HEAT TRANSFER
IN THE THERMAL ENTRANCE REGION
OF AN ANNULUS

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by

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1. Introduction

The processing of non-Newtonian fluids is important in many industries. Among these industries are nuclear energy, minerals, petroleum, rocket propellants, plastics and the synthetic fiber industry. Non-Newtonian fluids are characterized by a non-linear shearing stress-strain rate relationship. Suspensions such as thorium oxide in water, emulsions, molten polymers, high molecular weight polyatomic and polymeric fluids and solutions of polymers are, for example, often non-Newtonian. The shear stress-rate of strain relationship for many fluids can often by represented by the power-law model. This model has proved to be a very useful two parameter model for a wide variety of non-Newtonian fluids. The model, in complex geometry, is expressed as

$$\tau_{ij} = -\left\{ m \left| \sqrt{\tfrac{1}{2} \sum_k \sum_l \Delta_{kl} \Delta_{lk}} \right|^{n-1} \right\} \Delta_{ij} \qquad (1\text{-}1)$$

where τ_{ij} is the shear stress and Δ_{ij} is the symmetrical rate of deformation tensor with components $\Delta_{ij} = \frac{\partial V_i}{\partial X_j} + \frac{\partial V_j}{\partial X_i}$. The parameters m and n are constants for a particular fluid at a given temperature and pressure. When $n < 1$ the fluid is called pseudoplastic, when $n > 1$ the fluid is called dilatant, and when $n = 1$ the expression reduces to the Newtonian relation:

$$\tau_{ij} = -\mu \Delta_{ij} \qquad (1\text{-}2).$$

Studies of the heat transfer to these non-Newtonian fluids have been restricted almost exclusively to tubular flow. Other geometries are of engineering importance also. The concentric annulus is an especially useful geometry to analyze because flow between parallel plates and in a tube are limiting forms of the annular problem. When the ratio of the inner to the outer radius approaches zero, the tubular flow problem is approached, while the parallel plate problem is approached as the ratio nears one. The concentric annular heat transfer problem is also of direct interest in concentric tube heat exchanger design.

In the analysis below, it is assumed that the fluid with constant physical properties enters the annulus with a uniform temperature and a fully developed laminar velocity profile and, up to some point ($z = 0$) the fluid is isothermal. Four distinct problems with different values of the ratio of the inner to the outer radius and different indices of the power law model are considered here:

I. For $z > 0$, uniform heat input at the inner wall and insulation at the outer wall.

II. For $z > 0$, equal wall temperatures are prescribed at both the inner and the outer walls.

III. For $z > 0$, the outer wall is insulated and a temperature is prescribed at the inner wall.

IV. For $z > 0$, different wall temperatures are prescribed at both the inner and outer walls.

The purpose of this work is to determine the variation of the Nusselt number with distance from the inlet. The analytical treatment of the problems utilizes the technique of separation of variables. This technique reduces the energy equation to a Sturm - Liouville problem and a first order ordinary differential equation. After the eigenvalues and corresponding eigenfunctions of the Sturm - Liouville problem have been determined, the heat transfer parameters of interest can be readily calculated. The accuracy of the results depends on the number and accuracy of the eigenvalues. An increasing number of eigenvalues is required to obtain accurate results as the distance from the entrance is decreased. The limiting Nusselt number as the distance from $z = 0$ approaches infinity requires only one eigenvalue. An iterative method and an asymptotic solution are introduced to solve the Sturm - Liouville problem. The asymptotic method used is known as the WKB method after G. Wentzel, H.A. Kramers and L. Brillouin who independently discovered the procedure.

2. Literature Survey

Though there are no solutions or data with which this work can be compared, there are several papers which are especially pertinent to the work. In the discussion below these are divided into four groups which are concerned with (i) the velocity profile, (ii) non-Newtonian heat transfer in a tube, (iii) Newtonian heat transfer in annuli, and (iv) mathematical methods.

Fredrickson and Bird (1) presented the analytical solutions of the equation of motion for steady axial flow of Bingham and power law fluids in a long cylinderical annulus. From their solutions, they prepared tables showing values of the dimensionless radial coordinate for which the shear stress is zero and values of the ratio of maximum velocity to average velocity. This solution was attacked by Metzner (2). He noted that power law solutions required that the parameters be constant over the entire range of shear stress under consideration. Metzner showed that this could not occur for non-Newtonian fluids and that the power law solution will, at best, be an approximation. The power law model predicted infinite apparent viscosity at zero shear stress; however, real non-Newtonian fluids exhibited a finite and constant viscosity at zero shear stress. Vaughn and Bergman (3) presented experimental data confirming the failure of the power law model to predict pressure loss and flow rate in concentric annuli. Recently though, McEachern (4) has demonstrated that the solution of the annulus problem given by Fredrickson and Bird (1) to estimate flow curves for the annulus can be used if the power law parameters are evaluated in the range of shear stresses found at the outside wall of the annulus.

Laminar flow heat transfer for the cases of the circular tube and of infinite parallel planes represent limiting forms of the annulus. These simple cases have received considerable attention, but only a few publications have treated non-Newtonian fluids. Metzner et al, (5) presented the first theoretical analysis combined with an experimental study of the variables controlling heat transfer rates to non-Newtonian fluids in the laminar flow region. A review on the laminar flow work has also been given by Metzner (6). Lyche and Bird (7) showed how the Graetz - Nusselt problem in heat transfer theory may be extended to power law fluids. Temperature profiles were obtained and used to calculate average outlet temperature as well as Nusselt numbers for several degrees of non-Newtonian behavior. Schenk and Van Laar (8) used the Prandtl - Eyring formula to calculate the heat transfer parameters which were then compared with those obtained by other workers assuming the power law model. Christiansen (9) (10), using the same model, presented generalized plots of the Nusselt number versus the Graetz number. The temperature dependency of the viscosity was also included.

Until recently the annulus problem, even for Newtonian fluids, had received much less attention than the tubular and infinite parallel plate problems. Reynolds et al. (11) and Hatton et al. (12) have presented the results of an extensive four year study of annular heat transfer to Newtonian fluids. Included in their study is a bibliography of pertinent publications. Jakob and Rees (13) obtained the temperature distribution as axial distance tends to infinity for the solution of problem II in this work. Murakawa (14) (15) presented an integral equation formulation as well as some experimental results for water heated from the inside wall with the outside wall of the annulus being insulated. The case where arbitrary peripheral variations were allowed was also considered. He expanded the boundary conditions in a Fourier series and compared the coefficients of both sides of the energy equation. Unfortunately a general recurrence formula could not be obtained, so the coefficients had to be evaluated individually. Murakawa carried his solutions to the point of numerical calculation only for problem III and for one value of the radius ratio. Viskanta (16) (17) has presented complete thermal entry length solutions of the last three problems. He utilized the method of superposition to determine the temperature distribution for problem IV. Some numerical results for heat fluxes, mixing cup temperatures and Nusselt numbers were presented graphically. Analog computation seemed to be rather convenient, but of limited accuracy. Lundberg et al. (18) (19) have also presented thermal entry length solutions. This included evaluation of the four fundamental solutions, which are basically the same as in this work, by a solution of the eigenvalue problem. The analytical predictions were also substantiated by their agreement with careful experimental measurements. Hatton and Quarmby (20) gave the solutions to problems I and III. The case of parallel plates with one side insulated was included for comparison.

Siegel et al. (21) suggested the method of making the boundary conditions homogeneous by subtraction of the fully developed solutions. Berry and de Prima (22) developed the simple iterative method used for the determination of the eigenvalues and eigenfunctions of the Sturm - Liouville problem. Their method is particularly useful when the coefficients of the differential equation are not expressed in analytical form. Sellars, Tribus and Klein (23) first applied the WKB method of evaluating the higher eigenvalues to heat transfer problems in tubes. This method also has been applied by Lundberg et al. (18) (19) and by Ziegenhagen (24) to the annular problem.

3. The Velocity Profile

The equations describing the motion of the fluid are the equations of continuity and motion:

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \overline{V}) = 0 \qquad (3\text{-}1)$$

$$\rho\left[\frac{\partial \overline{V}}{\partial t} + (\overline{V} \cdot \nabla)\overline{V}\right] = -\nabla p - (\nabla \cdot \overline{\overline{\tau}}) + \rho g \qquad (3\text{-}2).$$

In the developments which follow, the flow between two coaxial cylinders using the coordinate system and notation shown in Figure 1 is considered. The solution of this problem was first given by Fredrickson

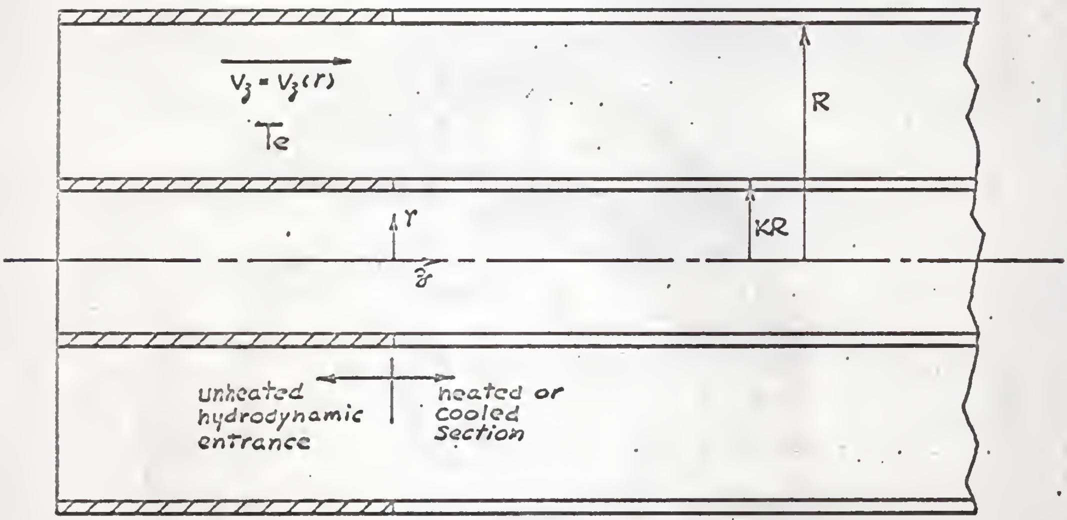

Fig. 1 Diagram of the coordinate system

and Bird (1). The following assumptions are made:

(1). The fluid is incompressible,

(2). The flow is in steady-state,

(3). The flow is laminar,

(4). The cylinders are sufficiently long that end effects may be neglected.

For the specific system under consideration, Equations (1-1), (3-1) and (3-2) may be written in cylindrical coordinates as

$$\tau_{rz} = -m\left|\frac{dV_z}{dr}\right|^{n-1} \frac{dV_z}{dr} \tag{3-3}$$

$$\frac{d}{dz}(\rho V_z) = 0 \tag{3-4}$$

$$V_z \frac{dV_z}{dz} = -\frac{dp}{dz} - \frac{1}{r}\frac{d}{dr}(r\,\tau_{rz}) + \rho g_z \tag{3-5}.$$

Combining and simplifying Equations (3-4) and (3-5) leads to

$$\frac{1}{r}\frac{d}{dr}(r\,\tau_{rz}) = \frac{p_o - p_L}{L} + \rho g_z \tag{3-6}$$

in which p_o and p_L are the static pressure at $z = 0$ and $z = L$, respectively, and g_z is the component of gravitational acceleration g in the direction of flow. This first order differential equation, valid over the entire annular region for any fluid, may be integrated to give

$$\tau_{rz} = \frac{P}{2}\left[r - \frac{(\lambda R)^2}{r}\right] \tag{3-7}$$

in which λ is the constant of integration and P is the sum of forces per unit volume on the right hand side of Equation (3-6). The radial position $r = \lambda R$ represents that position at which $\tau_{rz} = 0$.

Substituting Equation (3-3) into Equation (3-7) and introducing the dimensionless variable $\zeta = \frac{r}{R}$, yields

$$\frac{PR^{n+1}}{2}\left(\zeta - \frac{\lambda^2}{\zeta}\right) = -m\left|\frac{dV_z}{d\zeta}\right|^{n-1}\left(\frac{dV_z}{dr}\right) \quad (3\text{-}8).$$

For $K \leq \zeta \leq \lambda$, $\frac{dV_z}{d\zeta}$ is positive and

$$\frac{PR^{n+1}}{2}\left(\zeta - \frac{\lambda^2}{\zeta}\right) = -m\left(\frac{dV_z}{d\zeta}\right)^n \quad (3\text{-}9).$$

For $\lambda \leq \zeta \leq 1$, $\frac{dV_z}{d\zeta}$ is negative and

$$\frac{PR^{n+1}}{2}\left(\zeta - \frac{\lambda^2}{\zeta}\right) = m\left(-\frac{dV_z}{d\zeta}\right)^n \quad (3\text{-}10).$$

setting $s = \frac{1}{n}$, integrating Equations (3-9) and (3-10), and rearranging leads to

$$V_z = R\left(\frac{PR}{2m}\right)^s \int_K^\zeta \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta, \; K \leq \zeta \leq \lambda \quad (3\text{-}11)$$

$$V_z = R\left(\frac{PR}{2m}\right)^s \int_\zeta^1 \left(\zeta - \frac{\lambda^2}{\zeta}\right)^s d\zeta, \; \lambda \leq \zeta \leq 1 \quad (3\text{-}12).$$

The boundary conditions $V_z = 0$ at $\zeta = K$ and $\zeta = 1$ have been used. Obviously, the above two equations must give the same value of the velocity at $\zeta = \lambda$ where the shear stress is zero and the velocity is a maximum. Then

$$\int_K^\lambda \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta = \int_\lambda^1 \left(\zeta - \frac{\lambda^2}{\zeta}\right)^s d\zeta \quad (3\text{-}13)$$

and

$$V_{max} = R\left(\frac{PR}{2m}\right)^s \int_K^\lambda \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta \quad (3\text{-}14).$$

From Equation (3-13), values of λ at different values of K and s can be determined. These values have been tabulated by Fredrickson and Bird (1). In order to eliminate $R\left(\frac{PR}{2m}\right)^s$ from the velocity profile, Equation (3-14) may be rewritten as

$$R\left(\frac{PR}{2m}\right)^s = \frac{V_{avg}\left(\frac{V_{max}}{V_{avg}}\right)}{\int_K^\lambda \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta} \qquad (3\text{-}15).$$

Therefore, the expression for the velocity is

$$\overline{V_z} = \frac{V_z}{V_{avg}} = \frac{(V_{max}/V_{avg})}{\int_K^\lambda \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta} \cdot \int_K^\zeta \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta\,, \quad K \leq \zeta \leq \lambda \qquad (3\text{-}16)$$

$$\overline{V_z} = \frac{V_z}{V_{avg}} = \frac{(V_{max}/V_{avg})}{\int_K^\lambda \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta} \cdot \int_\zeta^1 \left(\zeta - \frac{\lambda^2}{\zeta}\right)^s d\zeta\,, \quad \lambda \leq \zeta \leq 1 \qquad (3\text{-}17)$$

where $\overline{V_z}$ is the dimensionless velocity, which may be calculated numerically. Results for values of n of 0.2, 0.5, and 0.8 are shown in Figure 2.

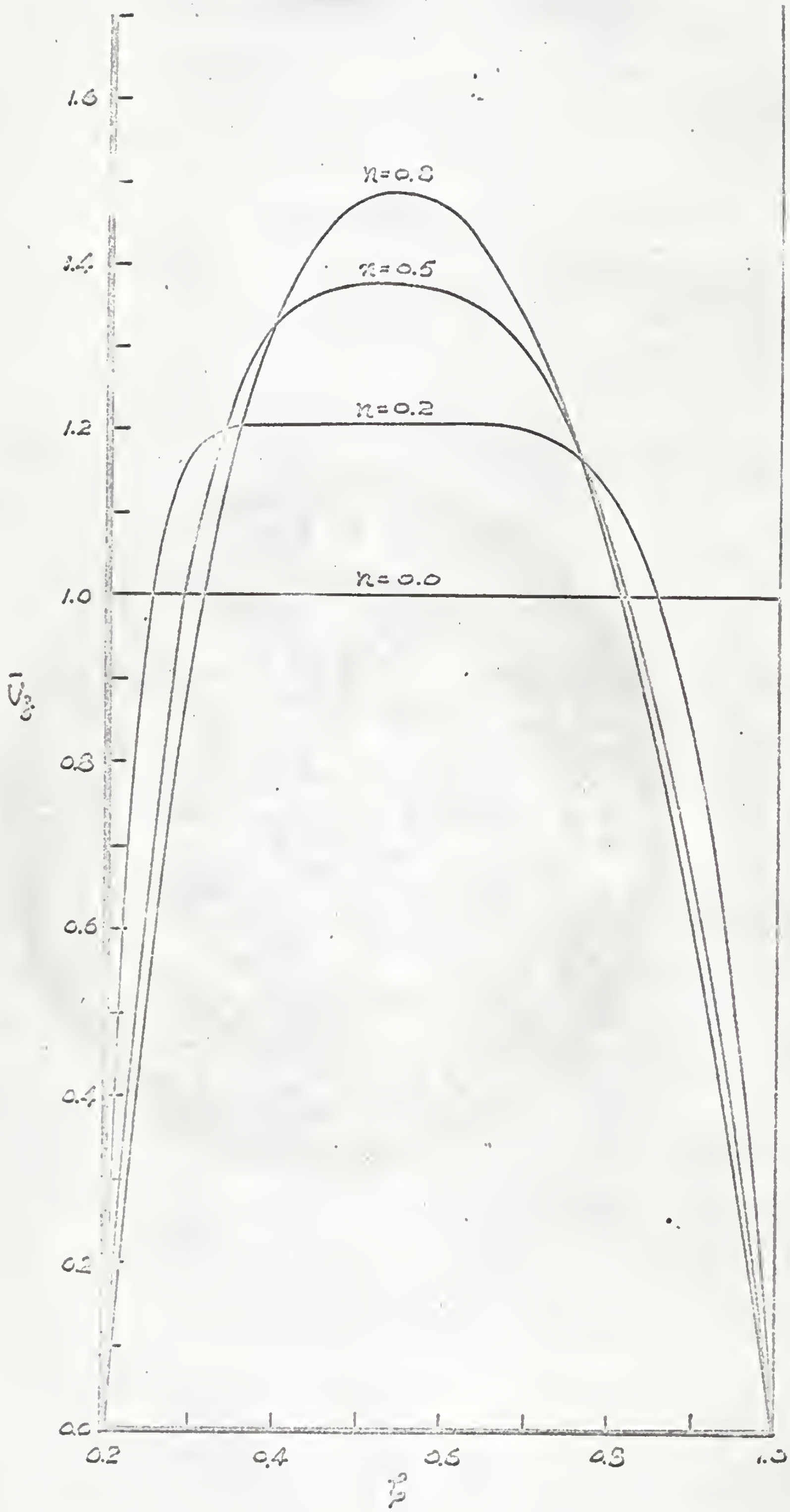

Fig. 2. Velocity profile for K=0.2

4. Heat Transfer to Power-Law Fluids in Laminar Flow through an Annulus

4.1 Case I: Heat Transfer in an Annulus with Uniform Heat Input at the Inner Wall and Insulation at the Outer Wall

4.1.1 Mathematical Statement of the Problem

The non-Newtonian fluid flows with a fully-developed laminar velocity profile in the +z direction in a concentric annulus. The coordinates and geometry of the system are shown in Fig. 1. In the region $z < 0$, the fluid and both walls are maintained at a uniform temperature T_e. In the region $z > 0$, the inner wall is prescribed with a uniform heat flux, $-q$, and the outer wall is insulated. The problem is to find the temperature distribution and the variation of the heat transfer coefficient on the inner surface with distance down the duct.

Subject to the limitations mentioned below, the energy equation describing the problem is

$$\rho C_p V_z \frac{\partial T}{\partial z} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{4.1-1}$$

and the boundary conditions are

$$\begin{aligned} T &= T_e && \text{for } z \leq 0 \\ \left.\frac{\partial T}{\partial r}\right|_{r=R} &= 0 && \text{for } z > 0 \\ -k \left.\frac{\partial T}{\partial r}\right|_{r=KR} &= q && \text{for } z > 0 \end{aligned} \tag{4.1-2}$$

The following assumptions are made.

(1) Steady state has been obtained.

(2) Heat conduction in the z-direction is negligible in comparison with heat transport in the z-direction by the bulk fluid motion.

(3) The physical properties ρ, C_p, and k are constant.

(4) Viscous dissipation is negligible.

4.1.2 Solution of the Problem

With the introduction of the following dimensionless variables,

$$\zeta = \frac{r}{R}$$

$$\overline{V_z} = \frac{V_z}{V_{avg}}$$

$$\theta = \frac{T - T_e}{qR/k} \qquad (4.1\text{-}3),$$

$$\xi = \frac{z}{z_d}$$

$$z_d = \frac{Re\ Pr\ R}{2(1 - K)}$$

the energy equation and the boundary conditions become

$$\overline{V}_z \frac{\partial \theta}{\partial \xi} = \frac{1}{\zeta} \frac{\partial}{\partial \zeta} (\zeta \frac{\partial \theta}{\partial \zeta}) \qquad (4.1\text{-}4),$$

$$\theta\ (0,\zeta) = 0$$

$$\left.\frac{\partial \theta}{\partial \zeta}\right|_{\zeta=1} = 0 \quad \text{for } \xi > 0 \qquad (4.1\text{-}5).$$

$$\left.\frac{\partial \theta}{\partial \zeta}\right|_{\zeta=K} = -1 \quad \text{for } \xi > 0$$

Because of the nature of the boundary conditions in this problem, the general method of the separation of variables cannot be used. The procedure used in this work was first introduced by Siegal et al. (21). The solution is divided into two parts, a fully-developed solution and a solution valid near the entrance which disappears far downstream from the entry. As will be shown, this procedure results in homogeneous boundary conditions.

Thus,

$$\theta\ (\xi,\zeta) = \theta_\infty\ (\xi,\zeta) + \theta_d\ (\xi,\zeta) \qquad (4.1\text{-}6)$$

in which θ_∞ is the asymptotic solution for large ξ, and θ_d is the solution which is valid near the entrance and which will be damped out exponentially with ξ. Thus, Equations (4.1-4) and (4.1-5) are divided into two parts,

$$\overline{V}_z \frac{\partial\theta_d}{\partial\xi} = \frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta \frac{\partial\theta_d}{\partial\zeta}\right) \qquad (4.1\text{-}7)$$

with boundary conditions

$$\left.\frac{\partial\theta_d}{\partial\zeta}\right|_{\zeta=1} = 0$$

$$\left.\frac{\partial\theta_d}{\partial\zeta}\right|_{\zeta=K} = 0 \qquad (4.1\text{-}8),$$

and

$$\overline{V}_z \frac{\partial\theta_\infty}{\partial\xi} = \frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta \frac{\partial\theta_\infty}{\partial\zeta}\right) \qquad (4.1\text{-}9)$$

with boundary conditions

$$\left.\frac{\partial\theta_\infty}{\partial\zeta}\right|_{\zeta=1} = 0$$

$$\left.\frac{\partial\theta_\infty}{\partial\zeta}\right|_{\zeta=K} = -1 \qquad (4.1\text{-}10),$$

and

$$\theta\ (0,\zeta) = \theta_\infty\ (0,\zeta) + \theta_d\ (0,\zeta) = 0 \qquad (4.1\text{-}11).$$

In order to solve Equations (4.1-7) and (4.1-8) by the method of separation of variables, we set

$$\theta\ (\xi,\zeta) = Z\ (\xi)\ E\ (\zeta) \qquad (4.1\text{-}12)$$

where Z and E are, respectively, functions of ξ and ζ only.

Substituting Equation (4.1-12) into Equation (4.1-7) and rearranging yields $$\frac{1}{Z}\frac{dZ}{d\xi} = \frac{1}{\overline{V}_z E}\frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dE}{d\zeta}\right) \qquad (4.1\text{-}13).$$

Because the right hand side is a function of ζ only and the left hand side is a function of ξ only, both must be equal to a constant, say $-\alpha^2$. Thus,

$$\frac{dZ}{d\xi} = -\alpha^2 Z \qquad (4.1\text{-}14),$$

$$\frac{d}{d\zeta}\left(\zeta\frac{dE}{d\zeta}\right) + \alpha^2\zeta\overline{V}_z E = 0 \qquad (4.1\text{-}15),$$

with boundary conditions

$$E(\zeta)Z(0) = 1$$

$$\left.\frac{\partial E}{\partial\zeta}\right|_{\zeta=1} = 0 \qquad (4.1\text{-}16).$$

$$\left.\frac{\partial E}{\partial\zeta}\right|_{\zeta=K} = 0$$

Equation (4.1-15) with the last two boundary conditions of Eq. (4.1-16) belongs to the well known class of differential equations of the Sturm - Liouville type. It can be shown (25) that there is a countable infinity of values α_1^2, α_2^2 . . . of the parameter α^2 for each of which the Sturm - Liouville problem, Eq. (4.1-15), has a solution that is not identically zero. The numbers α_n^2 are the eigenvalues of the problem and the corresponding solutions $E_n(\zeta)$ are the eigenfunctions. By exploitation of Berry and de Prima's (22) iterative method, Eq. (4.1-15) can be solved. A discussion of the method of Berry and de Prima is provided in Appendix 9.1.

Combination of the solutions of Eqs. (4.1-14) and (4.1-15) yields

$$\theta_d(\xi,\zeta) = \sum_{n=1}^{\infty} C_n E_n \exp(-\alpha_n^2\xi) \qquad (4.1\text{-}17).$$

For the solution of Eqs. (4.1-9) and (4.1-10), it is expected intuitively that after the fluid is far downstream from the beginning of the heated section the constant heat flux through the wall will result in a rise in the fluid temperature that is linear in ξ. Furthermore, the shape of the radial temperature profile will ultimately undergo no further change with increasing ξ. Hence, a solution of the following form is quite reasonable for large ξ.

$$\theta_\infty(\xi,\zeta) = C_g\xi + G(\zeta) \tag{4.1-18}$$

in which C_g is a constant to be determined presently and G is a function of the variable ζ only.

By an energy balance between the inlet and an arbitrary distance from the conduit, it is found that

$$\int_0^{2\pi}\int_{KR}^{R} \rho C_p V_z (T - T_e) r\,dr\,d\beta = 2\pi KRzq \tag{4.1-19}$$

Introducing the dimensionless variables and simplifying yields

$$\int_0^{2\pi}\int_K^1 \theta \overline{V}_z \zeta\,d\zeta\,d\beta = 2\pi K\xi \tag{4.1-20}$$

Therefore,

$$\int_0^{2\pi}\int_K^1 \left[C_g\xi + G(\zeta)\right] \overline{V}_z \zeta\,d\zeta\,d\beta = 2\pi K\xi \tag{4.1-21}$$

Setting

$$\int_K^1 G(\zeta)\overline{V}_z\zeta\,d\zeta = 0 \tag{4.1-22}$$

gives

$$C_g = \frac{2\pi K}{\int_0^{2\pi}\int_K^1 \overline{V}_z \zeta\,d\zeta\,d\beta}$$

$$= \frac{2K}{1 - K^2} \tag{4.1-23}$$

Equation (4.1-9) and its boundary conditions now become

$$\frac{d^2G}{d\zeta^2} + \frac{1}{\zeta}\frac{dG}{d\zeta} - C_g\overline{V}_z = 0 \tag{4.1-24}$$

and

$$\left.\frac{dG}{d\zeta}\right|_{\zeta=1} = 0 \qquad \text{for } \xi > 0$$

$$\left.\frac{dG}{d\zeta}\right|_{\zeta=K} = -1 \qquad \text{for } \xi > 0 \tag{4.1-25}$$

$$\int_K^1 \overline{V}_z G\zeta d\zeta = 0 \qquad \text{for } \xi > 0$$

which may be solved numerically. The computer flow sheet and program for this solution are provided in Appendix 9.2.

