Generalized Pak-Stanley labeling for Multigraphical Hyperplane Arrangements for n=3

by

Joshua Miller

B.S., Sam Houston State University, 2014

M.S., Sam Houston State University, 2016

M.S., Kansas State University, 2019

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Manhattan, Kansas

2021

Abstract

In 1998 Pak and Stanley defined the original Pak-Stanley labeling as a bijective map from the set of regions of an extended Shi arrangement to the set of parking functions. This map was later generalized to other arrangements: Sam Hopkins and David Perkinson considered Pak-Stanley labeling on bigraphical arrangements ([1]), and Mikhail Mazin generalized the labeling to arrangements associated with directed multigraphs ([2]). In this generalized setting the labeling always provides a surjective map from the set of regions of the arrangement to the set of graphical parking functions. However, this map often failed to be injective.

This lead to a natural question, what graphs admit arrangements with a bijective labeling? In this paper we present a necessary, but not sufficient, condition for the injectivity of the generalized Pal-Stanley labeling. Moreover, for $n = 3$ we show that even if an arrangement has duplicate labels, then the closure of the union of regions with the duplicate label is connected. Lastly, we present ways to construct bijective arrangements for several families of graphs in n=3, and present examples showing that the conditions are not sufficient.

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This lead to a natural question, what graphs admit arrangements with a bijective labeling? In this paper we present a necessary, but not sufficient, condition for the injectivity of the generalized Pal-Stanley labeling. Moreover, for $n = 3$ we show that even if an arrangement has duplicate labels, then the closure of the union of regions with the duplicate label is connected. Lastly, we present ways to construct bijective arrangements for several families of graphs in n=3, and present examples showing that the conditions are not sufficient.

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Finally I would like to thank my parents for their continued love and support throughout my academic years. Thank you for being in my life.

Dedication

There have been many people that have wandered through my life and left an impact on me and this thesis is dedicated to them.

This thesis is for all the friends, human, furry, and scaly that spent many nights keeping me company through this arduous process. This thesis is dedicated to, in no particular order, Sonya, Hiccup, Titus, and Toph who kept me company through everything. This thesis is for Bear, Richie, Ani, and so many others because without you all, none of this would be possible.

Lastly this thesis is dedicated to my loving wife, Morgan, and son Peyton, who decided to join me on this crazy journey called life. Without you two, I would not have had the courage to fly towards my dreams. Thank you.

Chapter 1

Introduction

For the introductory topics, we will first begin with a look at posets and their connection to general hyperplane arrangements. Then we will look at parking functions and the Sandpile model, some of its history, and how the model motivated the concept of G-parking functions. In the last part of the introduction we will talk about the Pak-Stanley labeling and some results for a particular family of hyperplane arrangements called multigraphical hyperplane arrangements.

1.1 Posets and General Hyperplane Arrangements

A **poset** is defined as

Definition 1.1. A partially ordered set, **poset**, is a set P together with a relation $\leq$ that is

1. Reflexive, $x \leq x$ for all $x \in P$
2. Antisymmetric, if $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in P$
3. Transitive, if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in P$.

An element $m \in P$ is called minimal if there does not exist an $x \in P$ such that $x < m$, however if m is a unique minimal element then it is called the least element, usually denoted $\hat{0}$. We say that y covers x, $x \lessdot y$, in poset P if $x < y$ and there does not exist an element $z \in P$ that satisfies $x < z < y$. Moreover, any finite poset is fully determined by the covering relations and can be represent pictorially by the Hasse diagram.

Definition 1.2. Given a poset P, the **Hasse diagram** is graph, $G = (V, E)$ constructed as follows. The vertex set of the diagram is the set of elements in the poset P where x is drawn lower than y if $x < y$. The edge set of the graph is given by, the edge $(x \to y) \in E$ if $x \lessdot y$. See Figure 1.1 for an example.

Beside the relation, some posets have the property that for every pair of $x, y \in P$ there is a largest common lower bound called the **meet**, denoted $x \wedge y$, and a unique smallest common upper bound called **join**, denoted $x \vee y$. These posets are called:

Definition 1.3. A poset L is a **lattice** if for every pair $x, y \in L$ there exists unique

$$(1) \quad x \text{ meet } y,\ x \wedge y = \max\{z \in L : z \leq x, z \leq y\},$$

$$(2) \quad x \text{ join } y,\ x \vee y = \min\{z \in L : z \geq x, z \geq y\}.$$

Inside a poset, we can have a **chain of length** l which is a subset of the elements in our poset P, say $\{x_0, x_1, \ldots, x_l\}$ that satisfy $x_0 < x_1 < \cdots < x_l$. Further, we say that the chain is **saturated** if the relations along the chain are all covering relations, i.e. $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l$.

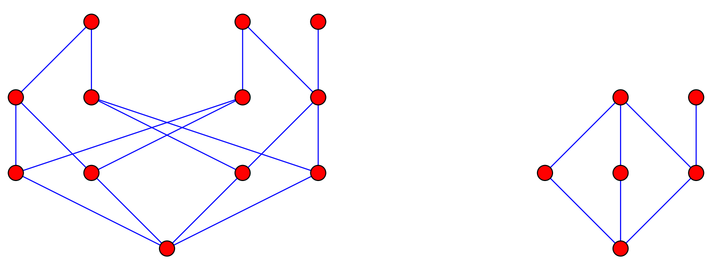


Figure 1.1: *Shown are two Hasse diagrams for posets that have a least element, located at the bottom of the diagram. Both posets are graded, where the left poset is of rank 3 and the right is of rank 2.*

If for every maximal chain in the poset P the length of the chains is l, then P is said to be graded and of rank l. Naturally, we can define a rank function $\mathrm{rk} : P \to \mathbb{N}$ by

$$\text{for every minimal element } \mathrm{rk}(\hat{0}) = 0, \qquad \mathrm{rk}(y) = \mathrm{rk}(x) + 1 \text{ if } x \lessdot y \text{ in } P.$$

Definition 1.4. Given two elements $x, y \in P$ with $x \leq y$, the interval from x to y, denoted $[x, y]$, is defined as

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

Moreover, the length of the interval can be determined using the rank function where $\mathrm{rk}(x, y) = \mathrm{rk}(y) - \mathrm{rk}(x)$ is the length of $[x, y]$.

Let P be a locally finite poset and let $\mathrm{Int}(P)$ denote the set of all closed intervals of P. For locally finite posets, there is a fundamental invariant that can be defined. Consider the following function $\mu : \mathrm{Int}(P) \to \mathbb{Z}$, called the **Möbius function** of P, where the following conditions are met:

$$\begin{aligned} &(1) \quad \mu(x, x) = 1 \text{ for all } x \in P \\ &(2) \quad \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z), \text{ for all } x < y \text{ in } P. \end{aligned}$$

If the poset has $\hat{0}$, then we consider $\mu(x) = \mu(\hat{0}, x)$. See Figure 1.2 for a poset together with the Möbius function values for each of the elements.

The Möbius function is an incidence function in an incidence algebra which is an associative alegbra that is defined for any locally finite poset. For a field $\mathbb{K}$, let $\mathcal{J}(P) = \mathcal{J}(P, \mathbb{K})$ be the vector space of all functions of the form $f : \mathrm{Int}(P) \to \mathbb{K}$ where the multiplication in $\mathcal{J}(P)$ is defined by

$$(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y) \qquad \text{for } f, g \in \mathcal{J}(P).$$

Let V be a vector space, then a finite hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes contained in V. A linear hyperplane is defined as subspace $H \subset V$ of co-dimension one. More precisely, it is defined as follows. Let $f \in V^* \setminus \{0\}$ be a non-zero linear functional on V, then

$$H_f = \{\omega \in V : f(\omega) = 0\}.$$

Further, an affine hyperplane is a translated linear hyperplane, i.e.

$$H^a = \{\omega \in V : f(\omega) = a\}$$

for some fixed non-zero $w \in V$ and a in any field $\mathbb{K}$. For any finite arrangement $\mathcal{A} \subset V$, the dimension of the arrangement $\dim(\mathcal{A})$ is equal to the dimension of the vector space V, while the $\mathrm{rank}(\mathcal{A})$ of the arrangement is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. Further, we say that $\mathcal{A}$ is essential if $\mathrm{rank}(\mathcal{A}) = \dim(\mathcal{A})$.

For hyperplane arrangements, we want to consider a special poset called the intersection poset due to its use in the application of the Möbius function.

Definition 1.5. Let $\mathcal{A}$ be an arrangement in the vector space V. Let $L(\mathcal{A})$ be the set of all non-empty intersections of hyperplanes in $\mathcal{A}$, this includes V itself as the intersection over the empty set. The relation is defined by $x \leq y$ if and only if $x \supseteq y$ as subsets of V. The poset $L(\mathcal{A})$ ordered by reverse inclusion is called the **intersection poset of** $\mathcal{A}$. See Figure

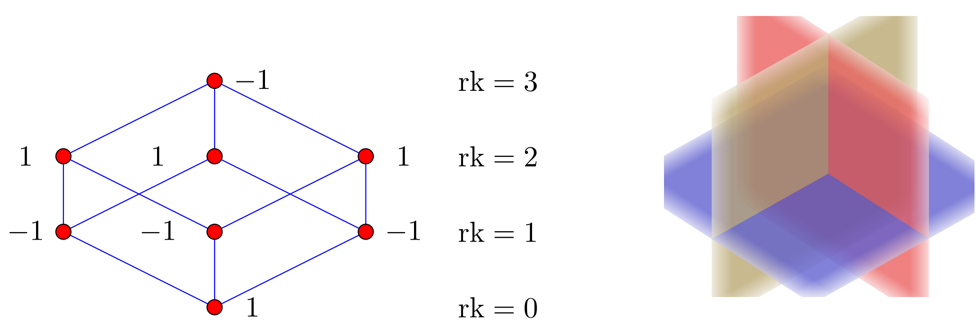


Figure 1.2: *Shown is the intersection poset $L(\mathcal{A})$ for an arrangement $\mathcal{A}$ together with the corresponding Möbius values and rank values for the elements in $L(\mathcal{A})$.*

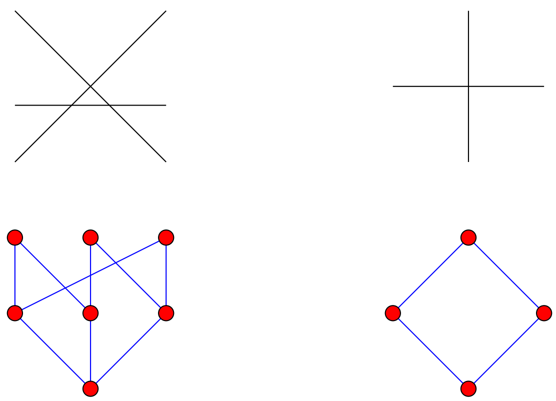


Figure 1.3: *Shown are two hyperplanes together with their corresponding intersection posets.*

1.3 for an example.

Now that we have the intersection poset defined, we can discuss the following polynomial

Definition 1.6. Given an arrangement $\mathcal{A}$ with corresponding intersection poset $L(\mathcal{A})$, the characteristic polynomial $\chi_{\mathcal{A}}(t)$ is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}.$$

Example 1.7. *Consider the arrangement given in Figure 1.2, then the corresponding characteristic polynomial is given by*

$$\chi_{\mathcal{A}}(t) = t^3 - 3t^2 + 3t - 1.$$

Let V now be a vector space over $\mathbb{R}$. Consider the complementary space Y in $\mathbb{R}^n$ to the space X which is spanned by the vectors normal to the hyperplanes in $\mathcal{A}$. Also, consider the following space

$$W = \{\omega \in V : \omega \cdot y = 0, \forall y \in Y\},$$

i.e. the space W is the set of vectors in V that are orthogonal to every vector in Y. Note,

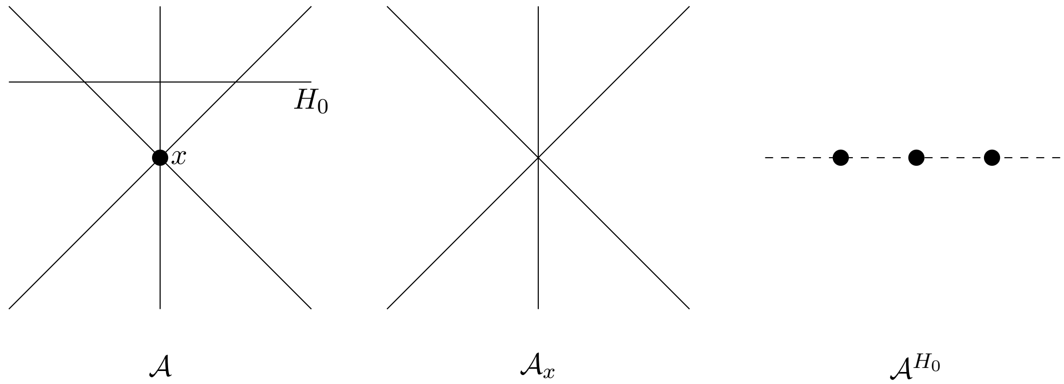


Figure 1.4: *Given the arrangement $\mathcal{A}$, in the center we see $\mathcal{A}_x$ where x is the intersection point of three hyperplanes in $\mathcal{A}$. On the far left, we see $\mathcal{A}^{H_0}$, where H_0 is a hyperplane in the arrangement $\mathcal{A}$. Note that the ambient space of $\mathcal{A}^{H_0}$ is the hyperplane H_0.*

since we are taking $\mathbb{K} = \mathbb{R}$, then we can take $W = X$. Moreover. for any $H \in \mathcal{A}$ we have that $H \cap W$ is of co-dimension one i.e. a hyperplane in W. Now consider the following arrangement, called the **essentialization of** $\mathcal{A}$, defined by

$$\text{ess}(\mathcal{A}) := \{H \cap W : H \in \mathcal{A}\}.$$

Definition 1.8. Given an arrangement $\mathcal{A}$, a **region** of $\mathcal{A}$ is a connected component of the complement of $\mathcal{A}$. Moreover, we say a region R is **relatively bounded** if $R \cap W$ is bounded. Further, for an arrangement $\mathcal{A}$, we let $r(\mathcal{A})$ be the number of regions of $\mathcal{A}$ while $b(\mathcal{A})$ is the number of bounded regions in $\mathcal{A}$.

For any arrangement $\mathcal{A}$ in the vector space V, one can define the subarrangement of $\mathcal{A}$ as a subset $\mathcal{B} \subset \mathcal{A}$. For our purposes, we are interested in two specific arrangements which are defined as follows. Let $x \in L(\mathcal{A})$, then we define the following

$$\mathcal{A}_x = \{H \in \mathcal{A} : x \subset H\},$$

$$\mathcal{A}^x = \{x \cap H \neq \emptyset : H \in \mathcal{A} \setminus \mathcal{A}_x\},$$

where A_x is a subarrangement of $\mathcal{A}$ and $\mathcal{A}^x$ is an arrangement in the affine space X.

Now, for any $H_0 \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \mathcal{A}^{H_0}$, then the number of regions and bounded regions for $\mathcal{A}$ is given by the following lemma.

Lemma 1.9. *Given an arrangement $\mathcal{A}$ and $H_0 \in \mathcal{A}$. Then*

$$\begin{aligned} r(\mathcal{A}) &= r(\mathcal{A}') + r(\mathcal{A}''), \\ b(\mathcal{A}) &= \begin{cases} b(\mathcal{A}') + b(\mathcal{A}''), & \text{if } rank(\mathcal{A}) = rank(\mathcal{A}'), \\ 0, & \text{if } rank(\mathcal{A}) = rank(\mathcal{A}') + 1. \end{cases} \end{aligned}$$

It turns out that the number of regions and bounded regions can be calculated using the characteristic polynomial, but one needs a few more pieces of machinery first. The first piece is that the characteristic polynomial obeys a recursive property.

Lemma 1.10. *Given an arrangement $\mathcal{A}$ and $H_0 \in \mathcal{A}$. Then*

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

The other piece that is needed was first proven for linear arrangements by Whitney, but was extended to arbitrary arrangements in [4]. Note, that for this theorem to hold, we must extend the definition of what it means for an arrangement to be central. Normally we say an arrangement $\mathcal{A}$ is **central** if $\cap_{H \in \mathcal{A}} H \neq \emptyset$, however, we can extend it slightly by saying that a subset $\mathcal{B}$ of $\mathcal{A}$ is central if $\cap_{H \in \mathcal{B}} H \neq \emptyset$.

Theorem 1.11 (Whitney's)**.** *Let $\mathcal{A}$ be any real hyperplane arrangement, then*

$$\chi_{\mathcal{A}}(t) = \sum_{\text{central } \mathcal{B} \subset \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim \mathcal{A} - rank\, \mathcal{B}}.$$

Theorem 1.11 and Lemma 1.10 are used to prove a major theorem for hyperplane arrangement that relates the characteristic polynomial to the number of regions and bounded regions.

Theorem 1.12 (Zaslavsky's)**.** *Let $\mathcal{A}$ be a real hyperplane arrangement. Then*

$$\begin{aligned} r(\mathcal{A}) &= (-1)^{\dim \mathcal{A}} \chi_{\mathcal{A}}(-1) \\ b(\mathcal{A}) &= (-1)^{rank(\mathcal{A})} \chi_{\mathcal{A}}(1). \end{aligned}$$

1.2 Parking Functions and the Sandpile Model

The classic notion of the parking function was introduced in [5] and the parking problem can be stated as follows. On a one-way street with ordered parking spaces $0, 1, \ldots, n-1$ we have n cars, $c_1, \ldots, c_n$, that want to park. The driver of car c_i prefers to park in space s_i. The cars then proceed to enter the street in the order $c_1, \ldots, c_n$ and try to park in their preferred spot first, but if the space is occupied then they park in the next available. If no spot exists, then the driver leaves. Therefore, if all drivers are parked, then the sequence $(s_1, \ldots, s_n)$ is a parking function of length n.

In [6] a new generalization of parking functions called G-parking functions which were associated with a general connected digraph G. More specifically, $G = (V, E)$ was a directed graph on the vertex set with $V = \{0, 1, \ldots, n\}$, with multiple edges and loops allowed, and a vertex called a sink represented by vertex 0. To establish the notation that will be used, we say that G has the directed edge $(i \to j)$ where i is the tail and j is the head of the edge. Further, for any subset of $U \subseteq V$ and vertex i, we define $\mathrm{outdeg}_U(i)$ to be $\#\{(i \to j) \in E : j \notin U\}$. Postnikov and Shapiro defined a G-parking function as a function $f : V \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ where the condition that for each subset $U \subseteq V$ with $0 \notin U$, there exists a vertex $i \in U$ such that $f(i) < \mathrm{outdeg}_U(i)$. For our purposes we will use a different definition that does not use the designated vertex.

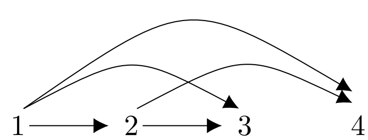
Definition 1.13. Given a graph G on vertex set $V = \{1, \ldots, n\}$, a function $f : \{1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$ is called a ***G*-parking function** if for any non-empty subset $I \subseteq \{1, \ldots, n\}$ there exists an $i \in I$ such that the number of edges $(i \to j) \in E$ such that $j \notin I$ is greater than or equal to $f(i)$.

Lemma 1.14. *Given a graph G on the vertex set $V = \{1, \ldots, n\}$ and a G-parking function f, then there exists at least one $i \in V$ such that $f(i) = 0$.*

To show the relation between Definition 1.13 and the Portnikov-Shapiro G-parking function, we proceed as follows. Let G be a graph on the vertex set $\{1, \ldots, n\}$, and let G^{cone} be the cone over G, i.e. create the rooted graph from G by adding the root vertex 0 and an edge from every vertex in G to 0. Then f is a G-parking function, according to Definition 1.13, if and only if it is a G^{cone}-parking function according to Postnikov-Shapiro. Note, that in their definition the sink is not required to have a zero out-degree and to be connected to every other vertex so our definition is more restrictive.

Remark 1.15. One is able to view the classical parking function of length n as a G-parking function by letting G be the complete graph on n vertices with edge directed in both directions, i.e. for vertices $i, j \in V$, then $(i \to j), (j \to i) \in E$.

Example 1.16. *Consider the following graph $G = (V, E)$ where the vertex and edge sets are given by $V = \{1, 2, 3, 4\}$ and edge set $\{(1 \to 2), (1 \to 3), (1 \to 4), (2 \to 3), (2 \to 4)\}$. We will show that the four-tuples $\lambda_0 = \langle 0, 0, 0, 0 \rangle$ and $\lambda_1 = \langle 2, 1, 0, 0 \rangle$ are G-parking functions, while $\lambda_2 = \langle 1, 1, 1, 1 \rangle$ is not a G-parking function.*



By the definition of a G-parking function, one must check every non-empty subset $I \subset V$ and find a vertex $i \in I$ such that $\sum_{j \notin I} m_{ij} \geq \lambda(i)$. Indeed, consider the following table:

Since for every non-empty $I \subset V$, there exists an $i \in I$ and $\sum_{j \notin I} m_{ij} \geq \lambda(i)$, then $\lambda_1 = \langle 2, 1, 0, 0 \rangle$ is a G-parking function according to Table 1.1. For the four-tuple $\lambda_0 = \langle 0, 0, 0, 0 \rangle$, one can see that it is a G-parking function by the following observation. Indeed, for any $I \subset V$ and any $i \in I$, then

$$\sum_{j \notin I} m_{ij} \geq 0.$$

$I \subset V$	$\lambda_1(i)$	$\sum_{j \notin I} m_{ij}$	$I \subset V$	$\lambda_1(i)$	$\sum_{j \notin I} m_{ij}$	$I \subset V$	$\lambda_1(i)$	$\sum_{j \notin I} m_{ij}$
$\{1\}$	$\lambda_1(1) = 2$	3	$\{1,2\}$	$\lambda_1(1) = 2$	2	$\{1,2,3\}$	$\lambda_1(3) = 0$	0
$\{2\}$	$\lambda_1(2) = 1$	2	$\{1,3\}$	$\lambda_1(1) = 2$	2	$\{2,3,4\}$	$\lambda_1(3) = 0$	0
$\{3\}$	$\lambda_1(3) = 0$	0	$\{1,4\}$	$\lambda_1(1) = 2$	2	$\{1,3,4\}$	$\lambda_1(3) = 0$	0
$\{4\}$	$\lambda_1(4) = 0$	0	$\{2,3\}$	$\lambda_1(3) = 0$	0	$\{1,2,4\}$	$\lambda_1(4) = 0$	0
			$\{2,4\}$	$\lambda_1(4) = 0$	0	V	$\lambda_1(3) = 0$	0
			$\{3,4\}$	$\lambda_1(4) = 0$	0			

Table 1.1: *Contains all possible choices of $I \subset V$ along with satisfactory choices for $i \in I$, in the form of $\lambda_1(i)$, and the calculation $\sum_{j \notin I} m_{ij}$ that shows $\lambda = \langle 2, 1, 0, 0 \rangle$ is a G-parking function.*

Consider the four-tuple $\lambda_2 = \langle 1, 1, 1, 1 \rangle$. To show that it is not a G-Parking function one must find a subset $I \subset V$ where regardless of the $i \in I$ one has $\sum_{j \notin I} m_{ij} < 1$. If $I = \{3\}$, then one has $m_{31} + m_{32} + m_{34} = 0 < 1 = \lambda_2(3)$. Therefore it is not a G-parking function.

There is a relation between G-parking functions and an automaton model created by Bak, Tang, and Wiesenfeld. This model is a dynamical system used to showcase self-organized criticality which refers to the proclivity of a system to show states over varying fluctuations. In a natural setting, one can see examples of this from earthquakes, coastlines, and mountains ([7]).

In 1990, [8] adapted the cellular automaton describing this phenomenon to a rectangular grid of cells in which the system evolves of discrete time. For every time interval a random cell is selected and a grain is added; after four pieces of grain have been accumulated the cell becomes unstable. To resolve the instability, the cell topples and sends one grain to each of the neighboring cells which either remain stable or becomes unstable and restarts the toppling process. In the case that the cell is located on the boundary of the grid, then grains are sent to the neighboring cells and one grain falls off (into the sink) or two grains fall off if the toppling cell is in a corner. This process continues over the entire time sand has been added and continues until all of the cells are stable. To illustrate the process, consider the following example.

Example 1.17. *Consider the* 4×4 *grid with the following grain multiplicities contained in each cell.*

1	0	3	0
0	1	3	1
0	3	3	0
0	2	1	2

According to the model, a grain is randomly added to a cell, for our purposes we will say the grain is added to row three column three. Once the grain is add the cell becomes unstable and must topple by way of the following sequence.

$$\begin{array}{|c|c|c|c|}\hline 1&0&3&0\\\hline 0&1&3&1\\\hline 0&3&\color{red}{4}&0\\\hline 0&2&1&2\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1&0&3&0\\\hline 0&1&\color{red}{4}&1\\\hline 0&\color{red}{4}&0&1\\\hline 0&2&2&2\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1&0&\color{red}{4}&0\\\hline 0&3&0&2\\\hline 1&0&2&1\\\hline 0&3&2&2\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1&1&0&1\\\hline 0&3&1&2\\\hline 1&0&2&1\\\hline 0&3&2&2\\\hline\end{array}$$

For the toppling process it does not matter the order in which unstable cells are toppled or whether or not multiple unstable cells are toppled simultaneously, the resulting stable configuration is the same.

For the sandpile model, we are interested in knowing whether or not a stable sand configuration is **recurrent**, i.e. if the configuration can be obtained from any other configuration by a sequence of grain additions and topplings. Finding recurrent configurations is not simple, however Dhar [8] was able to produce a method called the Burning Algorithm. Similarly to the sandpile model, it is defined on a grid and is stated as follows. For a stable configuration on a grid, all cells are labeled unburnt, and then each cell whose number of grains is greater than or equal to the number of unburnt neighbors is burned. If this process ends with all cells burned, then the configuration is recurrent. Otherwise the configuration is called **transient**, i.e. not recurrent.

Example 1.18. *Consider the 4×4 grid with the following grain multiplicities contained in each cell. The cells that are slated to be burned will appear in red, and burnt cells will have their entries removed.*

1	0	3	1
0	1	3	1
0	3	3	0
0	2	1	2

This is the stable configuration that we are testing to see if the configuration is recurrent. By Dhar's burning algorithm the following occurs:

1	0		1
0	1	3	1
0	3	3	0
0	2	1	

$\Rightarrow$

1	0		
0	1		1
0	3	3	0
0	2	1	

$\Rightarrow$

1	0		
0	1		
0	3		0
0	2	1	

$\Rightarrow$

1	0		
0	1		
0			
0			

When the algorithm ended, there are remaining unburnt cells which implies that the original configuration was not recurrent. Moreover, the remaining sub-configuration is called ***forbidden*** *since any configuration containing this sub-configuration cannot be recurrent.*

In a similar scope to the abelian sandpile model, many combinatorists studied chip-firing games which are defined on finite, connected graphs that have no loops. A version of the game is described as follows. Given a graph G on the vertex set $V = \{1, \ldots, n\}$, a configuration is a string of non-negative integers, $\vec{w}$ where $\vec{w}(i)$ is the number of chips located on vertex i, and a move in the game is done by choosing a vertex $v \in V$ such that $\vec{w}(i) \geq \deg(i)$. After the vertex is fired, chips are sent along each of the edges connected to

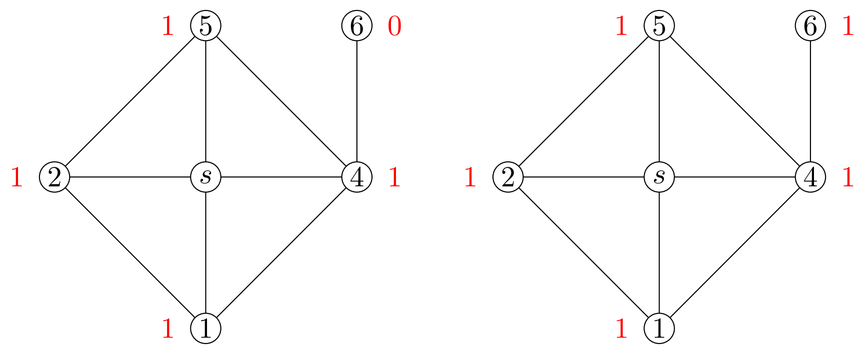


Figure 1.5: *Shown are graphs on six vertices with the sink marked by the label s. In red next to each vertex is an integer that corresponds to the number of chips located on the corresponding vertex. The configuration on the left is superstable while the configuration on the right is not since $S = \{6\}$ breaks the definition of superstable.*

i to the adjacent vertices. The game is complete when no more vertices can be chosen and the resulting configuration is called stable.

