Solenoids in Holomorphic Dynamics

by

Ismael A. Zeidan

BA, Morehead State University, 2024

A Report

submitted in partial fulfillment of the
requirements for the degree

Master Of Science

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2026

Approved by:

Major Professor
Dr. Tatiana Firsova

Copyright

Abstract

This Master's thesis is a report on the work of Hubbard and Oberste-Vorth[7]. We study the emergence of solenoidal structures in the dynamics of complex Hénon maps, with particular emphasis on their realization through inductive limit constructions. Starting from a class of solenoidal mappings $\tau_d : T \to T$ of the solid torus, we construct an associated invariant set

$$\Sigma_d = \bigcap_{m \geq 0} \tau_d^{\circ m}(T),$$

and extend τ_d to an orientation-preserving homeomorphism $h_d : \mathbb{S}^3 \to \mathbb{S}^3$ admitting Σ_d as an invariant solenoid.

We show that the inductive limit $\varinjlim(T, \tau_d)$ is naturally homeomorphic to the complement $\mathbb{S}^3 \setminus \Sigma_d$, providing a global topological model for the dynamics. This construction is then related to the dynamics of complex Hénon maps by analyzing the escaping set U_+ and its filtration by preimages of a forward-invariant region V^+. Using a Böttcher-type coordinate ϕ^+, we obtain a conjugacy between the dynamics of the Hénon map in U_+ and the solenoidal model.

As a consequence, the level sets of the associated Green's function G^+ are shown to be foliated by Riemann surfaces whose topology is governed by the solenoidal structure, and whose leaves are biholomorphic to $\mathbb{C}$ and dense in U_+. This establishes a direct connection between inductive limit topology and the global organization of dynamics in $\mathbb{C}^2$.

Contents

List of Figures

List of Nomenclature

- $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$

- $\mathbb{S}^3 = \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 2\}$

- $T = \mathbb{S}^1 \times \mathbb{D}, (\zeta, z) \in \mathbb{S}^1 \times \mathbb{D}$

- $\tau_d(\zeta, z) = (\zeta^d, \zeta + \epsilon z \zeta^{1-d})$

- $\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix} = \begin{pmatrix} x^2 + q(x) - ay \\ x \end{pmatrix}.$

- $K_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \sup_n \left\| F^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| < \infty \right\} \quad \text{and} \quad U_+ = \mathbb{C}^2 - K_+,$

- $K_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \sup_n \left\| F^{\circ -n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| < \infty \right\} \quad \text{and} \quad U_- = \mathbb{C}^2 - K_-.$

- $J_\pm = \partial K_\pm, \quad K = K_+ \cap K_-, \quad \text{and} \quad J = J_+ \cap J_-.$

- $\alpha \geq \max\{|x| : |x|^d - |q(x)| - (|a| + 2)|x| = 0\}$

- $V_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |y| \leq |x| \text{ and } |x| \geq \alpha \right\}$
- $V_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |x| \leq |y| \text{ and } |x| \geq \alpha \right\}$
- $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |x| \leq \alpha \text{ and } |y| \leq \alpha \right\}$
- $\varphi_\pm \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \left[(F^{\circ\pm n})_1 \begin{pmatrix} x \\ y \end{pmatrix} \right]^{1/d^n}$
- $G_\pm \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{d^n} \log_+ \|F^{\circ\pm n}(x)\|$

Acknowledgments

I would like to thank my Major professor Dr. Tatiana Firsova for all her time and patience in the process of this thesis. I would also like to thank Dr. Hrant Hakobyan and Dr. Dan Volok who were on my committee. And for all those who have helped me in my journey at Kansas State University.

Dedication

To my family and friends with love.

Chapter 1

Introduction

1.1 1-dimensional Holomorphic Dynamics

Holomorphic dynamics represents one of the earliest areas of dynamical systems that developed independently of classical mechanics. The origins of complex dynamical systems can be traced to the works of Pierre Fatou[3] and Gaston Julia[9] between 1917 and 1920. In a series of papers published during this period, they explored the iteration of rational functions on the Riemann sphere. Their work introduced what is known as the Fatou set, consisting of points with stable local dynamics, and the Julia set, which captures the boundary of chaotic behavior. For polynomial maps, these sets can be defined as follows:

Definition 1.1.1. *Let f be a polynomial. Then the filled-in Julia set is defined as:*

$$K(f) = \{z \in \mathbb{C} : \sup_k |f^{\circ k}(z)| < \infty\}.$$

The complement of the filled-in Julia set is the Fatou set $F(f)$. The Julia set is defined as

$$J(f) = \partial K(f)$$

.

