

A DETERMINISTIC STUDY OF DELTA MODULATION

by

JOHN STEPHEN TRIPP

B. S., Kansas State University, 1961

A MASTER'S REPORT

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1962

TABLE OF CONTENTS

INTRODUCTION	1
PREVIOUS WORK	1
F. de Jager's Paper	1
L. H. Zetterberg's Paper	3
B. L. Barber's Report	5
Other Papers	8
DETERMINISTIC ANALYSIS	10
Z-transform Description of Modulator-Demodulator Model	10
Recursion Formula for Model's Output	11
Discussion of Results for Step Inputs	13
Computer Outputs	13
Errors and Signal-to-Noise Ratios	14
Signum Function with Dead Space	17
Delta Modulator Outputs for Step Inputs	19
Review of Results for Step Inputs	21
Performance and System Modification for Ramp Inputs	21
$(1 - \lambda) + \lambda z$ Feedback Transfer Function	21
Computer Results for Various Combinations of Parameters with $(1 - \lambda) + \lambda z$ Feedback	24
Recursion Formula and Results for $1/(1 + \tau s)$ in Feedback Link	33
Derivative Compensation of Error	41
Conclusions on Admission of Ramps	42

Review of the Goals and Capabilities of Probabilistic and Deterministic Analyses	43
GENERALIZATION OF DELTA MODULATION	45
SUMMARY	53
ACKNOWLEDGMENT	55
REFERENCES	56

INTRODUCTION

Delta modulation is a hybrid system composed of pulse duration and pulse amplitude modulation in conjunction with feedback which encodes a signal into a sequence of quantized pulses.

This report reviews previous work on the subject which ranges from descriptive presentations, to probabilistic analyses from the information standpoint, to newer concepts of the character of delta modulation. It then presents an ideal mathematical model for a delta modulation system and a resulting deterministic method of analysis. The results of this analysis are given and are followed by a discussion of improvement techniques on the basic system. A comparison of the deterministic and probabilistic approaches is made, and it is shown that the two approaches are complementary in scope. Finally a generalization of delta modulation to a system having five-level output quantization is made.

PREVIOUS WORK

F. de Jager's Paper

F. de Jager (4) gave a qualitative description of the delta modulation system. No extensive mathematical analysis was attempted.

The physical system (see Fig. 1) consists of a comparator, D, which controls a pulse modulator, PM, fed by a pulse generator, PG. The signal from the modulator output passes through an

integrating network, F_1 , and back to the negative side of the comparator. The delta modulated signal is taken from the pulse modulator output, and after passage through a channel, the original signal is reconstructed by means of an integrating network, F_2 , similar to F_1 .

de Jager found that using a single integrating network for F_1 gave satisfactory results. In an effort to improve the signal-to-noise ratio he tried using a second integrator in cascade with the first. This resulted in noise reduction but gave rise to stability problems and difficulties in transmitting the higher frequencies. He compromised by using a single integrator followed by a low pass filter with cutoff at 3800 cps. Actually, the single integrator was a shallow-skirted low pass filter with cutoff at 200 cps.

The difference between the original and the reconstructed signals was defined as quantizing noise. Using this definition signal-to-noise ratio formulas were calculated for both single and double integration. For single integration,

$$S/N = C_1 f_i^{\frac{3}{2}} / f f_o^{\frac{1}{2}} \quad (1)$$

For double integration,

$$S/N = C_2 f_i^{\frac{5}{2}} / f f_o^{\frac{3}{2}} \quad (2)$$

where f is the frequency of the input, f_i is the pulse frequency, and f_o is the cutoff frequency of the low pass filter. The C_i 's must be determined by numerical methods. A comparison of these theoretical results with experimental values

showed that the improvement in signal to noise ratio as pulse frequency increased was in fair agreement with the $5/2$ and $3/2$ power of the pulse frequency for double and single integration respectively.

The necessary bandwidth for delta modulation was stated to be about 50% greater than for pulse code modulation. The quality of DM was said to be roughly the same as that of an eight-unit code PCM.

L. H. Zetterberg's Paper

L. H. Zetterberg (9) compared delta modulation and pulse code modulation from the point of view of principle as well as performance. He concluded that they are similar because both represent a signal by means of a sequence of pulses and furthermore because the systems may be characterized by the same parameters. This comparison may not have been completely valid, but at least PCM served as a standard, and Zetterberg was able to obtain a measure of the performance of DM.

In accordance with Shannon's general method of analyzing a communication system, delta modulation was viewed as a discrete, noiseless channel and channel capacity was calculated using this assumption. The source of information was considered to be a set of functions generated by finite Markov chains. A Markov chain is a stochastic process in which a future state is determined only by the previous state and the transition probabilities remaining fixed. For speech signals, or any signal likely to be transmitted, there will be a much stronger dependence between

the various states. But, according to Shannon, a stronger dependence between the values of the functions gives a smaller entropy for the process and thereby a lower rate of information. Therefore, use of the Markov chain will give only an upper bound on the channel capacity, but even this much will be very useful.

Zetterberg's results showed that

$$C = \frac{1}{T} \log 2 \quad (3)$$

for PCM, and

$$C = \frac{1}{T} \log (2 \cos \pi/n+1) \quad (4)$$

for DM, where C = channel capacity, T = pulse interval, and n = number of quantizing levels, i.e., the quantized output magnitudes.

It can be observed that as n increases to 10 or greater, the channel capacity of a DM system approaches that of a PCM system so closely that the difference is almost negligible. Since in a practical application the number of levels is likely to be much greater than ten, delta modulation can be seen to compare favorably with regard to channel capacity.

The paper continued with spectral calculations from which few conclusions can be drawn. Then analyses of the error signal were undertaken. Two major causes of error are the sampling process and the quantization of the output. Inherently sampling results in error which can be reduced mainly by increasing the sampling rate. Zetterberg divided the effects of quantization into those caused by overloading and those caused by granulation.

For DM, overloading appears when the slope of the input signal is greater than the height of one DM increment per unit sampling time. For PCM, overloading appears when the amplitude of the signal becomes greater than the maximum of the quantizing code. However, Zetterberg failed to point out directly that essentially the same effect occurs in DM when the input amplitude becomes greater than the saturation level. The other effect, granulation, is an interpolation error.

Values of noise power were calculated for both overloading and granulation effects. Signal-to-noise ratios were then computed for the various cases and the results for DM were compared with those for PCM. It was found that in certain cases the noise power due to granulation errors is very much less with delta modulation than with PCM. Furthermore, for higher frequencies, overloading in DM will be less troublesome than in PCM, and it occurs much more quickly in PCM.

Considering bandwidth requirements, it was found that the channel bandwidth for DM must be approximately double that for PCM.

B. L. Barber's Report

Barber's master's report (1) gave different insight into the nature of delta modulation. This report established the basis for a deterministic analysis. Previous writers have compared DM to PCM using the latter for a yardstick. Although they did not represent DM as actually being a form of PCM, they maintained that the systems are quite "similar". However, Barber

has shown DM in reality to be derived from pulse duration modulation and pulse amplitude modulation.

The pulse duration modulator is constructed as shown in Fig. 2. The input signal is added to a triangular wave and the sum is fed into a signum function, whose output is the PDM signal. Demodulation consists of averaging the PDM signal over the period of the triangular wave. It can be shown that this average is proportional to the modulator input level up to a saturation level. As will be shown later, such a demodulator can be realized by using a holding filter with holding time equal to the period of the triangular wave.

Pulse amplitude modulation is accomplished by multiplicative sampling of the input by means of a periodic delta function. A PAM demodulator is ideally a steeply-skirted low pass filter. If the period of the sampling function is T , the cut-off frequency of the low pass filter will be $1/T$.

Barber proposed to let the PDM and its demodulator be separated by a PAM and its demodulator. (See Fig. 3.) He called this combination of PDM and PAM "pulse duration-amplitude modulation". The new PDAM modulator consists of the cascaded PDM and PAM modulators, and the new demodulator is a low pass filter in cascade with a holding filter.

If the triangular wave is removed from the PDM modulator the weakened modulator can be somewhat improved by negative feedback so that the output of the holding filter replaces the triangular wave. This arrangement is shown in Fig. 4. It can be seen that this hybrid system is indeed delta modulation.

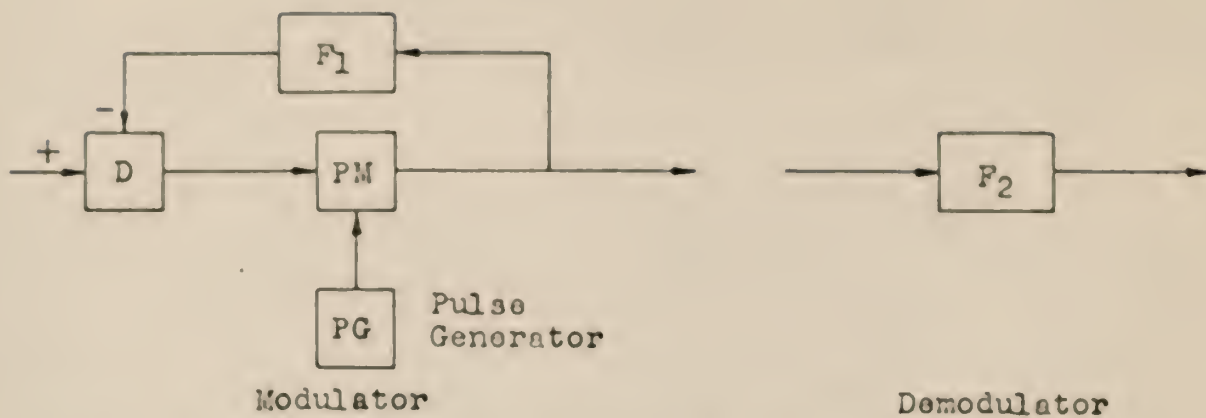


Fig.1. A delta modulation system as described by de Jager.

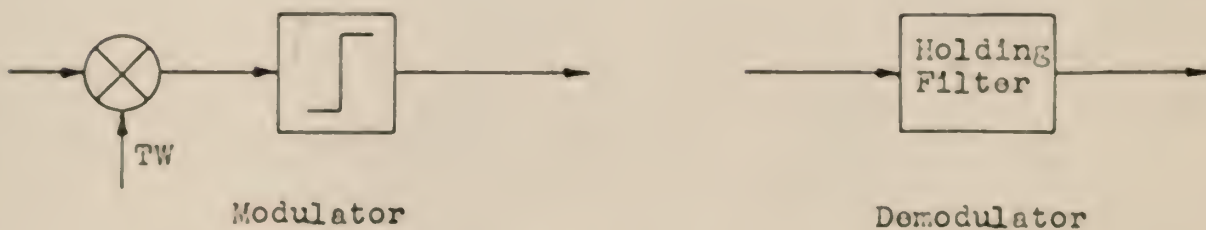


Fig. 2. Pulse duration modulation.

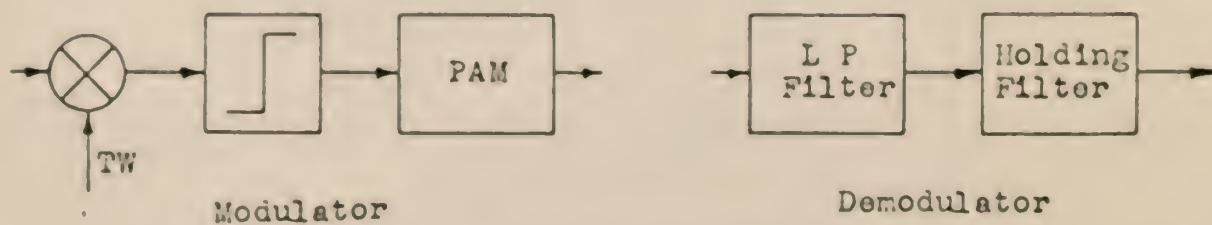


Fig. 3. Pulse duration-amplitude modulation.

Other Papers

E. E. David (3) developed interesting results from studies of delta modulated encodings for transmission of pictures. He also called this "differential quantization". The quantizer used for the picture encoding was eight-leveled with the levels tapered and equivalent to 3 bits per picture point. The performance of this was compared to that of a 3-bit pulse-code modulation. Delta modulation produced a superior reproduction of the picture. PCM reproduction had objectionable, ragged contours in the picture. Here is a case where the quality of delta modulation outstrips that of its standard, pulse code modulation.

A paper by Lender and Kozuch (6) provided experimental data on the operation of delta modulation. A discussion of the hardware used in the construction of an experimental model was given, and an evaluation of the performance was made. This evaluation was made by means of signal-to-noise ratios for sinusoidal inputs and by means of subjective tests with listeners for voice inputs. These results were compared to similar ones for pulse code modulation over a range of sampling rates. The comparison showed that delta modulation is better than PCM for sampling rates of 40,000 bits per second or less and that above this rate PCM is better. However, if bandwidth is no problem, DM can be made to match 7-bit PCM by using a DM bandwidth 40% greater than the bandwidth required for the PCM.