The question now is, what happens if we fire a subset of the non-sink vertices of our graph? For this question we want to talk about superstable configurations.

Definition 1.19. Given a configuration σ on G, we say that σ is **superstable** if the entries of σ are all non-negative and for all $S \subset V$, there exists a $v \in S$ such that

$$\sigma(v) < \mathrm{outdeg}_S(v) = \#\{(i \to j) : j \notin S\}.$$

Note, for our purposes, the configurations we are considering are always non-negative, this is because of the type of chip-firing game we are considering. For an example of a superstable configuration see Figure 1.5.

In the case of superstable and recurrent configurations, there is a relation between the two. Let σ_{max} be the maximal stable configuration, i.e. adding a grain to any vertex would make it unstable. Then for any configuration σ, σ is superstable if and only if $\sigma_{max} - \sigma$ is recurrent. Note σ_{max} and σ_0 (empty configuration) are both superstable.

The superstable configurations on a graph G are precisely the G-parking function defined in Definition 1.13.

1.3 Shi and Multigraphical Arrangements

Let $V \subset \mathbb{R}^n$ given by $x_1 + \cdots + x_n = 0$, and further we will use the following notation. For any $i, j \in \{1, \ldots, n\}$ and $a \in \mathbb{R}_{>0}$, let

$$H_{ij}^a := \{x_i - x_j = a\} \subset V.$$

The original Shi arrangements were first introduced by Shi in [9] during his study of Kazhgan-Lusztig polynomials and cells of affine symmetric groups. These arrangements were defined as:

Definition 1.20. The **Shi arrangement** is the arrangement consisting of the hyperplanes

$$Sh_n := \{H_{ij}^a : 0 < j < i \leq n, a = 0, 1\}.$$

These Shi arrangements were studied by Stanley where he looked at the enumeration of regions of the arrangement with respect to the distance from a specified base region. Here the distance between two regions is defined as the number of hyperplanes in the arrangement that separate them. Stanley was able to create a bijection between the set of regions and the set of parking functions, where the sum of the values of the parking function equals the distance from the base region to the corresponding region. This bijection is refered to as the Pak-Stanley labeling. Furthermore, this construction also applies to k-Shi arrangements where these are arrangements of the form

$$Sh_n^k := \{H_{ij}^l : 0 < j < i \leq n, -k < l \leq k\}.$$

While the original bijectivity proof of the Pak-Stanley labeling is complicated, one can see the bijectivity for the Shi and k-Shi arrangements by comparing the cardinality of the set of regions to the set of parking functions and proving either injectivity or surjectivity of the map. For 1-Shi arrangements, Shi calculated the number of regions in [9] and showed that it is equal to $(n+1)^{n-1}$. It was then shown in [5] and [10] that the number of parking functions was also $(n+1)^{n-1}$. In the case of k-Shi arrangements, with $k>1$, Stanley in [11] calculated the number of k-parking functions and showed that they totaled $(kn+1)^{n-1}$. The k-parking functions are defined as follows:

Definition 1.21. A **k-parking function of length** n is a sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfying the following condition. If $b_1 \leq b_2 \leq \cdots \leq b_n$ is a monotonic rearrangement of the terms $a_1, \ldots, a_n$, then $b_i \leq k(i-1)$.

In [12] and [13] the number of regions for the k-Shi arrangement was shown to also be $(kn+1)^{n-1}$.

Remark 1.22. One is able to view the k-parking function of length n as a G-parking function by letting G be the complete graph on n vertices with k copies of each edge directed in both directions.

Let $\mathcal{A}$ be any finite arrangement of hyperplanes of the form $H_{ij}^a = \{x_i - x_j = a\} \subset V$ where $a > 0$. For each of these hyperplanes we are able to associate an edge in a graph, and therefore for an arrangement there is the associated oriented multigraph $G_{\mathcal{A}}$ which is defined as:

Definition 1.23. Given an arrangement $\mathcal{A}$, the **associated directed graph** $G_{\mathcal{A}}$ is the graph with set of vertices $\{1, \ldots, n\}$ and directed edges $i \to j$ whose multiplicity is given by

$$m_{ij} := \#\{a \in \mathbb{R}_{>0} : H_{ij}^a \in \mathcal{A}\}.$$

Notice that one gets $m_{ij} + m_{ji}$ hyperplanes parallel to $\{x_i = x_j\}$ in the arrangement $\mathcal{A}$ where m_{ij} are located on one side of the origin, and m_{ji} are located on the other side of the

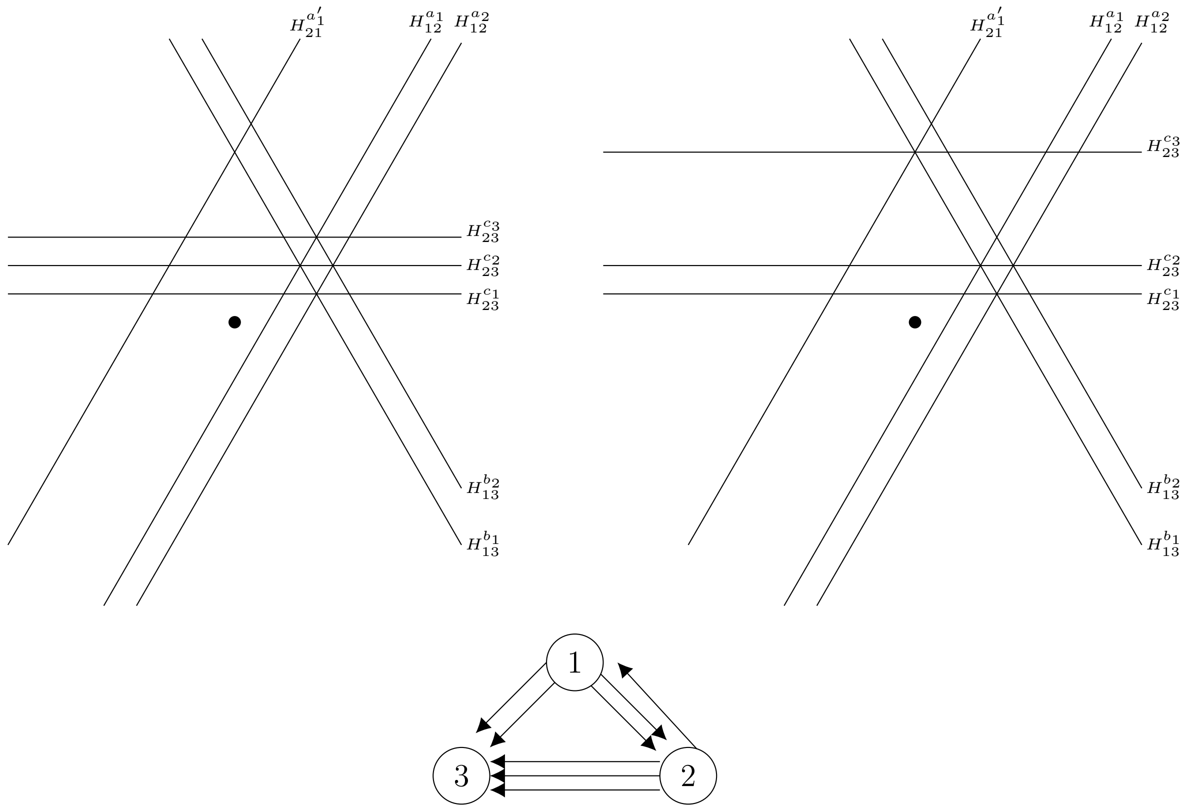


Figure 1.6: *Both arrangements correspond to the graph G that is given. Note that by changing the coefficients, in this case c_3 was changed, one is able to create and collapse regions without affecting the graph.*

origin. The combinatorial type of the arrangement is not determined by the multigraph $G_{\mathcal{A}}$ since one is able to shift the hyperplanes by changing the corresponding constants without changing the graph. See Figure 1.6 for an example.

Definition 1.24. We will call arrangements of this type **multigraphical arrangements**.

In [2] the generalized Pak-Stanley labeling for the regions of a multigraphical arrangement $\mathcal{A}$ was defined:

Definition 1.25. Let R be a region of the arrangement $\mathcal{A}$. Let $\mathcal{A}_R \subset \mathcal{A}$ be the subset containing all the hyperplanes that separate the region R from the origin. The **label,** λ_R is defined to be the function $\lambda_R : \{1, \dots, n\} \to \mathbb{Z}_{\geq 0}$ given by the following formula:

$$\lambda_R(i) := \#\{(a, j) : a \in \mathbb{R}_{>0}, j \in \{1, \dots, n\}, \text{ and } H_{ij}^a \in \mathcal{A}_R\}.$$

In particular, $\lambda_R(i)$ is the number of hyperplanes in the arrangement $\mathcal{A}$, of the form H_{ij}^a, that separate the region R from the origin. One should note that here, i is fixed, but j and a may vary.

For a label λ we will use the notation $\langle \lambda(1), \dots, \lambda(n) \rangle$. The region R_0, containing the origin, is the only region labeled by $\langle 0, \dots, 0 \rangle$. Moreover, the labeling of the regions can be defined inductively by: as one crosses the hyperplane $H_{ij}^a = \{x_i - x_j = a > 0\}$ in a direction away from the origin, the ith component of the label is increased by one, and the rest of the components remain unchanged.

Theorem 1.26 ([2, 1])**.** *Let R be any region of a multigraphical arrangement $\mathcal{A}$. Then the corresponding label λ_R is a $G_{\mathcal{A}}$-parking function.*

Theorem 1.27 ([2, 1])**.** *Let $\mathcal{A}$ be a multigraphical arrangement, and let λ be any $G_{\mathcal{A}}$-parking function. Then there exists a region R of $\mathcal{A}$ such that $\lambda_R = \lambda$.*

The previous two results were proved in [1] and [2] and together they imply that the generalize Pak-Stanley labeling gives a surjective map from the set of regions of $\mathcal{A}$ to the set of $G_{\mathcal{A}}$-parking functions. In [1], Hopkins and Perkinson showed that the surjectivity of the map holds for a restricted family of graphs, called bigraphical arrangements. However, it was in [2] that Mazin was able to generalize it to multigraphical arrangements. While the bijectivity results have been extended to other families of arrangements besides extended Shi arrangements, in general, the generalized Pak-Stanley labelings often fails to be injective.

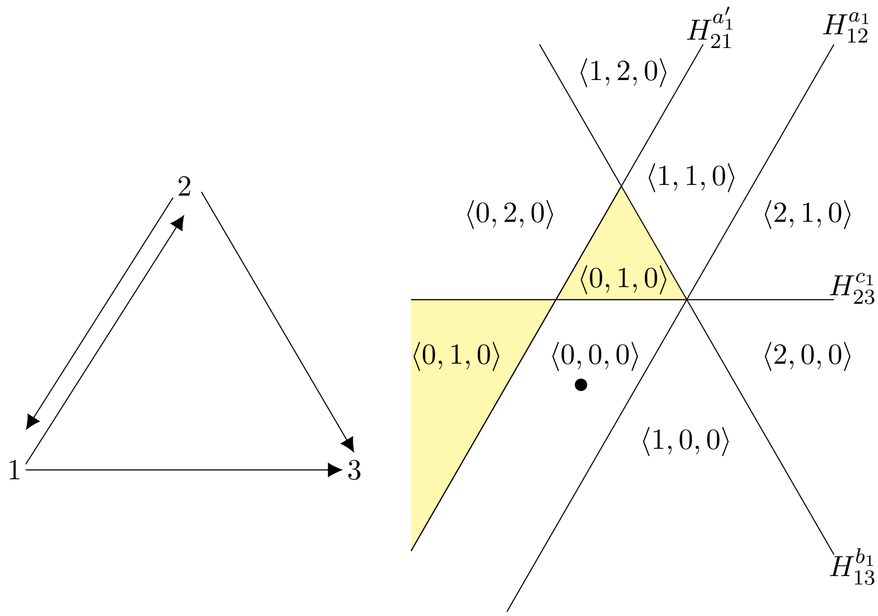


Figure 1.7: *We consider the multigraphical arrangement that corresponds with the digraph G given on the left. The regions of the arrangement are labeled with the corresponding generalized Pak-Stanley labels and the regions with the duplicate labels are colored in yellow.*

1.4 Main Results

Naturally, one might ask the question; is there a way to characterize the directed multigraphs for which there exist arrangements with a bijective labelings? In this thesis we will first discuss the special case of central hyperplane arrangements, meaning arrangements for which all hyperplanes pass through a common point in Section 2.

While working with central arrangements we realized that the graphs associated with these types of hyperplane arrangements are simple and acyclic. Moreover, we realized that the condition that guaranteed the existence of arrangements with bijective labelings can be stated as follows.

Theorem 1.28. *([14]) Let $V = \{1, 2, \ldots, n\}$ and $G = (V, E)$ be an acyclic digraph on n vertices with edges oriented in the increasing way. Then the hyperplane arrangement* *corresponding to G produces duplicate Pak-Stanley labelings if and only if there exists $1 \leq k < i < j \leq n$ such that $(k \to i), (k \to j) \in E$ and $(i \to j) \notin E$.*

For the third section, we will consider general hyperplanes in dimension $n = 3$, and we will look to expand on Baker's necessary condition for the multigraphs to have a bijective labelings where her condition is stated as follows.

Theorem 1.29. *([3]) Suppose $\mathcal{A}$ is a multigraphical hyperplane arrangement with a bijective Pak-Stanley labeling and corresponding graph $G_{\mathcal{A}} = (V, E)$. For a fixed $i, j, k \in V$, if $m_{ij} \neq 0$ and $m_{ik} \neq 0$, then $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$.*

During our research, we noticed that if one expands the aforementioned condition, then we are able expand the necessary condition as follows.

Theorem 1.30. *Let $\mathcal{A}$ be a multigraphical arrangement and let $G_{\mathcal{A}}$ be the corresponding digraph. If $\mathcal{A}$ has a bijective Pak-Stanley labeling, then for $i, j, k \in V$ with $m_{ij} \neq 0$, $m_{ik} \neq 0$, then $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$. Furthermore, if also $m_{jk} \neq 0$ and $m_{ji} \neq 0$, then at least one of the following inequalities is strict.*

1. $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$.

2. $m_{ik} + m_{ki} \geq m_{jk} + m_{ji} - 1$.

However even though this is only a necessary condition, it is a sufficient condition when the multigraphs have less than five distinct edge types. Also in Section 3 we will show that if an arrangement is injective "locally," then it is injective globally. More precisely, we have the following theorem in dimension $n = 3$:

Theorem 1.31. *Let $\mathcal{A} \subset V$ be a multigraphical arrangement in $\mathbb{R}^3$. The generalized Pak-Stanley map from the set of regions of $\mathcal{A}$ to the set of G-parking functions is injective if and only if it is injective locally.*

In Section 4 we will show that while the criteria discussed in Section 3 is necessary, it is not sufficient to guarantee a bijective arrangement. This is done by presenting families of graphs in the cases of five and six edge types that produce a bijective arrangement and do not produce a bijective arrangement.

Chapter 2

Central Multigraphical Arrangements

In this section we will considered central multigraphical arrangements, which are defined as arrangements $\mathcal{A}$ where all hyperplanes intersect at a common point. For central multigraphical arrangements, the arrangement is determined by the corresponding digraph up to a global shift. In the case of central multigraphical arrangements the graphs that correspond to these arrangements are easily classified. For the sake of completeness, the following theorems are provided with proofs from [14].

Theorem 2.1. *([14]) Let $\mathcal{A}$ be a central multigraphical hyperplane arrangement, then the corresponding multi-digraph is simple and acyclic. Further, if G is a simple acyclic digraph, then there exists a central multigraphical arrangement $\mathcal{A}$ such that $G_{\mathcal{A}} = G$.*

Proof. Let $\mathcal{A}$ be a central multigraphical arrangement such that all hyperplanes intersect at the point $c = (c_1, c_2, \ldots, c_n)$. Since all hyperplanes H_{ij}^a intersect at c, then we can have at most one H_{ij}^a for each pair i, j. Moreover, if we have a hyperplane H_{ij}^a then we cannot have a hyperplane of the form H_{ji}^b, because they would also be parallel. Thus the digraph $G_{\mathcal{A}}$ is simple.

Assume that $G_{\mathcal{A}}$ contains the cycle $i_0 \to i_1 \to \cdots \to i_k \to i_0$. It then follows that the hyperplanes corresponding to the edges in the cycle exhibit

$$
\begin{aligned}
x_{i_0} - x_{i_1} &= a_1 > 0 \\
x_{i_1} - x_{i_2} &= a_2 > 0 \\
\vdots \quad & \vdots \quad \vdots \\
x_{i_{k-1}} - x_{i_k} &= a_k > 0 \\
x_{i_k} - x_{i_0} &= a_{k+1} > 0
\end{aligned}
$$

Since each hyperplane passes through the point c all these equations are satisfied at $\mathbf{x} = \mathbf{c}$. After taking the sum of the above equations we see that $0 = \sum_{i=1}^{k+1} a_i$ which contradicts the assumption that the $a_i > 0$ for all i. Thus $\mathcal{G}_{\mathcal{A}}$ is acyclic.

Now, given a simple acyclic digraph $G = (V, E)$, with $V = \{1, \ldots, n\}$, one can assume without loss of generality that the edges are oriented in an increasing way. We create the corresponding arrangement $\mathcal{A}$ by: for every edge $(i \to j) \in E$ create the hyperplane $H_{i,j}^{j-i} = \{x_i - x_j = j - i\}$. Consider the following point $c \in V$:

$$
c = \left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right) - (1, 2, \ldots, n)
$$

We immediately see that the point c lies in the intersection of all the hyperplanes since $c_i - c_j = j - i$ for all $1 \leq i < j \leq n$. Therefore the graph G has a corresponding central multigraphical arrangement. □

Further, let $\mathcal{A}$ be any central multigraphical arrangement and consider the linear arrangement $\mathcal{A}'$ that is obtained from $\mathcal{A}$ by shifting all the hyperplanes so that they all pass through the origin. By shifting $\mathcal{A}$ to the origin, the corresponding associated multigraph $G_{\mathcal{A}}$ becomes the simple graph G where the orientation on the edges is removed. It is well-known that the acyclic reorientations of G are in one to one correspondence with the regions of the linear arrangement $\mathcal{A}'$ ([15]). To give orientations to the edges of G consider the following. For a region R of $\mathcal{A}'$ and an edge $(i - j)$ of G, we then orient the edge $(i \to j)$ if and only if $x_i < x_j$ at every point of R.

This construction shows that the regions of the original multigraphical arrangement $\mathcal{A}$

are precisely the regions of $\mathcal{A}'$ shifted by a vector. Moreover, this shows that there is a bijection between the acyclic orientations of G and the acyclic reorientations of the graph $G_{\mathcal{A}}$. More precisely, we get the following theorem:

Theorem 2.2. *([14]) The fundamental region of $\mathcal{A}$ corresponds to the original orientation of $G_{\mathcal{A}}$, and crossing a hyperplane $H_{ij}^a \in \mathcal{A}$ switches the orientation of the corresponding edge between i and j.*

Proof. Let R_0 be the fundamental region of the arrangement $\mathcal{A}$, and let $\mathcal{A}'$ be the corresponding linear arrangement. Let $c = (c_1, \ldots, c_n)$ be in the intersection of all the hyperplanes of the arrangement $\mathcal{A}$.Then it follows that $-c$ belongs to the corresponding region $R' = R_0 - c$ of $\mathcal{A}'$. Therefore, if $H_{i,j}^a \in \mathcal{A}$ and the edge $i \to j$ is the corresponding edge in $\mathcal{G}_{\mathcal{A}}$, then at c we have $c_i - c_j = a$, in particular we have that $c_i > c_j$. It then follows that at $-c \in R'$ that we have $-c_i < -c_j$. Thus, in the orientation corresponding to R' we also get the edge oriented as $i \to j$.

Finally, crossing a hyperplane $H_{i,j}^a$ corresponds to crossing the hyperplane $x_i = x_j$ of the linear arrangement $\mathcal{A}'$, which switches the orientation of the corresponding edge. □

The bijection between the regions of a multigraphical arrangement $\mathcal{A}$ and the acyclic reorientations of the corresponding graph $G_{\mathcal{A}}$, implies that one is actually able to calculate the Pak-Stanley labels for the regions in terms of the acyclic reorientations of $G_{\mathcal{A}}$.

Lemma 2.3. *The Pak-Stanley labels for the arrangement $\mathcal{A}$ can be computed in terms of acyclic reorientations of the graph G. More precisely, for a region R of $\mathcal{A}$, the label $\lambda_R(i)$ equals to the number of edges of G leading from i such that their orientations got switched in the reorientation corresponding to R.*

See Figure 2.1 for an example of how the labels of the regions are calculated in terms of the reorientations of $G_{\mathcal{A}}$.

Now, for any central multigraphical arrangement, the following condition on the graph $G_{\mathcal{A}}$ guarantees that the arrangement has a bijective Pak-Stanley labeling.

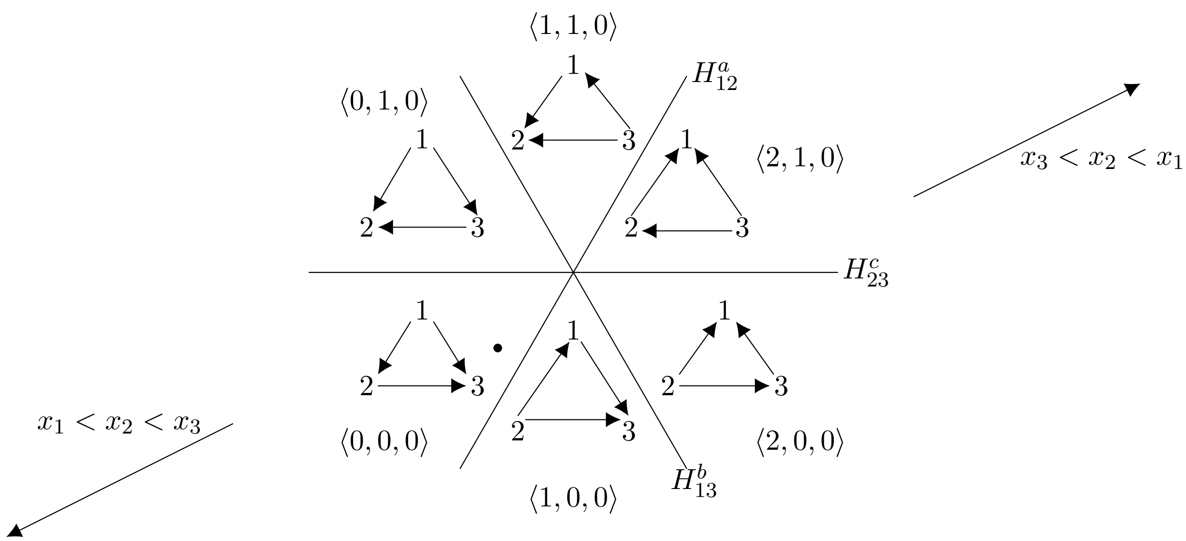


Figure 2.1: *We consider the central arrangement corresponding to the digraph* $G_{\mathcal{A}} = \{(1 \to 2, 1 \to 3, 2 \to 3)\}$. *The regions of the arrangement are labeled by the corresponding reorientations and the generalized Pak-Stanley labels. Note that the fundamental region is labeled by* $G_{\mathcal{A}}$ *and* $\langle 0,0,0 \rangle$, *and as we cross the hyperplanes the orientations of the corresponding edges switch. Moreover, as we cross the hyperplane* H_{ij}^a *in a direction away from the origin, the ith entry of the Pak-Stanley label increases by 1.*

Figure 2.2: *Here we see the two reorientations of the graph G, G' and G'', and the corresponding cycles created depending on the orientation of the edge $i \to j$.*

Theorem 2.4. *([14]) Let $V = \{1, 2, \ldots, n\}$ and $G = (V, E)$ be an acyclic digraph on n vertices with edges oriented in the increasing way. Then the hyperplane arrangement corresponding to G produces duplicate Pak-Stanley labelings if and only if there exists $1 \leq k < i < j \leq n$ such that $(k \to i), (k \to j) \in E$ and $(i \to j) \notin E$.*

Proof. $\Rightarrow$) Assume that G produces duplicate Pak-Stanley labelings and for the sake of contradiction assume that no such i, j, k exists. Since labelings correspond to acyclic reorientations of G, let $G' = (V, E')$ and $G'' = (V, E'')$ be such reorientations.

Since reorientations are in correspondence with labelings then there is an edge $k \to i$ of $\mathcal{G}_{\mathcal{A}}$ that is reoriented as $i \to k$ in G' but not in G''. Moreover since the labels are equal, then there must also be another edge emanating from k, say edge $k \to j$, such that it is reoriented as $j \to k$ in G'' but not in G'. In other words, the duplicate labeling implies that we have edges $(i \to k), (k \to j) \in E'$ and $(k \to i), (j \to k) \in E''$.

Let k be the largest integer such that this occurs. Since k is the largest possible, it follows that all edges between vertices p, q where $p, q > k$ are oriented in the same way in both reorientations. Without loss of generality we can assume that $i < j$. This gives arise to two cases depending on whether or not the edge from $i \to j$, is oriented as $i \to j$ or $j \to i$ in both G' and G''. If we have the edge $i \to j$ then in G'' we have the cycle $k \to i \to j \to k$, a contradiction since G-parking functions rise from acyclic reorientations. Otherwise we have the edge $j \to i$, but as before we have the cycle $k \to j \to i \to k$ in G' (see Figure 2).

$\Leftarrow$) The easiest way to produce the acyclic reorientations, G' and G'', is reordering the vertices and reorienting the edges so that they point in the increasing direction after considering the new vertex order. For the reoriented graph $G' = (V, E')$ we reorder the vertices of G' as follows

$$1 \prec \cdots \prec k-1 \prec k+1 \prec \cdots \prec i-1 \prec i+1 \prec \cdots \prec j-1 \prec i \prec k \prec j \prec \cdots \prec n.$$

In other words, for G' we move the vertices $k+1, \ldots, i-1, i+1, \ldots, j-1$ to the left so that they precede vertex k, and then switch vertices k and i. Note that as we reorder the vertices, the only edges that are reversed are

$$\begin{array}{lll}
(1) & (k \to p) \in E \text{ such that :} & \\
& & p \in \{k+1, \ldots, i-1\}, \text{ or} \\
& & p \in \{i+1, \ldots, j-1\}, \text{ or} \\
& & p = i \\
(2) & (i \to p) \in E \text{ such that:} & p \in \{i+1, \ldots, j-1\}.
\end{array}$$

To produce the reorientation that corresponds to $G'' = (V, E'')$ we reorder the vertices of G'' as follows:

$$1 \prec \cdots \prec k-1 \prec k+1 \prec \cdots \prec i-1 \prec i+1 \prec \cdots \prec j-1 \prec j \prec k \prec i \prec j+1 \prec \cdots \prec n.$$

In other words, for G'' we move the vertices $k+1, \ldots, i-1, i+1, \ldots, j-1$ so that they precede vertex k, but now we move vertex j two places to the left so that it precedes k instead of switching vertices k and i. This time the following edges are reoriented

$$\begin{array}{lll}
(1) & (k \to p) \in E \text{ such that :} & \\
& & p \in \{k+1, \ldots, i-1\}, \text{ or} \\
& & p \in \{i+1, \ldots, j-1\}, \text{ or} \\
& & p = j \\
(2) & (i \to p) \in E \text{ such that:} & p \in \{i+1, \ldots, j-1\}.
\end{array}$$

Note that $(i \to j) \notin E$ by assumption, therefore it does not need to be reoriented.

