# LARGE AMPLITUDE FREE VIBRATIONS OF CIRCULAR PLATE WITH STEPWISE THICKNESS

by

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# NOMENCLATURE

| | |
|---|---|
| $r, \theta, z$ | Cylindrical coordinates used to describe the initial configuration of the plate |
| $h(r)$, $a$ | Thickness function and radius of the plate |
| $h_0$ | Minimum thickness |
| $t$ | Time variable |
| $u, w$ | Radial and transverse displacements of the middle plane |
| $\epsilon_r^0, \epsilon_\theta^0$ | Radial and circumferential strains of the middle plane |
| $\epsilon_r, \epsilon_\theta$ | Radial and circumferential strains |
| $\sigma_r, \sigma_\theta$ | Radial and circumferential stresses |
| $a_{11}, a_{12}, a_{22}$ | Stress-strain relation coefficients |
| $N_r, N_\theta$ | Middle plane forces per unit length |
| $M_r, M_\theta$ | Bending moments per unit length |
| $Q$ | Transverse shearing force per unit length |
| $P(r, t)$ | Loading intensity |
| $\nu = -a_{12}/a_{22}$ | Poisson's ratio |
| $c = a_{11}/a_{22}$ | |
| $D(r)$ | Flexural rigidity |
| $T, V$ | Kinetic and potential energy |
| $U$ | Potential energy due to loading |
| $V_1, V_2$ | Potential energy due to stretching and bending of the plate |
| $\xi, \tau$ | Dimensionless space and time variables |
| $h(\xi)$ | Dimensionless thickness function |

| | |
|---|---|
| $A$, $\alpha$ | Amplitude parameters |
| $g$, $F$ | Dimensionless shape functions |
| $\Psi$ | Stress function |
| $\omega$ | Dimensionless angular frequency |
| $\lambda$ | Nonlinear eigenvalue |
| $\overline{Y},\overline{H},\overline{U},\overline{V}$, $\overline{Z},\overline{W},\overline{X}$ | (6x1) vector functions |
| $\overline{0}$ | (6x1) null vector |
| $[A]$, $[B]$, $[C]$, $[D]$ | Coefficient matrices |
| $C,S,M,V,N$ | Initial values |
| $T_0$, $T_1$ | Linear and nonlinear period |
| $W_0$ | Deflection at the center of the plate |

# I. INTRODUCTION

The linear theory for the motion of elastic plates is based on the assumption that the deflections are small in comparison with the thickness of plates. However, light weight structures of thin plates may be required to withstand large amplitudes of vibration when subjected to severe dynamic loading conditions. If the amplitude of motion is of the same order of magnitude as the thickness of the plate, it is necessary to modify the mathematical model from linear plate theory to include deformation of the middle plane.

In 1956, Chu and Herrmann (1) studied the large amplitude free vibrations of a rectangular plate. By applying a perturbation method, they showed that the in-plane inertial force and buoyancy terms can be neglected and obtained equations which are dynamic analogs of the von Karman (2) equations of static equilibrium. With appropriate choices for the displacement functions the space variables were eliminated, and the remaining ordinary differential equation in terms of the time variable was solved in the form of an elliptic cosine function.

Nowinski (3) utilized von Karman's dynamic equations in an investigation of the free nonlinear vibrations of a circular plate built in at the boundary. He represented the deflection as a series of separable terms and used an orthogonalization procedure to eliminate the space variable. By confining the study to one term of the series, the solution in the time variable was found in the form of elliptic functions.

In 1970, Huang and Sandman (4) utilized von Karman's dynamic equations to describe the large amplitude axisymmetric oscillations of a circular plate with a clamped and immovable boundary. They used a different approach from the previous investigators. Harmonic vibrations were assumed and the time variable was eliminated by applying a Kantorovich averaging method (10). The remaining nonlinear eigenvalue problem was solved numerically by considering the related initial value problem (5, 6).

In this report the general governing partial differential equations of large amplitude axisymmetric oscillations of a variable thickness circular plate are derived. Following the same approach as Huang and Sandman, harmonic vibrations are assumed and the time variable is eliminated by applying a Kantorovich averaging method. For simplicity only examples of stepwise thickness plates under free vibration are investigated. This is a nonlinear eigenvalue problem solved numerically by considering the related initial value problems. The influence of amplitude on the shape function of vibration is illustrated. The induced stresses for different amplitude conditions are calculated and the response curves are investigated.

## II. PROBLEM FORMULATION

Consider a thin elastic circular plate with variable thickness as shown in Fig. 1, which has cylindrical coordinates r, θ, and z.

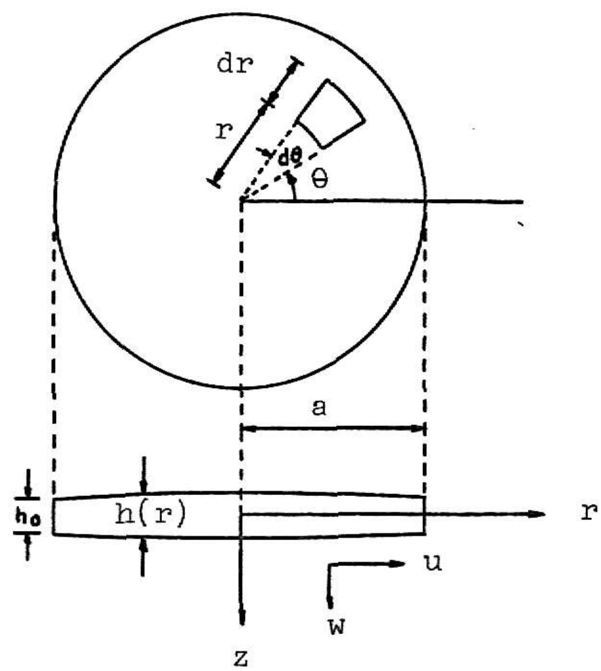


Fig. 1.

The basic assumptions regarding the flexural deformation of the plate are:

1. Loads and deflections of the plate are symmetrical with respect to the z-axis.
2. Middle plane is a plane of symmetry.
3. Slopes produced by flexure are moderately large, but small in comparison to unity.
4. Hooke's law applies.
5. Normals to the middle plane remain normal to the deformed middle plane.

6. Normal stresses in the transverse direction z are negligible.

From these assumptions, the nonvanishing components of strain are (2)

$$\epsilon_r = \frac{\partial u}{\partial r} - z \frac{\partial^2 w}{\partial r^2} + \frac{1}{2} \left(\frac{\partial w}{\partial r}\right)^2 ,$$

$$\epsilon_\theta = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r} , \tag{1}$$

where $u(r, t)$ and $w(r, t)$ are the radial and transverse components of mid-plane displacement, respectively. The stress-strain relations for cylindrically orthotropic media are

$$\epsilon_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r ,$$

$$\epsilon_r = a_{12}\sigma_\theta + a_{22}\sigma_r . \tag{2}$$

The stress resultants and moments per unit length are

$$N_r = \int_h \sigma_r \, dz ,$$

$$N_\theta = \int_h \sigma_\theta \, dz ,$$

$$M_r = \int_h \sigma_r \, z \, dz , \tag{3}$$

$$M_\theta = \int_h \sigma_\theta \, z \, dz .$$

From equations (1) and (2), the stresses can be written as functions of displacements. Upon substitution of expressions

(1) and (2) into (3), equations for force and couple resultants in terms of displacements can be obtained as follows:

$$
\begin{aligned}
N_\theta &= \frac{h(r)}{a_{22}\left[c-\nu^2\right]}\left\{\frac{u}{r}+\nu\frac{\partial u}{\partial r}+\frac{\nu}{2}\left(\frac{\partial w}{\partial r}\right)^2\right\} \\
N_r &= \frac{h(r)}{a_{22}\left[c-\nu^2\right]}\left\{c\left[\frac{\partial u}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]+\nu\frac{u}{r}\right\} \\
M_r &= -D(r)\left\{c\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}\right\} \\
M_\theta &= -D(r)\left\{\frac{1}{r}\frac{\partial w}{\partial r}+\nu\frac{\partial^2 w}{\partial r^2}\right\} ,
\end{aligned}
\tag{4}
$$

where

$$
D(r) = \frac{a_{22}\,h^3(r)}{12(a_{11}a_{22}-a_{12})^2} , \quad \nu = -\frac{a_{12}}{a_{22}} , \quad c = \frac{a_{11}}{a_{22}} .
$$

The equations of motion can be obtained by referring to Fig. 2, the free body diagram of a small element of the plate. The equilibrium condition for moments about the θ direction gives

$$
\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} = Q \tag{5}
$$

where Q represents the transverse shearing force per circumferential unit length. Consideration of the equilibrium of forces in the z direction yields

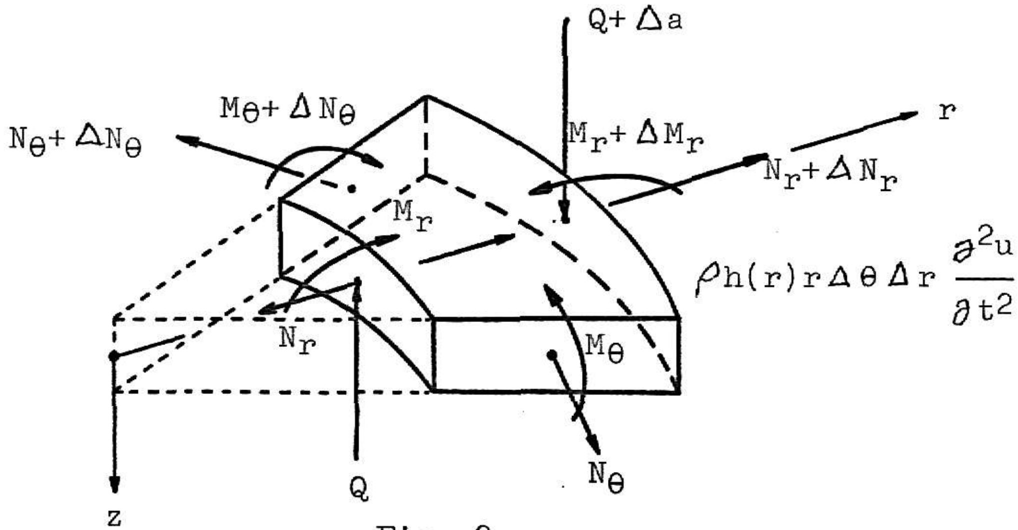


Fig. 2.

$$P(r)r + \frac{\partial}{\partial r}(rQ) + \frac{\partial}{\partial r}\left(rN_r \frac{\partial w}{\partial r}\right) = \rho h(r) r \frac{\partial^2 w}{\partial t^2} \qquad (6)$$

where $\rho$ is the mass density of the plate.

By combining equations (4), (5), and (6), the governing differential equation for an oscillating circular plate of variable thickness is obtained.

$$D(r)\left[c\frac{\partial^4 w}{\partial r^4} + \frac{2c}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r}\right]$$

$$+ D'(r)\left[2c\frac{\partial^3 w}{\partial r^3} + \frac{(2c+\nu)}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right] + D''(r)\left[c\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right]$$

$$= P(r) - \rho h(r)\frac{\partial^2 w}{\partial t^2}$$

$$+ \frac{1}{a_{22}\left[c-\nu^2\right]}\frac{1}{r}\frac{\partial}{\partial r}\left\{h(r)r\frac{\partial w}{\partial r}\left[c\frac{\partial u}{\partial r} + \frac{c}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\nu u}{r}\right]\right\}. \qquad (7)$$

The equilibrium equation for forces in the r direction yields

$$\frac{\partial N_r}{\partial r} + \frac{N_r - N_\theta}{r} = \rho h(r) \frac{\partial^2 u}{\partial t^2} . \tag{8}$$

For a moderately large oscillation, it is reasonable to neglect the in-plane inertial force; therefore

$$\frac{\partial N_r}{\partial r} + \frac{N_r - N_\theta}{r} = 0 . \tag{8'}$$

The differential equation of displacement in r direction is obtained by substitution of equation (4) into equation (8'), i.e.,

$$h(r) \left\{ c\left(\frac{\partial^2 u}{\partial r^2}\right) + \frac{c}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} \right\} + h'(r) \left\{ c\,\frac{\partial u}{\partial r} + \frac{\nu}{r}\,u \right\}$$

$$+ h(r) \left\{ c\left(\frac{\partial w}{\partial r}\right)\left(\frac{\partial^2 w}{\partial r^2}\right) + \frac{(c - \nu)}{2r}\left(\frac{\partial w}{\partial r}\right)^2 \right\}$$

$$+ \frac{1}{2}\, h'(r) \left\{ c \left(\frac{\partial w}{\partial r}\right)^2 \right\} = 0 . \tag{9}$$

Equations (7) and (9) form the basic governing differential equations for a cylindrically orthotropic circular plate with moderately large amplitude oscillation. These equations are a set of nonlinear coupled partial differential equations. Equations (7) and (9) can also be obtained by a variational method, and the associated boundary conditions can be obtained simultaneously.