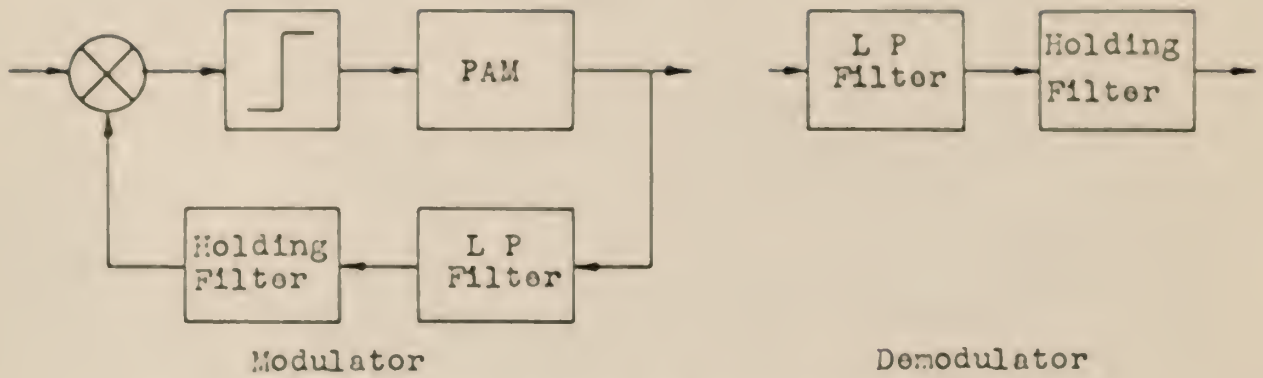


Fig. 4. Delta modulation as derived from PDAM.

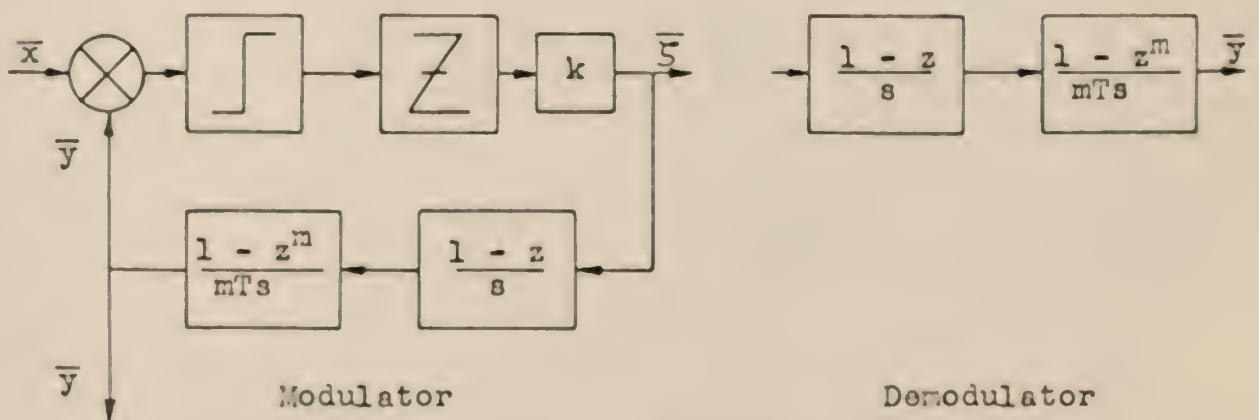


Fig. 5. Z-transform model of delta modulation system.

DETERMINISTIC ANALYSIS

Z-transform Description of Modulator- Demodulator Model

The description of the delta modulation system given by Barber provides a basis for devising a mathematical model in terms of Z-transforms by means of which one can make a deterministic analysis. The same degenerate pulse duration modulator may be used for a model. The pulse amplitude modulator is the sampling operator, Z.

The PDM demodulator is easy to represent. It has been mentioned that the requirement for this demodulator is an integrating network which averages over the period of the triangular wave. Let this period be mT , where m is an integer and T is the period of the sampling operator. If $g(t)$ is the function to be averaged over mT , then the average at time t is

$$\frac{1}{mT} \int_{t-mT}^t g(\tau) d\tau . \text{ Assuming that } g(t) \text{ is piecewise continuous,}$$

taking the Laplace transform of the averaged function gives

$$\frac{1}{mTs} \left\{ L [g(t)] \right\} (1-e^{-mTs}) = \frac{1-e^{-mTs}}{mTs} L [g(t)] \quad (5)$$

Therefore the required operator is $\frac{1-e^{-mTs}}{mTs} = \frac{1-z^m}{mTs}$, (where

$z = e^{-Ts}$), which is the transfer function of a holding filter.

The PAM demodulator, a steeply-skirted low pass filter, is not satisfactory for stable operation of a feedback device, and a filter with shallow skirts must be used. This simply degrades the original configuration a bit further. Another holding filter

of cutoff frequency $1/T$ can be substituted. Its transfer function is $(1-e^{-Ts})/s = (1-z)/s$. To show that this is a shallow-skirted low pass filter, replace s by $j\omega$ in the transfer function and find the spectrum.

$$T(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = \frac{2}{\omega} \sin \frac{\omega T}{2} = T \frac{\sin \theta}{\theta} \quad (6)$$

(where $\theta = \frac{\omega T}{2}$). This function has a shallow, low pass characteristic with cutoff occurring at $\theta = \pi$ or $f_c = 1/T$. The phase angle, $\arg T(j\omega) = \omega T/2$, is linear, assuring a constant delay for all frequencies. The PDM holding filter has similar response characteristics with a cutoff frequency $f_c = 1/mT$.

The Z-transform model of the delta modulation system now appears as shown in Fig. 5, and is in suitable form for analysis.

Recursion Formula for Model's Output

The objective of a deterministic analysis is to determine the output of the demodulator as a function of time, $y(t)$, given the modulator input, $x(t)$. A solution yielding the complete function, $y(t)$, will be very difficult to obtain, but by means of z-transform methods it is possible to find the value of $y(t)$ at each of the sampling times, $t = nT$, where $n = 1, 2, \dots$. For notational purposes let the Laplace transforms of x and y be \bar{x} and \bar{y} respectively. The derivation proceeds as follows: the input to the signum function is $(\bar{x} - \bar{y})$. Then

$$\bar{y} = \left[\frac{1-z^m}{mTs} \right] \left[\frac{1-z}{s} \right] kZ \left[\text{sgn} (\bar{x} - \bar{y}) \right] \quad (7)$$

See the definition of $\text{sgn} \xi$ on page 13.

Now take the Z-transform of both sides of the above relation, observing that Z may be taken inside the signum function since the order of sampling and quantization is immaterial.

$$Z\bar{y} = \frac{k}{mT} Z \left\{ \frac{1}{s^2} \left[(1-z^m)(1-z) \text{sgn} (Z\bar{x} - Z\bar{y}) \right] \right\} \quad (8)$$

By use of the relation,

$$Z \left\{ F(s) Z [G(s)] \right\} = Z [F(s)] Z [G(s)] \quad (9)$$

it follows that

$$Z\bar{y} = \frac{k}{mT} \left[Z \left(\frac{1}{s^2} \right) \right] \left[(1-z^m)(1-z) \text{sgn} (Z\bar{x} - Z\bar{y}) \right] \quad (10)$$

$$\text{Now } Z \left(\frac{1}{s^2} \right) = \frac{Tz}{(1-z)^2} .$$

Then

$$Z\bar{y} = \frac{k}{m} \left(\frac{1-z^m}{1-z} \right) z \text{sgn} (Z\bar{x} - Z\bar{y}) \quad (11)$$

or

$$Z\bar{y} - zZ\bar{y} = \frac{k}{m} \left[z \text{sgn} (Z\bar{x} - Z\bar{y}) - z^{m+1} \text{sgn} (Z\bar{x} - Z\bar{y}) \right] \quad (12)$$

But since z is a delay operator it may be taken inside the signum function giving

$$Z\bar{y} - zZ\bar{y} = \frac{k}{m} \left[\text{sgn} (zZ\bar{x} - zZ\bar{y}) - \text{sgn} (z^{m+1} Z\bar{x} - z^{m+1} Z\bar{y}) \right] \quad (13)$$

By writing this equation in terms of the sampled and delayed values of x and y , one obtains the recursion relation,

$$y_n - y_{n-1} = \frac{k}{m} \left[\text{sgn} (x_{n-1} - y_{n-1}) - \text{sgn} (x_{n-m-1} - y_{n-m-1}) \right] \quad (14)$$

Discussion of Results for Step Inputs

Computer Outputs. The previous recursion formula is particularly well suited for computation on a digital computer. It is necessary only to specify the values of x_n and the value of y_0 , and the computer will do the rest.

This particular set of calculations was carried out on an IBM 650 digital computer. Provision was made in the programming to compute data for various types of input functions and for different values of system parameters. The answers were supplied by the machine at the rate of approximately 500 per minute.

The following system parameters were used:

$$k = 2, \quad m = 256, \quad T = 1$$

The signum function was such that

$$\operatorname{sgn} \xi = \begin{cases} 1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0 \end{cases} \quad (15)$$

This does not agree completely with the models of de Jager and Zetterberg, which were binary in character. They were such that the output for zero input was a series of alternating positive and negative pulses. Such a characteristic could be obtained here by assigning either +1 or -1 to $\operatorname{sgn} \xi$ when $\xi = 0$. The performance of a three-level device can be expected to be somewhat better than for the two-level case.

Results of the calculations showed the overall input-output characteristic to be almost that of a servo for values of x between k/m and k . Below $x = k/m$ there was no response.

Above $x = k$ saturation occurred. In the useful range the output, after reaching the input level, contained fluctuations. For the particular case calculated, y built up linearly to x and remained equal to x until sample number 257. Thereafter it dropped to $(x - \frac{k}{m})$ and remained at that value for a number of intervals depending on the value of x . It then returned to x and remained until sample number 514 where it again dropped. This pattern appeared to repeat indefinitely. See Table 1 for rise times and durations of the drops (or holes) for different inputs, and see Fig. 6 for a typical curve. Since the period of the degenerate triangular wave was mT , it can be seen that the period of the sets of holes is about the same as that of the triangular wave. The reason for the fluctuation could be explained as the need for the feedback system to generate its own "triangular wave".

Errors and Signal-to-Noise Ratios. For inputs, x , restricted to powers of 2 an empirical relation between the duration of the drop and the input magnitude can be made. From the tabulation it can be seen that the number of drop intervals is an integral power of 2. Then if $x = 1/2^n$ for $k/m \leq x \leq k$, then ν , the number of drop intervals, is equal to 2^{7-n} . For example, for $x = 1/4$, $n = 2$ and $\nu = 2^{7-2} = 32$, which is in agreement with the tabulated values. Now the magnitude of y is $\frac{1}{2^n} - \frac{k}{m} = \frac{1}{2^n} - \frac{1}{128}$ for 2^{7-n} intervals and $\frac{1}{2^n}$ for the remaining $257 - 2^{7-n}$ intervals in each period. Hence the average value of y can be computed as follows:

$$\langle y \rangle = \frac{1}{257} \left[\left(\frac{1}{2^n} - \frac{1}{128} \right) (2^{7-n}) + \frac{1}{2^n} (257 - 2^{7-n}) \right] = \frac{256}{257} \left(\frac{1}{2^n} \right) = \frac{256}{257} x \quad (16)$$

From this it can be seen that the average value of the output could be made equal exactly to the value of the input by means of a suitable gain, attaining a servo response. (This has been shown only for values of x equal to 2^n .)

Subtracting the average signal from the total signal yields a noise signal which will enable one to compute a signal-to-noise ratio. Let the amplitude of the positive portion of the noise signal be called u . Then $u = x - \langle y \rangle = \frac{1}{2^n} - \frac{256}{257} \cdot \frac{1}{2^n} = \frac{1}{257} \cdot \frac{1}{2^n}$. The amplitude of the negative portion will be $\frac{1}{128} - u = \frac{1}{128} \left(1 - \frac{2^{7-n}}{257} \right)$. Then the rms value of the noise signal, N , is given by the following equation:

$$\sqrt{\frac{1}{257} \left\{ \left[\frac{1}{128} \left(1 - \frac{2^{7-n}}{257} \right) \right]^2 \cdot \frac{128}{2^n} + \left(\frac{1}{257} \cdot \frac{1}{2^n} \right)^2 \left(257 - \frac{128}{2^n} \right) \right\}} \quad (17)$$

Simplifying yields

$$N = \frac{1}{257 \cdot 2^n} \sqrt{257 \cdot 2^{n-7} - 1} \quad (18)$$

The signal to noise ratio is then

$$S/N = \sqrt{\frac{256}{257 \cdot 2^{n-7} - 1}} \quad (19)$$

Expressed in decibels this is

$$S/N \text{ (db)} = 48.16 - 10 \log_{10} (257 \cdot 2^{n-7} - 1) \quad (20)$$

One must remember that the preceding are empirical relationships used to expedite noise calculations. See Fig. 7 for a curve of calculated signal to noise ratios plotted versus step input magnitudes.