The complete solution may now be written

$$\theta = \theta_\infty(\xi,\zeta) + \theta_d(\xi,\zeta)$$

$$= C_g\xi + G(\zeta) + \sum_{n=1}^{\infty} C_n E_n \exp(-\alpha_n^2\xi) \tag{4.1-26}.$$

Multiplying both sides of Eq. (4.1-26) by $\zeta\overline{V}_z E_m$, integrating with respect to ζ from K to 1, and utilizing Eq. (4.1-11) yields

$$\int_K^1 \sum_{n=1}^{\infty} C_n \zeta \overline{V}_z E_n E_m d\zeta = -\int_K^1 \zeta\overline{V}_z G E_m d\zeta \tag{4.1-27}.$$

For $n \neq m$, $$\int_K^1 C_n \zeta \overline{V}_z E_n E_m d\zeta = 0 \tag{4.1-28},$$

because of the orthogonality of the eigenfunctions, and Eq. (4.1-27) reduces to

$$\int_K^1 C_n \zeta \overline{V}_z E_n^2 d\zeta = -\int_K^1 \zeta \overline{V}_z G E_n d\zeta \tag{4.1-29}.$$

Therefore,

$$C_n = -\frac{\int_K^1 \zeta \overline{V}_z G E_n d\zeta}{\int_K^1 \overline{V}_z \zeta E_n^2 d\zeta} \tag{4.1-30}.$$

4.1.3 Expression for the Nusselt Number

The determination of the variation of the Nusselt number with distance from the inlet is the main purpose of this work. But before the expression for the Nusselt number can be derived, the mixing-cup temperature must first be determined. By definition

$$\theta_{avg} = \frac{\int_0^{2\pi}\int_K^1 \theta\overline{V}_z\zeta d\zeta d\beta}{\int_0^{2\pi}\int_K^1 \overline{V}_z\zeta d\zeta d\beta}$$

$$= \frac{2}{1 - K^2}\int_K^1 \theta\overline{V}_z\zeta d\zeta$$

$$= \frac{2}{1 - K^2}\Big[\int_K^1 C_g\xi\overline{V}_z\zeta d\zeta + \int_K^1 G\overline{V}_z\zeta d\zeta$$

$$+ \sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2\xi)\int_K^1 E_n\overline{V}_z\zeta d\zeta\Big] \quad (4.1\text{-}31).$$

Furthermore, from Eq. (4.1-15), it can be shown that

$$\int_K^1 \zeta\overline{V}_z E_n d\zeta = -\frac{1}{\alpha_n^2}\left(\zeta\frac{dE_n}{d\zeta}\right)\Bigg|_K^1 = 0 \quad (4.1\text{-}32).$$

Substituting into Eq. (4.1-31) leads to

$$\theta_{avg} = \frac{2}{1 - K^2}\int_K^1 C_o\xi\overline{V}_z\zeta d\zeta$$

$$= C_g\xi\frac{2\int_K^1 V_z\zeta d\zeta}{(1 - K^2)V_{avg}}$$

$$= C_g\xi \quad (4.1\text{-}33).$$

The Nusselt number is defined as

$$Nu = \frac{D_e h_i}{k} \quad (4.1\text{-}34)$$

where $$h_i(T - T_{avg}) = -k \frac{\partial T}{\partial r}\bigg|_{r=KR} \qquad (4.1\text{-}35).$$

Thus,

$$Nu_i = -\frac{D_e \frac{\partial T}{\partial r}\Big|_{r=KR}}{T - T_{avg}}$$

$$= -\frac{2(1-K)\frac{\partial \theta}{\partial \zeta}\Big|_{\zeta=K}}{\theta - \theta_{avg}}$$

$$= \frac{2(1-K)\left[G'(K) + \sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2 \xi) E_n'(K)\right]}{G(K) + \sum_{n=1}^{\infty} C_n E_n(K) \exp(-\alpha_n^2 \xi)}$$

$$= \frac{2(1-K)}{G(K) + \sum_{n=1}^{\infty} C_n E_n(K) \exp(-\alpha_n^2 \xi)} \qquad (4.1\text{-}36).$$

This completes the solution of the problem. Results are presented in Tables 1 and 2 and in Figs. 3–8.

4.1.4 Asymptotic Solution by the WKB Method

The computation of tho higher modes of Eq. (4.1-15) becomes increasingly difficult due to the fact that the eigenfunctions oscillato (undergo a sign change) n times in the interval $K \leq \zeta \leq 1$. To follow these oscillations the spacing of the net for the numerical calculations, either by the method of Runge – Kutta or finite differences, must be reduced. This ontails considerable time expense for many eigenvalues and functions of different boundary conditions. In addition, it is desired to check tho solutions obtained from the iterative method. Accordingly, it is advantageous to develop an asymptotic solution valid as α_n becomes large. Following the method of Sellars, Tribus and Klein (22), the so-called WKB method solution of Eq. (4.1-15) can be obtained.

Table 1. Functions in the solution of problem I for n=0.5

Iterative Method

Radius ratio	Eigenvalue α_n	Expansion Coeff. C_n	$C_n E_n(K)$
0.2	4.5254	0.02716686	-0.07568741
	8.5177	0.00860408	-0.02700806
	12.4744	0.00429905	-0.01461511
	16.4744	0.00260579	-0.00933038
	20.3522	0.00172230	-0.00642637
	24.2753	0.00125292	-0.00482587
	28.1888	0.00091030	-0.00359735
	32.0880	0.00072802	-0.00294076
0.5	7.1455	0.02662257	-0.07248851
	13.7433	0.00779512	-0.02301125
	20.2084	0.00381384	-0.01200757
	26.66138	0.00228768	-0.00751367
	33.09439	0.00150830	-0.00512679
	39.51390	0.00109488	-0.00382661
	45.91941	0.00080289	-0.00287281
	52.30940	0.00064127	-0.00234039

WKB Method

Radius ratio	Eigenvalue α_n	Expansion Coeff. C_n	$C_n E_n(K)$
0.2	4.6140615	-0.1343204	-0.08750703
	8.5689715	-0.04787027	-0.03118655
	12.523882	-0.02543207	-0.01656850
	16.523882	-0.01609675	-0.01048672
	20.433702	-0.01124695	-0.00732717
	24.388612	-0.00837468	-0.00545594
	28.345220	-0.00651850	-0.00424667
	32.298432	-0.00524378	-0.00341622
0.5	7.515493	-0.09639799	-0.62394420
	13.957344	-0.03435518	-0.02223669
	20.399196	-0.01825191	-0.01181370
	26.841048	-0.01155220	-0.00747726
	33.282899	-0.00807164	-0.00522444
	39.724751	-0.00601028	-0.00389021
	46.166602	-0.00467861	-0.00302827
	52.608454	-0.00376332	-0.00243584

Table 2. Functions in the solution of problem I for n=0.8

Iterative Method

Radius ratio	Eigenvalue α_n	Expansion Coeff. C_n	$C_n E_n(K)$
0.2	4.62768	0.02567792	-0.07086307
	8.63452	0.00855808	-0.02734728
	12.60923	0.00433102	-0.01512950
	16.57412	0.00266460	-0.00986245
	20.53212	0.00177207	-0.00685938
	24.47825	0.00130180	-0.00521646
	28.41486	0.00095422	-0.00393300
	32.33733	0.00076834	-0.00324269
0.5	7.33398	0.02545143	-0.06969242
	13.91432	0.00779740	-0.02367462
	20.41371	0.00388651	-0.01261984
	26.90379	0.00231617	-0.00787358
	33.52579	0.00155049	-0.00546708
	39.83944	0.00113102	-0.00410468
	46.28813	0.00081306	-0.00302409
	52.72142	0.00068791	-0.00261042

WKB Method

Radius ratio	Eigenvalue α_n	Expansion Coeff. C_n	$C_n E_n(K)$
0.2	4.645519	-0.14166713	-0.09778348
	8.627392	-0.05048862	-0.03484896
	12.609266	-0.02682312	-0.01851423
	16.591140	-0.01697718	-0.01171823
	20.573014	-0.01186213	-0.00818764
	24.554888	-0.00883275	-0.00609667
	28.536760	-0.00687572	-0.00474586
	32.518635	-0.00553060	-0.00381741
0.5	7.569564	-0.10012111	-0.06762953
	14.057762	-0.03568206	-0.02410242
	20.545960	-0.01895684	-0.01280492
	27.034159	-0.01199837	-0.00810463
	33.522357	-0.00838338	-0.00566278
	40.010555	-0.00624241	-0.00421661
	46.498753	-0.00485931	-0.00328236
	52.986951	-0.00390867	-0.00264022

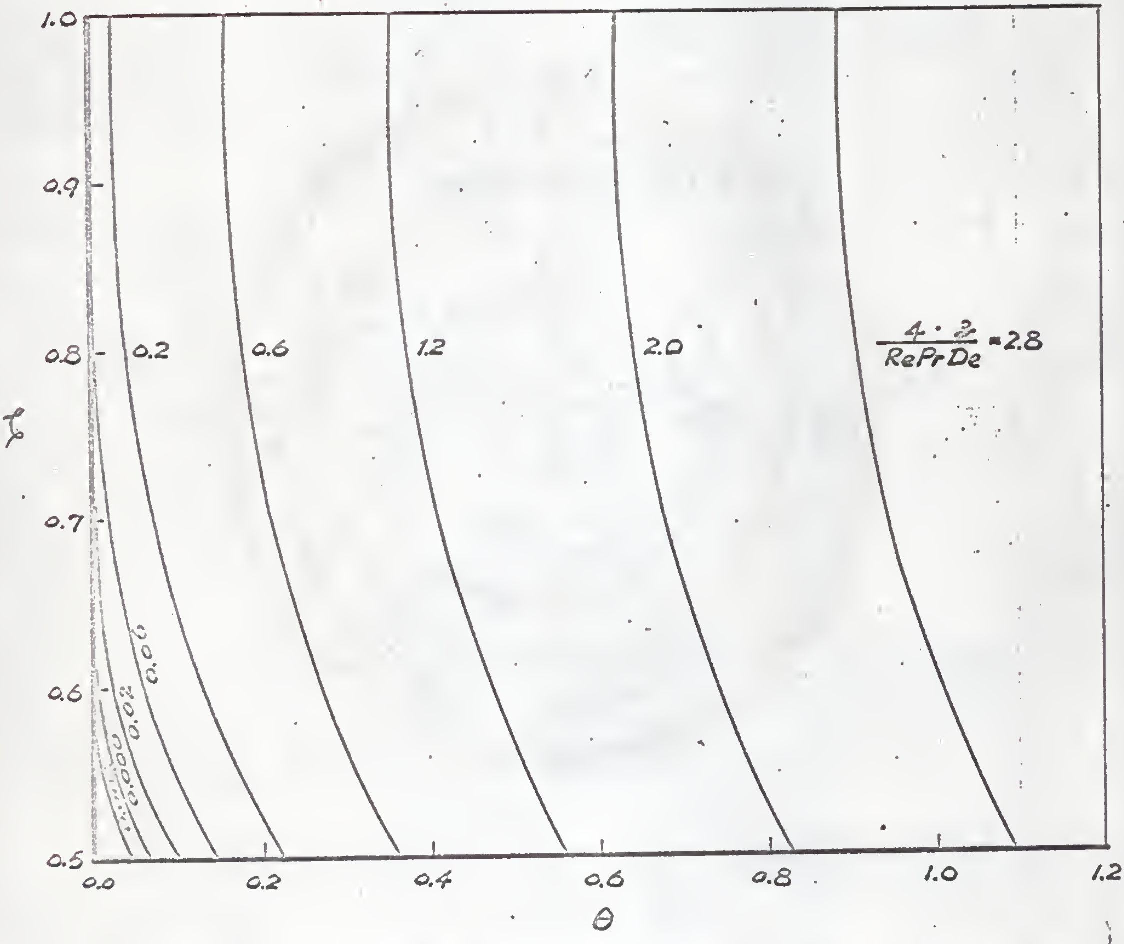

Fig. 3 Temperature profile development, problem I, K = 0.5, n = 0.5

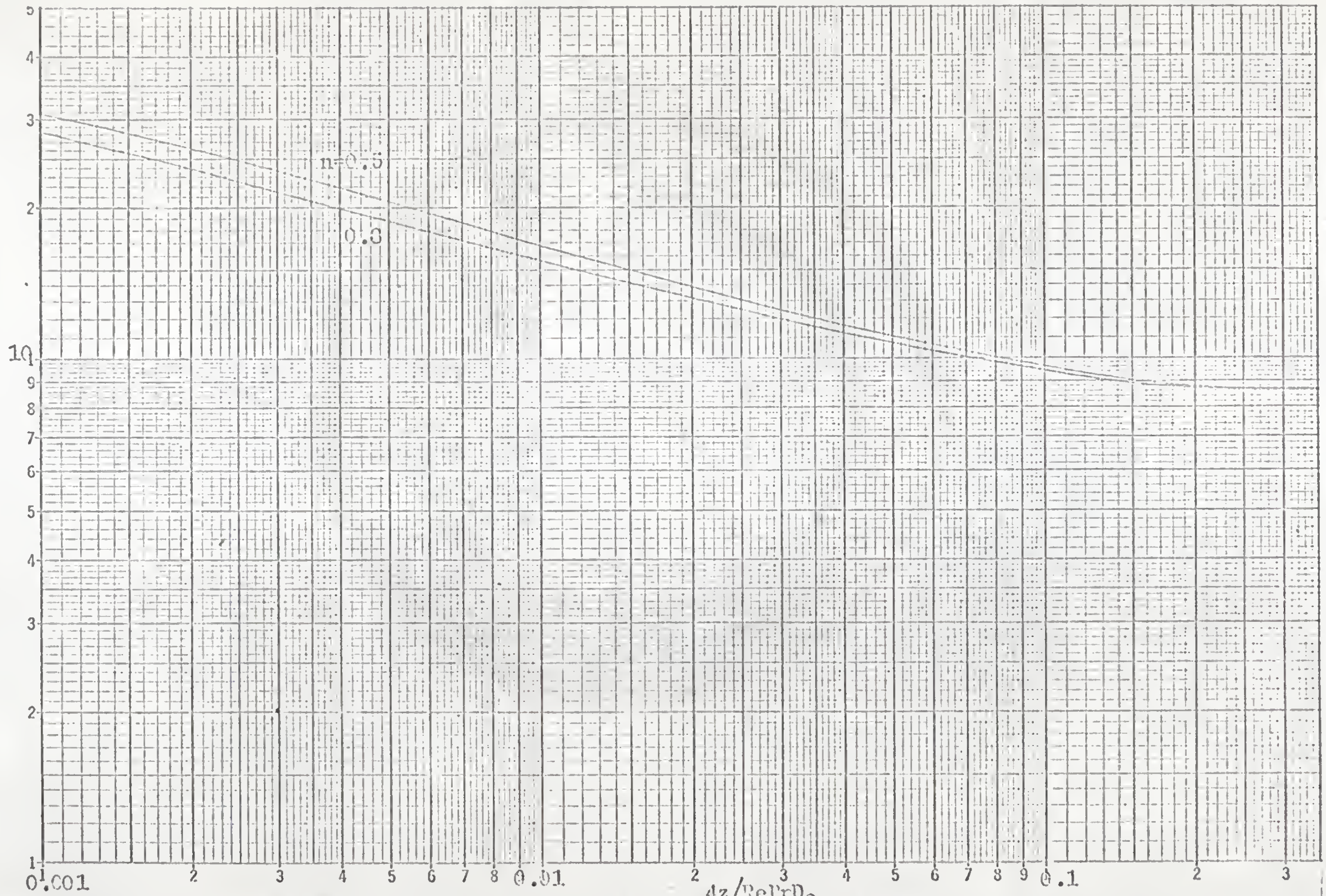

Fig. 4. Nusselt numberversus axial distance, problem I, K=0.2

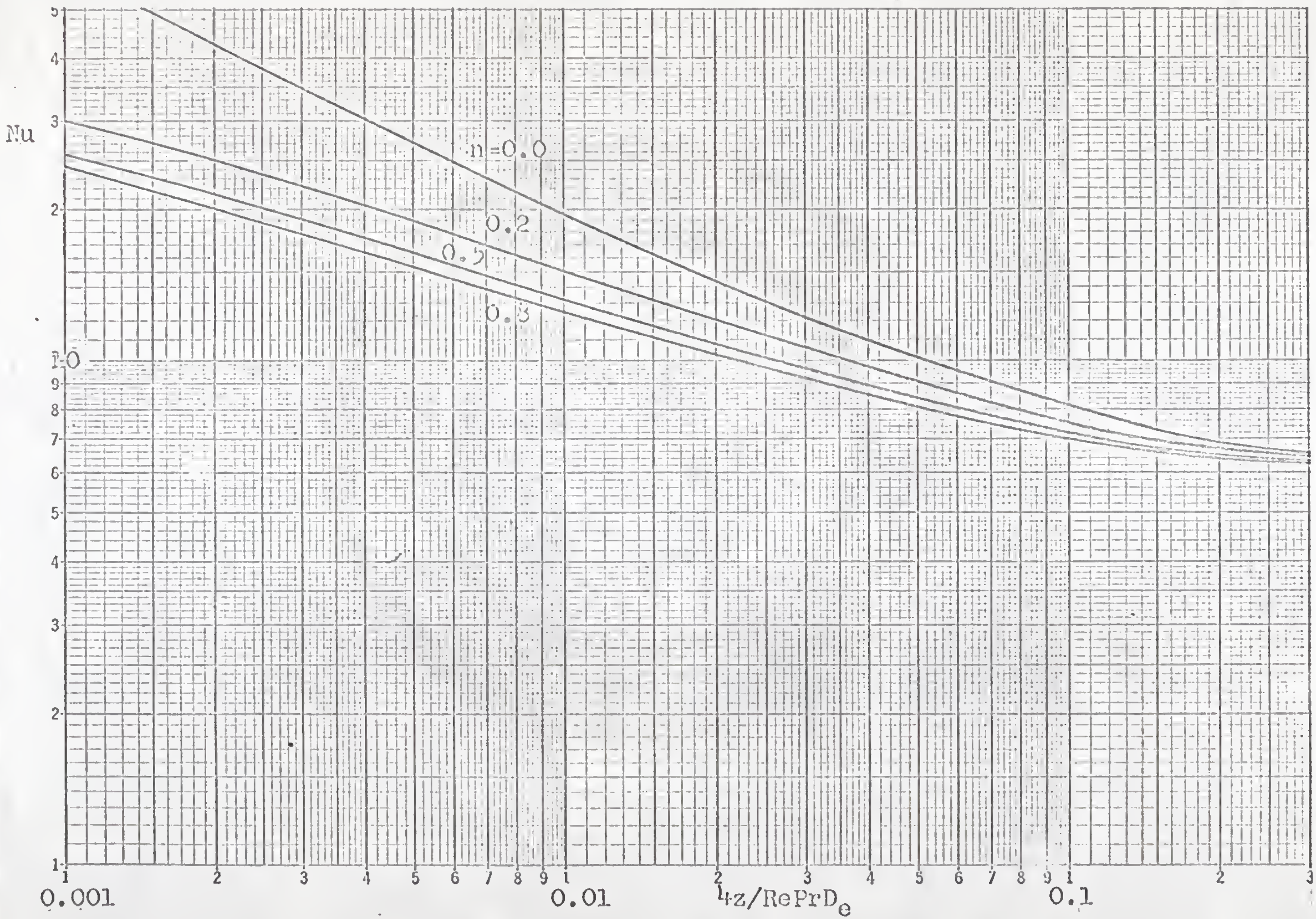

Fig. 5. Nusselt number versus axial distance, problem I, k=0.5.

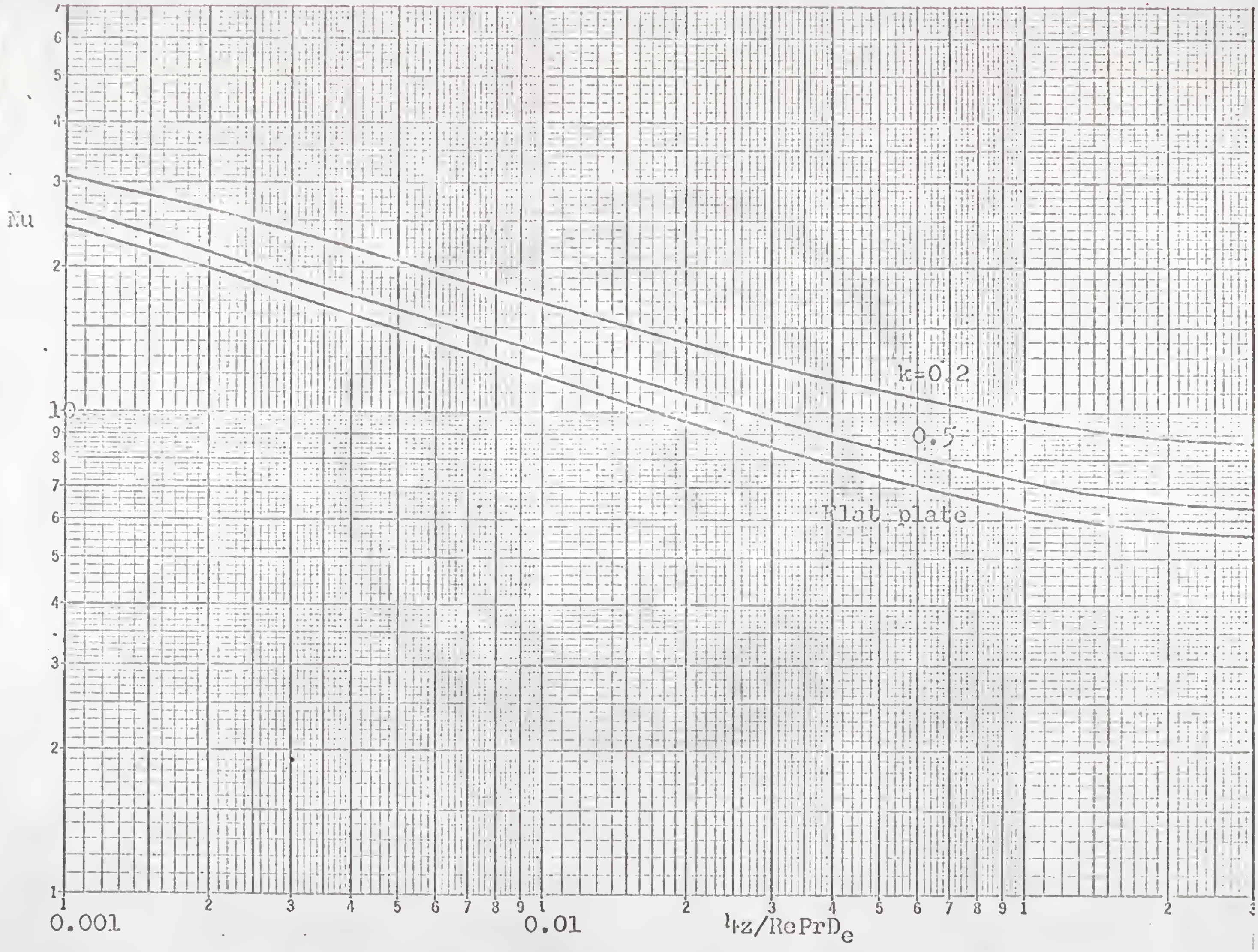

Fig. 6. Nusselt number versus axial distance, problem I, n=0.5.

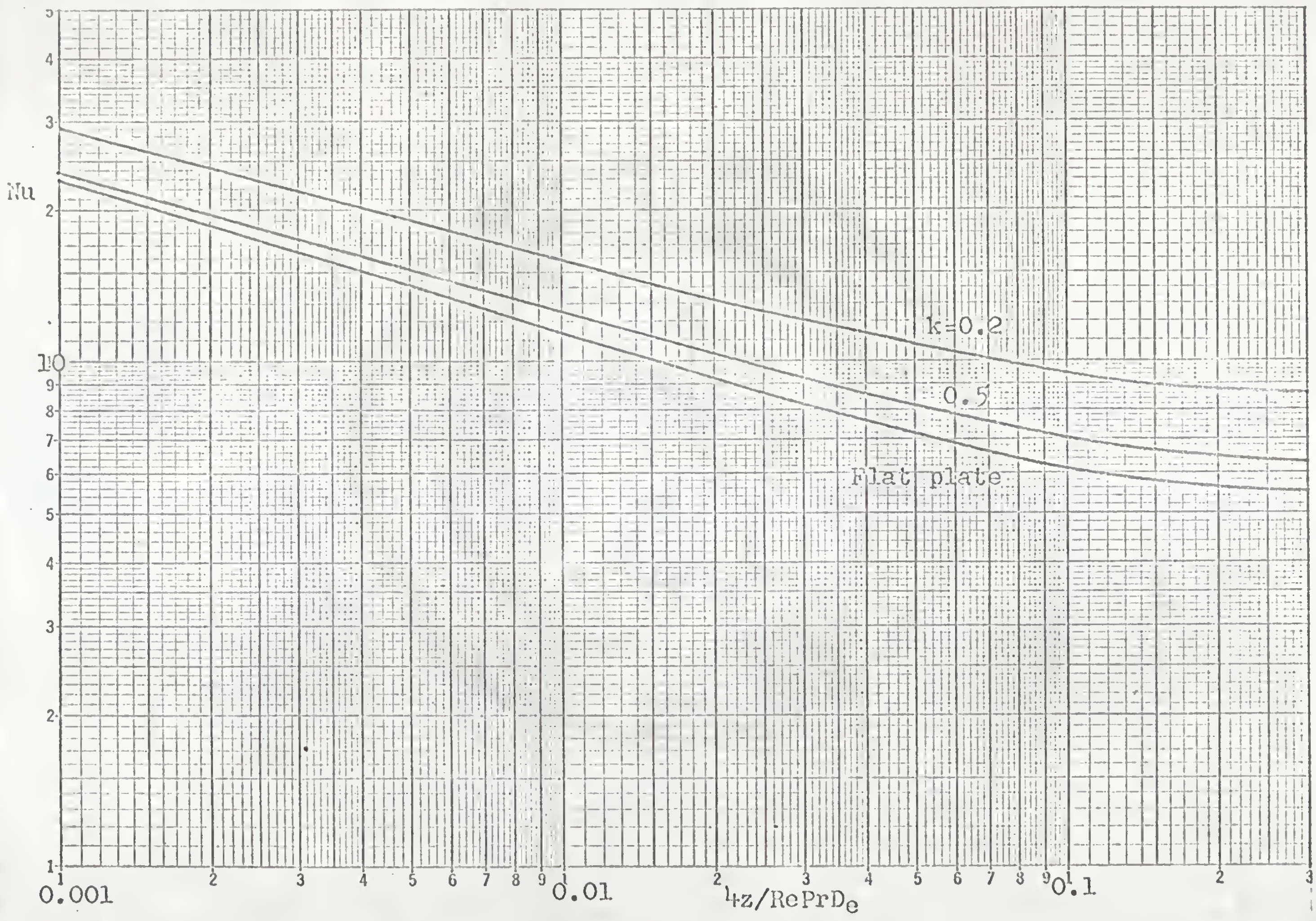

Fig. 7. Nusselt number versus axial distance, problem I, n=0.8.