For a conservative system, Hamilton's principle requires that the integral (7)

$$H = \int_t (T - V)dt \tag{10}$$

be a minimum, where T is the kinetic energy and V is the potential energy of the system. The kinetic energy of the system is

$$T = \frac{1}{2}\int_h\int_0^{2\pi}\int_0^a \rho w_t^2 \, rdrd\theta dz = \pi\int_0^a \rho h(r) w_t^2 \, rdr \tag{11}$$

where $\rho$ represents the constant mass density for homogeneous media. The potential energy due to the loading intensity p(r, t) is

$$U = -\int_s P(r,t)wrdrd\theta = -2\pi \int_0^a P(r,t)wrdr \tag{12}$$

As defined previously in equation (3), $N_r$ and $N_\theta$ are the membrane stresses integrated over the entire thickness h(r) of the plate. $\epsilon_r^0$ and $\epsilon_\theta^0$ represent strains of the middle plane. Thus the energy stored in a plate element with thickness h(r) and area $(dr \cdot rd\theta)$ is

$$\frac{1}{2}(N_r \epsilon_r^0 + N_\theta \epsilon_\theta^0) \, rdrd\theta.$$

The strain energy due to stretching of the plane is

$$V_1 = \frac{1}{2}\int_0^{2\pi}\int_0^a \left[N_r\,\epsilon_r^{\,0} + N_0\,\epsilon_0^{\,0}\right] r\,dr\,d\theta$$

$$= \frac{\pi}{\beta}\int_0^a \left\{cu_r^{\,2} + c\,u_r w_r^{\,2} + \frac{c}{4}\,w_r^{\,4} + 2\nu\frac{u}{r}\,u_r + \nu\frac{u}{r}\,w_r^{\,2} + \left(\frac{u}{r}\right)^2\right\} h(r)\,r\,dr \qquad (13)$$

where $\beta = a_{22}\,(c - \nu^2)$.

For an element of thickness $h(r)$ and area $r\,dr\,d\theta$, with radial moment $M_r$ and tangential moment $M_\theta$, the energy stored is

$$-\frac{1}{2}\left(M_r\frac{\partial^2 w}{\partial r^2} + \frac{M_\theta}{r}\frac{\partial w}{\partial r}\right) r\;dr\,d\theta\ .$$

The strain energy due to bending of the plate is

$$V_2 = -\frac{1}{2}\int_0^{2\pi}\int_0^a \left[M_r\frac{\partial^2 w}{\partial r^2} + \frac{M_\theta}{r}\frac{\partial w}{\partial r}\right] r\,dr\,d\theta$$

$$= \pi\int_0^a D(r)\left\{\left(\sqrt{c}\;w_{rr} + \frac{w}{r}\right)^2 + \frac{2}{r}\left(\nu - \sqrt{c}\right)w_{rr}w_r\right\} r\,dr \qquad (14)$$

Combination of equations (11), (12), (13), and (14), and integration from time $t_1$ to $t_2$ yields the integral H, i.e.,

$$H = \int_{t_1}^{t_2}(T - V)\,dt = \int_{t_1}^{t_2}(T - V_1 - V_2 - U)\,dt\ ,$$

or

$$H = \int_t\int_0^a f_1(t,\ r;\ w,\ w_t,\ w_r,\ w_{rr})\,dr\,dt \qquad (15)$$

where

$$f_1 = \pi \Big\{ \rho h(r) r\, w_t^2 - \frac{h(r)}{\beta} \Big[ cru_r^2 + cru_r w_r^2 + \frac{c}{4} rw_r^4 + 2\nu uu_r + \nu uw_r^2 + \frac{u^2}{r} \Big] - D(r) \Big[ crw_{rr}^2 + 2\nu w_{rr} w_r + \frac{w_r^2}{r} \Big] + 2\,P(r,\,t) rw \Big\} .$$

The extremization of the functional H will be considered. After substitution of $f_1$ from equation (15) into Euler's equation (8), the basic governing differential equations, which are the same as equations (7) and (9), are obtained again. Meanwhile the boundary conditions can also be obtained. They are

$$\eta_r^{(1)} \left( c\, w_{rr} + \frac{\nu}{r}\, w_r \right) = 0$$

$$\eta^{(1)} \left\{ rD(r) \left( cw_{rrr} + \frac{c}{r}\, w_{rr} - \frac{w_r}{r^2} \right) + rD'(r) \left( cw_{rr} + \frac{\nu}{r}\, w_r \right) \right\} = 0 \tag{16}$$

$$\eta^{(2)} \left[ rh(r) \left( cu_r + \frac{\nu}{r} . u + \frac{c}{2}\, w_r^2 \right) \right] = 0$$

where $\eta^{(1)}$ represent the variation related to w and $\eta^{(2)}$ to u.

# III. ANALYSIS OF THE PROBLEM

By introducing the dimensionless quantities

$$\eta = u/a, \quad \xi = r/a, \quad \chi = w/a, \quad h(\xi) = h(r)/h_0$$
$$D_\theta = h_\theta^3/(12\beta), \quad D(\xi) = D(r)/D_0, \quad \tau = \sqrt{D_0/(\rho h_0 a^4)}\, t \tag{17}$$

equations (7) and (9) can be rewritten as

$$\begin{aligned}
&h^3(\xi)\left[c\frac{\partial^4\chi}{\partial\xi^4} + \frac{2c}{\xi}\frac{\partial^3\chi}{\partial\xi^3} - \frac{1}{\xi^2}\frac{\partial^2\chi}{\partial\xi^2} + \frac{1}{\xi^3}\frac{\partial\chi}{\partial\xi}\right] \\
&+ \left[h^3(\xi)\right]_\xi\left[2c\frac{\partial^3\chi}{\partial\xi^3} + \frac{(2c+\nu)}{\xi}\frac{\partial^2\chi}{\partial\xi^2} - \frac{1}{\xi^2}\frac{\partial\chi}{\partial\xi}\right] \\
&+ \left[h^3(\xi)\right]_{\xi\xi}\left[c\frac{\partial^2\chi}{\partial\xi^2} + \frac{\nu}{\xi}\frac{\partial\chi}{\partial\xi}\right] = \frac{P(\xi,\tau)a^3}{D_0} - h(\xi)\frac{\partial^2\chi}{\partial\tau^2} \\
&+ \frac{12a^2}{h_0^2}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left\{h(\xi)\,\xi\frac{\partial\chi}{\partial\xi}\left[c\frac{\partial\eta}{\partial\xi} + \frac{c}{2}\left(\frac{\partial\chi}{\partial\xi}\right)^2 + \frac{\nu\eta}{\xi}\right]\right\}
\end{aligned} \tag{18A}$$

$$\begin{aligned}
&h(\xi)\left\{c\frac{\partial^2\eta}{\partial\xi^2} + \frac{c}{\xi}\frac{\partial\eta}{\partial\xi} - \frac{\eta}{\xi^2}\right\} + \left[h(\xi)\right]_\xi\left\{c\frac{\partial\eta}{\partial\xi} + \frac{\nu\eta}{\xi}\right\} \\
&+ h(\xi)\left\{c\frac{\partial\chi}{\partial\xi}\frac{\partial^2\chi}{\partial\xi^2} + \frac{c-\nu}{2\xi}\left(\frac{\partial\chi}{\partial\xi}\right)^2\right\} + \left[h(\xi)\right]_\xi\frac{c}{2}\left(\frac{\partial\chi}{\partial\xi}\right)^2\Big\} = 0\,.
\end{aligned} \tag{18B}$$

Let the plate be subjected to a time varying load of the form

$$P(\xi, \tau) = P(\xi)\sin\omega\tau \tag{19}$$

and assume that the steady-state responses are

$$\chi = A\ g(\xi)\ \sin \omega\tau$$
$$\eta = A^2 f(\xi)\ \sin^2 \omega\tau \tag{20}$$

where A is an amplitude parameter, and $g(\xi)$ and $f(\xi)$ are shape functions. Since expressions (19) and (20) cannot satisfy equation (18A) for all $\tau$, the Kantorovich averaging method used, and the integral

$$I = \int_0^1 \left\{ \left\{ h^3(\xi) \left[ c\frac{\partial^4 \chi}{\partial \xi^4} + \frac{2c}{\xi}\frac{\partial^3 \chi}{\partial \xi^3} - \frac{1}{\xi^2}\frac{\partial^2 \chi}{\partial \xi^2} + \frac{1}{\xi^3}\frac{\partial \chi}{\partial \xi} \right] \right. \right.$$
$$+ \left[h^3(\xi)\right]_{\xi} \left[ 2c\frac{\partial^3 \chi}{\partial \xi^3} + (2c+\nu)\frac{1}{\xi}\frac{\partial^2 \chi}{\partial \xi^2} - \frac{1}{\xi^2}\frac{\partial \chi}{\partial \xi} \right] + \left[h^3(\xi)\right]_{\xi\xi}$$
$$\left[ c\frac{\partial^2 \chi}{\partial \xi^2} + \frac{\nu}{\xi}\frac{\partial \chi}{\partial \xi} \right] - \frac{P(\xi,\tau)a^3}{D_0} + h(\xi)\frac{\partial^2 \chi}{\partial \tau^2} - \frac{12a^2}{h_0^2}\frac{1}{\xi}\frac{\partial}{\partial \xi}$$
$$\left. \left. \left\{ h(\xi)\,\xi\,\frac{\partial \chi}{\partial \xi} \left[ c\frac{\partial \eta}{\partial \xi} + \frac{c}{2}\left(\frac{\partial \chi}{\partial \xi}\right)^2 + \frac{\nu\eta}{\xi} \right] \right\} \right\} \right\} \delta\chi\,\xi\,d\xi \tag{21}$$

is employed to obtain a governing equation which closely approximates (18A) within the limits of the assumed form of motion given by equation (20). For a given $\tau$, the above integral is equivalent to the virtual work of all the transverse forces as they move through a virtual displacement $\delta\chi$. Substituting equations (19) and (20) into equation (21) and equating the average virtual work over one period of oscillation to zero, i.e.,

$$I_A = \int_0^{2\pi/\omega} I \, d\tau = 0 \quad , \tag{21'}$$

one obtains

$$h^3(\xi) \left[ cg^{iv} + \frac{2c}{\xi} g''' - \frac{1}{\xi^2} g'' + \frac{1}{\xi^3} g' \right] + \left[ h^3(\xi) \right]_\xi$$

$$\left[ 2cg''' + \frac{(2c+\nu)}{\xi} g'' - \frac{1}{\xi^2} g' \right] + \left[ h^3(\xi) \right]_{\xi\xi} \left[ cg'' + \frac{\nu}{\xi} g' \right]$$

$$- \omega^2 g h(\xi)$$

$$= \frac{1}{\sqrt{\alpha}} \left( \frac{a^4}{D_0 h_0} P(\xi) \right) + \frac{9\alpha}{\xi} \frac{d}{d\xi} \left[ h(\xi) \xi g'(cf' + \frac{c}{2} g'^2 + \frac{\nu}{\xi} f) \right] \tag{22A}$$

where $\alpha = \left(\frac{Aa}{h_0}\right)^2$ .

