DIGRAPH APPLICATIONS

by

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INTRODUCTION

Graph theory was launched in 1736 by the famous Swiss mathematician Euler. Early problems in graph theory usually dealt with entertaining puzzles. More recently graph theory has been applied to practical problems in a wide variety of fields. Graphs are quite common and we use them every day without realizing that we are dealing with a graph. A few common examples of graphs are family relationships, road maps, computer flow charts, and pipeline networks. In this paper we shall present a few examples of the application of graph theory to practical problems.

FUNDAMENTALS

In order that conventions, notation, and operations may be clearly understood, it will be advantageous to review some fundamentals of graph theory. To a large extent we will adopt the definitions of Carpenter (3). A graph is a topological structure composed of zero dimensional elements called nodes, and one dimensional elements called links connecting a pair of nodes. A link may have a direction associated with it, in which case we refer to the graph as a directed graph or simply as a digraph.

If a link is directed from a node i to a node j we say the link is emergent from i and incident on j. We refer to the number of distinct nodes as the order of a graph. If a single link is incident and emergent on the same node, we call that link a loop. A path is a sequence of links such that the terminal node of each link in sequence is the initial node of the following link. The length of a path is the number of links in the path. A path incident and emergent on the same node is called a circuit. A graph is said to be strongly connected if there is a path joining every pair of distinct nodes.

An n^{th} order digraph may be completely represented by an nxn connection matrix C where

$$c_{ij} = \begin{cases} 1 & \text{if a link exists from node i to node j} \\ 0 & \text{otherwise} \end{cases}$$

Diagonal elements of a connection matrix must necessarily be zero if the digraph is to be loop free.

Graphs may be combined in three ways, by a logical "or", by a logical "and", or by concatenation. It is necessary that two digraphs be of the same order before they may be combined by any of these three methods.

Juxtaposition (logical "or" or cupping) of matrices is defined as element-wise cupping in the two-valued Boolean algebra.

For conformable Boolean matrices A and B, $A \cup B = (a_{ij} \cup b_{ij})_{nxn}$. This operation is called juxtaposition since the corresponding digraph operation is superimposing digraph A on digraph B with corresponding nodes coincident.

The logical product (logical "and" or capping) of matrices is defined as the matrix obtained by capping together like elements. $A \cap B = (a_{ij} \cap b_{ij})_{nxn}$. The digraph of $A \cap B$ is formed by connecting corresponding links of A and B in series.

Concatenation (matrix multiplication) is defined by $AB = (\bigvee_{k=1}^{m} a_{ik} \cap b_{kj})$ where the symbol $\bigvee_{k=1}^{m}$ means all terms obtained by substitution of k = 1, 2,...n are cupped together. The significance of concatenation is that the ij^{th} element of AB is 1 if and only if a path of length two from i to j having its first link in A and its second link in B is possible.

MATRIX | DIGRAPH

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Digraph A

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Digraph B

$$A \cup B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Juxtaposition of digraphs A and B

$$A \cap B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Series connection of digraphs A and B

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Concatenation of digraphs A and B

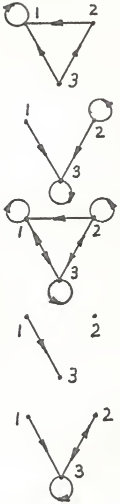

Fig. 1. Examples of binary operations on matrices and digraphs.

If we let A = C and B = C, then $AB = C^2$. The ij^{th} element of C^2 is 1 if and only if a path of length two exists from i to j in C. If we now let A = C and $B = C^2$, then $AB = C^3$. This corresponds to the connections for paths of length three in C. Similarly, if ${}_nC$ is the connection matrix for paths of length n in C, ${}_nC = C^n$.

A <u>directed cut set</u> is a set of links having the property that removal of the set from the graph leaves the graph with two disjoint subgraphs such that the terminal nodes of the links are in one subgraph and the initial nodes are in the other and such that no proper subset of the set links has this property.

