)}(\tau^{(1)}(f,k), g.l) \\
&- \alpha_{ABC} ab \otimes \mu^{(1)}(f.k, \tau^{(1)}(g,l)) - \alpha_{ABC} ab \otimes \mu^{(1)}(f,g).\mu^{(1)}(k,l) \\
&- \alpha_{ABC} ab \otimes \tau^{(1)}(\mu^{(1)}(f,g), kl) - \alpha_{ABC} ab \otimes \tau^{(1)}(fg, \mu^{(1)}(k,l) \\
&+ \tau^{(1)}(a.f,k)\tau^{(1)}(b.g,l)\alpha_{XYZ} + \mu^{(1)}(\tau^{(1)}(a.f,k), b.g \otimes l)\alpha_{XYZ} \\
&+ \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b.g,l))\alpha_{XYZ} - \mu^{(1)}(a.f, b.g) \otimes \mu^{(1)}(k,l)\alpha_{XYZ} \\
&- \tau^{(1)}(\mu^{(1)}(a.f, b.g), kl)\alpha_{XYZ} - \tau^{(1)}(ab.fg, \mu^{(1)}(k,l))\alpha_{XYZ} \\
&- \alpha_{ABC}\tau^{(1)}(a, f.k)\tau^{(1)}(b, g.l) - \alpha_{ABC}\mu^{(1)}(\tau^{(1)}(a, f.k), b \otimes g.l) \\
&- \alpha_{ABC}\mu^{(1)}(a \otimes f.k), \tau^{(1)}(b, g.l) + \alpha_{ABC}\mu^{(1)}(a,b) \otimes \mu^{(1)}(f.k, g.l) \\
&+ \alpha_{ABC}\tau^{(1)}(\mu^{(1)}(a,b)), fg.kl) + \alpha_{ABC}\tau^{(1)}(ab, \mu^{(1)}(f.k, g.l) \\
&+ \tau^{(1)}((a,f)\tau^{(1)}(b,g) \otimes kl\alpha_{XYZ} + \mu^{(1)}(\tau^{(1)}((a,f), b.g) \otimes kl\alpha_{XYZ} \\
&+ \mu^{(1)}(a.f, \tau^{(1)}(b,g)) \otimes kl\alpha_{XYZ} - \mu^{(1)}(a,b).\mu^{(1)}(f,g) \otimes kl\alpha_{XYZ} \\
&- \tau^{(1)}(\mu^{(1)}(a,b), fg) \otimes kl\alpha_{XYZ} - \tau^{(1)}(ab, \mu^{(1)}(f,g) \otimes kl\alpha_{XYZ} \\
&- \mu^{(1)}(\mu^{(1)}(\alpha_{ABC}, a \otimes f.k), b \otimes g.l) +^{[2]} \mu^{(1)}(\alpha_{ABC}, \mu^{(1)}(a \otimes f.k, b \otimes g.l)) \\
&+ \mu^{(1)}(\mu^{(1)}(a.f \otimes k, \alpha_{UVW}), b \otimes g.l) - \mu^{(1)}(a.f \otimes k, \mu^{(1)}(\alpha_{UVW}, b \otimes g.l)) \\
&- \mu^{(1)}(\mu^{(1)}(a.f \otimes k, b.g \otimes l), \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k, \mu^{(1)}(b.g \otimes l, \alpha_{XYZ})) \\
&+^{[11]} [\mu^{(1)}(a.f \otimes k, b.g \otimes l\alpha^{(1)}_{XYZ})] -^{[12]} [\mu^{(1)}(\alpha^{(1)}_{ABC}, ab \otimes fg.kl)] \\
&+^{[13]} [\mu^{(1)}(a.f \otimes k\alpha^{(1)}_{UVW}, b \otimes g.l) - \mu^{(1)}(a.f \otimes k, \alpha^{(1)}_{UVW} b \otimes g.l)]
\end{aligned}
$$