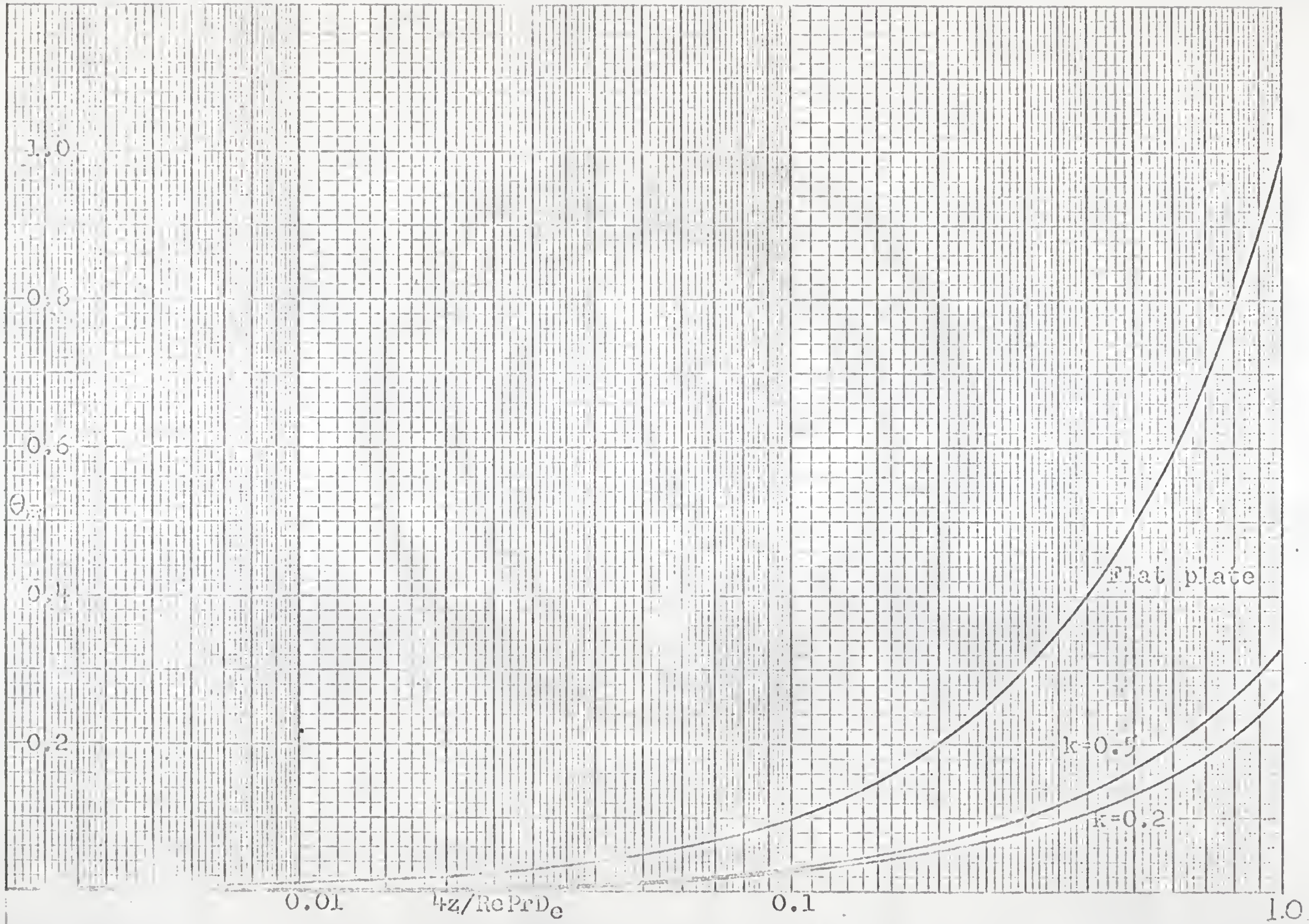

Fig. 8. Average temperature versus axial distance, problem I, n=0.5.

Let $E_n = e^{g(\zeta)}$, (4.1-37)

and an asymptotic solution is sought in the form

$$g(\zeta) = \alpha_n g_0 + g_1 + \alpha_n^{-1} g_2 + \ldots \quad (4.1\text{-}38).$$

Substituting Eqs. (4.1-37) and (4.1-38) into Eq. (4.1-15) and equating powers of α_n gives

$$({g_0'}^2 + \overline{V}_z)\alpha_n^2 + (2g_0'g_1' + g_0'' + \frac{1}{\zeta} g_0')\alpha_n + ({g_1'}^2 + 2g_0'g_2' + g_1'' + \frac{g_1}{\zeta}) + \ldots = 0 \quad (4.1\text{-}39),$$

where the primes indicate differentiation with respect to ζ. Since α_n is assumed to be large, only the first two terms of Eq. (4.1-38) are retained. Therefore,

$$g_0 = \pm i\int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta \quad (4.1\text{-}40)$$

$$g_1 = - \ln\sqrt{g_0' \zeta} \quad (4.1\text{-}41).$$

Substituting the above two equations into Eq. (4.1-15) gives for E_n,

$$E_n = e^{g(\zeta)}$$

$$= \frac{A''\exp\left\{i\lambda_n \int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta\right\} + B''\exp\left\{-i\lambda_n \int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta\right\}}{\sqrt{g_0'\zeta}}$$

$$= \frac{A'\exp\left\{i\lambda_n \int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta\right\} + B'\exp\left\{-i\lambda_n \int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta\right\}}{\sqrt{\zeta}\, \overline{V}_z^{\frac{1}{4}}}$$

$$= \frac{1}{\sqrt{\zeta}\, \overline{V}_z^{\frac{1}{4}}} \left\{A \cos \left(\lambda_n \int_K^{\zeta} \sqrt{\overline{V}_z}\, d\zeta - \sigma\right)\right\}. \quad (4.1\text{-}42).$$

Equation (4.1-42) is the so-called WKB solution. It must be patched to the regular solution of Eq. (4.1-15) for $K < \zeta < 1$ for sufficiently large α_n.

If it is assumed that very near the walls the velocity profile can be expressed as a linear equation, Eq. (4.1-15) can be reduced to two simpler equations. Let

$$\eta_1 = \alpha_n^{2/3} (\zeta - K) \qquad (4.1\text{-}43).$$

Then

$$\overline{V}_z = \frac{(V_{max}/V_{avg})}{\int_K^{\lambda} (\frac{\lambda^2}{\zeta} - \zeta)^s \, d\zeta} \cdot \int_0^{\eta_1} \left[\frac{\lambda^2}{\frac{\eta_1}{\alpha_n^{2/3}} + K} - (\frac{\eta_1}{\alpha_n^{2/3}} + K) \right]^s d\eta_1 \rightarrow D_i \eta_1$$

as $\eta_1 \rightarrow 0$

where

$$D_i = \frac{(V_{max}/V_{avg})}{\int_K^{\lambda} (\frac{\lambda^2}{\zeta} - \zeta)^s \, d\zeta} \cdot \left[\frac{\lambda^2}{K} - K\right]^s \qquad (4.1\text{-}44).$$

Equation (4.1-15) then becomes

$$\frac{d^2E_n}{d\eta_1^2} + \frac{1}{\alpha_n^{2/3} \, (\frac{\eta_1}{\alpha_n^{2/3}} + K)} \frac{dE_n}{d\eta_1} + D_i \eta_1 E_n = 0 \qquad (4.1\text{-}45).$$

When α_n is large, Eq. (4.1-45) reduces to

$$\frac{d^2E_n}{d\eta_1^2} + D_i \eta_1 E_n = 0 \qquad (4.1\text{-}46),$$

which is a form of Bessel's equation. The solution is

$$E_n = \eta_1^{\frac{1}{2}} \left\{ G_1 J_{1/3}\left[\frac{2\sqrt{D_i}}{3} \eta_1^{3/2}\right] + H_1 J_{-1/3}\left[\frac{2\sqrt{D_i}}{3} \eta_1^{3/2}\right] \right\} \qquad (4.1\text{-}47).$$

By a similar procedure, the solution near the outer wall is found to be

$$E_n = \eta_2^{\frac{1}{2}} \left\{ G_2 J_{1/3}\left[\frac{2\sqrt{D_o}}{3} \eta_2^{3/2}\right] + H_2 J_{-1/3}\left[\frac{2\sqrt{D_o}}{3} \eta_2^{3/2}\right] \right\} \qquad (4.1\text{-}48)$$

where
$$D_o = \frac{(V_{max}/V_{avg})}{\int_K^{\lambda} \left(\frac{\lambda^2}{\zeta} - \zeta\right)^s d\zeta} \cdot \left[1 - \lambda^2\right]^s \qquad (4.1\text{-}49).$$

The patching of Eq. (4.1-42) - (4.1-47) and (4.1-48) can be performed by appropriately linearizing the velocity profile in Eq. (4.1-42) and by noting that, for large values of the argument, the Bessel functions appearing in Eqs. (4.1-47) and (4.1-48) can be expressed as cosine functions. This results in the following equations for the constants G_1, H_1, G_2, and H_2;

$$\begin{aligned}
G_1 \cos\frac{5}{12}\pi + H_1 \cos\frac{\pi}{12} &= \cos\sigma \\
G_1 \sin\frac{5}{12}\pi + H_1 \sin\frac{\pi}{12} &= \sin\sigma \\
G_2 \cos\frac{5}{12}\pi + H_2 \cos\frac{\pi}{12} &= K^{\frac{1}{2}} \cos(\alpha_n\gamma - \sigma) \\
G_2 \sin\frac{5}{12}\pi + H_2 \sin\frac{\pi}{12} &= K^{\frac{1}{2}} \sin(\alpha_n\gamma - \sigma)
\end{aligned} \qquad (4.1\text{-}50)$$

where
$$\gamma = \int_K^1 \sqrt{\overline{V}_z}\, d\zeta \qquad (4.1\text{-}51).$$

The derivation of these equations may be found in Appendix 9.3.1 Values of γ for different values of K and n are shown in Table 3.

Table 3. Constants in the Asymptotic Solution

n	K	γ	n	K	γ
0.5	0.2	0.794354	0.8	0.2	0.788975
0.5	0.5	0.487686	0.8	0.5	0.484202

After evaluating the constants, it is found that near the inner wall

$$E_n = \eta_1^{\frac{1}{2}} \left\{ \frac{2}{\sqrt{3}} \sin\left(\sigma - \frac{\pi}{12}\right) J_{1/3}\left[\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right] - \frac{2}{\sqrt{3}} \sin\left(\sigma - \frac{5\pi}{12}\right) J_{-1/3}\left[\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right] \right\} \qquad (4.1\text{-}52),$$

and near the outer wall

$$E_n = \eta_2^{\frac{1}{2}} \left\{ \frac{2K^{\frac{1}{2}}}{\sqrt{3}} \sin(\alpha_n \gamma - \sigma - \frac{\pi}{12}) J_{1/3}(\frac{2\sqrt{D_o}}{3} \eta_1^{3/2}) \right.$$

$$\left. - \frac{2K^{\frac{1}{2}}}{\sqrt{3}} \sin(\alpha_n \gamma - \sigma - \frac{5\pi}{12}) J_{-1/3}\left[\frac{2\sqrt{D_o}}{3} \eta_2^{3/2}\right] \right\} \qquad (4.1\text{-}53).$$

Using the boundary conditions, Eq. (4.1-16) yields

$$\sin(\sigma - \frac{\pi}{12}) = 0$$

$$\qquad (4.1\text{-}54).$$

$$\sin(\alpha_n \gamma - \sigma - \frac{\pi}{12}) = 0$$

$$\therefore \quad \alpha_n = (n + \frac{1}{6})\pi/\gamma \quad . \quad n=1,2,3... \qquad (4.1\text{-}55)$$

This is the asymptotic expression for the eigenvalues. The results are shown in Tables 1 and 2 and a derivation of Eq. (4.1-55) is provided in Appendix 9.3.2.

For computational purposes, particularly to establish the asymptotic values of the C_n, it is necessary to provide a more convenient form for the integrals appearing in Eq. (4.1-30). It is desired to evaluate the integral, $\int_K^1 \overline{V}_z \zeta E_n^2 d\zeta$, which is the norm of the eigenfunction. Taking the derivation of Eq. (4.1-15) with respect to α_n yields

$$\frac{\partial}{\partial \alpha_n}\left[\frac{\partial}{\partial \zeta}(\zeta \frac{\partial E_n}{\partial \zeta})\right] + 2\alpha_n \overline{V}_z \zeta E_n + \alpha_n^2 \overline{V}_z \zeta \frac{\partial E_n}{\partial \alpha_n} = 0 \qquad (4.1\text{-}56).$$

Since the order of partial differentiation may be reversed, this may be written as

$$\frac{\partial}{\partial \zeta}\left[\zeta \frac{\partial}{\partial \zeta}(\frac{\partial E_n}{\partial \alpha_n})\right] + 2\alpha_n \overline{V}_z \zeta E_n + \alpha_n^2 \overline{V}_z \zeta \frac{\partial E_n}{\partial \alpha_n} = 0 \qquad (4.1\text{-}57).$$

Multiplying Eq. (4.1-57) by E_n and integrating between K and 1 leads to

$$\int_K^1 E_n \frac{\partial}{\partial\zeta}\left[\zeta \frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\zeta}\right)\right]d\zeta + 2\alpha_n \int_K^1 \overline{V}_z \zeta E_n^2 d\zeta + \alpha_n^2 \int_K^1 \overline{V}_z \zeta \frac{\partial E_n}{\partial \alpha_n} E_n d\zeta = 0$$

(4.1-58).

Integrating by parts twice yields

$$\int_K^1 \overline{V}_z \zeta E_n^2 d\zeta = \frac{1}{2\alpha_n}\left\{-\left[\zeta E_n \frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\right]_K^1 + \left[\left(\frac{\partial E_n}{\partial\alpha_n}\right)\cdot \zeta\left(\frac{\partial E_n}{\partial\zeta}\right)\right]_K^1\right\} \quad (4.1\text{-}59)$$

$$= -\frac{1}{2\alpha_n}\left\{E_n(1)\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\Big|_{\zeta=1} - KE_n(K)\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\Big|_{\zeta=K}\right\} \quad (4.1\text{-}60).$$

The numerator of Eq. (4.1-30) has the form, $\int_K^1 \zeta \overline{V}_z G E_n d\zeta$. Multiplying Eq. (4.1-15) by $G(\zeta)$ and integrating by parts yields

$$\int_K^1 \zeta \overline{V}_z G E_n d\zeta = \frac{1}{\alpha_n^2}\left\{\left[\zeta \frac{dG}{d\zeta} E_n\right]_K^1 - \int_K^1 E_n \frac{d}{d\zeta}\left(\zeta \frac{dG}{d\zeta}\right)d\zeta\right\} \quad (4.1\text{-}61).$$

Recalling Eqs. (4.1-24) and (4.1-15), it is found that

$$\frac{d}{d\zeta}\left(\zeta \frac{dG}{d\zeta}\right) = C_g \zeta \overline{V}_z \quad (4.1\text{-}62),$$

$$\zeta \overline{V}_z E_n = -\frac{1}{\alpha_n^2}\frac{d}{d\zeta}\left(\zeta \frac{dE_n}{d\zeta}\right) \quad (4.1\text{-}63).$$

Then
$$\int_K^1 E_n \frac{d}{d\zeta}\left(\zeta \frac{dG}{d\zeta}\right)d\zeta = C_g \int_K^1 \zeta \overline{V}_z E_n d\zeta$$

$$= -\frac{C_g}{\alpha_n^2}\left[\zeta \frac{dE_n}{d\zeta}\right]_K^1$$

$$= 0 \quad (4.1\text{-}64).$$

Therefore,
$$\int_K^1 \zeta \overline{V}_z G E_n d\zeta = \frac{1}{\alpha_n^2}\left[\zeta \frac{dG}{d\zeta} E_n\right]_K^1$$

$$= \frac{KE_n(K)}{\alpha_n^2} \quad (4.1\text{-}65).$$

Thus, the coefficients of the infinite series may be written

$$C_n = \frac{2KE_n(K)}{\alpha_n\left\{E_n(1)\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\Big|_{\zeta=1} - KE_n\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\Big|_{\zeta=K}\right\}} \qquad (4.1\text{-}66).$$

Further simplification can be made by substituting appropriate terms derived from Eq. (4.1-47) and (4.1-48) into Eq. (4.1-66). Differentiating Eq. (4.1-52) and introducing the condition of Eq. (4.1-54) yields

$$\frac{\partial E_n}{\partial\alpha_n} = -\frac{2}{\sqrt{3}}\sin\left(\sigma - \frac{5}{12}\pi\right)\frac{\partial}{\partial\alpha_n}\left[\eta_1^{\frac{1}{2}}J_{-1/3}\left(\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right)\right]$$

$$= -\alpha_n^{1/3}\,\frac{2}{3}\sqrt{D_i}\,(\zeta - K)^2\,J_{2/3}\left[\frac{2\sqrt{D_i}}{3}\alpha_n(\zeta - K)^{3/2}\right] \qquad (4.1\text{-}67).$$

Differentiating again with respect to ζ leads to

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right) = -\frac{2}{3}\alpha_n^{1/3}\sqrt{D_i}\left\{2(\zeta - K)J_{2/3}\left[\frac{2\sqrt{D_i}}{3}\alpha_n(\zeta - K)^{3/2}\right]\right.$$

$$\left. + (\zeta - K)^2\frac{\partial}{\partial\zeta}J_{2/3}\left[\frac{-2\sqrt{D_i}}{3}\alpha_n(\zeta - K)^{3/2}\right]\right\} \qquad (4.1\text{-}68)$$

as $\zeta \to K$, $(\zeta - K)J_{2/3} \to 0$ and $(\zeta - K)^2\frac{\partial}{\partial\zeta}J_{2/3} \to 0$.

Therefore,

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial E_n}{\partial\alpha_n}\right)\Big|_{\zeta=K} = 0 \qquad (4.1\text{-}69).$$

Differentiating Eq. (4.1-53) and introducing the condition of Eq. (4.1-54) yields

$$\frac{\partial E_n}{\partial\alpha_n} = \frac{2}{\sqrt{3}}K^{\frac{1}{2}}\left\{\gamma\cos\left(\alpha_n\gamma - \sigma - \frac{\pi}{12}\right)\eta_2^{1/2}J_{1/3}\left(\frac{2\sqrt{D_o}}{3}\eta_2^{3/2}\right)\right.$$

$$-\gamma\cos\left(\alpha_n\gamma - \sigma - \frac{5\pi}{12}\right)\eta_2^{\frac{1}{2}}J_{-1/3}\left(\frac{2\sqrt{D_o}}{3}\eta_2^{3/2}\right)$$

$$\left.-\sin\left(\alpha_n\gamma - \sigma - \frac{5\pi}{12}\right)\frac{\partial}{\partial\alpha_n}\left[\eta_2^{\frac{1}{2}}J_{-1/3}\left(\frac{2\sqrt{D_o}}{3}\eta_2^{3/2}\right)\right]\right\} \qquad (4.1\text{-}70).$$

For small η_2,

$$\frac{\partial E_n}{\partial \alpha_n} = \frac{2}{\sqrt{3}} K^{\frac{1}{2}} \left\{ \gamma \cos(\alpha_n \gamma - \sigma - \frac{\pi}{12}) \cdot \frac{D_o^{1/6}}{(\frac{1}{3})! 3^{1/3}} - \gamma \cos(\alpha_n \gamma - \sigma - \frac{5\pi}{12}) \cdot \frac{3^{1/3}}{(-\frac{1}{3})! D_o^{1/6}} \right.$$

$$\left. - \sin(\alpha_n \gamma - \sigma - \frac{5\pi}{12}) \cdot \frac{\partial}{\partial \alpha_n} \left[\frac{3^{1/3}}{(-\frac{1}{3})! D_o^{1/6}} \right] \right\} \qquad (4.1\text{-}71).$$

Therefore,

$$\frac{\partial}{\partial \zeta}\left(\frac{\partial E_n}{\partial \alpha_n}\right)\Bigg|_{\zeta=1} = (-1)^{n+1} \frac{2\gamma D_o^{1/6} \alpha_n^{2/3} K^{\frac{1}{2}}}{\Gamma(\frac{4}{3}) \cdot 3^{5/6}} \qquad (4.1\text{-}72).$$

Again from Eq. (4.1-47) and (4.1-48),

$$E_n(K) = \eta_1^{\frac{1}{2}} J_{-1/3} \left(\frac{2\sqrt{D_i}}{3} \eta_1^{3/2} \right) \Bigg|_{\zeta=K}$$

$$= \eta_1^{\frac{1}{2}} \frac{2^{1/3}}{(-\frac{1}{3})!} \left(\frac{2\sqrt{D_i}}{3} \eta_1^{3/2} \right)^{-1/3} \Bigg|_{\zeta=K}$$

$$= \frac{3^{1/3}}{\Gamma(\frac{2}{3}) \cdot D_i^{1/6}} \qquad (4.1\text{-}73).$$

$$E_n(1) = \frac{(-1)^n \cdot 3^{1/3} K^{\frac{1}{2}}}{\Gamma(\frac{2}{3}) \cdot D_o^{1/6}} \qquad (4.1\text{-}74).$$

Consequently,

$$C_n = -\frac{3^{5/6}\, \Gamma(\frac{4}{3})}{\alpha_n^{5/3} \cdot \gamma \cdot D_i^{1/6}} \qquad (4.1\text{-}75).$$

Eq. (4.1-55) and Eq. (4.1-75) may be solved numerically. Eigenvalues, expansion coefficients and combined function, $C_n E_n$, calculated by the both methods are shown in Tables 1 and 2. Temperature profiles and variation of the Nusselt number and average temperature with axial distance based on the data calculated from the iterative method are shown in Figs. 3-8.

For small η_2,

$$\frac{\partial E_n}{\partial \alpha_n} = \frac{2}{\sqrt{3}} K^{\frac{1}{2}} \left\{ \gamma \cos(\alpha_n \gamma - \sigma - \frac{\pi}{12}) \cdot \frac{D_o^{1/6}}{(\frac{1}{3})!\,3^{1/3}} - \gamma \cos(\alpha_n \gamma - \sigma - \frac{5\pi}{12}) \cdot \frac{3^{1/3}}{(-\frac{1}{3})!\,D_o^{1/6}} \right.$$

$$\left. - \sin(\alpha_n \gamma - \sigma - \frac{5\pi}{12}) \cdot \frac{\partial}{\partial \alpha_n} \left[\frac{3^{1/3}}{(-\frac{1}{3})!\,D_o^{1/6}} \right] \right\} \qquad (4.1\text{-}71).$$

Therefore,

$$\frac{\partial}{\partial \zeta}\left(\frac{\partial E_n}{\partial \alpha_n}\right)\bigg|_{\zeta=1} = (-1)^{n+1} \frac{2\gamma D_o^{1/6} \alpha_n^{2/3} K^{\frac{1}{2}}}{\Gamma(\frac{4}{3}) \cdot 3^{5/6}} \qquad (4.1\text{-}72).$$

Again from Eq. (4.1-47) and (4.1-48),

$$E_n(K) = \eta_1^{\frac{1}{2}} J_{-1/3} \left(\frac{2\sqrt{D_i}}{3} \eta_1^{3/2}\right)\bigg|_{\zeta=K}$$

$$= \eta_1^{\frac{1}{2}} \frac{2^{1/3}}{(-\frac{1}{3})!} \left(\frac{2\sqrt{D_i}}{3} \eta_1^{3/2}\right)^{-1/3}\bigg|_{\zeta=K}$$

$$= \frac{3^{1/3}}{\Gamma(\frac{2}{3}) \cdot D_i^{1/6}} \qquad (4.1\text{-}73).$$

$$E_n(1) = \frac{(-1)^n \cdot 3^{1/3} K^{\frac{1}{2}}}{\Gamma(\frac{2}{3}) \cdot D_o^{1/6}} \qquad (4.1\text{-}74).$$

Consequently,

$$C_n = -\frac{3^{5/6}\,\Gamma(\frac{4}{3})}{\alpha_n^{5/3} \cdot \gamma \cdot D_i^{1/6}} \qquad (4.1\text{-}75).$$

Eq. (4.1-55) and Eq. (4.1-75) may be solved numerically. Eigenvalues, expansion coefficients and combined function, $C_n E_n$, calculated by the both methods are shown in Tables 1 and 2. Temperature profiles and variation of the Nusselt number and average temperature with axial distance based on the data calculated from the iterative method are shown in Figs. 3-8.

4.2 Case II: Heat Transfer in an Annulus with Equal Wall Temperatures at Both Walls; also Constant Temperature at the Inner Wall, Insulation at the Outer Wall

In these two problems, the flow conditions are the same as that in Section 4.1, but the boundary conditions [illegible] equations are different. Instead of having a uniform heat flux on the inner wall, a fixed constant temperature is maintained on it. On the outer wall, the conditions of constant temperature, or of insulation, may be treated simultaneously. Since the boundary conditions in these problems can be made homogeneous with respect to the Sturm - Liouville problem, the energy equation is readily solved by the method of separation of variables.

4.2.1 Solution of the Problem

The energy equation describing the problem is

$$\rho C_p V_z \frac{\partial T}{\partial z} = k \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \qquad (4.2\text{-}1).$$

The following boundary conditions are considered:

Problem 2

$$T(o,r) = T_e$$

$$T(z,k) = T_o \qquad \text{for } z > 0$$

$$T(z,KR) = T_o \qquad \text{for } z > 0$$

Problem 3 $\qquad (4.2\text{-}2).$

$$T(o,r) = T_e$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = 0 \qquad \text{for } z > 0$$

$$T(z,KR) = T_o \qquad \text{for } z > 0$$

Introducing the following dimensionless variables,

$$\zeta = \frac{r}{R}$$

$$\xi = \frac{z}{z_d}$$

$$\bar{\theta} = \frac{T - T_o}{T_e - T_o} \tag{4.2-3}$$

$$\bar{V}_z = \frac{V_z}{V_{avg}}$$

$$z_d = \frac{R_e P_r R}{2(1 - K)}$$

yields
$$\bar{V}_z \frac{\partial \bar{\theta}}{\partial \xi} = \frac{1}{\zeta} \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial \bar{\theta}}{\partial \zeta}\right) \tag{4.2-4}$$

with boundary conditions

Problem 2

$$\bar{\theta}(o,\zeta) = 1$$

$$\bar{\theta}(\xi,1) = 0 \qquad \text{for } \xi > 0$$

$$\bar{\theta}(\xi,K) = 0 \qquad \text{for } \xi > 0$$

Problem 3 (4.2-5).

$$\bar{\theta}(0,\zeta) = 1$$

$$\left.\frac{\partial \bar{\theta}}{\partial \zeta}\right|_{\zeta=1} = 0 \qquad \text{for } \xi > 0$$

$$\bar{\theta}(\xi,K) = 0 \qquad \text{for } \xi > 0$$

Equation (4.2-4), with the last two boundary conditions of each problem, is a differential equation with homogeneous boundary conditions on the ζ variable. Using the method of the separation of variables in the same way as in Sec. 4.1.2, set $\bar{\theta}(\xi,\zeta) = Z(\xi)\cdot E(\zeta)$, thereby obtaining the two equations with the boundary conditions shown.

$$\frac{dZ_n}{d\xi} = -\alpha_n^2 Z_n \tag{4.2-6}$$

$$\frac{d}{d\zeta}\left(\zeta \frac{dE_n}{d\zeta}\right) + \alpha_n^2 \zeta \overline{V}_z E_n = 0 \tag{4.2-7}$$

with boundary conditions

Problem 2

$$E_n(1) = 0 \qquad \text{for } \xi > 0$$

$$E_n(K) = 0 \qquad \text{for } \xi > 0$$

Problem 3 (4.2-8)

$$\left.\frac{\partial E_n}{\partial \zeta}\right|_{\zeta=1} = 0 \qquad \text{for } \xi > 0$$

$$E_n(K) = 0 \qquad \text{for } \xi > 0$$

and $\overline{\theta} = E_n(\zeta)Z_n(0) = 1$ (4.2-9).