This procedure, as outlined by expressions (21) and (21'), is analogous to the approach used in reference (9) for the single degree of freedom Duffing's equation. Substitution of equations (20) into equation (18B) yields

$$h(\xi) \left\{ cf'' + \frac{c}{\xi} f' - \frac{f}{\xi^2} + cg'g'' + \frac{c-\nu}{2\xi} (g')^2 \right\}$$

$$+ h'(\xi) \left\{ cf' + \frac{\nu}{\xi} f + \frac{c}{2} (g')^2 \right\} = 0. \tag{22B}$$

Equations (22) are the final forms of the governing differential equations in the displacement formulation. For the stress formulation, the stress function $\psi$ is defined by

$$N_\theta = \frac{\partial}{\partial r} (\psi)$$

$$N_r = \frac{\psi}{r} \quad ,$$

which satisfy equation (8') identically. By introducing the function

$$F(\xi) = \frac{a_{22}(c - \nu^2)}{h_0} \phi(\xi)$$

with

$$\phi(\xi) = \frac{\psi}{aA^2 \sin^2 \omega\tau} \quad ,$$

equations (22) can be rewritten as

$$h^3(\xi) \left[ cg^{iv} + \frac{2c}{\xi} g''' - \frac{1}{\xi^2} g'' + \frac{1}{\xi^3} g' \right] + \left[ h^3(\xi) \right]_\xi$$

$$\left[ 2cg''' + \frac{(2c+\nu)}{2} g'' - \frac{1}{\xi^2} g' \right] + \left[ h^3(\xi) \right]_{\xi\xi} \left[ cg'' + \frac{\nu}{\xi} g' \right]$$

$$- \omega^2 gh(\xi)$$

$$= \frac{1}{\sqrt{\alpha}} \left( \frac{a^4}{D_0 h_0} P(\xi) \right) + \frac{9\alpha}{\xi} \frac{d}{d\xi} \left[ g'F(\xi) \right] , \tag{23A}$$

$$(cF'' + \frac{c}{\xi} F' - \frac{F}{\xi^2}) - \frac{h'(\xi)}{h(\xi)} (cF' - \nu \frac{F}{\xi})$$

$$= - (c - \nu^2) \frac{h(\xi)}{2\xi} (g')^2 . \tag{23B}$$

This set of equations is the final form of the governing differential equations in the stress formulation.

## IV. NUMERICAL ANALYSIS AND COMPUTATION

The governing differential equations (23), with a given set of boundary conditions, can be solved approximately by a numerical approach. For this purpose the governing differential equations are changed to a system of six first-order differential equations. They are

$$\frac{dY_1}{d\xi} = Y_2$$

$$\frac{dY_2}{d\xi} = Y_3$$

$$\frac{dY_3}{d\xi} = Y_4$$

$$\begin{aligned}\frac{dY_4}{d\xi} &= \frac{1}{ch^3(\xi)} \frac{1}{\sqrt{\alpha}} \left(\frac{a^4}{D_0 h_0}\right) P(\xi)) + \frac{\omega^2}{ch^2(\xi)} Y_1 \\ &+ \frac{-h^3(\xi) + \xi\left[h^3(\xi)\right]_{\xi} - \nu\xi^2\left[h^3(\xi)\right]_{\xi\xi}}{c\,\xi^3\,h^3(\xi)} Y_2 \\ &+ \frac{h^3(\xi) - (2c+\nu)\xi\left[h^3(\xi)\right]_{\xi} - c\,\xi^2\left[h^3(\xi)\right]_{\xi\xi}}{c\,\xi^2\,h^3(\xi)} Y_3 \\ &+ \frac{-2h^3(\xi) - 2\xi\left[h^3(\xi)\right]_{\xi}}{\xi\,h^3(\xi)} Y_4 + \frac{9\alpha}{c\,\xi\,h^3(\xi)} (Y_2Y_6+Y_3Y_5)\end{aligned} \tag{24}$$

$$\frac{dY_5}{d\xi} = Y_6$$

$$\frac{dY_6}{d\xi} = \frac{h(\xi) - \nu\xi\left[h(\xi)\right]_{\xi}}{c\,\xi^2 h(\xi)}\, Y_5 + \frac{\xi\left[h(\xi)\right]_{\xi} - h(\xi)}{\xi\, h(\xi)}\, Y_6$$

$$+ \frac{-(c - \nu^2)h(\xi)}{2c\,\xi}\,(Y_2)^2$$

for $0 < \xi \leq 1$,

where

$$\overline{Y}(\xi) = \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{Bmatrix} = \begin{Bmatrix} g \\ g' \\ g'' \\ g''' \\ F \\ F' \end{Bmatrix}.$$

For simplicity, a clamped isotropic circular plate with stepwise thickness under free vibration will be studied. Refer to Fig. 3A. A radially stepwise thickness circular plate with $h = 2h$, when $0 \leq r \leq a/2$, and $h = h$, when $a/2 \leq r \leq a$, is illustrated. For isotropic media under free vibration, $c = 1$ and $p(\xi) = 0$. Since the thickness h is constant within each interval of $\xi$, its first derivative and the higher order derivatives with respect to $\xi$ vanish. Then equations (24) are reduced to

$$\frac{dY_1}{d\xi} = Y_2$$

$$\frac{dY_2}{d\xi} = Y_3$$

$$\frac{dY_3}{d\xi} = Y_4$$

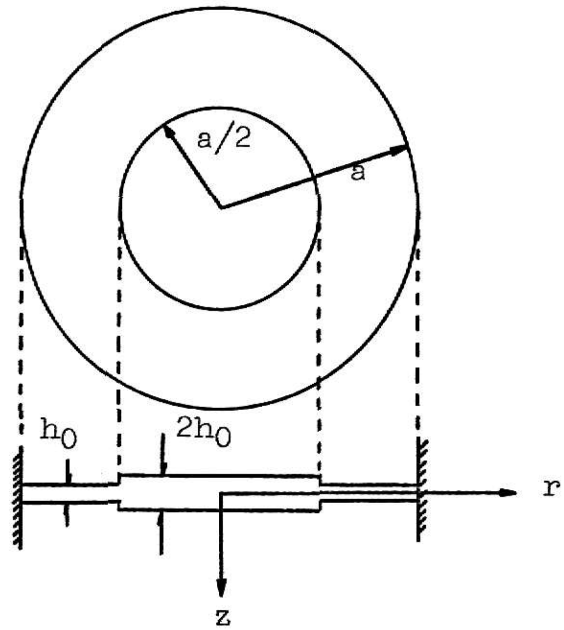


Fig. 3A.

$$\frac{dY_4}{d\xi} = \frac{\lambda}{h^2} Y_1 - \frac{1}{\xi^3} Y_2 + \frac{1}{\xi^2} Y_3 - \frac{2}{\xi} Y_4 + \frac{9\alpha}{\xi h^3} Y_2 Y_6$$

$$+ \frac{9\alpha}{\xi h^3} Y_3 Y_5 \qquad (25)$$

$$\frac{dY_5}{d\xi} = Y_6$$

$$\frac{dY_6}{d\xi} = \frac{1}{\xi^2} Y_5 - \frac{1}{\xi} Y_6 - \frac{(1 - \nu^2)}{2\xi} h(Y_2)^2$$

for $0 < \xi \leq 1/2$ and $1/2 \leq \xi \leq 1$ with a discontinuity of the function $\overline{Y}(\xi)$ at $\xi = 1/2$, where $\lambda = \omega^2$.

The singular point located at = 0 causes unboundedness unless certain continuity conditions are satisfied. Arguing on a physical basis, g and F and their derivatives are continuous and bounded functions if there is no concentrated load acting at the origin. Hence $Y_1$ and $Y_5$ can be expanded in the Maclaurin series

$$Y_1 = (Y_1)_0 + (Y_2)_0 \xi + \frac{(Y_3)_0}{2!} \xi^2 + \frac{(Y_4)_0}{3!} \xi^3 + \ldots$$

and

$$Y_5 = (Y_5)_0 + (Y_6)_0 \xi + \ldots .$$

Substituting these expansions into equations (25) and demanding that derivatives remain bounded in the limit as $\xi$ approaches zero, one finds

$$\left[\frac{dY_1}{d\xi}\right]_0 = 0$$

$$\left[\frac{dY_2}{d\xi}\right]_0 = C$$

$$\left[\frac{dY_3}{d\xi}\right]_0 = 0$$

$$\left[\frac{dY_4}{d\xi}\right]_0 = \frac{3}{32} \lambda + \frac{27}{32} \dot{\alpha} (C)(S) \qquad (26)$$

$$\left[\frac{dY_5}{d\xi}\right]_0 = S$$

$$\left[\frac{dY_6}{d\xi}\right]_0 = 0$$

at $\xi = 0$, where $(Y_1)_0 = 1$ and C and S are certain undetermined constants.

Consider $\overline{Y}$ at $\xi = 1/2$. Refer to Fig. 4, a free body diagram of an element of the plate in the vicinity of $\xi = 1/2$.

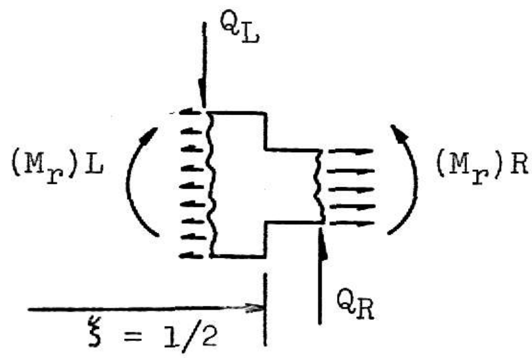


Fig. 4.

For no concentrated torque or concentrated load acting upon this element, equilibrium conditions require that $(M_r)_L = (M_r)_R$ and $Q_L = Q_R$. From these relations, the following equations are derived; i.e.,

$$(w_{rr})_L = \frac{1}{8}(w_{rr})_R + \frac{1}{12}(w_r)_R - \frac{2}{3}(w_r)_L \, ,$$

$$(w_{rrr})_L = \frac{1}{8}(w_{rrr})_R - \frac{2}{3}(w_r)_R + \frac{16}{3}(w_r)_L \tag{27}$$

with $\nu = 1/3$.

After changing to a dimensionless form, a relation between $\overline{Y}(1/2 - 0)$ and $\overline{Y}(1/2 + 0)$ is obtained; i.e.,

$$[A]\,\overline{Y}\,(1/2 - 0) - [B]\,\overline{Y}\,(1/2 + 0) = \overline{0} \, , \tag{28}$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1 & 0 & 0 & 0 \\ 0 & -16/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 1/8 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Combining equations (25) and (26) and writing in vector form, one obtains

$$\frac{d\overline{Y}}{d\xi} = \overline{H}(\xi, \overline{Y}; \alpha, \lambda) \tag{29}$$

for $0 \leq \xi \leq 1$, where $\overline{H}$ is the appropriately defined vector function as stated in equations (25) and (26). The associated forms for the boundary conditions in terms of $\overline{Y}(\xi)$ are

$$[C]\,\overline{Y}(0) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

and (30)

$$[D]\,\overline{Y}(1) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

with

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad [D] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & 1 \end{bmatrix} .$$

To solve this boundary value problem, as given in equations (28), (29), and (30), the related initial value problems are considered.

