TESTING THE HOMOGENEITY OF THE TWO VARIANCES OF A NORMAL BIVARIATE POPULATION

by

YOLANDA T. JUICO

B.S. in Chemical Engineering, Mapua Institute of Technology, 1972

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1979

Approved by:

Dallas E. Johnson

Major Professor

To my parents, this work is lovingly dedicated.

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

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Suppose that $U_i = \begin{bmatrix} X_i \\ Y_i \end{bmatrix}$ have independent bivariate normal distributions with mean$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and non-singular covariance matrix $\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ for $i = 1,2,\ldots,n$. A test of the hypothesis that $\sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $\sigma_1^2 \neq \sigma_2^2$ is desired.

Three tests statistics were used in this study namely the likelihood ratio test, the F-test and modified F-test. Derivations of the likelihood ratio test and modified F-test were shown. A Monte Carlo Study was done to compare the power of the three test statistics. A procedure to generate a sample covariance matrix was introduced by P. L. Odell and H. H. Feivenson. The generation of sample covariance matrix was done for different combinations of $\rho = 0, .2, .4, .6, .8, .9, .95$, $\sigma_2^2 = 2, 4, 8$ and $n = 10, 20, 40$ while σ_1^2 is kept constant at 1. The technique requires the generation of only $p(p+1)/2$ random numbers while a straightforward one would require the generation of $Np - p$ random numbers, the greatest saving in computing time occurs when the sample size N is large.

The results of the study show that the likelihood ratio test has observed significance level close to the "theoretical" value with slight differences for whatever values of ρ and sample size when the null hypothesis is true. When the null hypothesis is false, the observed power increases with increasing values of ρ. As sample size is increased the observed power increases. The observed power is large when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. The F-test has observed significance level close to the "theoretical" value when $\rho = .2$ for all sample sizes. For greater values of ρ the F-test is inappropriate. When the null hypothesis is true the modified F-test shows that the observed significance level is close to the "theoretical" value as sample size is increased with slight differences among values of ρ. When the null hypothesis is false the observed power of the modified F-test increases with increasing values of ρ. The increased in power is large as sample size is increased. The observed power is larger when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. Among the three tests used the likelihood ratio test is the best. The F-test is good only when $\rho = 0$ but the test still holds when $\rho = .2$. The modified F-test is good at sample size 40 since the observed significance level is close to the "theoretical" value and observed power is large.

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CHAPTER 1

INTRODUCTION

Suppose that $U_i = \begin{bmatrix} X_i \\ Y_i \end{bmatrix}$ have independent bivariate normal distribution with mean $\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and non-singular covariance matrix $\begin{bmatrix} \sigma_1^2 & \rho\sigma\ \sigma \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ for $i = 1,2,\ldots,n$. A test of the hypothesis that $\sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $\sigma_1^2 \neq \sigma_2^2$ is desired.

Pitman (1939) and Morgan (1939), Biometrika, independently proposed the same test for $\sigma_1^2 = \sigma_2^2$. Three test procedures namely the likelihood ratio test, the F-test, and modified F-test are considered in this study. Chapter 2 gives the three test statistics. Derivations of the likelihood ratio test and modified F-test are shown. The F-test is appropriate only when the population correlation coefficient is zero, that is when the two variates are independent. A Monte Carlo Study comparing the three tests is given in Chapter 3. Tables 1, 2, 3 show the observed power for .05, .01 and .1 significance level of the three test statistics evaluated at varying values of population variances, population correlation coefficient and for sample sizes 10, 20 and 40. Bargraphs are plotted for each of the three test statistics at .05 significance level for each of the three sample sizes. Chapter 4 gives the summary and conclusion of the study. It shows that the likelihood ratio test is superior to the other two tests. The modified F-test has observed significance level close to the "theoretical" value and high power at sample size 40.

CHAPTER 2

DERIVATION OF THE LIKELIHOOD RATIO TEST

The likelihood ratio method of Neyman and Pearson has proved of service in a number of instances where the appropriate criterion was not immediately obvious. In general, a generalized likelihood ratio test will be a good test; although there are examples where the generalized likelihood ratio test makes a poor showing compared to other tests. One possible drawback of the test is that it is sometimes difficult to find the maximum of the likelihood function; another is that it can be difficult to find the distribution of the test statistic. This distribution is required in order to evaluate the power of the test. Summarized briefly, it involves the following steps.

Suppose (X_i, Y_i) $i = 1,2,\ldots,n$ are n independent observations from Bivariate Normal distribution.

$$\binom{X_i}{Y_i} \sim N\,(\mu,\ V)$$

where $\mu = \binom{\mu_1}{\mu_2}$ and $V = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ is a positive definite matrix.

Then $Z = \begin{bmatrix} X_1 \\ Y_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \\ Y_n \end{bmatrix} \sim N\,(\mu \otimes j_n,\ \Sigma)$ where $\Sigma = V \otimes I_n$

It is of interest to test the hypothesis

$$\text{Ho: } \sigma_1^2 = \sigma_2^2 = \sigma^2 \quad \text{vs.} \quad \text{Ho: } \sigma_1^2 \neq \sigma_2^2$$

The likelihood function of the parameters given the n pairs of observations is

$$L_1 \; (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho; (X_1, Y_1), \ldots, (X_n, Y_n))$$

$$= \frac{1}{(2\pi)^n |V|^{n/2}} \exp \left[- \tfrac{1}{2} \sum_{i=1}^{n} (Z_i - \mu)' V^{-1} (Z_i - \mu)\right] \qquad \ldots\ldots. (1)$$

Tha parameter space is $\Omega = \{ (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \;\; \rho); -\infty \leq \mu_1, \mu_2 \leq \infty,$

$0 \leq \sigma_1^2, \sigma_2^2 \leq \infty, -1 \leq \rho \leq 1\}$

Taking the log of expression (1) one gets

$$\log L_1 = -n \log (2\pi) - \frac{n}{2} \log |V| - \tfrac{1}{2} \sum_{i=1}^{n} (Z_i - \mu)' V^{-1} (Z_i - \mu) \qquad \ldots\ldots. (2a)$$

$$= -n \log (2\pi) - \frac{n}{2} \log \sigma_2^2 - \frac{n}{2} \log (1 - \rho^2) - \frac{n}{2} \log \sigma_1^2$$

$$- \tfrac{1}{2} \frac{\sum_{i=1}^{n} (X_i - \mu_1)^2}{\sigma_1^2 (1 - \rho^2)} + \rho \frac{\sum_{i=1}^{n} (X_i - \mu_1)(Y_i - \mu_2)}{\sigma_1 \sigma_2 (1 - \rho^2)}$$

$$- \tfrac{1}{2} \frac{\sum_{i=1}^{n} (Y_i - \mu_2)^2}{\sigma_2^2 (1 - \rho^2)} \qquad \ldots\ldots. (2b)$$

A determination of those values of the five unknown parameters as functions of the observations, which jointly maximize expression (1) are:

$$\frac{\partial \log L_1}{\partial \mu_1} = \frac{\sum_{i=1}^{n} X_i}{\sigma_1^2(1-\rho^2)} - \frac{n\mu_1}{\sigma_1^2(1-\rho^2)} - \rho\,\frac{\sum_{i=1}^{n} Y_i}{\sigma_1\sigma_2(1-\rho^2)}$$

$$+ \frac{\rho n\mu_2}{\sigma_1\sigma_2(1-\rho^2)} = 0 \qquad \ldots\ldots (3)$$

$$\frac{\partial \log L_2}{\partial \mu_2} = \frac{\sum_{i=1}^{n} Y_i}{\sigma_2^2(1-\rho^2)} - \frac{n\mu_2}{\sigma_2^2(1-\rho^2)} - \rho\,\frac{\sum_{i=1}^{n} X_i}{\sigma_1\sigma_2(1-\rho^2)}$$

$$+ \frac{\rho n\mu_1}{\sigma_1\sigma_2(1-\rho^2)} = 0 \qquad \ldots\ldots (4)$$

Solving equations (3) and (4) simultaneously, we get maximum likelihood estimators of μ_1 and μ_2.

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X} \qquad \hat{\mu}_2 = \frac{\sum_{i=1}^{n} Y_i}{n} = \bar{Y} \qquad \ldots\ldots (5)$$

Taking derivatives of $\log L_1$ with respect to the other parameters and replacing μ_1 and μ_2 by $\bar{X}$ and $\bar{Y}$ respectively gives

$$\frac{\partial \log L_1}{\partial \rho} = \frac{n\rho}{(1-\rho^2)} - \rho\,\frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_1^2(1-\rho^2)^2} + \frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sigma_1\sigma_2(1-\rho^2)}$$

$$+\,2\,\rho^2\,\frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sigma_1\sigma_2(1-\rho^2)^2} - \rho\,\frac{\sum\limits_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma_2^2(1-\rho^2)^2} = 0 \qquad \ldots\ldots(6)$$

$$\frac{\partial \log L_1}{\partial \sigma_1^2} = -\frac{n}{2\,\sigma_1^2} + \frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})^2}{2\,\sigma_1^4(1-\rho^2)} - \rho\,\frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{2\,\sigma_1^3\sigma_2(1-\rho^2)} = 0 \qquad \ldots\ldots(7)$$

$$\frac{\partial \log L_1}{\partial \sigma_2^{2=}} = -\frac{n}{2\,\sigma_2^2} + \frac{\sum\limits_{i=1}^{n}(Y_i-\bar{Y})}{2\,\sigma_2^4(1-\rho^2)} - \rho\,\frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{2\,\sigma_1\sigma_2^3(1-\rho^2)} = 0 \qquad \ldots\ldots(8)$$

Solving equations (6), (7) and (8) simultaneously, we get maximum likelihood estimators of σ_1^2, σ_2^2 and ρ.