We conclude that both $G' = (V, E')$ and $G'' = (V, E'')$ produce the labeling

$$\tau = \langle 0, \ldots, 0, \overset{k\text{th}}{(N+1)}, 0, \ldots, 0, \overset{i\text{th}}{(K)}, 0, \ldots, 0 \rangle$$

where

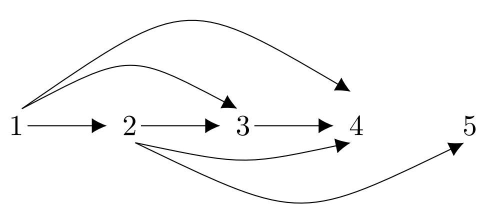
$$N = \#\{(k \to p) \in E : p \in \{k+1, \ldots, i-1\} \cup \{i+1, \ldots, j-1\}\}$$

and

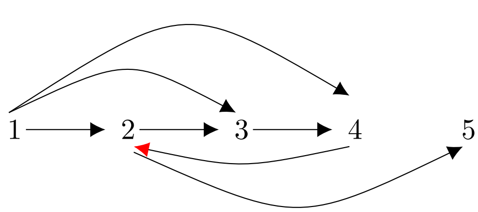
$$K = \#\{(i \to p) \in E : p \in \{i+1, \ldots, j-1\}\}.$$

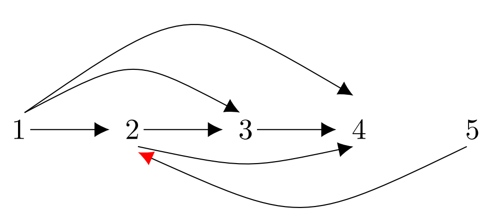
□

Example 2.5. *Consider the following graph $G = (V, E)$ where the vertex and edge sets are given by $E = \{(1 \to 2), (1 \to 3), (2 \to 3), (2 \to 4), (2 \to 5), (3 \to 4)\}$ on the vertex set $V = \{1, 2, 3, 4, 5\}$.*



In this example we see that the graph contains the edges $(2 \to 4)$ and $(2 \to 5)$, but $(4 \to 5) \notin E$. It then follows that Theorem 2.4 implies that there should exist at least two reorientations G' and G" of G that produce the same Pak-Stanley labeling. Consider the following two reorientations





These two reorientations of $G_{\mathcal{A}}$ produce the label $\langle 0,1,0,0,0\rangle$. Similarly for $(2 \to 3),(2 \to 5) \in E$, but $(3 \to 5) \notin E$ there will be duplicates.

These two reorientations of G produce the duplicate label $\langle 3,2,1,0,0\rangle$. Actually, this graph produces fourteen more duplicate labelings.

Theorem 2.4 provided motivation for the idea of local injectivity of the labeling since examples for general multigraphical arrangements in $n = 3$ always had the duplicates "close" to one another. We define local injectivity as follows:

Definition 2.6. Let $\mathcal{A}$ be a multigraphical arrangement and $p \in V$ be any point. The Pak-Stanley labeling for $\mathcal{A}$ is **locally injective near** p if all of labels of R such that $p \in \overline{R}$ are distinct. Further, if this holds for all $p \in V$, then we say that $\mathcal{A}$ is **locally injective**.

In the case of central multigraphical arrangements, local injectivity and "global" injectivity are the same.

Chapter 3

General Hyperplanes in Dimension n=3

In [3] Baker worked on generalizing the requirements on the associated multigraph in the $n = 3$ case. During her research she noticed that arrangements with a bijective labeling satisfied the following:

Theorem 3.1. *([3]) Suppose $\mathcal{A}$ is a multigraphical hyperplane arrangement with a bijective Pak-Stanley labeling and corresponding graph $G_{\mathcal{A}} = (V, E)$. For a fixed $i, j, k \in V$, if $m_{ij} \neq 0$ and $m_{ik} \neq 0$, then $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$.*

However, later on in her thesis she shows that while this is a necessary condition, it was not sufficient. See Figure 3.1 for an example of a graph that does not admit an arrangement with an injective label despite satisfying Theorem 3.1. In the case of central multigraphical arrangements Theorem 3.1 reduces as follow, if the edges $(i \to j)$ and $(i \to k)$ are in the associated graph $G_{\mathcal{A}}$, then $1 \leq m_{jk} + m_{kj}$. Meaning that either the edge $(j \to k)$ or $(k \to j)$ is present in $G_{\mathcal{A}}$. Moreover, in the case of central arrangement, Theorem 2.4, the condition is not only necessary but sufficient for the arrangement to emit a bijective labeling.

Examples in the $n = 3$ case show that when an arrangement yields duplicates that the duplicate labels are "close" to each other. More precisely, for a duplicate label λ in an

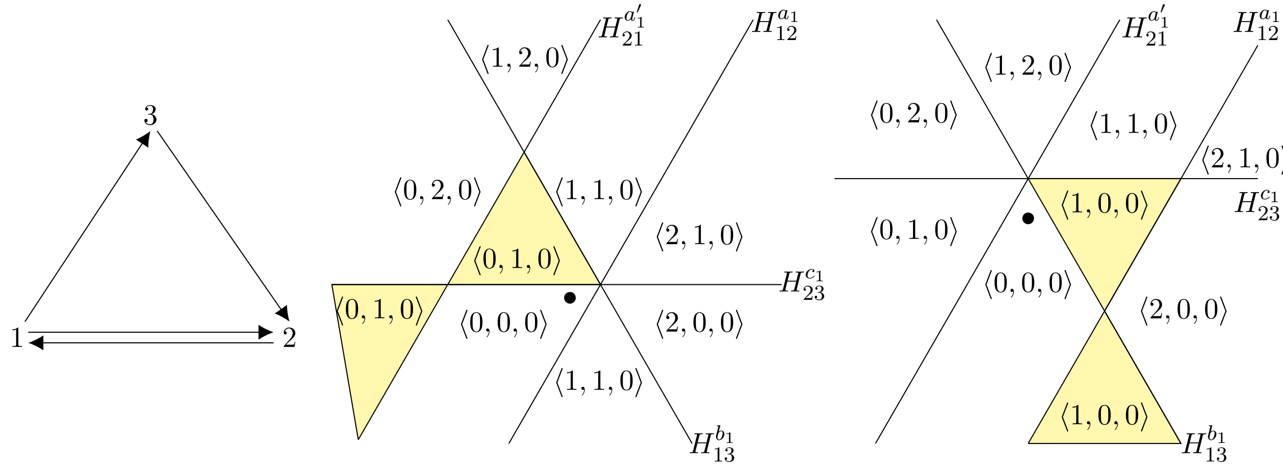


Figure 3.1: *Despite satisfying the conditions listed in Theorem 3.1, this graph does not emit an arrangement with a bijective labeling. Further, this is the smallest such graph. We illustrate this with two arrangements (center and right) corresponding to the graph. In the first arrangement (center) the label $\langle 0, 1, 0\rangle$ appears twice, while in the second arrangement (right) the label $\langle 1, 0, 0\rangle$ appears twice. One can alter the arrangements by changing the positive constants a_1, a_2, b_1, and c_1 on the right hand sides of the equations of $H_{12}^{a_1}, H_{21}^{a_1'}, H_{12}^{b_1}$, and $H_{12}^{c_1}$, but one cannot get rid of both duplicates at the same time (see [3] for details).*

arrangement $\mathcal{A}$, then the closure of the union of all the regions labeled by λ is connected. This means that in the case of general arrangements in $n = 3$, that if the arrangement is locally injective at every point, then the labeling is injective. The following theorem shows that this holds in our special case.

3.1 Injectivity: Local to Global

Recall from Chapter 2 that the definition of local injectivity is defined as follows. Let $\mathcal{A}$ be a multigraphical arrangement and $p \in V$ be any point. The Pak-Stanley labeling for $\mathcal{A}$ is **locally injective near** p if all of labels of R such that $p \in \overline{R}$ are distinct. Further, if this holds for all $p \in V$, then we say that $\mathcal{A}$ is **locally injective**.

Theorem 3.2. *Let $\mathcal{A} \subset V = \{x_1+x_2+x_3 = 0\} \subset \mathbb{R}^3$ be a multigraphical arrangement. The generalized Pak-Stanley map from the set of regions of $\mathcal{A}$ to the set of G-parking functions* *is injective if and only if it is injective locally.*

Proof. If $\mathcal{A}$ is injective, then there are no duplicates which means locally there are no duplicates. Therefore the arrangement is locally injective.

Now assume that $\mathcal{A}$ is locally injective and for the sake of contradiction that it is not injective. Let R_1 and R_2 be the two regions containing the duplicate label λ. Now consider the subarrangement $\mathcal{B} \subset \mathcal{A}$ that consists of the hyperplanes that separate R_1 from R_2. Note, since R_1 and R_2 have duplicate labels, then after removing all non-separating hyperplanes the labels for the new regions containing R_1 and R_2 in $\mathcal{B}$ will also have duplicate labels in $\mathcal{B}$. Let $\overline{R_1}$ and $\overline{R_2}$ be the regions of $\mathcal{B}$ such that $R_1 \subset \overline{R_1}$ and $R_2 \subset \overline{R_2}$.

First, we claim that if $\lambda(R_1) = \lambda(R_2)$ and a hyperplane of the type H_{12} separates R_1 from R_2, then there exists a hyperplane of the type H_{13} that also separates them. Indeed, all hyperplanes of the type H_{12} that separates R_1 from R_2 have to separate one (without loss of generality assume R_1) from the origin and not separate the other from the origin. Since $\lambda(R_1) = \lambda(R_2)$, then there has to be hyperplanes of the type H_{13} that separate R_2 from the origin and not separate R_1 from the origin (first entry of each label is equal). Similarly, for any $i, j, k \in \{1, 2, 3\}$, if two regions have the same label and a hyperplane of the form H_{ij} separates R_1 from R_2, then there must exist a H_{ik} that separates the two regions.

Assume now, that hyperplanes of the type H_{12} and H_{21} separate the regions and let H_{12}^a and $H_{21}^{a'}$ be hyperplanes of each type that separate R_1 and R_2. Note, if R_1 and R_2 are separated by hyperplanes of types H_{12} and H_{21}, then all of the hyperplanes of type H_{12} separate one (without loss of generality say R_1) from the origin, and all of the H_{21} hyperplanes separate the other (say R_2) from the origin. The assumption that the labels of each region are equal implies that there is at least one hyperplane of type H_{13}^b that separates R_2 from the origin (and not R_1) and at least one hyperplane of type H_{23}^c that separates R_1 from the origin (and not R_2). Let (y_1, y_2, y_3) and (z_1, z_2, z_3) be points in R_1 and R_2, respectively. It follows that each point satisfies the following

$$
\begin{array}{ll}
(y_1, y_2, y_3) \in R_1 & (z_1, z_2, z_3) \in R_2 \\
y_1 - y_2 > a & z_2 - z_1 > a' \\
y_2 - y_3 > c & z_1 - z_3 > b \\
y_1 - y_3 < b & z_2 - z_3 < c
\end{array}
$$

From region R_1 we get the inequality $c+a-b < 0$, while R_2 yields the inequality $a'+b-c < 0$. After adding both inequalities, one sees that $a' + a < 0$ which is a contradiction since a' and a are positive. Therefore hyperplanes of the types H_{12} and H_{21} cannot both separate the regions. In general, hyperplanes of the type H_{ij} and H_{ji}, for some i, j cannot both separate the regions.

Assume now that the hyperplanes $H_{12}^a, H_{23}^c, H_{31}^{b'}$ separate region R_1 from R_2. It follows that two hyperplanes separate one region from the origin while the third separates the other region from the origin. Since the labels are equal, then all three entries for λ are non-zero. This is a contradiction since λ is a G-parking function and therefore has to have at least one zero entry by Lemma 1.14.

We have now shown that the only two types of hyperplanes that can separate R_1 from R_2 are of the form H_{ij} and H_{ik} for some $i, j, k \in \{1, 2, 3\}$. Lastly, assume that the hyperplane H_{12}^a separates R_1 from the origin and R_2 while the hyperplane H_{13}^b separates R_2 from the origin and R_1. Since the intersection of the hyperplane H_{12}^a with H_{13}^b creates a bad intersection and since $\mathcal{A}$ is locally injective, then there exists either a H_{32}^{a-b} or H_{23}^{b-a}, depending on the sign of $a - b$, that rectifies this bad intersection. More precisely, if $a - b > 0$, then $H_{32}^{a-b} \in \mathcal{A}$ while if $a - b < 0$, then $H_{23}^{a-b} \in A$. One cannot have that $a = b$ since these are affine hyperplanes. Let (y_1, y_2, y_3) and (z_1, z_2, z_3) be points in R_1 and R_2, respectively. It follows that each point satisfies the following

$$
\begin{array}{ll}
(y_1, y_2, y_3) \in R_1 & (z_1, z_2, z_3) \in R_2 \\
y_1 - y_2 > a & z_1 - z_2 < a \\
y_1 - y_3 < b & z_1 - z_3 > b
\end{array}
$$

From region R_1 the inequalities yield $y_3 - y_2 > a - b$ while region R_2 yields $z_3 - z_2 < a - b$.

This implies that either H_{32}^{a-b} and H_{23}^{b-a} depending on the sign of $a-b$ separates the two regions, i.e. is in $\mathcal{B}$. This is a contradiction of the previous claim that only two types of hyperplanes that can separate R_1 from R_2 are of the form H_{12} and H_{13}. Therefore if $\mathcal{A}$ is injective locally, then the map from the set of regions of $\mathcal{A}$ to the set of G-parking functions is injective. □

Even though a proof is provided in the case of $n = 3$, it is believed that the following holds for general n.

Conjecture 3.3. *Let $\mathcal{A} \subset V = \{x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$ be a multigraphical arrangement. The generalized Pak-Stanley map from the set of regions of $\mathcal{A}$ to the set of G-parking functions is injective if and only if it is injective locally.*

In [2] Mazin proved that the Pak-Stanley labeling is always a surjective map. From a set-theoretic standpoint, since it is surjective, the number of regions is bounded from below by the number of parking functions. Therefore we will always consider arrangements with the fewest number of regions.

The following theorem provides a necessary condition for a directed multigraph to yield a multigraphical arrangement that admits an injective Pak-Stanley labeling.

3.2 Necessary Condition for a Bijective Labeling

Theorem 3.4. *Let $\mathcal{A}$ be a multigraphical arrangement and let $G_{\mathcal{A}}$ be the corresponding digraph. If $\mathcal{A}$ has a bijective Pak-Stanley labeling, then for any $i, j, k \in V$ with $m_{ij} \neq 0$, $m_{ik} \neq 0$, one has $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$. Furthermore, if also $m_{jk} \neq 0$ and $m_{ji} \neq 0$, then at least one of the following inequalities is strict.*

1. $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$.

2. $m_{ik} + m_{ki} \geq m_{jk} + m_{ji} - 1$.

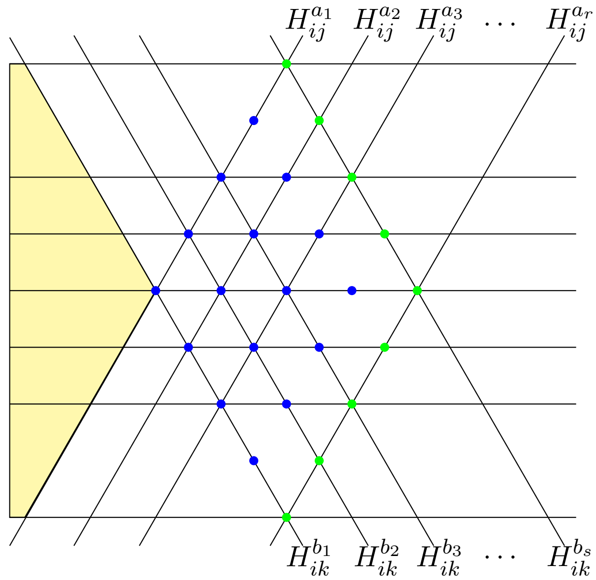


Figure 3.2: *General multigraphical arrangement for Theorem 3.4 where the blue points represent rectified points between hyperplanes of type H_{ij} and H_{ik} and the green points represent the points $P_1, \ldots, P_{r+s-1}$.*

Proof. The proof of the first part of the statement is covered by Theorem 3.1 and can be seen in more detail in [3]. The premise of Theorem 3.1 is that for an arrangement $\mathcal{A}$ with $m_{ij} > 0$ and $m_{ik} > 0$ hyperplanes of type H_{ij} and H_{ik}, respectively, that one needs at least $m_{ij} + m_{ik} - 1$ hyperplanes of type H_{jk} and H_{kj} to rectify the bad intersections.

Let the arrangement $\mathcal{A}$ have a bijective labeling where the edge multiplicities of $G_{\mathcal{A}}$ are given by $m_{ij} = r > 0$, $m_{ik} = s > 0$, $m_{jk} = t > 0$, $m_{ji} = u > 0$, $m_{ki} = v > 0$ and $m_{kj} = w > 0$. For the sake of contradiction assume that $m_{jk} + m_{kj} = m_{ij} + m_{ik} - 1$ and $m_{ik} + m_{ki} = m_{jk} + m_{ji} - 1$. It follows that the arrangement $\mathcal{A}$ contains the potential bad intersection points created from the hyperplanes

$$H_{ij}^{a_1}, \ldots H_{ij}^{a_r} \text{ with } H_{ik}^{b_1}, \ldots, H_{ik}^{b_s}$$

$$H_{jk}^{c_1}, \ldots H_{jk}^{c_t} \text{ with } H_{ji}^{a_1'}, \ldots, H_{ji}^{a_u'}.$$

By our assumption, there is exactly enough hyperplanes to rectify all of the potential bad intersections. Assume without loss of generality that $a_1 < \cdots < a_r$, $b_1 < \cdots < b_s$, $c_1 < \cdots < c_t$, and $a'_1 < \cdots < a'_u$. Furthermore, define the points $P_1, P_2, \ldots, P_{r+s-1}$ to be the intersection points of the hyperplanes

$$H_{ij}^{a_1}, \ldots H_{ij}^{a_r} \text{ with } H_{ik}^{b_s}, \text{ and}$$

$$H_{ik}^{b_1}, \ldots H_{ik}^{b_s} \text{ with } H_{ij}^{a_r}.$$

Consider the following differences of the coefficients for hyperplanes of type H_{ij} and H_{ik}

$$d_1 = b_s - a_1, d_2 = b_s - a_2, \ldots d_r = b_s - a_r, d_{r+1} = b_{s-1} - a_r, d_{r+2} = b_{s-2} - a_r, \ldots, d_{r+s-1} = b_1 - a_r.$$

Note that $d_1 > d_2 > \cdots > d_{r+s-1}$ and to avoid having bad intersections at points $P_1, \ldots, P_{r+s-1}$ one must have the hyperplanes $H_{jk}^{d_1}, \ldots, H_{jk}^{d_{r+s-1}} \in \mathcal{A}$. Since $r + s - 1 = t + w$, it follows that $d_1 > \cdots > d_t > 0 > d_{t+1} > \cdots > d_{t+w}$.

To rectify the bad intersections, let $c_t = d_1, c_{t-1} = d_2, \ldots, c_1 = d_t$ and $c'_1 = -d_{t+1}, \ldots, c'_w = -d_{t+w}$. These coefficients place the hyperplanes of types H_{jk} and H_{kj}; further the bad intersections have been rectified.

Consider the points $Q_1, \ldots, Q_{t+u-1}$ to be the intersection points created from the hyperplanes

$$H_{ji}^{a'_1}, \ldots H_{ji}^{a'_u} \text{ with } H_{jk}^{c_t}, \text{ and}$$

$$H_{jk}^{c_1}, \ldots H_{jk}^{c_t} \text{ with } H_{ji}^{a'_u}.$$

The potential bad intersections are rectified in a similar way, however, $m_{ji} + m_{jk} - 1 = m_{ik} + m_{ki}$ implies that all hyperplanes of the form H_{ik} and H_{ki} are used to rectify the potential bad intersections. More precisely, $H_{ik}^{b_s}$ must be used, and by the previous argument the coefficient is defined to be $b_s = c_t - a'_1$. However, the coefficient c_t was defined as $c_t = b_s - a_1$, so adding together yields $0 = -a_1 - a'_1$. This is a contradiction since $a_1, a'_1 > 0$.

Therefore at least one of the equations is a strict inequality. $\square$

3.3 Graphs that emit a Bijective Labeling

For some families of hyperplane arrangements, Theorem 3.4 is not only necessary, but also sufficient for the multigraph to emit an arrangement that has a bijective labeling. In the $n = 3$ case, there are six different types of hyperplanes of the form H_{ij} for $i, j \in \{1, 2, 3\}$, and the set of families will be broken up by the number of different types that appear in the arrangement. We begin with one type chosen as follows:

Theorem 3.5. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If there is exactly one $m_{ij} \neq 0$, then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$.*

Proof. In the instance that only one edge multiplicity is non-zero, say $m_{ij} = r > 0$, there are no conditions required to guarantee a bijective arrangement. It follows that the arrangement $\mathcal{A}$ is given by

$$H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \quad \text{for } 1 \leq \alpha \leq r.$$

See Figure 3.3 for a picture of the arrangement with corresponding labels. $\square$

In the second family we have that two different hyperplane types are chosen, and in this case there are multigraphs with two different types of edges that do not produce bijective arrangements. First, lets address the multigraphs that have corresponding arrangements that emit bijective labeling.

Theorem 3.6. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If there exists exactly two non-zero edge multiplicities and the non-zero multiplicities are not of the form m_{ij}, m_{ik} for some $i, j, k \in V$, then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$.*

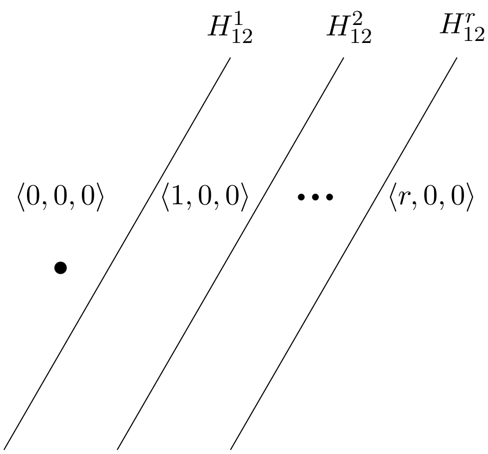


Figure 3.3: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicity $m_{12} = r$.*

Proof. In the instance that two edge multiplicities are non-zero, there are three different scenarios depending which multiplicities are non-zero. However, for each case there are several graphs that only differ up to a rearrangement of the vertices, so without loss of generality we will consider the following three cases.

Assume that the graph G has non-zero multiplicities $m_{ij} = r$ and $m_{ji} = u$. In this case, there are no conditions required to guarantee a bijective arrangement, and it then follows that the arrangement $\mathcal{A}$ is given by

$$\begin{aligned} &H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \quad \text{for } \alpha \in \{1,\ldots,r\}, \\ &H_{ji}^{a'_\beta}, \quad \text{where } a'_\beta = \beta, \quad \text{for } \beta \in \{1,\ldots,u\}. \end{aligned}$$

See Figure 3.4 for a picture of the arrangement with corresponding labels.

In the second scenario, assume that G has non-zero multiplicities $m_{ji} = u$ and $m_{ki} = v$. Similar to the previous scenario, there are no conditions required to guarantee a bijective arrangement, and it follows that the arrangement $\mathcal{A}$ is given by

$$\begin{aligned} &H_{ji}^{a'_\alpha}, \quad \text{where } a'_\alpha = \alpha, \quad \text{for } \alpha \in \{1,\ldots,u\}, \\ &H_{ki}^{b'_\beta}, \quad \text{where } b'_\beta = \beta, \quad \text{for } \beta \in \{1,\ldots,v\}. \end{aligned}$$

See Figure 3.5 for a picture of the arrangement with corresponding labels.

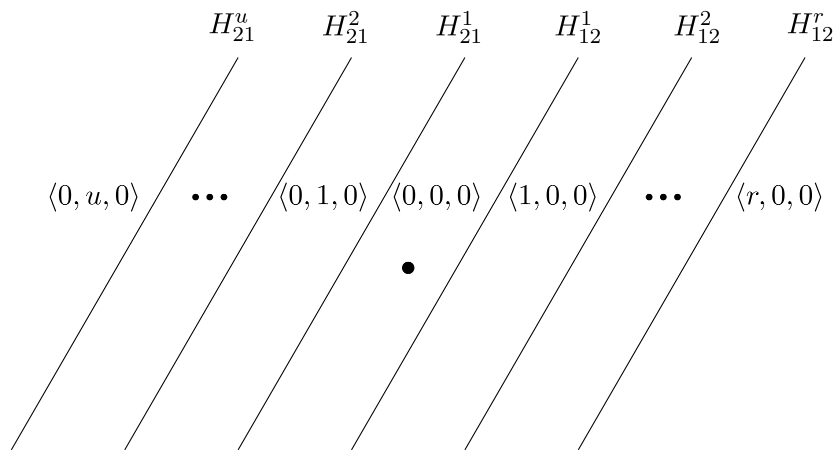


Figure 3.4: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities* $m_{12} = r$ *and* $m_{21} = u$*.*

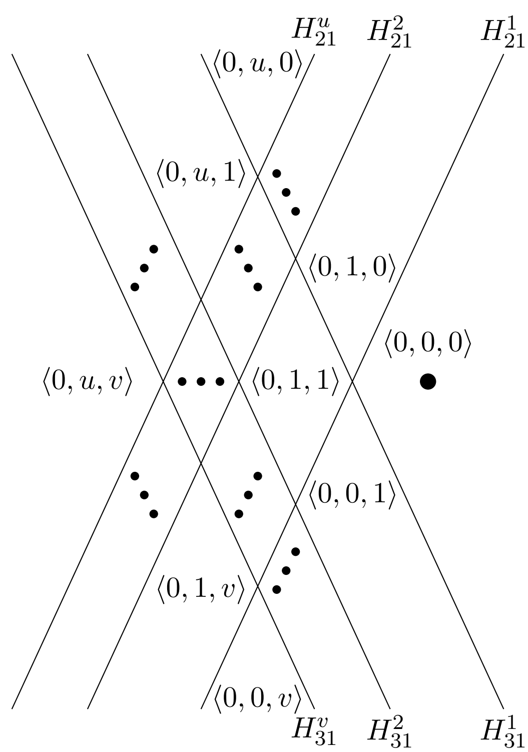


Figure 3.5: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities* $m_{21} = u$ *and* $m_{31} = v$*.*

In the third scenario, assume that G has non-zero multiplicities $m_{ji} = u$ and $m_{ki} = v$. Similar to the previous scenario, there are no conditions required to guarantee a bijective arrangement, and it follows that the arrangement $\mathcal{A}$ is given by

$$\begin{aligned} &H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \quad \text{for } \alpha \in \{1, \ldots, r\}, \\ &H_{jk}^{c_\beta}, \quad \text{where } c_\beta = \beta, \quad \text{for } \beta \in \{1, \ldots, t\}. \end{aligned}$$

See Figure 3.6 for a picture of the arrangement with corresponding labels.

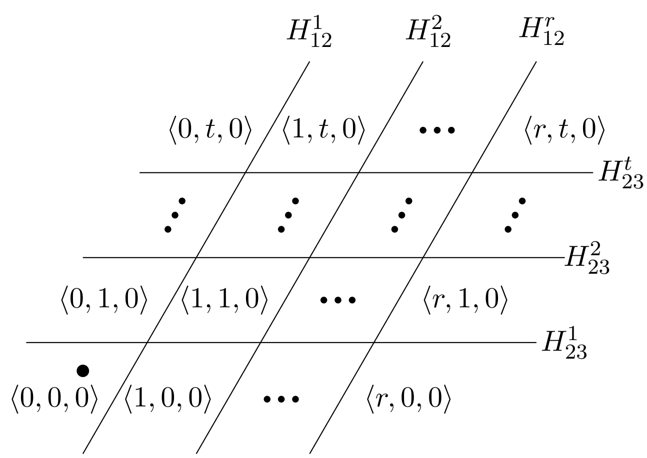


Figure 3.6: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = r$ and $m_{23} = t$.*

□

In the case that two edge types are chosen, there is a forbidden graph that does not emit any arrangements with a bijective labeling. This graph occurs when m_{ij} and m_{ik} are non-zero for some $i, j \in \{1, 2, 3\}$, and it fails Theorem 3.1.