$$
\begin{aligned}
&= [a.f \otimes k\mu^{(1)}(\alpha_{UVW}, \tau^{(1)}(b,g.l)) + a.f \otimes k\mu^{(1)}(\alpha_{UVW}, b \otimes \tau^{(1)}(g,l)) \\
&+ a.f \otimes k\alpha^{(1)}_{UVW} b \otimes \tau^{(1)}(g,l) + a.f \otimes k\alpha^{(1)}_{UVW}\tau^{(1)}(b,g.l) \\
&+ a.f \otimes k\alpha_{UVW}\tau^{(1)}(b, \tau^{(1)}(g,l)) - a.f \otimes k\tau^{(1)}(b, \tau^{(1)}(g,l))\alpha_{XYZ} \\
&- a.f \otimes k\mu^{(1)}(\tau^{(1)}(b.g,l), \alpha_{XYZ}) - a.f \otimes k\mu^{(1)}(\tau^{(1)}(b,g) \otimes l, \alpha_{XYZ}) \\
&- \alpha^{(1)}_{ABC} a \otimes f.k\tau^{(1)}(b,g.l) - \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a,f.k)b \otimes g.l) \\
&- \mu^{(1)}(\alpha_{ABC}, a \otimes f.k\tau^{(1)}(b,g.l)) - \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f,k)b \otimes g.l) \\
&- \mu^{(1)}(\alpha_{ABC}, a \otimes f.kb \otimes \tau^{(1)}(g,l)) - \alpha^{(1)}_{ABC} a \otimes f.kb \otimes \tau^{(1)}(g,l)) \\
&- \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f,k))b \otimes g.l - \alpha_{ABC}\mu^{(1)}(a \otimes \tau^{(1)}(f,k)b \otimes g.l) \\
&- \alpha_{ABC} a \otimes \tau^{(1)}(f,k))\tau^{(1)}(b,g.l) + \alpha_{ABC}\mu^{(1)}(a,b) \otimes \tau^{(1)}(f,k)g.l
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_{ABC} ab \otimes \mu^{(1)}(\tau^{(1)}(f,k), g.l) - \alpha_{ABC} \tau^{(1)}(ab, \mu^{(1)}(f.k, g.l)) \\
&- \alpha_{ABC} \tau^{(1)}(a, f.k) b \otimes \tau^{(1)}(g,l) - \alpha_{ABC} \mu^{(1)}(a \otimes f.k, b \otimes \tau^{(1)}(g,l)) \\
&- \alpha_{ABC} a \otimes f.k \tau^{(1)}(b, \tau^{(1)}(g,l)) + \alpha_{ABC} \mu^{(1)}(a,b) \otimes f.k \tau^{(1)}(g,l) \\
&+ \alpha_{ABC} ab \otimes \mu^{(1)}(f.k, \tau^{(1)}(g,l)) + \tau^{(1)}(\tau^{(1)}(a,f), k) b.g \otimes l \alpha_{XYZ} \\
&+ \mu^{(1)}(\tau^{(1)}(a,f) \otimes k, b.g \otimes l) \alpha_{XYZ} + \tau^{(1)}(a,f) \otimes k \tau^{(1)}(b.g, l) \alpha_{XYZ} \\
&- \mu^{(1)}(\tau^{(1)}(a,f), b.g) \otimes kl \alpha_{XYZ} - \tau^{(1)}(a,f) b.g \otimes \mu^{(1)}(k,l) \alpha_{XYZ} \\
&+ \tau^{(1)}(a.f, k) \tau^{(1)}(b,g) \otimes l \alpha_{XYZ} + a.f \otimes k \tau^{(1)}(\tau^{(1)}(b,g), l) \alpha_{XYZ} \\
&+ \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b,g) \otimes l) \alpha_{XYZ} - \tau^{(1)}(a.f, \tau^{(1)}, \tau^{(1)}(b,g)) \otimes kl \alpha_{XYZ} \\
&- a.f \tau^{(1)}(b,g) \otimes \mu^{(1)}(k,l) \alpha_{XYZ} - \tau^{(1)}(\mu^{(1)}(a.f, b.g), kl) \alpha_{XYZ} \\