Since $\bar{Y}$ is discontinuous at $\xi = 1/2$, it is convenient to introduce two initial value functions. They are

$$\frac{d\bar{U}}{d\xi} = \bar{H}(\xi, \bar{U} ; \alpha, \lambda) \tag{31}$$

for $0 \leq \xi \leq 1/2$, with the initial condition

$$\bar{U}(0) = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}_{\xi=0} = \begin{Bmatrix} 1 \\ 0 \\ C \\ 0 \\ 0 \\ S \end{Bmatrix} = \bar{r}_1$$

and

$$\frac{d\bar{V}}{d\xi} = \bar{H}(\xi, \bar{V} ; \alpha, \lambda) \tag{32}$$

for $1/2 \leq \xi \leq 1$, with the initial condition

$$\overline{V}(1) = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{Bmatrix}_{\xi=1} = \begin{Bmatrix} 0 \\ 0 \\ M \\ V \\ N \\ \nu N \end{Bmatrix} = \overline{r}_2 \quad .$$

Among the components of the initial values in equations (31) and (32), C, S, M, V, and N are adjustable parameters, and the remaining components correspond to those boundary values at $\xi = 0$ and $\xi = 1$ found in equation (30).

By integrating the first initial value problem, equation (31), simultaneously with a fourth-order Runge-Kutta method, the solutions can be obtained as

$$\overline{U} = \overline{U}\ (\xi\ ;\ \overline{\eta_1}\ ,\ \alpha)\ ;\quad \overline{\eta_1} = \begin{Bmatrix} C \\ S \\ \lambda \end{Bmatrix} \quad \text{for } 0 \leq \xi \leq 1/2. \qquad (33)$$

In the same manner, equation (32) yields

$$\overline{V} = \overline{V}\ (\xi\ ;\ \overline{\eta_2}\ ,\ \alpha)\ ;\quad \overline{\eta_2} = \begin{Bmatrix} M \\ V \\ N \\ \lambda \end{Bmatrix} \quad \text{for } 1/2 \leq \xi \leq 1. \qquad (34)$$

The secondary arguments discussed here indicate that the solution depends upon the set of parameters $\eta_1$, $\eta_2$, and $\alpha$. From these solutions the six equations of continuity

$$\overline{Z}(\frac{1}{2}\ ;\ \overline{\eta},\ \alpha) = [A]\ \overline{U}(\frac{1}{2}\ ;\ \overline{\eta_1}\ ,\ \alpha) - [B]\ \overline{V}(\frac{1}{2}\ ;\ \overline{\eta_2}\ ,\ \alpha) = \overline{0} \tag{35}$$

where

$$\overline{\eta} = \begin{Bmatrix} C \\ S \\ \lambda \\ M \\ V \\ N \end{Bmatrix}$$

may be constructed.

It is now apparent that solving the boundary-value problem is equivalent to obtaining a functional relation $\overline{\eta} = \overline{\eta}(\alpha)$ which satisfies (35); i.e.,

$$\overline{Z}(1/2;\ \overline{\eta}(\alpha),\ \alpha) \equiv \overline{0}\ . \tag{36}$$

Thus for a fixed value of $\alpha$, say $\alpha = \alpha^0$ ,

$$\begin{aligned} \overline{Y}(\xi) &= \overline{U}(\xi\ ;\ \overline{\eta_1}^0\ ,\ \alpha^0)\ , \quad 0 \leq \xi \leq 1/2; \\ &= \overline{V}(\xi\ ;\ \overline{\eta_2}^0\ ,\ \alpha^0)\ ,\ 1/2 \leq \xi \leq 1 \end{aligned} \tag{37}$$

is a solution to the boundary-value problem, where

$$\overline{\eta_1}^0 = \begin{Bmatrix} C \\ S \\ \lambda \end{Bmatrix}\ , \qquad \overline{\eta_2}^0 = \begin{Bmatrix} M \\ V \\ N \\ \lambda \end{Bmatrix}$$

are vectors corresponding to

$$\bar{\eta}^0 = \begin{Bmatrix} C \\ S \\ \lambda \\ M \\ V \\ N \end{Bmatrix}$$

in

$$\bar{Z}\,(1/2\ ;\ \bar{\eta}^0\ ,\ \alpha_0) = \bar{0}\ .$$

The solution of the six equations (36) for the six unknowns, $\bar{\eta}$, can be accomplished by a direct application of Newton's method. Starting from an initial guess $\overline{\eta_1}$, the iterative sequence

$$\overline{\eta_{k+1}} = \overline{\eta_k} + \Delta\overline{\eta_k}\ ;\quad k = 1,\ 2,\ .\ .\ . \tag{38}$$

is generated. Rejection of the higher order terms of a Taylor series expansion about $\bar{\eta}_k$ provides the linear correction

$$\Delta\bar{\eta}_k = -\left[(J)_k\right]^{-1}\bar{Z}(1/2\ ;\ \bar{\eta}_k\ ,\ \alpha^0) \tag{39}$$

at the $k^{th}$ step of iteration. The Jacobian matrix (J) is defined as

$$(J) = \left[\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right]_{\xi=1/2} \tag{40}$$

and can be interpreted physically as the change of final values with respect to a change of the initial data $\bar{\eta}$. If $\bar{\eta}_1$ is chosen to be in a sufficiently small neighborhood of $\bar{\eta}^0$, the convergence of the sequence, equations (38) and (39), to the root $\bar{\eta}^0$ can be expected. Since an explicit solution of the initial-value problem does not avail itself due to the nonlinear character of the vector function $\bar{H}$, the expression for (J) in equation (40) cannot be evaluated directly. Therefore a method of constructing the Jacobian matrix at any given step of iteration must be devised. Differentiation of the initial-value problems with respect to $\bar{\eta}$ results in the variational problems

$$\frac{d}{d\xi}\left[\frac{\partial \bar{U}}{\partial \bar{\eta}}\right] = \frac{\partial \bar{H}}{\partial \bar{\eta}} + \left[\frac{\partial \bar{H}}{\partial \bar{U}}\right]\left[\frac{\partial \bar{U}}{\partial \bar{\eta}}\right], \quad 0 \leq \xi \leq 1/2 \; ;$$

$$\left[\frac{\partial \bar{U}}{\partial \bar{\eta}}\right]_{\xi = 0} = \frac{\partial \bar{r}_1}{\partial \bar{\eta}} \tag{41}$$

and

$$\frac{d}{d\xi}\left[\frac{\partial \bar{V}}{\partial \bar{\eta}}\right] = \frac{\partial \bar{H}}{\partial \bar{\eta}} + \left[\frac{\partial \bar{H}}{\partial \bar{V}}\right]\left[\frac{\partial \bar{V}}{\partial \bar{\eta}}\right], \quad 1/2 \leq \xi \leq 1 \; ;$$

$$\left[\frac{\partial \bar{V}}{\partial \bar{\eta}}\right]_{\xi = 1} = \frac{\partial \bar{r}_2}{\partial \bar{\eta}} \quad .$$

In long-hand notation, define

$$\bar{W}_i = \begin{Bmatrix} \partial U_i/\partial C \\ \partial U_i/\partial S \\ \partial U_i/\partial \lambda \\ \partial U_i/\partial M \\ \partial U_i/\partial V \\ \partial U_i/\partial N \end{Bmatrix} \quad \text{and } \bar{X}_i = \begin{Bmatrix} \partial V_i/\partial C \\ \partial V_i/\partial S \\ \partial V_i/\partial \lambda \\ \partial V_i/\partial M \\ \partial V_i/\partial V \\ \partial V_i/\partial N \end{Bmatrix}, \quad i = 1, 2, \ldots, 6. \tag{42}$$

The variational problem will be

$$\frac{d\overline{W}_1}{d\xi} = \overline{W}_2$$

$$\frac{d\overline{W}_2}{d\xi} = \overline{W}_3$$

$$\frac{d\overline{W}_3}{d\xi} = \overline{W}_4$$

$$\frac{d\overline{W}_4}{d\xi} = -\frac{2}{\xi}\overline{W}_4 + \frac{1}{\xi^2}\overline{W}_3 - \frac{1}{\xi^3}\overline{W}_2 + \frac{\lambda}{4}\overline{W}_1 + \frac{U_1}{4}\begin{Bmatrix}0\\0\\1\\0\\0\\0\end{Bmatrix}$$

$$+ \frac{9\alpha}{\partial\xi}\left[U_3\overline{W}_5 + U_5\overline{W}_3 + U_2\overline{W}_6 + U_6\overline{W}_2\right]$$

$$\frac{d\overline{W}_5}{d\xi} = \overline{W}_6$$

$$\frac{d\overline{W}_6}{d\xi} = -\frac{\overline{W}_6}{\xi} + \frac{\overline{W}_5}{\xi^2} - \frac{2(1-\nu^2)}{\xi}U_2\overline{W}_2 \tag{43}$$

when $0 < \xi \leq 1/2$,

$$\left[\frac{d\overline{W}_1}{d\xi}\right]_0 = \begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}$$

$$\left[\frac{d\overline{W}_2}{d\xi}\right]_0 = \begin{Bmatrix}1\\0\\0\\0\\0\\0\end{Bmatrix}$$

$$\left[\frac{d\overline{W}_3}{d\xi}\right]_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left[\frac{d\overline{W}_4}{d\xi}\right]_0 = \frac{3}{32} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{27}{32} \alpha \left\{ C \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + S \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \right\}$$

$$\left[\frac{d\overline{W}_5}{d\xi}\right]_0 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left[\frac{d\overline{W}_6}{d\xi}\right]_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

when $\xi = 0$, with the initial value matrix

$$\left[W\right]_{\xi=0} = \left[\{W_i\}\right]_{\xi=0} = \left[\{W_1\} \cdots \{W_6\}\right]_{\xi=0} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$i = 1, 2, \ldots, 6$

$$\frac{d\overline{X}_1}{d\xi} = \overline{X}_2$$

$$\frac{d\overline{X}_2}{d\xi} = \overline{X}_3$$

$$\frac{d\overline{X}_3}{d\xi} = \overline{X}_4$$

$$\frac{d\overline{X}_4}{d\xi} = -\frac{2}{\xi}\overline{X}_4 + \frac{1}{\xi^2}\overline{X}_3 - \frac{1}{\xi^3}\overline{X}_2 + \lambda\overline{X}_1 + V_1\begin{Bmatrix}0\\0\\1\\0\\0\\0\end{Bmatrix}$$

$$+\frac{9\alpha}{\xi}\left[V_3\overline{X}_5 + V_5\overline{X}_3 + V_2\overline{X}_6 + V_6\overline{X}_2\right] \tag{44}$$

$$\frac{d\overline{X}_5}{d\xi} = \overline{X}_6$$

$$\frac{d\overline{X}_6}{d\xi} = -\frac{\overline{X}_6}{\xi} + \frac{\overline{X}_5}{\xi^2} - \frac{1-\nu^2}{\xi}V_2\overline{X}_2$$

when $1/2 \leq \xi \leq 1$, with the initial value matrix

$$\left[X\right]_{\xi=1} = \left[\{X_i\}\right]_{\xi=1} = \begin{bmatrix}0&0&0&0&1&\nu\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}, \quad i = 1, 2, \ldots, 6.$$

For a given vector $\overline{\eta}$ and $\alpha = \alpha^0$, these derived problems and equations (31) and (32) may be solved simultaneously on both intervals $[0, 1/2]$ and $[1/2, 1]$. Evaluation of the solution to the variational problem, equation (41), at $\xi = 1/2$ yields $(J)$, where

$$(J) = [A]\left[\{W_i\}\right]^T - [B]\left[\{X_i\}\right]^T, \quad i = 1, 2, \ldots, 6.$$

Thus Newton's method in conjunction with the initial-value problems (31), (32), and (41) provide an operational technique for solving the boundary-value problem (29) and (30) for a fixed value of $\alpha$.

The analysis of the two-point boundary-value problem is completed when the functional relation $\bar{\eta} = \bar{\eta}(\alpha)$ is established. Using the above procedure this can be accomplished in a discrete manner, i.e.,

$$\bar{\eta}^i = \bar{\eta}^i(\alpha^i) \quad ; \quad i = 0, 1, 2, \ldots, m.$$

Having obtained a root $\bar{\eta}^j$ corresponding to $\alpha = \alpha^j$, the value of $\alpha$ may be perturbed,

$$\alpha = \alpha^{j+1} = \alpha^j + \Delta\alpha^j \quad . \tag{45}$$

For this value of $\alpha$, iteration is reinstated starting from $\bar{\eta} = \bar{\eta}^j$. If $\Delta\alpha^j$ is not exceedingly large, the $\bar{\eta}^j$ is contained in the new contraction domain of Newton's method, and iteration converges to the root $\bar{\eta}^{j+1}$, which corresponds to $\alpha = \alpha^{j+1}$.