$$\hat{\sigma}_1^2 = \frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})^2}{n}, \quad \hat{\sigma}_2^2 = \frac{\sum\limits_{i=1}^{n}(Y_i-\bar{Y})^2}{n}, \quad \hat{\rho} = \frac{\sum\limits_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum\limits_{i=1}^{n}(X_i\bar{X})^2\sum\limits_{i=1}^{n}(Y_i-\bar{Y})^2}} \qquad \ldots\ldots(9)$$

The maximum value of the probability function defined in (1) then becomes

$$L_1 (\hat{\Omega}) = \{ 2 e\pi\hat{\sigma}_1 \hat{\sigma}_2 \sqrt{(1 - \hat{\rho}^2)} \}^{-n} \qquad \ldots\ldots(10)$$

The likelihood function of n pairs of observations under the null hypothesis is obtained by putting $\sigma_1^2 = \sigma_2^2$ expression (1).

$$L_2(\mu_1, \mu_2, \sigma^2, \rho: (X_1, Y_1),\ldots, (X_n, Y_n))$$

$$= \frac{1}{(2\pi)^n(\sigma^4(1-\rho^2))^{n/2}}$$

$$\times \exp [- \frac{1}{2 \sigma^4(1-\rho^2)} \{ \sigma^2 \sum_{i=1}^{n} (X_i-\mu_1)^2 - 2\rho\sigma^2 \sum_{i=1}^{n} (X_i-\mu_1)(Y_i-\mu_2)$$

$$+ \sigma^2 \sum_{i=1}^{n} (Y_i-\mu_2)^2\}] . \qquad \ldots\ldots(11)$$

The parameter space under Ho is $\Omega o = \{ \mu_1, \mu_2, \sigma^2,\rho \}; - \infty \leq \mu_1, \mu_2 \leq \infty$, $o \leq \sigma^2 \leq \infty, -1 \leq \rho \leq 1 \}$.

Taking log of expression (11) one gets

$$\log L_2 = -n \log (2\pi) - \frac{n}{2} \log \sigma^4 - \frac{n}{2} \log (1-\rho^2)$$

$$- \frac{\sum_{i=1}^{n}(X_i-\mu_1)^2}{2\,\sigma^2(1-\rho^2)} + \rho\,\frac{\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sigma^2(1-\rho^2)} - \frac{\sum_{i=1}^{n}(Y_i-\mu_2)^2}{2\sigma^2(1-\rho^2)} \qquad \ldots\ldots(12)$$

The value of μ which maximizes (11) is, as before

$$\tilde{\mu} = \begin{bmatrix} \dfrac{\sum_{i=1}^{n} X_i}{n} \\ \dfrac{\sum_{i=1}^{n} Y_i}{n} \end{bmatrix} = \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}. \qquad \ldots\ldots(13)$$

Taking derivatives of $\log L_2$ with respect to the other parameters and replacing μ_1 and μ_2 by $\bar{X}$ and $\bar{Y}$ respectively gives

$$\frac{\partial \log L_2}{\partial \rho} = \frac{n\rho}{(1-\rho^2)} - \rho\,\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2(1-\rho^2)^2} + \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sigma^2(1-\rho^2)}$$

$$+ 2\,\rho^2\,\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sigma^2(1-\rho^2)^2} - \rho\,\frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma^2(1-\rho^2)^2} = 0 \text{ and} \qquad \ldots\ldots(14)$$

$$\frac{\partial \log L_2}{\partial \sigma^2} = \frac{-n}{\sigma^2} + \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{2\,\sigma^4(1-\rho^2)} - \rho\,\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sigma^4(1-\rho^2)} + \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{2\,\sigma^4(1-\rho^2)} = 0 \qquad \ldots\ldots(15)$$

Solving equations (14) and (15) simultaneously, we get the values of likelihood estimators of σ^2 and ρ which maximize (11) are

$$\tilde{\sigma}^2 = \left(\frac{\sum_{i=1}^{n} (X_i-\bar{X})^2}{n} + \frac{\sum_{i=1}^{n} (Y_i-\bar{Y})^2}{n}\right)/2 \text{ and } \tilde{\rho} = \frac{\sum_{i=1}^{n} (X_i-\bar{X}(Y_i-\bar{Y})}{\dfrac{\sum_{i=1}^{n} (X_i-\bar{X})^2 + \sum_{i=1}^{n} (Y_i-\bar{Y})^2}{2}} \qquad \ldots.(16)$$

It is interesting to note that $\tilde{\sigma}^2 = \frac{1}{2} (\hat{\sigma}_1^2 + \hat{\sigma}_2^2)$ and that $\tilde{\rho} = \dfrac{\hat{\rho}\ \hat{\sigma}_1\hat{\sigma}_2}{(\hat{\sigma}_1^2+\hat{\sigma}_2^2)/2}$.

The maximum value of the probability function defined in expression (11) then becomes

$$L_2\ (\hat{\omega}) = \{ e\pi \sqrt{(\hat{\sigma}_1^2 + \hat{\sigma}_2^2)^2 - 4\ \hat{\rho}^2\ \hat{\sigma}_1^2\ \hat{\sigma}_2^2}\}^{-n} \qquad \ldots\ldots(17)$$

The likelihood ratio criterion is then

$$\lambda = \frac{L_2\ (\hat{\omega})}{L_1\ (\hat{\Omega})} = \left\{ \frac{4\ \hat{\sigma}_1^2\ \hat{\sigma}_2^2(1-\hat{\rho}^2)}{(\hat{\sigma}_1^2 + {}_2^2) - 4\ \hat{\rho}^2\ \hat{\sigma}_1^2\ \hat{\sigma}_2^2} \right\}^{\frac{n}{2}} \qquad \ldots\ldots.(18a)$$

$$= \left\{ 1 - \frac{(\hat{\sigma}_1^2-\hat{\sigma}_2^2)^2}{(\hat{\sigma}_1^2+\hat{\sigma}_2^2)^2 - 4\ \hat{\rho}^2\ \hat{\sigma}_1^2\ \hat{\sigma}_2^2} \right\}^{\frac{n}{2}} \qquad \ldots\ldots.(18b)$$

To complete the test it is necessary to know the sampling distribution of λ, or of a single valued function of λ when the hypothesis being tested is true.

Pitman derived the same test by using a different argument. It holds for any value ρ of the population correlation between X and Y. However, if ρ is zero, the ordinary F-test should be used, since it is slightly more powerful. Pitman's derivation is described as follows.

If $\binom{X}{Y} \sim N\left(\binom{\mu_1}{\mu_2}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right)$,

Then let

$$D = X - Y \qquad \text{......(19)}$$

$$S = X + Y \qquad \text{......(20)}$$

$$\sigma_D^2 = \sigma_1^2 + \sigma_2^2 - 2\,\rho\sigma_1\sigma_2\,, \qquad \text{......(21)}$$

$$\sigma_S^2 = \sigma_1^2 + \sigma_2^2 + 2\,\rho\sigma_1\sigma_2\,, \text{ and} \qquad \text{......(22)}$$

$$\mathrm{Cov}(D, S) = \mathrm{Cov}\,[(X-Y), (X+Y)] \qquad \text{......(23a)}$$

$$= \sigma_1^2 - \sigma_2^2 \quad \text{thus} \qquad \text{......(23b)}$$

$$\rho_{DS} = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2+\sigma_2^2)^2 - 4\rho^2\sigma_1^2\sigma_2^2}}\,. \qquad \text{......(24)}$$

The necessary and sufficient condition that the hypothesis tested is true, or that $\sigma_1^2 = \sigma_2^2$ is that

$$\rho_{DS} = 0 \qquad \text{......(25)}$$

Since X and Y are normally distributed variables, the appropriate criterion to test the hypothesis, $\rho_{DS} = 0$, is the sample correlation coefficient between the transformed variables, i.e. r_{DS}. If the hypothesis is true, it is well-known that

$$t = \frac{\sqrt{n-2}\, r_{DS}}{\sqrt{1 - r_{DS}^2}}$$

has the t-distribution with n-2 degrees of freedom.

From equations (19) and (20), the sample correlation coefficient between the variables D and S is given by

$$r_{DS} = \frac{\sum_{i=1}^{n} (D_i-\bar{D})(S_i-\bar{S})}{\sqrt{\sum_{i=1}^{n} (D_i-\bar{D})^2 \sum_{i=1}^{n} (S_i-\bar{S})^2}} \qquad .....(26)$$

Substituting $D_i = X_i-Y_i$, $\bar{D} = \sum_{i=1}^{n} \frac{D_i}{n}$, $S_i = X_i+Y_i$, and $\bar{S} = \sum_{i=1}^{n} \frac{S_i}{n}$ in equation (26) one gets

$$r_{DS} = \frac{\hat{\sigma}_1^2 - \hat{\sigma}_2^2}{\{(\hat{\sigma}_1^2+\hat{\sigma}_2^2)^2 - 4\, \hat{\rho}^2\hat{\sigma}_1^2\hat{\sigma}_2^2\}^{\frac{1}{2}}} \qquad .....(27)$$

Hence, the likelihood criterion of expression (18b) is

$$\lambda = (1 - r_{DS}^2)^{\frac{n}{2}} \qquad .....(28)$$

and as the hypothesis tested becomes less and less likely, $\lambda \to 0$ or $r^2_{DS} \to 1$. The test may therefore be carried out by referring

$$t = \frac{r_{DS}\sqrt{n-2}}{\sqrt{1 - r^2_{DS}}} \qquad \text{.......(29)}$$

to "students'" distribution with degrees of freedom $\nu = n-2$ and rejecting the hypothesis at 100α% significance level if $|t| > t_{\alpha/2}\ \nu$.

F-TEST

Although the usual F-test for testing $\sigma_1^2 = \sigma_2^2$ is not appropriate for the problem being considered here, it is of interest to examine how well it performs in comparison to other tests. The usual F-test statistic can be given as $F = S^2_{max}/S^2_{min}$ where S^2_{max} is the larger of the two variances and S^2_{min} is the smaller. It is known that if the population correlation coefficient, $\rho = 0$; i.e., if two variates are independent then the F-test is the uniformly most powerful unbiased test of Ho.