With the above background material we shall now proceed to examine several interesting problems. The problems have been kept simple to illustrate the method of attack and the procedure in obtaining the solution rather than to weigh the reader down with undue detail.

APPLICATIONS

Bipartite Graphs

Suppose that a number of positions are open and exactly enough people apply to fill all of the positions. The positions all have different qualifications, but some of the applicants are qualified for more than one job. The problem is to determine assignment of people to jobs so that all positions are filled.

If we draw a graph where each person is represented by a node, and each job by a node, and if we draw a link from the node representing a person to a node representing a job if the person is qualified for the job, we may obtain a graph as shown in Fig. 2.

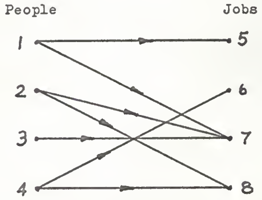

Fig. 2. Digraph indicating job qualifications.

The corresponding connection matrix C will be

$$C = \begin{bmatrix} 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

We note that this matrix may be characterized as

$$\underset{(8x8)}{C} = \begin{bmatrix} \underset{(4x4)}{A} & \underset{(4x4)}{F} \\ \underset{(4x4)}{R} & \underset{(4x4)}{B} \end{bmatrix} = \begin{bmatrix} \underset{(4x4)}{0} & \underset{(4x4)}{F} \\ \underset{(4x4)}{0} & \underset{(4x4)}{0} \end{bmatrix}$$

This type of graph is called a <u>bipartite graph</u>. This bipartite graph may be completely characterized by F, a submatrix of C.

To assign one and only one applicant to each job and to assign each applicant a job, we require an assignment matrix D such that D is a permutation matrix, i.e. only one 1 in each row and column. D will be of the same order as F. For this problem we see that

$$F = \begin{bmatrix} 1&0&1&0 \\ 0&0&1&1 \\ 0&0&1&0 \\ 0&1&0&1 \end{bmatrix}$$

If P is a 4x4 permutation matrix that is a solution of this problem, then,

$$F \cap P = P$$

We know that the links in the assignment graph, D, are a subset of the links in the qualification graph, C. Therefore, if we take $F \cap D$ we must get D as the result. We see that for this problem, there is only one choice for D since there is only one 4x4 permutation matrix with the property that $F \cap P = P$.

Thus

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Suppose, however, that we have a problem such that

$$F = P_1 \cup P_2 \cup P_3 \cup \cdots\cdots \cup P_n \cup X_r$$

where P_1, P_2,....P_n are all permutation matrices and X_r is that portion of F that is not representable by any permutation matrix. Now take $F \cap P_i$.

$$F \cap P_i = P_i \cap (P_1 \cup P_2 \cup P_3 \cup \cdots\cdots \cup P_i \cup \cdots\cdots \cup P_n \cup X_r)$$

$$(F \cap P_i)_{ij} = p_{iij} \cap (p_{1ij} \cup p_{2ij} \cup \cdots\cdots \cup p_{iij} \cup \cdots\cdots \cup p_{nij} \cup x_{rij})$$

If $p_{iij} = 1$

$$(F \cap P_i)_{ij} = 1 \cap (p_{1ij} \cup p_{2ij} \cup \cdots\cdots \cup 1 \cup \cdots\cdots \cup p_{nij} \cup x_{rij})$$

$$= 1 \cap 1$$

$$(F \cap P_i)_{ij} = 1$$

If $p_{iij} = 0$

$$(F \cap P_i)_{ij} = 0 \cap (p_{1ij} \cup p_{2ij} \cup \cdots\cdots \cup 0 \cup \cdots\cdots \cup p_{nij} \cup x_{rij})$$

$$= 0 \cap 0 \quad \text{or} \quad 0 \cap 1$$

$$(F \cap P_i)_{ij} = 0$$

Therefore, $F \cap P_i = P_i$.