&+ \mu^{(1)}(\tau^{(1)}(a,f) \otimes kb.g \otimes l, \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b,g) \otimes l, \alpha_{XYZ}) \\
&+ \mu^{(1)}(\tau^{(1)}(a.f,k) b.g \otimes l, \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k \tau^{(1)}(b.g, l), \alpha_{XYZ}) \\
&+ \tau^{(1)}(a,f) \otimes kb.g \otimes l \alpha^{(1)}_{XYZ} + \tau^{(1)}(a.f,k) b.g \otimes l \alpha^{(1)}_{XYZ}) \\
&+ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k) b \otimes g.l + \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f,k) b \otimes g.l \\
&+ \alpha_{ABC} \tau^{(1)}(a, \tau^{(1)}(f,k)) b \otimes g.l - \tau^{(1)}(\tau^{(1)}(a,f), k) \alpha_{UVW} b \otimes g.l \\
&- \mu^{(1)}(\tau^{(1)}(a,f) \otimes k, \alpha_{UVW}) b \otimes g.l - \tau^{(1)}(a,f) \otimes k \alpha^{(1)}_{UVW} b \otimes g.l \\
&- \mu^{(1)}(\tau^{(1)}(a.f,k), \alpha_{UVW}) b \otimes g.l - \tau^{(1)}(a.f,k) \alpha^{(1)}_{UVW} b \otimes g.l \\
&- \alpha_{ABC} a \otimes \tau^{(1)}(f,k) b \otimes \tau^{(1)}(g,l) - \alpha_{ABC} ab \otimes \mu^{(1)}(\tau^{(1)}(f,k), g.l) \\
&- \alpha_{ABC} ab \otimes \mu^{(1)}(f.k, \tau^{(1)}(g,l)) + \tau^{(1)}(a.f,k) \tau^{(1)}(b.g,l) \alpha_{XYZ} \\
&+ \mu^{(1)}(\tau^{(1)}(a.f,k), b.g \otimes l) \alpha_{XYZ} + \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b.g,l)) \alpha_{XYZ} \\
&- \mu(a.f, b.g) \mu^{(1)}(k,l) \alpha_{XYZ} - \tau^{(1)}(\mu^{(1)}(a.f, b.g), kl) \alpha_{XYZ} \\
&- \alpha_{ABC} \tau^{(1)}(a, f.k) \tau^{(1)}(b, g.l) - \alpha_{ABC} \mu^{(1)}(\tau^{(1)}(a, f.k), b \otimes g.l) \\
&- \alpha_{ABC} \mu^{(1)}(a \otimes f.k, \tau^{(1)}(b, g.l)) + \alpha_{ABC} \mu^{(1)}(a,b) \otimes \mu^{(1)}(f.k, g.l) \\
&+ \alpha_{ABC} \tau^{(1)}(ab, \mu^{(1)}(f.k, g.l) + \mu^{(1)}(a.f, \tau^{(1)}(b,g)) \otimes kl \alpha_{XYZ} \\
&- \alpha_{ABC} \mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g).kl + \tau^{(1)}(a,f) \otimes k \tau^{(1)}(b,g) \otimes l \alpha_{XYZ} \\
&+ \mu^{(1)}(\tau^{(1)}(a,f), b.g) \otimes kl \alpha_{XYZ} + \alpha_{ABC} ab \otimes \mu^{(1)}(f,g).\mu^{(1)}(k,l) \\
&- \mu^{(1)}(\mu^{(1)}(\alpha_{ABC}, a \otimes f.k), b \otimes g.l) + \mu^{(1)}(\mu^{(1)}(a.f \otimes k, \alpha_{UVW}), b \otimes g.l) \\
&- \mu^{(1)}(a.f \otimes k, \mu^{(1)}(\alpha_{UVW}, b \otimes g.l)) + \mu^{(1)}(a.f \otimes k, \mu^{(1)}(b \otimes g.l, \alpha_{XYZ})
\end{aligned}
$$