In the third family we have that three different types of hyperplanes are chosen, and the ones that emit an arrangement with a bijective labeling are as follow.

Theorem 3.7. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If there exists exactly three non-zero edge multiplicities and if the graph satisfies one of the following two cases:*

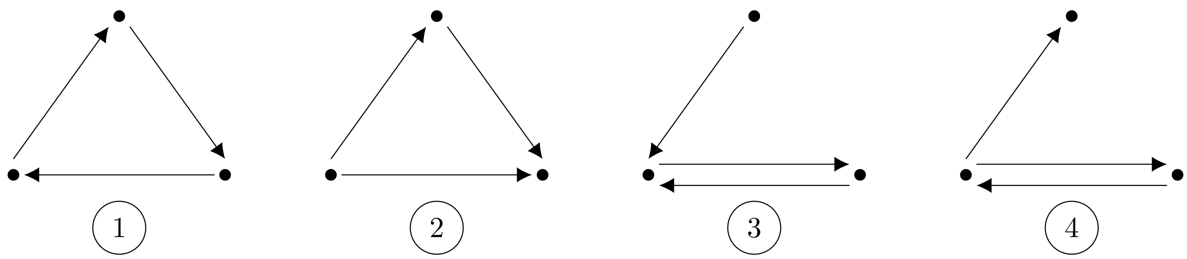


Figure 3.7: *In the case that three edge types are chosen, these four graphs are the only choices up to a relabeling of the vertices. For Theorem 3.7, graphs 1 and 2 satisfy case one while graph 3 satisfies case 2. The remaining graph, 4, fails to produce a bijective labeling.*

1. *There does not exist an $i \in V$ such that m_{ij} and m_{ik} are non-zero*

2. *There does exist an $i \in V$ such that m_{ij} and m_{ik} are non-zero and the edge-multiplicities satisfy $m_{ij} + m_{ik} - 1 \leq m_{jk}$.*

Then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$.

Proof. In the instance that three edge multiplicities are non-zero, there are three different scenarios depending on which multiplicities are non-zero. However, for each case there are several graphs that only differ up to a rearrangement of the vertices, so without loss of generality we will consider the following three cases.

Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ji} = u$, and $m_{kj} = w$. In this case, there are no conditions required to guarantee a bijective arrangement, and it follows that the arrangement $\mathcal{A}$ is given by

$$\begin{aligned} &H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \quad \text{for } \alpha = \{1, \dots, r\}, \\ &H_{ji}^{a'_\beta}, \quad \text{where } a'_\beta = \beta, \quad \text{for } \beta \in \{1, \dots, u\}, \\ &H_{kj}^{c'_\gamma}, \quad \text{where } c'_\gamma = \gamma, \quad \text{for } \gamma \in \{1, \dots, w\}. \end{aligned}$$

See Figure 3.8 for a picture of the arrangement with corresponding labels.

In the second scenario, assume that G has non-zero multiplicities $m_{ij} = r$, $m_{jk} = t$, and $m_{ki} = v$. Similar to the previous scenario, there are no conditions required to guarantee a bijective arrangement, and it follows that the arrangement $\mathcal{A}$ is given by

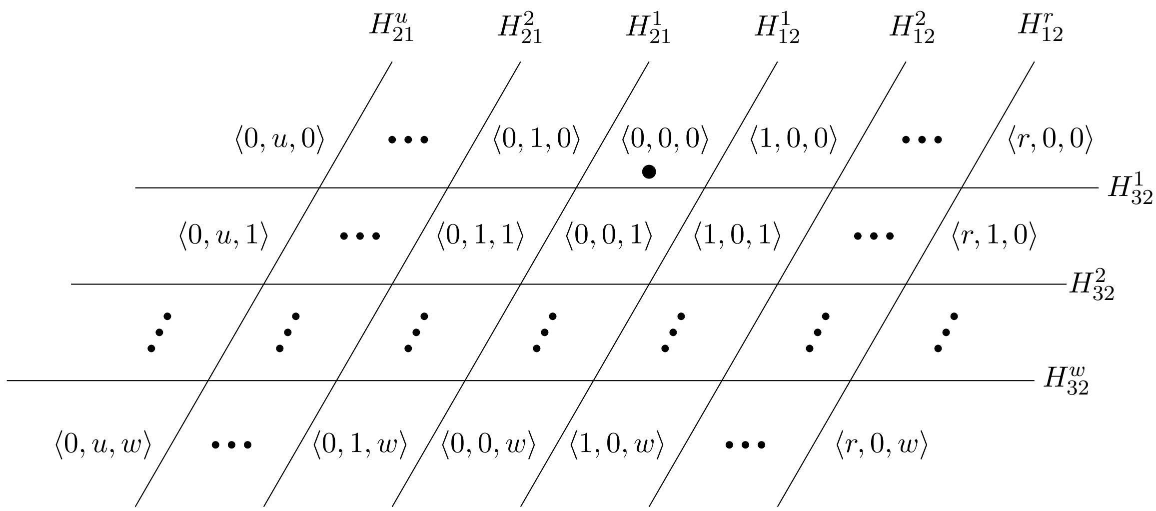


Figure 3.8: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = r$, $m_{21} = u$, and $m_{32} = w$.*

$$\begin{aligned} &H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \quad \text{for } \alpha \in \{1, \dots, r\}, \\ &H_{jk}^{c_\beta}, \quad \text{where } c_\beta = \beta, \quad \text{for } \beta \in \{1, \dots, t\}, \\ &H_{ki}^{b'_\gamma}, \quad \text{where } b'_\gamma = \gamma, \quad \text{for } \gamma \in \{1, \dots, v\}. \end{aligned}$$

See Figure 3.9 for a picture of the arrangement with corresponding labels.

In the third scenario, assume that G has non-zero multiplicities $m_{ij} = r$ and $m_{ik} = s$. For this scenario the graph does have conditions required to have a bijective arrangement, namely that either m_{jk} or m_{kj} is non-zero and $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$. Without loss of generality, assume that $m_{jk} = t$ is non-zero since the case that m_{kj} is non-zero is the same case with the vertex labels of j and k switched. The arrangement $\mathcal{A}$ is given by

$$\begin{aligned} &H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, \qquad\quad \text{for } \alpha \in \{1, \dots, r\}, \\ &H_{ik}^{b_\beta}, \quad \text{where } b_\beta = r + \beta, \quad \text{for } \beta \in \{1, \dots, s\}. \end{aligned}$$

When the hyperplanes of the form H_{ij} and H_{ik} intersect they create bad intersections that need to be rectified. This is done by intersecting each bad intersection with a hyperplane of the form H_{jk}. Furthermore, the coefficients can be found in terms of the a_α's and b_β's by considering the following differences

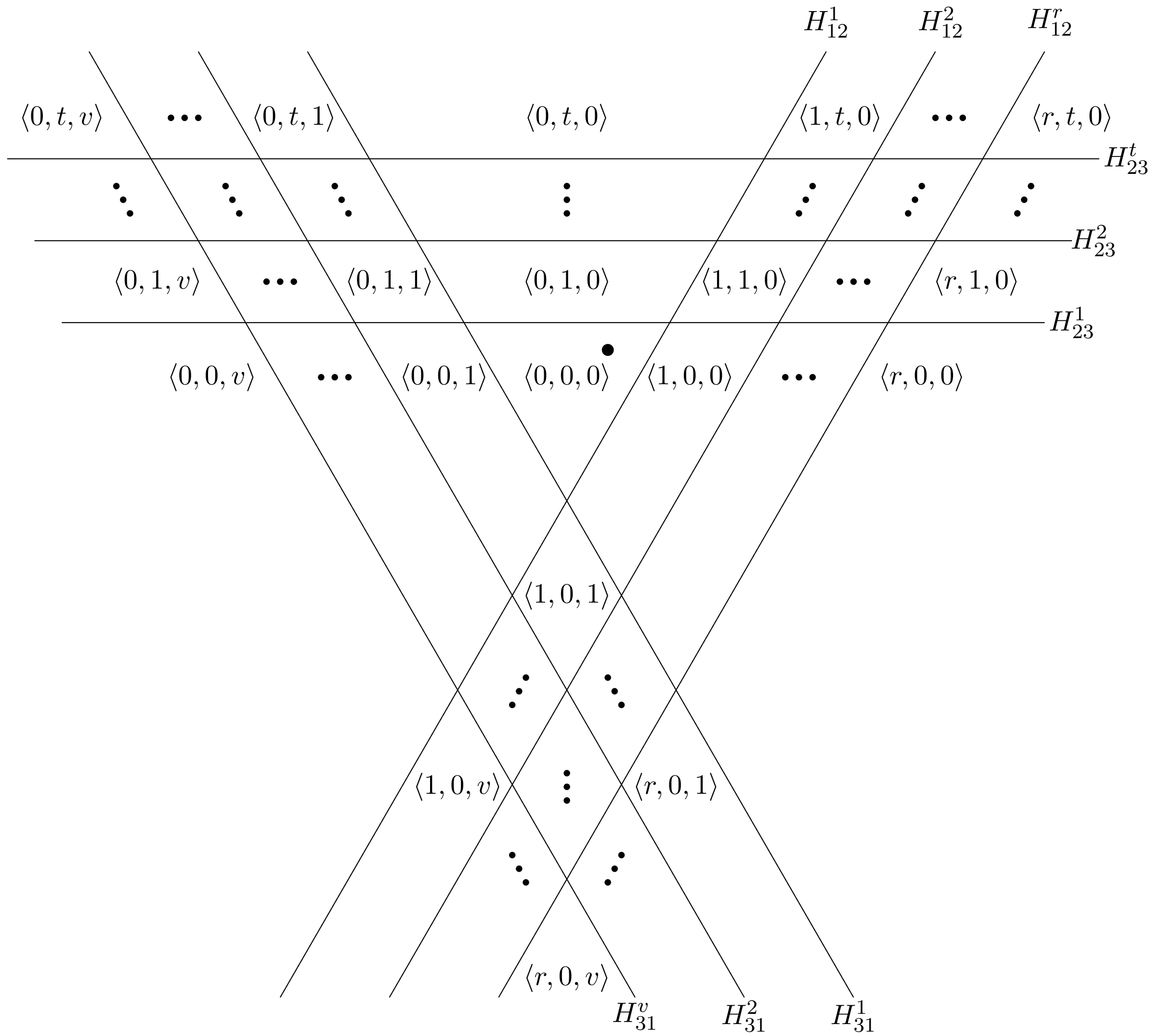


Figure 3.9: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = r$, $m_{31} = v$, and $m_{23} = t$.*

$$
\begin{aligned}
&d_1 = b_1 - a_r = 1\\
&d_2 = b_2 - a_r = 2\\
&\vdots\\
&d_s = b_s - a_r = s\\
&d_{s+1} = b_s - a_{r-1} = s+1\\
&\vdots\\
&d_{s+r-1} = b_s - a_1 = s+r-1
\end{aligned}
$$

Note, by the construction of coefficients, all of the differences are positive since $b_1 > a_r$ and correspond to the ordering needed for hyperplanes of the form H_{jk}. Therefore the hyperplanes can be placed as follows

$$
\begin{array}{lll}
H_{jk}^{c_\gamma}, & \text{where } c_\gamma = d_\gamma, & \text{for } \gamma \in \{1, \ldots, r+s-1\}.\\
H_{jk}^{c_{r+s-1\omega}}, & \text{where } c_{s+\omega} = c_{r+s-1} + \gamma, & \text{for } \omega \in \{1, \ldots, t-r-s+1\}.
\end{array}
$$

See figure 3.10 for a picture of the arrangement with corresponding labels.

□

In the case that three edges types are chosen, there are two forbidden graphs that do not emit any arrangements with a bijective labeling. The two graphs are similar up switching the order of j and k, therefore the graphs are given by the non-zero multiplicities m_{ij}, m_{ik}, and m_{ji}. This graphs fail to emit a bijective labeling since they fail Theorem 3.1, i.e. there are no hyperplanes to rectify the bad intersections between the hyperplanes of the type H_{ij} and H_{ik}.

In the fourth family we have that four different types of hyperplanes are chosen, and the ones that emit an arrangement with a bijective labeling are as follow.

Theorem 3.8. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If there exists four non-zero directed edge multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, and $m_{ki} = v$ where the inequality $m_{ij} + m_{ik} - 1 \leq m_{jk}$ is satisfied, then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$..*

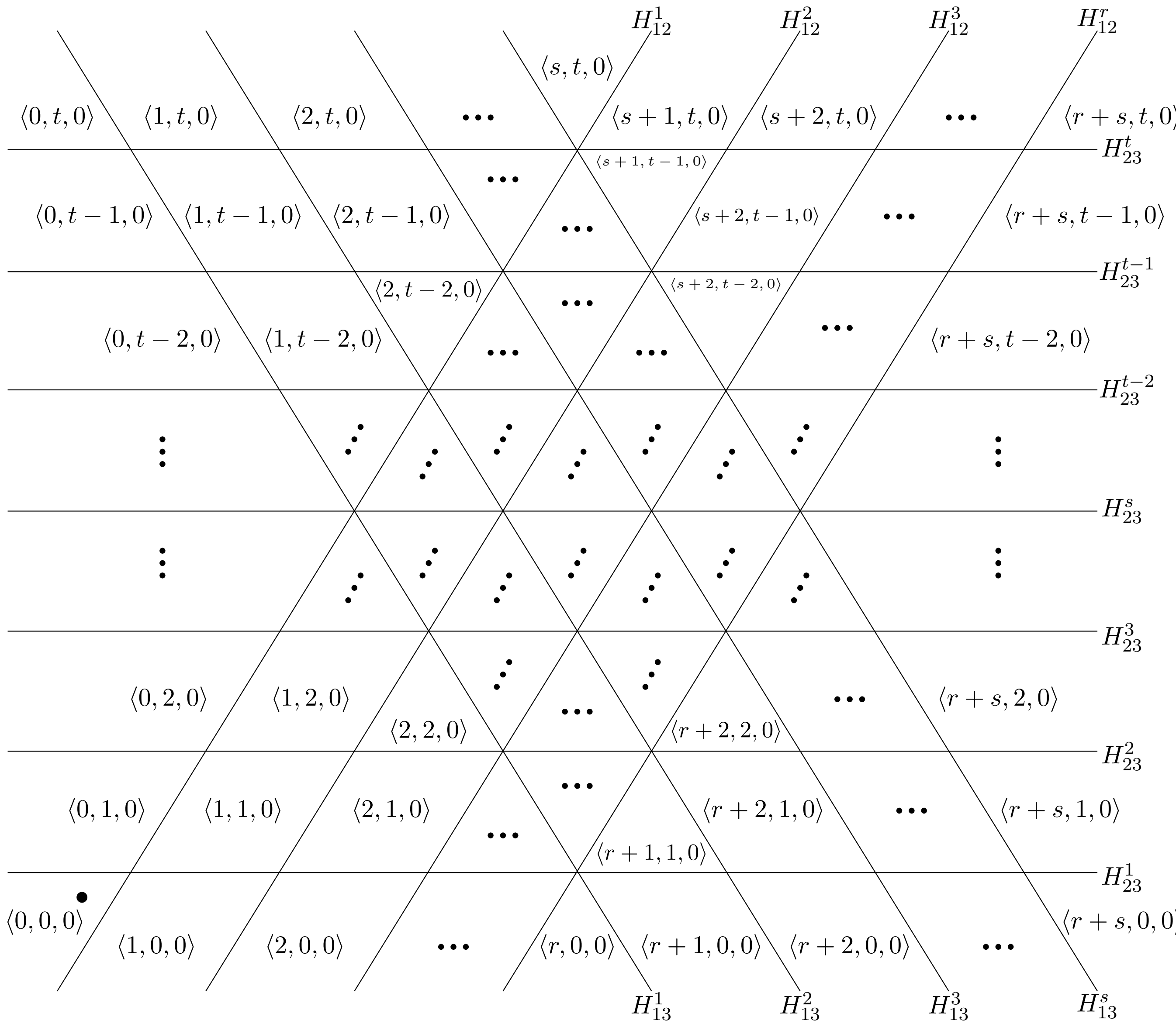


Figure 3.10: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = r$, $m_{13} = s$, and $m_{23} = t$. Note that in the figure that we take that $t = r + s - 1$. Note that the labels of the regions between the kth and $(k + 1)$th horizontal lines have the second entry equal to k and the first entry of these labels grows monotonically from left to right.*

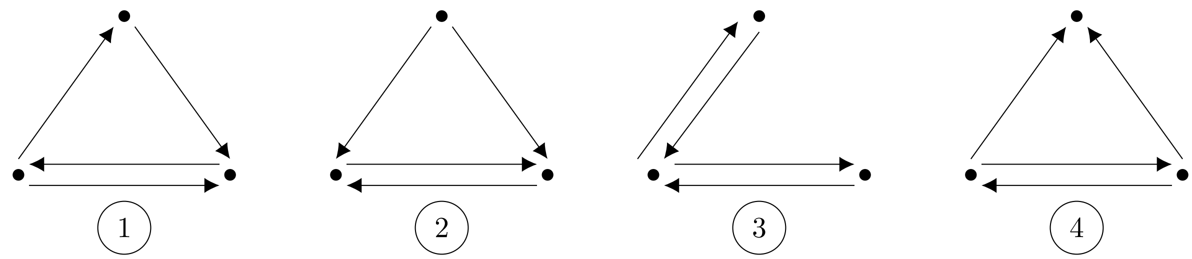


Figure 3.11: *In the case that four edge types are chosen, these four graphs are the only choices up to a relabeling of the vertices. For these graphs, graph* 1 *satisfies Theorem 3.8 while graph* 2 *satisfies Theorem 3.9. The remaining graphs,* 3 *and* 4*, these fail to produce a bijective labeling.*

Proof. In the instance that four edge multiplicities are non-zero, there are two different scenarios depending on which multiplicities are non-zero, this case corresponds the graph labeled 1 in Figure 3.11. However, for this case there are several graphs that only differ up to a rearrangement of the vertices, so without loss of generality we will consider the following case.

Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, and $m_{ki} = v$. In this case, there is one condition required to guarantee a bijective arrangement, namely that $m_{ij} + m_{ik} - 1 \leq m_{jk}$. Consider the arrangement given by the following hyperplanes of the form H_{ij}, H_{ik}, and H_{jk}

$$
\begin{array}{lll}
H_{ij}^{a_\alpha}, & \text{where } a_\alpha = \alpha, & \text{for } \alpha \in \{1, \ldots, r\}, \\
H_{ik}^{b_\beta}, & \text{where } b_\beta = \beta + r, & \text{for } \beta \in \{1, \ldots, s\}, \\
H_{jk}^{c_\gamma}, & \text{where } c_\gamma = b_\gamma - a_r = \gamma, & \text{for } \gamma \in \{1, \ldots, s-1\}, \\
H_{jk}^{c_{s+\omega}}, & \text{where } c_{s+\omega-} = b_s - a_{r+1-\omega} = s + \omega - 1, & \text{for } \omega \in \{1, \ldots, r\}, \\
H_{jk}^{c_{s+r-1+\delta}}, & \text{where } c_{s+r-1+\delta} = c_{s+r-1} + \delta = s + r - 1 + \delta, & \text{for } \delta \in \{1, \ldots, t - r - s + 1\}.
\end{array}
$$

Where the coefficients were defined in Theorem 3.7, and now the only hyperplanes that need to be placed are of the form H_{ki}. These can be placed using the coefficients

$$H_{ki}^{b'_\epsilon}, \quad \text{where } b'_\epsilon = \epsilon, \quad \text{for } \epsilon \in \{1, \ldots, v\}.$$

□

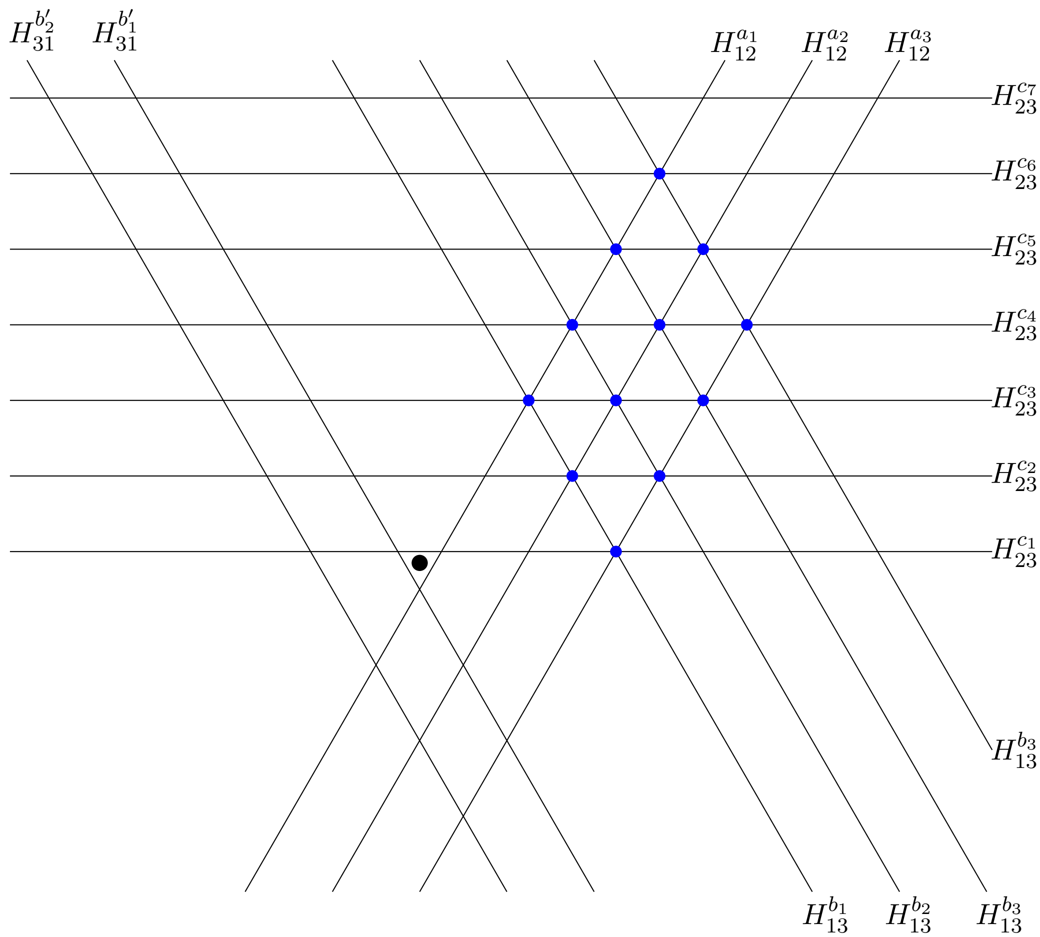


Figure 3.12: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = 4$, $m_{13} = 5$, $m_{23} = 7t$, and $m_{31} = 2$, and the blue intersection points indicate bad intersections that have been rectified. Moreover, in this case there are no restrictions on the number of hyperplanes of type H_{31}, so we are able to add as many hyperplanes of type H_{31} as we want by assigning coefficients larger than b_2'.*

Theorem 3.9. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If there exists four non-zero directed edge multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, and $m_{kj} = w$ and the inequality $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$ is satisfied, then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$.*

Proof. In the second case that four edge multiplicities are chosen, corresponding to the graph labeled two in Figure 3.11, assume that G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, and $m_{kj} = w$. In this case, like the previous one, the only condition required to guarantee a bijective arrangement is that $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$. However, in this scenario there is more freedom of choice depending on whether $m_{ij} > m_{kj}$, $m_{ij} \leq m_{kj}$, and whether the inequality is strict or not. To address this freedom consider the following method for placing the hyperplanes of the form H_{ij} and H_{ik}.

First, assume that $r + s - 1 = t + w$, this will form the base case for the strict inequality. This is since any extra hyperplanes of type H_{jk} and H_{kj} can be placed anywhere above the last H_{jk} or below the last H_{kj} that is used to rectify a bad intersection between the hyperplanes of type H_{ij} and H_{ik}. Now, let the hyperplanes of the form H_{ij} be placed using the following coefficients

$$H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = 2\alpha - 1, \quad \text{for } \alpha \in \{1, \ldots, r\}.$$

For the hyperplanes of the form H_{ik}, the placement is dependent on the number of hyperplanes of the form H_{jk} and H_{kj}. If $m_{ij} > m_{kj}$, i.e. $r > w$, then the hyperplanes of the form H_{ik} can be placed in the following manner

$$\begin{aligned}
&H_{ik}^{b_1}, \quad \text{where } b_1 = a_{|w-s|} + 1 = 2(r - w),\\
&H_{ik}^{b_\beta}, \quad \text{where } b_\beta = b_1 + 2\beta - 2 = 2(r - w) + 2\beta - 2, \quad \text{for } \beta \in \{2, \ldots, s\}.
\end{aligned}$$

By this construction, there are precisely t hyperplanes of the type H_{jk} and w hyperplanes of the type H_{kj} needed to rectify the bad intersections. Since these bad intersections are rectified by intersecting either a hyperplane of type H_{jk} or H_{kj}, then the coefficients can be found in terms of the a_α's and b_β's by considering the following differences.

$$d_1 = b_1 - a_r = -2w + 1$$
$$d_2 = b_2 - a_r = -2w + 3$$
$$\vdots$$
$$d_s = b_s - a_r = 2(s - w) - 1$$
$$d_{s+1} = b_s - a_{r-1} = 2(s - w) + 3$$
$$\vdots$$
$$d_{s+r-1} = b_s - a_1 = 2(r - w) + 2s - 3$$

By construction the first w differences are negative and the next t are positive, therefore they can be used to place the last two types of hyperplanes as follow

$$H_{kj}^{c'_\gamma}, \quad \text{where } c_\gamma = d_\gamma, \qquad \text{for } \gamma\{1, \ldots, w\},$$
$$H_{jk}^{c_\delta}, \quad \text{where } c_\delta = d_{w+\delta}, \quad \text{for } \delta \in \{1, \ldots, t - w\}.$$

In the case that $m_{ij} \leq m_{kj}$, i.e. $r \leq w$, the coefficients for the hyperplanes will follow a similar argument as the above case with the exception that the hyperplanes of the form H_{ik} will be placed first. More precisely, the indices i and j are switched and the coefficients are assigned in the exact same manner.

For the case when $m_{ij} + m_{ik} - 1 < m_{jk} + m_{kj}$, it follows that for some $m_1 \leq m_{jk}$ and $m_2 \leq m_{kj}$ that $m_{ij} + m_{ik} - 1 = m_1 + m_2$. Therefore, we can use one of the above scenarios to create the bijective arrangement for G' with multiplicities m_{ij}, m_{ik}, $m_{jk} = m_1$ and $m_{kj} = m_2$. The remaining $m_{jk} - m_1$ and $m_{kj} - m_2$ hyperplanes of type H_{jk} and H_{kj}, respectively, can by placed by assigning unique coefficients c and c' such that $c > c_{m_1}$ and $c' > c'_{m_2}$.

$\square$

In the case that four edge types are chosen, there are two forbidden graphs that do not emit any arrangements with a bijective labelings. The first graph has non-zero edge multiplicities m_{ij}, m_{ji}, m_{ik} and m_{ki} which fails Theorem 3.1, this corresponds to the graph labeled 3 in Figure 3.11.