The hypothesis $\sigma_1^2 = \sigma_2^2$ is rejected at the $100\alpha\%$ significance level if $F > F_{\alpha/2}, \nu_{max}, \nu_{min}$ where ν_{max} is the degrees of freedom of S^2_{max} and ν_{min} is the degrees of freedom of S^2_{min}.

A MODIFIED F-TEST

The Bartlett test is used for testing homogeneity of variance of several populations. If we have p populations of equal sample size n Bartlett's statistics is then

$$M_1 = -(n-1)p \log p + (n-1)p \log \left(\sum_{i=1}^{p} S_i^2\right)$$

$$- (n-1) \sum_{i=1}^{p} \log S_i^2 \qquad \ldots\ldots(1)$$

For our study we consider only 2 populations of equal sample size. From equation (1) our test statistics becomes

$$M_1 = -(n-1)\ 2 \log 2 + (n-1)\ 2 \log \left(\sum_{i=1}^{2} S_i^2\right)$$

$$- (n-1) \sum_{i=1}^{2} \log S_i^2 \qquad \ldots\ldots(2)$$

We can replace (n-1) by n when the sample size n is large. Using the Taylor series, we expand M_1 around σ_i^2. The terms with power greater than 2 are omitted for large sample size since the log S_i^2 are consistent estimators for log σ_i^2. When the null hypothesis is true, that is $\sigma_1^2 = \sigma_2^2 = \sigma^2$, equation (2) can be written as

$$M_1 = \frac{n}{4} [(\log S_1^2 - \log \sigma^2)^2 + (\log S_2^2 - \log \sigma^2)^2$$

$$- 2 (\log S_1^2 - \log \sigma^2)(\log S_2^2 - \log \sigma^2)] \qquad \ldots\ldots(3)$$

To complete the test we must know the sampling distribution of M_1 or a single valued function of M_1 when the hypothesis being tested is true.

If we let $U_i = \log S_i^2$ and ......(4)

$$\bar{U} = \frac{\sum_{i=1}^{2} \log S_i^2}{2} \qquad \text{......(5)}$$

Equation (3) can be written as

$$M_1 = \frac{n}{2} \sum_{i=1}^{2} (U_i - \bar{U})^2 \qquad \text{......(6)}$$

Using the following lemma (Rao, 1965, p. 158) the large sample distribution of equation (6) can be achieved.

Lemma. If U_i, $i = 1,2,\ldots,p$ are normally distributed and have common variance V, and every pair has covariance C, then $\Sigma(U_i - \bar{U})^2$ has a (V-C) χ^2 distribution with p-1 degrees of freedom.

The distribution of U_i is asymptotically multivariate normal with

$$E(U_i) = \log \sigma_i^2,\ \text{Var}\ (U_i) = \frac{2}{n} \text{ and Cov } (U_i, U_j) = \frac{2\rho^2}{n}$$

Therefore by the lemma, $\sum_{i=1}^{2} (U_i - \bar{U})^2$ has $2(1 - \rho^2)n^{-1} \chi^2$ (1) distribution which implies that $M_1/(1 - \rho^2)$ is χ^2 (1) asymptotically. We call $M_1/(1 - \rho^2)$ the modified F-test.

Equation (3) can be written as

$$\frac{M_1}{(1-\rho^2)} = \frac{n}{4(1-\rho^2)} \quad (\log S_1^2 - \log S_2^2)^2 \qquad \ldots\ldots(7)$$

The above equation has a χ^2 distribution with one degree of freedom. In practice ρ^2 is usually unknown and is replaced by r^2 where r is the sample correlation coefficient.

Taking the square root of both sides of equation (7) and replacing ρ^2 by r^2 we have

$$\left[\frac{M_1}{(1-r^2)}\right]^{\frac{1}{2}} = \frac{\log S_1^2 - \log S_2^2}{\sqrt{\frac{4(1-r^2)}{n}}} \qquad \ldots\ldots(8)$$

where $S_1^2 = S_{max}^2$ and $S_2^2 = S_{min}^2$ and substituting in the above equation we have

$$\left[\frac{M_1}{(1-r^2)}\right]^{\frac{1}{2}} = \frac{\log S_{max}^2 - \log S_{min}^2}{\sqrt{\frac{4(1-r^2)}{n}}} \qquad \ldots\ldots(9)$$

The above statistic is approximately distributed with the standard normal distribution. We reject the hypothesis when the statistic in (9) is greater than $Z_{\alpha/2}$.

CHAPTER 3

MONTE CARLO STUDY

A procedure to generate a sample covariance matrix was introduced by P. L. Odell and H. H. Feivenson [Odell, P. L. and Feivenson, A. H. (1966). A Numerical Procedure to Generate a Sample Covariance Matrix. Amer. Statist. Assoc. 61, pp. 196-203].

The method can be summarized by the following:

Algorithm: Given a p x n covariance matrix R in a factored form $R = CC^T$, a sample of size N, a sequence of independent standardized normal random variates $\{ N_{ij};\ i = 1,2,\ldots,p,\ j = 1,2,\ldots,p,\ i < j \}$ and a sequence of independent Chi-square variates $\{ V_j,\ j = 1,2,\ldots,p \}$ where for each j, V_j has N-j degrees of freedom; then a sample covariance matrix is given by

$$S^* = A^*/N \quad \text{where} \quad A^* = CB^*C^T \quad \text{and}$$

the elements b^*_{ij} of the p x p symmetric matrix B* can be generated by computing

$$b^*_{ii} = V_1$$

$$b^*_{jj} = V_j + \sum_{i=1}^{j-1} N^2_{ij} \qquad j = 2,3,\ldots,p$$

$$b^*_{ij} = N_{ij}\sqrt{V_1}$$

$$b^*_{ij} = N_{ij}\sqrt{V_i} + \sum_{k=1}^{i-1} N_{ki} N_{kj} \quad i < j = 2,3,\ldots,p$$

One method [Odell, P. L. and Feivenson, A. H. (1966). A Numerical Procedure to Generate a Sample Covariance Matrix. Amer. Statist. Assoc. 61, pp. 196-203] of obtaining V_j is to generate a standard normal variate N_j and substitute it into the equation below which is called the Wilson-Hifferty Chi-square approximation. The approximation can be written

$$V_j = [N-j]\ [1-2/9\ (N-j) + N_j\ (2/9\ \{N-j\})^{\frac{1}{2}}]^3$$

Since this technique requires the generation of only p(p+1)/2 random numbers while a straightforward one would require the generation of Np-p random numbers, the greatest saving in computing time occurs when the sample size N is large. The remaining arithmetic, if the Wilson-Hilferty approximation is used, to highest order in p, amounts to about p^2 additions, and multiplications, and p square roots, which is again less work than the remaining arithmetic for the direct method mentioned above.

Using the above procedure, a sample covariance matrix was generated and three tests statistics were computed. First 5000 sets of samples of size 10 were generated; all the three tests statistics namely the likelihood ratio statistics, F-statistics, and modified F-statistics were computed for each set. The procedure was repeated for 2500 and 1250 sets of samples of size 20 and 40 respectively. The rejection rate of each of the three tests statistics was computed for .05, .01 and .1 significance level. The above process was carried out for various combinations of σ_2^2 = 1, 4, 8 and ρ = 0, .2, .4, .6, .8, .9, .95 while

σ_1^2 was kept constant at 1. The results are shown in Tables 1, 2, and 3.

Table 1 whows the rejection rate for test of significance level, $\alpha = .05$ when $\sigma_1^2 = 1$. Column (1) shows the sample size used for the different tests for the varying values of ρ shown in column (2). Columns (3), (4) and (5) show the rejection rate of the three tests statistics when $\sigma_2^2 = 1$. From column (3) the observed significance level of the likelihood ratio test differs slightly from the .05 "theoretical" value with slight differences for different values of ρ and sample size. It can be seen in column (4) that the observed significance level of the F-test is close to the "theoretical" value when $\rho = 0$ and $\rho = .2$ for all sample sizes. For greater values of ρ the observed significance level decreases to zero with increasing values of ρ for all sample sizes. The observed significance level of the modified F-test in column (5) is twice as large as the "theoretical" value for $n = 10$ with slight differences among values of ρ. As the sample size is increased the observed significance level approaches the "theoretical" value. Columns (6), (7) and (8) show the rejection rate of the three tests statistics when $\sigma_2^2 = 4$. The observed power of the likelihood ratio test in column (6) increases with increasing values of ρ for all sample sizes. The observed power increases with increasing sample size. When $n = 40$ the observed power increases to almost one. The observed power of the F-test in column (7) does not vary much for different values of ρ when $n = 10$. As the sample size is increased the observed power increases with increasing values of ρ and the power approaches to one when $n = 40$. In column (8) the modified F-test is seen to behave similarly as the likelihood ratio test although the observed power of the modified F-test is larger for each values of ρ than the F-test when n = 10 and 20. Columns (9),(10), and (11) show the rejection rate of the three tests statistics when $\sigma_2^2 = 8$. The observed power of the three tests increases with increasing values of ρ and increasing sample size. The three tests increases with increasing values of ρ and increasing sample size. The three tests have the same observed power of almost one when n = 40 for all values of ρ. The observed power is larger for all of the three tests when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$.