A revival of interest in holomorphic dynamics began in the late 1970's and early 1980's[1], driven in part by advances in computer graphics and the visualization of fractal structures. For the map $z \mapsto z^2$, the Julia set is a unit circle $\{z \in \mathbb{C} : |z| = 1\}$, see Fig. 1.1. Apart from a few notable examples, such as $f(z) = z^2$ Julia sets produce intricate fractal patterns on the boundary, see Fig. 1.2.

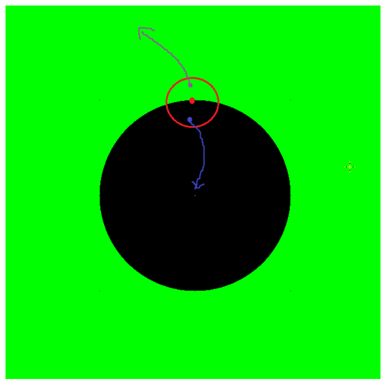

Figure 1.1: Julia set for $f(z) = z^2$

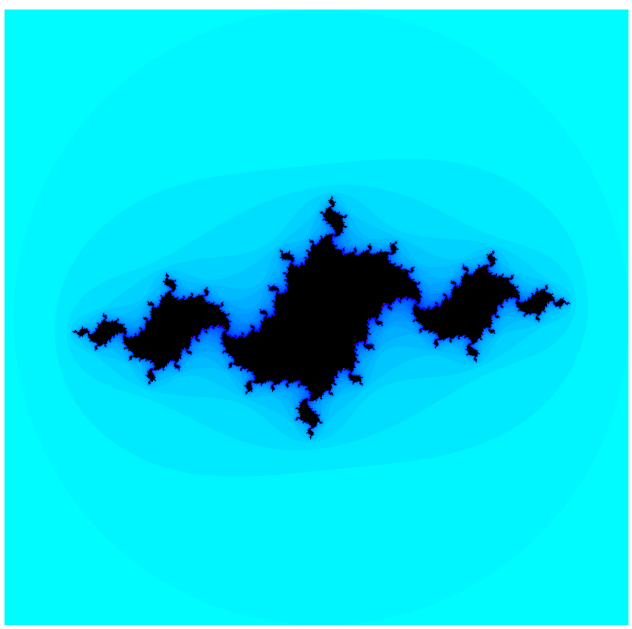

Figure 1.2: Julia set for $f(z) = z^2 + (-1 + 0.2i)$

Even in the seemingly simple case of quadratic polynomials $z \mapsto z^2 + c$, the resulting dynamics can be remarkably complex, with behavior that varies dramatically depending on the parameter c. The set of parameters encoding this behavior forms the well-known *Mandelbrot set*:

$$\mathbb{M} = \left\{c \in \mathbb{C} : z_{n+1} = z_n^2 + c \text{ is bounded as } n \to \infty \text{ and } z_0 = 0\right\}$$

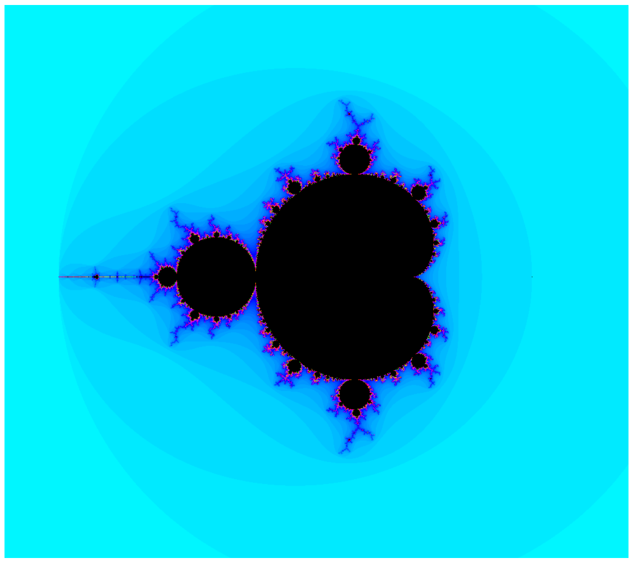

Figure 1.3: Mandelbrot Set

The work of Benoît Mandelbrot in the late 1970's[1], particularly his study of the Mandelbrot set, brought renewed attention to the intricate geometry arising from complex iteration.