Thus we see that if we $\cup$-factor F into permutation matrices and a remainder then we have obtained the desired result. Let us consider an illustrative example. Suppose

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

If we now $\cup$-factor F into permutation matrices and a remainder, we obtain

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$F = P_1 \cup P_2 \cup P_3 \cup 0_r$$

Thus we have three solutions corresponding to P_1, P_2, and P_3.

Now suppose the problem is changed slightly. Suppose that there are more applicants than there are positions in which to place them. This situation is illustrated by the graph in Fig. 3.

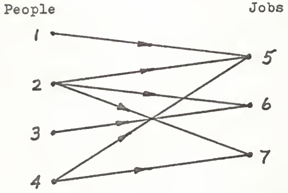

Fig. 3. Job qualification graph with fewer jobs than applicants.

The connection matrix, C, for the graph in Fig. 3 is

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Since one person must be omitted from the assignments, let us eliminate each one in turn and observe the possible assignments.

Number 1 eliminated yields

$$F_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number 2 eliminated yields

$$F_2 = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&1\end{bmatrix}$$

Number 3 eliminated yields

$$F_3 = \begin{bmatrix}1&0&1\\1&1&1\\1&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0\\1&0&1\\0&0&0\end{bmatrix}$$

Number 4 eliminated yields

$$F_4 = \begin{bmatrix}1&0&1\\1&1&1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

Thus we see that we now have eight possible ways to fill the positions. If we assume that each person is equally well qualified if he is qualified at all, then all eight possible assignments have equal merit. In reality this is seldom true. In that case some weighting function must be applied to each possible assignment to determine the best one. This, however, is beyond the scope of this paper.

If we pose the problem of fewer applicants than positions, we must proceed in a fashion similar to that used in the case of more applicants than positions, except that we must eliminate positions in sequence and then obtain all possibilities with that position unfilled.

Determining Connectedness of Communications Networks

An n^{th} order Hamilton circuit is a system of n nodes and n links connecting all nodes into a single circuit. Suppose that we consider a system of six nodes and we form two subsets from this set. Let the two subsets be (1, 2, 3) and (3', 4, 5). Now form a Hamilton circuit on each subset and connect node 3 to node 3' with a bidirectional link as shown in Fig. 4a. A Hamilton circuit is strongly connected since start-

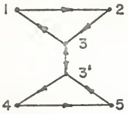

Fig. 4a.

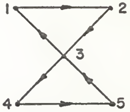

Fig. 4b.

Fig. 4. 5^{th} order graph with two Hamilton circuits.

ing at any node, it is possible to reach any other node with a path length at most n-1. In particular paths exist from nodes 1 or 2 to node 3 in the first Hamilton circuit and from 4 or 5 to 3' in the second circuit, and conversely. Since a bidirectional link exists between 3 and 3' paths exist connecting nodes 4 or 5 to nodes 1 and 2. Now merge nodes 3 and 3' into one node, 3. The graph, shown in Fig. 4b, now consists of two Hamilton circuits with one node in common. Thus a path exists between every pair of distinct nodes and the graph is strongly connected. We may show the strong connectedness very simply by forming the connection matrix for paths of length one, two, three, and four, taking the juxtaposition of these four matrices.

$$C = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} \quad C^2 = \begin{bmatrix} 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&1&0&1&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \end{bmatrix} \quad C^3 = \begin{bmatrix} 1&0&0&0&1 \\ 0&1&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&1&0 \\ 1&0&0&0&1 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&1&0&1&0 \end{bmatrix} \qquad C \cup C^2 \cup C^3 \cup C^4 = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}$$

Thus we arrive at the fundamental lemma of this report: a graph is strongly connected if two Hamilton circuits have one node or more in common.

We may use this lemma to determine if a graph is strongly connected. The same idea can be used to determine the directed cut set and superfluous links.