Table 2 shows the rejection rate when $\alpha = .01$ and $\sigma_1^2 = 1$. Columns (3), (4), and (5) show the rejection rate of the three tests statistics. The likelihood ratio test in column (3) shows that the observed significance level is close to the "theoretical" .01 value with slight differences for varying values of ρ and sample size. In column (4) the observed significance level of the F-test is close to the "theoretical" value when $\rho = 0$ and $\rho = .2$ for all sample sizes. For greater values of ρ the observed significance level decreases to zero with increasing values of ρ for all sample sizes. The modified F-test in column (5) shows that the observed significance level increases up to five times the "theoretical" value with increasing values of ρ when n = 10. However, when n = 20 and 40 the observed significance level is closer to the "theoretical" value with slight differences for different values of ρ. Columns (6), (7), and (8) show the rejection rate of the three tests statistics when $\sigma_2^2 = 4$. The likelihood ratio test in column (6) shows that the observed power increases with increasing values of ρ for all sample sizes. The increase in power is large among different values of ρ when n = 10. The observed power increases with increasing sample size and when $n = 40$ the observed power increases to one. The F-test in column (7) shows that the observed power decreases with increasing values of ρ when $n = 10$. For $n = 20$ and $n = 40$ the observed power increases with increasing values of ρ. The observed power increases to one when $n = 40$. The observed power of the modified F-test in column (8) increases with increasing values of ρ and sample size. When $n = 40$ the observed power increases to one. Columns (9), (10), and (11) show the rejection rate when $\sigma_2^2 = 8$. The observed power of the three tests increases with increasing values of ρ for all sample sizes. As the sample size is increased the observed power is also increased. When $n = 40$ the observed power of the three tests is almost one for all values of ρ. The observed power is larger for all of the three tests when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$.

Table 3 shows the rejection rate of the three tests when $\alpha = .10$ and $\sigma_1^2 = 1$. Columns (3), (4), and (5) show the rejection rate of the three tests statistics when $\sigma_2^2 = 1$. The likelihood ratio test in column (3) shows that the observed significance level is close to the "theoretical" .10 value for all sample sizes and for all values of ρ. The F-test in column (4) shows that the observed significance level decreases to zero with increasing values of ρ for all sample sizes. When $\rho = 0$ and $\rho = .2$ the observed significance level is close to the "theoretical" value for all three sample sizes. The modified F-test in column (5) shows that the observed significance level is closer to the "theoretical" value for larger sample sizes for all values of ρ. Columns (6), (7), and (8) show the rejection rate of the three tests statistics when $\sigma_2^2 = 4$. The three tests show that the observed power increases with increasing values of ρ and sample size. When n = 40 the observed power is very close to one. Columns (9), (10), and (11) show the rejection rate of the three tests statistics when $\sigma_2^2 = 8$. The observed power of the three tests increases with increasing values of ρ and sample size. When n = 40 all the three tests have observed power of one for all values of ρ. The observed power is larger for all of the three tests when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$.

TABLE 1

Rejection Rate for $\alpha = .05$ $\sigma_1^2 = 1$

		$\sigma_2^2 = 1$			$\sigma_2^2 = 4$			$\sigma_2^2 = 8$		
		Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
n=10	ρ									
	0	.05400	.05600	.10000	.48960	.49500	.62520	.82520	.83240	.89960
	.2	.05200	.04860	.10080	.49980	.49700	.64140	.84280	.84760	.91380
	.4	.04840	.03500	.10040	.53880	.48860	.67760	.88040	.86600	.93680
	.6	.04920	.01960	.10700	.63740	.49040	.76740	.92780	.88800	.96180
	.8	.04920	.00420	.11180	.83800	.49180	.91600	.98540	.94100	.99140
	.9	.04820	.00080	.11260	.95200	.48580	.97920	.99820	.98260	.99940
	.95	.05280	.00020	.11880	.99380	.48920	.99720	1.0	1.0	.99940
n=20										
	0	.05480	.05400	.07800	.82700	.82800	.86840	.99120	.99120	.99480
	.2	.05000	.04640	.07000	.86600	.86120	.89560	.99320	.99200	.99640
	.4	.05280	.03720	.07280	.88160	.85400	.90760	.99400	.99440	.99640
	.6	.04800	.01560	.07160	.94800	.89280	.96520	1.0	.99800	1.0
	.8	.05000	.00120	.07480	.99560	.94840	.99760	1.0	1.0	1.0
	.9	.05240	0.0	.08080	.99880	.98320	.99960	1.0	1.0	1.0
	.95	.04280	0.0	.06920	1.0	.99720	1.0	1.0	1.0	1.0
n=40										
	0	.05120	.05040	.05840	.98480	.98640	.98800	.99920	.99920	.99920
	.2	.06240	.05440	.07120	.98960	.98800	.99200	1.0	1.0	1.0
	.4	.03760	.02400	.04400	.99520	.99120	.99600	1.0	1.0	1.0
	.6	.05280	.01760	.07040	.99920	.99920	1.0	1.0	1.0	1.0
	.8	.04640	.00240	.05920	1.0	1.0	1.0	1.0	1.0	1.0
	.9	.05040	0.0	.06080	1.0	1.0	1.0	1.0	1.0	1.0
	.95	.05120	0.0	.06000	1.0	1.0	1.0	1 0	1.0	1.0

TABLE 2

Rejection Rate For $\alpha = .01$ $\sigma_1^2 = 1$

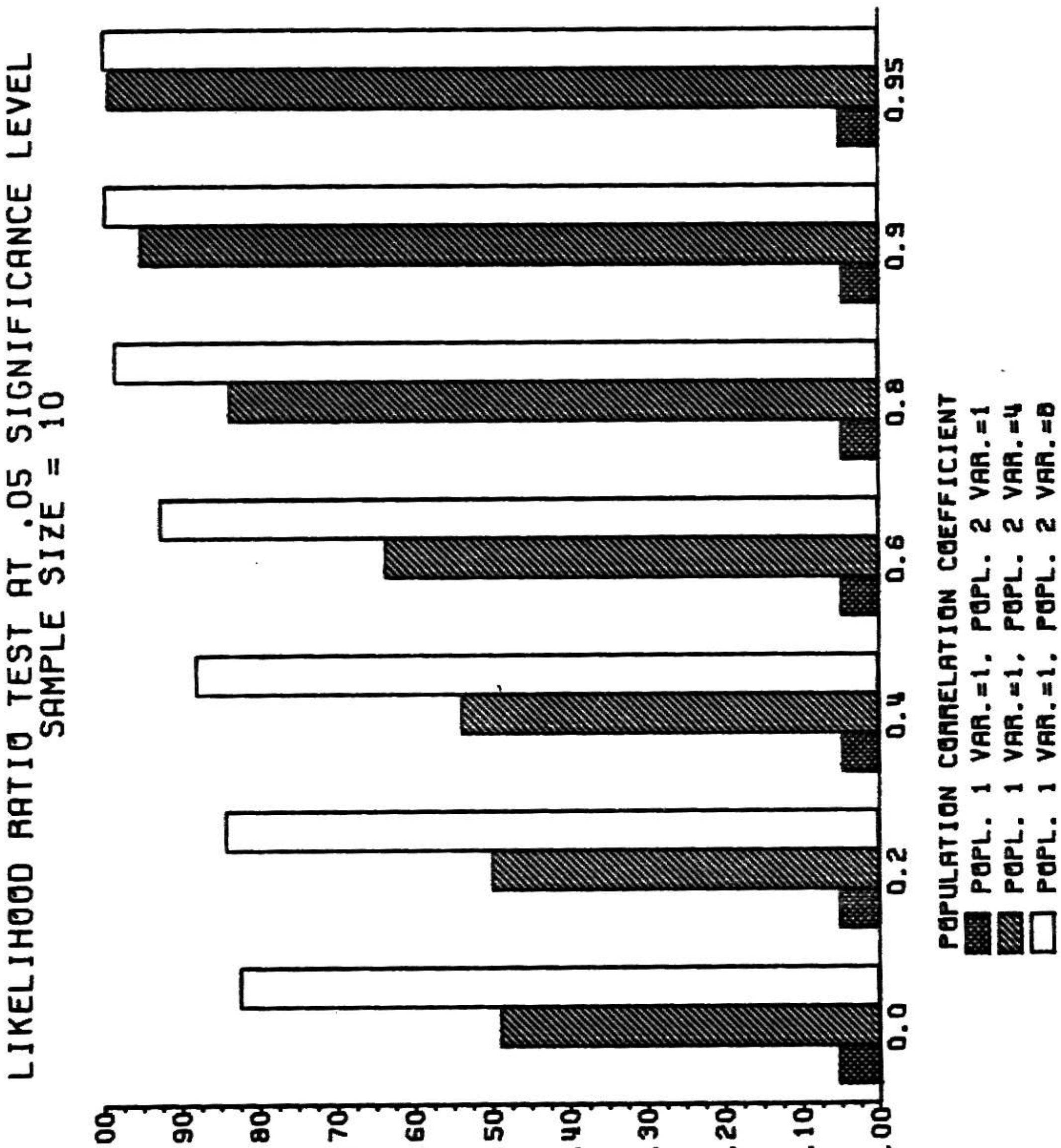
		$\sigma_2^2 = 1$			$\sigma_2^2 = 4$			$\sigma_2^2 = 8$		
		Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
n=10	ρ									
	0	.01140	.01220	.03760	.22000	.23260	.41520	.59680	.62020	.78160
	.2	.01200	.01140	.03840	.23500	.23600	.43620	.61120	.61400	.79740
	.4	.01240	.00680	.03700	.26860	.21520	.47460	.66500	.62740	.84740
	.6	.01060	.00380	.04020	.35860	.18100	.58960	.76760	.64380	.90780
	.8	.00940	.00040	.04480	.60940	.12580	.81820	.92700	.68120	.98200
	.9	.00940	.00020	.04640	.83700	.05780	.94520	.98960	.74860	.99800
	.95	.01080	0.0	.05260	.96620	.02040	.99320	1.0	1.0	.97940
n=20										
	0	.01120	.01200	.02320	.61080	.61440	.70640	.96400	.96520	.97840
	.2	.01000	.00880	.01800	.65080	.63240	.74440	.96280	.96240	.98080
	.4	.01400	.00720	.02440	.70360	.63080	.79360	.97840	.97280	.98720
	.6	.00840	.00080	.01880	.83040	.66240	.89120	.99280	.98240	.99640
	.8	.01120	0.0	.02600	.96840	.70960	.98520	.99960	.99720	1.0
	.9	.01240	0.0	.02760	.99640	.76200	.99800	1.0	1.0	1.0
	.95	.00640	0.0	.02440	1.0	.85200	1.0	1.0	1.0	1.0
n=40										
	0	.01200	.01120	.01280	.94160	.94160	.95760	.99840	.99840	.99920
	.2	.01120	.00800	.02080	.95840	.95520	.96960	1.0	1.0	1.0
	.4	.00880	.00480	.01040	.97600	.96320	.98240	1.0	1.0	1.0
	.6	.00720	.00080	.01360	.99600	.97840	.99920	1.0	1.0	1.0
	.8	.00720	0.0	.01680	1.0	.99600	1.0	1.0	1.0	1.0
	.9	.00960	0.0	.01520	1.0	.99920	1.0	1.0	1.0	1.0
	.95	.00800	0.0	.01760	1.0	1.0	1.0	1.0	1.0	1.0