Around the same time, mathematicians such as Adrien Douady and John H. Hubbard provided rigorous foundations for these phenomena, culminating in their influential work during the early 1980s on polynomial dynamics and the structure of parameter spaces. These developments firmly established complex dynamics as a central area of modern mathematical research.

1.2 Multidimensional Dynamics

While classical theory focuses primarily on one-dimensional systems, the extension to higher-dimensional complex dynamical systems emerged as a natural and challenging direction of inquiry.

In 1963 Lorenz[10] developed the following system as a simplified version of atmospheric convection:

$$\frac{dx}{dt} = \sigma(y - x), \tag{1.1}$$
$$\frac{dy}{dt} = x(\rho - z) - y, \tag{1.2}$$
$$\frac{dz}{dt} = xy - \beta z. \tag{1.3}$$

He observed that the orbits converge to the "butterfly" looking attractor:

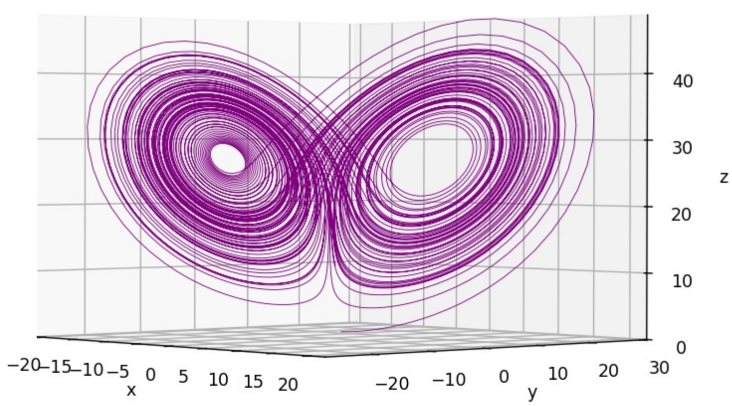

Figure 1.4: Lorenz attractor

for

$$\sigma = 10, \qquad \rho = 28, \qquad \beta = \frac{8}{3},$$

In 1969 M. Hénon[5] introduced what is now known as Hénon maps as a simplified version of the first return map to the Poincare section of the Lorenz attractor. These maps are of the form:

$$\begin{cases} x_{n+1} = x_n^2 + c - ay_n \\ y_{n+1} = x_n \end{cases}$$

$$a \neq 0$$

Numerically he was able to show the existence of strange attractors for certain values of parameters[5]. Benedicks and Carleson[7] rigorously proved their existence. Since then Hénon maps have been extensively studied. As complex methods proved to be very useful in understanding one-dimensional phenomenon, Hénon maps were complexified and studied by means of multidimensional complex analysis.

A key development in this direction has been the analysis of escape rates and growth estimates for iterated maps. By the late twentieth century, it was understood that the behavior of polynomial dynamical systems at large scales is governed by dominant terms, allowing for the identification of regions in which orbits escape to infinity. This analytic framework provides the foundation for defining invariant sets, which generalize the classical Julia set to higher dimensions and serve as the primary objects of study in complex dynamics.
Parallel to these analytic advances, important connections with topology began to emerge.

In 1967, S. Smale[12] introduced the concept of the solenoid as an example of a hyperbolic invariant set, arising as an inverse limit of expanding maps on the circle. These spaces, which are compact, connected, and not locally simply connected, provided early examples of the intricate structures that can arise in dynamical systems.

Solenoidal mappings for Hénon maps play much the same role as multiplying angles by d for polynomial dynamics.

Chapter 2

Solenoidal Mappings

2.1 Geometric Definition of a Solenoid

Let us first consider the general picture of a solenoid. Intuitively, we can picture a solid torus which is stretched by some factor such that it is longer in the major axis d times and smaller along the minor axis. Then we embed the new torus into original torus by wrapping our new torus d times inside the original torus. We then repeat this process infinitely many times. This limiting object is called a solenoid.