Example 1. Consider the graph in Fig. 5. We wish to determine

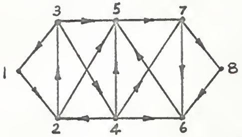

Fig. 5. Digraph considered in Example 1.

whether this graph is strongly connected; if it is not strongly connected we wish to know the directed cut set and any superfluous links that may exist. The rule we shall follow is to draw all possible Hamilton circuits on subsets of the nodes of the given graph. If, however, it is possible to draw a Hamilton circuit on a subset of the nodes contained in another Hamilton circuit, we shall omit that circuit. For this example it is possible to draw three Hamilton circuits, as illustrated in Fig. 6. We note that it is also possible to draw a Hamilton

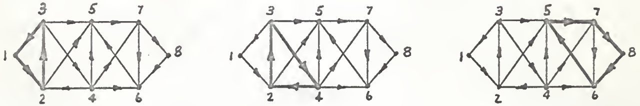

Fig. 6a. Fig. 6b. Fig. 6c.

Fig. 6. Hamilton circuits drawn on the digraph of Fig. 5.

circiut including nodes 5, 6, and 7, but these nodes are a subset of the nodes 5, 6, 7, and 8 which are already included in a Hamilton circuit. If we now superimpose the graphs in Figures 6a, 6b, and 6c we will obtain the graph shown in Fig. 7.

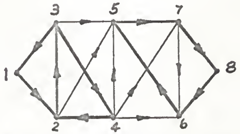

Fig. 7. Juxtaposition of the digraphs in Fig. 6.

We may now identify the directed cut set. The links (2,5), (3,5), (4,5), (4,6), and (4,7) form the directed cut set. One superfluous link (7,6) is present.

Example 2. Given the graph in Fig. 8, determine whether the graph is strongly connected; if not determine the directed cut set and any superfluous links.

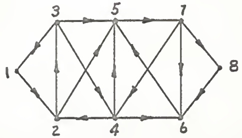

Fig. 8. Digraph considered in Example 2.

We proceed in a manner similar to that used in the previous example. In this example, four Hamilton circuits may be drawn. They are illustrated in Fig. 9.

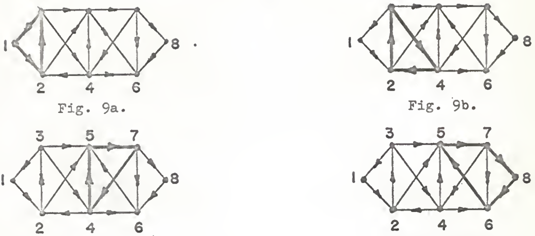

Fig. 9a. Fig. 9b. Fig. 9c. Fig. 9d.

Fig. 9. Hamilton circuits drawn on the digraph of Fig. 8.

Superimposing the Hamilton circuits in Fig. 9 we obtain the result shown in Fig. 10. We note that this graph is strongly connected and that the

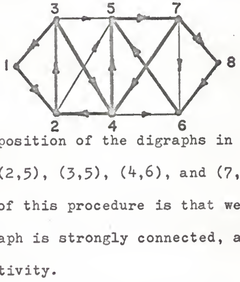

Fig. 10. Juxtaposition of the digraphs in Fig. 9.

superfluous links are (2,5), (3,5), (4,6), and (7,6).

The significance of this procedure is that we may simultaneously determine whether a graph is strongly connected, and determine a minimum form for strong connectivity.

Suppose that we are interested in examining a communications network to determine if it is strongly connected. If the graph in Fig. 5 represents such a network, we note that it is impossible to establish communications from a member of the subset (5,6,7,8) to a member of the subset (1,2,3,4). The graph in Fig. 8 represents a system that is strongly connected, but it contains several links that are not necessary to the strong connectivity. These links are eliminated because parallel perfect circuits are not desired for reasons of economy. The dual problem of paralleling imperfect circuits for increased communication reliability will not be considered.