The solution is

$$\overline{\theta} = \sum_{n=1}^{\infty} B_n E_n \exp(-\alpha_n^2 \xi) \tag{4.2-10}$$

with coefficients

$$B_n = \frac{\int_K^1 \zeta \overline{V}_z E_n d\zeta}{\int_K^1 \zeta \overline{V}_z E_n^2 d\zeta} \tag{4.2-11}$$

Eigenvalues and corresponding expansion coefficients and combined functions are shown in Tables 4-7.

4.2.2 Expressions for the Nusselt Numbers

The Nusselt numbers are defined by $Nu = \frac{D_e h}{k}$

where $h_o(T_o - T_{avg}) = +k \left.\frac{\partial T}{\partial r}\right|_{r=R}$

and $h_i(T_i - T_{avg}) = -k \left.\frac{\partial T}{\partial r}\right|_{r=KR}$ (4.2-12).

Table 4. Functions in the solution of problem II by the iterative method for n=0.5

Radius ratio	Eigenvalue α_n	Expansion Coeff., B_n	$B_n E'_n(K)$	$B_n E'_n(1)$	$B_n\left[E'_n(1)-KE'_n(K)\right]$
0.2	3.3749	-0.652722	8.187139	-3.208594	-4.846022
	7.2493	0.083455	-2.023264	-0.772690	-0.368037
	11.1914	-0.161466	5.595233	-2.155558	-3.274605
	15.1376	0.035658	-1.598649	-0.612785	-0.293055
	19.0857	-0.086441	4.730517	-1.808780	-2.754883
	23.0425	0.021799	-1.409237	-0.535454	-0.253607
	26.8886	-0.057558	4.300519	-1.618970	-2.479074
0.5	5.5836	-0.580416	8.935970	-6.033153	-10.501138
	11.8406	0.034367	-1.015515	-0.673411	-0.165653
	18.2676	-0.147369	6.226329	-4.150868	-7.264032
	24.6865	0.014698	-0.801174	-0.532289	-0.131702
	31.1164	-0.079116	5.241128	-3.496911	-6.117475

Table 5. Functions in the solution of problem II by the iterative method for n=0.8

Radius ratio	Eigenvalue α_n	Expansion Coeff., B_n	$B_n E'_n(K)$	$B_n E'_n(1)$	$B_n\left[E'_n(1)-KE'_n(K)\right]$
0.2	3.3122	-0.656781	7.960136	-3.130316	-4.722344
	7.2652	0.084536	-1.954267	-0.770919	-0.380066
	11.2477	-0.155744	5.148884	-2.045140	-3.074917
	15.2264	0.035863	-1.531163	-0.606550	-0.300317
	19.2064	-0.083197	4.325253	-1.712811	-2.577862
	23.1919	0.021906	-1.342649	-0.529185	-0.260655
	27.1820	-0.055337	3.912067	-1.529902	-2.312315
0.5	5.4844	-0.582620	8.680105	-5.871877	-10.211930
	11.8880	0.034998	-0.997143	-0.676870	-0.178298
	18.3764	-0.141961	5.800221	-3.919578	-6.819689
	24.8497	0.014852	-0.782219	-0.527490	-0.136480
	31.3333	-0.076107	4.870570	-3.286380	-5.721665

Table 6. Functions in the solution of problem III by the iterative method for n=0.5

Radius ratio	Eigenvalue α_n	Expansion Coeff. B'_n	$B'_nE'_n(K)$
0.2	1.4632	-0.673244	4.80837590
	5.7299	-0.123214	2.49363260
	9.7283	-0.066002	2.04800080
	13.7159	-0.043251	1.78785030
	17.6868	-0.032926	1.69100360
	21.6128	-0.025059	1.53566560
	25.6319	-0.021547	1.53646000
0.5	2.7747	-0.585164	5.26671000
	9.4404	-0.138666	3.45031730
	15.9337	-0.074649	2.82480400
	22.4204	-0.049671	2.50305580
	28.8651	-0.037380	2.32838520

Table 7. Functions in the solution of problem III by the iterative method for n=0.8

Radius ratio	Eigenvalue α_n	Expansion Coeff. B'_n	$B'_nE'_n(K)$
0.2	1.4555	-0.676644	4.82575730
	5.7442	-0.117378	2.27100010
	9.7874	-0.062326	1.84624880
	13.8046	-0.040782	1.60756830
	17.8051	-0.030992	1.51375930
	21.8067	-0.023619	1.37338690
	25.8088	-0.020275	1.36893650
0.5	2.7698	-0.587556	5.29570090
	9.5026	-0.132201	3.16349060
	16.0538	-0.071165	2.60591990
	22.5730	-0.047374	2.30801620
	29.0795	-0.035617	2.14290390

Thus,

$$Nu_o = + \frac{D_o \left.\frac{\partial T}{\partial r}\right|_{r=R}}{T_o - T_{avg}} = - \frac{2(1 - K)\left.\frac{\partial \bar{\theta}}{\partial \zeta}\right|_{\zeta=1}}{\bar{\theta}_{avg}} \quad (4.2\text{-}13),$$

and

$$Nu_i = \frac{2(1 - K)\left.\frac{\partial \bar{\theta}}{\partial \zeta}\right|_{\zeta=K}}{\bar{\theta}_{avg}} \quad (4.2\text{-}14).$$

It can be shown that

$$\bar{\theta}_{avg} = \frac{2}{1 - K^2} \int_K^1 \bar{V}_z \bar{\theta} \zeta d\zeta$$

$$= \frac{2}{1 - K^2} \sum_{n=1}^{\infty} B_n \exp(-\alpha_n^2 \xi) \int_K^1 \zeta \bar{V}_z E_n d\zeta \quad (4.2\text{-}15).$$

Integrating Eq. (4.2-7) with respect to ζ from K to 1 results in

$$\int_K^1 \zeta \bar{V}_z E_n d\zeta = - \frac{1}{\alpha_n^2} \left[E_n'(1) - K E_n'(K) \right] \quad (4.2\text{-}16).$$

Therefore, $$\bar{\theta}_{avg} = - \frac{2}{1 - K^2} \sum_{n=1}^{\infty} B_n \exp(-\alpha_n^2 \xi) \cdot \frac{1}{\alpha_n^2} \left[E_n'(1) - K E_n'(K) \right] \quad (4.2\text{-}17),$$

and Nu can be expressed as

$$Nu_o = \frac{(1 - K)(1 - K^2) \sum_{n=1}^{\infty} B_n E_n'(1) \exp(-\alpha_n^2 \xi)}{\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} B_n \left[E_n'(1) - K E_n'(K) \right] \exp(-\alpha_n^2 \xi)} \quad (4.2\text{-}18)$$

$$Nu_i = - \frac{(1 - K)(1 - K^2) \sum_{n=1}^{\infty} B_n E_n'(K) \exp(-\alpha_n^2 \xi)}{\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} B_n \left[E_n'(1) - K E_n'(K) \right] \exp(-\alpha_n^2 \xi)} \quad (4.2\text{-}19).$$

When $\zeta \rightarrow 0$, $Nu \rightarrow \infty$. For values above a certain $\xi = \xi_e$, Nu will not differ by more than a few percent from the final asymptotic value of the Nusselt number, Nu_a. The region between 0 and ξ_e is called the thermal entrance region. In this region, Nu decreases from an infinitely large value at $\xi = 0$ to Nu_a for $\xi > \xi_e$. For large value of ξ, the first term of these series for Nu dominates, so that

$$Nu_{a,o} = \frac{(1 - K)(1 - K^2)\alpha_1^2 E_1'(1)}{\left[E_1'(1) - KE_1'(K)\right]} \qquad (4.2\text{-}20),$$

and

$$Nu_{a,i} = -\frac{(1 - K)(1 - K^2)\alpha_1^2 E_1'(K)}{\left[E_1'(1) - KE_1'(K)\right]} \qquad (4.2\text{-}21)$$

are the asymptotic or fully developed Nusselt numbers at the outer and the inner walls, respectively. Temperature profile development and variation of the Nusselt number and average temperature with axial distance are shown in Figs. 9-16.

4.2.3 Asymptotic Solution by the WKB Method

The WKB method was presented in Sec. (4.1.4). To solve the present problems, we have only to substitute the boundary conditions of Eq. (4.2-8) into Eq. (4.1-52) and 4.1-53). Thus,

$$\begin{aligned} &\sin(\sigma - \tfrac{5}{12}\pi) = 0 \\ &\sin(\alpha_n\gamma - \sigma - \tfrac{5}{12}\pi) = 0 \end{aligned} \qquad (4.2\text{-}22)$$

for problem 2, therefore

$$\alpha_n = (n + \tfrac{5}{6})\pi /\gamma \qquad (4.2\text{-}23),$$

and

$$\begin{aligned} &\sin(\sigma - \tfrac{5}{12}\pi) = 0 \\ &\sin(\alpha_n\gamma - \sigma - \tfrac{\pi}{12}) = 0 \end{aligned} \qquad (4.2\text{-}24)$$

for problem 3, therefore

$$\alpha_n = (n + \tfrac{1}{2})\pi /\gamma \qquad (4.2\text{-}25).$$

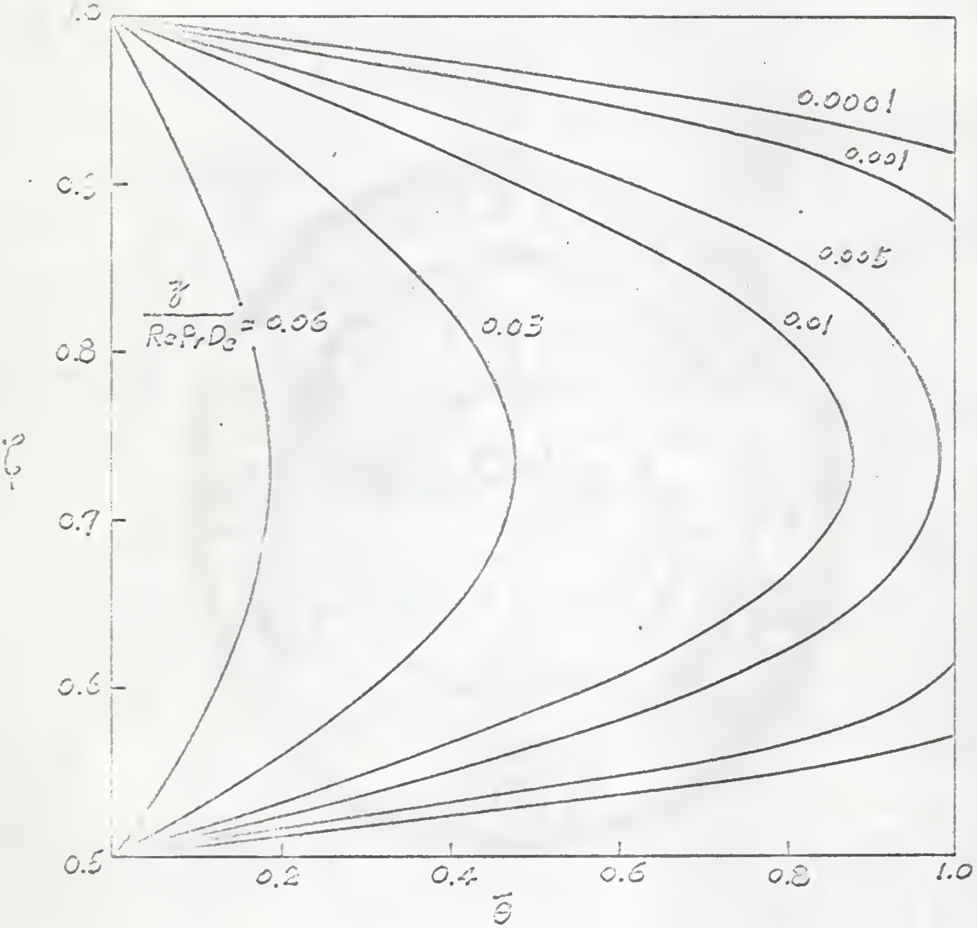

Fig. 9 Temperature profile development, problem II, K=0.5, n=0.5

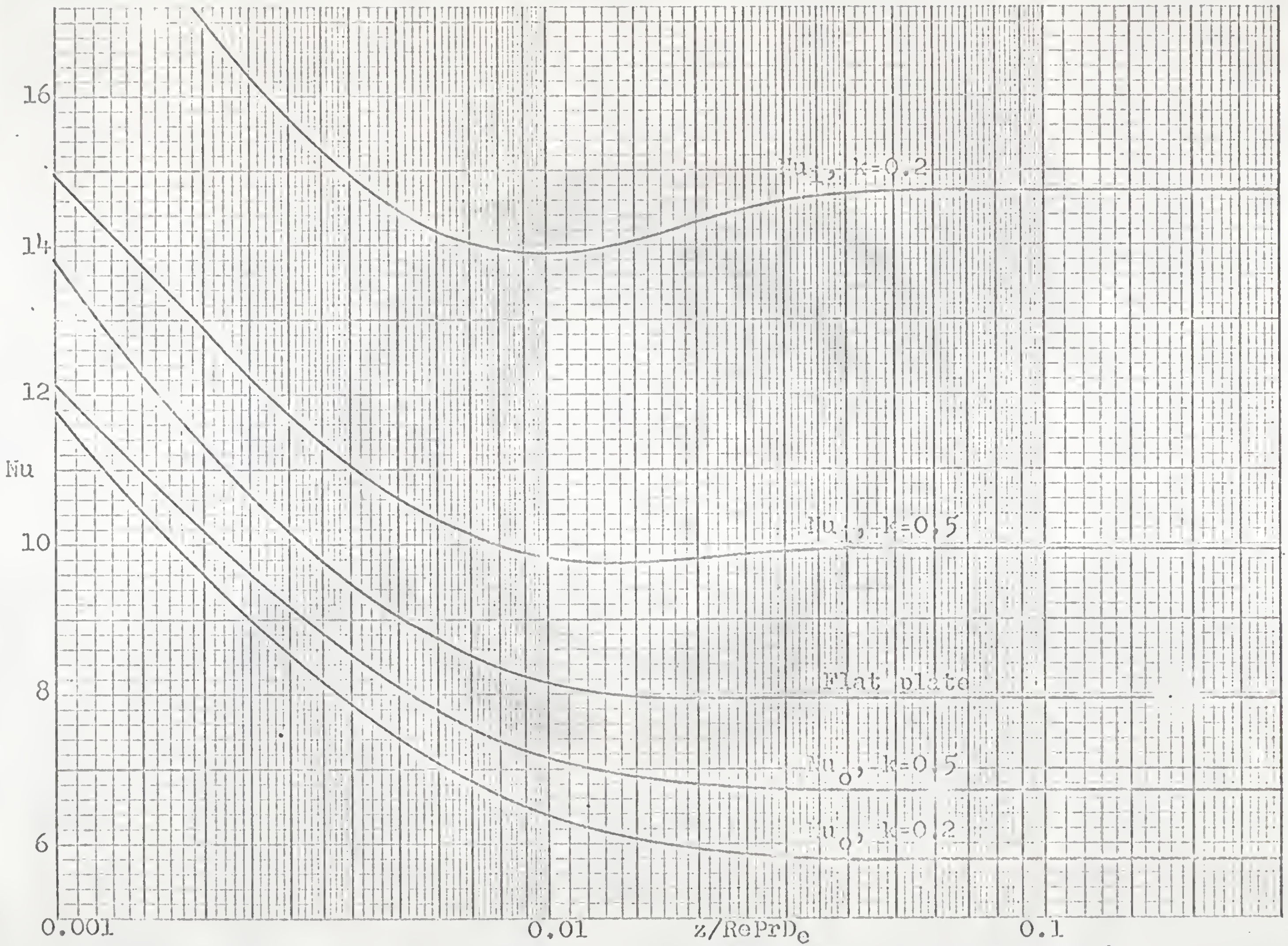

Fig. 10. Nusselt number versus axial distance, problem II, n=0.5.

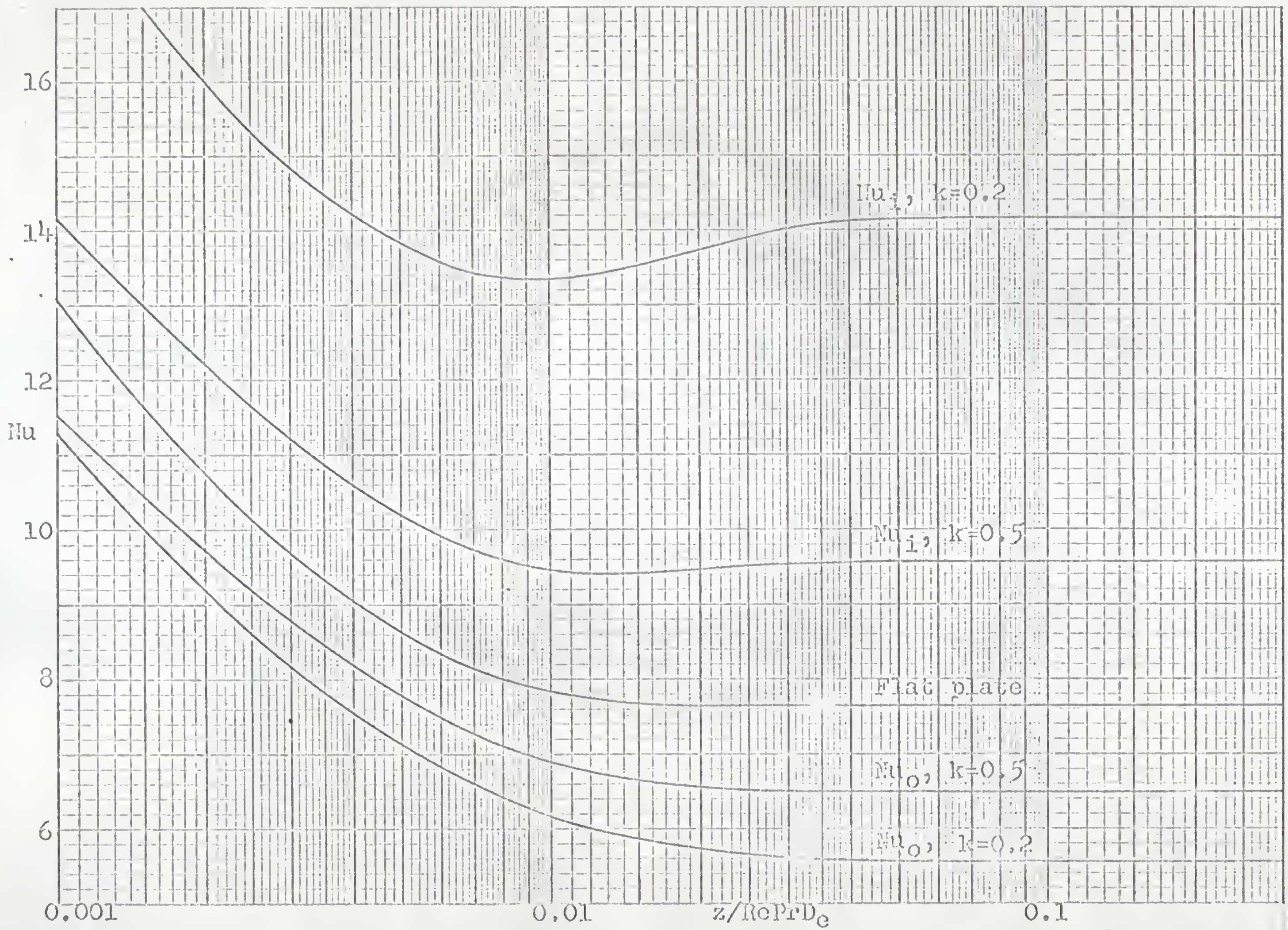

Fig. 11. Nusselt number versus axial distance, problem II, n=0.8.

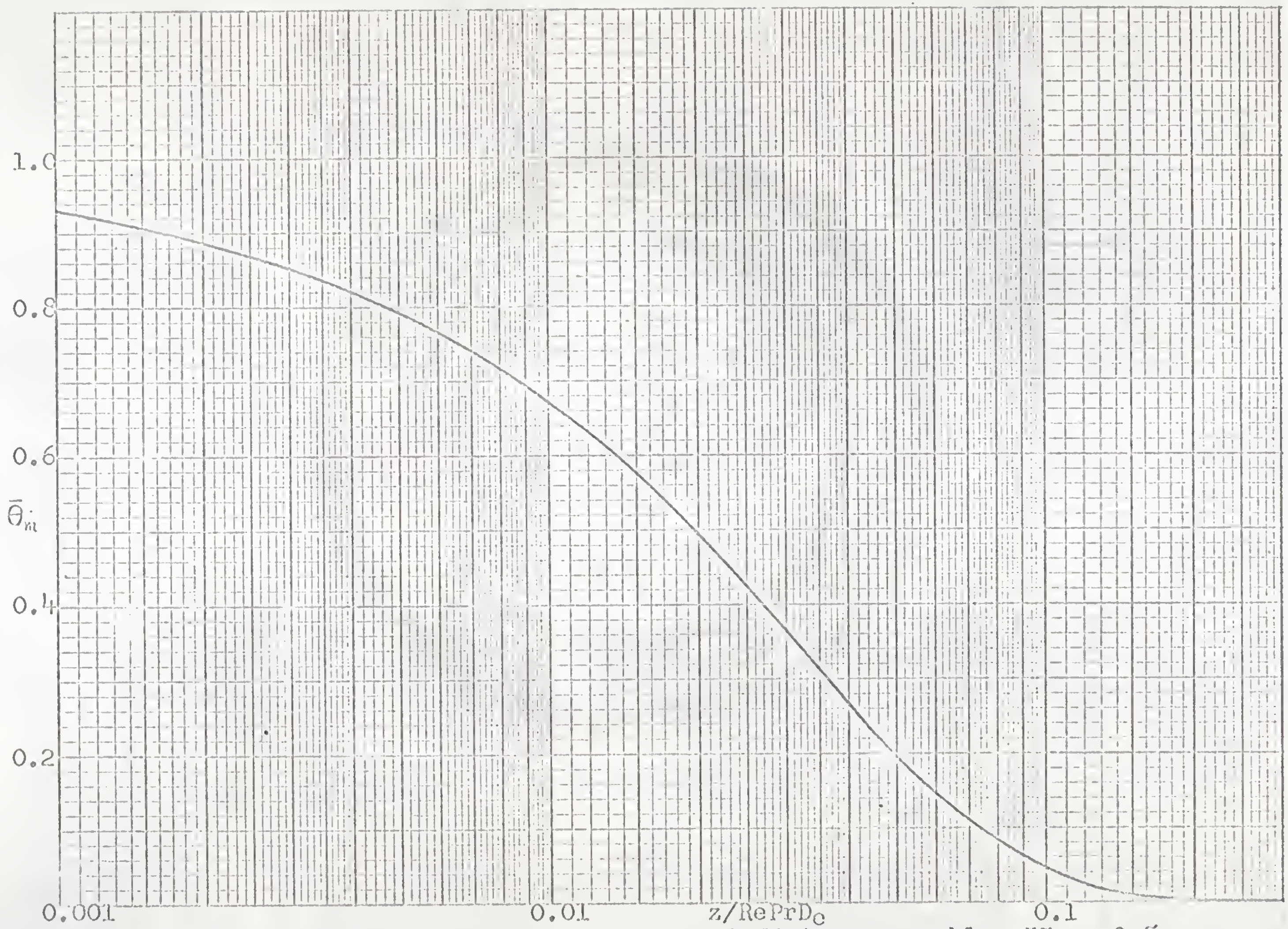

Fig. 12. Average temperature versus axial distance, problem II, n=0.5.