$$+ \mu^{(1)}(a.f \otimes kb \otimes g.l), \alpha^{(1)}_{XYZ}) - \mu^{(1)}(\alpha^{(1)}_{ABC}, a \otimes f, kb \otimes g.l)$$
$$+ \mu^{(1)}(a.f \otimes k\alpha^{(1)}_{UVW}, b \otimes g.l) - \mu^{(1)}(a.f \otimes k, \alpha^{(1)}_{UVW} b \otimes g.l)$$
$$- \alpha_{ABC}\mu^{(1)}(a, b) \otimes \tau^{(1)}(fg, kl) + \tau^{(1)}(ab, fg) \otimes \mu^{(1)}(k, l)\alpha_{XYZ}$$

$$= a.f \otimes k\mu^{(1)}(\alpha_{UVW}, \tau^{(1)}(b, g.l) + a.f \otimes k\mu^{(1)}(\alpha_{UVW}, b \otimes \tau^{(1)}(g, l))$$
$$+^{[7]} a.f \otimes k\alpha_{UVW}\tau^{(1)}(b, \tau^{(1)}(g, l)) -^{[8]} a.f \otimes k\tau^{(1)}(\tau^{(1)}(b, g), l)\alpha XYZ$$
$$- a.f \otimes k\mu^{(1)}(\tau^{(1)}(b.g, l), \alpha_{XYZ}) - a.f \otimes k\mu^{(1)}(\tau^{(1)}(b, g) \otimes l, \alpha_{XYZ})$$
$$- \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k)b \otimes g.l) - \mu^{(1)}(\alpha_{ABC}, a \otimes f.k\tau^{(1)}(b, g.l))$$
$$- \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f, k)b \otimes g.l) - \mu^{(1)}(\alpha_{ABC}, a \otimes f.kb \otimes \tau^{(1)}(g, l))$$
$$-^{[6]} \alpha_{ABC}\tau^{(1)}(a, \tau^{(1)}(f, k))b \otimes g.l - \alpha_{ABC}\mu^{(1)}(a \otimes \tau^{(1)}(f, k), b \otimes g.l)$$
$$-^{[10]} \alpha_{ABC}a \otimes \tau^{(1)}(f, k)\tau^{(1)}(b, g.l) + \alpha_{ABC}\mu^{(1)}(a, b) \otimes \tau^{(1)}(f, k)g.l$$
$$+^{[22]} \alpha_{ABC}ab \otimes \mu^{(1)}(f, k), g.l) -^{[27]} \tau^{(1)}(ab, \mu^{(1)}(f.k, g.l))$$
$$-^{[13]} \alpha_{ABC}\tau^{(1)}(a, f.k)b \otimes \tau^{(1)}g, l) - \alpha_{ABC}\mu^{(1)}(a \otimes f.k, b \otimes \tau^{(1)}(g, l))$$
$$-^{[7]} \alpha_{ABC}a \otimes f.k\tau^{(1)}(b, \tau^{(1)}(g, l)) + \alpha_{ABC}\mu^{(1)}(a, b) \otimes f.k\tau^{(1)}(g, l)$$
$$+^{[11]} \alpha_{ABC}ab \otimes \mu^{(1)}(f.k, \tau^{(1)}(g, l)) + [5]\tau^{(1)}\tau^{(1)}(a, f), k)b.g \otimes l\alpha_{XYZ}$$
$$+ \mu^{(1)}(\tau^{(1)}(a, f) \otimes k, b.g \otimes l)\alpha_{XYZ} +^{[23]} \tau^{(1)}(a, f) \otimes k\tau^{(1)}(b.g, l)\alpha_{XYZ}$$
$$-^{[21]} \mu^{(1)}(\tau^{(1)}(a, f), b.g) \otimes kl\alpha_{XYZ} - \tau^{(1)}(a, f)b.g \otimes \mu^{(1)}(k, l)\alpha_{XYZ}$$
$$+^{[25]} \tau^{(1)}(a.f, k)\tau^{(1)}(b, g) \otimes l\alpha_{XYZ} +^{[8]} a.f \otimes k\tau^{(1)}(\tau^{(1)}(b, g), k)\alpha_{XYZ}$$
$$+ \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b, g) \otimes l)\alpha_{XYZ} -^{[20]} \mu^{(1)}(a.f, \tau^{(1)}(b, g)) \otimes kl\alpha_{XYZ}$$
$$- a.f\tau^{(1)}(b, g) \otimes \mu^{(1)}(k, l)\alpha_{XYZ} -^{[26]} \tau^{(1)}(\mu^{(1)}(a.f, b.g), kl)\alpha_{XYZ}$$
$$+ \mu^{(1)}(\tau^{(1)}(a, f) \otimes kb.g \otimes l, \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k\tau^{(1)}(b, g) \otimes l, \alpha_{XYZ})$$
$$+ \mu^{(1)}(\tau^{(1)}(a.f, k)b.g \otimes l, \alpha_{XYZ}) + \mu^{(1)}(a.f \otimes k\tau^{(1)}(b.g, l), \alpha_{XYZ})$$
$$+ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k)b \otimes g.l + \mu^{(1)}(\alpha_{ABC}, a \otimes \tau^{(1)}(f, k))b \otimes g.l$$
$$+^{[6]} \alpha_{ABC}\tau^{(1)}(a, tau^{(1)}(f, k))b \otimes g.l -^{[5]} \tau^{(1)}(\tau^{(1)}(a, f), k)b \otimes g.l$$
$$- \mu^{(1)}(\tau^{(1)}(a, f) \otimes k, \alpha_{UVW})b \otimes g.l - \mu^{(1)}(\tau^{(1)}(a.f, k), \alpha_{UVW})b \otimes g.l$$
$$-^{[9]} \alpha_{ABC}a \otimes \tau^{(1)}(f, k)b \otimes \tau^{(1)}(g, l) -^{[22]} \alpha_{ABC}ab \otimes \mu^{(1)}(\tau^{(1)}(f, k), g.l)$$
$$-^{[11]} \alpha_{ABC}ab \otimes \mu^{(1)}(f.k, \tau^{(1)}(g, l)) +^{[24]} \tau^{(1)}(a.f, k)\tau^{(1)}(b.g, l)\alpha_{XYZ}$$