Chen and Wing (4) have given an algorithm for determining the strong connectivity of a graph by operations on the connection matrix. The procedure of applying the algorithm is given below.

Start from the first row of the given connection matrix:

a. If all elements of the first row are ones, the node associated with this row is a supervisor[1]. Skip to the next row.

b. If the condition of (a) does not exist, but the associated node of this row is connected to a supervisor, this node is a supervisor. Set every element of this row equal to one and work on the next row.

c. If conditions of (a) and (b) do not exist, the following steps are taken:

Step 1. Let C_{1i}, C_{1j},...C_{1m} be the non-zero elements in the first row of the connection matrix. Add (Boolean sum) the i^{th}, j^{th},...and m^{th} rows to the first row.

Step 2. Suppose that there are k additional non-zero elements C_{1p}, C_{1q},...C_{1r} in the first row generated by the first step, then add (Boolean sum) the p^{th}, q^{th},...r^{th} rows to the first row.

Step 3. Repeat step 2 until one or both of the following two conditions occurs:

---

1 A node is called a supervisor if there exists a path from that node to every other node.

1. The whole row contains no zero element.

2. No additional non-zero elements is generated by the preceeding step.

This procedure is repeated on every row of the connection matrix. The final matrix is called the <u>terminal connection matrix</u> associated with the graph.

If we are only interested in testing whether the graph is strongly connected and do not care to determine the directed cut set, it is not necessary to construct the associated terminal connection matrix of the graph. For this case a more efficient algorithm was formulated by Chen and Wing (4); it is

<u>Step 1</u>. Construct the transpose C' of the connection matrix C.

<u>Step 2</u>. Apply the Boolean sum operation only to the first row of the connection matrix C and the first row of C'.

The graph is strongly connected if and only if the first row of C and the first row of C' after the Boolean sum operation of step 2 contain no zero elements.

Suppose we consider the graph in Fig. 8 and test whether the graph is strongly connected. The connection matrix C is given as

$$C = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 1&0&0&1&1&0&0&0 \\ 0&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&1&0&0 \end{bmatrix} \qquad C' = \begin{bmatrix} 0&0&1&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&1&1&1&0&1&0&0 \\ 0&0&0&1&0&0&1&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

Applying the Boolean sum operation to the first row of C and the first row of C', we obtain C_s and C'_s as given below.

$$
C_s = \begin{bmatrix}
1&1&1&1&1&1&1&1\\
0&0&1&0&1&0&0&0\\
1&0&0&1&1&0&0&0\\
0&1&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0\\
0&0&0&1&0&1&0&1\\
0&0&0&0&0&1&0&0
\end{bmatrix}
\qquad
C'_s = \begin{bmatrix}
1&1&1&1&1&1&1&1\\
1&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0\\
0&1&1&1&0&1&0&0\\
0&0&0&1&0&0&1&1\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0
\end{bmatrix}
$$

After this operation the first row of C_s and the first row of C'_s contain no zero element. Therefore the graph is strongly connected. This result is in agreement with the result obtained earlier to the same problem.

Suppose we consider the graph in Fig. 5 and identify the strongly connected components and all directed cut sets. The associated connection matrix is given below.

$$
C = \begin{bmatrix}
0&1&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0\\
1&0&0&1&1&0&0&0\\
0&1&0&0&1&1&1&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&1\\
0&0&0&0&0&1&0&0
\end{bmatrix}
$$

The terminal connection matrix T constructed from C is given below.

$$
T = \left[\begin{array}{cccc:cccc}
1&1^*&1&1&1&1&1&1\\
1&1&1^*&1&1^*&1&1&1\\
1^*&1&1&1^*&1^*&1&1&1\\
1&1^*&1&1&1^*&1^*&1^*&1\\
\hdashline
0&0&0&0&1&1&1^*&1\\
0&0&0&0&1^*&1&1&1\\
0&0&0&0&1&1^*&1&1^*\\
0&0&0&0&1&1^*&1&1
\end{array}\right]
$$

1^* -- Elements of the connection matrix C

From T nodes 1, 2, 3, and 4 form a strongly connected subset, while nodes 5, 6, 7, and 8 also form a strongly connected subset. The links (2,5), (3,5), (4,5), (4,6), and (4,7) are the elements of the directed cut set separating the two strongly connected subsets. Again this is the same result as was obtained earlier to the same problem.