Initially, $\alpha$ is set equal to zero, $\alpha^0 = 0$. By assuming initial values, $C_1$, $S_1$, $\lambda_1$, $M_1$, $V_1$, and $N_1$ properly, the initial-value problems are integrated numerically with step size $\Delta\mu = 1/Z_0$ on $[0, 1/2]$ and $[1/2, 1]$. Correction and integration are carried out until the components of the error vector in (35) are within a range of allowable error. The corresponding values of $C^0$, $N^0$, $\lambda^0$, $M^0$, $V^0$, and $N^0$ are stored and the solution is recorded.

By successive repetition of this analytic continuation given in expression (45), the resonance curve and accompanying solutions are found. This procedure is illustrated schematically in Fig. 5.

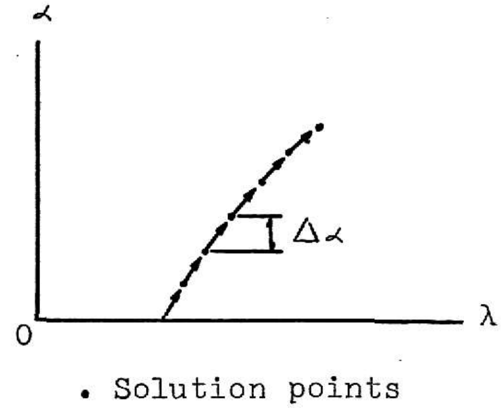


Fig. 5.

A second example (Fig. 3B) is also solved by the method stated above. The numerical results of these two examples are put together for comparison.

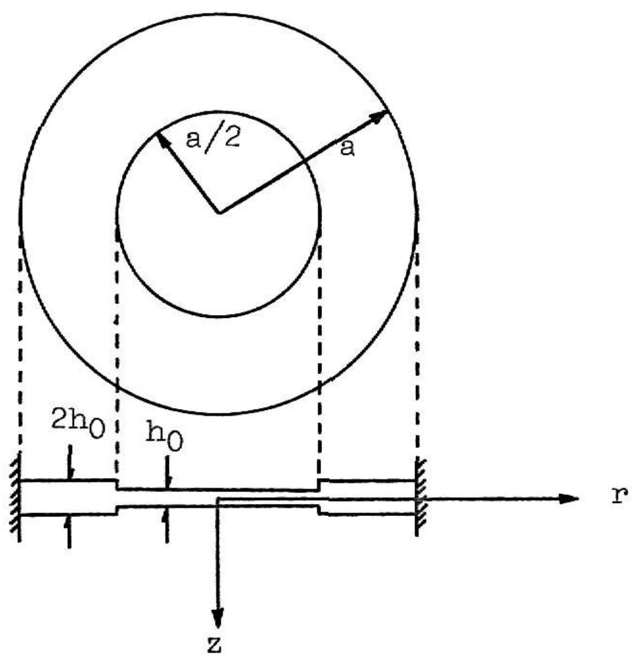


Fig. 3B.

# V. NUMERICAL RESULTS

The numerical results are illustrated in Figs. 6 through 12. In Fig. 6, the nondimensional amplitude at the center of the plate is plotted against the ratio of nonlinear period to linear period. By a decrease of period with increasing amplitude, the hard-spring behavior is evidenced.

Figures 7A and 7B illustrate the influence of amplitude upon the shape function of transverse vibration. The mode shapes for $W_0/h_0 = 2$ differ greatly from those for the linear solution. It is expected from these results that the influence of amplitude on the distribution of bending stress will be of greater significance since the bending stresses are related to the derivatives of the transverse shape function.

In Figs. 8 to 11, nondimensional graphs of radial stress versus amplitude are given. The expressions

$$\sigma_r^{\,b} = \frac{6\ M_r}{h^2}$$

and

$$\sigma_r^{\,m} = \frac{N_r}{h}$$

are used in the calculations of bending and membrane stresses respectively. In terms of the previous assumptions,

$$M_r = -D\left\{c\,\frac{d^2w}{dr^2} + \frac{\nu}{r}\,\frac{dw}{dr}\right\} \quad ; \quad r = a\xi \quad ; \; w = a\chi = aAg(\xi)\sin\omega\tau ;$$

$$D = \frac{h^3}{12a_{22}(1-\nu^2)} \quad ; \quad N_r = \frac{\psi}{r} = \frac{aA^2\,\phi(\xi)\sin^2\omega\tau}{r} \quad ;$$

$$\phi = \frac{F\,h_0}{a_{22}(c-\nu^2)} \quad .$$

When the time $\tau$ is equal to $\pi/2\omega^2$, the maximum amplitude occurs and

$$\frac{\sigma_r{}^b a^2}{E\,h_0{}^2} = -\,\frac{h\sqrt{\alpha}}{2(1-\nu^2)}\left(\frac{d^2g}{d\xi^2} + \frac{\nu}{\xi}\,\frac{dg}{d\xi}\right) ;$$

$$\frac{\sigma_r{}^m a^2}{E\,h_0{}^2} = \frac{1}{h}\,\frac{\alpha}{1-\nu^2}\,\frac{F}{\xi} \quad ,$$

where $E = 1/a_{22}$.

With the aid of L'Hospital's rule, the stresses at $\xi = 0$ are found to be

$$\left[\frac{\sigma_r{}^b a^2}{E\,h_0{}^2}\right]_0 = -\,\frac{h\sqrt{\alpha}}{2(1-\nu)}\left[\frac{d^2g}{d\xi^2}\right]_{\xi=0}$$

$$\left[\frac{\sigma_r{}^m a^2}{E\,h_0{}^2}\right]_0 = \frac{1}{h}\,\frac{\alpha}{1-\nu^2}\left[\frac{dF}{d\xi}\right]_{\xi=0} .$$

In Figs. 8A and 8B, the rapid increase of radial membrane stress with increasing amplitude is shown. In Fig. 8A, in contrast to the case of a uniform thickness plate under the same dynamic condition, the membrane stress at the edge is greater than that at the center. This is due to the variable thickness effect.

The nonlinear behavior of radial bending stress is shown by Figs. 9 to 11. It is shown that the nonlinearity in the stresses of the second plate is greater than that of the first one. In Fig. 11, the discontinuity of stresses occurring at $\xi = 0.5$ is due to the assumption that the stresses are distributed over the entire thickness of the plate at that section.

The free resonance characteristics are established in Fig. 12. A comparison is made between the first stepwise thickness plate and a uniform thickness plate under the same dynamic conditions (4). The higher frequency in the linear case and the lower frequency in the nonlinear case of the first plate again signifies the influence of the variable thickness effect. Under the same amplitude, the greater frequency of the second plate shows its higher stiffness.

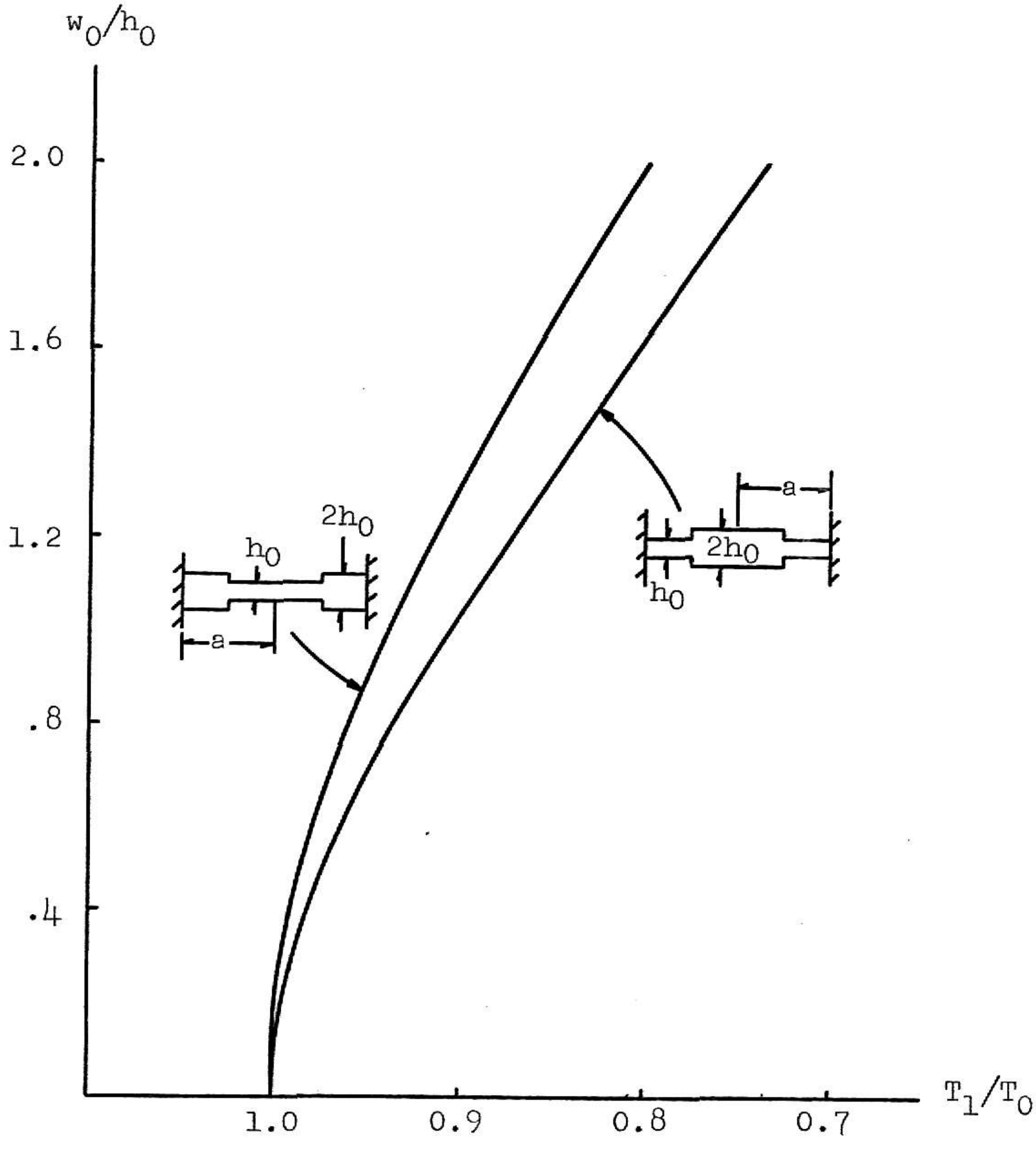


Fig. 6. Free nonlinear period as a function of relative amplitude.