$+\mu^{(1)}(\tau^{(1)}(a.f,k),b.g\otimes l)\alpha_{XYZ}+\mu^{(1)}(a.f\otimes k,\tau^{(1)}(b.g,l))\alpha_{XYZ}$

$-\mu^{(1)}(a.f,b.g)\mu^{(1)}(k,l)\alpha_{XYZ}-^{[26]}\tau^{(1)}(\mu^{(1)}(a.f,b.g),kl)\alpha_{XYZ}$

$-^{[14]}\alpha_{ABC}\tau^{(1)}(a,f.k(\tau^{(1)}(b,g.l)-\alpha_{ABC}\mu^{(1)}(\tau^{(1)}(a,f.k),b\otimes g.l)$

$-\alpha_{ABC}\mu^{(1)}(a\otimes f.k,\tau^{(1)}(b,g.l))+\alpha_{ABC}\mu^{(1)}(a,b).\mu^{(1)}(f.k,g.l)$

$+^{[27]}\tau^{(1)}(ab,\mu^{(1)}(f.k,g.l))+^{[20]}\mu^{(1)}(a.f,\tau^{(1)}(b,g))\otimes kl\alpha_{XYZ}$

$+[19]\tau^{(1)}(a,f)\otimes k\tau^{(1)}(b,g)\otimes l\alpha_{XYZ}+[21]\mu^{(1)}(\tau^{(1)}(a,f),b.g)\otimes kl\alpha_{XYZ}$

$-\mu^{(1)}(a,b).\mu^{(1)}(f,g)\otimes kl\alpha_{XYZ}+\alpha+ABXab\otimes\mu^{(1)}(f,g).\mu^{(1)}(k,l)$

$-^{[3]}\mu^{(1)}(\mu^{(1)}(\alpha_{ABC},a\otimes f.k)b\otimes g.l+^{[4]}\mu^{(1)}(\mu^{(1)}(a.f\otimes k,\alpha_{UVW}),b\otimes g.l)$

$-^{[1]}\mu^{(1)}(a.f\otimes k,\mu^{(1)}(\alpha_{UVW},b\otimes g.l))+^{[2]}\mu^{(1)}(a.f\otimes k,\mu^{(1)}(b\otimes g.l,\alpha_{XYZ}))$

$-\alpha_{ABC}\mu^{(1)}(a,b)\mu^{(1)}(fg,kl)+\tau^{(1)}(ab,fg)\otimes\mu^{(1)}(k,l)\alpha_{XYZ}$

$+^{[1]}[\mu^{(1)}(a.f\otimes k,b.g\otimes l)\alpha^{(1)}_{XYZ}-\mu^{(1)}(a.f\otimes k,\alpha^{(1)}_{UVW}b\otimes g.l)]$

$+^{[2]}[\mu^{(1)}(a.f\otimes k\alpha^{(1)}_{UVW},b\otimes g.l)-\mu^{(1)}(\alpha^{(1)}_{ABC}a\otimes f.k,b\otimes g.l)]$

$+^{[3]}[a.f\otimes k\alpha^{(1)}_{UVW}\tau^{(1)}(b,g.l)-\alpha^{(1)}_{ABC}a\otimes f.k\tau^{(1)}(b,g.l)]$

$+^{[4]}[a.f\otimes k\alpha^{(1)}_{UVW}b\otimes\tau^{(1)}(g,l)-\alpha^{(1)}_{ABC}a\otimes f.kb\otimes\tau^{(1)}(g,l)]$

$+^{[5]}[\tau^{(1)}(a,f)\otimes b.g\otimes l\alpha^{(1)}_{XYZ}-\tau^{(1)}(a,f)\otimes\alpha^{(1)}_{UVW}b\otimes g.l]$