Determining Reducibility of Matrices

Varga (22) has defined a reducible matrix af follows:

Definition: For $n \geq 2$ an nxn complex matrix[1] A with elements over a field is reducible if there exists an nxn permutation matrix P such that

$$PAP' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an rxr submatrix and A_{22} is an (n-r)x(n-r) submatrix, where $1 \leq r \leq n$. If no such permutation matrix exists, than A is irreducible.

Suppose we have a system of equations given by

$$Ax = k$$

where A is nxn matrix with elements over a field. If A is reducible, then

$$\bar{A}\bar{x} = \bar{k}$$

where

$$\bar{A} = PAP'$$
$$\bar{x} = Px$$
$$\bar{k} = Pk$$

---

1 A complex matrix is a matrix with complex elements. It is obvious, however, that a complex matrix may be expressed as the sum of the real part matrix and the imaginary part matrix and each part may be handled separately.

But we may also write this as

$$A_{11}\bar{x}_1 + A_{12}\bar{x}_2 = \bar{k}_1$$
$$A_{22}\bar{x}_2 = \bar{k}_2$$

Now we may first solve an (n-r)x(n-r) system and then solve an rxr system of equations. This represents a distinct advantage in ease of solution.

To determine if it is possible to reduce a matrix, Varga proceeds to transform the matrix A into a matrix, C, where

$$c_{ij} = 1 \qquad a_{ij} \neq 0$$
$$c_{ij} = 0 \qquad a_{ij} = 0$$

and then considers that C is the connection matrix for a digraph. If the graph is strongly connected then the matrix is irreducible, otherwise it is reducible. Varga's technique is then to draw the graph of C and by inspection determine whether it is strongly connected. As it has already been shown, this is by no means a simple task even for a system of relative low order. We have already presented two ways for determining whether a graph is strongly connected. However, this only reveals the existence or non-existence of reducibility. The problem of reducing the matrix remains.

If we investigate just what the operation

$$PAP'$$

does, we may gain some insight into the problem.

Let us first consider

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

We see that pre-multiplication by the permutation matrix effects a simple interchange of rows. If we consider

$$AP' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{13} & a_{14} & a_{12} \\ a_{21} & a_{23} & a_{24} & a_{22} \\ a_{31} & a_{33} & a_{34} & a_{32} \\ a_{41} & a_{43} & a_{44} & a_{42} \end{bmatrix}$$

We see that this operation interchanges the columns in a manner similar to the previous interchange of rows. If we are considering a system of equations

$$Ax = k$$

and we now premultiply by the permutation matrix, we obtain

$$PAx = Pk$$

This is the row interchange. If we now post-multiply A by P' we obtain

$$(PAP')(Px) = (Pk)$$

or

$$\bar{A}\bar{x} = \bar{k}$$

since for a permutation matrix $P' = P^{-1}$ (Carpenter, 3). The total operation has now interchanged rows in the same manner as the columns have been interchanged. Since $P_1P_2 = P_3$ (Ledley, 12) we may permute the matrix again without altering the basic operation. Observe, however, that this is often done in a very simple manner almost without thought, by the use of the augmented matrix.