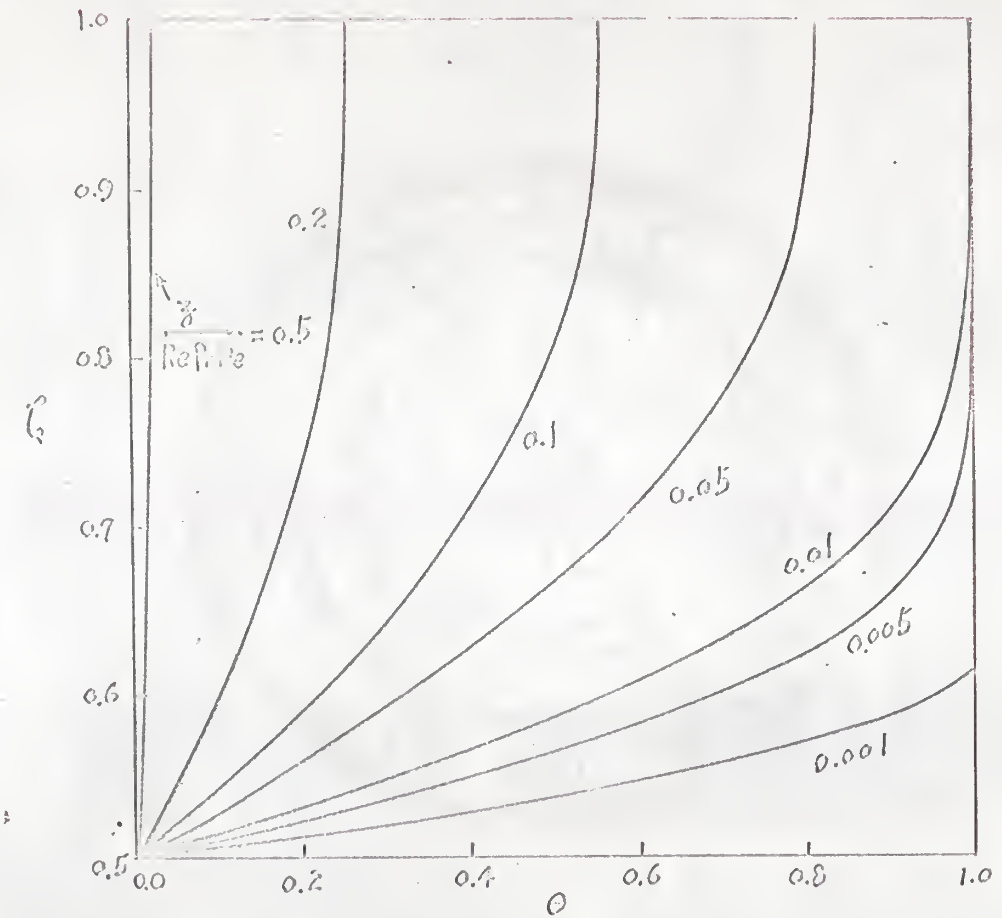

Fig. 13 Temperature profile development, problem III, K=0.5, n=0.5

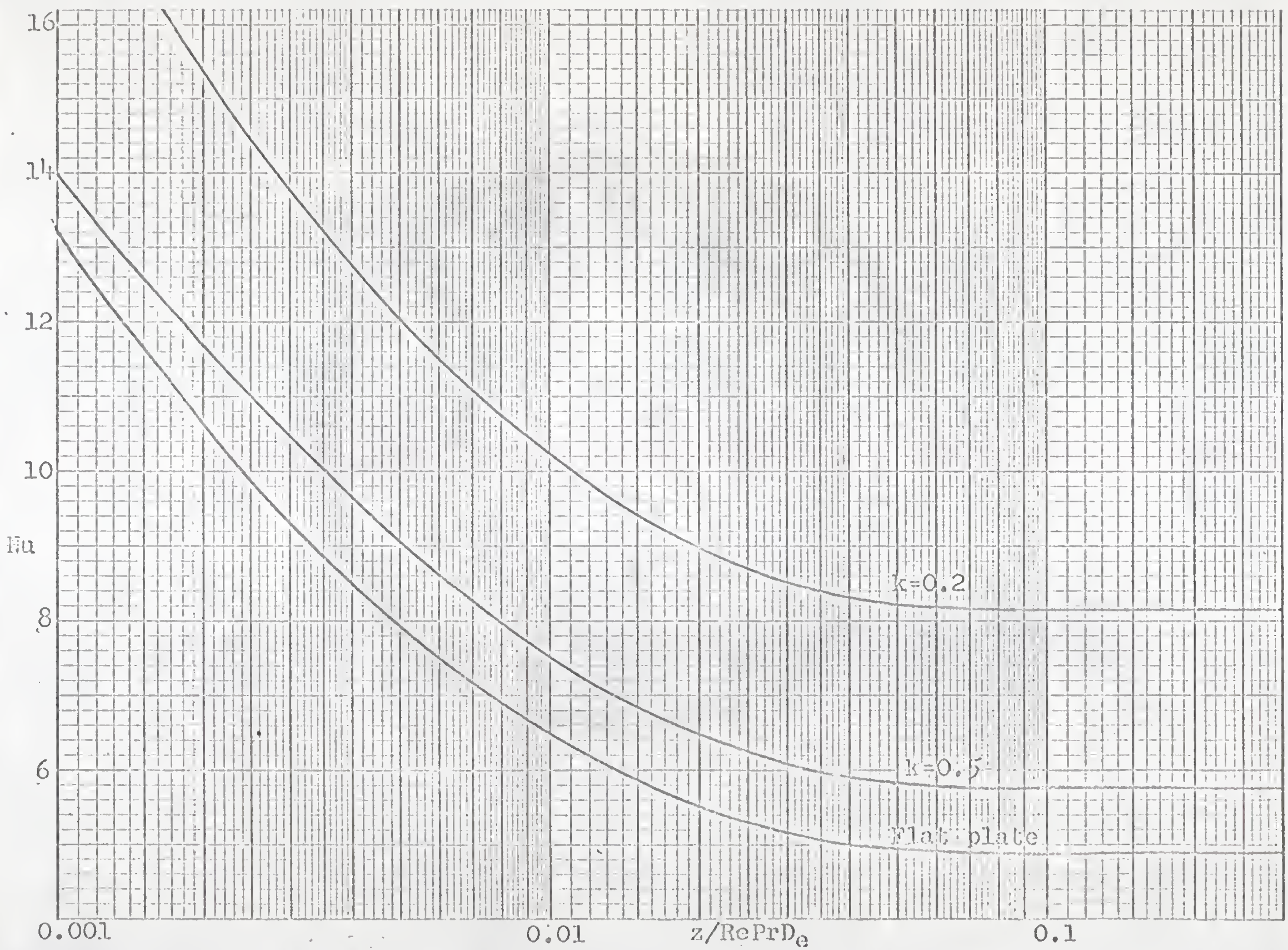

Fig. 14. Nusselt number versus axial distance, problem III, n=0.5

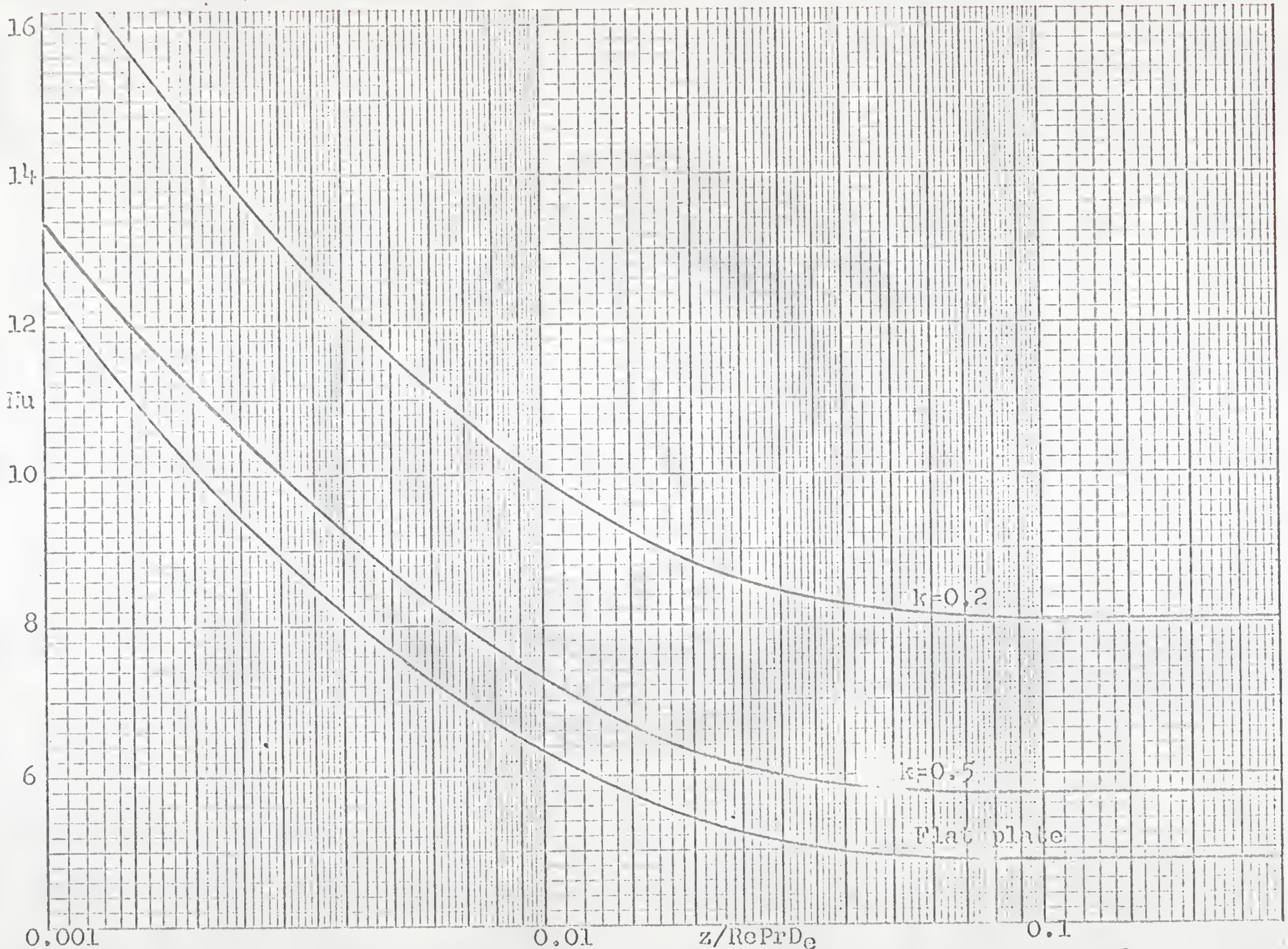

Fig. 15. Nusselt number versus axial distance, problem III, n=0.8.

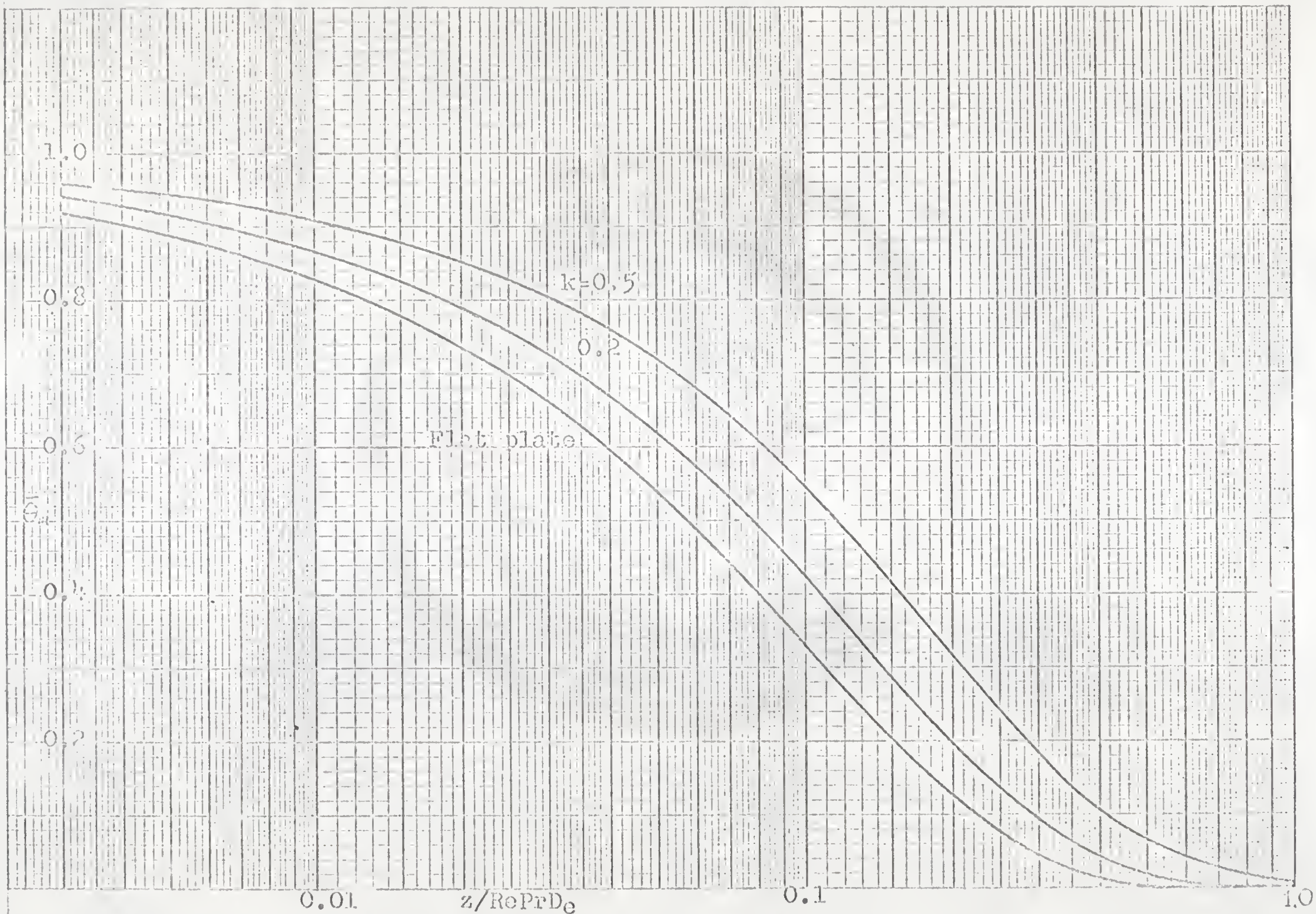

Fig. 16. Average temperature versus axial distance, problem III, n=0.5.

Equations (4.2-23) and (4.2-25) are the asymptotic expressions for the eigenvalues and are easy to evaluate. The results are shown in Tables 8-11. These results should be compared with those obtained by the iterative method and are presented in Tables 4-7.

For the evaluation of the coefficients of the infinite series, it is necessary to rewrite Eq. (4.2-11) in terms of expressions which are obtainable from the WKB method. Substituting the appropriate boundary conditions into Eq. (4.1-59) yields

$$\int_K^1 \overline{V}_z \zeta E_n^2 d\zeta = \frac{1}{2\alpha_n}\left[\frac{\partial E_n}{\partial \alpha_n} \cdot \zeta \cdot \frac{\partial E_n}{\partial \zeta}\right]_K^1 \qquad (4.2\text{-}26)$$

for problem 2, and

$$\int_K^1 \overline{V}_z \zeta E_n^2 d\zeta = -\left\{\frac{1}{2\alpha_n} E_n(1)\frac{\partial}{\partial \zeta}\left(\frac{\partial E_n}{\partial \alpha_n}\right)\bigg|_{\zeta=1} + \frac{\partial E_n}{\partial \alpha_n}\zeta\frac{\partial E_n}{\partial \zeta}\bigg|_{\zeta=K}\right\} \qquad (4.2\text{-}27)$$

for problem 3.

Furthermore, from Eq. (4.2-16),

$$\int_K^1 \zeta \overline{V}_z E_n d\zeta = -\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] \qquad (4.2\text{-}28)$$

for problem 2, and

$$\int_K^1 \zeta \overline{V}_z E_n d\zeta = \frac{1}{\alpha_n^2} KE_n'(K) \qquad (4.2\text{-}29)$$

for problem 3.

Therefore,

$$B_n = -\frac{2\left[E_n'(1) - KE_n'(K)\right]}{\alpha_n\left[\left(\frac{\partial E_n}{\partial \alpha_n}\cdot\zeta\cdot\frac{\partial E_n}{\partial \zeta}\right)\bigg|_{\zeta=1} - \left(\frac{\partial E_n}{\partial \alpha_n}\cdot\zeta\cdot\frac{\partial E_n}{\partial \zeta}\right)\bigg|_{\zeta=K}\right]} \qquad (4.2\text{-}30)$$

for problem 2, and

$$B_n = - \frac{K E_n'(K)}{\alpha_n^2 \left\{ \frac{1}{2\alpha_n} \left[E_n \cdot \frac{\partial}{\partial \zeta} \left(\frac{\partial E_n}{\partial \alpha_n} \right) \right] \Big|_{\zeta=1} + \left(\frac{\partial E_n}{\partial \alpha_n} \cdot \zeta \cdot \frac{\partial E_n}{\partial \zeta} \right) \Big|_{\zeta=K} \right\}} \tag{4.2-31}$$

for problem 3.

Simplifications similar to those shown in Sec. 4.1.4 can be made yielding

$$B_n = \frac{\left[(-1)^n D_o^{1/6} + K^{\frac{1}{2}} D_i^{1/6} \right] \cdot 3^{1/6}\, \Gamma(\frac{2}{3})}{\alpha_n \gamma K^{\frac{1}{2}}} \tag{4.2-32}$$

for problem 2, and

$$B_n = \frac{D_i^{1/6}\, \Gamma(\frac{2}{3}) \cdot 3^{1/6}}{\alpha_n \gamma} \tag{4.2-33}$$

for problem 3.

Thus, the coefficients are ready to evaluate. Eigenvalues, expansion coefficients and some combined functions calculated by the WKB method are shown in Table 8-11.

Table 8. Functions in the solution of problem II by the WKB method for n=0.5

Radius ratio	Eigenvalue α_n	Expansion Coeff., B_n	$B_n E'_n(K)$	$B_n E'_n(1)$	$B_n\left[E'_n(1)-KE'_n(K)\right]$
0.2	3.295758	2.969854	8.349079	-3.213493	-4.883309
	7.250668	-0.426731	-2.029270	-0.781049	-0.375195
	11.205578	0.873486	5.552376	-2.137065	-3.247540
	15.160489	-0.204089	-1.586943	-0.610801	-0.293413
	19.115399	0.512043	4.646891	-1.788551	-2.717929
	23.070309	-0.134115	-1.379689	-0.531031	-0.255093
	27.025219	0.362177	4.140320	-1.593576	-2.421640
	30.980129	-0.099873	-1.250560	-0.481330	-0.231218
0.5	5.368209	2.384888	9.342101	-6.227518	-10.898567
	11.810061	-0.154816	-1.025828	-0.683825	-0.170911
	18.251912	0.701437	6.212763	-4.141477	-7.247858
	24.693764	-0.074042	-0.802225	-0.534769	-0.133657
	31.135615	0.411187	5.199583	-3.466083	-6.065873
	37.577467	-0.048656	-0.697454	-0.464929	-0.116201
	44.019318	0.290840	4.432671	-3.088235	-5.404614
	50.461170	-0.036233	-0.632178	-0.421415	-0.105326

Table 9. Functions in the solution of problem II by the WKB method for n=0.8

Radius ratio	Eigenvalue α_n	Expansion Coeff., B_n	$B_n E'_n(K)$	$B_n E'_n(1)$	$B_n\left[E'_n(1)-KE'_n(K)\right]$
0.2	3.318228	2.865589	7.638174	-3.039447	-4.567082
	7.300101	-0.431171	-1.944050	-0.773593	-0.384782
	11.281975	0.842820	5.079604	-2.021319	-3.037250
	15.263849	-0.206212	-1.520299	-0.604970	-0.300910
	19.245723	0.494067	4.251219	-1.691681	-2.541925
	23.227596	-0.135511	-1.321749	-0.525961	-0.261611
	27.209470	0.349462	3.787781	-1.507266	-2.264822
	31.191344	-0.100912	-1.198043	-0.476735	-0.237126
0.5	5.406831	2.302810	8.685150	-5.867408	-10.209982
	11.895030	-0.156326	-0.997318	-0.673756	-0.175097
	18.383228	0.677297	5.775834	-3.901994	-6.789930
	24.871426	-0.074764	-0.779930	-0.526895	-0.136930
	31.359624	0.397036	4.833940	-3.265654	-5.682624
	37.847822	-0.049131	-0.678071	-0.458083	-0.119047
	44.336021	0.280830	4.306979	-2.909656	-5.063145
	50.824219	-0.036587	-0.614609	-0.415210	-0.107905

Table 10. Functions in the solution of problem III by the WKB method for n=0.5

Radius ratio	Eigenvalue α_n	Expansion Coeff. B'_n	$B'_n E'_n(K)$
0.2	1.977455	1.69253790	3.38487410
	5.932365	0.56417931	2.34694090
	9.887275	0.33850759	1.97948680
	13.842185	0.24179114	1.76947010
	17.797095	0.18805977	1.62727790
	21.752005	0.15386709	1.52198950
	25.706915	0.13019522	1.43955450
	29.661825	0.11283586	1.37249940
0.5	3.220925	1.70357730	4.74721700
	9.662777	0.56785910	3.29153640
	16.104629	0.34071546	2.77618970
	22.546480	0.24336818	2.48164530
	28.988332	0.18928636	2.28222420
	35.430183	0.15487067	2.13455900
	41.872035	0.13104441	2.01894580
	48.313886	0.11357182	1.92490240

Table 11. Functions in the solution of problem III by the WKB method for n=0.8

Radius ratio	Eigenvalue α_n	Expansion Coeff. B'_n	$B'_n E'_n(K)$
0.2	1.990936	1.59750970	3.02914450
	5.972810	0.53250324	2.10029150
	9.954684	0.31950195	1.77145470
	13.936558	0.22821567	1.58350950
	17.918431	0.17750109	1.45626090
	21.900305	0.14522816	1.36203760
	25.882179	0.12288536	1.28826620
		0.10650065	1.22825810
0.5	3.244099	1.63240980	4.37974810
	9.732297	0.54413660	3.03674750
	16.220495	0.32648197	2.56129250
	22.708693	0.23320140	2.28954810
	29.196892	0.18137886	2.10556350
	35.685090	0.14840089	1.96932860
	42.173288	0.12556999	1.86266470
	48.661486	0.10882732	1.77590090

4.3 Case III: Heat Transfer in an Annulus with Different but Constant Wall Temperatures at the Inner and Outer Walls

The solutions which were presented in the preceding section apply only when the two walls of the annulus are held at the same constant temperature. In this section, the problem is generalized to the situation in which the inner and outer walls of the annulus are at different but constant wall temperatures. The method used is that of superposition, so that the eigenvalues obtained in the preceding section can be used here. The results obtained are such that one wall of the annulus can be at either a higher or a lower temperature than the other. This technique has been used by Viskanta (15). The energy equation and boundary conditions describing the problem are

$$\begin{aligned} \rho C_p V_z \frac{\partial T}{\partial z} &= k \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \\ T(o,r) &= T_o \\ T(z,R) &= Tw_o \\ T(z,KR) &= Tw_i \end{aligned} \qquad (4.3\text{-}1).$$

4.3.1 Method of Superposition

To solve the energy equation, Eq. (4.3-1), it is convenient to split the problem into two simpler ones. Since the energy equation is linear, the general solution can be obtained by superposition of the two simpler solutions.

Let U denote the general solution of Eq. (4.3-1) with the boundary conditions

$$U(o,r) = T_e$$

$$U(z,R) = T_e \qquad (4.3\text{-}2)$$

$$U(z,KR) = Tw_i$$

and let V denote the general solution of Eq. (4.3-1) with the boundary conditions

$$V(o,r) = T_e$$

$$V(z,R) = Tw_o \qquad (4.3\text{-}3).$$

$$V(z,KR) = T_e$$

Figure 17 provides a graphical description of these boundary conditions.

Because of the linearity of Eq. (4.3-1), any linear combination of solutions is also a solution, and a proper addition of solutions U and V will yield a temperature distribution satisfying the boundary conditions of the general problem. Combining solutions U and V yields

$$T = U + V - T_e \qquad (4.3\text{-}4).$$

This equation can be rewritten in the form

$$T = \left(\Phi + \frac{\zeta - \zeta_i}{1 - \zeta_i}\right)(T_e - Tw_i) + Tw_i + \left(\psi + \frac{1 - \zeta}{1 - \zeta_i}\right)(T_e - Tw_o) + Tw_o - T_o \qquad (4.3\text{-}5),$$

where $$\Phi(\xi,\zeta) = \frac{U - Tw_i}{T_e - Tw_i} - \frac{\zeta - \zeta_i}{1 - \zeta_i} \qquad (4.3\text{-}6),$$

and $$\psi(\xi,\zeta) = \frac{V - Tw_o}{T_e - Tw_o} - \frac{1 - \zeta}{1 - \zeta_i} \qquad (4.3\text{-}7).$$

The solution $\Phi(\xi,\zeta)$ satisfies the energy equation

$$\overline{V}_z \frac{\partial \Phi}{\partial \xi} = \frac{1}{\zeta}\frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial \Phi}{\partial \zeta}\right) + \frac{1}{\zeta(1 - \zeta_i)} \qquad (4.3\text{-}8)$$

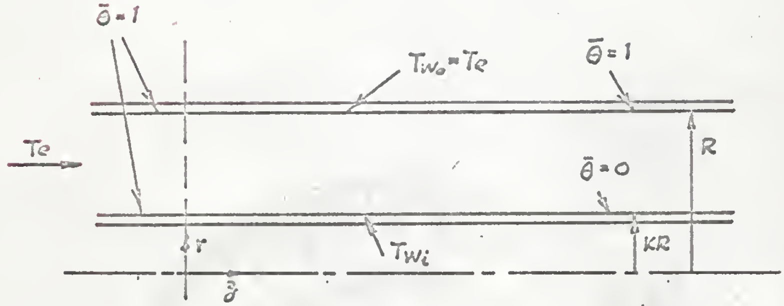

Fig. 17a Step change at the inner wall

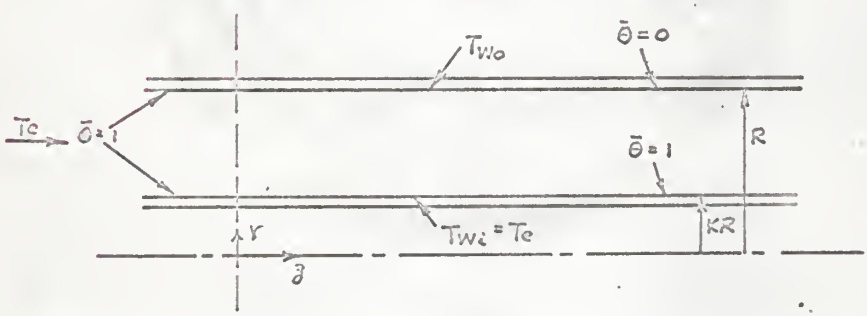

Fig. 17b Step change at the outer wall

with boundary conditions

$$\begin{aligned} \phi(o,\zeta) &= \frac{1-\zeta}{1-\zeta_i} \\ \phi(\xi,1) &= 0 \\ \phi(\xi,K) &= 0 \end{aligned} \qquad (4.3\text{-}9).$$

Similarly, the solution ψ satisfies the energy equation

$$\overline{V}_z \frac{\partial\psi}{\partial\xi} = \frac{1}{\zeta}\frac{\partial}{\partial\zeta}(\zeta\frac{\partial\psi}{\partial\zeta}) - \frac{1}{\zeta(1-\zeta_i)} \qquad (4.3\text{-}10)$$

with boundary conditions

$$\begin{aligned} \psi(o,\zeta) &= \frac{\zeta-\zeta_i}{1-\zeta_i} \\ \psi(\xi,1) &= 0 \\ \psi(\xi,K) &= 0 \end{aligned} \qquad (4.3\text{-}11).$$

The validity of the temperature distribution given by Eq. (4.3-5) can be demonstrated as follows:

At $\xi = 0$, $T = T_e$

$$T = (\frac{1-\zeta}{1-\zeta_i} + \frac{\zeta-\zeta_i}{1-\zeta_i})(T_e - Tw_i) + Tw_i + (\frac{\zeta-\zeta_i}{1-\zeta_i} + \frac{1-\zeta}{1-\zeta_i})(T_e - Tw_o)$$

$$+ Tw_o - T_e$$

$$= (T_e - Tw_i) + Tw_i + (T_e - Tw_o) + Tw_o - T_e$$

$$= T_e \quad .$$

At $\xi > 0$, $\zeta = \zeta_i = K$, $T = Tw_i$

$$T = (0 + 0)(T_e - Tw_i) + Tw_i + (0 + 1)(T_e - Tw_o) + Tw_o - T_e$$

$$= Tw_i \quad .$$

At $\xi > 0$, $\zeta = 1$, $T = Tw_o$

$$T = (0 + 1)(T_e - Tw_i) + Tw_i + (0 + 0)(T_e - Tw_o) + Tw_o - T_e$$

$$= Tw_o \quad .$$

Thus, the boundary conditions are satisfied and Eq. (4.3-5) can represent the general problem of Eq. (4.3-1).

4.3.2 Solution of the Problem

Before the method of separation of variables can be used to solve Eqs.(4.3-8) and (4.3-10), it is necessary to define new functions to change the non-homogeneous partial differential equations into a homogeneous partial differential equations.