Table 1. Rise time and number of hole intervals for step inputs, $m = 256$.

Input	Rise Time	Number of holes per 257 samples
2	256T	256
1	128T	128
1/2	64T	64
1/4	32T	32
1/8	16T	16
1/16	8T	8
1/32	4T	4
1/64	2T	2
1/128	T	1
1/256	-	-

Table 2. Rise time and number of hole intervals for step inputs, $m = 16$.

Input	Rise Time	Number of holes per 17 samples
2	16T	16
1	8T	8
1/2	4T	4
1/4	2T	2
1/8	T	1
1/16	-	-

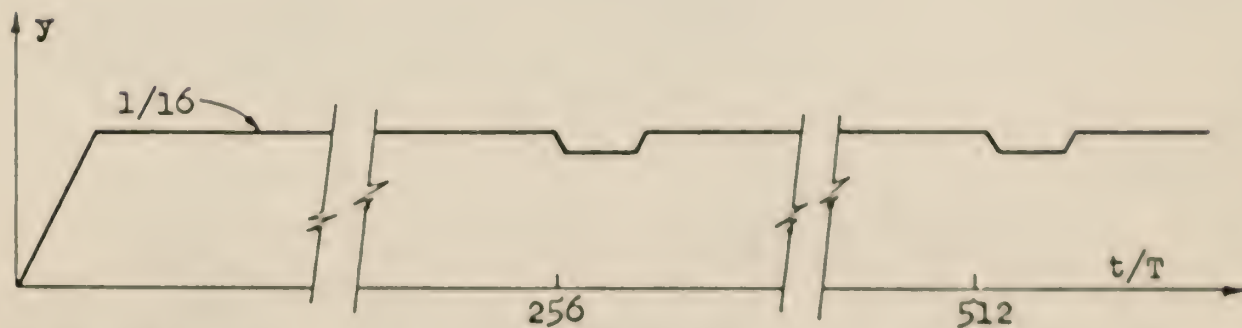


Fig. 6. Output, $y(t)$, for step input, $x(t) = 1/16$, $m = 256$.

A short set of calculations for step inputs with $m = 16$ and $k = 2$ was made. The results are tabulated in Table 2, showing rise times and durations of drops. It can be seen that the results were very similar to those for $m = 256$, with corresponding outputs having similar properties. The period of output oscillation was 17 samples. A more general set of empirical formulas can now be determined. The period of output oscillation is equal to $(m + 1)$. The duration of the drops is equal to $xm/2$, where x and m are both integral powers of 2. Formulas for the average output magnitude, the rms noise value, and the signal-to-noise ratio could be derived from these more general relations.

Signum Function with Dead Space. A few calculations using a dead space of magnitude δ in the signum function were made. This new function was defined as

$$\text{sgn}_{\delta} \xi = \begin{cases} 1, & \xi > \delta \\ 0, & |\xi| \leq \delta \\ -1, & \xi < -\delta \end{cases} \quad (21)$$

In the delta modulator, the old $\text{sgn} \xi$ was replaced by $\text{sgn}_{\delta} \xi$ with no other system changes.

For $\delta < k/m$ no change was observed in response to step inputs. For $\delta = k/m$ the magnitude of y was reduced by an amount equal to k/m . Furthermore the number of hole intervals per $(m+1)$ samples was reduced by 1. As might be expected, $y = 0$ for $x = k/m$. These results indicated that there is no advantage to a dead space alone. A modification of the dead space to produce a five-level signum function and the necessary changes in the system for such a modification will be explored later in the paper.

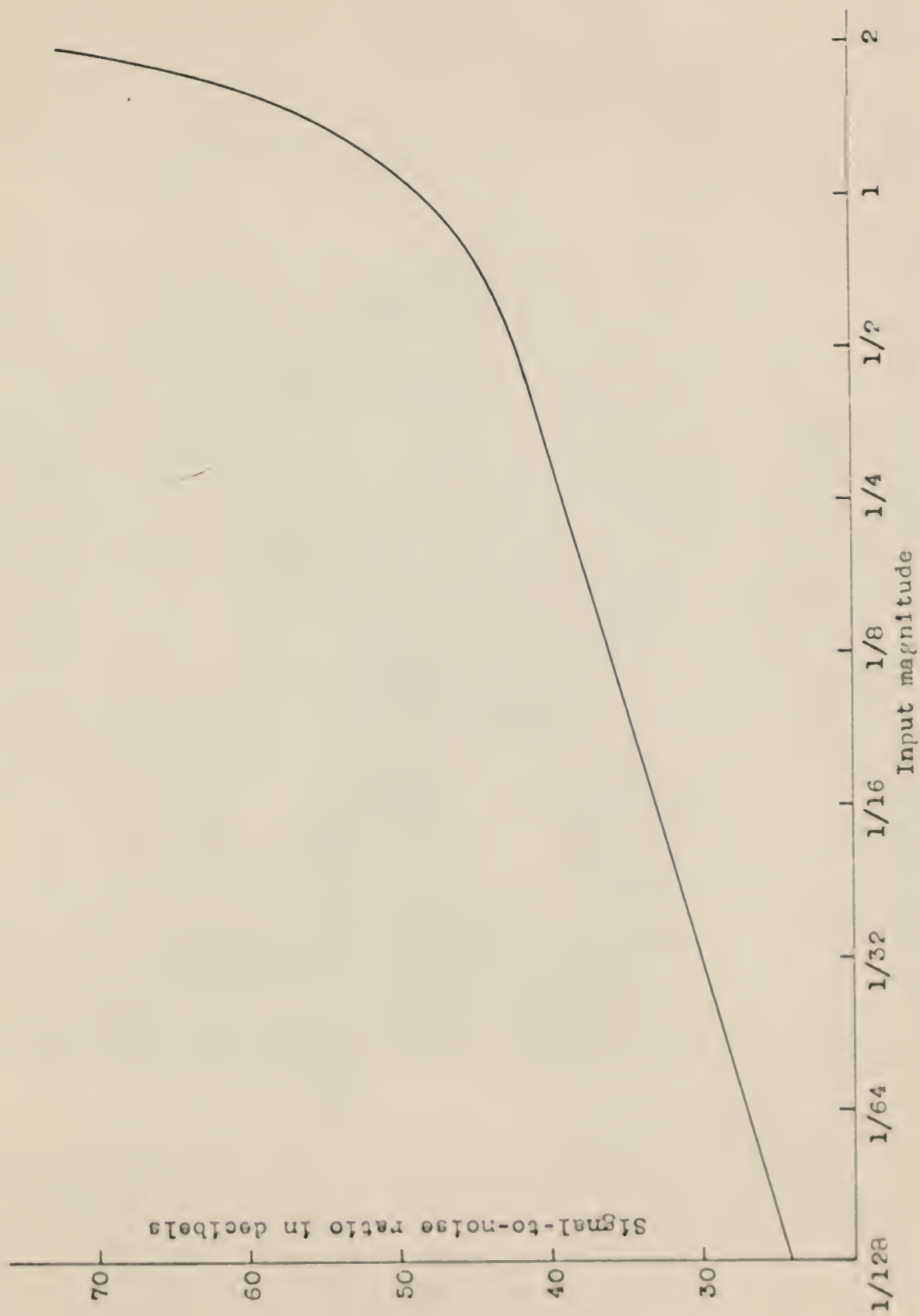


Fig. 7. Signal-to-noise ratio vs. stop input magnitude.

Delta Modulator Outputs for Step Inputs. To make the investigation more nearly complete, a calculation of the modulator output function was made. This is known to be a sequence of positive and negative pulses. With the signum function defined to be equal to 0 when its argument is zero, or with a dead space, the modulator output becomes a sequence of positive, negative, and null pulses.

To obtain a relation between the modulator output and the input consider the modulator shown in Fig. 5.

Let k be placed in the feedback return for calculation convenience. The equation of this system is

$$\bar{\zeta} = Z \operatorname{sgn} \left[\bar{x} - \left(\frac{1-z^m}{mTs} \right) \left(\frac{1-z}{s} \right) k \bar{\zeta} \right] \quad (22)$$

where ζ is the modulator output function. But this is equivalent to

$$\bar{\zeta} = \operatorname{sgn} \left[Z\bar{x} - \frac{k}{mT} (1-z^m)(1-z) Z\left(\frac{1}{s^2} \bar{\zeta}\right) \right] \quad (23)$$

Now $Z\left(\frac{1}{s^2} \bar{\zeta}\right)$ is difficult to express as a function of the z -transforms of $\frac{1}{s^2}$ and $\bar{\zeta}$. But Halijak (5) has shown by his integrator substitution program that such an operation may be approximated by

$$Z \left[\frac{1}{s^n} \bar{f}(s) \right] \doteq T Z \left(\frac{1}{s^n} \right) \left[Z \bar{f}(s) - \frac{1}{2} f(t) \Big|_{t=0} \right] \quad (24)$$

and hence

$$Z \left[\frac{1}{s^2} \bar{\zeta} \right] \doteq \frac{T^2 z}{(1-z)^2} \left[Z \bar{\zeta} - \frac{1}{2} \zeta_0 \right] \quad (25)$$

Therefore the substitution can be made

$$\bar{\zeta} = \operatorname{sgn} \left[Z\bar{x} - \frac{k}{mT} (1-z^m)(1-z) \frac{T^2 z}{(1-z)^2} \left(Z \bar{\zeta} - \frac{1}{2} \zeta_0 \right) \right] \quad (26)$$

Now by taking the z -transform of both sides and observing that $\zeta_0 = 0$, the following relation results

$$z \bar{\zeta} = \operatorname{sgn} \left[z \bar{x} - \frac{kT}{m} \frac{(1-z^m)z}{1-z} z \bar{\zeta} \right] \quad (27)$$

$$z \bar{\zeta} = \operatorname{sgn} \left[z \bar{x} - \frac{kT}{m} \left(\sum_{r=1}^m z^r \right) z \bar{\zeta} \right] \quad (28)$$

Writing this equation in terms of the sampled values of x and ζ produces the recursion formula.

$$\zeta_n = \operatorname{sgn} \left[x_n - \frac{kT}{m} \sum_{r=1}^m \zeta_{n-r} \right] \quad (29)$$

This recurrence relationship was programmed for the IBM 650, and modulator outputs were computed for several values of x , with and without dead spaces in the signum function. Calculation shows that a positive pulse is emitted for each interval until y equals x , on the initial build-up. Then no pulses are emitted until sample 258 when a sequence of consecutive pulses is emitted. The number of pulses in this sequence turns out to be exactly equal to the number of hole intervals in the output y . This sequence of positive pulses is followed by another sequence of 0 pulses. The next pulse occurs at sample 515 with the same pattern as before repeated. The entire pattern is repeated indefinitely with a period of 257 samples. Consider the samples of y_n which are not equal to x . If these be delayed by an amount equal to T , they will occur coincidentally with the positive pulses from the modulator. This seems reasonable when one studies the system and observes that the future output of the signum function depends on the present value of the error.

Review of Results for Step Inputs. A brief review of the deterministic analysis shows that a mathematical model of the system can be constructed, from which a formal relationship between output and input can be derived. By programming a digital computer to solve for the outputs corresponding to various step inputs, it is possible to make an evaluation of the performance of the delta modulator. Between certain input amplitude limits the system functions as a servo with the output equaling the input after a finite delay. A measurement of the noise power generated in the system is also available allowing a computation of signal-to-noise ratios from the various inputs. Finally, the form of the modulated signal is computed by simple changes in the formula.

Performance and System Modifications for Ramp Inputs

$(1-\lambda)+\lambda z$ Feedback Transfer Function. Since the delta modulation system can be made to act like a servo for properly chosen step inputs, an obvious extension of the investigation is to determine the response for appropriate ramp inputs. One can be sure that the slope of the ramp can never exceed $2k/mT$ since $|y_n - y_{n-1}| \leq 2k/mT$. Also the magnitude of the output cannot exceed k since this is the saturation level of delta modulation output. To show this, consider equation (11), and let

$$z \operatorname{sgn} (Z\bar{x} - Z\bar{y}) = \operatorname{sgn} Z \bar{\xi}$$

where $\bar{\xi}_n = (x_{n-1} - y_{n-1})$.