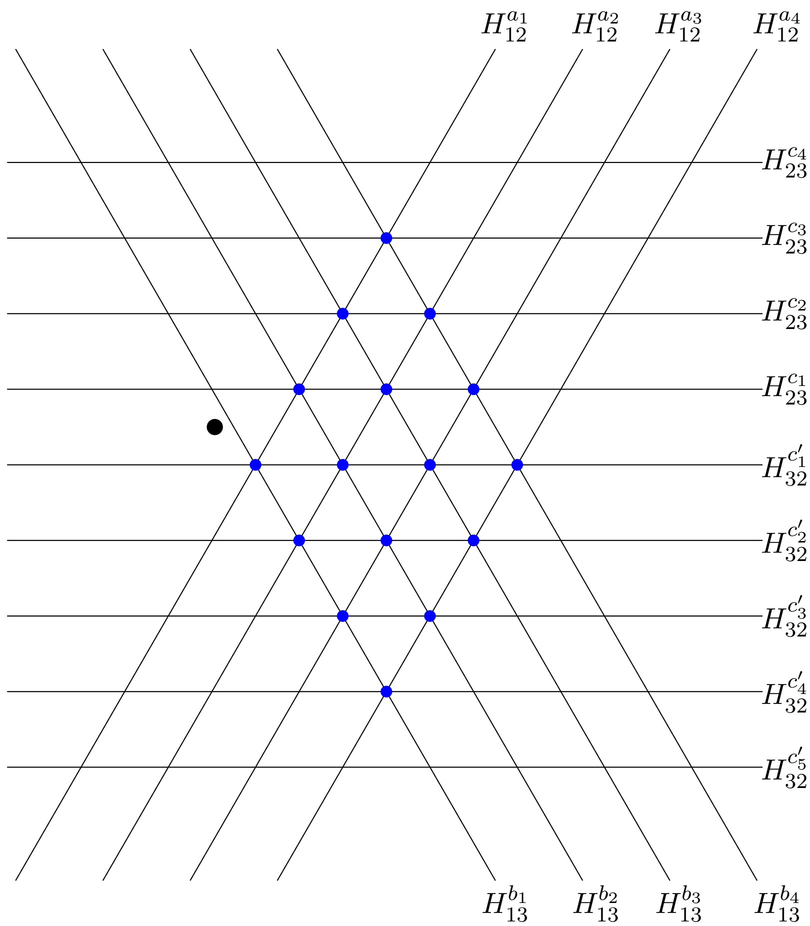


Figure 3.13: *In this example we see the bijective arrangement with labels for a graph that has non-zero multiplicities $m_{12} = 4$, $m_{13} = 4$, $m_{23} = 4$, and $m_{32} = 5$, and the blue intersection points indicate bad intersections that have been rectified.*

For the second forbidden graph, corresponding to graph labeled 4 in Figure 3.11, assume that m_{ij}, m_{ik}, m_{ji}, and m_{jk} are the non-zero edge multiplicities. In this case there are two families of bad intersections, the first between the hyperplanes of types H_{ij} and H_{ik}, and the second between the hyperplanes of types H_{ji} and H_{jk}. For both families to be rectified, the following two inequalities must be satisfied:

$$m_{ij} + m_{ik} - 1 \leq m_{jk} \quad \text{and} \quad m_{ji} + m_{jk} - 1 \leq m_{ik}$$

where at least one of them is strict. Regardless of which is strict, adding the two inequalities together will yield

$$m_{ij} + m_{ji} - 2 < 0$$

which cannot happen since $m_{ij}, m_{ji} > 0$. Therefore this is indeed a forbidden graph.

Chapter 4

Necessary but Not Sufficient

For the cases of five and six types of edges the conditions in Theorem 3.4 are necessary, but not sufficient to guarantee the existence of an arrangement with a bijective labeling. We will first discuss the families of graphs that do emit a bijective labeling in the case of five types of hyperplanes.

In this case since we have one edge multiplicity being zero, without loss of generality we will assume that $m_{kj} = 0$, then we have two equations according to Theorem 3.4,

$$m_{ij} + m_{ik} - 1 \leq m_{jk} \quad \text{and} \quad m_{ji} + m_{jk} - 1 \leq m_{ik} + m_{ki}.$$

By our conditions at least one of these equations must be strict, however if one of these is equality then there is a forced rigidity on how hyperplanes must be placed since there just enough hyperplanes to rectify all of the bad intersections. Another situation unique to this case is that there is no inequality creating an upper bound to the number of hyperplanes of the type H_{ki} that is in our arrangement. The first family of graphs that produce an arrangement with a bijective labeling is based on the fact that there are enough hyperplanes of the type H_{ki} to rectify all of the bad intersections between the H_{ji} and H_{jk} hyperplanes.

4.1 Five Edge Types with a Bijective Labeling

Theorem 4.1. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. For $i, j, k \in V$ with the m_{kj} being the only edge multiplicity that is zero, and the following inequalities where at least one is strict*

1. $m_{ij} + m_{ik} - 1 \leq m_{jk}$,

2. $m_{jk} + m_{ji} - 1 \leq m_{ki}$,

then there exists an arrangement $\mathcal{A}$ with bijective labeling such that $G = G_{\mathcal{A}}$.

Proof. In the instance of five edge multiplicities are non-zero, there is one situation up to a rearrangement of the vertices. Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, and $m_{ki} = v$. The assumption that $m_{ji} + m_{jk} - 1 \leq m_{ki}$ implies that there are enough hyperplanes of the type H_{ki} to rectify all of the bad intersections between the hyperplanes of type H_{jk} and H_{ji}. Let the hyperplanes of the type H_{ij} and H_{ik} be placed using the following coefficients

$$\begin{aligned}
&H_{ij}^{a_\alpha}, \quad \text{where } a_\alpha = \alpha, && \text{for } \alpha \in \{1, \dots, r\}, \\
&H_{ik}^{b_\beta}, \quad \text{where } b_\beta = a_r + \beta = r + \beta, && \text{for } \beta \in \{1, \dots, s\}.
\end{aligned}$$

Since there are no hyperplanes of the type H_{kj}, then the bad intersections are rectified solely with hyperplanes of the form H_{jk}. This is done by using the coefficients defined by

$$\begin{aligned}
&H_{jk}^{c_\gamma}, && \text{where } c_\gamma = \gamma, && \text{for } \gamma \in \{1, \dots, r+s-1\}, \\
&H_{jk}^{c_{s+r-1+\omega}}, && \text{where } c_{s+r-1+\omega} = c_{s+r-1} + \omega, && \text{for } \omega \in \{1, \dots, t-r-s+1\}.
\end{aligned}$$

Note that for the first $r + s - 1$ H_{jk} hyperplanes to rectify all of the bad intersections between the hyperplanes of type H_{ij} and H_{ik} one requires the coefficients to satisfy

$$\gamma \in \{b_1 - a_r, \dots, b_{s-1} - a_r, b_s - a_r, \dots, b_s - a_1\} \quad \text{for } \gamma \in \{1, \dots, r+s-1\}.$$

By substituting the corresponding integer for each coefficient one yields

$$\{b_1 - a_r, \ldots, b_{s-1} - a_r, b_s - a_r, \ldots, b_s - a_1\} \quad \Leftrightarrow \quad \{1, \ldots, r + s - 1\}.$$

Therefore all of the bad intersections, between the hyperplanes H_{ij} and H_{ik}, are rectified by the first $r + s - 1$ H_{jk} hyperplanes. The remaining to hyperplanes, those of type H_{ji} and H_{ki}, can be placed in the following manner.

$$\begin{aligned} &H_{ki}^{b'_\zeta}, \quad \text{where } b'_\zeta = \zeta, && \text{for } \zeta \in \{1, \ldots, v\}, \\ &H_{ji}^{a'_\eta}, \quad \text{where } a'_\eta = c_t + b'_\eta = t + \eta, && \text{for } \eta \in \{1, \ldots, u\}. \end{aligned}$$

□

Theorem 4.2. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with $m_{ij} = 1$, $m_{ik} \neq 0$, $m_{jk} \neq 0$, $m_{ji} \neq 0$, $m_{ki} \neq 0$, $m_{kj} = 0$, and the following inequalities where at least one is strict*

1. $m_{ik} \leq m_{jk}$,

2. $m_{ji} + m_{jk} - 1 \leq m_{ik} + m_{ki}$,

then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$

Proof. In the instance that five edge multiplicities are non-zero, and fitting the inequalities above. For both cases there are several graphs that only differ up to a rearrangement of the vertices, so without loss of generality we will consider the following two cases. For either case, assume that the graph G has non-zero multiplicities $m_{ij} = 1$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, and $m_{ki} = v$.

In the first case, assume that the inequality $m_{jk} + m_{ji} - 1 \leq m_{ki}$ is satisfied, then by Theorem 4.1 there exists an arrangement $\mathcal{A}$ such that $G = G_{\mathcal{A}}$.

In the second case, assume that $m_{jk} + m_{ji} - 1 > m_{ki}$. In other words, hyperplanes of the types H_{ki} and H_{ik} are needed to rectify the bad intersections between the hyperplanes H_{jk}

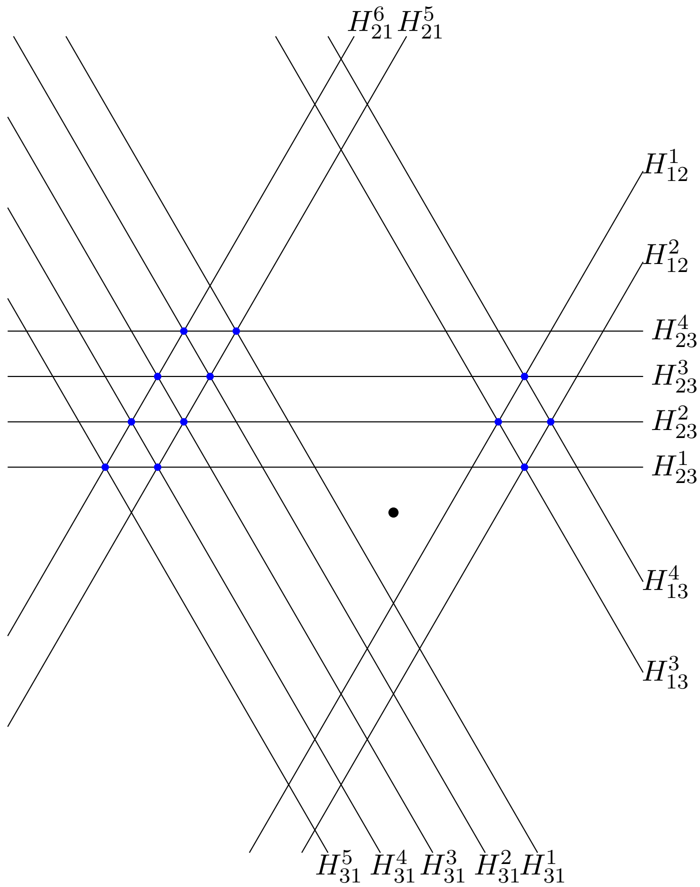


Figure 4.1: *In this example we see a graph on three vertices with edge multiplicities* $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 2$, *and* $m_{31} = 5$. *This graph satisfies the conditions of Theorem 4.1 since there are just enough hyperplanes of type* H_{31} *rectify all of the bad intersections between the hyperplanes of type* H_{21} *and* H_{23}. *All of the bad intersections that have been rectified are represented by blue.*

and H_{ji}. Let the hyperplanes of the type H_{ij}, H_{ik}, and H_{jk} be placed with the following coefficients.

$$\begin{aligned}
&H_{ij}^{a_1}, \quad \text{where } a_1 = 1, \\
&H_{ik}^{b_\beta}, \quad \text{where } b_\beta = 3\beta - 1, \quad \text{for } \beta \in \{1, \ldots, s\}, \\
&H_{jk}^{c_\gamma}, \quad \text{where } c_\gamma = 3\gamma - 2 \quad \text{for } \gamma \in \{1, \ldots, t\}.
\end{aligned}$$

For the hyperplanes of type H_{jk} to rectify a bad intersection between those of type H_{ij}

and H_{ik} the coefficient c_γ, for some $\gamma \in \{1, \ldots, t\}$, must be contained in the set

$$\{b_1 - a_1, \ldots, b_{s-1} - a_1, b_s - a_1\} \quad \text{for } \gamma \in \{1, \ldots, r+s-1\}.$$

By substituting the corresponding integer for each coefficient one yields

$$\{b_1 - a_1, \ldots, b_s - a_r, \ldots, b_s - a_1\} \quad \Leftrightarrow \quad \{1, 4, 7, \ldots, 3(r+s-1) - 2\}.$$

Therefore the first $r+s-1$ hyperplanes of type H_{jk} are rectifying the desired bad intersections, and further the remaining hyperplanes are placed at the same intervals.

Now, let the hyperplanes of type H_{ki} and H_{ji} be placed in the following manner

$$\begin{aligned} &H_{ki}^{b'_\delta}, \quad \text{where } b'_\delta = 3\delta - 2, && \delta \in \{1, \ldots, v\}, \\ &H_{ji}^{a'_\epsilon}, \quad \text{where } a'_\epsilon = 3\max\{1, t-s\} - 1 + 3\epsilon - 3 && \text{for } \epsilon \in \{1, \ldots, u\}. \end{aligned}$$

Similarly to the hyperplanes of type H_{jk}, the coefficients for the H_{ik} and H_{ki} hyperplanes must be of the form $a' - c$ or $c - a'$ for some a' and c defined above. Let $X = 3\max\{1, t-s\} - 1$ and consider the following

$$\begin{aligned} x_2 - x_1 &= q, \quad \in \{X, X+3, \ldots, X+3u-3\}, \\ x_2 - x_3 &= p, \quad \in \{1, 4, 7, \ldots, 3t-2\}. \end{aligned}$$

It then follows that the coefficients for H_{ik} and H_{ki} satisfies

$$\begin{aligned} q > p \quad x_3 - x_1 &= q - p, \quad \in \{1, 4, 7, \ldots, X+3u-4\}, \\ p < q \quad x_1 - x_3 &= p - q, \quad \in \{2, 5, 8, \ldots, 2t-2-X\}. \end{aligned}$$

Therefore the coefficient are indeed in the same sequence of integers that were defined earlier. Now, all that remains is to show that the following two inequalities are satisfied

(1) $X + 3(u-1) - 1 \leq 3v - 2$ and

(2) $3t - 2 - X \leq 3s - 1$.

If the inequalities are satisfied, then the furthest intersections from the origin of the hyperplanes of type H_{jk} and H_{ji} are rectified by a H_{ki}, inequality (1), and a H_{ik}, inequality (2). Indeed, inequality (1) is satisfied as follows

$$\begin{aligned} X + 3(u-1) - 1 &\leq 3v - 2 \\ 3\max\{1, t-s\} - 1 + 3(u-1) - 1 &\leq 3v - 2 \end{aligned}$$

If $\max\{1, t-s\} = 1$, then $t = s$ and the inequality becomes $u \leq v$. Which is true, since $t + u - 1 < v + s$ simplifies to $u - 1 < v$. The second case is if $\max\{1, t-s\} = t - s$, then the inequality simplifies to $t - s + u - 1 \leq v$, i.e. the inequality $m_{jk} + m_{ji} - 1 \leq m_{ki} + m_{ik}$. For the second inequality

$$\begin{aligned} 3t - 2 - X &\leq 3s - 1 \\ 3t - 2 - 3\max\{1, t-s\} + 1 &\leq 3s - 1 \\ t - \max\{1, t-s\} &\leq s \end{aligned}$$

Similarly to the previous inequality, there are two cases depending on $\max\{1, t-s\}$. If $\max\{1, t-s\} = 1$, then $t = s$ and $t - 1 \leq s$. In the other case, if $\max\{1, t-s\} = t - s$, then the inequality becomes $t - (t - s) \leq s$ which is trivial. Therefore, both inequalities (1) and (2) are satisfied and all bad intersections between hyperplanes of type H_{ji} and H_{jk} are rectified.

□

The following three cases of five edge types with a bijective labeling depends on the behavior of how the hyperplanes of types H_{ji} and H_{ik} intersect. See Figure 4.3 for examples of the three different types of lattices these hyperplane types create when they intersect. However, all three cases are covered in Theorem 4.3.

In the case of example 1 in Figure 4.3, since $m_{ji} = 1$ then one is able to use every intersection of the hyperplane H_{ji} with the hyperplanes of type H_{ik} provided other conditions are met. Note, if $m_{ik} = s$, then there are at most s intersection points, denoted $P_1, \ldots, P_s$, that can be utilized. The first condition is that $m_{ij} + m_{ik} - 1 < m_{jk}$, this gives us the

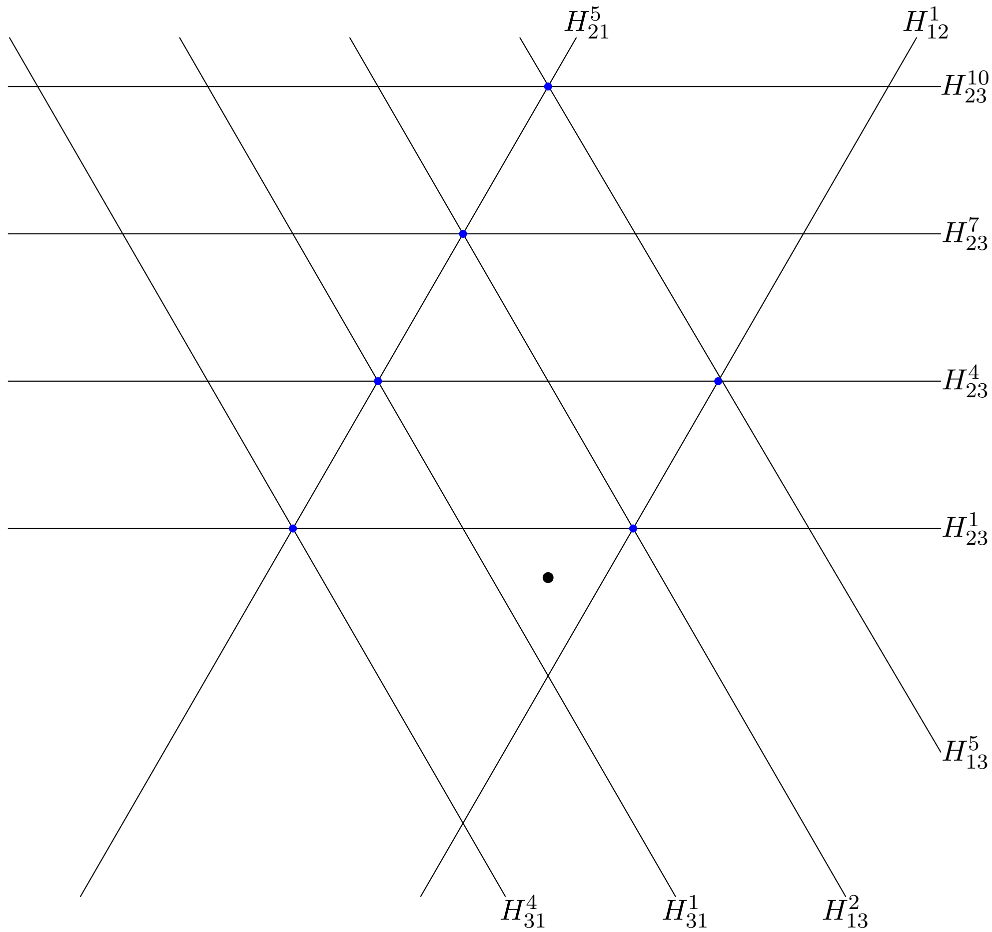


Figure 4.2: *In this example we see a graph on three vertices with edge multiplicities* $m_{12} = 1$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 1$, *and* $m_{31} = 2$. *This graph satisfies the conditions of Theorem 4.2 since* $m_{12} = 1$. *All of the bad intersections that have been rectified are represented by blue. Note in this example that the hyperplanes of type* H_{23} *are split into to groups to utilize both the hyperplanes of type* H_{13} *and* H_{31} *to rectify bad intersections with the hyperplane* H_{21}.

freedom to place the hyperplanes of type H_{jk} that are not used to rectify bad intersections between those of type H_{ij} and H_{ik} in a manner that allows them to intersect a point P_α for some $\alpha \in \{1, \ldots, s\}$.

The first condition in Theorem 4.3 that must be met is

$$(1)\ m_{ji} - 2 \leq m_{ki} - m_{ik} - m_{ij} \quad \iff \quad m_{ji} + (m_{ik} + m_{ij} - 1) - 1 \leq m_{ki}.$$

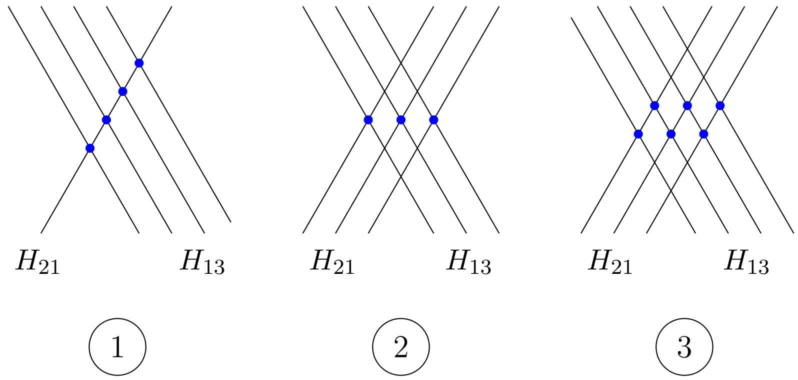


Figure 4.3: *This figure shows examples of the three types of intersection lattices created between the hyperplanes of types H_{ji} and H_{jk} that are utilized in Theorem . In each example the blue intersection points represent points that can be utilized by intersecting a hyperplane of type H_{jk} through the lattice.*

Since $m_{ij} + m_{ik} - 1$ is the number of hyperplanes of type H_{jk} that are needed to rectify all of the bad intersections between those of type H_{ij} and H_{ik}, then the condition can be interpreted in the following way. Condition (1) says that there are at least enough hyperplanes of the type H_{ki} to rectify all of the bad intersections, $m_{ji} + (m_{ik} + m_{ij} - 1) - 1$ in total, between the hyperplane H_{ji} and the $m_{ik} + m_{ij} - 1$ hyperplanes of type H_{jk} that are used rectify the intersections between the H_{ij} and H_{ik} hyperplanes.

The second condition is $m_{jk} + m_{ji} \leq m_{ij} + 2m_{ik}$ which is derived as follows. First, we count the number of hyperplanes of the type H_{jk} that can intersect the lattice and have all of the intersections with the hyperplanes H_{ji} occur within the lattice, this is given by $m_{ik} - m_{ji} + 1$. For the graph to have an arrangement with a bijective labeling we require that

$$m_{ik} - m_{ji} - 1 \geq m_{jk} - (m_{ij} + m_{ik} - 1),$$

i.e. that the number of hyperplanes of the type H_{jk} that are not being used to rectify bad intersections with those of type H_{ij} and H_{ik} is less than or equal to the number that can pass through the lattice without creating non-rectified bad intersections. Note, combining the above like terms yields the desired inequality.

Theorem 4.3. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with $m_{ij} > 1$, $m_{ik} \neq 0$, $m_{jk} \neq 0$, $m_{ji} \neq 0$, $m_{ki} \neq 0$, $m_{kj} = 0$, and the following inequalities where at least one is strict*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,
2. $m_{ji} + m_{jk} - 1 \leq m_{ik} + m_{ki}$,
3. $m_{ji} + m_{jk} - 1 > m_{ki}$.

If G also satisfies $m_{ji} - 2 \leq m_{ki} - m_{ij} - m_{ik}$ and $m_{jk} + m_{ji} \leq m_{ij} + 2m_{ik}$, then there exists an arrangement $\mathcal{A}$ with a bijective labeling such that $G = G_{\mathcal{A}}$

Proof. In the instance that five edge multiplicities are non-zero, and satisfy the inequalities above there is one case. However there are several graphs that only differ up to a rearrangement of the vertices, so without loss of generality we will consider the following case. Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, and $m_{ki} = v$. Let the hyperplanes of type H_{ij} and H_{ik} be placed as follows

$$\begin{aligned} &H_{ij}^{a_\alpha} \quad \text{where } a_\alpha = \alpha, && \text{for } \alpha \in \{1, \ldots, r\}, \\ &H_{ik}^{b_\beta} \quad \text{where } b_\beta = r + \beta, && \text{for } \beta \in \{1, \ldots, s\}. \end{aligned}$$

For the hyperplanes of type H_{jk}, these will be split into two groups where the first $r + s - 1$ are used to rectify the bad intersections between those of type H_{ij} and H_{ik} and the second group which will utilize the hyperplanes of type H_{ik} to rectify the bad intersections they create with those of type H_{ji}. Consider the following placement of the first $r + s - 1$ hyperplanes of type H_{jk}

$$H_{jk}^{c_\gamma} \quad \text{where } c_\gamma = \gamma, \quad \text{for } \gamma \in \{1, \ldots, r + s - 1\}.$$

We will now place the hyperplanes of type H_{ji} and H_{ki} in the following manner, then we will place the remainder of the H_{jk} hyperplanes. Consider the following placement

$$H_{ji}^{a'_\delta} \quad \text{where } a'_\delta = r+s-1+\delta, \quad \text{for } \delta \in \{1,\dots,u\}$$
$$H_{ki}^{b'_\epsilon} \quad \text{where } b'_\epsilon = \epsilon, \qquad\qquad \text{for } \epsilon \in \{1,\dots,v\}.$$

The remaining hyperplanes of H_{jk} must be placed in a manner that the hyperplanes of type H_{ik} rectify the bad intersections between them and the hyperplanes of type H_{ji}. Let the remaining hyperplanes be placed as follows

$$H_{jk}^{c_{r+s-1+\zeta}} \quad \text{where } c_{r+s-1+\zeta} = 2r+s+u-1+\zeta, \quad \text{for } \zeta \in \{1,\dots,t-r-s+1\}.$$

Since these hyperplanes of type H_{jk} are placed one unit apart, it suffice to check that $b_1 \leq c_{r+s} - a'_u$ and $b_s \geq c_t - a'_1$. Indeed,

$$b_1 = c_{r+s} - a'_u = 2r+s+u-(r+s+u-1) = r+1.$$

By a similar argument we have that

$$c_t - a'_1 = r+v+t-(r+s) = v+t-s.$$

To see that $v+t-s \leq r+s$, it follows from the condition that

$$u+t = m_{21}+m_{23} \leq m_{12}+2m_{13} = r+2s \quad \Leftrightarrow \quad u+t-s = m_{21}+m_{23}-m_{13} \leq m_{12}+m_{13} = r+s.$$

Therefore all of the bad intersections are rectified and $\mathcal{A}$ is bijective. $\square$

4.2 Five Edge Types with a Non-Bijective Labeling

In the following theorems we will assume that the only multiplicity that is zero is m_{kj} which means that the only equations this graph must obey to have a bijective labeling are

$$m_{ij} + m_{ik} - 1 \leq m_{jk},$$

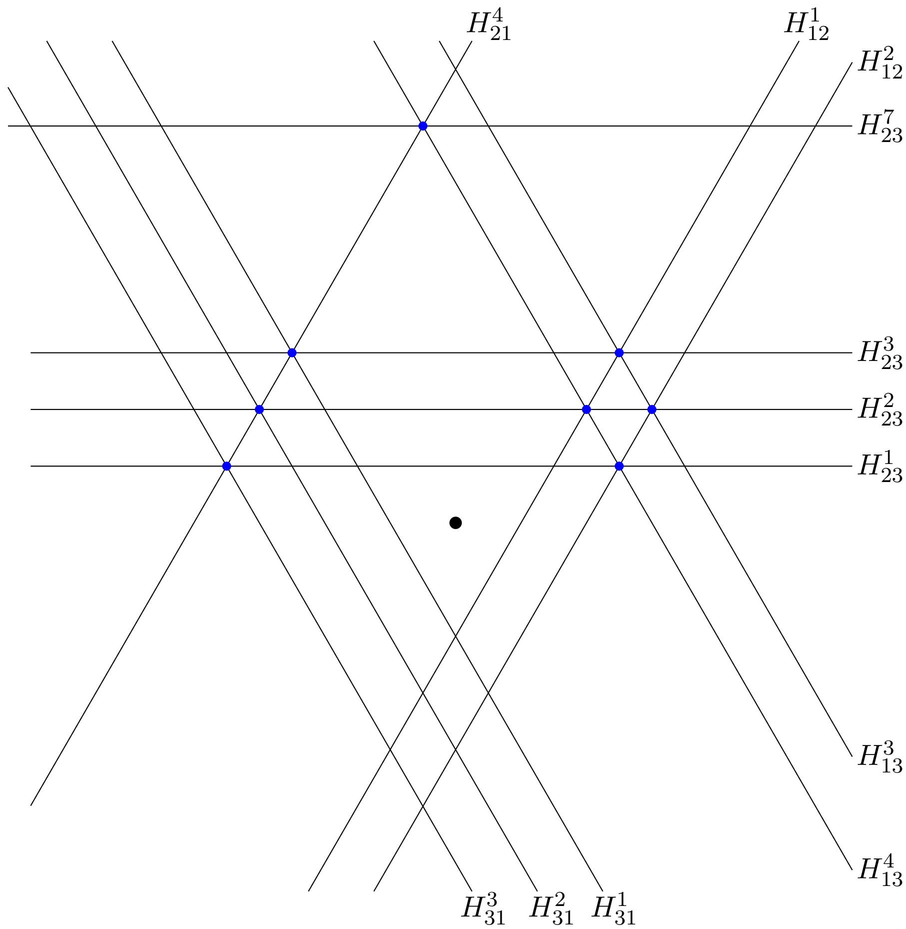


Figure 4.4: *In this example we see a graph on three vertices with edge multiplicities* $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 1$, *and* $m_{31} = 3$. *This graph satisfies the conditions of Theorem 4.3 and the lattice is similar to example 1 in Figure 4.3. All of the bad intersections that have been rectified are represented by blue. Note in this example that the hyperplanes of type* H_{23} *are split into to groups to utilize both the hyperplanes of type* H_{13} *and* H_{31} *to rectify bad intersections with the hyperplane* H_{21}.