TABLE 3

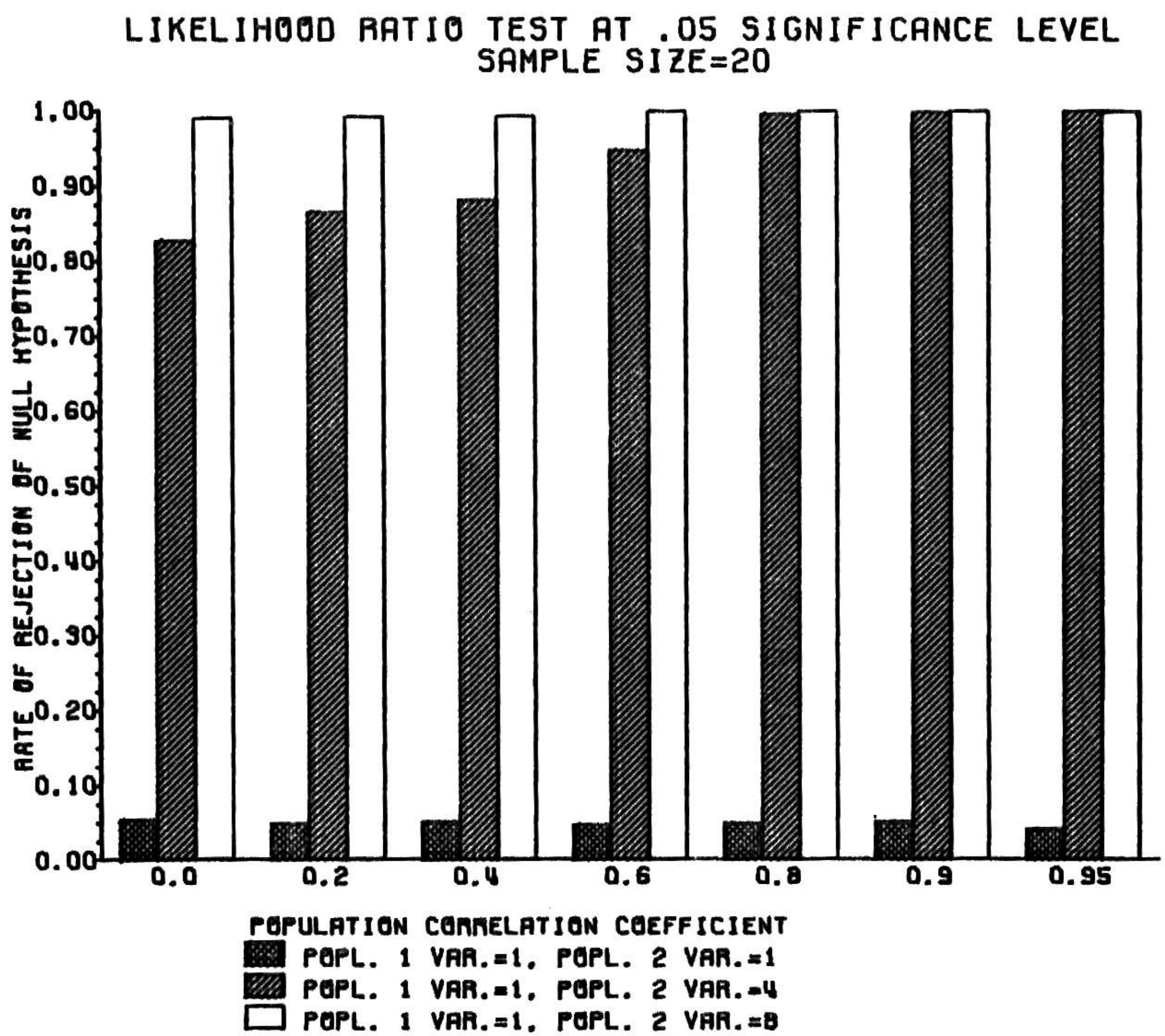
Rejection Rate For $\alpha = .10$ $\sigma_1^2 = 1$

		$\sigma_2^2 = 1$			$\sigma_2^2 = 4$			$\sigma_2^2 = 8$		
		Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test	Likelihood Ratio Test	F-Test	M-F-Test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
n=10	ρ									
	0	.0996	.09740	.16380	.62440	.63340	.71700	.90000	.90360	.93640
	.2	.09920	.09400	.16500	.64040	.63800	.73580	.91280	.91160	.94760
	.4	.09640	.07460	.16480	.67280	.63360	.76780	.93600	.92780	.96120
	.6	.10000	.04380	.16540	.75880	.65680	.83820	.96120	.94420	.98020
	.8	.10040	.01240	.17000	.91000	.70760	.94940	.99120	.97840	.99620
	.9	.10000	.00200	.17120	.97640	.77480	.98820	.99900	.99620	.99960
	.95	.10400	.00040	.17780	.99700	.84920	.99880	0.0	1.0	.99960
n=20										
	0	.10560	.10040	.12840	.90120	.90080	.92480	.99600	.99640	.99680
	.2	.09800	.08880	.12600	.92200	.91680	.93880	.99720	.99720	.99840
	.4	.09840	.07120	.12360	.93080	.91680	.94600	.99720	.99720	.99840
	.6	.09520	.04240	.12720	.97560	.94720	.98280	1.0	.99960	1.0
	.8	.09680	.00800	.12200	.99880	.98320	.99960	1.0	1.0	1.0
	.9	.10280	0.0	.13840	.99960	.99600	.99960	1.0	1.0	1.0
	.95	.09200	0.0	.12760	1.0	.99960	1.0	1.0	1.0	1.0
n=40										
	0	.09440	.09280	.10400	.99680	.99600	.99680	1.0	1.0	1.0
	.2	.11040	.10080	.11920	.99680	.99600	.99680	1.0	1.0	1.0
	.4	.08640	.05600	.10640	.99840	.99520	.99840	1.0	1.0	1.0
	.6	.10880	.04240	.12240	1.0	1.0	1.0	1.0	1.0	1.0
	.8	.10560	.00480	.11440	1.0	1.0	1.0	1.0	1.0	1.0
	.9	.08640	0.0	.10400	1.0	1.0	1.0	1.0	1.0=	1.0
	.95	.09280	0.0	.10320	1.0	1.0	1.0	1.0	1.0	1.0

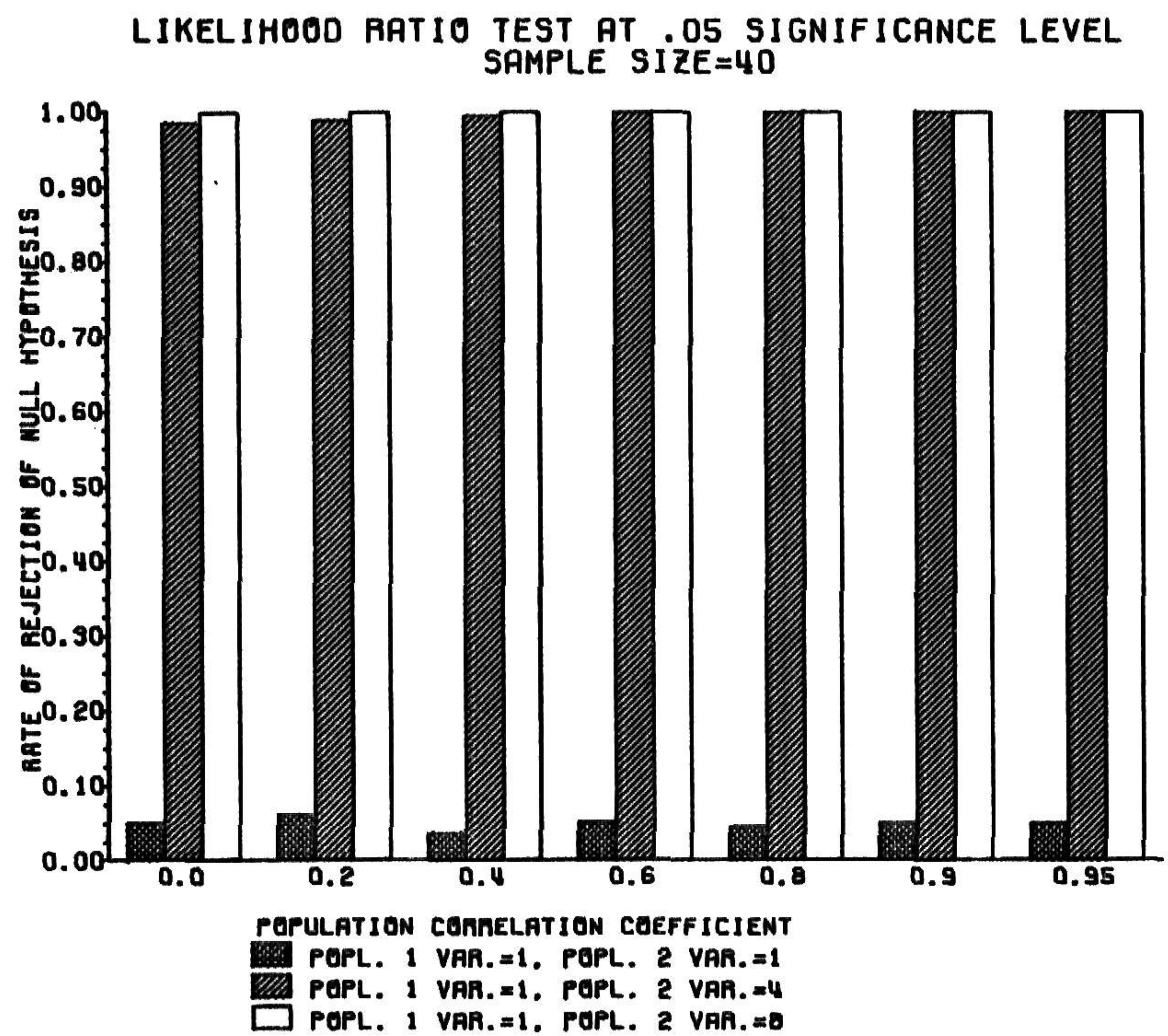
BARGRAPH 1


LIKELIHOOD RATIO TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE = 10
RATE OF REJECTION OF NULL HYPOTHESIS
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