Then

$$z\bar{y} = \frac{k}{m} \frac{1-z^m}{1-z} \operatorname{sgn} z \bar{\xi} = \frac{k}{m} \sum_{r=0}^{m-1} z^r \operatorname{sgn} z \bar{\xi} \quad (30)$$

In terms of sampled values this is

$$y_n = \frac{k}{m} \sum_{r=0}^{m-1} \operatorname{sgn} \xi_{n-r} \quad (31)$$

But $-1 \leq \operatorname{sgn} \xi \leq 1$

Hence $-k \leq y_n \leq k$ and since y lies between these bounds it cannot follow x beyond them.

It was suspected that the output would tend to lag the input, and compensation for this was attempted before any calculations were made. In order to bring the magnitude of the output closer to the input, the difference signal, ξ , seen at the signum function could be increased or be made positive more frequently. Since ramps are the inputs, the introduction of a delay into the loop return would result in earlier values of y being subtracted from x , and thus increasing the average of ξ . Such a delay can be introduced by placing the delay operator, z , in the loop return. ξ_n would then become $(x_{n-1} - y_{n-2})$. Even better might be the subtraction of a weighted sum of y_{n-1} and y_{n-2} from x_{n-1} , so that ξ_n would become

$$\xi_n = x_{n-1} - \frac{ay_{n-1} + by_{n-2}}{a+b} = x_{n-1} - \left[(1-\lambda)y_{n-1} + \lambda y_{n-2} \right] \quad (32)$$

where $|\lambda| \leq 1$. The operator for the return link is then of the form $(1-\lambda) + \lambda z$. See Fig. 8 for the new system.

The relation between \bar{x} and \bar{y} may be written as

$$\bar{y} = \frac{(1-z^m)}{mTs} \frac{(1-z)}{s} k z \operatorname{sgn}_6 \left[\bar{x} - (1-\lambda + \lambda z)\bar{y} \right] \quad (33)$$

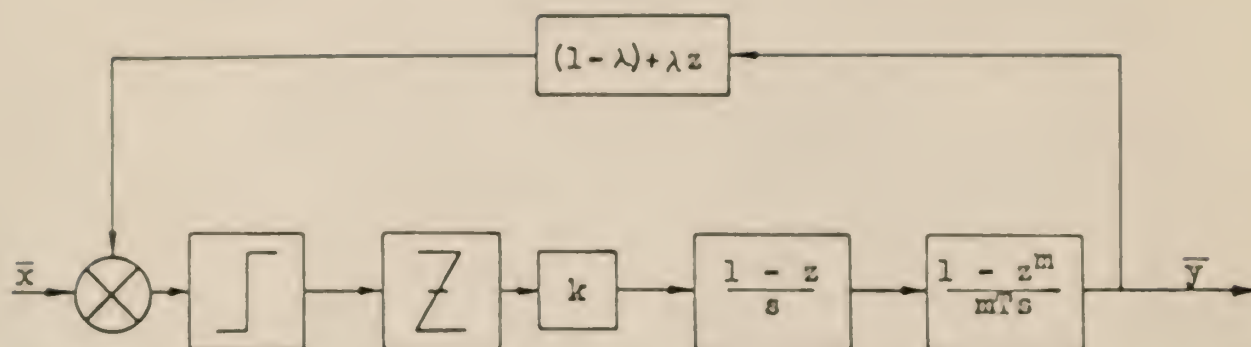


Fig. 8. Proposed delta modulation for admission of ramps using delay in feedback return.

A recursion relation may be obtained from this equation in the same manner that equation (14) was obtained from equation (7) and which is

$$y_n = y_{n-1} + \frac{k}{m} \left\{ \text{sgn}_{\delta} [x_{n-1} - (1-\lambda)y_{n-1} - \lambda y_{n-2}] - \text{sgn}_{\delta} [x_{n-m-1} - (1-\lambda)y_{n-m-1} - \lambda y_{n-m-2}] \right\} \quad (34)$$

By setting $\lambda = 0$, one may calculate the unity feedback case. Various values of λ between 0 and 1 may then be tried to find conditions for optimum performance corresponding to best agreement of the output with the input.

Computer Results for Various Combinations of Parameters with $(1-\lambda)+\lambda z$ Feedback. For the particular case considered the slope of x was set equal to $1/512T$. The values of λ used were 0, 0.5, and 1, with dead spaces in the signum function of 0, $1/512$, and $1/128$ for each value of λ . This combination thus included the unity feedback case with and without signum dead spaces and covered the extremes and the midpoint for λ in the $(1-\lambda)+\lambda z$ compensation.

See Figs. 9-17, Plates I through III, for plots of the output for $\lambda = 0, 0.5$, and 1, with $\delta = 0$. The output fluctuated about the input in a regular pattern with abrupt pattern changes every 257th sample. Therefore only segments of each consecutive set of 257 samples were plotted. Analysis of the output shows that in general the average error remained constant during each 257 sample section and increased from one section to the next. The standard deviation of the error, a measure of the noise, also remained constant during each section and increased substantially from one section to the next.

EXPLANATION OF PLATE I

Delta modulation with unity feedback. Output for input, $x(t) = t/512T$.

Fig. 9. Output between samples 500 to 560.

Fig. 10. Output between samples 240 to 300.

Fig. 11. Output between samples 0 to 80.

PLATE I

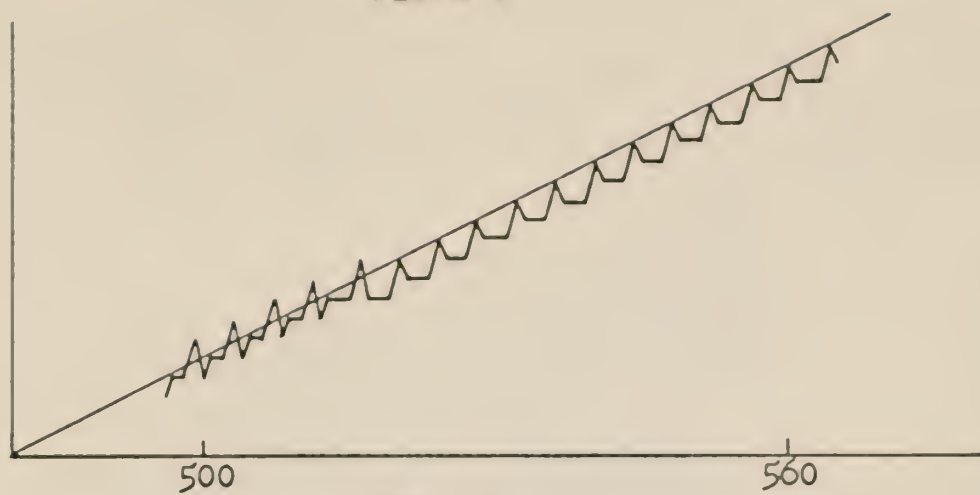


Fig. 9.

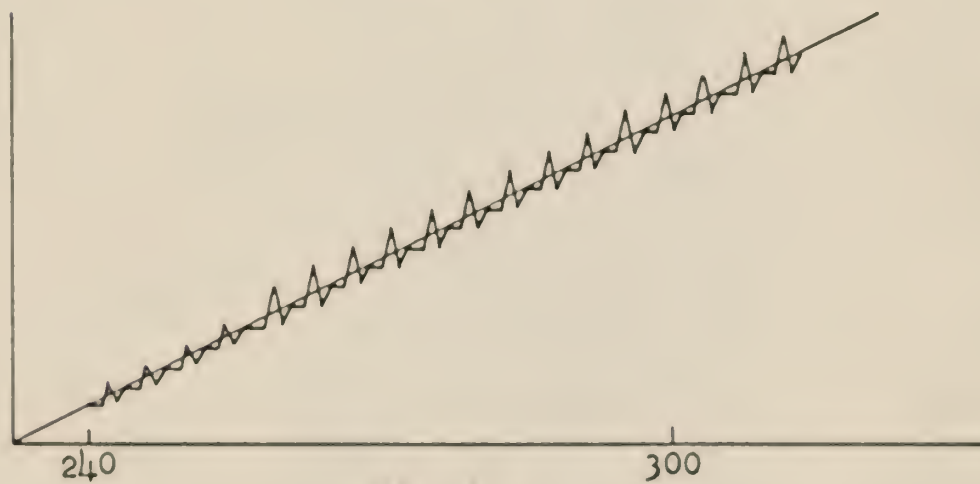


Fig. 10.

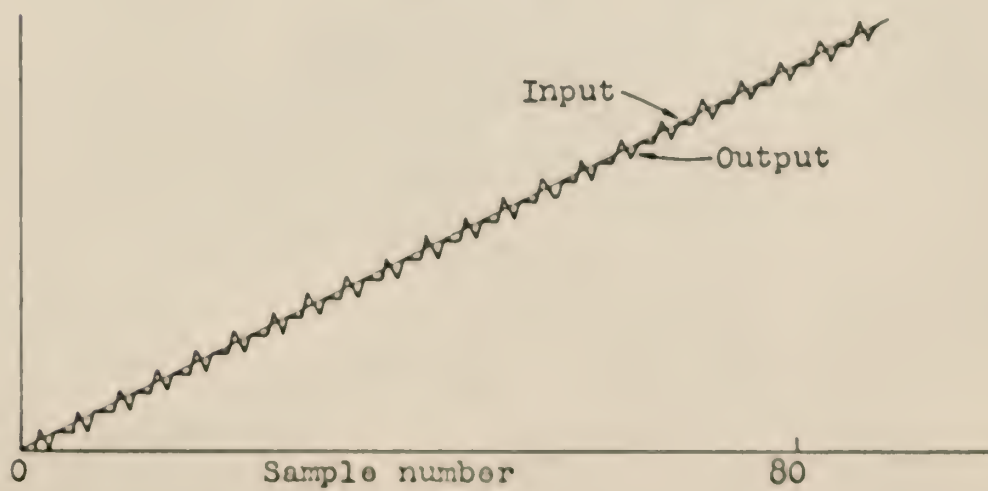


Fig. 11.

EXPLANATION OF PLATE II

Delta modulation with $(1-\lambda)+\lambda z$ feedback, where $\lambda = 0.5$.

Output for input $x(t) = t/512T$.

Fig. 12. Output between samples 500 to 560.

Fig. 13. Output between samples 240 to 300.

Fig. 14. Output between samples 0 to 80.

PLATE II

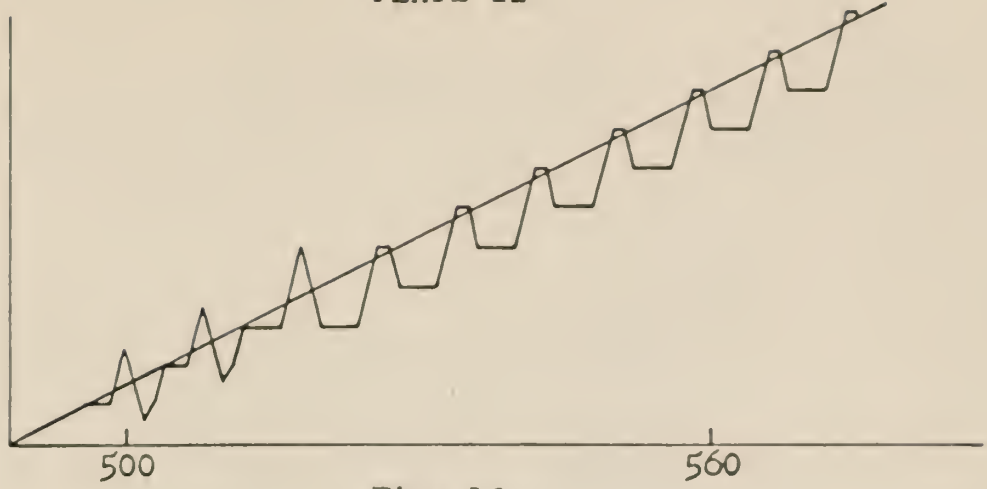


Fig. 12.

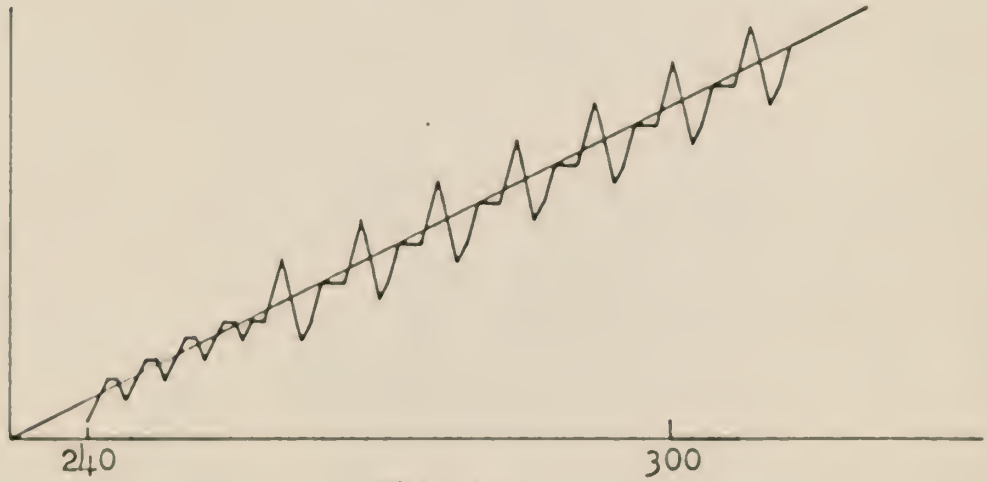


Fig. 13.