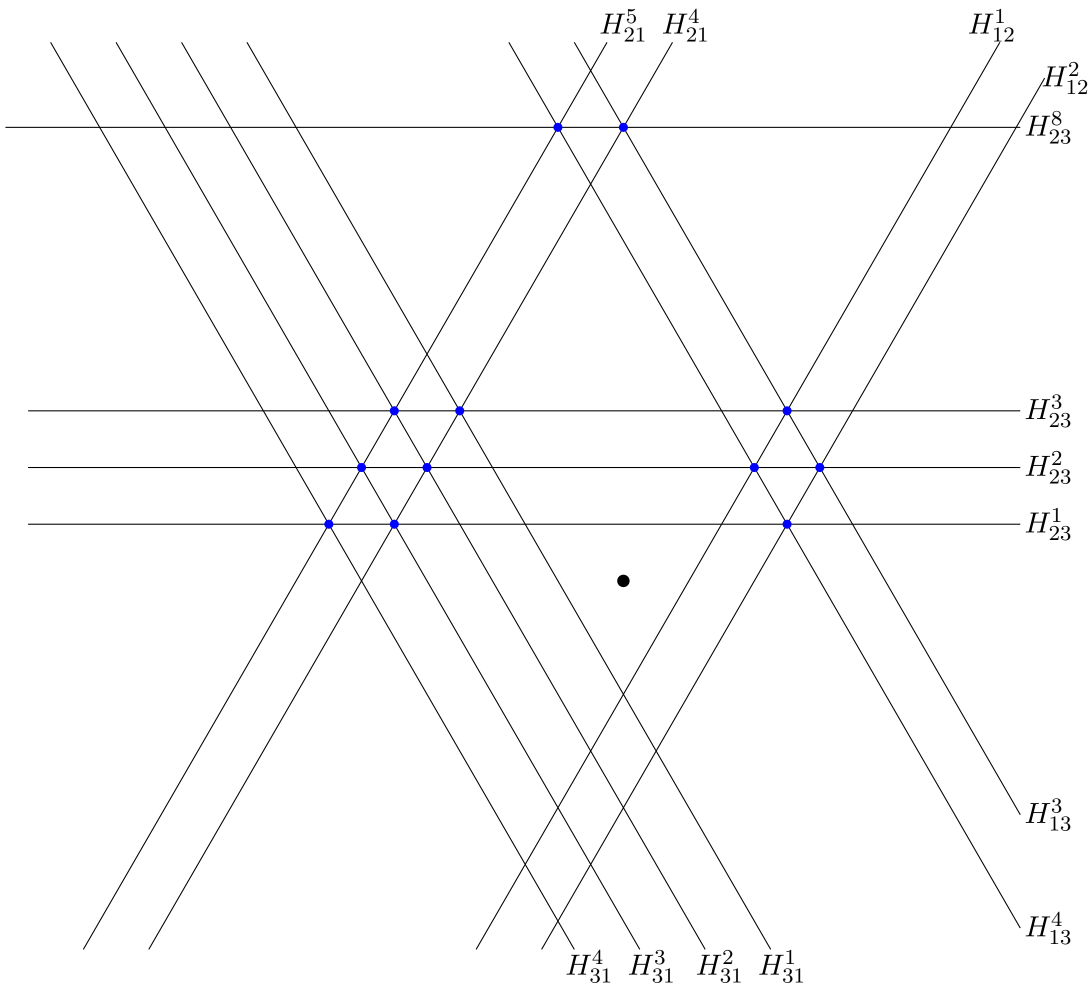


Figure 4.5: *In this example we see a graph on three vertices with edge multiplicities* $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 2$, *and* $m_{31} = 4$. *This graph satisfies the conditions of Theorem 4.3 since* $m_{13} = m_{21}$. *All of the bad intersections that have been rectified are represented by blue. Note in this example that the hyperplanes of type* H_{23} *are split into to groups to utilize both the hyperplanes of type* H_{13} *and* H_{31} *to rectify bad intersections with the hyperplanes* H_{21}.

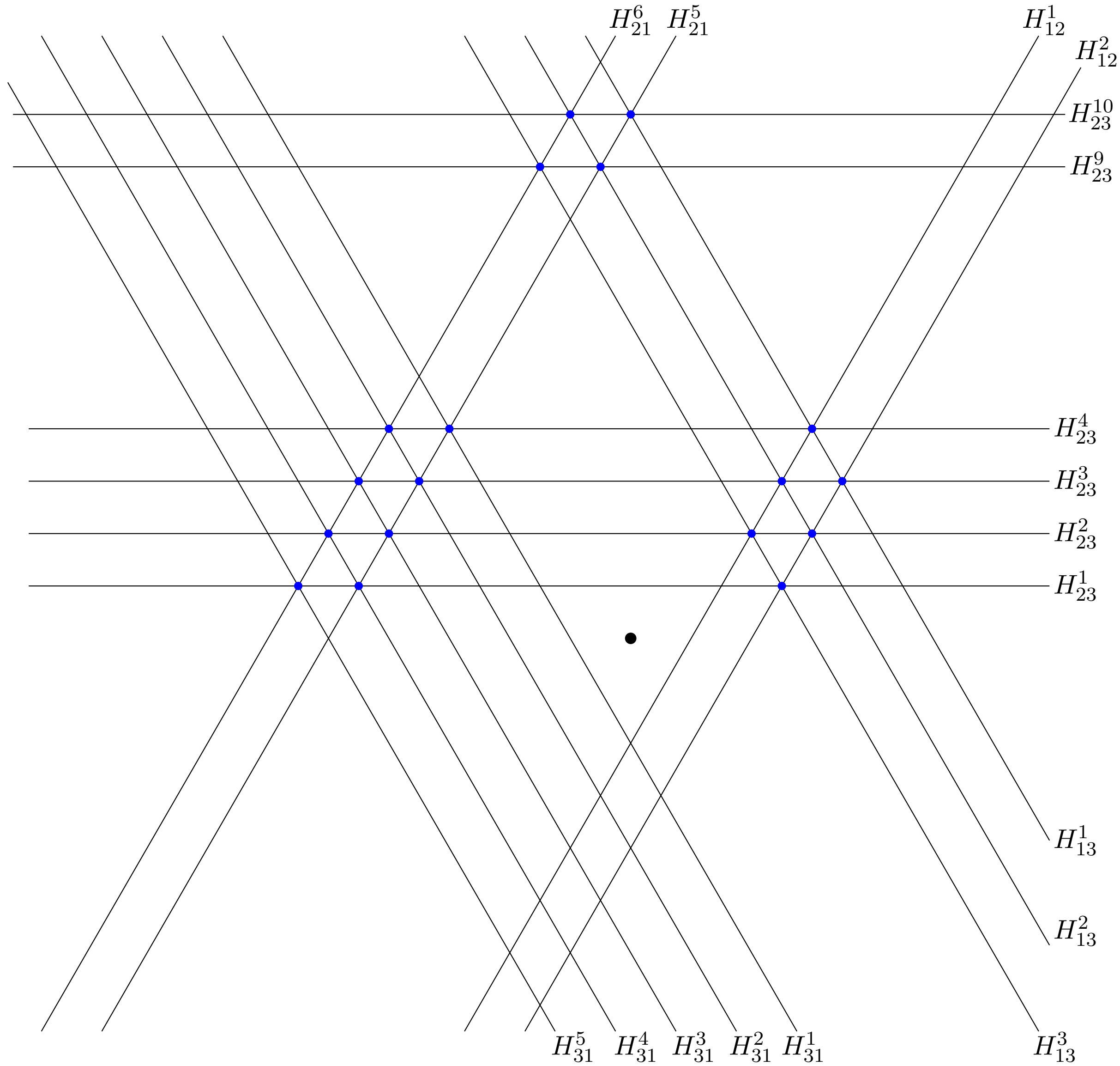


Figure 4.6: *In this example we see a graph on three vertices with edge multiplicities $m_{12} = 2$, $m_{13} = 3$, $m_{23} = 6$, $m_{21} = 2$, and $m_{31} = 5$. This graph satisfies the conditions of Theorem 4.3 since $1 < m_{21} < m_{13}$. All of the bad intersections that have been rectified are represented by blue. Note in this example that the hyperplanes of type H_{23} are split into to groups to utilize both the hyperplanes of type H_{13} and H_{31} to rectify bad intersections with the hyperplanes H_{21}.*

$$m_{ji} + m_{jk} - 1 \leq m_{ik} + m_{ki}.$$

Moreover, if one of these inequalities is actually an equality, then there is forced rigidity on the hyperplane arrangement. Specifically, consecutive hyperplanes are spaced at equal intervals for all hyperplanes of type H_{ij}, H_{ik}, H_{ji}, and H_{ki}. The only exception to this is the hyperplanes of type H_{jk} since they must utilize both hyperplanes H_{ik} and H_{ki} to rectify bad intersections with those of type H_{ji}.

Theorem 4.4. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 = m_{jk}$,
2. $m_{ki} < m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $m_{ij} > 1$, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Proof. In this instance of five edge multiplicities are non-zero, there is one situation up to a rearrangement of the vertices. Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, and $m_{ki} = v$. The assumption that $m_{ij} + m_{ik} - 1 = m_{jk}$ implies that there are just enough hyperplanes of the type H_{jk} to rectify all of the bad intersections between the hyperplanes of type H_{ij} and H_{ik}. Consider the following placement of the hyperplanes of type H_{ij}, H_{ik}, and H_{jk}.

$$\begin{aligned}
&H_{ij}^{a}, H_{ij}^{a+1}, \ldots, H_{ij}^{a+r-1}\\
&H_{ik}^{b}, H_{ik}^{b+1}, \ldots, H_{ik}^{b+s-1}\\
&H_{jk}^{b-a-r+1}, H_{jk}^{b-a-r+2}, \ldots, H_{jk}^{b-a}, \ldots, H_{jk}^{b+s-1-a}
\end{aligned}$$

Note that $b > a + r - 1 > 1$ since all of the bad intersections between the H_{ij} and H_{ik} hyperplanes must be rectified by a hyperplane of type H_{jk}. If this inequality did not hold, then one of the bad intersection points would be rectified by a hyperplane of type H_{kj}.

Since $m_{ji} > 0$, then there exists a $H_{ji}^{a'}$ such that its intersection with $H_{jk}^{b-a+s-1}$ is rectified by a hyperplane of type H_{ik}. Let such a hyperplane be $H_{ik}^{b+\alpha}$ for some $\alpha \in \{0, \ldots, s-1\}$.

One can solve for a' by

$$b - a + s - 1 - a' = b + \alpha \qquad \Leftrightarrow \qquad s - a - a' - 1 - \alpha = 0,$$

i.e. that $a' = s - a - 1 - \alpha > 0$. Once the hyperplane $H_{ji}^{a'}$ is placed, the bad intersections created when it intersects $H_{jk}^{b-a+s-1}, H_{jk}^{b-a+s-2}, \ldots, H_{jk}^{b-a+s-\alpha}$ are all rectified by H_{ik} hyperplanes, namely $H_{ik}^{b+\alpha}, H_{ik}^{b+\alpha-1}, \ldots, H_{ik}^{b}$.

However, the intersection with $H_{jk}^{b-a+s-\alpha-1}$ must be rectified by H_{ik}^{b-1}. Moreover, since $b > a + r - 1 > 1$ then $b - 1 > 0$, but $H_{ik}^{b-1} \notin \mathcal{A}$. Therefore there does not exist a bijective arrangement. $\square$

Example 4.5. *In this example we consider the graph G that has multiplicities $m_{12} = 2, m_{13} = 2, m_{23} = 3, m_{21} = 1$, and $m_{31} = 2$, see Figure 4.7 for two arrangements that correspond to the graph G.*

On the left, the hyperplanes are placed in a manner similar to how the hyperplanes in Section 4.1 however one is not able to rectify the bad intersection point between $H_{23}^{c_3}$ and $H_{21}^{a'}$. The highlighted regions correspond to the duplicate label $\langle 0, 3, 0\rangle$.

For the arrangement on the right, the hyperplanes are placed in a manner similar to Theorem 4.4. Recall that in this theorem we have that $m_{12} + m_{13} = m_{23}$ which forces the hyperplanes of type H_{12}, H_{13}, and H_{23} in the following grid.

Following the proof of Theorem 4.4, one sees that $H_{13}^{b_1}$ rectifies the intersection between $H_{23}^{c_3}$ and $H_{21}^{a'}$, however to rectify the next intersection, shown in red, one requires one more hyperplane of the type H_{13} to rectify the intersection. However, that is not possible and the only option to rectify the bad intersection is with the hyperplane $H_{31}^{b_1'}$ which can be done by letting $b_1' < 1$ which cannot happen due to the graph or let $b_1' = c_1 = a_1 = 0$ which also cannot happen since our hyperplanes are all affine.

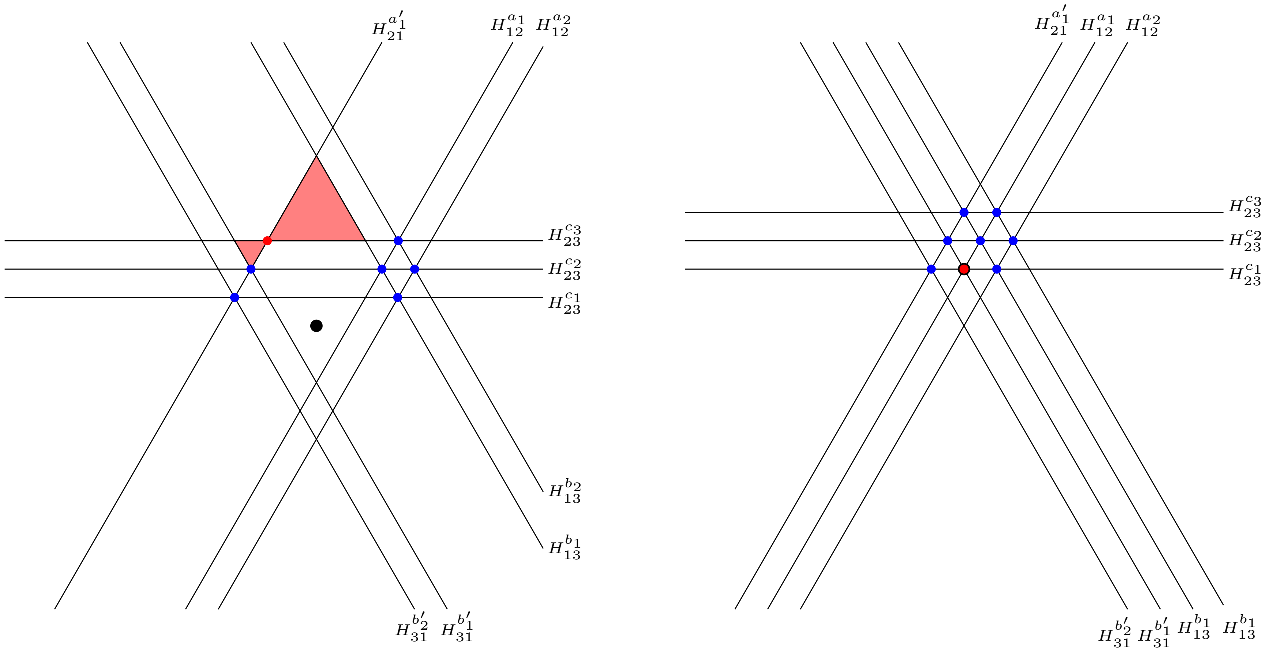


Figure 4.7: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 3$, $m_{21} = 1$, and $m_{31} = 2$. This graph satisfies Theorem 4.4.*

Theorem 4.6. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,

2. $m_{ji} + m_{jk} - 1 = m_{ik} + m_{ki}$.

Further, if $m_{ij}, m_{ji} > 1$, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Proof. In the instance of five edge multiplicities are non-zero, there is one situation up to a rearrangement of the vertices. Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, and $m_{ki} = v$. The assumption that $m_{ji} + m_{jk} - 1 = m_{ik} + m_{ki}$ implies that there are just enough hyperplanes of the types H_{ik} and H_{ki} to rectify all of the bad intersections between the hyperplanes of type H_{ji} and H_{jk}. Consider the following placement of the hyperplanes of type H_{ji} and H_{jk}:

$$H_{jk}^{c}, H_{jk}^{c+1}, \ldots, H_{jk}^{c+t-1}$$
$$H_{ji}^{a'}, H_{ji}^{a'+1}, \ldots, H_{ji}^{a'+u-1}.$$

Where c and a' are positive real numbers. Now, each of the intersections between the hyperplanes of type H_{jk} and H_{ji} need to be rectified by either a hyperplane of type H_{ik} and H_{ki}. Moreover, since the distance between consecutive hyperplanes of type H_{jk} (or H_{ji}) is one, then so is the distance between consecutive hyperplanes of type H_{ik} or H_{ki}, and the distance between the hyperplanes H_{ik} and H_{ki} that are closest to the origin. Consider the following placement of the hyperplanes of type H_{ik} and H_{ki}:

$$H_{ik}^{c-a'+t-1}, H_{ik}^{c-a'+t-2}, \ldots, H_{ik}^{c-a'+t-s},$$
$$H_{ki}^{a'-c+s-t+1}, H_{ki}^{a'-c+s-t+2}, \ldots, H_{ki}^{a'-c+u-1}.$$

For these hyperplanes, note that the last hyperplane H_{ik} placed is the closest to the origin, and that has each H_{ki} is placed they are placed in a manner that is moving away from the origin. Moreover, note that the distance between the closest H_{ik} and H_{ki} to the origin are distance one from each other. This implies that

$$1 > c - a' + t - s > 0.$$

Consider now the first hyperplane of type H_{ij}^{a}. When this hyperplane is placed it creates bad intersections between the hyperplanes of type H_{ik} which must be rectified by a hyperplane of type H_{jk}. More precisely, consider the intersection between

$$H_{ij}^{a} \text{ and } H_{ik}^{c-a'+t-s},$$

which must be rectified by the hyperplane $H_{jk}^{c+\alpha}$ for some $\alpha \in \{0, \ldots, t-1\}$. To find α, one must solve

$$c - a' + t - s - a = c + \alpha \qquad \Leftrightarrow \qquad t - s - a' - a - \alpha = 0.$$

This implies that $a = t - s - a' - \alpha > 0$ since $a > 0$. Now, $t - s - a' - \alpha > 0$ and $1 - c > t - s - a' > -c$ imply that the only α that guarantees $a > 0$ is $\alpha = 0$. However, $m_{ij} > 1$ so we need to place another H_{ij} hyperplane, but since $m_{kj} = 0$ we cannot place the any $H_{ij}^{a_2}$ with coefficient $a_2 > a$ since that would require a hyperplane of type H_{kj} to rectify the bad intersection it creates with $H_{ik}^{c-a'+t-s}$. Lastly, since $\alpha = 0$, it implies that $a_2 \not< a$. Therefore no bijective arrangement exists.

$\square$

Example 4.7. *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 1$, $m_{23} = 4$, $m_{21} = 3$, and $m_{31} = 5$; see Figure 4.8 for two arrangements that correspond to the graph G.*

On the left, the hyperplanes are placed in a manner similar to how the hyperplanes in the previous section would be placed, however one is not able to rectify the red or green intersection points. The highlighted regions correspond to the repeated labels $\langle 0, 4, 0\rangle$, $\langle 0, 5, 0\rangle$, $\langle 0, 6, 0\rangle$, and $\langle 1, 4, 0\rangle$ for regions red, orange, yellow, and brown respectively. Note, there is some freedom in the placement of $H_{23}^{c_3}$, $H_{23}^{c_4}$, and H_{13}^{b}, however one is not able to rectify all of the bad intersections.

Following the proof of Theorem 4.6, one sees that the intersection between $H_{23}^{c_1}$ and $H_{13}^{b_1}$ is utilized by the hyperplane $H_{12}^{a_2}$. Moreover, one sees that the hyperplane $H_{12}^{a_1}$ cannot be placed to the right of $H_{12}^{a_2}$, i.e. letting the coefficients satisfy $a_1 > a_2$. Therefore the remaining hyperplane must be placed to the right and utilize the hyperplane $H_{23}^{c_2}$ to rectify the bad intersection it creates when intersecting the hyperplane $H_{12}^{b_1}$. However, since the distance between the the hyperplanes of type H_{23} is one it forces $a_1 = a_2 - 1 < 0$. This cannot happen since the coefficient must be positive for the hyperplane to be of the type H_{12}.

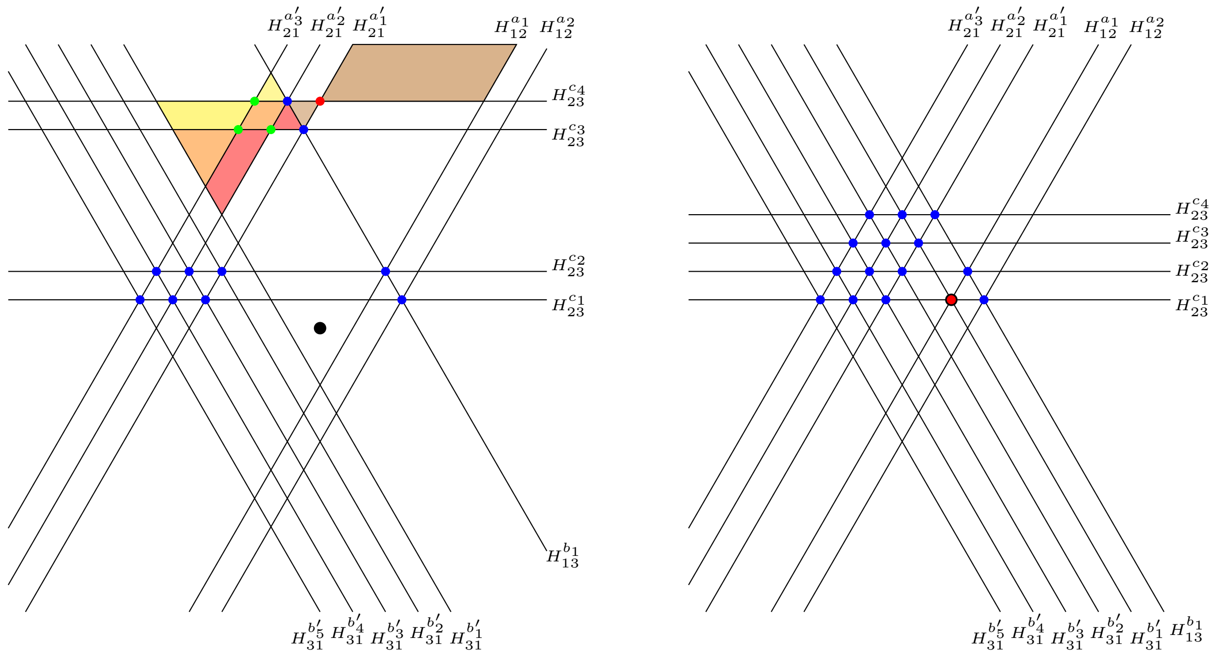


Figure 4.8: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 1$, $m_{23} = 4$, $m_{21} = 3$, and $m_{31} = 5$. This graph satisfies the conditions of Theorem 4.6.*

Example 4.8. *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 2$, and $m_{31} = 3$; see Figure 4.9 for two arrangements that correspond to the graph G.*

On the left, the hyperplanes are placed in a manner similar to how the hyperplanes in the previous section would be placed, however one is not able to rectify the red intersection points. The regions highlighted in red correspond to the repeated label $\langle 0, 3, 0\rangle$.

The arrangement on the right follows the proof of Theorem 4.6 where the intersection point between $H_{23}^{c_2}$ and $H_{13}^{b_1}$ is utilized by $H_{12}^{a_2}$. Further, one sees that the hyperplane $H_{12}^{a_1}$ cannot be placed with coefficient $a_1 > a_2$ since the intersection point between $H_{12}^{a_1}$ and $H_{13}^{b_1}$ would have to rectified by a hyperplane of type H_{31} which $\mathcal{A}$ does not have. Therefore $H_{12}^{a_1}$ must be placed with $a_1 < a_2$. However, similar to the previous example where we would have that $a_1 = a_2 - 1 < 0$.

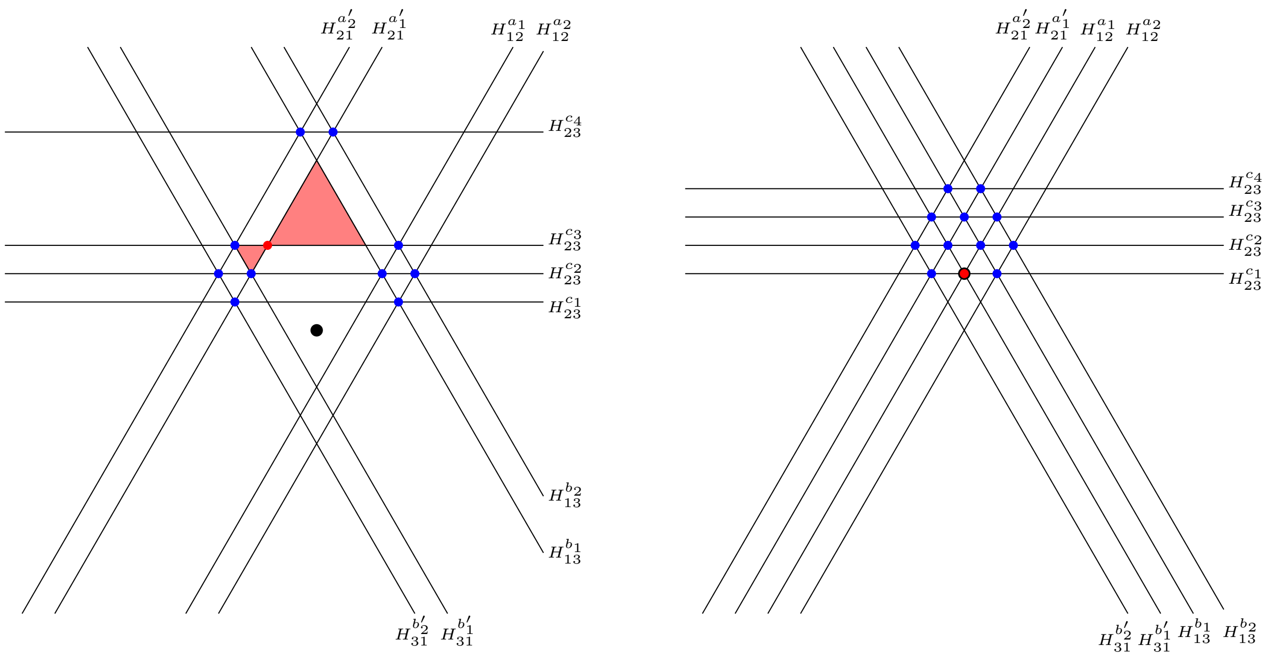


Figure 4.9: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 2$, and $m_{31} = 3$. This graph satisfies Theorem 4.6*

Example 4.9. *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 3$, $m_{23} = 7$, $m_{21} = 2$, and $m_{31} = 5$; see Figure 4.10 for two arrangements that correspond to the graph G.*

On the left, the hyperplanes are placed in a manner similar to how the hyperplanes in the previous section would be placed, however one is not able to rectify the red intersection points. The regions highlighted in red correspond to the repeated label $\langle 0, 6, 0\rangle$.