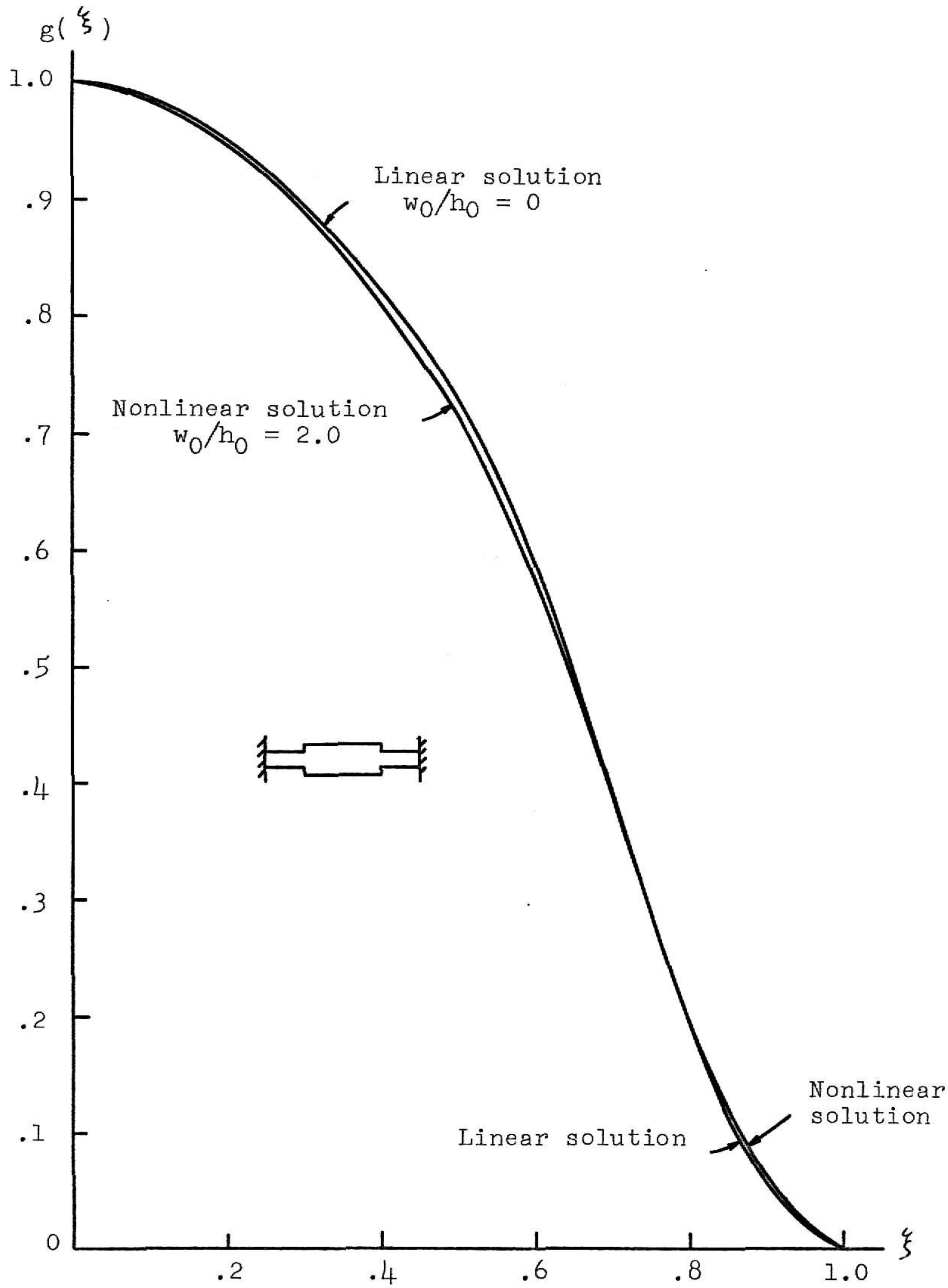


Fig. 7A. Transverse shape function of free vibration.

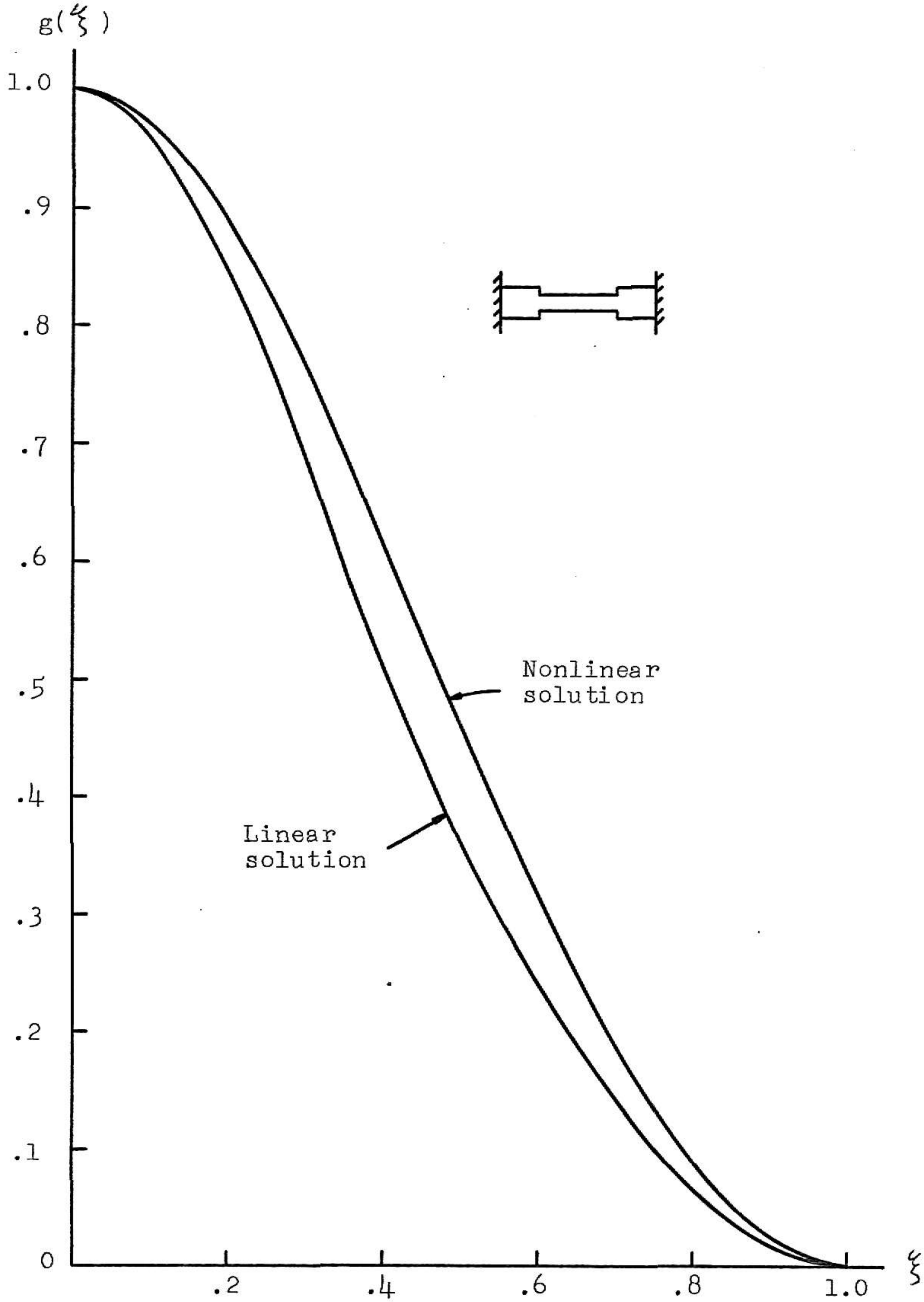


Fig. 7B. Transverse shape function of free vibration.

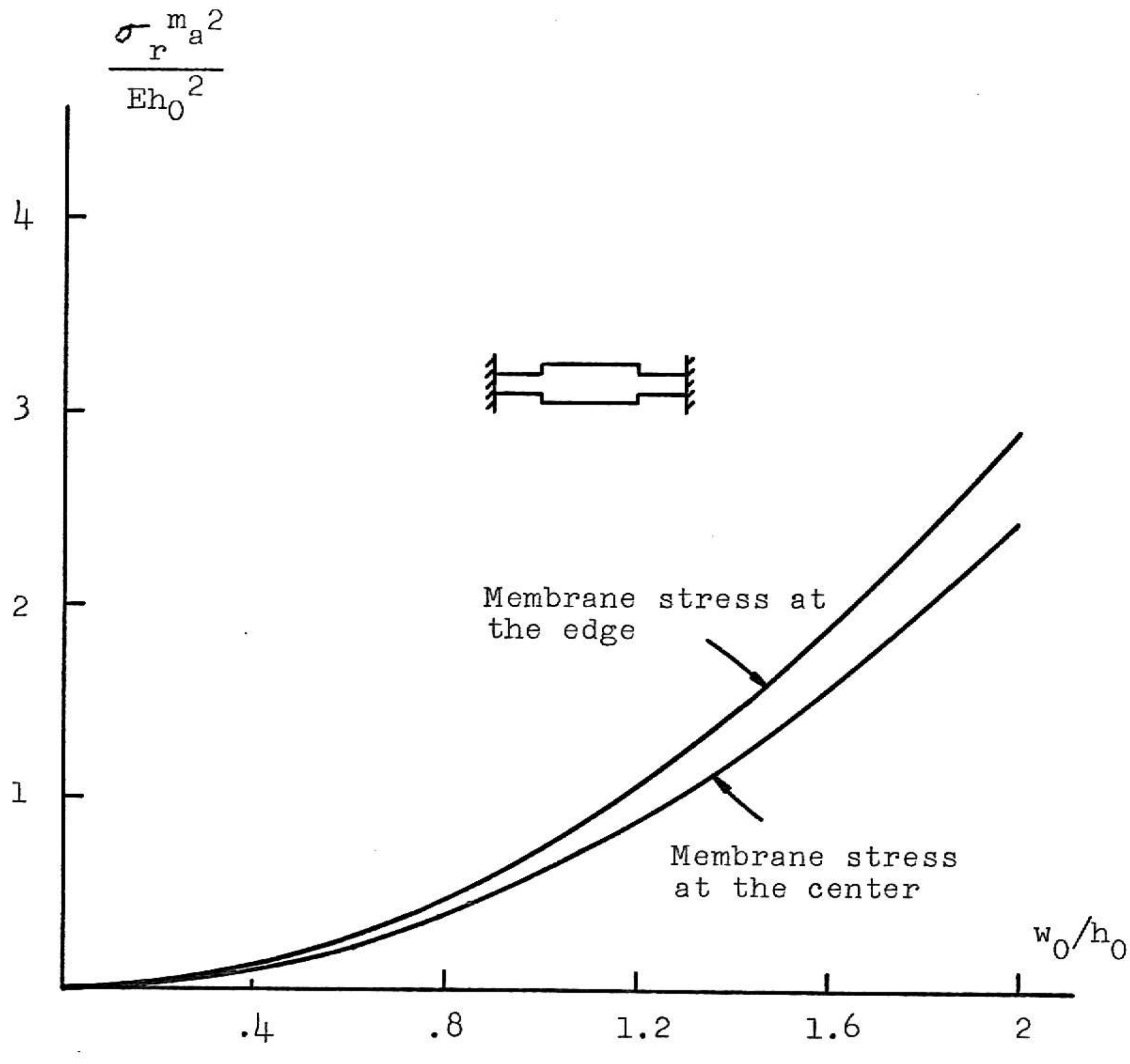


Fig. 8A. Nonlinear membrane stress.