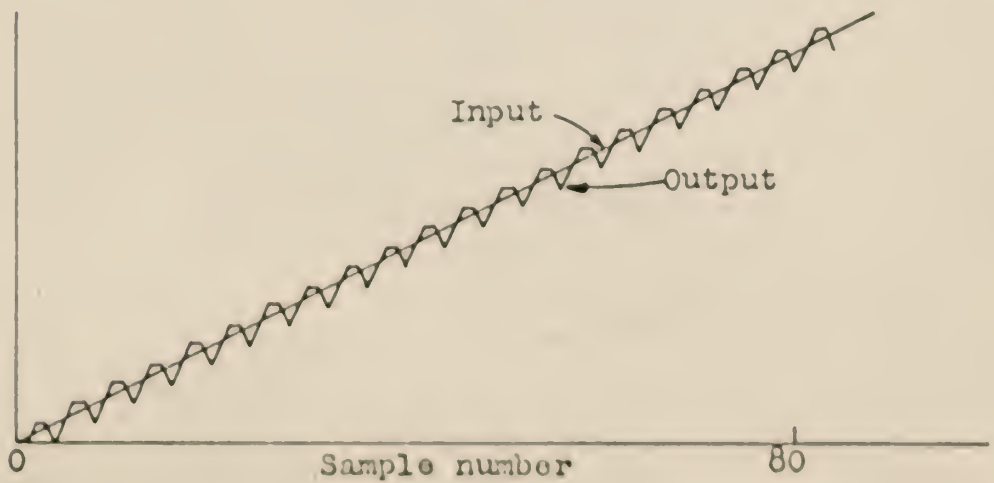


Fig. 14.

EXPLANATION OF PLATE III

Delta modulation with $(1-\lambda)+\lambda z$ feedback where $\lambda = 1.0$.

Output for input $x(t) = t/512T$.

Fig. 15. Output between samples 500 to 580.

Fig. 16. Output between samples 240 and 320.

Fig. 17. Output between samples 0 to 80.

PLATE III



Fig. 15.

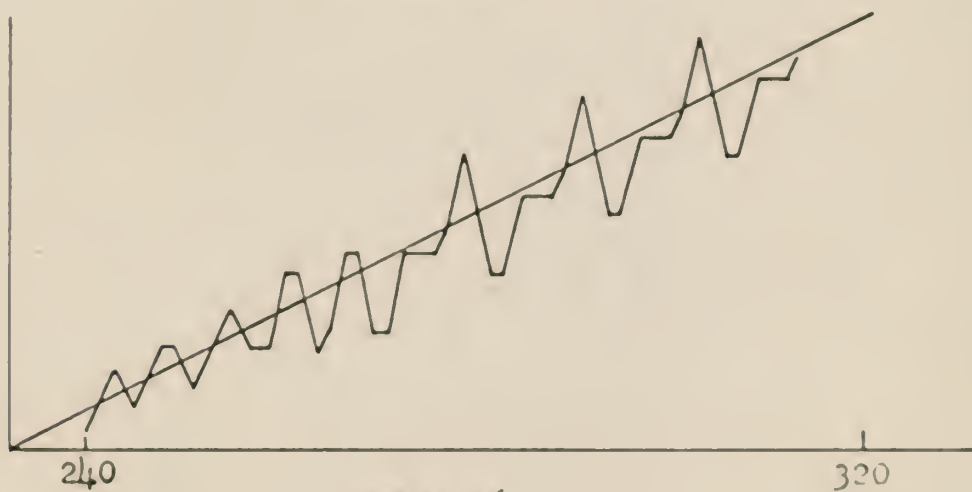


Fig. 16.

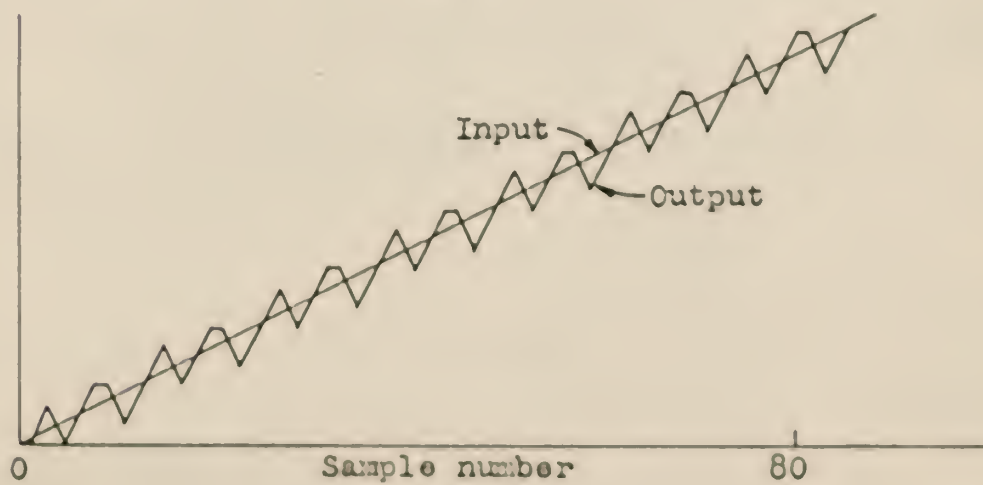


Fig. 17.

The average error was calculated by averaging the difference between x and y at each of the sampling intervals over the period of repetition of the fluctuations. That is

$$\langle \epsilon \rangle = \frac{1}{k} \sum_{i=1}^k (x_i - y_i) = \frac{1}{k} \sum_{i=1}^k \epsilon_i \quad (35)$$

The standard deviation of the error is

$$\sigma_{\epsilon} = \sqrt{\frac{1}{k} \sum_{i=1}^k \epsilon_i^2 - \langle \epsilon \rangle^2} \quad (36)$$

See Table 3 for a tabulation of average errors, maximum errors and standard deviations of the error for $\lambda = 0, 0.5$, and 1 , which are shown for three regular sections of 257 samples.

Table 3 shows that the effect of delay in the feedback return is to increase the noise, the maximum error, and the average error. Reference to the curves of output vs. input with $\lambda \neq 0$ shows that, as the number periods increased, the output continued to attempt to reach the input but could do so for only one or two sampling intervals. It then fell behind even further than for the unity feedback case and remained behind for a considerable time. This indicates why the average value of the output decreased for larger λ .

The results of the calculations with $\delta = \frac{1}{512}$ and $\frac{1}{128}$ showed that the effect of δ was to smooth out fluctuations at the expense of increased delay. With larger values of λ , the fluctuations increased considerably but remained less than those for $\delta = 0$. However the decreased noise did not seem to be worth the added delay except for the case with $\delta = 1/512$.

Table 3. Maximum error, average error, and standard deviation of error for input, $x(t) = t/512$, $(1-\lambda) + \lambda z$ in feedback link.

λ	n	Max. Error	Avg. Error	Std. Dev. Error
0.0	1-257	5.8594×10^{-3}	0.9766×10^{-3}	3.5210×10^{-3}
	258-514	9.7656×10^{-3}	0.9766×10^{-3}	6.5510×10^{-3}
	515-771	15.6250×10^{-3}	10.2539×10^{-3}	6.0001×10^{-3}
0.5	1-257	7.8125×10^{-3}	0.9766×10^{-3}	4.4752×10^{-3}
	258-514	19.5313×10^{-3}	2.9297×10^{-3}	9.8143×10^{-3}
	515-771	23.4375×10^{-3}	12.6553×10^{-3}	9.8143×10^{-3}
1.0	1-257	13.6719×10^{-3}	0.9766×10^{-3}	6.9283×10^{-3}
	258-514	25.3906×10^{-3}	4.8828×10^{-3}	14.778×10^{-3}
	515-771	37.1094×10^{-3}	21.8099×10^{-3}	13.362×10^{-3}

Table 4. Maximum error, average error, and standard deviation of error for input $x(t) = t/512$. Dead space = $1/512$ in signum function, and unity feedback.

n	Max. Error	Avg. Error	Std. Dev. Error
1-257	3.9063×10^{-3}	0.9766×10^{-3}	2.18×10^{-3}
258-514	5.8593×10^{-3}	2.9298×10^{-3}	2.18×10^{-3}
515-771	7.8125×10^{-3}	4.8830×10^{-3}	2.18×10^{-3}

Figs. 18-20, Plate IV, show the output for $\delta = 1/512$, which provided the smoothest output with the least standard deviation and least maximum error for any case. If one is willing to tolerate an average error, one will have a minimum amount of noise throughout the three periods for $\lambda = 0$, $\delta = 1/512$. Calculations show that the standard deviation of the error remained constant for the three periods. See Table 4.

This seems to be as nearly ideal as can be obtained using this particular arrangement. The conclusion drawn from this set of calculations is that the basic delta modulation system will almost follow properly chosen ramps. The amount of delay between input and output is small, but tends to increase with each degenerate triangular wave period. The amount of noise can be minimized by the use of a small dead space in the signum function. The use of a delay in the feedback link produces no beneficial effects, but does increase delay and noise.

Recursion Formula and Results for $1/(1+\tau s)$ in Feedback Link.

The next compensation scheme employed an approximation to the identity operator, $1/(1+\tau s)$, in place of the unity in the return link. This procedure is analogous to the compensation employed in continuous servos to admit ramp inputs. The resulting system is shown in Fig. 21.

The equation of this system is

$$\bar{y} = \frac{(1-z)(1-z^m)}{m\tau s^2} k Z \left\{ \text{sgn} \left[\bar{x} - \frac{\bar{y}}{1 + \tau s} \right] \right\} \quad (37)$$

EXPLANATION OF PLATE IV

Delta modulation with unity feedback and signum dead space of magnitude $1/512$. Output for input $x(t) = t/512T$.

Fig. 18. Output between samples 500 to 580.

Fig. 19. Output between samples 240 to 320.

Fig. 20. Output between samples 0 to 80.

PLATE IV

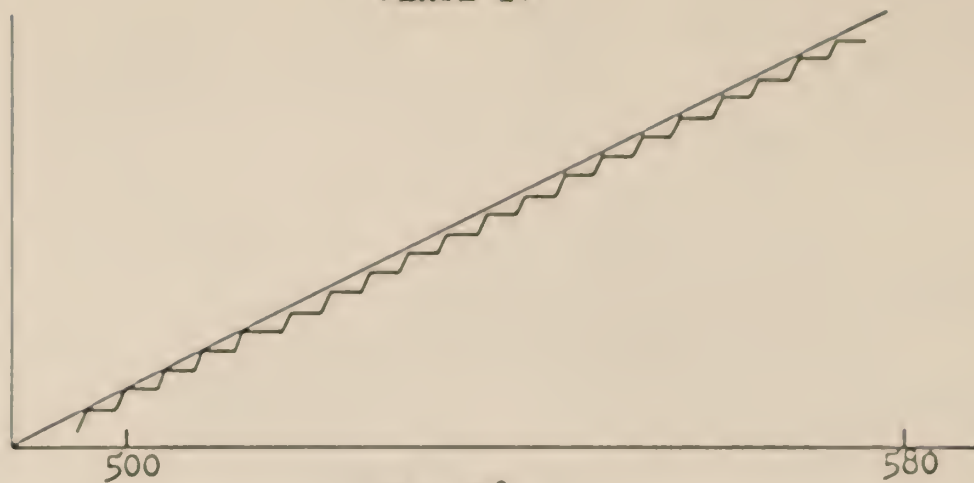


Fig. 18.