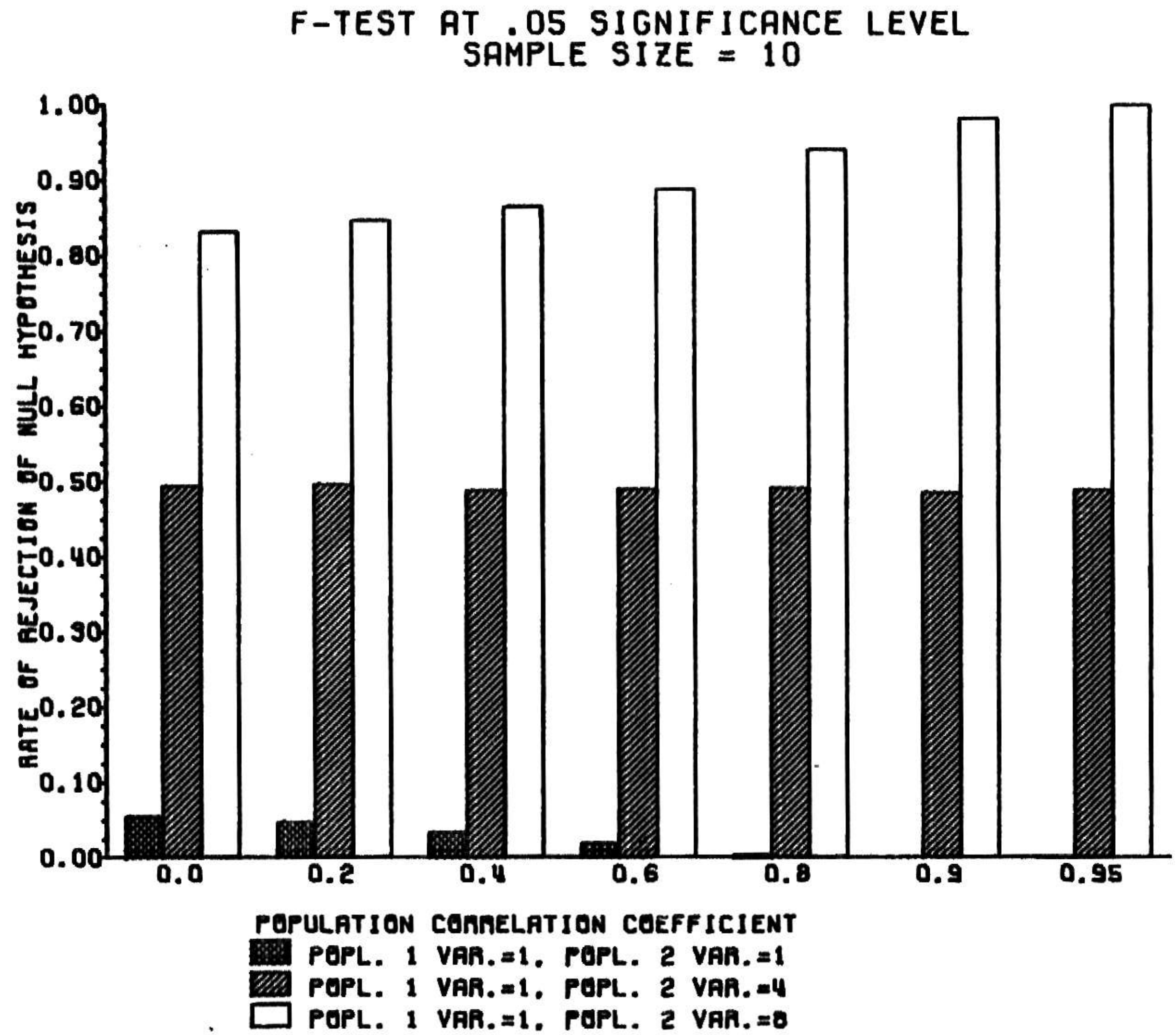
BARGRAPH 2


LIKELIHOOD RATIO TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=20
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

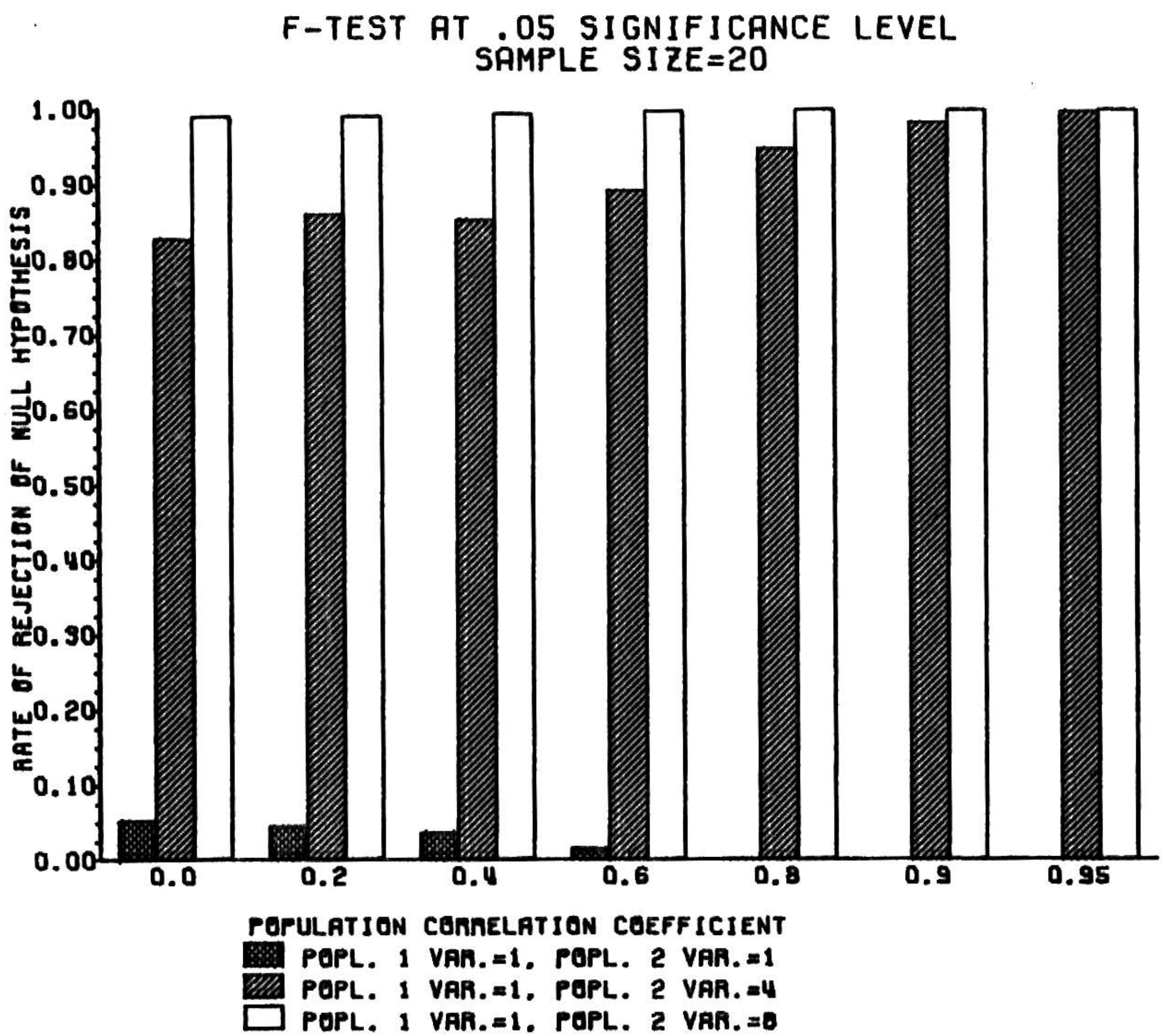
BARGRAPH 3


LIKELIHOOD RATIO TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=40
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