To solve Eq. (4.3-8) and its boundary condition, Eq. (4.3-9),

define $\quad \Phi(\xi,\zeta) = Y(\xi,\zeta) + W(\zeta) \qquad$ (4.3-12).

Substituting Eq. (4.3-12) into Eq. (4.3-8) yields

$$\overline{V}_z \frac{\partial Y}{\partial \xi} = \frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta \frac{\partial Y}{\partial \zeta}) + \frac{1}{\zeta}\frac{d}{d\zeta}(\zeta \frac{dW}{d\zeta}) + \frac{1}{\zeta(1 - \zeta_i)} \qquad (4.3\text{-}13).$$

Splitting the above equation into two equations and their corresponding boundary conditions yields

$$\overline{V}_z \frac{\partial Y}{\partial \xi} = \frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta \frac{\partial Y}{\partial \zeta}) \qquad (4.3\text{-}14)$$

with boundary conditions

$$\begin{aligned} Y(o,\zeta) &= \frac{1 - \zeta}{1 - \zeta_i} - W(\zeta) \\ Y(\xi,1) &= 0 \\ Y(\xi,K) &= 0 \end{aligned} \qquad (4.3\text{-}15)$$

and $$\frac{1}{\zeta}\frac{d}{d\zeta}(\zeta \frac{dW}{d\zeta}) + \frac{1}{\zeta(1 - \zeta_i)} = 0 \qquad (4.3\text{-}16)$$

with boundary conditions

$$\begin{aligned} W(1) &= 0 \\ W(K) &= 0 \end{aligned} \qquad (4.3\text{-}17).$$

Inspection of Eq. (4.3-14) and the last two boundary conditions of Eq. (4.3-15) shows that these are identical with Eq. (4.2-7) and Eq. (4.2-8) of the preceeding section. Therefore, the eigenvalues and eigenfunctions obtained in the preceeding section will be the same as those of Eqs. (4.3-14) and (4.3-15). The solution, therefore, is

$$Y(\xi,\zeta) = \sum_{n=1}^{\infty} C_n E_n(\zeta)\exp(-\alpha_n^2\xi) \qquad (4.3\text{-}18).$$

From the first condition of Eq. (4.3-15) and the orthogonality property of the eigenfunctions, the coefficients are found to be

$$C_n = \frac{\int_K^1 \left(\frac{\ln\zeta}{\ln\zeta_i}\right)\zeta\bar{V}_z E_n d\zeta}{\int_K^1 \zeta\bar{V}_z E_n^2 d\zeta} \qquad (4.3\text{-}19).$$

A similar procedure may be followed to solve Eq. (4.3-16). Setting

$$\psi(\xi,\zeta) = Y'(\xi,\zeta) + W'(\zeta) \qquad (4.3\text{-}20)$$

yields

$$W'(\zeta) = -\frac{1-\zeta}{1-\zeta_i} + \frac{\ln\zeta}{\ln\zeta_i} \qquad (4.3\text{-}21),$$

and

$$Y'(\xi,\zeta) = \sum_{n=1}^{\infty} D_n E_n \exp(-\alpha_n^2\xi) \qquad (4.3\text{-}22),$$

where

$$D_n = \frac{\int_{\zeta_i}^1 \left(1-\frac{\ln\zeta}{\ln\zeta_i}\right)\zeta\bar{V}_z E_n d\zeta}{\int_{\zeta_i}^1 \zeta\bar{V}_z E_n^2 d\zeta} \qquad (4.3\text{-}23).$$

Therefore,

$$\Phi(\xi,\zeta) = \sum_{n=1}^{\infty} C_n E_n(\zeta)\exp(-\alpha_n^2\xi) + \frac{1-\zeta}{1-\zeta_i} - \frac{\ln\zeta}{\ln\zeta_i} \qquad (4.3\text{-}24),$$

$$\psi(\xi,\zeta) = \sum_{n=1}^{\infty} D_n E_n(\zeta)\exp(-\alpha_n^2\xi) - \frac{1-\zeta}{1-\zeta_i} + \frac{\ln\zeta}{\ln\zeta_i} \qquad (4.3\text{-}25).$$

Substituting Eqs.(4.3-24) and (4.3-25) into Eq. (4.3-5) results in

$$T = \left[\sum_{n=1}^{\infty} C_n E_n(\zeta)\exp(-\alpha_n^2\xi) + 1 - \frac{\ln \zeta}{\ln \zeta_i}\right](T_e - Tw_i) + Tw_i$$

$$+ \left[\sum_{n=1}^{\infty} D_n E_n(\zeta)\exp(-\alpha_n^2\xi) + \frac{\ln \zeta}{\ln \zeta_i}\right](T_e - Tw_o) + Tw_o - T_e \tag{4.3-26}$$

This is the temperature profile of the problem. The expressions for the Nusselt numbers follow readily from their definitions and the temperature profile given by Eq. (4.3-26).

4.3.3 Expressions for tne Nusselt Numbers

For the case $Tw_o = T_e$, i.e. step change at inner wall, reducing and rearranging Eq. (4.3-26) yields

$$\overline{\theta} = \frac{T - Tw_i}{T_e - Tw_i} = \sum_{n=1}^{\infty} C_n E_n(\zeta)\exp(-\alpha_n^2\xi) + 1 - \frac{\ln \zeta}{\ln \zeta_i} \tag{4.3-27}$$

Proceeding as in Eq. (4.2-28) leads to

$$\int_K^1 \overline{V}_z \zeta E_n d\zeta = -\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] \tag{4.3-28}$$

Therefore,

$$\theta_{avg} = \frac{2}{1 - K^2}\int_K^1 \overline{V}_z \theta\zeta d\zeta$$

$$= \frac{2}{1 - K^2}\left[\sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2\xi)\int_K^1 \overline{V}_z E_n \zeta d\zeta + \int_K^1 \overline{V}_z \zeta d\zeta - \int_K^1 \overline{V}_z \frac{\ln \zeta}{\ln \zeta_i}\zeta d\zeta\right]$$

$$= -\frac{2}{1 - K^2}\left\{\sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] - \int_K^1 \overline{V}_z \zeta d\zeta\right.$$

$$\left. + \int_K^1 \overline{V}_z \frac{\ln \zeta}{\ln \zeta_i} d\zeta\right\} \tag{4.3-29}$$

and

$$\frac{\partial\theta}{\partial\zeta} = \sum_{n=1}^{\infty} C_n E_n'(\zeta)\exp(-\alpha_n^2\xi) - \frac{1}{\zeta \ln \zeta_i} \tag{4.3-30}$$

Thus, the Nusselt number can be expressed as

$$Nu_i = -\frac{(1-K)(1-K^2)\left[\sum_{n=1}^{\infty} C_n E_n'(K)\exp(-\alpha_n^2\xi) - \frac{1}{K \ln \zeta_i}\right]}{\left\{\sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] - \int_K^1 \overline{V}_z \zeta d\zeta + \int_K^1 \overline{V}_z \frac{\ln \zeta}{\ln \zeta_i}\zeta d\zeta\right\}} \tag{4.3-31}$$

$$Nu_o = \frac{(1-K)(1-K^2)\left[\sum_{n=1}^{\infty} C_n E_n'(1)\exp(-\alpha_n^2\xi) - \frac{1}{\ln \zeta_i}\right]}{\left\{\sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] - \int_K^1 \overline{V}_z \zeta d\zeta + \int_K^1 \overline{V}_z \frac{\ln \zeta}{\ln \zeta_i}\zeta d\zeta + \frac{1-K^2}{2}\right\}} \tag{4.3-32}$$

For the case $Tw_i = T_e$, i.e. step change at outer wall, reducing and rearranging of Eq. (4.3-26) yields

$$\overline{\theta} = \frac{T - Tw_o}{T_e - Tw_o} = \sum_{n=1}^{\infty} D_n E_n \exp(-\alpha_n^2\xi) + \frac{\ln \zeta}{\ln \zeta_i} \tag{4.3-33}$$

By procedures similar to that used in the former case, the expressions for the heat transfer parameters are found to be;

$$\overline{\theta}_{avg} = \frac{2}{1-K^2}\left\{-D_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] + \int_K^1 \frac{\ln \zeta}{\ln \zeta_i}\overline{V}_z \zeta d\zeta\right\} \tag{4.3-34}$$

$$Nu_i = -\frac{(1-K)(1-K^2)\left[\sum_{n=1}^{\infty} D_n E_n'(K)\exp(-\alpha_n^2\xi) + \frac{1}{K \ln K}\right]}{\left\{\sum_{n=1}^{\infty} D_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] - \int_K^1 \frac{\ln \zeta}{\ln \zeta_i}\overline{V}_z \zeta d\zeta + \frac{1-K^2}{2}\right\}} \tag{4.3-35}$$

$$Nu_o = \frac{(1-K)(1-K^2)\left[\sum_{n=1}^{\infty} D_n E_n'(1)\exp(-\alpha_n^2\xi) + \frac{1}{\ln K}\right]}{\left\{\sum_{n=1}^{\infty} D_n \exp(-\alpha_n^2\xi)\frac{1}{\alpha_n^2}\left[E_n'(1) - KE_n'(K)\right] - \int_K^1 \frac{\ln \zeta}{\ln \zeta_i}\zeta d\zeta\right\}} \tag{4.3-36}$$

Values of the expansion coefficients and combined functions are shown in Tables 12-15. Temperature profiles and variation of the Nusselt number and average temperature with the axial distance are shown in Figs. 18-23.

4.3.4 Asymptotic Solution by the WKB Method

Since, as we have mentioned, the eigenvalues of this problem are exactly the same as those of problem II, Eq. (4.2-23) is valid for the present problem. Now the problem is to rewrite Eqs.(4.3-19) and (4.3-23) in terms of the asymptotic solutions which were derived in Sec. (4.1.4).

For the case $Tw_o = T_e$, multiplying Eq. (4.2-7) by $\frac{\ln \zeta}{\ln \zeta_i}$ and integrating by parts yields

$$\int_K^1 \left(\frac{\ln \zeta}{\ln \zeta_i}\right)\zeta \bar{V}_z E_n d\zeta = \frac{1}{\alpha_n^2} K \left.\frac{dE_n}{d\zeta}\right|_{\zeta=K} \tag{4.3-37}$$

Combining Eq. (4.3-37) and Eq. (4.2-26) leads to

$$C_n = \frac{2K \left.\frac{dE_n}{d\zeta}\right|_{\zeta=K}}{\alpha_n\left\{\left(\frac{\partial E_n}{\partial \alpha_n}\right)\cdot\zeta\cdot\left(\frac{\partial E_n}{\partial \zeta}\right)\Big|_{\zeta=1} - \left(\frac{\partial E_n}{\partial \alpha_n}\right)\cdot\zeta\cdot\left(\frac{\partial E_n}{\partial \zeta}\right)\Big|_{\zeta=K}\right\}} \tag{4.3-38}$$

$$= \frac{D_i^{1/6} 3^{1/3} \Gamma(\frac{2}{3})}{\alpha_n \gamma} \tag{4.3-39}$$

For the case $Tw_i = T_e$, there results

$$D_n = - \frac{2 \left.\frac{dE_n}{d\zeta}\right|_{\zeta=1}}{\alpha_n\left\{\frac{\partial E_n}{\partial \alpha_n}\cdot\zeta\cdot\frac{\partial E_n}{\partial \zeta}\Big|_{\zeta=1} - \frac{\partial E_n}{\partial \alpha_n}\cdot\zeta\cdot\frac{\partial E_n}{\partial \zeta}\Big|_{\zeta=K}\right\}} \tag{4.3-40}$$

$$= (-1)^n \frac{3^{1/6}\Gamma(\frac{2}{3}) D_o^{1/6}}{\alpha_n \gamma K^{\frac{1}{2}}} \tag{4.3-41}$$

Table 12. Functions in the solution of problem IV, step change at the inner wall, by the iterative method for n=0.5

Radius ratio, K	Expansion Coeff., C_n	$C_nE(K)$	$C_nE'_n(1)$	$C_nE'_n(K)$
0.2	-0.22125546	-1.64267310	-1.08762850	2.77522310
	-0.09249535	0.40790546	0.85639275	2.24243640
	-0.05524702	-1.12043510	-0.73754340	1.91445870
	-0.03884531	0.31925064	0.66755975	1.74154560
	-0.02959992	-0.94335255	-0.61937912	1.61986720
	-0.02377399	0.27658431	0.58396717	1.53691420
	-0.01979819	-0.85272568	-0.55687630	1.47924690
0.5	-0.24697002	-4.46828900	-2.56713800	3.80230200
	-0.10528754	0.50750005	2.06307870	3.11115730
	-0.06313751	-3.11213980	-1.77836230	2.66755500
	-0.04449602	0.39870894	1.61142710	2.42543640
	-0.03393647	-2.62406490	-1.49998510	2.24815960

Table 13. Functions in the solution of problem IV, step change at the outer wall, by the iterative method for n=0.5

Radius ratio, K	Expansion Coeff., D_n	$D_nE(K)$	$D_nE'_n(K)$	$D_nE'_n(1)$
0.2	-0.43146680	-3.20335120	5.41191910	-2.12096730
	0.17595052	-0.77594364	-4.26570480	-1.62908460
	-0.10621944	-2.15417940	3.68079110	-1.41802120
	0.07450342	-0.61230725	-3.34020000	-1.28034720
	-0.05684178	-1.81155340	3.11068860	-1.18941570
	0.04557377	-0.53020086	-2.94620200	-1.11944120
	-0.03776020	-1.62636540	2.82130130	-1.06210510
0.5	-0.33344643	-6.03285780	5.13367580	-3.46601990
	0.13965548	-0.67315812	-4.12670070	-2.73650840
	-0.08423176	-4.15190610	3.55878550	-2.37251340
	0.05919442	-0.53041473	-3.22663240	-2.14373090
	-0.04517963	-3.49341820	2.99297540	-1.99693060

Table 14. Functions in the solution of problem IV, step change at the inner wall, by the iterative method for n=0.8

Radius ratio	Expansion Coeff., C_n	$C_nE_n(K)$	$C_nE_n'(1)$	$C_nE_n'(K)$
0.2	-0.22224438	-1.59796700	-1.05925000	2.69358520
	-0.08787225	0.39506572	0.80134439	2.03139330
	-0.05226240	-1.03183790	-0.68627978	1.72779100
	-0.03665041	0.30691108	0.61986746	1.56478190
	-0.02789862	-0.86443999	-0.57436074	1.45039620
	-0.02239542	0.26647895	0.54100840	1.37264720
	-0.01864399	-0.75905897	-0.51545039	1.31804290
0.5	-0.24762470	-4.34026660	-2.49566070	3.68921180
	-0.10053694	0.51218795	1.94440950	2.86444310
	-0.06036259	-2.89976900	-1.66662620	2.46628560
	-0.04230420	0.38846360	1.50249500	2.22806270
	-0.03241070	-2.43661130	-1.39952800	2.07416650

Table 15. Functions in the solution of problem IV, step change at outer wall, by the iterative method for n=0.8

Radius ratio, K	Expansion Coeff., D_n	$D_nE_n(K)$	$D_nE'(K)$	$D_n'E(1)$
0.2	-0.43453722	-3.12438120	5.26655850	-2.07106950
	0.17240889	-0.77513485	-3.98567540	-1.57226990
	-0.10348214	-2.04309030	3.42111170	-1.35886790
	0.07251433	-0.60723608	-3.09598480	-1.22643300
	-0.05529848	-1.71342580	2.87486290	-1.13845330
	0.04430177	-0.52713856	-2.71531860	-1.07020220
	-0.03669338	-1.53327190	2.59405040	-1.01446180
0.5	-0.33514706	-5.87432360	4.99315490	-3.37774610
	0.13603933	-0.69305576	-3.87595770	-2.63103460
	-0.08210059	-3.94404460	3.35445350	-2.26681780
	0.05790804	-0.53174781	-3.04988030	-2.05668800
	-0.04434224	-3.33361520	2.83774150	-1.91474440

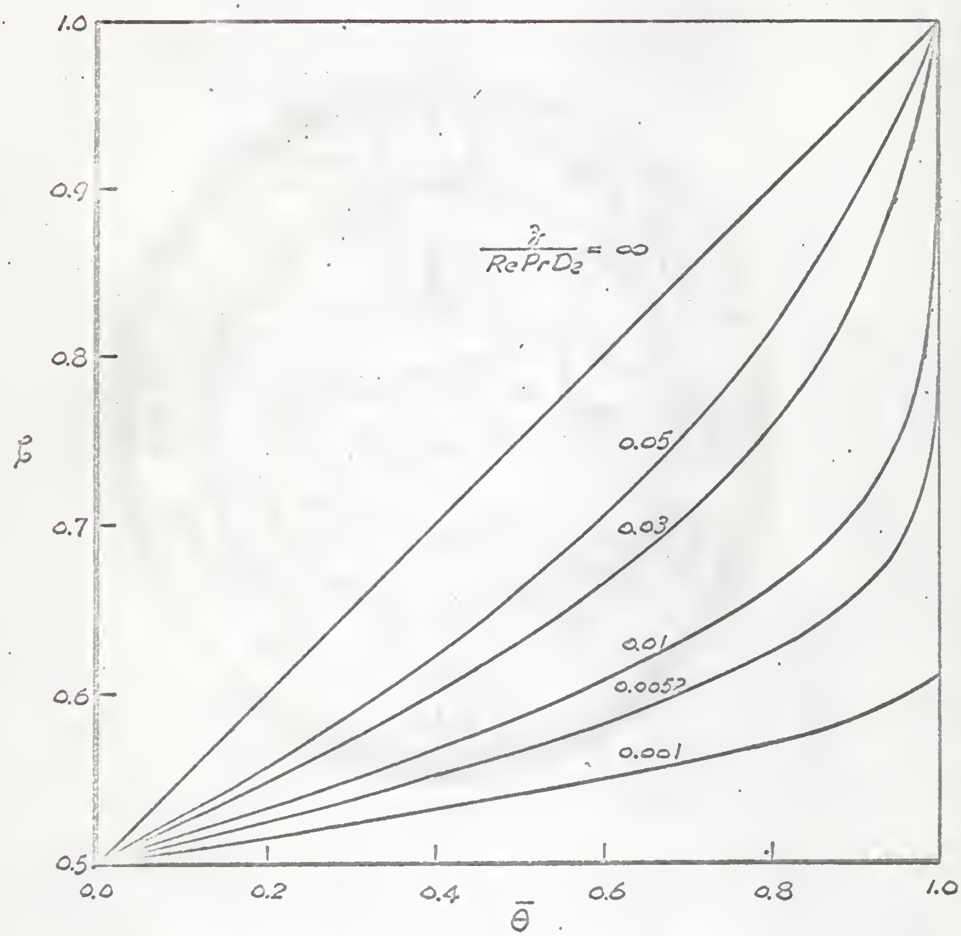

Fig. 18a Temperature profile development, problem IV, step change at the inner wall, K=0.5, n=0.5

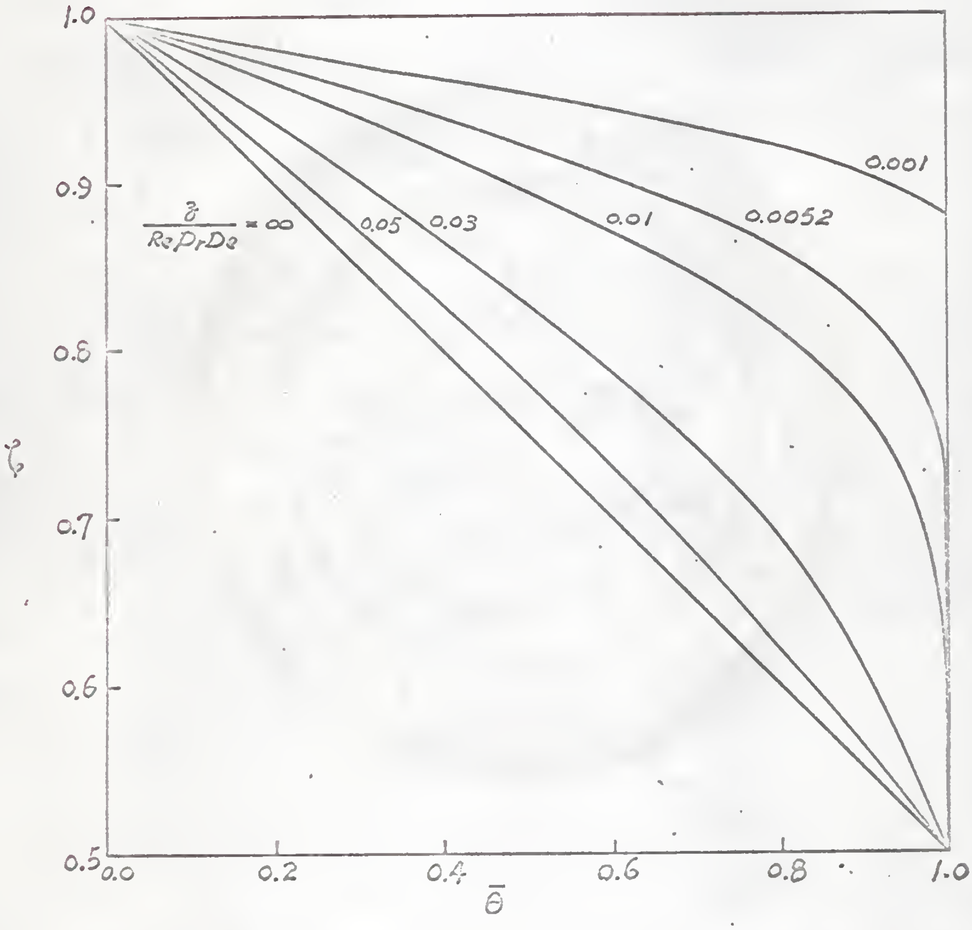

Fig 18b Temperature profile development, problem IV, step change at the outer wall, K = 0.5, n = 0.5

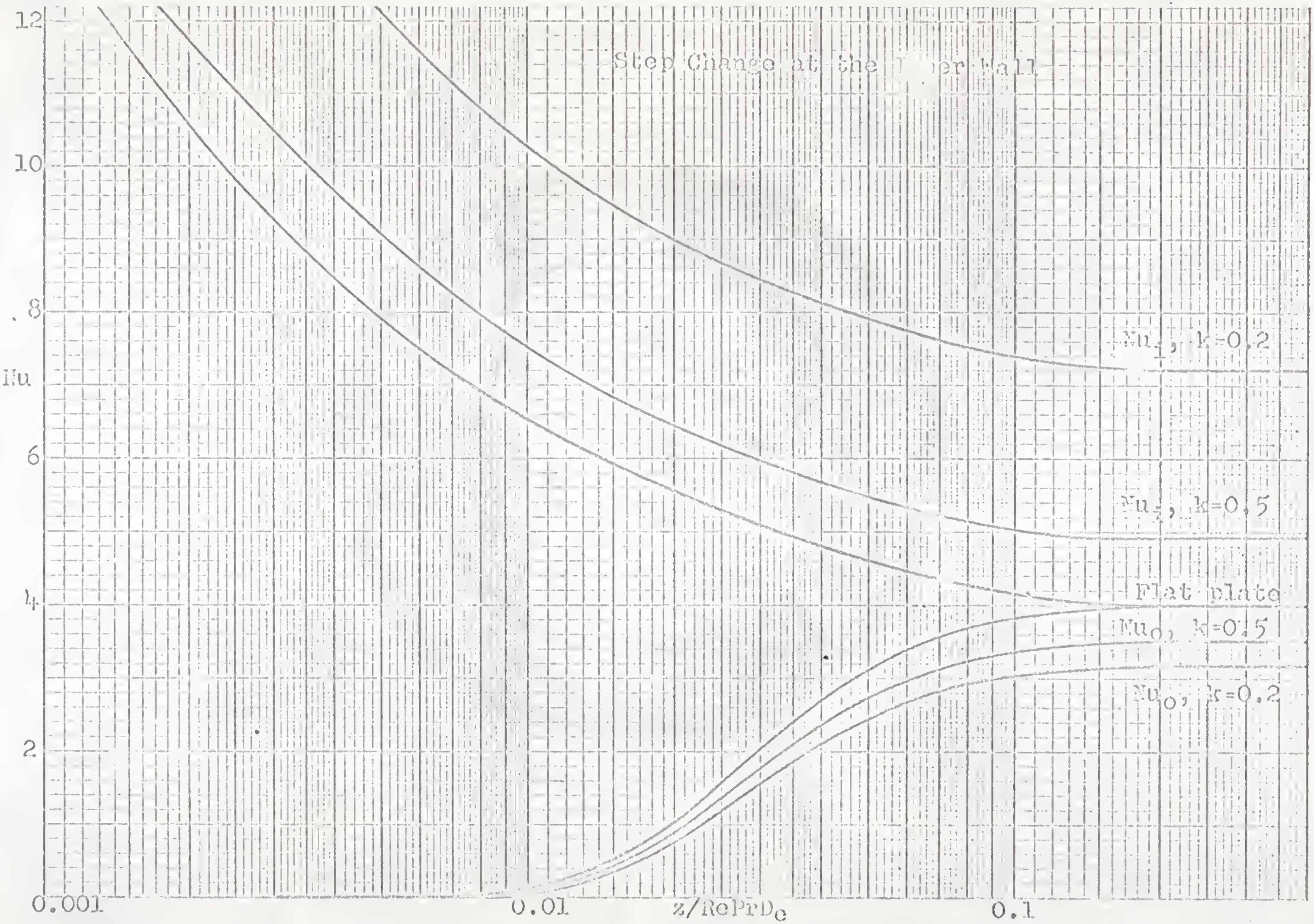

Fig. 19. Nusselt number versus axial distance, problem IV, n=0.5.