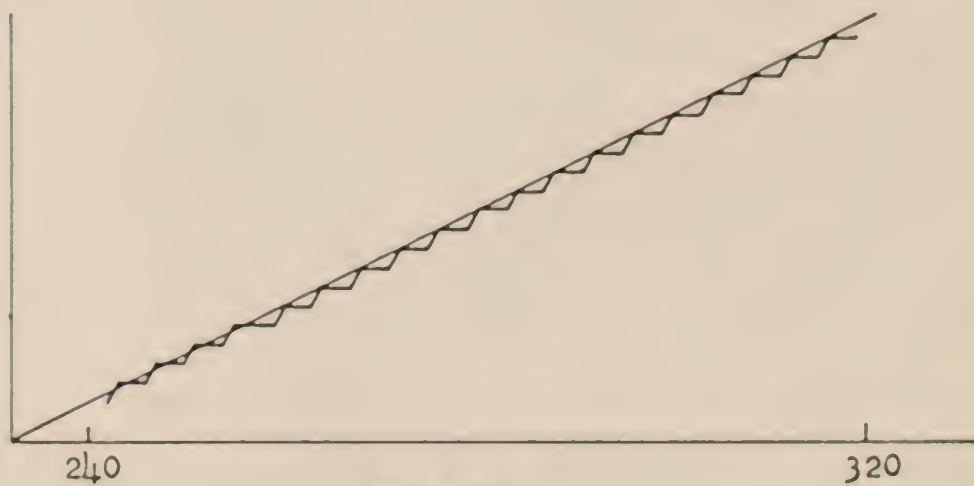


Fig. 19.

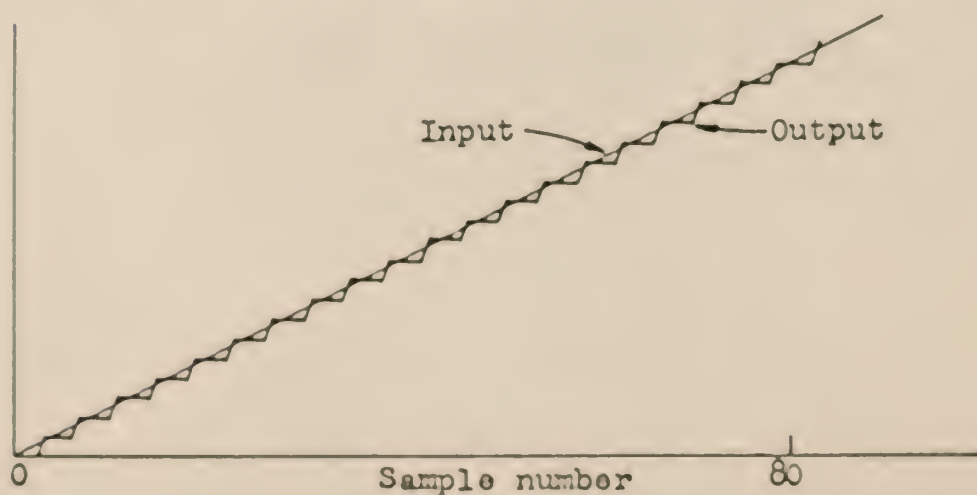


Fig. 20.

The z-transform of both sides of the equation is taken giving

$$Z\bar{y} = \frac{k}{m} \frac{1-z^m}{1-z} \operatorname{sgn} \left[zZ\bar{x} - zZ\left(\frac{\bar{y}}{1+\tau_s}\right) \right] \quad (38)$$

Use the trapezoidal approximation (Halijak (5)) to obtain

$$Z(\bar{f}\bar{g}) \doteq T(Z\bar{f})(Z\bar{g}) - \frac{1}{2}T \left[f(0)Z\bar{g} + g(0)Z\bar{f} \right] \quad (39)$$

Consider $Z(\bar{f}\bar{g}) = Z\left[\frac{1}{1+\tau_s}(\bar{y})\right]$

wherein $\bar{f} = \bar{y}$, $\bar{g} = \frac{1}{1+\tau_s}$, $f(0) = y_0 = 0$, and

$$g(0) = L^{-1} \left(\frac{1}{1+\tau_s} \right) \Big|_{t=0} = \frac{1}{\tau} e^{-t/\tau} \Big|_{t=0} = \frac{1}{\tau} \quad (40)$$

Also needed is

$$Z \left(\frac{1}{1+\tau_s} \right) = \frac{1}{\tau} \sum_{r=1}^{\infty} (e^{-T/\tau_z})^r \quad (41)$$

Hence

$$Z \left[\left(\frac{1}{1+\tau_s} \right) \bar{y} \right] = T \left[\frac{1}{\tau} \sum_{r=1}^{\infty} (e^{-T/\tau_z})^r Z\bar{y} - \frac{1}{2\tau} Z\bar{y} \right] \quad (42)$$

Finally

$$Z\bar{y} = \frac{k}{m} \left(\frac{1-z^m}{1-z} \right) \operatorname{sgn} \left\{ zZ\bar{x} - \frac{T}{\tau} z \left[\sum_{r=0}^{\infty} (e^{-T/\tau_z})^r - \frac{1}{2} \right] Z\bar{y} \right\} \quad (43)$$

And

$$\begin{aligned} y_n = & y_{n+1} + \frac{k}{m} \operatorname{sgn} \left[x_{n-1} + \frac{T}{2\tau} y_{n-1} - \frac{T}{\tau} \sum_{r=0}^{\infty} (e^{-T/\tau})^r y_{n-r-1} \right] \\ & - \frac{k}{m} \operatorname{sgn} \left[x_{n-m-1} + \frac{T}{2\tau} y_{n-m-1} - \frac{T}{\tau} \sum_{r=0}^{\infty} (e^{-T/\tau})^r y_{n-r-m-1} \right] \end{aligned} \quad (44)$$

In the calculation of outputs, it was not known what value of τ would provide an optimum response, so the following range of values was used: $\tau = 192, 128, 64, 32, 16, 8, 4$, and 2 . The

value of δ was 0, and the slope of the input was $1/512T$, with k and m unchanged.

The results were not without compromise. They showed that output delay can be eliminated at the expense of increased noise and the appearance of a delay during the later sampling intervals. The output pattern was similar to previous cases in that it formed sections of 257 samples in which the fluctuation remained about the same, although not as regular as before. The pattern changed from one section to the next, usually with sharply increasing noise.

The optimum value of τ appeared to be in the neighborhood of 4. See Figs. 23-25, Plate V. The fluctuation became violent for larger values of τ , and delay began to appear for τ less than 4. The minimum fluctuation for any case calculated was for $\tau = 2$, but was still greater than the corresponding deviation for the cases with $(1-\lambda) + \lambda z$ compensation. For example, with $\tau = 2$, $n < 257$; $\sigma_{\epsilon} = 9.00 \times 10^{-3}$ while the corresponding deviation for the worst case of the $(1-\lambda) + \lambda z$ compensation, where $\lambda = 1$, $n < 257$, was 6.93×10^{-3} . However the average error for both cases was equal to 0.9766×10^{-3} . Further decreases in τ would likely result in performance approaching that of the uncompensated case.

A tabulation of the maximum error, average error, and standard deviation for the output sections and for $\tau = 4$ is given in Table 5. It can be seen from the tabulation that the output tended to lead the input for the first 257 samples, and then began to fall behind, although not significantly until n became

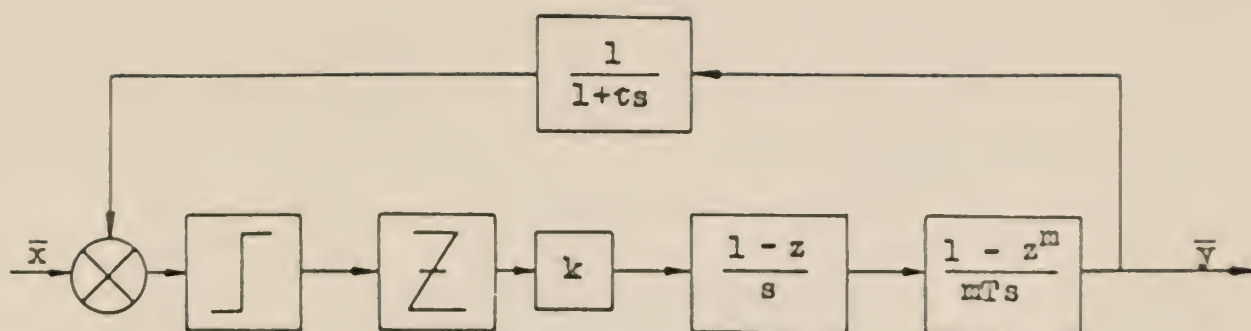


Fig. 21. Delta modulation with $\frac{1}{1+\tau s}$ operator in return link.

Table 5. Maximum error, average error, and standard deviation of error for input $x(t) = t/512T$, with $1/1+\tau s$ in feedback return, $\tau = 4$.

n	Max. Error	Avg. Error	Std. Dev. Error
1 - 257	21.4×10^{-3}	-6.84×10^{-3}	9.21×10^{-3}
296 - 344	41.0×10^{-3}	0.72×10^{-3}	17.58×10^{-3}
568 - 624	37.1×10^{-3}	8.86×10^{-3}	15.46×10^{-3}

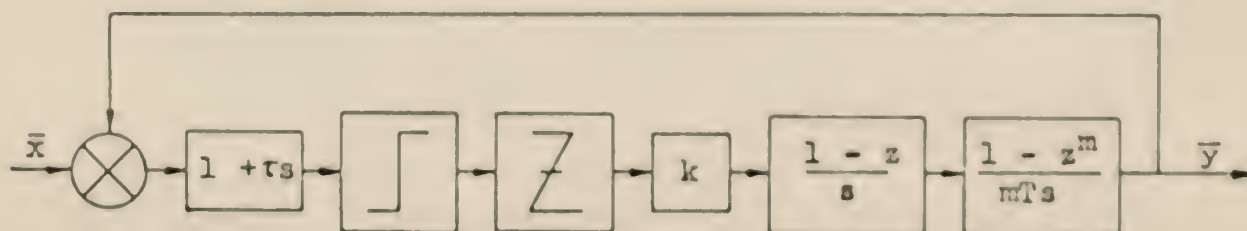


Fig. 22. Delta modulation with $1+\tau s$ operator feeding signum function.

EXPLANATION OF PLATE V

Delta modulation with $1/(1+\tau s)$ feedback where $\tau = 4$.

Output for input $x(t) = t/512T$.

Fig. 23. Output between samples 500 to 580.

Fig. 24. Output between samples 240 to 320.

Fig. 25. Output between samples 0 to 80.

greater than 514. Comparison with previous tabulations shows that the delay was greater for all previous cases than for $\tau = 4$ except for the uncompensated delta modulation with a small dead space in the signum function and $n > 514$. There the average error was only 4.88×10^{-3} . For the first two output sections the error for the $1/(1+4s)$ case was better. With regard to considerations of ε_{\max} and σ_{ε} , the uncompensated case was better. Its maximum error was $1/3$ that of the $1/(1+4s)$ case, and its noise was 17 db under that of the $1/(1+4s)$ case for $n > 514$.

If the great increase in noise can be allowed, an overall compromise on the delay can be reached by proper choice of τ , perhaps yielding a zero overall average delay. The amount of noise that can be tolerated and the final compromise made will depend on the application at hand.

Derivative Compensation of Error. Another method of compensation, insertion of a function of the form $(1+\tau s)$ between the comparator and the signum function, was attempted next. See Fig. 22. The equation of this system is

$$\bar{y} = \frac{k}{m} \frac{(1-z^m)}{Ts} \left(\frac{1-z}{s} \right) z \operatorname{sgn} [(1+\tau s)(\bar{x}-\bar{y})] \quad (45)$$

The solution of this equation yields the following recursion formula:

$$\begin{aligned} y_n = & y_{n-1} + \frac{k}{m} \operatorname{sgn} \left[\left(\frac{2\tau}{T} + 1 \right) (x_{n-1} - y_{n-1}) \right. \\ & - \frac{4\tau}{T} \sum_{r=2}^{\infty} (-1)^r (x_{n-r} - y_{n-r}) \left. \right] - \frac{k}{m} \operatorname{sgn} \left[\left(\frac{2\tau}{T} + 1 \right) (x_{n-m-1} \right. \\ & \left. - y_{n-m-1}) - \frac{4\tau}{T} \sum_{r=2}^{\infty} (-1)^r (x_{n-m-r} - y_{n-m-r}) \right] \end{aligned} \quad (46)$$

Again, proper values for τ were unknown, and values of 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ were used. Slopes of $1/512T$ and $1/256T$ for x were used, and δ was set equal to zero. Other parameters remained unchanged.

The effect of this modification was to introduce considerable delay, increasing with τ , with a certain amount of fluctuation. For $\tau \geq 1/4$, in every case, the output was very small, hovering slightly above zero, since the error was almost equal to the input. It could be said that the output consisted of the noise generated plus a small constant. For $\tau = 1/8, 1/16$, the output resembled a ramp with the error remaining more nearly constant. For $\tau = 1/8$ the average delay was approximately $175T$ while for $\tau = 1/16$ it was approximately $75T$. This much delay is intolerable and the fluctuation was still greater than is desired. It was concluded that this particular modification was unsuccessful.