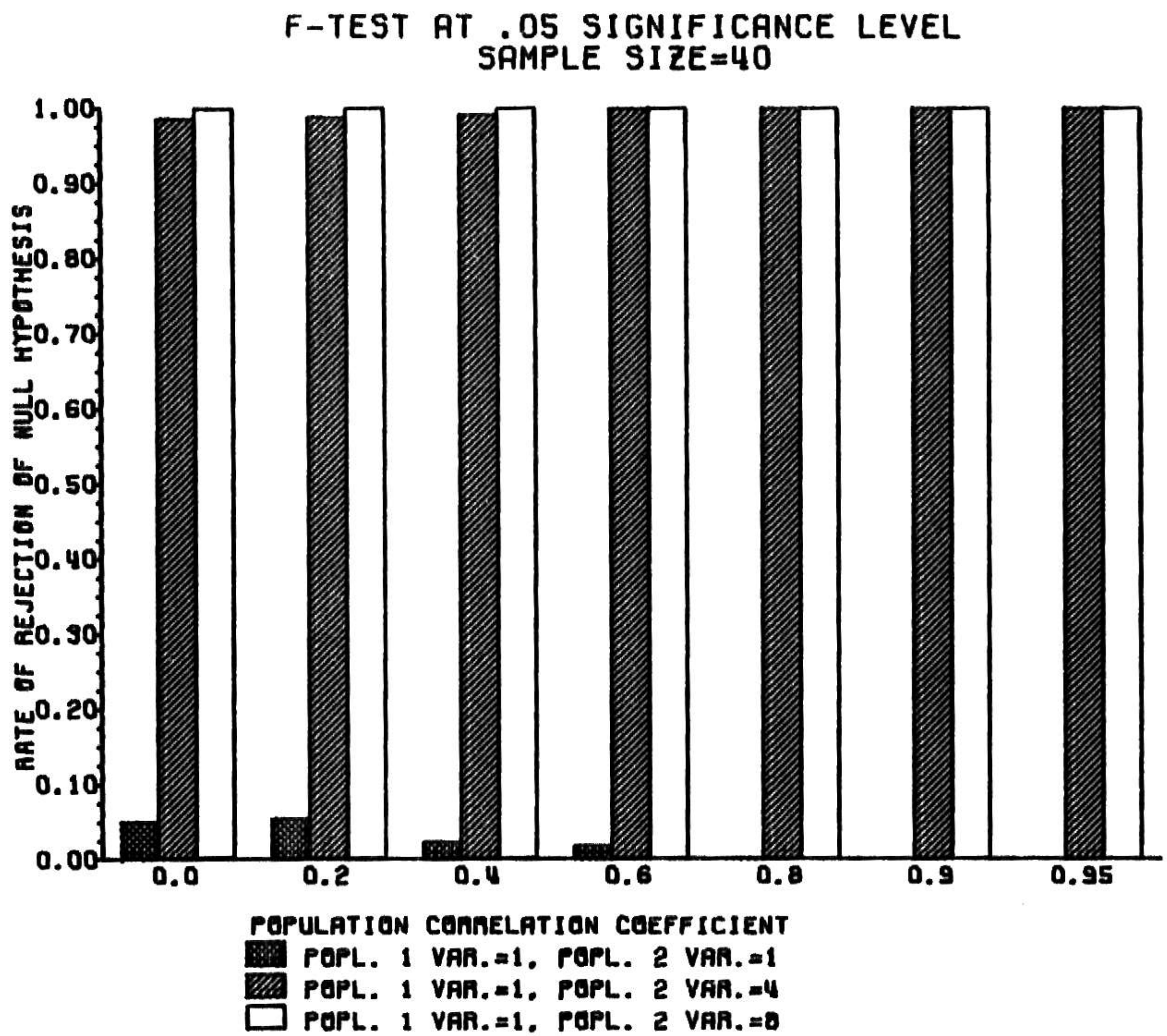
BARGRAPH 4


F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE = 10
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

BARGRAPH 5

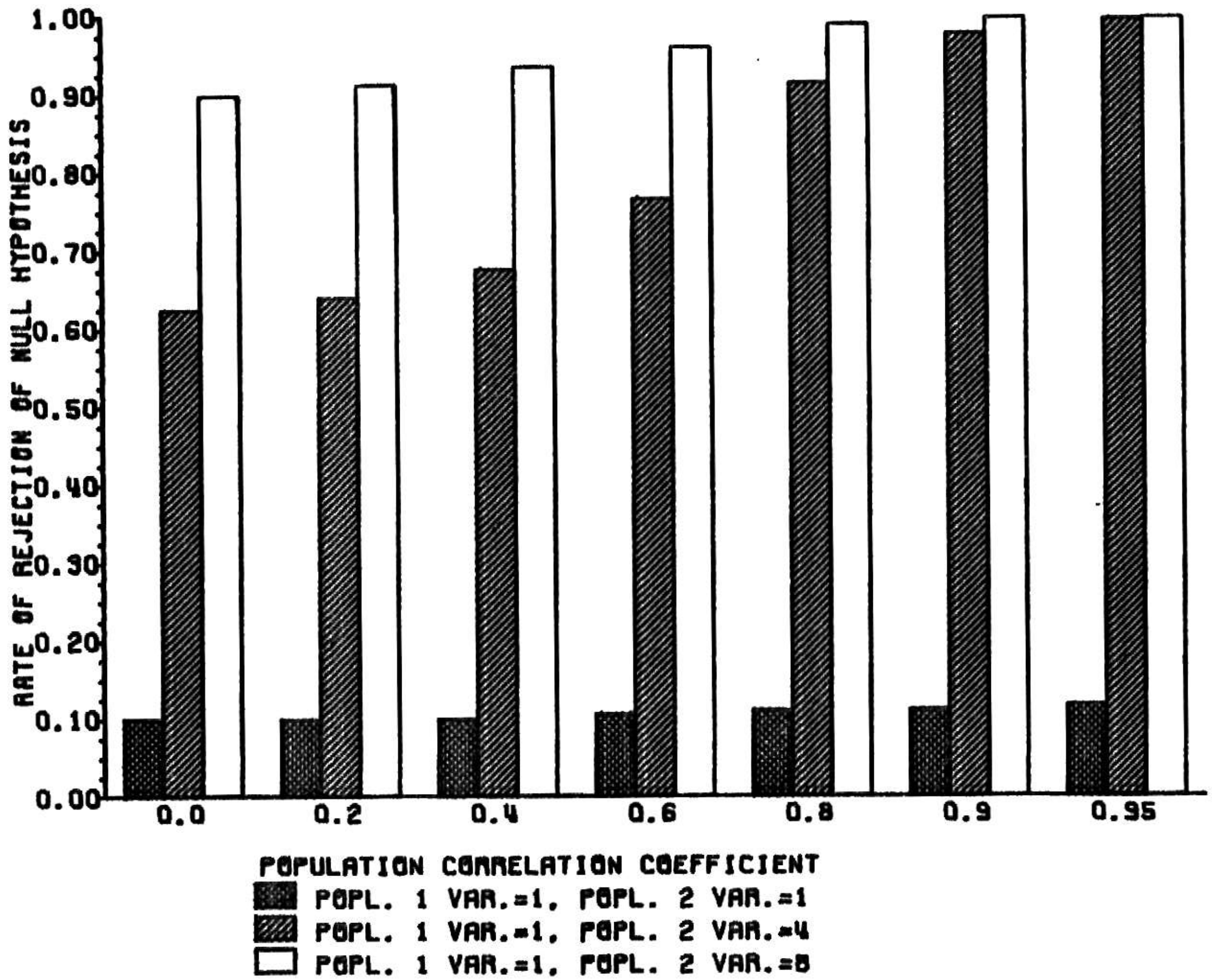

F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=20
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

BARGRAPH 6

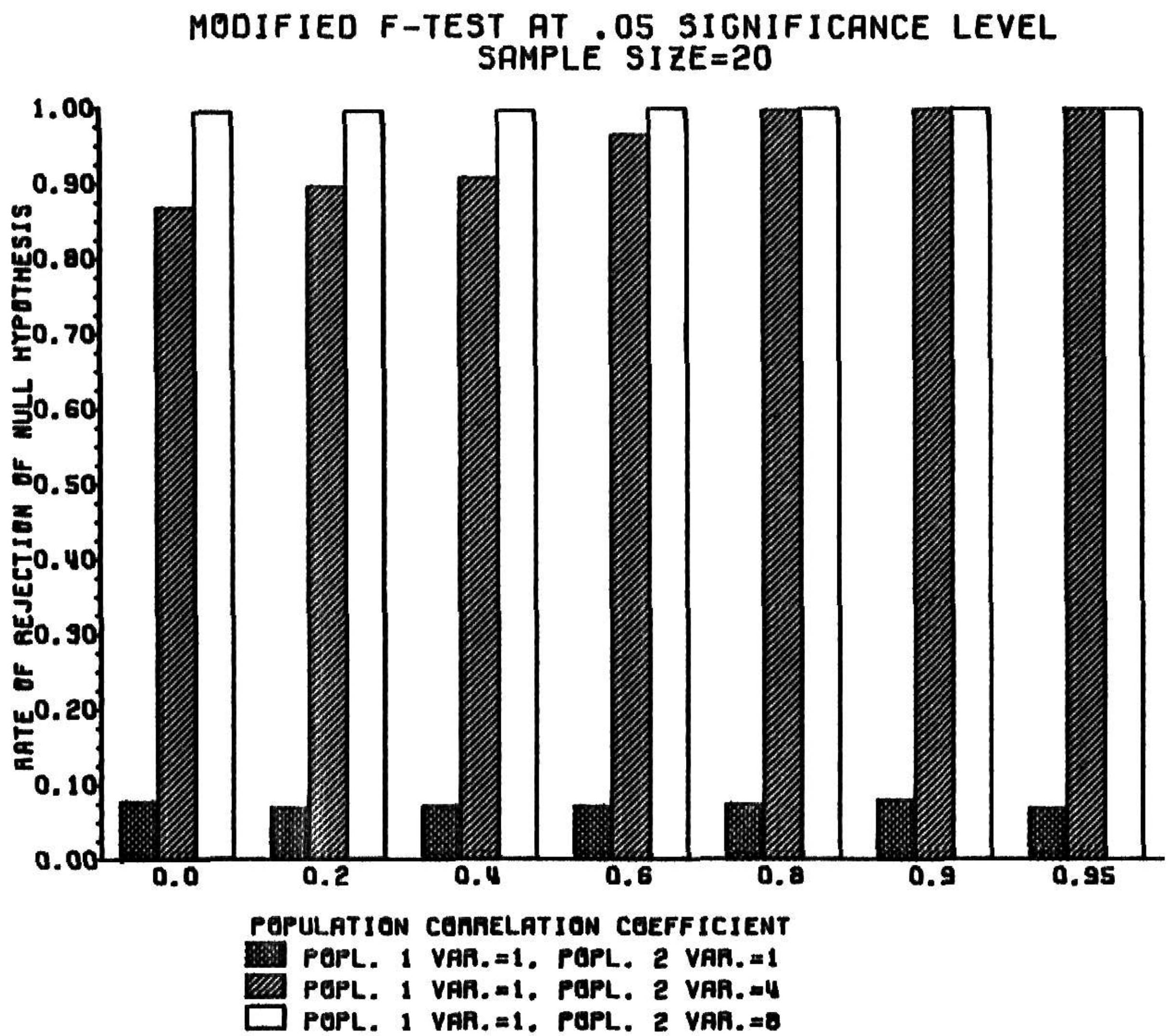

F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=40
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