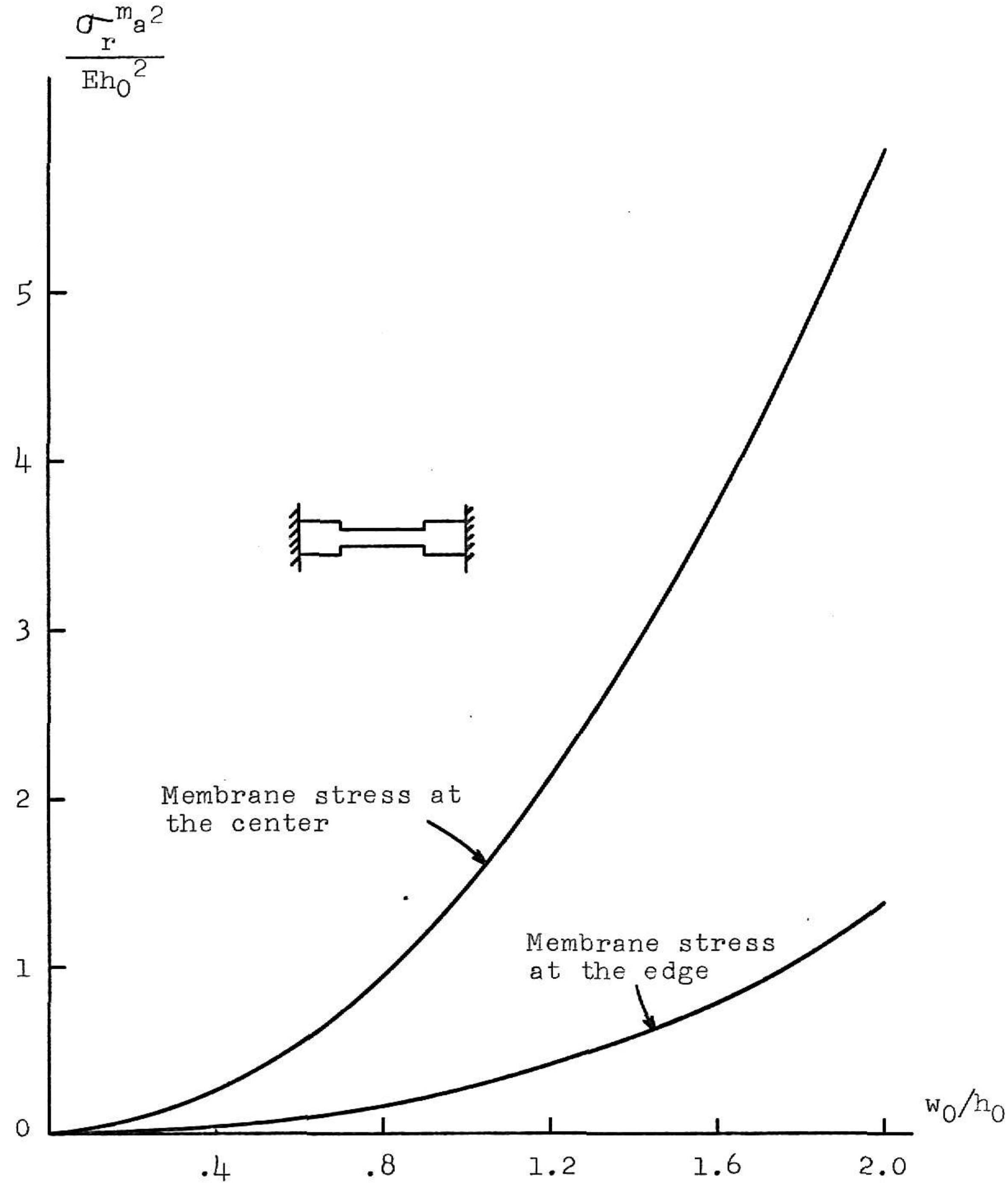


Fig. 8B. Nonlinear membrane stress.

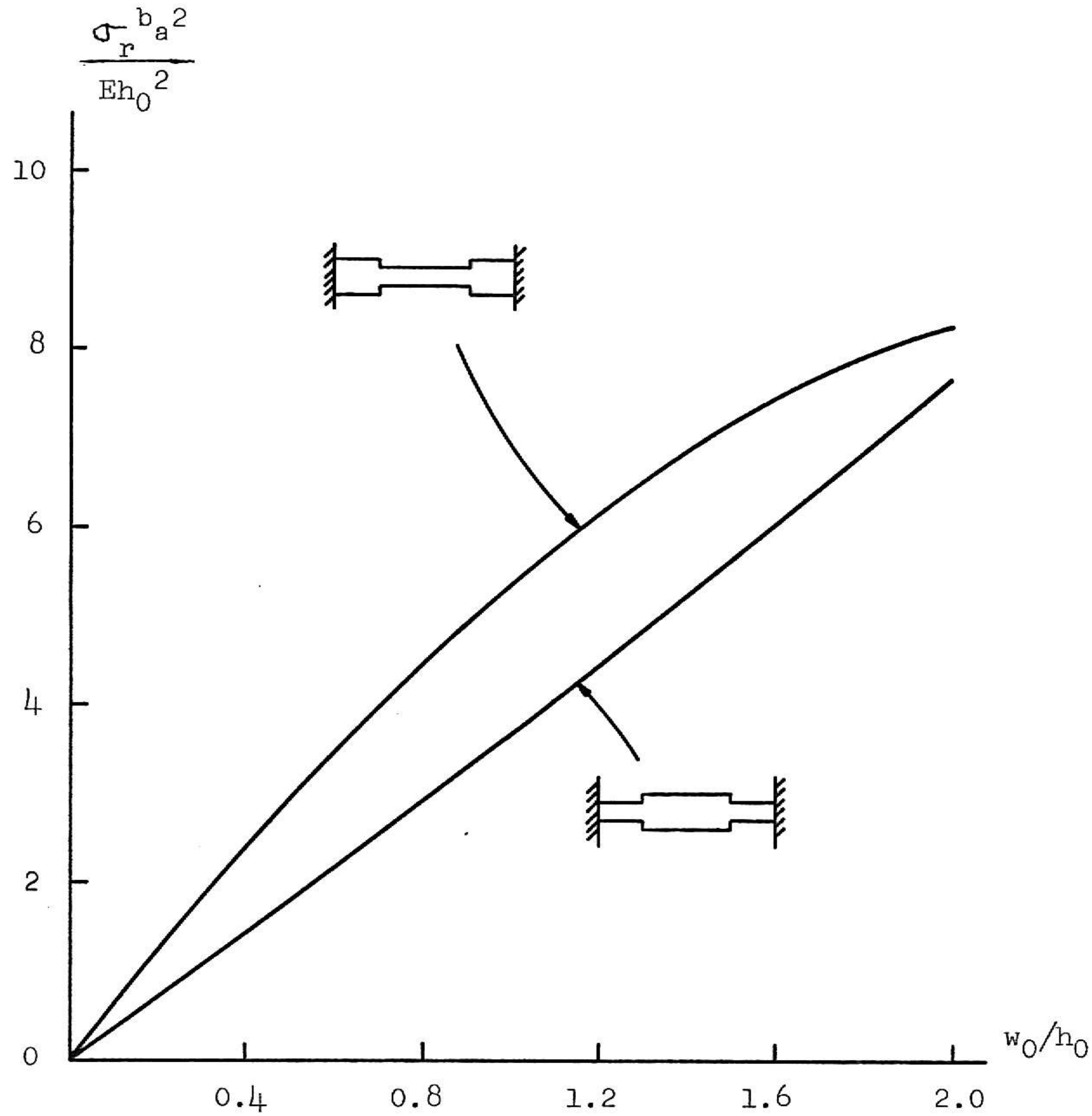


Fig. 9. Center bending stress.

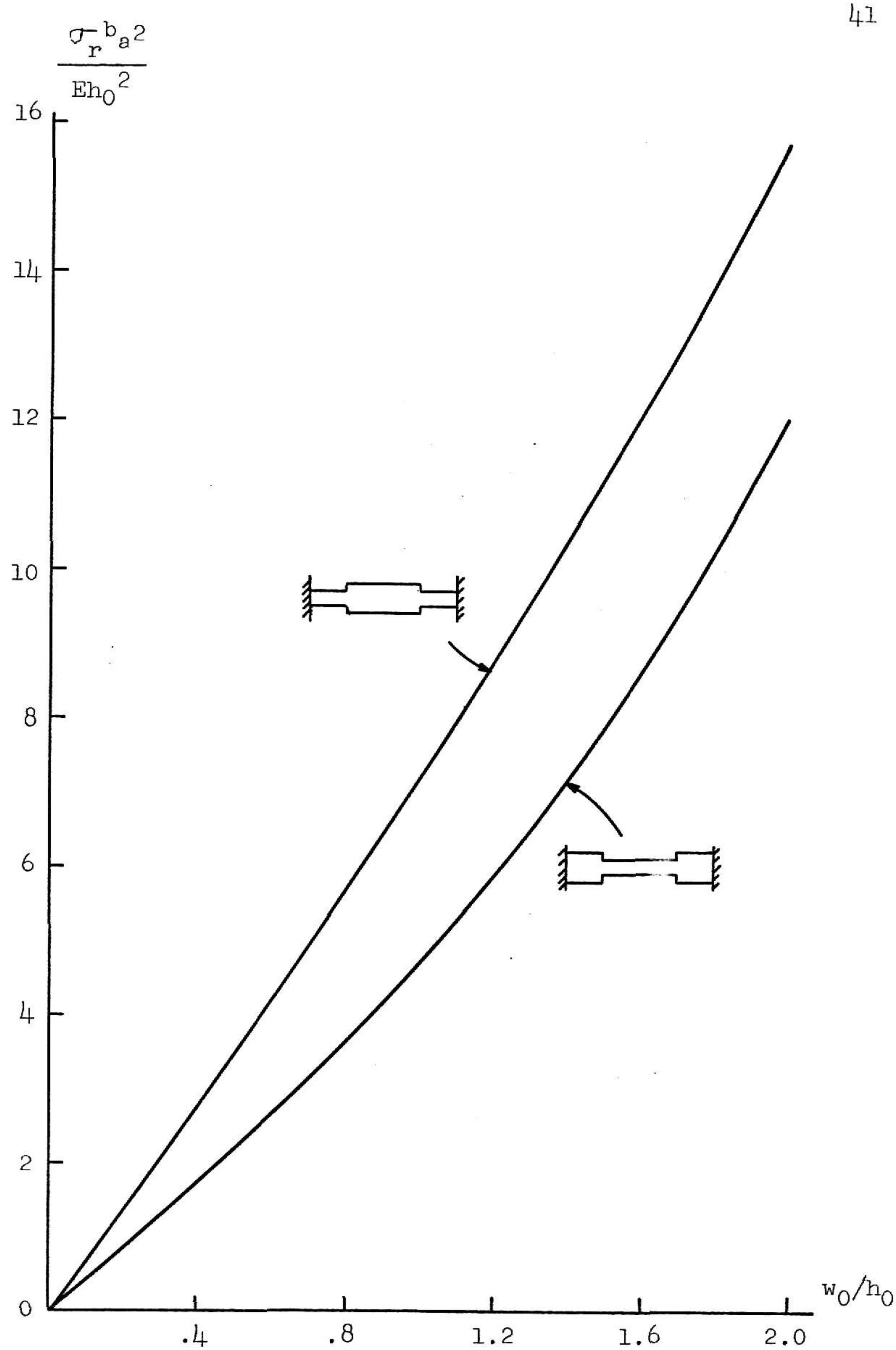


Fig. 10. Edge bending stress.