Conclusions on Admission of Ramps. The results of the investigation of delta modulation response to ramp inputs led to the following conclusion. A trade-off exists between error and noise, and an ideal output of zero error and constant minimized fluctuation is unattainable. If one desires minimal noise and is willing to tolerate small errors, a small dead space in the signum function of a delta modulation system with unity feedback will provide such performance. On the other hand, if zero average error is desired with little concern for fluctuation, then such performance can be obtained through the use of $1/(1 + \tau s)$ feedback with τ properly chosen.

Review of the Goals and Capabilities of Probabilistic and Deterministic Analyses

A deterministic analysis of the delta modulation system has been presented. It would be of interest to compare the accomplishments and capabilities of this approach with those of the probabilistic analysis of Zetterberg. Since de Jager's work was primarily descriptive and experimental, there is no need to discuss it further.

The goal of the probabilistic analysis was to determine the properties of delta modulation as an information encoding-decoding system from the viewpoint of information theory and to compare these properties to corresponding ones for pulse code modulation. Considerable emphasis was placed on this comparison rather than evaluating DM as a new system. Among the properties considered were channel capacity, signal power, optimum ensemble and its spectral properties, and noise power generated within the system.

The probabilistic approach has the capability of giving an indication of how well a communication system will perform with delta modulation as an integral part. This is carried out by computing the amount of information that can be handled, given requirements regarding quality of demodulated signals. The limitations on types of inputs that can be reproduced with desired accuracy and the optimal inputs, together with upper bounds on input frequencies, may be obtained. An idea of channel bandwidth requirement may also be computed. By this means, one can discover the types of applications for which DM is best suited along with

ideas on its incorporation into larger systems. The probabilistic analysis deals with the overall performance of DM and is incapable of predicting specific events at particular times. It is unable to determine least upper bounds on information capacities leaving the user uncertain as to how far away actual capacities are.

This uncertainty points out the need for other methods of studying the system, and the deterministic analysis of this paper fills a part of this need. The goal was to obtain the magnitude of the demodulated output as a function of time, given the input as a function of time, and to provide compensation schemes. Deterministic analysis is capable of dealing with absolute quantities at specific times, and furthermore a measure of the overall performance as well as performance in a larger system may be derived. Also problems encountered in the physical design of a delta modulation system require deterministic methods for adequate solution.

The exact form of the output of the demodulator can be calculated. From this one can ascertain the faithfulness of reproduction along with signal to noise ratios. Since the exact form of the modulator output is also available, channel requirements and bandwidths may be obtained. The digital computer makes it possible to carry out the arithmetic operations of the deterministic method very rapidly, relieving any time element drawback. This method also provides an excellent means for evaluating the effects of new modifications on the

system, as is illustrated by servo compensation techniques described earlier. In most cases a probabilistic analysis would provide no information of this sort. Although the functions of the two approaches overlap slightly, the fact remains that both are necessary for a complete picture.

GENERALIZATION OF DELTA MODULATION

It is possible to generalize a delta modulation system by replacing the three-level quantizer with a five-level quantizer. For basic development of the new system reconsider DM as the original combination of pulse amplitude and pulse duration modulators and demodulators. Let the input signal be fed into two separate PDAM modulators. Both of these modulators consist of the usual comparator with its triangular wave, followed by a signum function and a sampling operator. For section 1 modulator the signum function has no dead space, but for section 2 it has a dead space of magnitude D . The triangular waves are both of the same period but have different amplitudes. Now, let the two-pulse duration-amplitude modulated signals be added to yield a five-level sequence of pulses. This combination of PDAM modulators operating in tandem makes up the new modulator.

Since the PDAM demodulators for each section are identical, the use of a single demodulator for the added outputs is equivalent to demodulating each section separately and adding the resultants, and this justifies addition of the two signals. A diagram of the new arrangement is shown in Fig. 26.

An analysis of each section must be made in order to obtain a combined input-output characteristic which is linear. For simplicity it will be assumed that the PAM portion reproduces perfectly so that only the PDM part need be analyzed. Since the modulator in section 1 has no dead space in the signum function, when combined with the demodulator it gives an overall input-output characteristic which is linear, passes through the origin, and saturates at output level mkT , where mT is the period of the triangular wave and k is the amplification factor. A plot of this characteristic is shown in Fig. 27. Saturation occurs for an input magnitude equal to H_1 , where H_1 is the amplitude of the triangular wave of section 1. Hence the slope of the output versus input curve is equal to mkT/H_1 . It must be pointed out that these response characteristics apply only to step inputs.

It is desirable that the response characteristic of section 2 be such that it will complement the response of section 1 to produce a linear resultant when the two outputs are added. Figure 28 shows the desired output versus input curve for section 2. It has a dead space for which the output is zero for inputs of magnitude less than some value, Δ . For inputs greater than Δ the output rises linearly until an output saturation level is reached. An analysis similar to the one for section 1 shows that such a characteristic may be obtained only when the magnitude of the dead space, D , of the signum function is greater than the amplitude of the triangular wave, H_2 . The value of Δ , which is the value of input, x , for

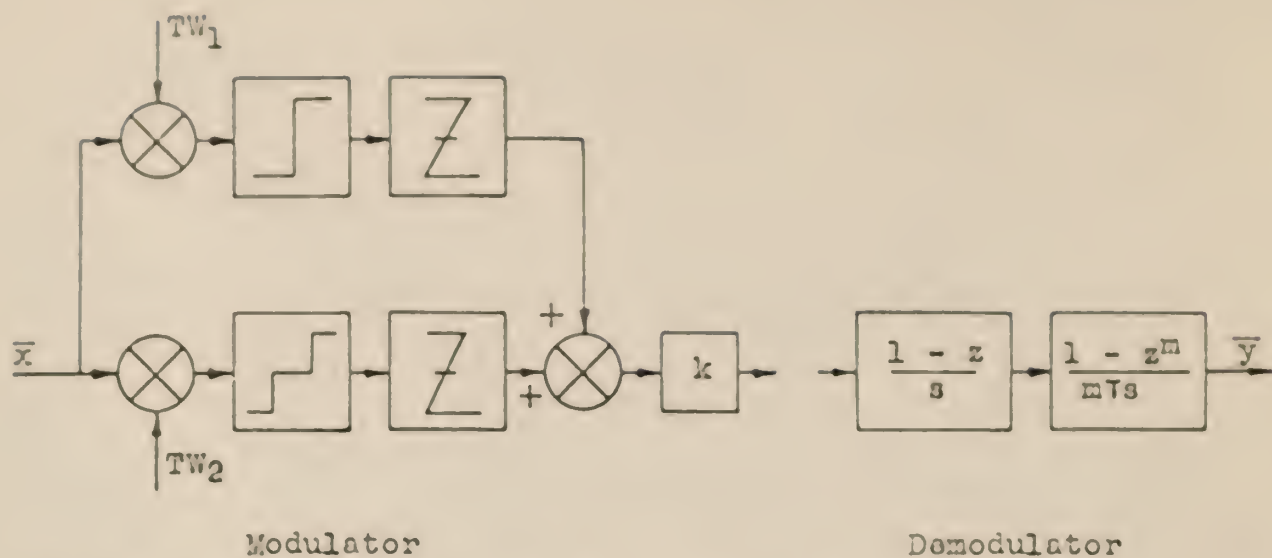


Fig. 26. Dual section pulse duration-amplitude modulation.

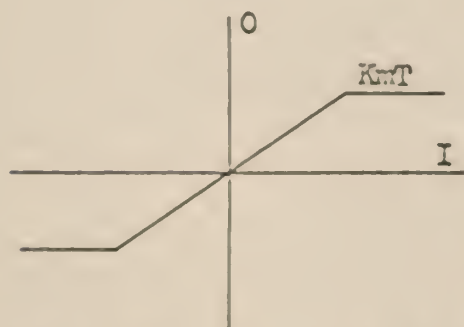


Fig. 27. Response of PDM without dead space.

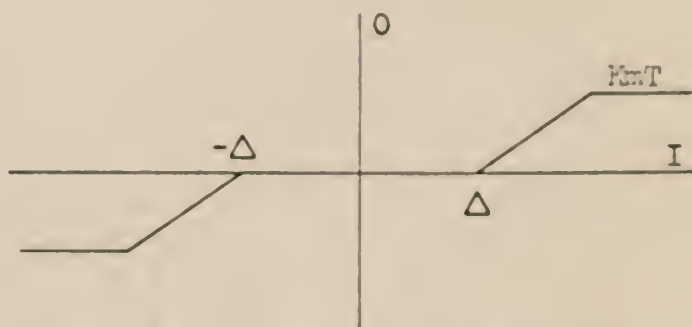


Fig. 28. Response of PDM with dead space.

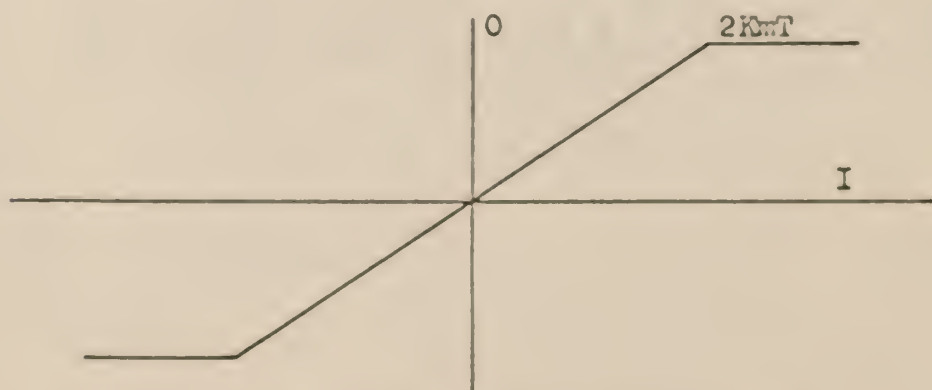


Fig. 29. Response of dual section PDM.

which output begins, equals the difference between the magnitude of signum dead space and the triangular wave amplitude, or $\Delta = D - H_2$. Saturation occurs for an input equal to $(D + H_2)$. The output saturation level is again mkT so that the slope of the output-versus-input line is $mkT/2H_2$. In order to obtain a linear overall characteristic, D and H_2 must be chosen so that section 2 output begins where section 1 output saturates, which implies that $(D - H_2)$ must equal H_1 . Furthermore the slopes of the two characteristics must be equal, or $mkT/2H_1 = mkT/2H_2$. Therefore $H_2 = H_1/2$, and $D = 3H_1/2$. The overall transfer characteristic is shown in Fig. 29. This combined system will now respond to inputs whose magnitudes range from zero to $(D + H_2) = 2H_1$, with a maximum output equal to $2mkT$. The effect has been to double the range of response of pulse amplitude duration modulation, while retaining the same accuracy of response, since the pulse duration modulator period, mT , and pulse amplitude modulator period T , have remained unchanged. If the same amplitude range were maintained the accuracy would double.

Now let the triangular waves be removed from both sections of the modulator, and let the output of the demodulator be substituted. Since the amplitude of the triangular wave for section 2 is to be $1/3$ the magnitude of the dead space, a limiting amplifier may be placed in the return to this section which will limit the amount fed back to a maximum magnitude of $D/3$. This new configuration is shown in Fig. 30. An even further weakening of the system would be the elimination of the limiting amplifier. This then would permit simplification of

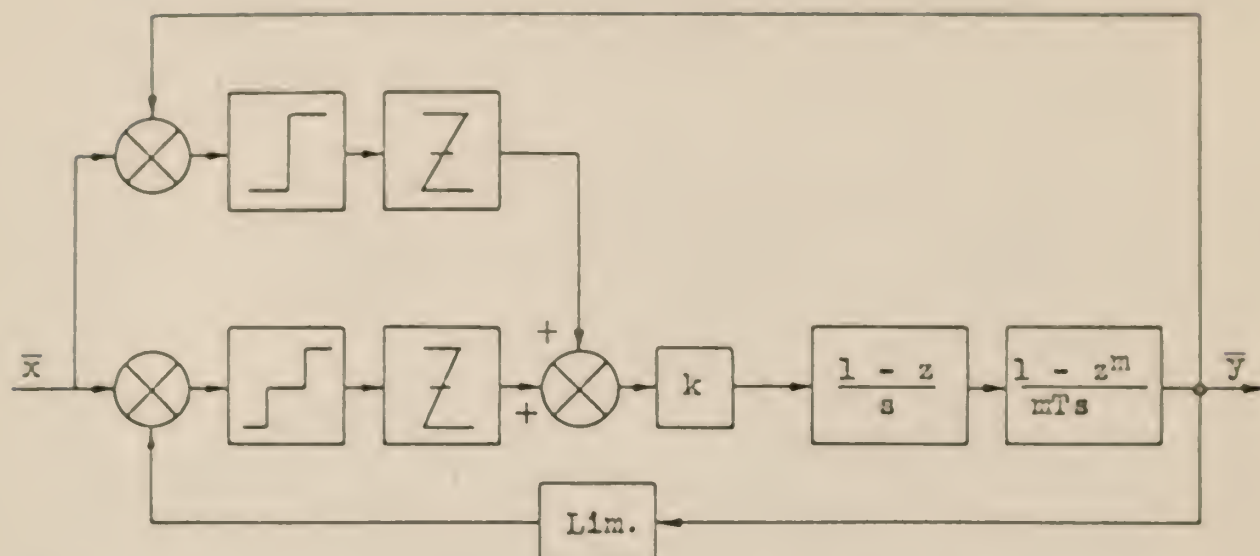


Fig. 30. Dual section PDAM weakened to a feedback device.