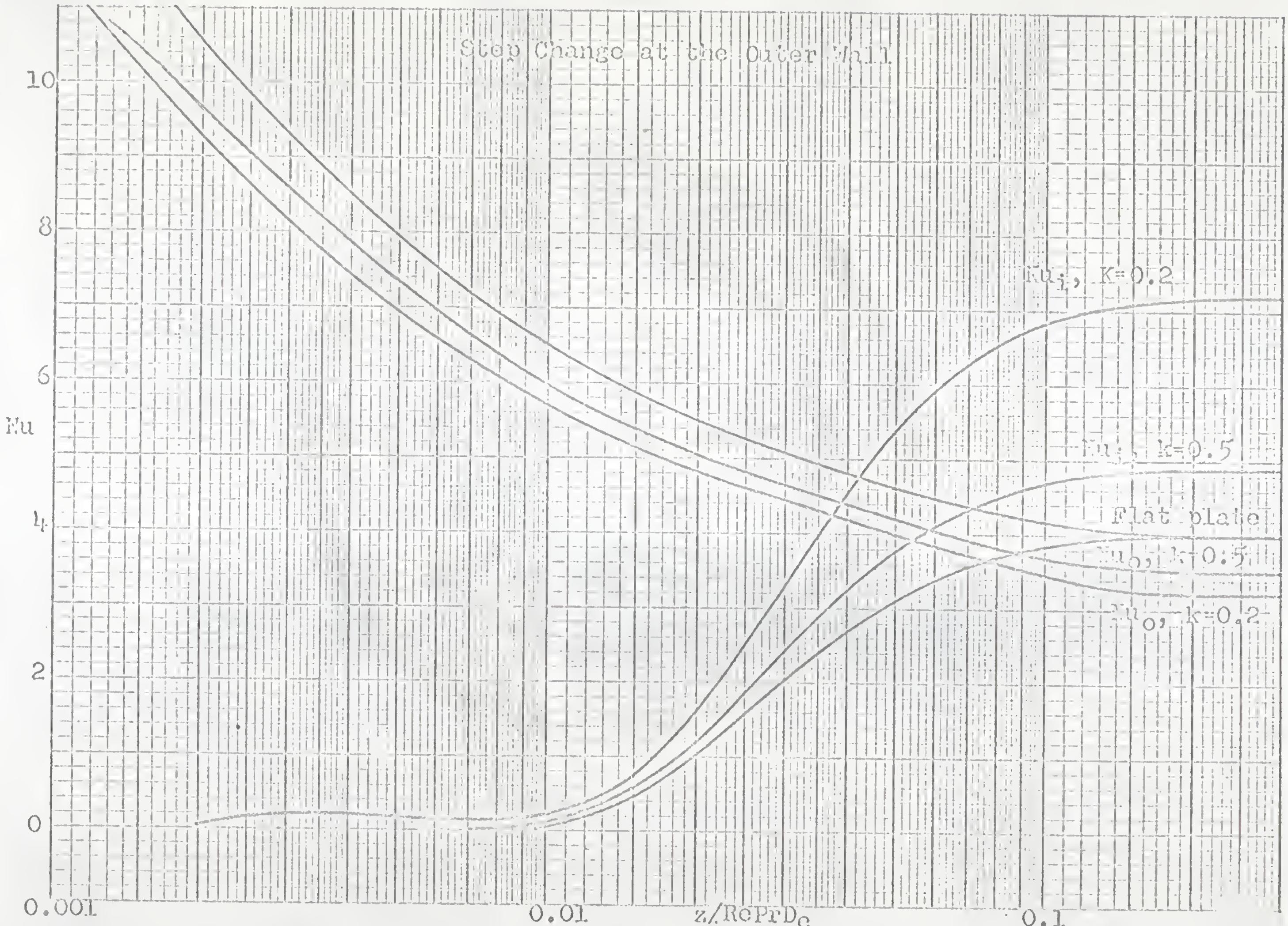

Fig. 20. Nusselt number versus axial distance, problem IV, n=0.5.

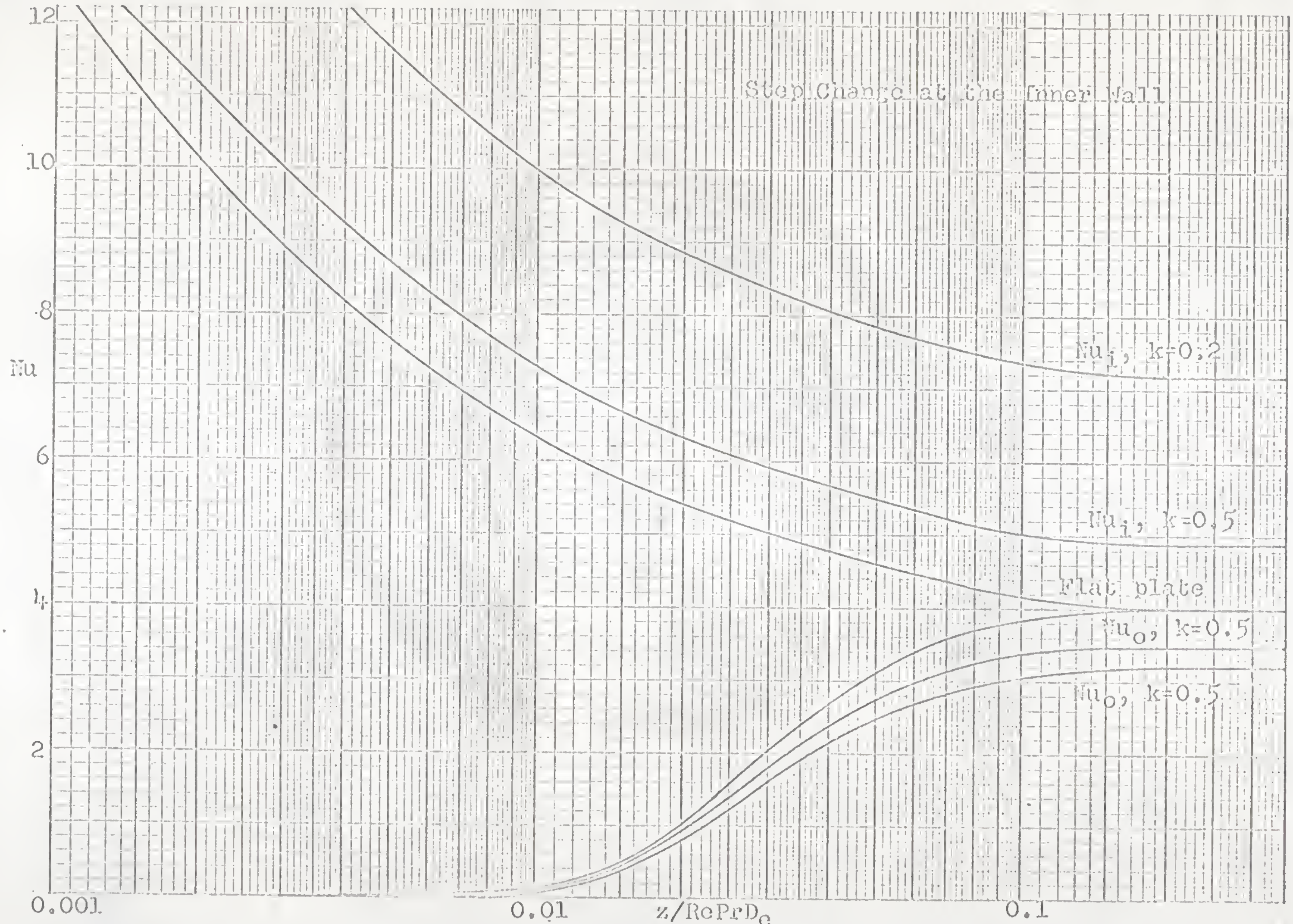

Fig. 21. Nusselt number versus axial distance, problem IV, n=0.8.

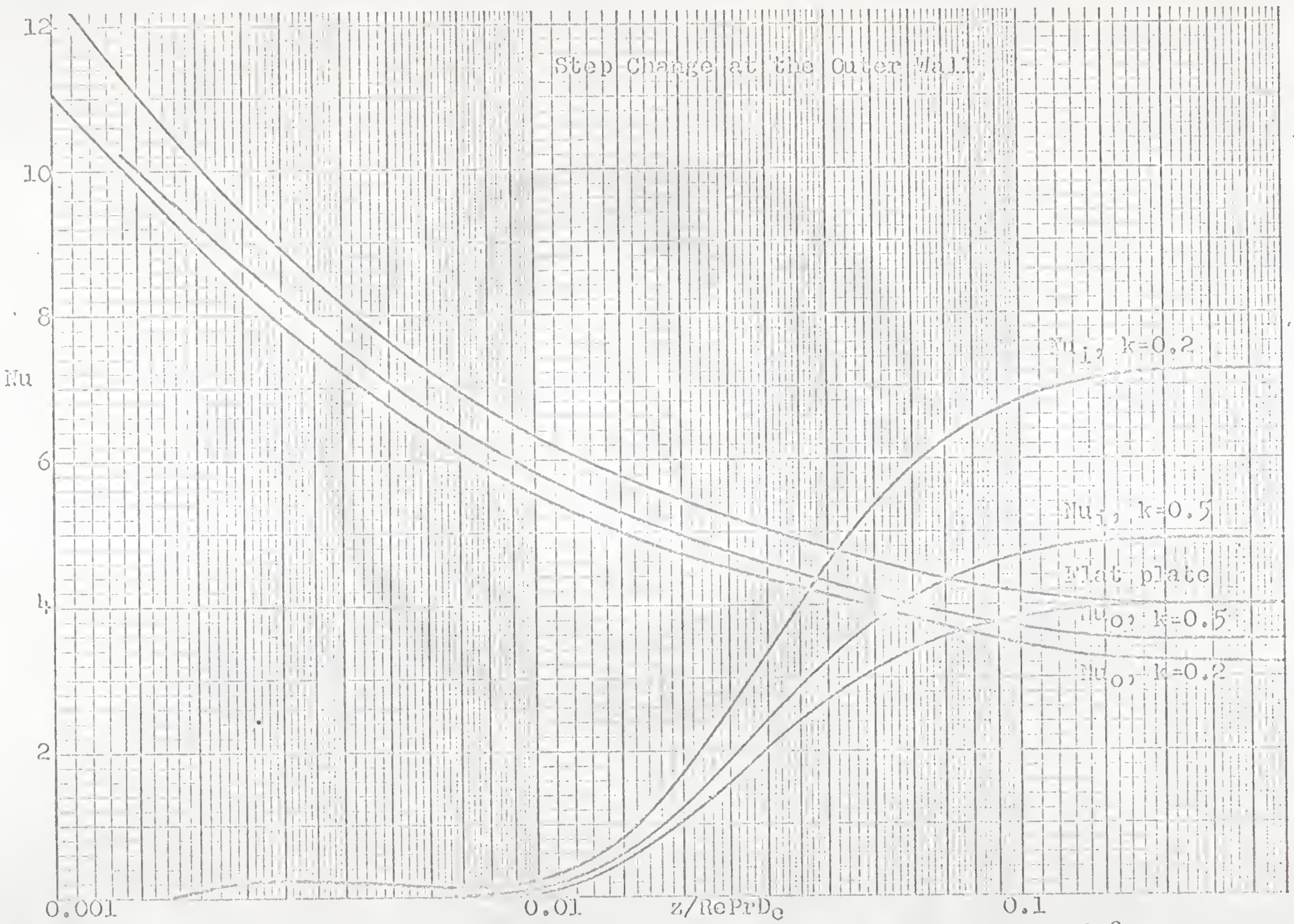

Fig. 22. Nusselt number versus axial distance, problem IV, n=0.8.

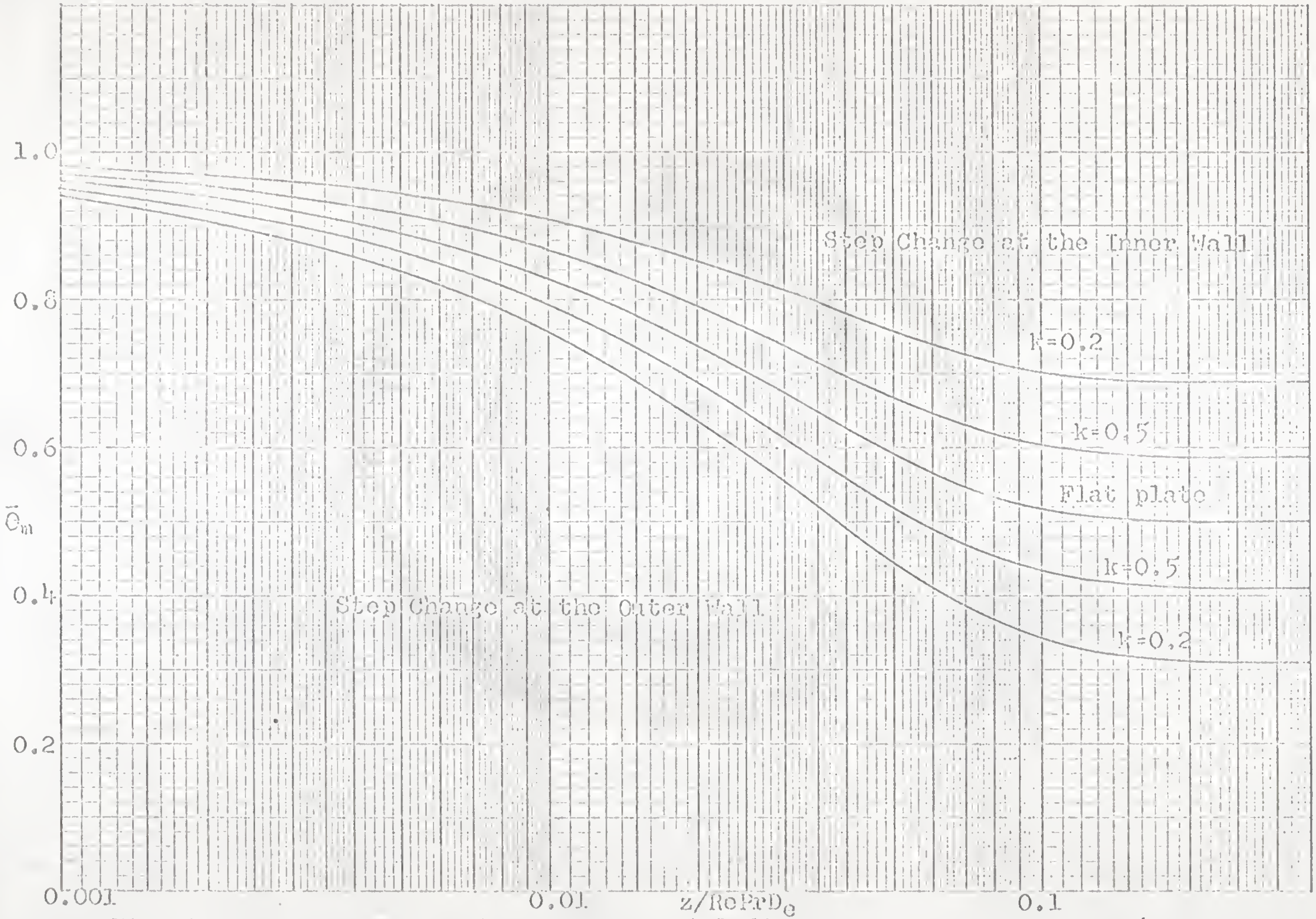

Fig. 23. Average temp rature versus axial distance, problem IV, n=0.5.

Eigenvalues, expansion coefficients and some combined functions calculated by the WKB method are shown in Tables 16-19.

Table 16. Functions in the solution of problem IV, step change at the inner wall, by the WKB method for n=0.5

Radius ratio	Expansion Coeff., C_n	$C_nE_n(K)$	$C_nE'_n(1)$	$C_nE'_n(K)$
0.2	1.01552270	-1.66981640	-1.09883340	2.85491410
	0.46160126	0.40585418	0.84487203	2.19508880
	0.29868318	-1.11047570	-0.73075568	1.89859950
	0.22076581	0.31738884	0.66071257	1.71661830
	0.17509012	-0.92937852	-0.61158352	1.58897450
	0.14507468	0.27593801	0.57442382	1.49242880
	0.12384424	-0.82806435	-0.54491307	1.41575590
	0.10803435	0.25011226	0.52066202	1.35274850
0.5	1.02214640	-4.67104910	-2.66907030	4.00395850
	0.46461198	0.51291397	2.05219690	3.07856680
	0.30063129	-3.10638100	-1.77500790	2.66274670
	0.22220573	0.40111250	1.60487310	[illegible].40752200
	0.17623213	-2.59979080	-1.48553870	2.22850460
	0.14602090	0.34872740	1.39527740	2.09310070
	0.12465200	-2.31637990	-1.32359570	1.98556870
	0.10873897	0.31608913	1.26468990	1.89720220

Table 17. Functions in the solution of problem IV, step change at the outer wall, by the WKB method for n=0.5

Radius ratio	Expansion Coeff., D_n	$D_nE(K)$	$D_nE'_n(K)$	$D_nE'_n(1)$
0.2	1.95433200	-3.21349300	5.49416410	-2.11466020
	-0.88833276	-0.78105020	-4.22435840	-1.62592180
	0.57480357	-2.13706480	3.65377670	-1.40630950
	-0.42485479	-0.61080230	-3.30356080	-1.27151440
	0.33695379	-1.78855090	3.305791700	-1.17696750
	-0.27919029	-0.53103160	-2.87211810	-1.10545520
	0.23833317	-1.59357570	2.72456360	-1.04866300
	-0.20790770	-0.48133100	-2.60330900	-1.00199280
0.5	1.36274220	-6.22751780	5.33814010	-3.55844770
	-0.61942820	-0.68382400	-4.10439620	-2.73602210
	0.40080649	-4.14147800	3.55001730	-2.36646930
	-0.29624827	-0.53476950	-3.20974660	-2.13964280
	0.23495552	-3.46608330	2.97107800	-1.98054430
	-0.19467743	-0.46492330	-2.79055610	-1.86020640
	0.16618806	-3.08823510	2.64719180	-1.76463920
	-0.14497256	-0.42141530	-2.52937950	-1.68610510

Table 18. Functions in the solution of problem IV, step change at the inner wall, by the WKB method for n=0.8

Radius ratio	Expansion Coeff. C_n	$C_nE_n(K)$	$C_nE'_n(1)$	$C_nE'_n(K)$
0.2	0.95850586	-1.52763550	-1.01665940	2.55487970
	0.43568447	0.38881022	0.78168999	1.96439830
	0.28191349	-1.01592130	-0.67610757	1.69906810
	0.20837084	0.30406002	0.61130255	1.53621220
	0.16525963	-0.85024418	-0.56584749	1.42198300
	0.13692940	0.26434995	0.53146677	1.33558370
	0.11689095	-0.75755663	-0.50416284	1.26696850
	0.10196870	0.23960872	0.48172535	1.21058280
0.5	0.97944587	-4.34257390	-2.49556290	3.69402280
	0.44520267	-0.49865930	1.91879040	2.84026320
	0.28807232	-2.88793580	-1.65962070	2.45663080
	0.21292302	0.38996502	1.50054580	2.22116240
	0.16886998	-2.41696960	-1.38896880	2.05600200
	0.13992084	0.33903576	1.30457520	1.93107960
	0.11944462	-2.15348890	-1.23755340	1.83187140
	0.10419637	0.30730455	1.18247670	1.75034490

Table 19. Functions in the solution of problem IV, step change at the outer wall, by the WKB method for n=0.8

Radius ratio	Expansion Coeff., D_n	$D_nE_n(K)$	$D_nE'_n(K)$	$D_nE'_n(1)$
0.2	1.90708320	-3.03944670	5.08329450	-2.02278780
	-0.86685597	-0.77359330	-3.90844830	-1.55528290
	0.56090682	-2.02131910	3.38053670	-1.34521180
	-0.41458330	-0.60497090	-3.05651080	-1.21627300
	0.32880743	-1.69168110	2.82923720	-1.12583370
	-0.27244044	-0.52596190	-2.65733280	-1.05742840
	0.23257111	-1.50726590	2.52081270	-1.00310340
	-0.20288117	-0.47673568	-2.40862560	-0.95846080
0.5	1.32336470	-5.86740820	4.99112600	-3.37184520
	-0.60152941	-0.67375550	-3.83758320	-2.59254710
	0.38922491	-3.90199500	3.31924290	-2.24237350
	0.28768799	-0.52689570	-3.00109190	-2.02744170
	0.22816633	-3.26565510	2.77793830	-1.87668590
	-0.18905211	-0.45808290	-2.60915160	-1.76265870
	0.16138594	-2.90965640	2.47510660	-1.67210310
	-0.14078348	-0.41521040	-2.36495320	-1.59768700

5. Discussion of Results

The variation of the Nusselt number with axial distance has been calculated for four sets of boundary conditions on the annular surfaces. Results for different values of the power-law model indices and two values of the ratio of the inner to the outer radius of the annulus are presented graphically in Figs. 4-7, 10, 11, 14, 15 and 19-22. The corresponding eigenvalues, α_n, coefficients, C_n, and other functions obtained in the investigation of the individual problems are given in Tables 1 and 2 and 4-19. The four different problems have also been evaluated for the limiting case of infinite parallel plates.

It cannot be said that this work completes the needed analysis of non-Newtonian annular heat transfer. Only one of many possible non-Newtonian models has been considered. Even for the one model considered, the power-law model, only a limited range of the parameter has been covered. Perhaps the greatest contribution made by this work is that it has shown how to extend the analytical procedures to problems involving the complex non-Newtonian velocity profiles. These same procedures can now be used to calculate the heat transfer rates for any velocity profile and hence for any non-Newtonian model.

A sufficient number of eigenvalues and eigenfunctions have been calculated by direct solution of the problems to prepare plots of Nusselt numbers to within a dimensionless distance of 0.001 of the entrance. Asymptotic solutions have also been presented and these can be used to extend the calculations to still closer to the entrance. Future calculations would employ the direct method for only about four eigenvalues and then switch to the simpler WKB method for the higher eigenvalues.

This is discussed in more detail below.

It can be seen from those figures which show the variation of the Nusselt number with the axial distance, that the Nusselt number at the inner wall always decreases with increasing radius ratio for a given power law model index while, at the outer wall, it always decreases with decreasing radius ratio. But for a given radius ratio, the Nusselt number, at either the inner or outer wall, decreases with increasing power law model index. These phenomena are expected from consideration of the basic fluid dynamics. Further investigation of these plots shows that the dependence of the Nusselt number on radius ratio is much greater than the dependence on the power law model index.

Another problem of considerable practical importance is the conditions under which entrance effects must be accounted for in heat transfer calculations. The thermal entrance length is defined here as that value of $\frac{1}{Pe}\frac{z}{De}$ for which the Nusselt number approaches to within 5% of its asymptotic (fully-developed) value. Because this value may be seen from the plots mentioned in the last paragraph, no additional plots have been prepared. One thing to note is that as K approaches unity (flat plate situation), both the Nusselt number and the thermal entrance length predicted for the heat transfer from the inside wall of the annulus only approach those for the heat transfer from the outside wall of the annulus only. The same conclusion can be reached from physical arguments.

It is not practical to give temperature distributions as functions of radial and axial coordinates for all of the problems solved. However, a plot has been given, Figures 3, 9, 13, and 18, for each kind of problem as an illustration of the development of the temperature profile. It is quite obvious that these results are consistent with what can be expected intuitively. Note especially in Fig. 3, in which the shape of the radial temperature profiles do not undergo further change with increasing axial coordinates after a certain distance from the entry. This is the basis for the assumption of the expression of Eq. (4.1-18).

For practical purposes, the mixing-cup temperature, as defined by Eq. (4.1-31) etc., is of greater interest than the transverse temperature distributions. Figures 8, 12, 16 and 23 are illustrative comparisons of the longitudinal change of θ_{avg} for various values of K for each problem. As the axial distance down the inlet increases, the temperature of the fluid approaches the surface temperature of annulus. Figure 12 is easily understood from energy balance considerations; for a given value of $\frac{1}{Pe}\frac{z}{De}$, with the symmetric boundary conditions, the variation of θ_m will trace the same curve in spite of different values of K. Furthermore, it is found that the change of θ_{avg} for a given parameter K and $\frac{1}{Pe}\frac{z}{De}$ are smaller in problem 2 than in problem 3 or 4. These trends in θ_{avg} are expected and can be readily be explained from the consideration of the energy balance on the coolant in the annulus.

In Tables 1 and 2 and 4-19, the corresponding eigenvalues and expansion coefficients of the series, as well as some other functions concerned with the calculation of the Nusselt number are tabulated. Comparing these results for the two methods of calculation, it is found that the expansion coefficients obtained are not the same. This difference arises because of differences in the procedure. The eigenvalues and the combined functions, however, have to be the same in order to have the same variation of the Nusselt number along the axial distance. The developed expressions from the WKB method are assumed valid only for large eigenvalues. Therefore, Eq. (4.1-46) is taken as an approximation to the actual equation only as α_n becomes large. It is apparent from Eq. (4.1-45) that if K is small, α_n must be very large in order to make Eq. (4.1-46) a reasonable approximation of the actual equation. A comparison of the eigenvalues predicted by the WKB method and those obtained directly exhibit very good agreement for the third and higher eigenvalues, even if K is small. The difference between them is within 1%. But the combined functions, such as $C_nE(K)$, $B_nE(K)$, $B_nE'(K)$ and $B_nE(1)$ etc., are less accurate, particularly those evaluated at the inner wall. For K = 0.2, the eigenvalues shown in the tables are not sufficiently large to remove the effect of the first derivative terms in Eq. (4.1-45). When α_n becomes very large, Eq. (4.1-45) approaches Eq. (4.1-46); but for small K, this value may be so large as to lie outside the range of practical interest.

NOMENCLATURE

Symbols

A, A', A''	Arbitrary constants
B, B', B''	Arbitrary constants
B_n	Expansion coefficient defined by Eq. (4.2-11)
C_g	Constant defined as $\frac{2K}{1 - K}$
C_p	Specific heat
C_n	Expansion coefficient defined by Eq.(4.1-30) or Eq.(4.3-19)
De	Equivalent diameter
D_i	$D_i = (V_{max} / V_{avg}) \cdot \left[\frac{\lambda^2}{K} - K\right]^s / \int_K^{\lambda} (\frac{\lambda^2}{\zeta} - \zeta)^s d\zeta$
D_n	Expansion coefficient defined by Eq.(4.3-23)
D_o	$D_o = (V_{max} / V_{avg}) \cdot \left[1 - \lambda^2\right]^s / \int_K^{\lambda} (\frac{\lambda^2}{\zeta} - \zeta)^s d\zeta$
$E_n(\zeta)$	Eigenfunction obtained from the solution of Eq.(4.1-15)
$G(\zeta)$	Function defined by Eq.(4.1-18)
G_1, G_2	Arbitrary constants
g	Gravitational acceleration
g_z	Gravitational acceleration in z direction
$g(\zeta)$	Function in the WKB method
H_1, H_2	Arbitrary constants
k	Thermal conductivity
K	Ratio of outer radius to inner radius
L	Length of annular region
m	Parameters of power law fluid
n	Parameters of power law fluid
Nu	Nusselt number defined by Eq.(4.1-34)

p_o, p_L	Static pressure at $z = o$ and $z = L$
P	Sum of forces per unit volume defined as $\frac{p_o - p_L}{L} + \rho g_z$
Pe	Peclet number defined as RePr
Pr	Prandtl number defined as $\frac{\mu C_p}{k}$
q	Heat flux
r	Radius
R	Outer radius
Re	Reynolds number defined as $\frac{\rho V_z De}{\mu}$
s	Defined as $\frac{1}{n}$
T	Temperature
T_e	Temperature at the inlet to the annulus
T_w	Temperature at wall
T_o	Temperature at wall
T_{avg}	Average temperature
U	Function satisfying Eq.(4.3-2)
V_z	Local velocity
$\overline{V}_z$	Dimensionless local velocity defined as $\frac{V_z}{V_{avg}}$
V	Function satisfying Eq. (4.3-3)
W	Function defined by Eq.(4.3-12)
W'	Function defined by Eq.(4.3-20)
Y	Function defined by Eq.(4.3-12)
Y'	Function defined by Eq.(4.3-20)
z	Axial coordinate
z_d	Function defined as $\frac{Re\ Pr\ R}{2(1 - K)}$
$Z(\xi)$	Function defined by Eq.(4.1-12)

Greek symbols

α_n	Eigenvalue satisfied Eq.(4.1-15) and boundary condition Eq.(4.1-16)
γ	$\gamma = \int_K^1 \sqrt{V_z}\, d\zeta$
ζ	Dimensionless radius variables defined as $\frac{r}{R}$
ξ	Dimensionless axial variables defined as $\frac{z}{z_d}$
β	Phase angle
σ	Phase shift in the WKB method
η_1	$\eta_1 = \alpha_n^{2/3}(\zeta - K)$
η_2	$\eta_2 = \alpha_n^{2/3}(1 - \zeta)$
θ	Dimensionless temperature defined as $(T - T_e)/\frac{q_R}{k}$
$\bar{\theta}$	Dimensionless temperature defined as $\frac{T - T_o}{T_e - T_o}$ or $\frac{T - T_w}{T_e - T_w}$
θ_{avg}	Dimensionless average temperature
Φ	Function defined as Eq.(4.3-6)
ψ	Function defined as Eq.(4.3-7)
ρ	Density
τ_{rz}	Shearing stress
λ	Dimensionless radio position represents the position at which $Z_{rz} = 0$

Subscripts

a	Designates the asymptotic value
i	Designates a value of a variable of a function evaluated at the inside surface of the annulus
n	Designates the n^{th} eigenvalue, eigenfunction or coefficient
o	Designates a value of a variable of a function evaluated at the outer surface of the annulus

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9. APPENDIX

9.1 The Method of Berry and de Prima for Determining the Eigenfunctions and Eigenvalues

The method developed by Berry and de Prima (21) is a simple iterative procedure for the determination of eigenfunctions and eigenvalues associated with the solution of Sturm – Liouville problems on a finite interval. The method is particularly useful when the coefficients of the differential equation are not expressed in analytical form. The iterative scheme of the calculations is presented here. For a complete discussion the reader is referred to the original paper.