$+^{[6]}[\tau^{(1)}(a.f,k)b.g\otimes l\alpha^{(1)}_{XYZ}-\tau^{(1)}(a.f,k)\alpha^{(1)}_{UVW}b\otimes g.l]$

Above last 5 boxes labeled 1,2,..,5, can be written as

$+^{[5]}[\tau^{(1)}(a,f)\otimes b.g\otimes l\alpha^{(1)}_{XYZ}-\tau^{(1)}(a,f)\otimes\alpha^{(1)}_{UVW}b\otimes g.l]$

$=\tau^{(1)}(a,f)\otimes k\mu^{(1)}(\alpha_{UVW},b\otimes g.l)+\tau^{(1)}(a,f)\otimes k\tau^{(1)}(b,g.l)$

$+\tau^{(1)}(a,f)\otimes k\alpha_{UVW}b\otimes\tau^{(1)}(g,l)-\tau^{(1)}(a,f\otimes\tau^{(1)}(b,g)\otimes l\alpha_{XYZ}$

$-\tau^{(1)}(a,f)\otimes k\tau^{(1)}(b.g,l)\alpha_{XYZ}-\tau^{(1)}(a,f)\otimes k\mu^{(1)}(b.g\otimes l,\alpha_{XYZ}$

and similar expressions for others. So, the last 5 boxes can be replaced by

$+^{[1]}\mu^{(1)}(a.f\otimes k,\mu^{(1)}(\alpha_{UVW},b\otimes g.l))+\mu^{(1)}(a.f\otimes k,\alpha_{UVW}\tau^{(1)}(b,g.l)$

$+\mu^{(1)}(a.f\otimes k,\alpha_{UVW}b\otimes\tau^{(1)}(g,l))-\mu^{(1)}(a.f\otimes k,\mu^{(1)}(b.g\otimes l,\alpha_{XYZ}))$

$-\mu^{(1)}(a.f\otimes k,\tau^{(1)}(b.g,l)\alpha_{XYZ})-^{[2]}\mu^{(1)}(a.f\otimes k,\mu^{(1)}(b.g\otimes l,\alpha_{XYZ}))$

$+^{[3]}\ \mu^{(1)}(\mu^{(1)}(\alpha_{ABC}, a\otimes f.k), b\otimes g.l) + \mu^{(1)}(\alpha_{ABC}\tau^{(1)}(a, f.k), b\otimes g.l)$

$+\ \mu^{(1)}(\alpha_{ABC}a\otimes\tau^{(1)}(f, g), b\otimes g.l) - \mu^{(1)}(\tau^{(1)}(a, f)\otimes k\alpha_{UVW}, b\otimes g.l)$

$-\ \mu^{(1)}(\tau^{(1)}(a.f, k)\alpha_{UVW}, b\otimes g.l) -^{[4]}\ \mu^{(1)}(\mu^{(1)}(a.f\otimes k, \alpha_{UVW}), b\otimes g.l)$

$+\ \mu^{(1)}(\alpha_{ABC}, a\otimes f.k)\tau^{(1)}(b, g.l) +^{[14]}\ \alpha_{ABC}\tau^{(1)}(a, f.k)\tau^{(10}(b, g.l)$

$+^{[10]}\ \alpha_{ABC}a\otimes\tau^{(1)}(f, k)\tau^{(1)}(b, g.l) -^{[18]}\ \tau^{(1)}(a, f)\otimes k\alpha_{UVW}\tau^{(1)}(b, g.l)$

$-^{[15]}\ \tau^{(1)}(a.f, k)\alpha_{UVW}\tau^{(1)}(b, g.l) - \mu^{(1)}(a.f\otimes k, \alpha_{UVW})\tau^{(1)}(b, g.l)$

$+\ \mu^{(1)}(\alpha_{ABC}, a\otimes f.k)b\otimes\tau^{(1)}(g, l) +^{[13]}\ \alpha_{ABC}\tau^{(1)}(a, f.k)b\otimes\tau^{(1)}(g, l)$

$+^{[9]}\ \alpha_{ABC}a\otimes\tau^{(1)}(f, k)b\otimes\tau^{(1)}(g, l) -^{[16]}\ \tau^{(1)}(a, f)\otimes k\alpha_{UVW}b\otimes\tau^{(1)}(g, l)$

$-^{[17]}\ \tau^{(1)}(a.f, k)\alpha_{UVW}b\otimes\tau(1)(g, l) - \mu^{(1)}(a.f\otimes k, \alpha_{UVW})b\otimes\tau^{(1)}(g, l)$