The arrangement on the right follows the proof of Theorem 4.6 where the intersection point between $H_{23}^{c_3}$ and $H_{13}^{b_3}$ is utilized by $H_{12}^{a_2}$. Further, one sees that the hyperplane $H_{12}^{a_1}$ cannot be placed with coefficient $a_1 > a_2$ since the intersection point between $H_{12}^{a_1}$ and $H_{13}^{b_1}$ would have to rectified by a hyperplane of type H_{31} which $\mathcal{A}$ does not have. Therefore $H_{12}^{a_1}$ must be placed with $a_1 < a_2$. However, similar to the previous example where we would have that $a_1 = a_2 - 1 < 0$.

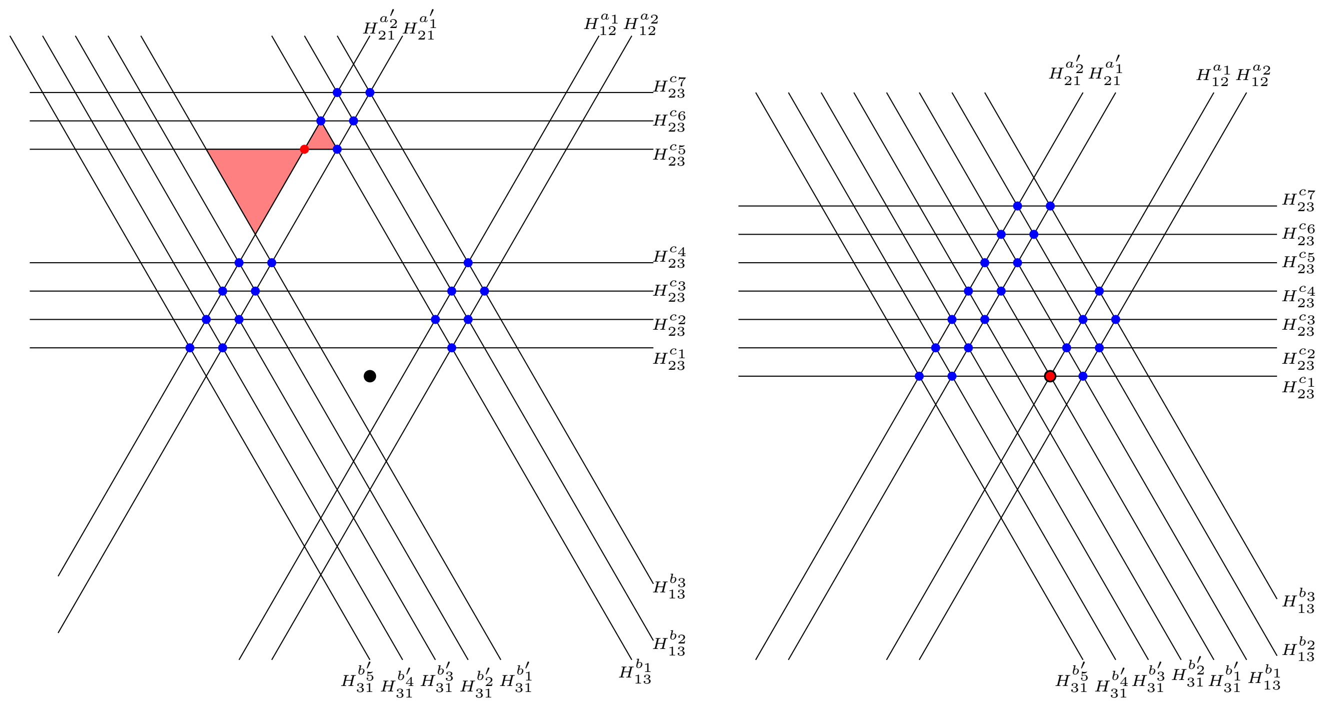


Figure 4.10: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 3$, $m_{23} = 7$, $m_{21} = 2$, and $m_{31} = 5$. This graph satisfies Theorem 4.6.*

4.3 Five Edge Type Conjectures

For the following conjecture, even though we have that $m_{ji} + m_{jk} - 1 = m_{ik} + m_{ki}$ which normally would require rigidity in the placement of the hyperplanes of type H_{ji} and H_{jk}, it is not the case for this family of graphs. Since $m_{ji} = 1$ for this family, then one is able to place the hyperplanes of type H_{jk} with any positive coefficient c.

Conjecture 4.10. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,
2. $m_{ji} + m_{jk} - 1 = m_{ik} + m_{ki}$,
3. $m_{ji} - 2 > m_{ki} - m_{ij} - m_{ik}$.

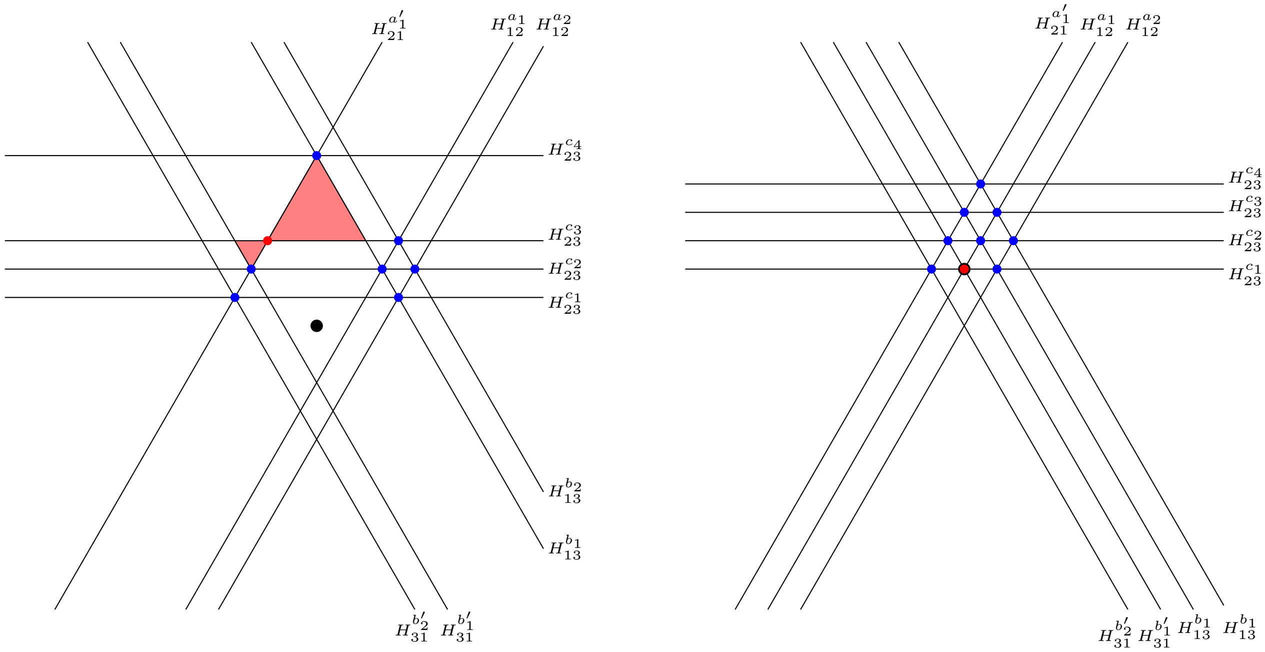


Figure 4.11: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph G has the multiplicities $m_{12} = 2$, $m_{13} = 2$, $m_{23} = 4$, $m_{21} = 1$, and $m_{31} = 2$. Similar to previous examples, the red intersection point is not rectified and the red regions correspond to the duplicate label $\langle 0, 3, 0 \rangle$. On the right is the same arrangement, but all points have been rectified. However, to rectify the remaining bad intersection, we collapsed the region containing the origin and had to let $a_1 = b'_1 = c_1 = 0$. This graph satisfies Conjecture 4.10.*

Further, if $m_{ij} > 1$ and $m_{ji} = 1$, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

See Figure 4.11 for the smallest example of a graph that satisfies the conditions of the previous conjecture.

For the following five conjectures the necessary conditions are strict inequalities

$$m_{ij} + m_{ik} - 1 < m_{jk} \quad \text{and} \quad m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}.$$

However, unlike in the previous section the arguments do not suffice since there is nothing that forces the spacing for these arrangements to be equally spaced.

Conjecture 4.11. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,

2. $m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $m_{ik} < m_{ji}$, $m_{ij} > 1$, and $m_{ji} + m_{jk} - 1 > m_{ki}$ is also satisfied, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Conjecture 4.12. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,

2. $m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $m_{ji} \leq m_{ik}$, $m_{ij} > 1$, and $m_{ji} + m_{jk} - 1 > m_{ki}$ is also satisfied, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Conjecture 4.13. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,

2. $m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $m_{ji} < m_{ik}$, $m_{ij} > 1$, $m_{ji} - 2 > m_{ki} - m_{ij} - m_{ik}$ and $m_{ji} + m_{jk} - 1 > m_{ki}$ is also satisfied, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Conjecture 4.14. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,

2. $m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $m_{ji} = m_{ik}$, $m_{ij} > 1$, and $m_{ji} + m_{jk} - 1 > m_{ki} + 1$ is also satisfied, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

Conjecture 4.15. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and the following inequalities*

1. $m_{ij} + m_{ik} - 1 < m_{jk}$,
2. $m_{ji} + m_{jk} - 1 < m_{ik} + m_{ki}$.

Further, if $1 < m_{ji} < m_{ik}$, $m_{ij} > 1$, and $m_{ji} + m_{jk} > m_{ij} + 2m_{ik}$ is also satisfied, then there does not exist an arrangement $\mathcal{A}_G$ with a bijective labeling.

The following theorem conjectures that in the case of five edge types that the theorems in Sections 4.1, 4.2, and 4.3 categorize all of the graphs where five edge multiplicities are nonzero. This conjecture is based on observations done in Sage, where for fixed number of edges, that each graph falls in one of the above families with no overlap. See Appendix A for the data through nineteen total edges.

Conjecture 4.16. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. If for $i, j, k \in V$ with m_{kj} being the only edge multiplicity that is zero, and G has a bijective labeling, then it satisfies either Theorem 4.1, 4.2, or 4.3.*

4.4 Six Edge Types with a Bijective Labeling

In this section we consider graphs where all six edge types are non-zero, and in this case there are three equations according to Theorem 3.4

1. $m_{ij} + m_{ik} - 1 \leq m_{jk} + m_{kj}$,
2. $m_{ji} + m_{jk} - 1 \leq m_{ik} + m_{ki}$,
3. $m_{ki} + m_{kj} - 1 \leq m_{ij} + m_{ji}$.

By our condition at least two of the inequalities must be strict. Similar to the case where five edge types were nonzero, the above conditions is necessary but not sufficient for a graph to have an arrangement with a bijective labeling. The following theorem is a unique family of graphs that produce an arrangement with a bijective labeling that has the property that all of the hyperplane are placed at the equal interval three.

Theorem 4.17. *Let $G = (V, E)$ be a graph on $n = 3$ vertices. For $i, j, k \in V$ with all non-zero edge multiplicities which obey the following inequalities*

1. $0 \leq m_{ij} - m_{kj} \leq 1$,
2. $0 \leq m_{jk} - m_{ik} \leq 1$,
3. $0 \leq m_{ki} - m_{ji} \leq 1$.

Then G there exists an arrangement $\mathcal{A}$ with bijective labeling such that $G = G = \mathcal{A}$.

Proof. In this instance of six edge multiplicities are non-zero, there is one situation up to a rearrangement of the vertices. Assume that the graph G has non-zero multiplicities $m_{ij} = r$, $m_{ik} = s$, $m_{jk} = t$, $m_{ji} = u$, $m_{ki} = v$, and $m_{kj} = w$. Consider the placement of the hyperplanes using the following coefficients

$$\begin{array}{lll} H_{ij}^{a_\alpha}, & \text{where } a_\alpha = 3\alpha - 2 & \text{for } \alpha \in \{1, \ldots, r\}, \\ H_{ik}^{b_\beta}, & \text{where } b_\beta = 3\beta - 1 & \text{for } \beta \in \{1, \ldots, s\}, \\ H_{jk}^{c_\gamma}, & \text{where } c_\gamma = 3\gamma - 2 & \text{for } \gamma \in \{1, \ldots, t\}, \\ H_{ji}^{a'_\delta}, & \text{where } a'_\delta = 3\delta - 1 & \text{for } \delta \in \{1, \ldots, u\}, \\ H_{ki}^{b'_\epsilon}, & \text{where } b'_\epsilon = 3\epsilon - 2 & \text{for } \epsilon \in \{1, \ldots, v\}, \\ H_{kj}^{c'_\zeta}, & \text{where } c'_\zeta = 3\zeta - 1 & \text{for } \zeta \in \{1, \ldots, w\}. \end{array}$$

Since the hyperplanes are all spaced at distance three, it suffices to check that the furthest bad intersections from origin are rectified for each family of bad intersections. Without loss of generality, consider the family of bad intersections that are created by hyperplanes of type H_{ij} and H_{ik} since the all families of bad intersections are checked in a similar manner.

To rectify the bad intersections for this family one must have the hyperplanes of type H_{jk} and H_{kj} have coefficients of the form $b-a$ or $a-b$, respectively, for some a and b as defined earlier. The first intersection we want to consider is between the hyperplane of type $H_{ij}^{a_1}$ and $H_{ik}^{b_s}$. It then follows that $1 < 3s-1$ since $s \geq 1$ that this intersection is rectified by a hyperplane of type H_{jk}, specifically it is rectified by the hyperplane with coefficient

$$H_{jk}^{c} \quad \text{where } c = b_s - a_1 = 3s-2.$$

Further, since $0 \leq t-s \leq 1$, then $3s-2 \leq 3t-2$ and it follows that there is a hyperplane with the coefficient to rectify the bad intersection.

By a similar argument we want to check that the intersection between the hyperplane of type $H_{ij}^{a_r}$ and $H_{ik}^{b_1}$ is also rectified. For this intersection we can have two cases, $s=1$ or $s>1$. In the case that $s=1$, then $a_s = a_1 = 1$ and $b_1 = 2$. This implies that the intersection must be rectified by the following hyperplane

$$H_{jk}^{c_1} \quad \text{where } c_1 = b_1 - a_1 = 1.$$

Since $t \neq 0$, then this hyperplane does exist and rectifies the bad intersection. If $s>1$, then the intersection must be rectified by the following hyperplane

$$H_{kj}^{c'} \quad \text{where } c' = a_s - b_1 = 3s-4.$$

However, since $0 \leq s-w \leq 1$, i.e. $s=w$ or $s-1=w$, then

$$3s-4 = 3(s-1)-1 \leq 3w-1.$$

Therefore there exists a hyperplane of type H_{kj} that rectifies this bad intersection. $\square$

Remark 4.18. In the case of Theorem 4.17 a choice was made on which group of hyperplanes would be placed closest to the origin, i.e. have a hyperplane whose coefficient is one. The two choices are H_{ij}, H_{jk}, H_{ki}, or H_{kj}, H_{ik}, H_{ji}. For Theorem 4.17, H_{ij}, H_{jk}, and H_{ki} where chosen, however one is able to place the second group first by just switching the coefficients between the pairs, i.e. switch the coefficients for H_{ij} and H_{kj}, H_{jk} and H_{ik}, and H_{ki} and H_{ji}.

By making the switch to H_{kj}, H_{ik}, H_{ji}, one is able to create a bijective arrangement for another graph family that has multiplicities $m_{ij}+1 = m_{kj}$, $m_{jk}+1 = m_{ik}$, and $m_{ki}+1 = m_{ji}$. This is done in a similar manner to the previous theorem.

The following graph is the smallest example of a graph that does not satisfy Theorem 4.17 or Remark 4.18, but still produces an arrangement with a bijective labeling.

Example 4.19. *Consider the graph $G = (V, E)$ with edge multiplicities given by $m_{12} = 3$, $m_{13} = 1$, $m_{23} = 2$, $m_{21} = 1$, $m_{31} = 2$, and $m_{32} = 1$. Despite not satisfying Theorem 4.17, the graph G does emit an arrangement with a bijective label. See Figure 4.13 for the bijective arrangement $\mathcal{A}_G$.*

4.5 Six Edge Types with a Non-Bijective Labeling

The following graph is the smallest example of a graph that does not satisfy Theorem 4.17 or Remark 4.18, and does not produce an arrangement with a bijective labeling.

Example 4.20. *Consider the graph $G = (V, E)$ with edge multiplicities given by $m_{12} = 3$, $m_{13} = 2$, $m_{23} = 3$, $m_{21} = 1$, $m_{31} = 2$, and $m_{32} = 1$. Even though this graph satisfies the necessary conditions of Theorem 3.4, it fails to emit an injective Pak-Stanley labeling.*

To show this, assume for the sake of contradiction that $\mathcal{A}$ is a bijective arrangement corresponding to G. Assume without loss of generality that $a_1 < a_2 < a_3$ and $b_1 < b_2$. Since $m_{ij} + m_{ik} - 1 = m_{jk} + m_{kj}$ we know that the hyperplanes of type H_{ij} and H_{ik} must be placed at equal intervals, i.e. $a_2 = a_1 + \alpha$ and $a_3 = a_2 + \alpha$ for some $\alpha > 0$. Further, since $m_{kj} = 1$,

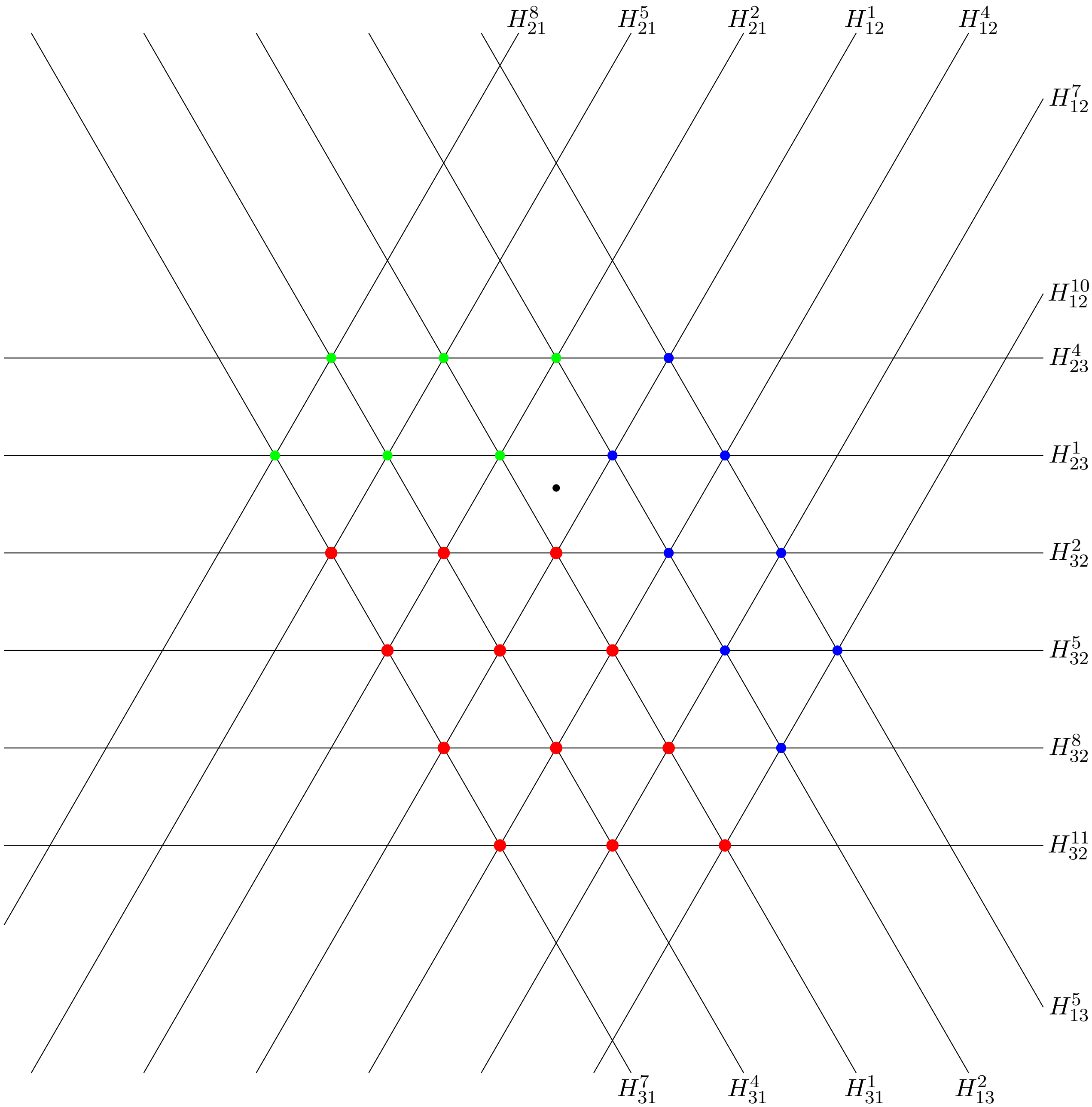


Figure 4.12: *In this example we see a bijective arrangement $\mathcal{A}_G$ where the graph has the multiplicities $m_{12} = m_{32} = 4$, $m_{13} = m_{23} = 2$, and $m_{21} = m_{31} = 3$. Further, in this example the blue, green, and red intersection points represent rectified points for hyperplanes H_{12} and H_{13}, H_{21} and H_{23}, and H_{31} and H_{32} respectively.*

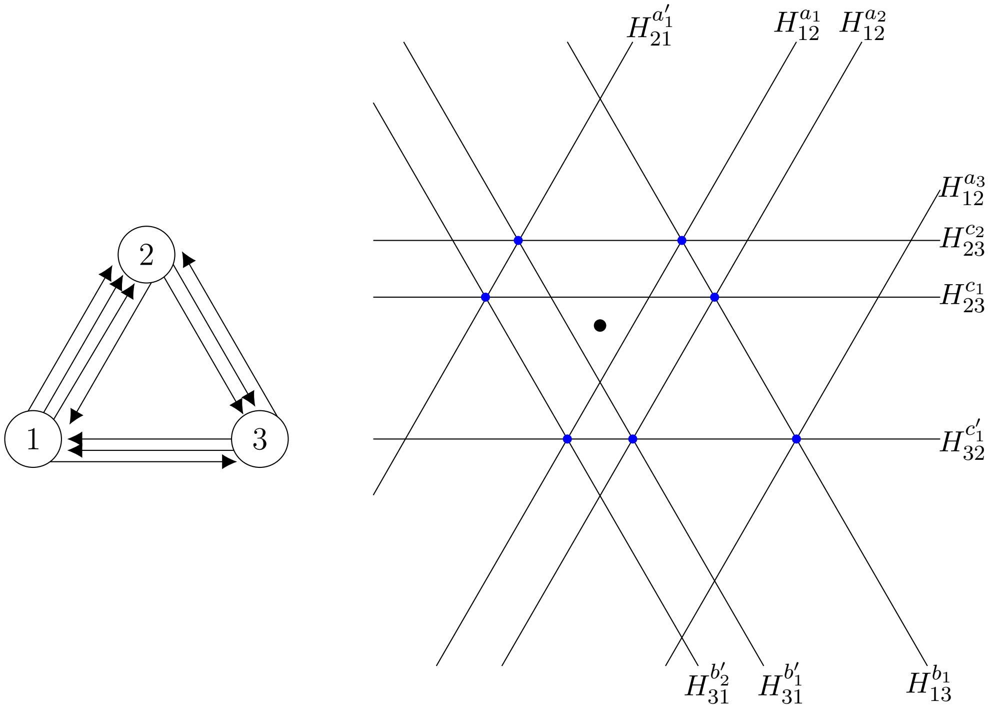


Figure 4.13: *In this example we see an a graph on three vertices with no non-zero edge multiplicities. The graph G, seen on the left, does not satisfy the conditions of Theorems 4.17, but still admits an arrangement with bijective labels. In the arrangement on the right, all potential bad intersections, shown in blue, have been rectified.*

then the coefficients must also obey the following condition

$$a_1 < a_2 < b_1 < a_3 < b_2.$$

To rectify the bad intersections between the hyperplanes of types H_{ij} and H_{ik}, all of the hyperplanes of type H_{jk} and H_{kj} must be used. Further, they must be placed in the following way:

$$c_3 = b_2 - a_1,$$

$$c_2 = b_2 - a_2,$$

$$c_1 = b_2 - a_3,$$

$$c_1' = a_3 - b_1.$$

Now, the next family of bad intersections is between those of type H_{ji} and H_{jk} which must be rectified with either a hyperplane of type H_{ik} or H_{ki}. However, since

$$m_{ji} + m_{jk} - 1 = 3 < 4 = m_{ik} + m_{ki},$$

we have two choices, either we use all of the hyperplanes of type H_{ki} or all the hyperplanes of type H_{ik} to rectify the the bad intersections. By an argument similar to Theorem 3.4 we know that we cannot use both H_{ik} hyperplanes to rectify the bad intersections. Therefore we must use both hyperplanes of type H_{ki} and hyperplane $H_{ik}^{b_1}$. Therefore the hyperplane H_{ji} must utilize the intersection point between $H_{ik}^{b_1}$ and $H_{jk}^{c_3}$, and that forces $H_{ji}^{a'}$ to be placed by $a' = c_3 - b_1$. This in turn forces the hyperplanes type H_{ki} to be placed by the coefficients

$$b'_1 = a' - c_2 \qquad \textit{and} \qquad b'_2 = a' - c_1.$$

Now, consider the following

$$\begin{aligned}
b'_1 &= a' - c_2 & \\
&= a' - b_2 + a_2 & \textit{Substitute } c_2 = b_2 - a_2 \\
&= c_3 - b_1 - b_2 + a_2 & \textit{Substitute } a' = c_3 - b_1 \\
&= b_2 - a_1 - b_1 - b_2 + a_2 & \textit{Substitute } c_3 = b_2 - a_1 \\
&= -a_1 - b_1 + a_2. &
\end{aligned}$$

However, since $a_2 < b_1$, then $b'_1 = -a_1 - b_1 + a_2 < 0$ which cannot happen since the coefficients must be positive. Therefore no such arrangement exists.