BARGRAPH 7

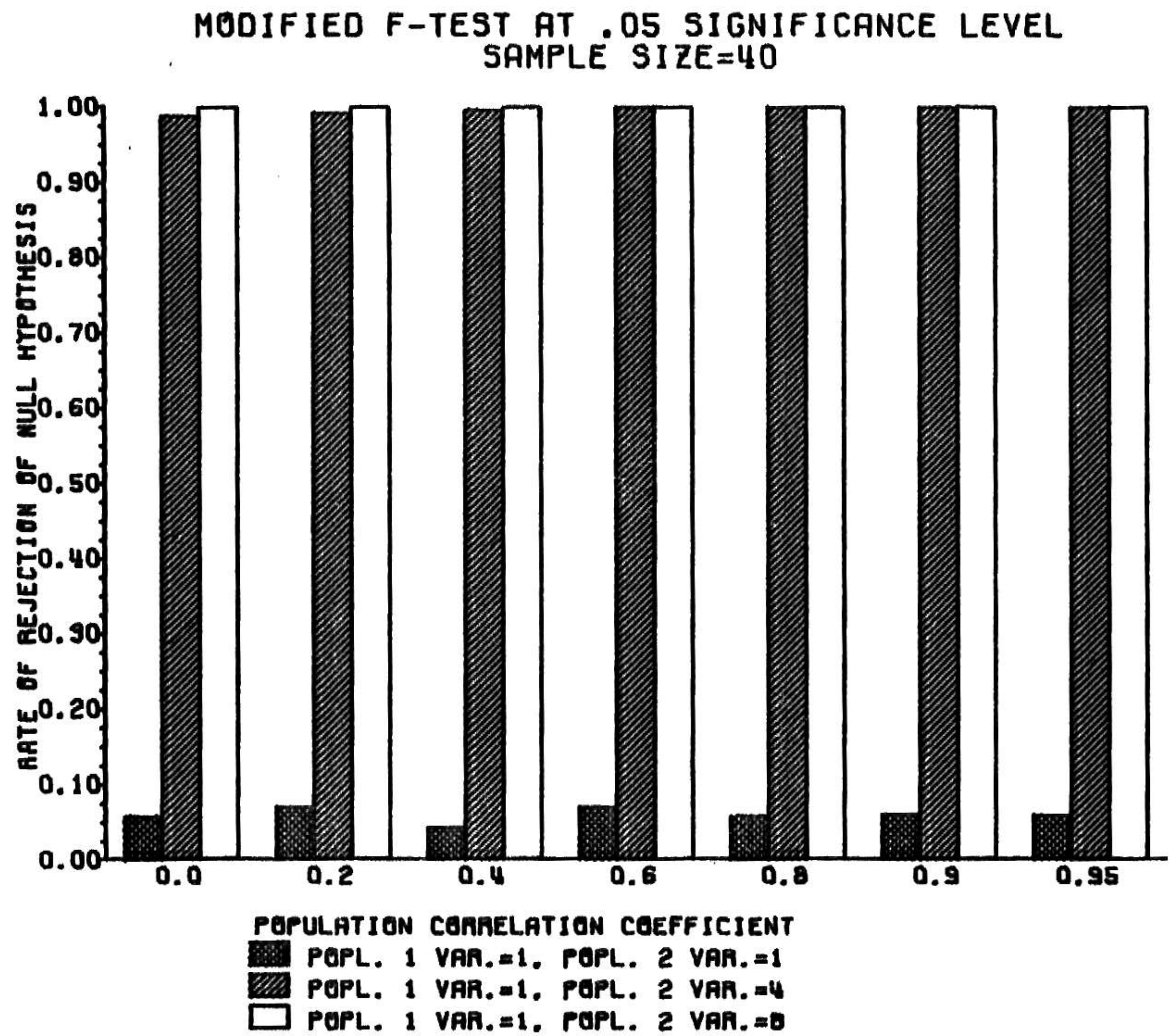
MODIFIED F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE = 10



BARGRAPH 8


MODIFIED F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=20
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

BARGRAPH 9


MODIFIED F-TEST AT .05 SIGNIFICANCE LEVEL
SAMPLE SIZE=40
RATE OF REJECTION OF NULL HYPOTHESIS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.0
0.2
0.4
0.6
0.8
0.9
0.95
POPULATION CORRELATION COEFFICIENT
POPL. 1 VAR.=1, POPL. 2 VAR.=1
POPL. 1 VAR.=1, POPL. 2 VAR.=4
POPL. 1 VAR.=1, POPL. 2 VAR.=8

CHAPTER 4

DISCUSSION OF RESULTS AND CONCLUSION

From Bargraphs 1, 2, and 3 when the null hypothesis is true, the rejection rate of the likelihood ratio test seems to be very close to the .05 significance level for whatever values of population correlation coefficient, ρ and sample size. When the null hypothesis is false, the test tends to reject more and the observed power increases as ρ and sample size increase. The observed power is larger when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. When $\sigma_2^2 = 8$ and $n = 40$ the power is almost one for whatever values of ρ.

Bargraphs 4, 5, and 6 describes the behavior of the power of the F-test at sample size 10, 20, and 40 at .05 significance level. When the hypothesis is true the rejection rate of the F-test seems to attain the .05 significance level when $\rho = 0$ and $\rho = .2$ for all of the three sample sizes. However, the test tends to accept more that is, less than the .05 significance level as ρ is increased for all of the sample sizes. When the null hypothesis is not true, the power of the test tends to increase with increasing values of ρ and sample size. The observed power is larger when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. When the sample size is 40 and $\sigma_2^2 = 8$, the power of the test approaches one for whatever values of ρ.

Bargraphs 7, 8, and 9 tells us about the modified F-test at .05 significance level for sample size 10, 20, and 40. When the null hypothesis is true at sample size 10, the modified F-test tends to reject more than the desired significance level with almost no differences for varying values of ρ. However, when the sample size is increased to

20 the observed significance level decreases and the desired significance level is almost attained at sample size 40 with no differences for whatever values of ρ. When the null hypothesis is false, the behavior of the power seems to be the same as for both the other two tests.

The results of the study from Tables (1), (2), and (3) show that the behavior of the rejection rate of the likelihood ratio test is the same for all of the significance levels of .05, .01, and .1. The F-test seems to behave similarly for all of the significance levels except when the null hypothesis is false that is, when $\sigma_2^2 = 4$ at sample size 10. From Table 1 when $\alpha = .05$ the observed power of the F-test vary slightly for different values of ρ. The observed power of the F-test from Table 2 when $\alpha = .01$ decreases with increasing values of ρ. From Table 3 when $\alpha = .10$ the observed power increases as ρ increases. The modified F-test exhibits the same behavior for all of the three significance levels.

From the study, the likelihood ratio test is the best among the three tests for any values of ρ and sample size. The power of the test seems to increase with increasing sample size. The F-test is found to be the uniformly most powerful unbiased test when $\rho = 0$, that is when the two variates are independent. The F-test seems to hold when $\rho = .2$. For greater values of ρ the F-test is inappropriate. The modified F-test has observed significance level close to the theoretical value at sample size 40 with very high observed power.

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TESTING THE HOMOGENEITY OF THE
TWO VARIANCES OF A NORMAL BIVARIATE POPULATION

by

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AN ABSTRACT OF A MASTER'S REPORT

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Suppose that $U_i = \begin{bmatrix} X_i \\ Y_i \end{bmatrix}$ have independent bivariate normal distributions with mean$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and non-singular covariance matrix $\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ for $i = 1,2,\ldots,n$. A test of the hypothesis that $\sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $\sigma_1^2 \neq \sigma_2^2$ is desired.

Three tests statistics were used in this study namely the likelihood ratio test, the F-test and modified F-test. Derivations of the likelihood ratio test and modified F-test were shown. A Monte Carlo Study was done to compare the power of the three test statistics. A procedure to generate a sample covariance matrix was introduced by P. L. Odell and H. H. Feivenson. The generation of sample covariance matrix was done for different combinations of $\rho = 0, .2, .4, .6, .8, .9, .95$, $\sigma_2^2 = 2, 4, 8$ and $n = 10, 20, 40$ while σ_1^2 is kept constant at 1. The technique requires the generation of only $p(p+1)/2$ random numbers while a straightforward one would require the generation of $Np - p$ random numbers, the greatest saving in computing time occurs when the sample size N is large.

The results of the study show that the likelihood ratio test has observed significance level close to the "theoretical" value with slight differences for whatever values of ρ and sample size when the null hypothesis is true. When the null hypothesis is false, the observed power increases with increasing values of ρ. As sample size is increased the observed power increases. The observed power is large when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. The F-test has observed significance level close to the "theoretical" value when $\rho = .2$ for all sample sizes. For greater values of ρ the F-test is inappropriate. When the null hypothesis is true the modified F-test shows that the observed significance level is close to the "theoretical" value as sample size is increased with slight differences among values of ρ. When the null hypothesis is false the observed power of the modified F-test increases with increasing values of ρ. The increased in power is large as sample size is increased. The observed power is larger when $\sigma_2^2 = 8$ than when $\sigma_2^2 = 4$. Among the three tests used the likelihood ratio test is the best. The F-test is good only when $\rho = 0$ but the test still holds when $\rho = .2$. The modified F-test is good at sample size 40 since the observed significance level is close to the "theoretical" value and observed power is large.