$+\ \tau^{(1)}(a, f)\otimes k\mu^{(1)}(\alpha_{UVW}, b\otimes g.l) +^{[18]}\ \tau^{(1)}(a, f)\otimes k\alpha_{UVW}\tau^{(1)}(b, g.l)$

$+^{[16]}\ \tau^{(1)}(a, f)\otimes k\alpha_{UVW}b\otimes\tau^{(1)}(g, l) -^{[19]}\ \tau^{(1)}(a, f)\otimes k\tau^{(1)}(b, g)\otimes l\alpha_{XYZ}$

$-^{[23]}\ \tau^{(1)}(a, f)\otimes k\tau^{(1)}(b.g, l)\alpha_{XYZ} - \tau^{(1)}(a, f)\otimes k\mu^{(1)}(b.g\otimes l, \alpha_{XYZ})$

$+\ \tau^{(1)}(a.f, k)\mu^{(1)}(\alpha_{UVW}, b\otimes g.l) + [15]\tau^{(1)}(a.f, k)\alpha_{UVW}\tau^{(1)}(b, g.l)$

$+^{[17]}\ \tau^{(1)}(a.f, k)\alpha_{UVW}b\otimes\tau^{(1)}(g, l) - \tau^{(1)}(a.f, k)\tau^{(1)}(b, g)\otimes l\alpha_{XYZ}$

$-^{[24]}\ \tau^{(1)}(a.f, k)\tau^{(1)}(b.g, l)\alpha_{XYZ} -^{[25]}\ \tau^{(1)}(a.f, k)\mu^{(1)}(b.g\otimes l, \alpha_{XYZ})$

Hence, using the lower order conditions and cancelation of the like terms with opposite signs, we get,

$=^{[4]}\ a.f\otimes k\mu^{(1)}(\alpha_{UVW}, \tau^{(1)}(b, g.l)) +^{[6]}\ a.f\otimes k\mu^{(1)}(\alpha_{UVW}, b\otimes\tau^{(1)}(g, l))$

$-^{[5]}\ a.f\otimes k\mu^{(1)}(\tau^{(1)}(b.g, l), \alpha_{XYZ}) +^{[15]}\ a.f\otimes k\mu^{(1)}(b\otimes\tau^{(1)}(g, l), \alpha_{XYZ})$

$-^{[8]}\ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a, f.k)b\otimes g.l) -^{[9]}\ \mu^{(1)}(\alpha_{ABC}, a\otimes f.k\tau^{(1)}(b, g.l))$

$-^{[7]}\ \mu^{(1)}(\alpha_{ABC}, a\otimes\tau^{(1)}(f, k)b\otimes g.l) -^{[14]}\ \mu^{(1)}(\alpha_{ABC}, a\otimes f, kb\otimes\tau^{(1)}(g, l))$

$-^{[7]}\alpha_{ABC}\mu^{(1)}(a\otimes\tau^{(1)}(f, k), b\otimes g.l) -^{[14]}\ \alpha_{ABC}\mu^{(1)}(a\otimes f.k, b\otimes\tau^{(1)}(g, l))$

$+^{[10]}\mu^{(1)}(\tau^{(1)}(a, f)\otimes k, b.g\otimes l)\alpha_{XYZ}$

$+^{[15]}\ \mu^{(1)}(a.f\otimes k, \tau^{(1)}(b, g)\otimes l)\alpha_{XYZ}$

$+^{[10]}\ \mu^{(1)}(\tau^{(1)}(a, f)\otimes k, b.g\otimes l\alpha_{XYZ})$

$+^{[15]}\ \mu^{(1)}(a.f \otimes k\tau^{(1)}(b,g) \otimes l, \alpha_{XYZ})$

$+^{[12]}\ \mu^{(1)}(\tau^{(1)}(a.f,k)b.g \otimes l, \alpha_{XYZ}) +^{[5]}\ \mu^{(1)}(a.f \otimes k\tau^{(1)}(b.g,l), \alpha_{XYZ})$

$+^{[8]}\ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(a,f.k)b \otimes g.l +^{[7]}\ \mu^{(1)}(\alpha_{ABC}, \tau^{(1)}(f,k)b \otimes g.l)$