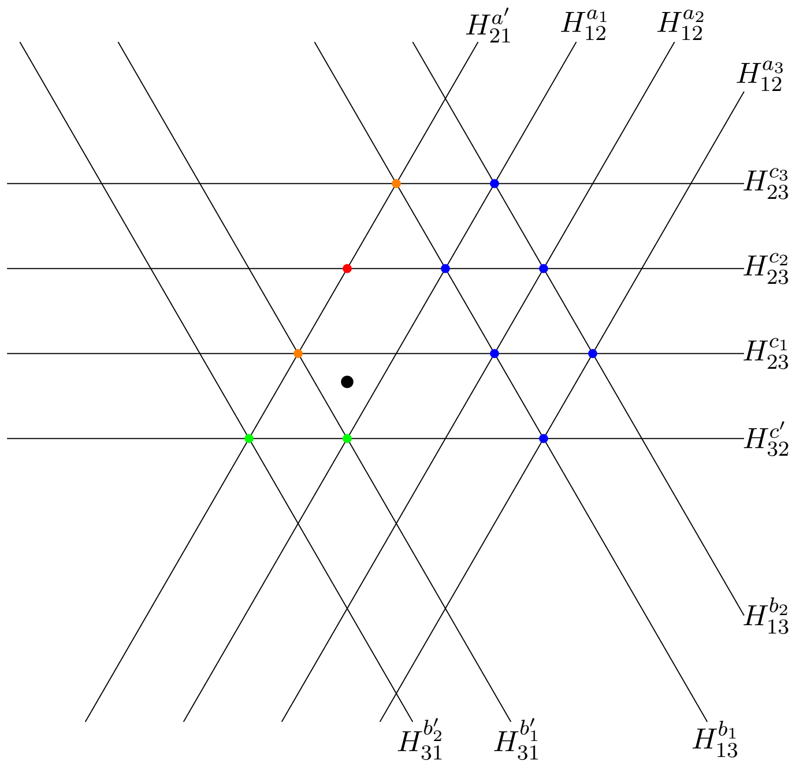


Figure 4.14: *In this example we see a non-bijective arrangement $\mathcal{A}_G$ where the graph has the multiplicities $m_{12} = 3$, $m_{13} = 2$, $m_{23} = 3$, $m_{21} = 1$, $m_{31} = 2$ and $m_{32} = 1$. Further, in this example the blue, orange , and green intersection points represent rectified points for hyperplanes H_{12} and H_{13}, H_{21} and H_{23}, and H_{31} and H_{32} respectively. However, notice that the red intersection point between hyperplanes $H_{23}^{c_2}$ and $H_{21}^{a'}$ cannot be rectified. One can see this in the following way, since $m_{12} + m_{13} - 1 = m_{23} + m_{32}$, then all of the hyperplanes are locked into a rigid lattice, i.e. all the hyperplanes are spaced equally. This in turn forces the remaining hyperplanes to follow. Therefore the only option that one can use to rectify the red intersection is to shift the hyperplanes of type H_{31}. However, if one does this we must have that $b'_1 < 0$ which cannot happen since b' must be positive.*

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Appendix A

Five Edge Types Classification Tables

This appendix contains a table relating to the case of five edge types are chosen where one is given the total number of edges in the graph, the edge multiplicities, and the theorem that tells us whether the graph has an arrangement with a bijective label or not. Note, the m_{32} is not listed and is assumed to be zero. Note that after eight hyperplanes, the list does not include those with $m_{12} = 1$.

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
5	1	1	1	1	1	Theorem 4.2	
6	1	1	1	1	2	Theorem 4.2	
6	1	1	2	1	1	Theorem 4.2	
7	1	1	1	1	3	Theorem 4.2	
7	1	1	1	2	2	Theorem 4.2	
7	1	1	2	1	2	Theorem 4.2	
7	1	2	2	1	1	Theorem 4.2	
8	1	1	1	1	4	Theorem 4.2	
8	1	1	1	2	3	Theorem 4.2	
8	1	1	2	1	3	Theorem 4.2	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
8	1	1	2	2	2	Theorem 4.2	
8	1	1	3	1	2	Theorem 4.2	
8	1	2	2	1	2	Theorem 4.2	
8	1	2	3	1	1	Theorem 4.2	
8	2	1	2	1	2	Theorem 4.1	
9	2	1	2	1	3	Theorem 4.1	
9	2	1	3	1	2	Theorem 4.3	
10	2	1	2	1	4	Theorem 4.1	
10	2	1	2	2	3	Theorem 4.1	
10	2	1	3	1	3	Theorem 4.1	
10	2	2	3	1	2		Theorem 4.4
11	2	1	2	1	5	Theorem 4.1	
11	2	1	2	2	4	Theorem 4.1	
11	2	1	3	1	4	Theorem 4.1	
11	2	1	3	2	3		Theorem 4.6
11	2	1	4	1	3	Theorem 4.3	
11	2	2	3	1	3	Theorem 4.1	
11	2	2	4	1	2		Conjecture 4.10
11	3	1	3	1	3	Theorem 4.1	
12	2	1	2	1	6	Theorem 4.1	
12	2	1	2	2	5	Theorem 4.1	
12	2	1	2	3	4	Theorem 4.1	
12	2	1	3	1	5	Theorem 4.1	
12	2	1	3	2	4	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
12	2	1	4	1	4	Theorem 4.1	
12	2	2	3	1	4	Theorem 4.1	
12	2	2	3	2	3		Theorem 4.4
12	2	2	4	1	3	Theorem 4.3	
12	2	3	4	1	2		Theorem 4.4
12	3	1	3	1	4	Theorem 4.1	
12	3	1	4	1	3	Theorem 4.3	
13	2	1	2	1	7	Theorem 4.1	
13	2	1	2	2	6	Theorem 4.1	
13	2	1	2	3	5	Theorem 4.1	
13	2	1	3	1	6	Theorem 4.1	
13	2	1	3	2	5	Theorem 4.1	
13	2	1	3	3	4		Theorem 4.6
13	2	1	4	1	5	Theorem 4.1	
13	2	1	4	2	4		Theorem 4.6
13	2	1	5	1	4	Theorem 4.3	
13	2	2	3	1	5	Theorem 4.1	
13	2	2	3	2	4	Theorem 4.1	
13	2	2	4	1	4	Theorem 4.1	
13	2	2	4	2	3		Theorem 4.6
13	2	2	5	1	3	Theorem 4.3	
13	2	3	4	1	3		Theorem 4.4
13	2	3	5	1	2		Conjecture 4.10
13	3	1	3	1	5	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
13	3	1	3	2	4	Theorem 4.1	
13	3	1	4	1	4	Theorem 4.1	
13	3	2	4	1	3		Theorem 4.4
14	2	1	2	1	8	Theorem 4.1	
14	2	1	2	2	7	Theorem 4.1	
14	2	1	2	3	6	Theorem 4.1	
14	2	1	2	4	5	Theorem 4.1	
14	2	1	3	1	7	Theorem 4.1	
14	2	1	3	2	6	Theorem 4.1	
14	2	1	3	3	5	Theorem 4.1	
14	2	1	4	1	6	Theorem 4.1	
14	2	1	4	2	5	Theorem 4.1	
14	2	1	5	1	5	Theorem 4.1	
14	2	2	3	1	6	Theorem 4.1	
14	2	2	3	2	5	Theorem 4.1	
14	2	2	3	3	4		Theorem 4.4
14	2	2	4	1	5	Theorem 4.1	
14	2	2	4	2	4	Theorem 4.3	
14	2	2	5	1	4	Theorem 4.3	
14	2	3	4	1	4	Theorem 4.1	
14	2	3	4	2	3		Theorem 4.4
14	2	3	5	1	3		Conjecture 4.13
14	2	4	5	1	2		Theorem 4.4
14	3	1	3	1	6	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
14	3	1	3	2	5	Theorem 4.1	
14	3	1	4	1	5	Theorem 4.1	
14	3	1	4	2	4		Theorem 4.6
14	3	1	5	1	4	Theorem 4.3	
14	3	2	4	1	4	Theorem 4.1	
14	3	2	5	1	3		Conjecture 4.10
14	4	1	4	1	4	Theorem 4.1	
15	2	1	2	1	9	Theorem 4.1	
15	2	1	2	2	8	Theorem 4.1	
15	2	1	2	3	7	Theorem 4.1	
15	2	1	2	4	6	Theorem 4.1	
15	2	1	3	1	8	Theorem 4.1	
15	2	1	3	2	7	Theorem 4.1	
15	2	1	3	3	6	Theorem 4.1	
15	2	1	3	4	5		Theorem 4.6
15	2	1	4	1	7	Theorem 4.1	
15	2	1	4	2	6	Theorem 4.1	
15	2	1	4	3	5		Theorem 4.6
15	2	1	5	1	6	Theorem 4.1	
15	2	1	5	2	5		Theorem 4.6
15	2	1	6	1	5	Theorem 4.3	
15	2	2	3	1	7	Theorem 4.1	
15	2	2	3	2	6	Theorem 4.1	
15	2	2	3	3	5	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
15	2	2	4	1	6	Theorem 4.1	
15	2	2	4	2	5	Theorem 4.1	
15	2	2	4	3	4		Theorem 4.6
15	2	2	5	1	5	Theorem 4.1	
15	2	2	5	2	4		Theorem 4.6
15	2	2	6	1	4	Theorem 4.3	
15	2	3	4	1	5	Theorem 4.1	
15	2	3	4	2	4		Theorem 4.4
15	2	3	5	1	4	Theorem 4.3	
15	2	3	5	2	3		Theorem 4.6
15	2	3	6	1	3		Conjecture 4.10
15	2	4	5	1	3		Theorem 4.4
15	2	4	6	1	2		Conjecture 4.10
15	3	1	3	1	7	Theorem 4.1	
15	3	1	3	2	6	Theorem 4.1	
15	3	1	3	3	5	Theorem 4.1	
15	3	1	4	1	6	Theorem 4.1	
15	3	1	4	2	5	Theorem 4.1	
15	3	1	5	1	5	Theorem 4.1	
15	3	2	4	1	5	Theorem 4.1	
15	3	2	4	2	4		Theorem 4.4
15	3	2	5	1	4	Theorem 4.3	
15	3	3	5	1	3		Theorem 4.4
15	4	1	4	1	5	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
15	4	1	5	1	4	Theorem 4.3	
16	2	1	2	1	10	Theorem 4.1	
16	2	1	2	2	9	Theorem 4.1	
16	2	1	2	3	8	Theorem 4.1	
16	2	1	2	4	7	Theorem 4.1	
16	2	1	2	5	6	Theorem 4.1	
16	2	1	3	1	9	Theorem 4.1	
16	2	1	3	2	8	Theorem 4.1	
16	2	1	3	3	7	Theorem 4.1	
16	2	1	3	4	6	Theorem 4.1	
16	2	1	4	1	8	Theorem 4.1	
16	2	1	4	2	7	Theorem 4.1	
16	2	1	4	3	6	Theorem 4.1	
16	2	1	5	1	7	Theorem 4.1	
16	2	1	5	2	6	Theorem 4.1	
16	2	1	6	1	6	Theorem 4.1	
16	2	2	3	1	8	Theorem 4.1	
16	2	2	3	2	7	Theorem 4.1	
16	2	2	3	3	6	Theorem 4.1	
16	2	2	3	4	5		Theorem 4.4
16	2	2	4	1	7	Theorem 4.1	
16	2	2	4	2	6	Theorem 4.1	
16	2	2	4	3	5		Conjecture 4.11
16	2	2	5	1	6	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
16	2	2	5	2	5	Theorem 4.3	
16	2	2	6	1	5	Theorem 4.3	
16	2	3	4	1	6	Theorem 4.1	
16	2	3	4	2	5	Theorem 4.1	
16	2	3	4	3	4		Theorem 4.4
16	2	3	5	1	5	Theorem 4.1	
16	2	3	5	2	4		Conjecture 4.13
16	2	3	6	1	4	Theorem 4.3	
16	2	4	5	1	4		Theorem 4.4
16	2	4	5	2	3		Theorem 4.4
16	2	4	6	1	3		Conjecture 4.13
16	2	5	6	1	2		Theorem 4.4
16	3	1	3	1	8	Theorem 4.1	
16	3	1	3	2	7	Theorem 4.1	
16	3	1	3	3	6	Theorem 4.1	
16	3	1	4	1	7	Theorem 4.1	
16	3	1	4	2	6	Theorem 4.1	
16	3	1	4	3	5		Theorem 4.6
16	3	1	5	1	6	Theorem 4.1	
16	3	1	5	2	5		Theorem 4.6
16	3	1	6	1	5	Theorem 4.3	
16	3	2	4	1	6	Theorem 4.1	
16	3	2	4	2	5	Theorem 4.1	
16	3	2	5	1	5	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
16	3	2	5	2	4		Theorem 4.6
16	3	2	6	1	4	Theorem 4.3	
16	3	3	5	1	4		Theorem 4.4
16	3	3	6	1	3		Conjecture 4.10
16	4	1	4	1	6	Theorem 4.1	
16	4	1	4	2	5	Theorem 4.1	
16	4	1	5	1	5	Theorem 4.1	
16	4	2	5	1	4		Theorem 4.4
17	2	1	2	1	11	Theorem 4.1	
17	2	1	2	2	10	Theorem 4.1	
17	2	1	2	3	9	Theorem 4.1	
17	2	1	2	4	8	Theorem 4.1	
17	2	1	2	5	7	Theorem 4.1	
17	2	1	3	1	10	Theorem 4.1	
17	2	1	3	2	9	Theorem 4.1	
17	2	1	3	3	8	Theorem 4.1	
17	2	1	3	4	7	Theorem 4.1	
17	2	1	3	5	6		Theorem 4.6
17	2	1	4	1	9	Theorem 4.1	
17	2	1	4	2	8	Theorem 4.1	
17	2	1	4	3	7	Theorem 4.1	
17	2	1	4	4	6		Theorem 4.6
17	2	1	5	1	8	Theorem 4.1	
17	2	1	5	2	7	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
17	2	1	5	3	6		Theorem 4.6
17	2	1	6	1	7	Theorem 4.1	
17	2	1	6	2	6		Theorem 4.6
17	2	1	7	1	6	Theorem 4.3	
17	2	2	3	1	9	Theorem 4.1	
17	2	2	3	2	8	Theorem 4.1	
17	2	2	3	3	7	Theorem 4.1	
17	2	2	3	4	6	Theorem 4.1	
17	2	2	4	1	8	Theorem 4.1	
17	2	2	4	2	7	Theorem 4.1	
17	2	2	4	3	6	Theorem 4.1	
17	2	2	4	4	5		Theorem 4.6
17	2	2	5	1	7	Theorem 4.1	
17	2	2	5	2	6	Theorem 4.1	
17	2	2	5	3	5		Theorem 4.6
17	2	2	6	1	6	Theorem 4.1	
17	2	2	6	2	5		Theorem 4.6
17	2	2	7	1	5	Theorem 4.3	
17	2	3	4	1	7	Theorem 4.1	
17	2	3	4	2	6	Theorem 4.1	
17	2	3	4	3	5		Theorem 4.4
17	2	3	5	1	6	Theorem 4.1	
17	2	3	5	2	5	Theorem 4.3	
17	2	3	5	3	4		Theorem 4.4

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
17	2	3	6	1	5	Theorem 4.3	
17	2	3	6	2	4		Theorem 4.6
17	2	3	7	1	4	Theorem 4.3	
17	2	4	5	1	5	Theorem 4.1	
17	2	4	5	2	4		Theorem 4.4
17	2	4	6	1	4		Conjecture 4.13
17	2	4	6	2	3		Theorem 4.6
17	2	4	7	1	3		Conjecture 4.10
17	2	5	6	1	3		Theorem 4.4
17	2	5	7	1	2		Conjecture 4.10
17	3	1	3	1	9	Theorem 4.1	
17	3	1	3	2	8	Theorem 4.1	
17	3	1	3	3	7	Theorem 4.1	
17	3	1	3	4	6	Theorem 4.1	
17	3	1	4	1	8	Theorem 4.1	
17	3	1	4	2	7	Theorem 4.1	
17	3	1	4	3	6	Theorem 4.1	
17	3	1	5	1	7	Theorem 4.1	
17	3	1	5	2	6	Theorem 4.1	
17	3	1	6	1	6	Theorem 4.1	
17	3	2	4	1	7	Theorem 4.1	
17	3	2	4	2	6	Theorem 4.1	
17	3	2	4	3	5		Theorem 4.4
17	3	2	5	1	6	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
17	3	2	5	2	5	Theorem 4.3	
17	3	2	6	1	5	Theorem 4.3	
17	3	3	5	1	5	Theorem 4.1	
17	3	3	5	2	4		Theorem 4.4
17	3	3	6	1	4		Conjecture 4.13
17	3	4	6	1	3		Theorem 4.4
17	4	1	4	1	7	Theorem 4.1	
17	4	1	4	2	6	Theorem 4.1	
17	4	1	5	1	6	Theorem 4.1	
17	4	1	5	2	5		Theorem 4.6
17	4	1	6	1	5	Theorem 4.3	
17	4	2	5	1	5	Theorem 4.1	
17	4	2	6	1	4		Conjecture 4.10
17	5	1	5	1	5	Theorem 4.1	
18	2	1	2	1	12	Theorem 4.1	
18	2	1	2	2	11	Theorem 4.1	
18	2	1	2	3	10	Theorem 4.1	
18	2	1	2	4	9	Theorem 4.1	
18	2	1	2	5	8	Theorem 4.1	
18	2	1	2	6	7	Theorem 4.1	
18	2	1	3	1	11	Theorem 4.1	
18	2	1	3	2	10	Theorem 4.1	
18	2	1	3	3	9	Theorem 4.1	
18	2	1	3	4	8	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
18	2	1	3	5	7	Theorem 4.1	
18	2	1	4	1	10	Theorem 4.1	
18	2	1	4	2	9	Theorem 4.1	
18	2	1	4	3	8	Theorem 4.1	
18	2	1	4	4	7	Theorem 4.1	
18	2	1	5	1	9	Theorem 4.1	
18	2	1	5	2	8	Theorem 4.1	
18	2	1	5	3	7	Theorem 4.1	
18	2	1	6	1	8	Theorem 4.1	
18	2	1	6	2	7	Theorem 4.1	
18	2	1	7	1	7	Theorem 4.1	
18	2	2	3	1	10	Theorem 4.1	
18	2	2	3	2	9	Theorem 4.1	
18	2	2	3	3	8	Theorem 4.1	
18	2	2	3	4	7	Theorem 4.1	
18	2	2	3	5	6		Theorem 4.4
18	2	2	4	1	9	Theorem 4.1	
18	2	2	4	2	8	Theorem 4.1	
18	2	2	4	3	7	Theorem 4.1	
18	2	2	4	4	6		Conjecture 4.11
18	2	2	5	1	8	Theorem 4.1	
18	2	2	5	2	7	Theorem 4.1	
18	2	2	5	3	6		Conjecture 4.11
18	2	2	6	1	7	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
18	2	2	6	2	6	Theorem 4.3	
18	2	2	7	1	6	Theorem 4.3	
18	2	3	4	1	8	Theorem 4.1	
18	2	3	4	2	7	Theorem 4.1	
18	2	3	4	3	6	Theorem 4.1	
18	2	3	4	4	5		Theorem 4.4
18	2	3	5	1	7	Theorem 4.1	
18	2	3	5	2	6	Theorem 4.1	
18	2	3	5	3	5		Conjecture 4.14
18	2	3	6	1	6	Theorem 4.1	
18	2	3	6	2	5	Theorem 4.3	
18	2	3	7	1	5	Theorem 4.3	
18	2	4	5	1	6	Theorem 4.1	
18	2	4	5	2	5		Theorem 4.4
18	2	4	5	3	4		Theorem 4.4
18	2	4	6	1	5	Theorem 4.3	
18	2	4	6	2	4		Conjecture 4.13
18	2	4	7	1	4		Conjecture 4.13
18	2	5	6	1	4		Theorem 4.4
18	2	5	6	2	3		Theorem 4.4
18	2	5	7	1	3		Conjecture 4.13
18	2	6	7	1	2		Theorem 4.4
18	3	1	3	1	10	Theorem 4.1	
18	3	1	3	2	9	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
18	3	1	3	3	8	Theorem 4.1	
18	3	1	3	4	7	Theorem 4.1	
18	3	1	4	1	9	Theorem 4.1	
18	3	1	4	2	8	Theorem 4.1	
18	3	1	4	3	7	Theorem 4.1	
18	3	1	4	4	6		Theorem 4.6
18	3	1	5	1	8	Theorem 4.1	
18	3	1	5	2	7	Theorem 4.1	
18	3	1	5	3	6		Theorem 4.6
18	3	1	6	1	7	Theorem 4.1	
18	3	1	6	2	6		Theorem 4.6
18	3	1	7	1	6	Theorem 4.3	
18	3	2	4	1	8	Theorem 4.1	
18	3	2	4	2	7	Theorem 4.1	
18	3	2	4	3	6	Theorem 4.1	
18	3	2	5	1	7	Theorem 4.1	
18	3	2	5	2	6	Theorem 4.1	
18	3	2	5	3	5		Theorem 4.6
18	3	2	6	1	6	Theorem 4.1	
18	3	2	6	2	5		Theorem 4.6
18	3	2	7	1	5	Theorem 4.3	
18	3	3	5	1	6	Theorem 4.1	
18	3	3	5	2	5		Theorem 4.4
18	3	3	6	1	5	Theorem 4.3	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
18	3	3	6	2	4		Theorem 4.6
18	3	3	7	1	4		Conjecture 4.10
18	3	4	6	1	4		Theorem 4.4
18	3	4	7	1	3		Conjecture 4.10
18	4	1	4	1	8	Theorem 4.1	
18	4	1	4	2	7	Theorem 4.1	
18	4	1	4	3	6	Theorem 4.1	
18	4	1	5	1	7	Theorem 4.1	
18	4	1	5	2	6	Theorem 4.1	
18	4	1	6	1	6	Theorem 4.1	
18	4	2	5	1	6	Theorem 4.1	
18	4	2	5	2	5		Theorem 4.4
18	4	2	6	1	5	Theorem 4.3	
18	4	3	6	1	4		Theorem 4.4
18	5	1	5	1	6	Theorem 4.1	
18	5	1	6	1	5	Theorem 4.3	
19	2	1	2	1	13	Theorem 4.1	
19	2	1	2	2	12	Theorem 4.1	
19	2	1	2	3	11	Theorem 4.1	
19	2	1	2	4	10	Theorem 4.1	
19	2	1	2	5	9	Theorem 4.1	
19	2	1	2	6	8	Theorem 4.1	
19	2	1	3	1	12	Theorem 4.1	
19	2	1	3	2	11	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
19	2	1	3	3	10	Theorem 4.1	
19	2	1	3	4	9	Theorem 4.1	
19	2	1	3	5	8	Theorem 4.1	
19	2	1	3	6	7		Theorem 4.6
19	2	1	4	1	11	Theorem 4.1	
19	2	1	4	2	10	Theorem 4.1	
19	2	1	4	3	9	Theorem 4.1	
19	2	1	4	4	8	Theorem 4.1	
19	2	1	4	5	7		Theorem 4.6
19	2	1	5	1	10	Theorem 4.1	
19	2	1	5	2	9	Theorem 4.1	
19	2	1	5	3	8	Theorem 4.1	
19	2	1	5	4	7		Theorem 4.6
19	2	1	6	1	9	Theorem 4.1	
19	2	1	6	2	8	Theorem 4.1	
19	2	1	6	3	7		Theorem 4.6
19	2	1	7	1	8	Theorem 4.1	
19	2	1	7	2	7		Theorem 4.6
19	2	1	8	1	7	Theorem 4.3	
19	2	2	3	1	11	Theorem 4.1	
19	2	2	3	2	10	Theorem 4.1	
19	2	2	3	3	9	Theorem 4.1	
19	2	2	3	4	8	Theorem 4.1	
19	2	2	3	5	7	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
19	2	2	4	1	10	Theorem 4.1	
19	2	2	4	2	9	Theorem 4.1	
19	2	2	4	3	8	Theorem 4.1	
19	2	2	4	4	7	Theorem 4.1	
19	2	2	4	5	6		Theorem 4.6
19	2	2	5	1	9	Theorem 4.1	
19	2	2	5	2	8	Theorem 4.1	
19	2	2	5	3	7	Theorem 4.1	
19	2	2	5	4	6		Theorem 4.6
19	2	2	6	1	8	Theorem 4.1	
19	2	2	6	2	7	Theorem 4.1	
19	2	2	6	3	6		Theorem 4.6
19	2	2	7	1	7	Theorem 4.1	
19	2	2	7	2	6		Theorem 4.6
19	2	2	8	1	6	Theorem 4.3	
19	2	3	4	1	9	Theorem 4.1	
19	2	3	4	2	8	Theorem 4.1	
19	2	3	4	3	7	Theorem 4.1	
19	2	3	4	4	6		Theorem 4.4
19	2	3	5	1	8	Theorem 4.1	
19	2	3	5	2	7	Theorem 4.1	
19	2	3	5	3	6	Theorem 4.3	
19	2	3	5	4	5		Theorem 4.6
19	2	3	6	1	7	Theorem 4.1	

#H	m_{12}	m_{13}	m_{23}	m_{21}	m_{31}	Injective	Not Injective
19	2	3	6	2	6	Theorem 4.3	
19	2	3	6	3	5		Theorem 4.6
19	2	3	7	1	6	Theorem 4.3	
19	2	3	7	2	5		Theorem 4.6
19	2	3	8	1	5	Theorem 4.3	
19	2	4	5	1	7	Theorem 4.1	
19	2	4	5	2	6	Theorem 4.1	
19	2	4	5	3	5		Theorem 4.4
19	2	4	6	1	6	Theorem 4.1	
19	2	4	6	2	5		Conjecture 4.13
19	2	4	6	3	4		Theorem 4.6
19	2	4	7	1	5	Theorem 4.3	
19	2	4	7	2	4		Theorem 4.6
19	2	4	8	1	4		Conjecture 4.10
19	2	5	6	1	5		Theorem 4.4
19	2	5	6	2	4		Theorem 4.4
19	2	5	7	1	4		Conjecture 4.13
19	2	5	7	2	3		Theorem 4.6
19	2	5	8	1	3		Conjecture 4.10
19	2	6	7	1	3		Theorem 4.4
19	2	6	8	1	2		Conjecture 4.10
19	3	1	3	1	11	Theorem 4.1	
19	3	1	3	2	10	Theorem 4.1	
19	3	1	3	3	9	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
19	3	1	3	4	8	Theorem 4.1	
19	3	1	3	5	7	Theorem 4.1	
19	3	1	4	1	10	Theorem 4.1	
19	3	1	4	2	9	Theorem 4.1	
19	3	1	4	3	8	Theorem 4.1	
19	3	1	4	4	7	Theorem 4.1	
19	3	1	5	1	9	Theorem 4.1	
19	3	1	5	2	8	Theorem 4.1	
19	3	1	5	3	7	Theorem 4.1	
19	3	1	6	1	8	Theorem 4.1	
19	3	1	6	2	7	Theorem 4.1	
19	3	1	7	1	7	Theorem 4.1	
19	3	2	4	1	9	Theorem 4.1	
19	3	2	4	2	8	Theorem 4.1	
19	3	2	4	3	7	Theorem 4.1	
19	3	2	4	4	6		Theorem 4.4
19	3	2	5	1	8	Theorem 4.1	
19	3	2	5	2	7	Theorem 4.1	
19	3	2	5	3	6		Conjecture 4.11
19	3	2	6	1	7	Theorem 4.1	
19	3	2	6	2	6	Theorem 4.3	
19	3	2	7	1	6	Theorem 4.3	
19	3	3	5	1	7	Theorem 4.1	
19	3	3	5	2	6	Theorem 4.1	

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
19	3	3	5	3	5		Theorem 4.4
19	3	3	6	1	6	Theorem 4.1	
19	3	3	6	2	5		Conjecture 4.13
19	3	3	7	1	5	Theorem 4.3	
19	3	4	6	1	5		Theorem 4.4
19	3	4	6	2	4		Theorem 4.4
19	3	4	7	1	4		Conjecture 4.13
19	3	5	7	1	3		Theorem 4.4
19	4	1	4	1	9	Theorem 4.1	
19	4	1	4	2	8	Theorem 4.1	
19	4	1	4	3	7	Theorem 4.1	
19	4	1	5	1	8	Theorem 4.1	
19	4	1	5	2	7	Theorem 4.1	
19	4	1	5	3	6		Theorem 4.6
19	4	1	6	1	7	Theorem 4.1	
19	4	1	6	2	6		Theorem 4.6
19	4	1	7	1	6	Theorem 4.3	
19	4	2	5	1	7	Theorem 4.1	
19	4	2	5	2	6	Theorem 4.1	
19	4	2	6	1	6	Theorem 4.1	
19	4	2	6	2	5		Theorem 4.6
19	4	2	7	1	5	Theorem 4.3	
19	4	3	6	1	5		Theorem 4.4
19	4	3	7	1	4		Conjecture 4.10

#H	$\mathbf{m_{12}}$	$\mathbf{m_{13}}$	$\mathbf{m_{23}}$	$\mathbf{m_{21}}$	$\mathbf{m_{31}}$	**Injective**	**Not Injective**
19	5	1	5	1	7	Theorem 4.1	
19	5	1	5	2	6	Theorem 4.1	
19	5	1	6	1	6	Theorem 4.1	
19	5	2	6	1	5		Theorem 4.4