Consider a Sturm – Liouville equation

$$\frac{d}{d\zeta}\left(p\,\frac{dE}{d\zeta}\right) - (q - \alpha^2 W)E = 0 \qquad (9.1\text{-}1)$$

with boundary conditions, for example,

$$\begin{aligned} E(K) &= 0 \\ E'(1) &= 0 \end{aligned} \qquad (9.1\text{-}2)$$

and orthogonality condition

$$\int_K^1 WE^2 d\zeta = 1 \qquad (9.1\text{-}3),$$

and where $p(\zeta)$, $\frac{d}{d\zeta}p(\zeta)$, $q(\zeta)$ and $W(\zeta)$ are continuous in $K \leq \zeta \leq 1$ and where $p(\zeta) > 0$ and $W(\zeta) > 0$ in $0 < \zeta \leq 1$. Then there exists a countable number of eigenvalues $\alpha_1, \alpha_2 \ldots \alpha_n \ldots$ and corresponding eigenfunctions $E_1, E_2 \ldots E_n \ldots$ such that E_n has precisely n-1 zeros in $0 \leq \zeta \leq 1$.

If $(\alpha_n^2)_K$ is the Kth approximation to the desired value α_n^2 and $E_n(K)$ is a solution to Eq. (9.1-1) with $\alpha_n^2 = (\alpha_n^2)_K$ such that $(E_n)_K$ satisfies the orthogonality conditions and the requisite boundary condition at $\zeta = K$ only, then the next approximation is given by

$$(\alpha_n^2)_{K+1} = (\alpha_n^2)_K \pm \left[E_n(1)\right]_K \left[E_n'(1)\right]_K \qquad (9.1\text{-}4).$$

This sequence of approximations converges monotonically to α_n^2. In Eq. (9.1-4), the plus (+) sign is associated with the condition of zero derivatives at the outer wall and the minus (–) sign with zero ordinate.

A value is assumed for either the slope or the ordinate at $\zeta = K$, whichever is not specified as zero by the boundary conditions, and Eq. (9.1-1) integrated numerically. Both the Runge – Kutta method and the method of finite differences have been used in different situations in this work. The outer wall values are adjusted in accordance with Eq. (9.1-3), then the value of α_n^2 is corrected by Eq. (9.1-4) and the process is repeated.

For the first approximation of the eiganvalue, $(\alpha_n)_1$, it is suggested by Berry and de Prima that the value given by

$$(\alpha_n)_1 = \left[(n-1)\pi \Big/ \int_K^1 \left(\frac{H}{P}\right)^{\frac{1}{2}} d\zeta\right]^2 \qquad n = 1, 2, \ldots \qquad (9.1\text{-}5)$$

be used.

The computer flow sheet and computer program for solving Eq. (4.1-15) with boundary conditions of Eq. (4.1-16) are shown on Pages 87 and 88.

9.2 Computer Flow Sheet and Computer Program for Calculation of the G Function and the Nusselt Number of Problem 1

In order to illustrate the numerical calculation of the iterative method, two more computer programs and their flow diagrams for solving problem 1 are given here. The first is for solving the ordinary differential

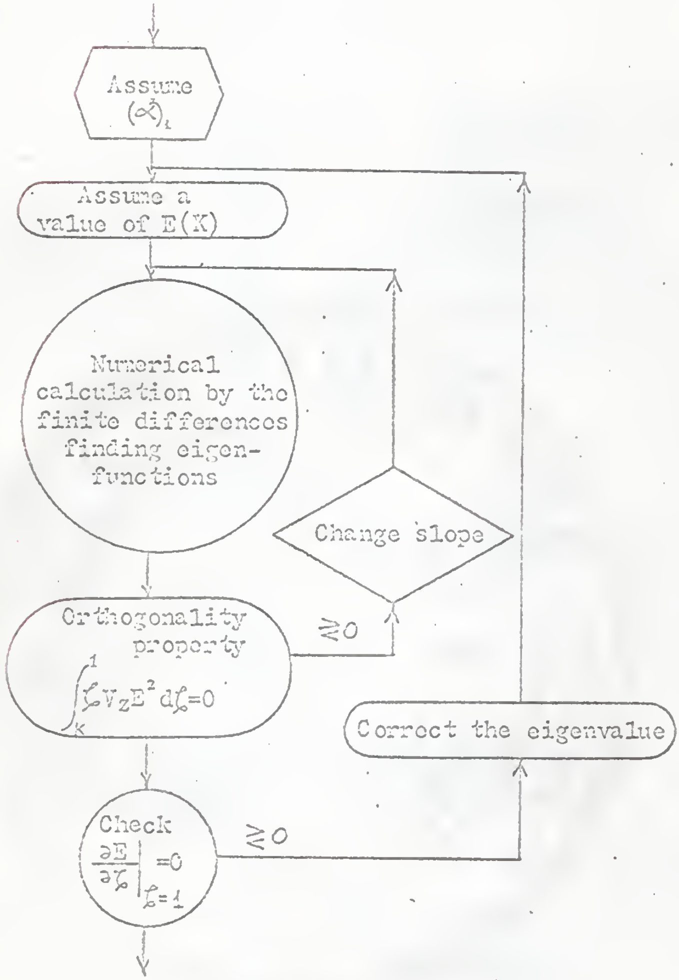

Fig.24 Computer flow sheet for solving Eq(4.1-15) and Eq(4.1-16)

```
C  C  N-N FOR UNIFORM HEAT INPUT BY FINITE DIFFERENCE
      DIMENSION Y(101),V(101)
  100 FORMAT(3I3,2F10.6,F12.6)
  101 FORMAT(F10.6)
  102 FORMAT(F10.6)
  200 FORMAT(5F12.8)
  201 FORMAT(3H S=F10.6)
  202 FORMAT(3H G=F15.10,4H AL=F15.10)
  203 FORMAT(4H AL=F15.10)
  205 FORMAT(6H DERY=F14.10)
      READ100,N,M,N1,DELX,X,AL
      READ101,(V(I),I=1,N1)
      READ102,Y(1)
    1 CONTINUE
      Y(2)=Y(1)
      DO 5 I=2,M
      AI=I
      XX=(X+(AI-1.)*DELX)
      A=2.*XX/(DELX*DELX)-AL*V(I)*XX
      B=1./(2.*DELX)-XX/(DELX*DELX)
      C=XX/(DELX*DELX)+1./(2.*DELX)
      Y(I+1)=A*Y(I)/C+B*Y(I-1)/C
    5 CONTINUE
      PUNCH200,(Y(I),I=1,N)
      S=0.0
      DO 10 I=1,N
      AI=I
      CC=V(I)*(X+(AI-1.)*DELX)*Y(I)*Y(I)*DELX
   10 S=S+CC
      PUNCH201,S
      B=S-1.0
      IF(ABS(B)-0.0015)6,6,7
    7 Y(1)=Y(1)+0.0001
      GO TO 1
    6 CONTINUE
      DERY=(Y(N)-Y(N-1))/DELX
      PUNCH205,DERY
      IF(ABS(DERY)-0.0001)8,8,9
    9 G=Y(N)*DERY
      AL=AL+G
      PUNCH202,G,AL
      GO TO 1
    8 CONTINUE
      PUNCH203,AL
      END
```

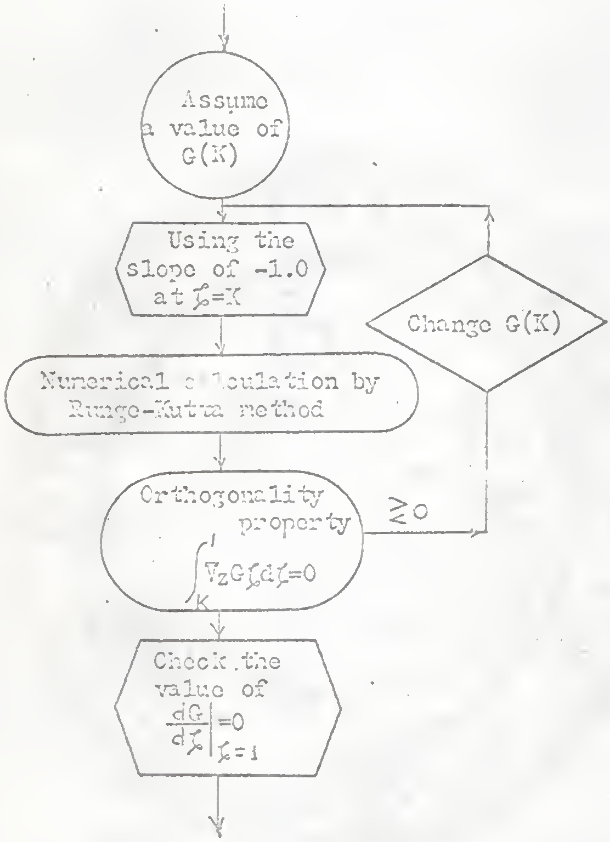

Fig. 25 . Computer flow sheet for solving Eq.(4.1-24)

```
C     FUNCTION OF G
      DIMENSION V(101),Y(51)
  100 FORMAT(2I3,I4,3F10.6)
  101 FORMAT(F10.6)
  102 FORMAT(F10.6)
  200 FORMAT(5F12.8)
  201 FORMAT(4H SS=F12.8)
  202 FORMAT(6H DERY=F12.6)
      READ100,N,M,N1,Z,DELX,X
      READ101,(V(I),I=1,N1)
      READ102,Y(1)
      C=2.*X/(1.-X*X)
    1 CONTINUE
      X=0.2
      Z=-1.0
      DO 5 I=1,M
      U1=Z*DELX
      K=2*I-1
      V1=-(Z/X-C*V(K))*DELX
      U2=(Z+V1/2.)*DELX
      J=2*I
      V2=-((Z+V1/2.)/(X+DELX/2.)-C*V(J))*DELX
      U3=(Z+V2/2.)*DELX
      V3=-((Z+V2/2.)/(X+DELX/2.)-C*V(J))*DELX
      U4=(Z+V3)*DELX
      L=2*I+1
      V4=-((Z+V3)/(X+DELX)-C*V(L))*DELX
      DELY=(U1+2.*U2+2.*U3+U4)/6.
      DELZ=(V1+2.*V2+2.*V3+V4)/6.
      Z=Z+DELZ
      Y(I+1)=Y(I)+DELY
      X=X+DELX
    5 CONTINUE
      PUNCH200,(Y(I),I=1,N)
      S=0.0
      DO 10 I=1,M
      X=0.2
      AI=I
      J=2*I-1
      CC=V(J)*(X+(AI-1)*DELX)*Y(I)*DELX
   10 S=S+CC
      SS=S
      PUNCH201,SS
      B=SS-0.0
      IF(ABS(B)-0.0001)6,6,7
    7 Y(1)=Y(1)+0.0001
      GO TO 1
    6 CONTINUE
      DERY=(Y(N)-Y(N-1))/DELX
      PUNCH202,DERY
      END
```

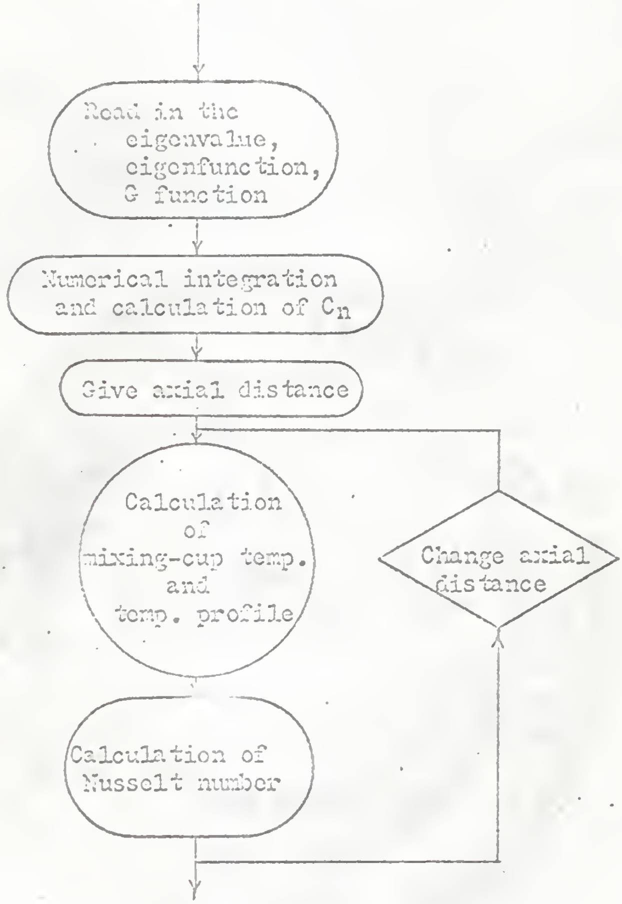

Fig. 26 Computer flow sheet for calculating the expansion coefficient, mixing-cup temp., Nusselt number and temp. profile.

```
                  FORTRAN LISTING                    1410-FO-970
C      NUSSELT NUMBER FOR UNIFORM HEAT INPUT
       DIMENSIONY(101,8),V(101),G(101),C(8),T(101),AL(8)
00100 FORMAT(3I5)
00101 FORMAT(2I5,2F10.6)
00102 FORMAT(5F10.6)
00103 FORMAT(F10.6)
00104 FORMAT(5F12.8)
00105 FORMAT(4F15.10)
00200 FORMAT(6H C(J)=F12.8)
00201 FORMAT(5F12.8)
00202 FORMAT(4H TB=F12.8,5H ANI=F12.8)
00203 FORMAT(3H Z=F10.6)
00106 FORMAT(5F12.8)
      READ(1,100)L,II,LL
      READ(1,101)N,M,DELX,X
      DO30J=1,2
      READ(1,102)(Y(I,J),I=1,N)
00030 CONTINUE
      DO45J=3,L
      READ(1,106)(Y(I,J),I=1,N)
00045 CONTINUE
      READ(1,103)(V(I),I=1,II)
      READ(1,104)(G(I),I=1,41)
      READ(1,105)(AL(I),I=1,L)
      C1=2.*X/(1.-X*X)
      DO10J=1,L
      S=0.0
      DO5I=1,M
      AI=I
      K=2*I-1
      S11=V(K)*G(I)*(X+(AI-1.)*DELX)*Y(K,J)*DELX
      S12=V(K+2)*G(I+1)*(X+AI*DELX)*Y(K+2,J)*DELX
      S1=-(S11+S12)/2.
      S=S+S1
00005 CONTINUE
      C(J)=S
      WRITE(3,200)C(J)
00010 CONTINUE
      Z=0.00001
00001 CONTINUE
      DO15I=1,LL
      T1=0.0
      K=2*I-1
      DO20J=1,L
      T2=C(J)*Y(K,J)*EXP(-AL(J)*Z)
00020 T1=T1+T2
      T(I)=C1*Z+G(I)+T1
00015 CONTINUE
      WRITE(3,203)Z
      WRITE(3,201)(T(I),I=1,LL)
```

```
      TB=C1*Z
      P=0.0
      DO25J=1,L
      P1=C(J)*Y(1,J)*EXP(-AL(J)*Z)
00025 P=P+P1
      ANI=2.*(1.-X)/(G(1)+P)
      WRITE(3,202)TB,ANI
      IF(Z-0.0002)6,7,7
00006 Z=Z+0.00001
      GOTO1
00007 IF(Z-0.002)8,9,9
00008 Z=Z+0.0001
      GOTO1
00009 IF(Z-0.02)11,12,12
00011 Z=Z+0.001
      GOTO1
00012 IF(Z-0.2)13,14,14
00013 Z=Z+0.01
      GOTO1
00014 IF(Z-1.)16,17,17
00016 Z=Z+0.1
      GOTO1
00017 IF(Z-5.0)21,22,22
00021 Z=Z+1.0
00022 CONTINUE
      END
```

equation of Eq. (4.1-24), and the other is for the calculation of the Nusselt number, the mixing-cup temperature and the temperature profile.

9.3 Derivation of Equation (4.1-50) and Equation (4.1-55)

9.3.1 Derivation of Equation (4.1-50)

As α_n becomes large,

$$J_{1/3}\left[\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right] \simeq \sqrt{\frac{3}{\pi\sqrt{D_i}\,\eta_1^{3/2}}}\,\cos\left(\frac{2\sqrt{D_i}}{3}\eta_1^{3/2} - \frac{5}{12}\pi\right) \qquad (9.3\text{-}1)$$

and

$$J_{-1/3}\left[\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right] \simeq \sqrt{\frac{3}{\pi\sqrt{D_i}\,\eta_1^{3/2}}}\,\cos\left(\frac{2\sqrt{D_i}}{3}\eta_1^{3/2} - \frac{\pi}{12}\right) \qquad (9.3\text{-}2).$$

Therefore, Eq. (4.1-47) becomes

$$E_n = \eta_1^{\frac{1}{2}}\sqrt{\frac{3}{\pi\sqrt{D_i}\,\eta_1^{3/2}}}\left\{\left[G_1\cos\frac{5}{12}\pi + H_1\cos\frac{\pi}{12}\right]\cos\left(\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right)\right.$$

$$\left. + \left[G_1\sin\frac{5}{12}\pi + H_1\sin\frac{\pi}{12}\right]\sin\left(\frac{2\sqrt{D_i}}{3}\eta_1^{3/2}\right)\right\} \qquad (9.3\text{-}3).$$

Furthermore, for large α_n,

$$\alpha_n\int_0^{\zeta}\sqrt{\overline{V}_z}\,d\zeta = \alpha_n\int_0^{\eta_1}(\overline{V}_z)^{\frac{1}{2}}\frac{d\eta_1}{\alpha_n^{2/3}} = \alpha_n^{1/3}\int_0^{\eta_1}\left[N\cdot\left(\frac{\lambda^2}{K} - K\right)^s\cdot\frac{1}{\alpha_n^{2/3}}\eta_1\right]^{\frac{1}{2}}d\eta_1$$

$$= N^{\frac{1}{2}}\cdot\left(\frac{\lambda^2}{K} - K\right)^{\frac{s}{2}}\cdot\frac{2}{3}\eta_1^{3/2} = \frac{2}{3}\sqrt{D_i}\,\eta_1^{3/2} \qquad (9.3\text{-}4).$$

Thus, Eq. (4.1-42) leads to

$$E_n = \frac{A}{\sqrt{\zeta}\,\overline{V}_z^{\frac{1}{4}}}\left\{\cos\sigma\cdot\cos\left(\frac{2}{3}\sqrt{D_i}\,\eta_1^{3/2}\right) + \sin\sigma\cdot\cos\left(\frac{2}{3}\sqrt{D_i}\,\eta_1^{3/2}\right)\right\} \qquad (9.3\text{-}5).$$

Comparing Eq. (9.3-3) and Eq. (9.3-5) yields

$$\begin{aligned} G_1\cos\frac{5}{12}\pi + H_1\cos\frac{\pi}{12} &= \cos\sigma \\ G_1\sin\frac{5}{12}\pi + H_1\sin\frac{\pi}{12} &= \sin\sigma \end{aligned} \qquad (4.1\text{-}50).$$

By a similar procedure, setting

$$\eta_2 = \alpha_n^{2/3}(1 - \zeta) \tag{9.3-6}$$

for large value of α_n, the Bessel's solution of Eq. (4.1-48) leads to

$$E_n = \eta_2^{\frac{1}{2}} \sqrt{\frac{3}{\pi\sqrt{D_o}\,\eta_2^{3/2}}} \left\{ \left[G_2\cos\frac{5}{12}\pi + H_2\cos\frac{\pi}{12} \right] \cos\left(\frac{2\sqrt{D_o}}{3}\eta_2^{3/2}\right) + \left[G_2\sin\frac{5}{12}\pi + H_2\sin\frac{\pi}{12} \right] \sin\left(\frac{2\sqrt{D_o}}{3}\eta_2^{3/2}\right) \right\} \tag{9.3-7}.$$

Expansion of Eq. (4.1-42) yields

$$E_n = \frac{A}{\sqrt{\zeta}\,\overline{V}_z^{\frac{1}{4}}} \left\{ \cos\left[\left(\alpha_n \int_K^1 \sqrt{\overline{V}_z}\, d\zeta - \sigma\right) - \alpha_n \int_\zeta^1 \sqrt{\overline{V}_z}\, d\zeta \right] \right\}$$

$$= \frac{A}{\sqrt{\zeta}\,\overline{V}_z^{\frac{1}{4}}} \left\{ \cos(\alpha_n\gamma - \sigma)\cos\left(\alpha_n \int_\zeta^1 \sqrt{\overline{V}_z}\, d\zeta\right) + \sin(\alpha_n\gamma - \sigma)\sin\left(\alpha_n \int_\zeta^1 \sqrt{\overline{V}_z}\, d\zeta\right) \right\} \tag{9.3-8}.$$

Furthermore, for large α_n,

$$\alpha_n \int_\zeta^1 \sqrt{\overline{V}_z}\, d\zeta = \alpha_n \int_{\eta_2}^{0} \left[N \cdot \int_0^{\eta_2} \left(1 - \frac{\eta_2}{\alpha_n^{2/3}} - \frac{\lambda^2}{1 - \frac{\eta_2}{\alpha_n^{2/3}}}\right)^s \frac{d\eta_2}{\alpha_n^{2/3}} \right]^{\frac{1}{2}} \left(- \frac{d\eta_2}{\alpha_n^{2/3}}\right)$$

$$= \frac{2}{3} N^{\frac{1}{2}} (1 - \lambda^2)^{s/2} \eta_2^{3/2}$$

$$= \frac{2}{3} \sqrt{D_o}\, \eta_2^{3/2} \tag{9.3-9}.$$

Comparing Eq. (9.3-8) and Eq. (9.3-7) leads to

$$\begin{aligned} G_2\cos\frac{5}{12}\pi + H_2\cos\frac{\pi}{12} &= K^{\frac{1}{2}}\cos(\alpha_n\gamma - \sigma) \\ G_2\sin\frac{5}{12}\pi + H_2\sin\frac{\pi}{12} &= K^{\frac{1}{2}}\sin(\alpha_n\gamma - \sigma) \end{aligned} \tag{4.1-47}.$$

9.3.2 Derivation of Equation (4.1-55)

Solving Eq. (4.1-50) yields

$$
\begin{aligned}
H_1 &= \frac{2}{\sqrt{3}} \sin(\sigma - \frac{\pi}{12}) \\
G_1 &= \frac{2}{\sqrt{3}} \sin(\sigma - \frac{5\pi}{12}) \\
H_2 &= \frac{2}{\sqrt{3}} K^{\frac{1}{2}} \sin(\alpha_n\gamma - \sigma - \frac{\pi}{12}) \\
G_2 &= \frac{2}{\sqrt{3}} K^{\frac{1}{2}} \sin(\alpha_n\gamma - \sigma - \frac{5\pi}{12})
\end{aligned}
\qquad (9.3\text{-}10).
$$

Substituting these constants into Eq. (4.1-47) and Eq. (4.1-48) and expanding the Bessel function in series form yields

$$
\Xi_n = \eta_1^{\frac{1}{2}} \left\{ \frac{2}{\sqrt{3}} \sin(\sigma - \frac{\pi}{12}) \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{2\sqrt{D_i}}{3} \frac{\eta_1^{3/2}}{2}\right)^{2K+1/3}}{K!\,(K+1/3)!} \right.
$$

$$
\left. - \frac{2}{\sqrt{3}} \sin(\sigma - \frac{5}{12}\pi) \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{2\sqrt{D_i}}{3} \frac{\eta_1^{3/2}}{2}\right)^{2K-1/3}}{K!\,(K-1/3)!} \right\} \qquad (9.3\text{-}11),
$$

and

$$
\Xi_n = \eta_2^{\frac{1}{2}} \left\{ \frac{2}{\sqrt{3}} \sin(\alpha_n\gamma - \sigma - \frac{\pi}{12}) \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{2\sqrt{D_o}}{3} \frac{\eta_2^{3/2}}{2}\right)^{2K+1/3}}{K!\,(K+1/3)!} \right.
$$

$$
\left. - \frac{2}{\sqrt{3}} \sin(\alpha_n\gamma - \sigma - \frac{5\pi}{12}) \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{2\sqrt{D_o}}{3} \frac{\eta_2^{3/2}}{2}\right)^{2K-1/3}}{K!\,(K-1/3)!} \right\}. \qquad (9.3\text{-}12).
$$

Changing the variables of the boundary conditions of Eq. (4.1-16) leads to

$$
\begin{aligned}
\left.\frac{d\Xi_n}{d\eta_1}\right|_{\eta_1=0} &= 0 \\
\left.\frac{d\Xi_n}{d\eta_2}\right|_{\eta_2=0} &= 0
\end{aligned}
\qquad (9.3\text{-}13).
$$

Applying Eq. (9.3-13) to Eq. (9.3-11) and Eq. (9.3-12) yields

$$\begin{aligned} \sin(\sigma - \tfrac{\pi}{12}) &= 0 \\ \sin(\alpha_n\gamma - \sigma - \tfrac{\pi}{12}) &= 0 \end{aligned} \qquad (9.3\text{-}14).$$

Therefore,

$$\alpha_n = (n + \tfrac{1}{6})\pi/\gamma \quad . \qquad n = 1, 2, \ldots$$

HEAT TRANSFER IN THE THERMAL ENTRANCE REGION OF AN ANNULUS

by

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AN ABSTRACT OF A MASTER'S THESIS

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ABSTRACT

Analytical solutions of the rates of heat transfer to non-Newtonian fluids in laminar flow through concentric annuli are presented. Four distinct problems are considered:

I. Constant heat flux at the inner surface, outer surface adiabatic,

II. Equal temperatures at both the inner and outer surface,

III. Prescribed temperature at the inner surface, outer surface adiabatic,

IV. The surfaces maintained at different temperatures.

An iterative method and an asymptotic "WKB" method have been used to calculate the eigenvalues and eigenfunctions for different values of the radius ratio and the power law model indices. The variation of the Nusselt number, the bulk temperature, and the temperature profile with axial distance are presented graphically.