$-^{[11]}\ \mu^{(1)}(\tau^{(1)}(a,f) \otimes k\alpha_{UVW})b \otimes g.l -^{[13]}\ \mu^{(1)}(\tau^{(1)}(a.f,k), \alpha_{UVW})b \otimes g.l$

$+^{[12]}\mu^{(1)}(\tau^{(1)}(a.f,k), b \otimes g.l)\alpha_{XYZ} +^{[5]}\ \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b.g,l))\alpha_{XYZ}$

$-^{[8]}\ \alpha_{ABC}\mu^{(1)}(\tau^{(1)}(a,f.k), b \otimes g.l) -^{[9]}\ \mu^{(1)}(a \otimes f.k, \tau^{(1)}(b,g.l))$

$-^{[3]}\ \mu^{(1)}(a,b).\mu^{(1)}(f,g) \otimes kl\alpha_{XYZ} +^{[2]}\ \alpha_{ABC}ab \otimes \mu^{(1)}(f,g).\mu^{(1)}(k,l)$

$+^{[4]}\ \mu^{(1)}(a.f \otimes k, \alpha_{UVW}\tau^{(1)}(b,g.l)) +^{[6]}\ \mu^{(1)}(a.f \otimes k, \alpha_{UVW}b \otimes \tau^(g,l))$

$+^{[15]}\mu^{(1)}(a.f \otimes k, \tau^{(1)}(b,g) \otimes l\alpha_{XYZ}) +^{[5]}\ \mu^{(1)}(a.f \otimes k, \tau^{(1)}(b.g,l)\alpha_{XYZ})$

$+^{[8]}\ \mu^{(1)}(\alpha_{ABC}\tau^{(1)}(a,f.k), b \otimes g.l) +^{[7]}\ \mu^{(1)}(\alpha_{ABC}a \otimes \tau^{(1)}(f,k), b \otimes g.l)$

$-^{[11]}\ \mu^{(1)}(\tau^{(1)}(a,f) \otimes k\alpha_{UVW}, b \otimes g.l)\ -^{[13]}\mu^{(1)}(\tau^{(1)}(a.f,k)\alpha_{UVW})b \otimes g.l$

$+^{[9]}\ \mu^{(1)}(\alpha_{ABC}, a \otimes f.k)\tau^{(1)}(b,g.l) -^{[4]}\ \mu^{(1)}(a.f,k,\alpha_{UVW})\tau^{(1)}(b,g.l)$

$+^{[14]}\ \mu^{(1)}(\alpha_{ABC}, a \otimes f.k)b \otimes \tau^{(1)}(g,l) -^{[6]}\ \mu^{(1)}(a.f \otimes k, \alpha_{XYZ})b \otimes \tau^{(1)}(g,l)$

$+^{[11]}\ \tau^{(1)}(a,f) \otimes k\mu^{(1)}(\alpha_{UVW}, b \otimes g.l) -^{[10]}\ \tau^{(1)}a,f) \otimes k\mu^{(1)}(b.g \otimes l, \alpha_{XYZ})$

$+^{[13]}\tau^{(1)}(a.f,k)\mu^{(1)}(\alpha_{UVW}, b \otimes g.l) -^{[12]}\ \tau^{(1)}(a.f,k)\mu^{(1)}(b.g \otimes l, \alpha_{XYZ})$

$+^{[3]}\ \alpha_{ABC}\mu^{(1)}(a,b) \otimes \mu^{(1)}(f,g).kl +^{[1]}\ \alpha_{ABC}\mu^{(1)}(a,b) \otimes fg.\mu^{(1)}(k,l)$

$-^{[1]}\ \mu^{(1)}(a,b).fg \otimes \mu^{(1)}(k,l)\alpha_{XYZ} -^{[2]}\ ab.\mu^{(1)}(f,g) \otimes \mu^{(1)}(k,l)\alpha_{XYZ}$

$+^{[9]}_{[4]}\mu^{(1)}(a.f \otimes k\alpha_{UVW}, \tau^{(1)}(b,g.l)) +^{[11]}_{[10]}\ \mu^{(1)}(\tau^{(1)}(a.f) \otimes k, b.g \otimes l\alpha_{XYZ})$

$-^{[13]}_{[12]}\ \mu^{(1)}(\tau^{(1)}(a.f,k), b.g \otimes l\alpha_{XYZ}) -^{[6]}_{[14]}\ \mu^{(1)}(\alpha_{ABC}a \otimes f.k, \tau^{(1)}(b,g.l))$

$= 0$ $\square$