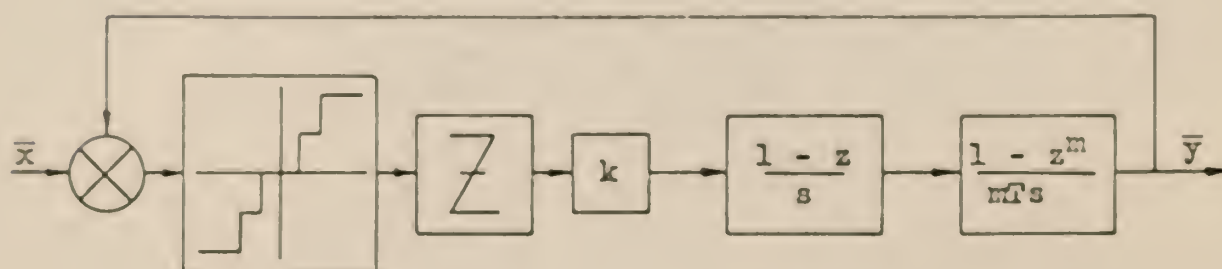


Fig. 31. 5-level delta modulation.

the new system to the old delta modulation with the signum function replaced by a new five-level quantizer. The quantizer is the result of adding the outputs of the ordinary signum function and the signum function with a dead space. See Fig. 31.

This weakened PDAM with its multilevel pulses is equivalent to a digital servo. However, accurate demodulation after passage through an imperfect transmission channel requires regeneration of pulses to unit height. This can be achieved by representing a pulse of the higher level by two pulses of a single level, with the second pulse occurring halfway between the original pulse and its following pulse. Hence, if the pulse to be encoded occurs at time $t = nT$, where $n = 1, 2, 3, \dots$, the first pulse of its coded representation occurs coincidentally at $t = nT$, and the second at $t = nT + T/2$. The same situation will apply to negative pulses.

An encoder for this purpose is shown in Fig. 32. The five-level sequence of pulses is fed into the device and passes into a signum function with a dead space such that it will produce an output for only the higher level positive and negative pulses. The output of the signum function is then delayed by time $t = T/2$ and added to the original input sequence. This sum is passed through a limiting amplifier (equivalent to another signum function) to obtain the final encoded output sequence of pulses with a uniform magnitude (and null pulses also).

The decoder is even simpler. It is assumed that it will receive a sequence of null or uniform magnitude pulses from

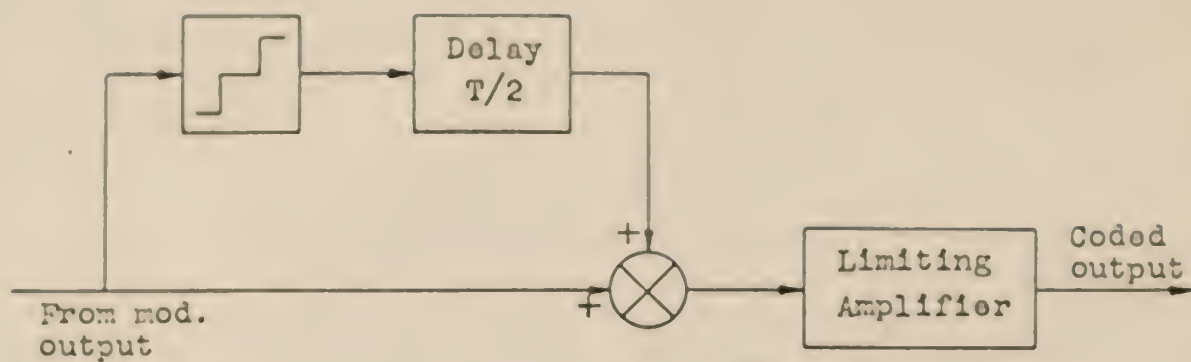


Fig. 32. Three-level encoder for five level delta modulation.

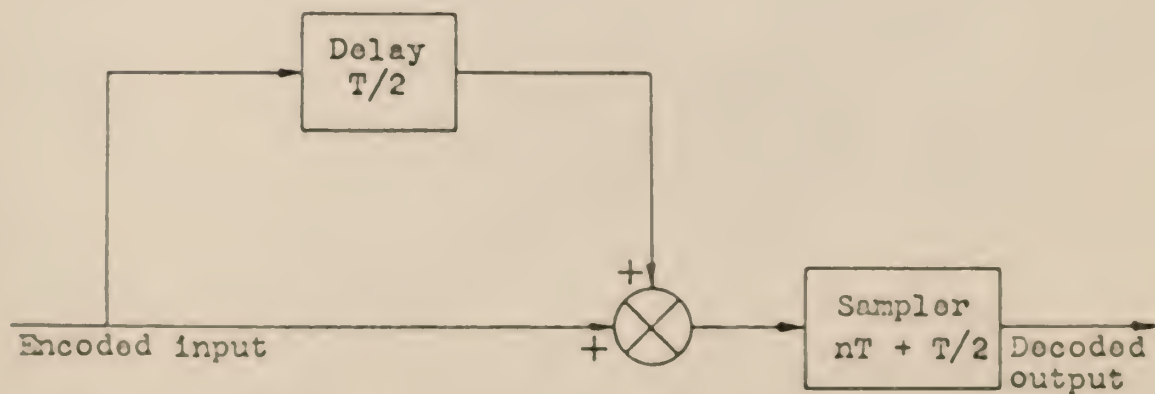


Fig. 33. Decoder for encoded five-level delta modulation.

suitable channel amplifiers. This sequence is delayed by time $t = T/2$ and added to the undelayed sequence. This sum is then amplitude sampled at times $t = nT + T/2$, ($n=1,2,3,\dots$), so that the original sequence of five-level pulses is reconstructed, although delayed by an amount $T/2$. See Fig. 33.

It has been shown that a five-level delta modulation is feasible, and that the sequence of pulses comprising the modulated signal can be constrained to have a single magnitude, which retains a desirable feature of DM. The accuracy of reproduction has been doubled for a given range of inputs. The price for this improvement is doubled bandwidth required for a given fundamental sampling frequency, or a halving of the fundamental sampling rate for a given bandwidth restriction of the channel. This is caused by insertion of the extra pulses in the coded output. An increased delay in the output has also been incurred. Further increases in the number of signum levels will lead to a pulse code modulation for encoding the delta modulated signal.

Analysis of the five-level delta modulation to determine how far the PDAM may be degraded and the best choice of parameters has not been carried out at this time. However, an idea of the performance of a multilevel system has been provided by David (3).

SUMMARY

A history of previous work in the field of delta modulation has been presented along with an interpretation of results. Past writers considered DM to be similar to pulse code modulation and used the latter as a yardstick to measure the performance of DM. The analyses were probabilistic in nature. An upper bound on channel capacities was derived as well as noise and spectral calculations.

Barber interpreted delta modulation as a combination of pulse duration and pulse amplitude modulation. This provided the basis for a deterministic analysis of DM using Z-transform methods and digital computer techniques. The results of these calculations showed the system to have the characteristics of a servo for certain ranges of step inputs. Signal-to-noise ratios were calculated from the computer results. Also the exact form of the modulator output, a sequence of positive, negative, and null pulses, was computed.

An investigation of the performance of DM for ramp inputs was made, and the effects of compensation techniques were studied. It was found that for DM with unity feedback, the most satisfactory performance could be obtained through the use of a signum function with a small dead space. This minimized output fluctuations although a small increasing error remained. By placing a function of the form $1/(1+\tau s)$ in the return link the average error could be eliminated at the expense of greatly increased noise. Further investigation showed that an idealized

response of minimal noise and zero error is unattainable so that a compromise will have to be made between the two.

A comparison of the goals and capabilities of the probabilistic and deterministic analyses showed that both were necessary for a more nearly complete picture. The deterministic approach was able to supply specific, concrete results and was best suited for use in analysis and design. The probabilistic approach was able only to set upper bounds on quantities and was better suited for evaluating DM as an information handling system.

Finally, an extension of delta modulation using five-level quantization was presented. The new system was developed from the rudimentary combination of pulse duration and pulse amplitude modulation. It should reproduce the input more accurately over the same range of input values. An encoder is needed to change the five-level pulses into positive and negative pulses of a single magnitude, effecting a generalization of delta modulation.

ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. Charles Halijak of the Department of Electrical Engineering for directing the investigation and for guidance in writing this report.

REFERENCES

- (1) Barber, Beryl L.
Delta modulation. Unpublished Master's report,
Kansas State University, Manhattan, Kansas, 1961.
- (2) Bowers, F. K.
What use is delta modulation to the transmission
engineer? Transactions AIEE. Part I, Communications
and Electronics, May 1957, 76:142-147.
- (3) David, Edward E., Jr.
Digital simulation in research on human communication.
Proceedings IRE. January, 1961, 49:319-328.
- (4) de Jager, F.
Delta modulation. Eindhoven, Netherlands: Philips
Research Reports, 1952, 7:442-466.
- (5) Halijak, Charles A.
Digital approximation of the solutions of differential
equations using trapezoidal convolution. Report ITM-64,
Bendix Systems Division, Ann Arbor, Michigan, August 26,
1960.
- (6) Lender, A., and M. Kozuch.
Single-bit delta modulating systems. Electronics,
November 17, 1961, 34:125-129.
- (7) Ragazzini, John R. and Gene F. Franklin.
Sampled data control systems. New York: McGraw Hill,
1958.
- (8) Tou, Julius T.
Digital and sampled data control systems. New York:
McGraw Hill, 1959.
- (9) Zetterberg, L. H.
A comparison between delta and pulse code modulation.
Stockholm, Sweden: Ericsson Technics, 1955, 2:95-154.

A DETERMINISTIC STUDY OF DELTA MODULATION

by

JOHN STEPHEN TRIPP

B. S., Kansas State University, 1961

AN ABSTRACT OF
A MASTER'S REPORT

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1962

Previous investigators have made probabilistic comparisons of delta modulation with pulse code modulation. However Barber has shown that delta modulation is actually a hybrid of pulse duration and pulse amplitude modulation. Let a PDM modulator and its demodulator be separated by a PAM modulator and demodulator. Then remove the triangular wave from the pulse duration modulator and replace it by the demodulated output. This weakened device is a delta modulation system.

An idealized mathematical model may be derived and leads to a deterministic analysis. The weakened PDM modulator is represented by a comparator followed by a signum function. Its demodulator is a holding filter whose transfer function is $(1-z^m)/mTs$. The PAM modulator is the sampling operator, Z . Its demodulator is another holding filter with the transfer function, $(1-z)/s$, which functions as a linear-phase low-pass filter. A Z -transform analysis leads to a recursion relation which enables one to compute the demodulator output at each sampling instant, given the input as a function of time. Digital computer techniques applied to the recursion formula show that the basic delta modulation system can be made to function as a servo for a range of step inputs. From these results one may compute signal-to-noise ratios. The exact form of the modulated signal may also be computed.

An extension of the investigation considers the response of a DM system to ramp inputs and provides proper modification for the admission of ramps. It is found that DM with unity

feedback will almost follow ramps with a considerable amount of noise and a small error, both of which increase periodically by jumps. The insertion of a small dead space into the signum function will minimize the noise to a constant value although the error increases slightly. A normalized delay in the return link to eliminate the error has the opposite effect of increasing both error and noise. An approximation to the identity operator, $1/(1 + \tau s)$, in the return link can be made to provide zero average output error over the interval of interest through the proper choice of τ . The price for this is increased noise.

Further study shows that a trade off exists between error and noise and that both may not be minimized at the same time. For minimal error, use $1/(1 + \tau s)$ in the feedback link, and for minimal noise use unity feedback with a small signum dead space.

The probabilistic and deterministic analyses are both needed for a complete study of a delta modulation system. The former provides a measure of the capacity of the system to handle information. The deterministic approach is necessary for specific analysis and the design of a system.

Delta modulation may be generalized to a form having five level quantization. This should result in improved and extended performance. Furthermore it is possible to encode the five-level delta modulated signal into one having three-level quantization by representing the highest level pulses by two lower level pulses. This, however, results in doubled channel bandwidth requirements.