Example 3. Suppose the given set of equations is

$$\begin{aligned} 3x + 5y - z + 2w &= 3 \\ 0x + y + 0z + w &= 4 \\ -5x - y + 2z + w &= 2 \\ 0x + y + 0z + 2w &= 1 \end{aligned}$$

Now perform the same operations as were done previously using PAP':

$$\begin{array}{cccc|c} x & y & z & w & \\ \hline 3 & 5 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ -5 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{array} \xrightarrow{\text{R.I.}} \begin{array}{cccc|c} x & y & z & w & \\ \hline 3 & 5 & -1 & 2 & 3 \\ -5 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 1 \end{array}$$

$$\xrightarrow{\text{C.I.}} \begin{array}{cccc|c} x & z & y & w & \\ \hline 3 & -1 & 5 & 2 & 3 \\ -5 & 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 1 \end{array}$$

where R.I. and C.I. are row and column interchange respectively. We have thus obtained the desired result. It is interesting to note that there is no real reason that identical row and column interchanges must be made as Varga seems to indicate.

In terms of graph theory, PCP' represents simply a renumbering of the nodes. If we observe that in a connection matrix of the form

$$C = \begin{bmatrix} A & F \\ R & B \end{bmatrix}$$

where A, F, B, and R are matrices, the original set of nodes is partitioned into two subsets say A and B. Thus F represents the directed connections between nodes in set A with those in B, while R represents the directed connections from B to A. The matrices A and B represent the connections within subsets A and B respectively. This situation is illustrated in Fig. 11.

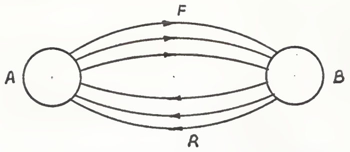

Fig. 11. Digraph for two sets of nodes.

If R is a zero matrix we have the reduced form that Varga wishes to obtain. We note that subsets A and B are connected only by the set of directed links F. Thus no node in B can be a supervisor, and according to Chen and Wing (4) the graph is not strongly connected. Therefore, we have but to choose the subsets A and B properly, irrespective of order within A and B, to obtain the desired result.

Consider our previous example. We found that the proper interchange of rows and columns yields

$$\begin{bmatrix} 3 & -1 & 5 & 2 \\ -5 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}$$

We could, however, write

$$\begin{bmatrix} -1 & 3 & 5 & 2 \\ 2 & 5 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & 3 & 2 & 5 \\ 2 & 5 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ x \\ w \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}$$

or any of several other interchanges without altering the basic result. This is due to the fact that it makes no difference in what order the nodes are considered as long as the subsets A and B always contain the same nodes.

We have shown that column interchanges can be made without the corresponding row interchanges. If we now construct the total of the permutation operations we find that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ -5 & -1 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 & 5 \\ 2 & 5 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

which is of the form P_1AP_2' where

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We see that there is not a unique solution to this problem as Varga seems to indicate. It is possible to apply the algorithm of Chen and Wing or the Hamilton circuit lemma to determine the strongly connected subsets and thus the proper division of the original set of nodes so that the original set of equations will be in the reduced form. In summary, this section contains proof and explanation of Varga's Theorem 1.6 that Varga states without proof.

CONCLUSIONS

We have illustrated several simple problems that lend themselves to convenient solution by graph theory. Many more complicated problems await solution by the use of digraphs. We have shown that some problems yield solutions more easily employing the connection matrix while some yield more easily to direct manipulation of the graph, although it is possible to perform the same operations on a problem either in terms of the graph or in terms of the connection matrix.

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DIGRAPH APPLICATIONS

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Early problems in graph theory dealt largely with entertaining puzzles, the solutions to which were reached by examination of the graph alone. More complicated problems do not yield so easily to simple examination of the graph; a more powerful tool is needed. This tool is the connection matrix. An n^{th} order directed graph (n nodes) may be completely represented by an nxn matrix of zeros and ones.

It is shown by a series of examples that it is possible to perform identical operations on a graph if it is considered in terms of the graph itself or in terms of the connection matrix. It is also shown that it is often convenient to employ both the graph and the connection matrix in the solution of a problem. It is shown that some operations carried out with the connection matrix may actually obscure a rather simple idea that may be carried out by inspection with the graph alone.