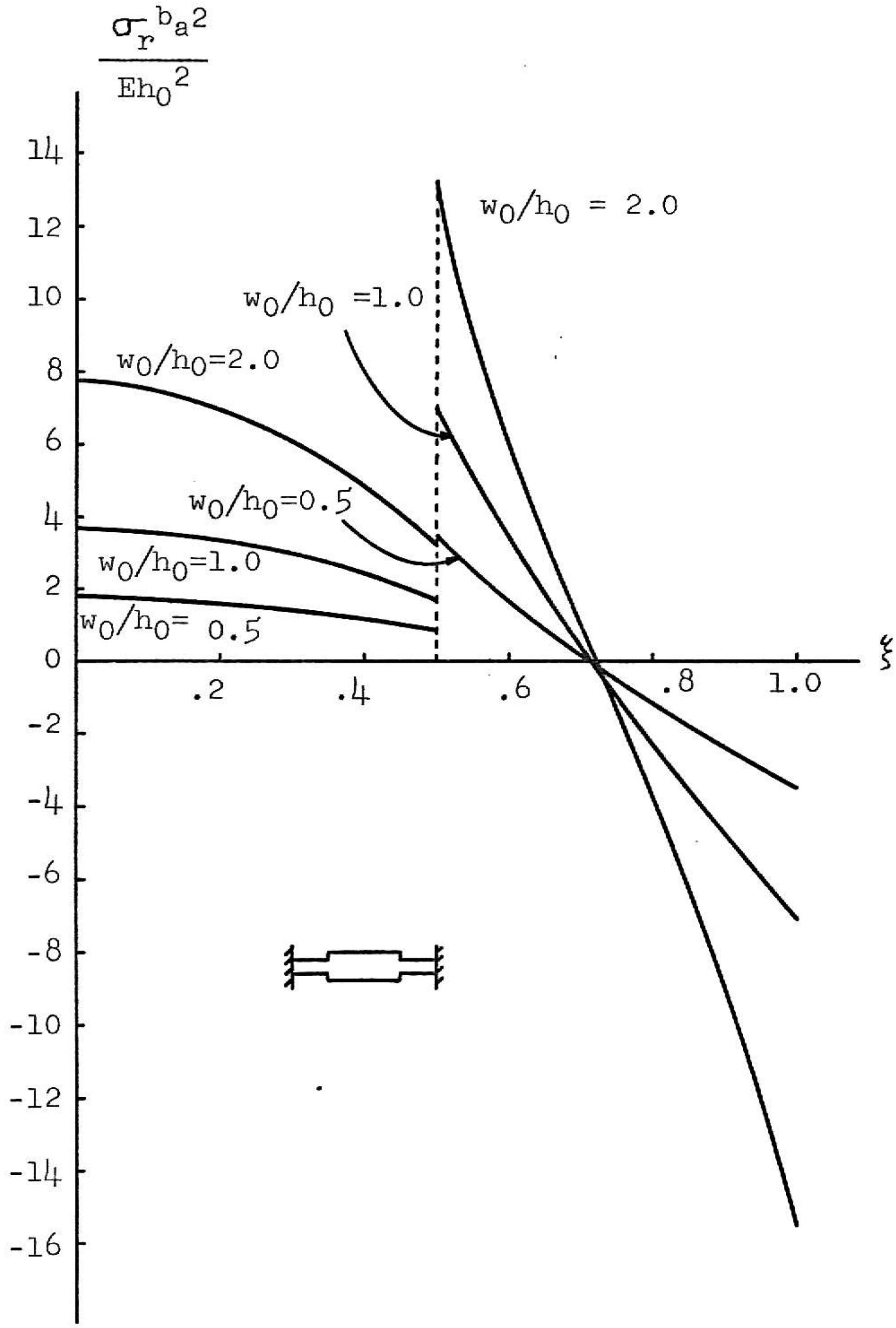


Fig. 11A. Bending stress as a function of radial position.

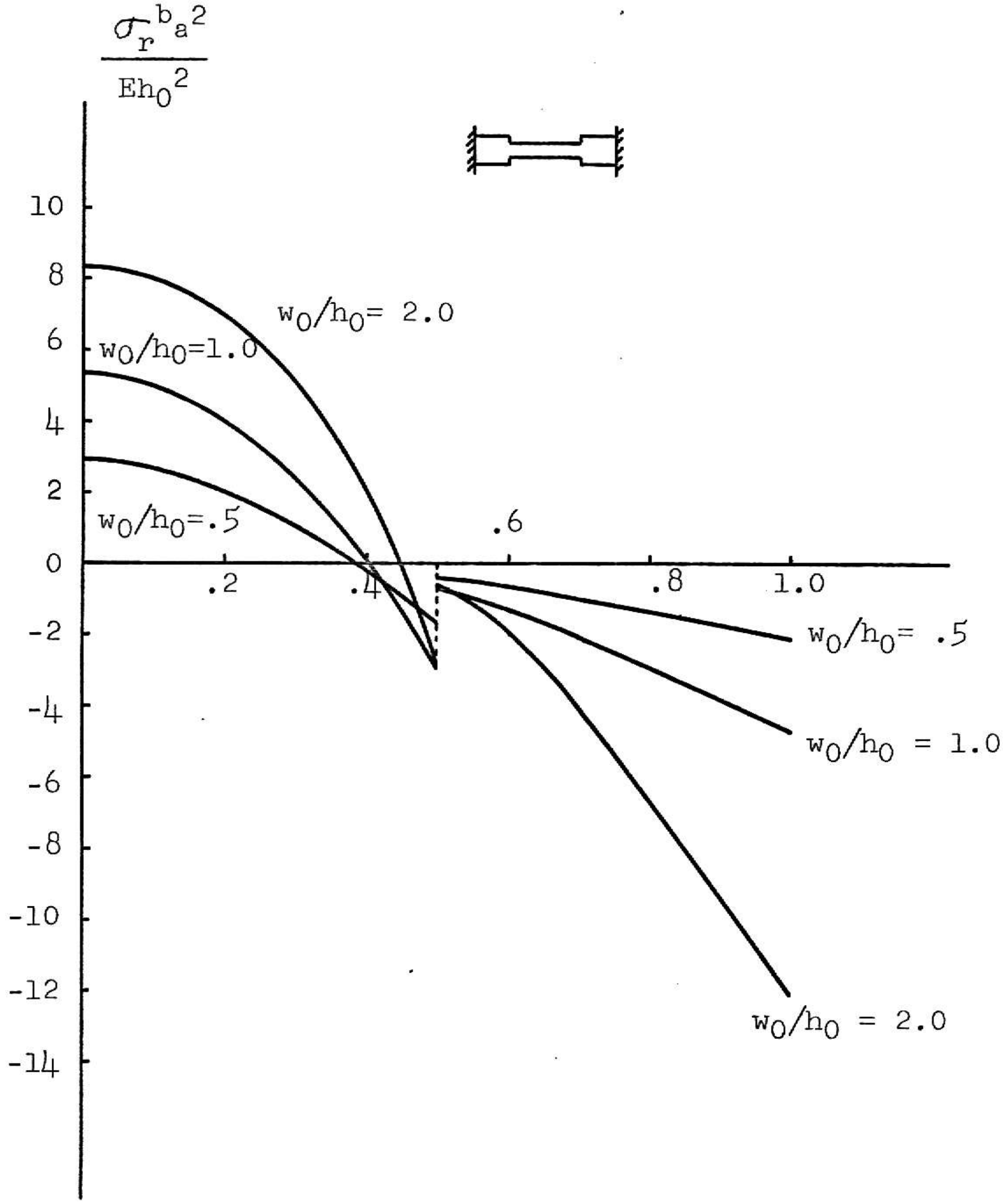


Fig. 11B. Bending stress as a function of radial position.

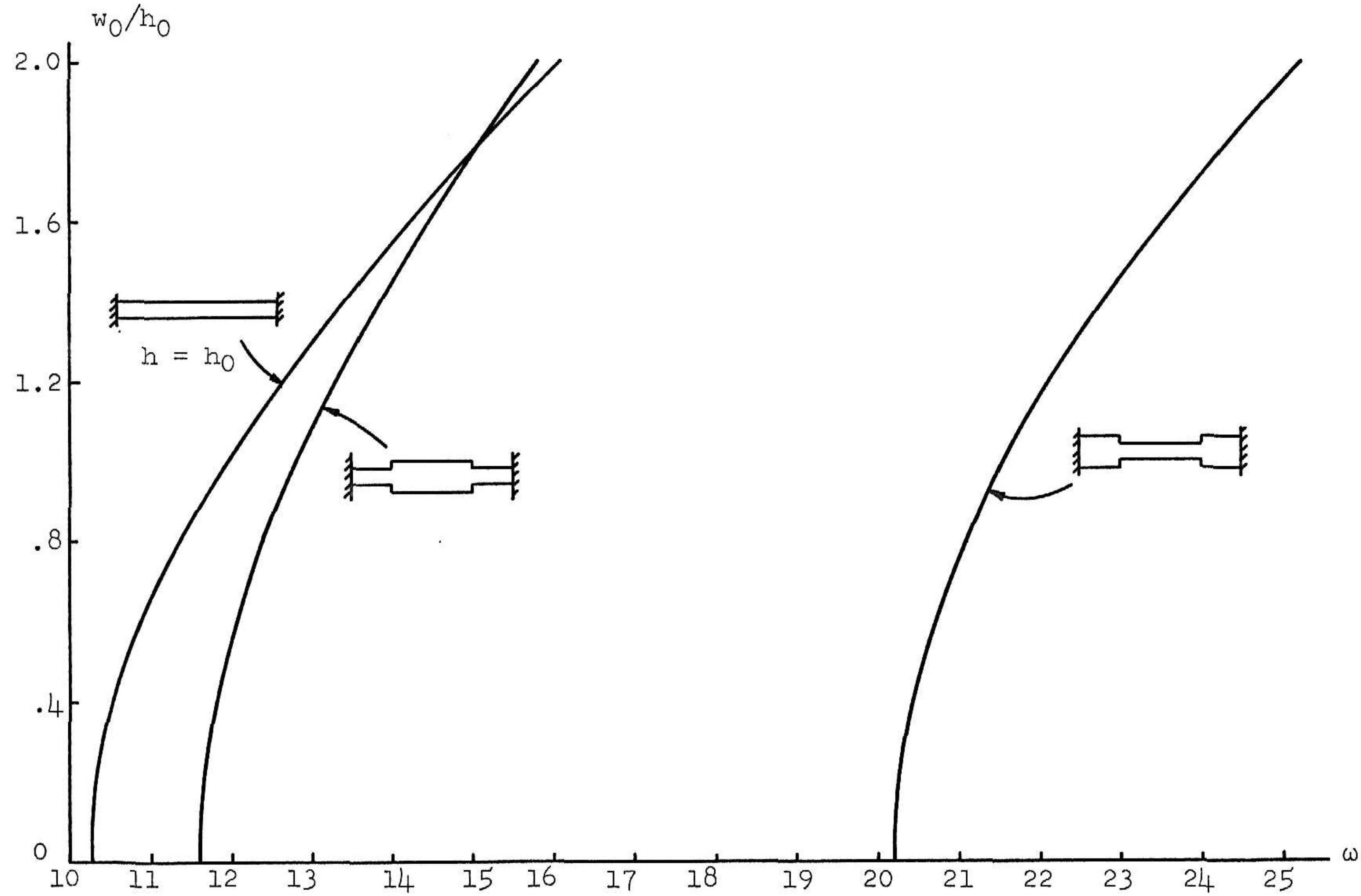


Fig. 12. Harmonic response of plates under free vibration.

## VI. CONCLUSIONS AND REMARKS

The study presented in this report is based on the critical supposition of harmonic oscillations. Although the assumptions (20) contradict the inseparability of modes in the governing partial differential equations, fundamental physical arguments may be used to justify the predicted results. By eliminating the time variable an infinite number of degrees of freedom in the space coordinate functions are maintained. The employment of numerical integration leads to a discrete approximation of the continuous system. The number of degrees of freedom and the accuracy of computation are governed by the choice of the integration step size. Thus by the principle of least constraint it is believed that the stress distributions and response curves for the plate obtained in this study closely approximate those that occur when the vibrations are nearly sinusoidal.

The two examples illustrated above show trends for two types of variable thickness plates under flexural vibrations; the first one is thicker and the second one is thinner in the middle part. From the numerical results, it is shown that the second type of plate is much stiffer for the given boundary conditions and causes higher frequencies than the first one. The bending stresses are lower for the second type of plate than the first kind, as given in Figs. 11A and 11B, especially at the point of discontinuity of thickness. As to the membrane stresses, the effect is contrary in general, as shown in Figs. 8A and 8B. The discontinuity of thickness leads to a sudden jump in stress curves which is caused by the assumption of the stress distributions. Practically, the stress curves shall be continuous. This kind of discontinuity also can be prevented if the continuous function of thickness is introduced. The part of continuous function of thickness will be of further interest in future research.

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# LARGE AMPLITUDE FREE VIBRATIONS OF CIRCULAR PLATE WITH STEPWISE THICKNESS

by

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AN ABSTRACT OF A MASTER'S REPORT

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The partial differential equations of large amplitude vibrations of an elastic circular plate with variable thickness are formulated. Steady-state sinusoidal oscillations are assumed. The time variable is eliminated by applying a Kantorovich averaging method. In this way the governing equations for the problem are reduced to a pair of ordinary differential equations which form a nonlinear eigenvalue problem. A simple example of a stepwise thickness isotropic plate under free vibration is solved by a numerical method. The effect of large amplitude upon the mode shapes corresponding to the fundamental natural frequency of vibration is studied and illustrated. The dynamic response curve is presented.