Solenoidal Mappings. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 2\}$, $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, $T = \mathbb{S}^1 \times \mathbb{D}$, and denote by (ζ, z) the coordinates in T.

Definition. — Let C_+ and C_- denote the constant families of cones

$$C_+(\zeta, z) = \{(\xi, w) \mid |\xi| \geq |w|\} \quad \text{and} \quad C_-(\zeta, z) = \{(\xi, w) \mid |\xi| \leq |w|\}$$

in the tangent bundle of T.

Definition. — A *solenoidal mapping* $\tau : T \to T$ of degree d is an injective C^1 immersion of degree d, such that, for all $(\zeta, z) \in T$ and for some constant $K > 1$,

$$d_{(\zeta,z)}\tau\big(C_+(\zeta, z)\big) \subset C_+(\tau(\zeta, z)), d_{(\zeta,z)} \text{ is the differential of the map } \tau.$$

$$(\xi, w) \in C_+(\zeta, z) \text{ and } d_{(\zeta,z)}\tau(\xi, w) = (\xi_1, w_1) \;\Rightarrow\; |\xi_1| > K|\xi|,$$

and

$$(\xi, w) \in C_-(\zeta, z) \text{ and } d_{(\zeta,z)}\tau(\xi, w) = (\xi_1, w_1) \;\Rightarrow\; |w_1| < \frac{1}{K}|w|.$$

Then a general solenoid can be defined as follows:

$$\Sigma = \bigcap_{i=0}^{\infty}(\tau)^{\circ i}(T)$$

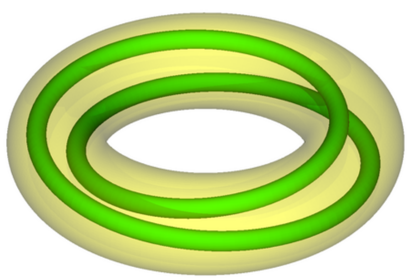

Figure 2.1: Solenoidal Map[13]

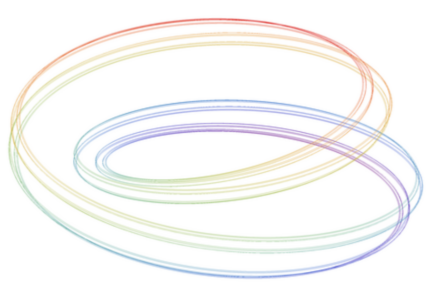

Figure 2.2: 3rd iteration of a solenoidal map[13]

Definition 2.1.1. *An injective mapping $\tau : T \to T$ of degree d is unbraided if there exists a fiber homeomorphism $\varphi : T \to T$ such that $\varphi \circ \tau$ sends the core circle $\mathbb{S}^1 \times \{0\}$ into the torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{D} \times \mathbb{S}^1$ as a $(d, 1)$-torus unknot.*

We will only consider unbraided solenoids.

2.2 Dynamical Definition of Solenoid

Given a space X with a mapping $f : X \to X$, then the *projective limit*

$$\widehat{X}_f = \varprojlim(X, f) = \{(\dots, x_2, x_1, x_0) \mid f(x_{i+1}) = x_i \text{ for } i = 0, 1, 2, \dots\}$$

The map f extends to the projective limit as $f((...x_1, x_0)) = (...f(x_1), f(x_0))$. Consider $\delta : \mathbb{S}^1 \to \mathbb{S}^1$ with $\delta(\zeta) = \zeta^d$, then we get another definition of unbraided solenoid

$$\Sigma = \varprojlim(\mathbb{S}^1, \delta)$$

.

Below we give a topological description of a solenoid. For simplicity, we restrict to the case $d = 2$. Solenoid Σ can be thought of as $I \times C$ with identifications at the boundary points $\{0\} \times C$ and $\{1\} \times C$, where $I = [0, 1]$,

$$C = \{0, 1\}^{\mathbb{N}} = \{x = (x_0, x_1, \dots) : x_j \in \{0, 1\}\}.$$

The point $(0, x)$ is identified with $(1, M(x))$, where M is defined as follows:
Let $x_k = 1$, $x_j = 0$ for $j < k$.

$$M(x_j) = \begin{cases} 0, & 0 \le j < k, \\ 1, & j = k, \\ x_j, & j > k, \end{cases} \qquad M(1, \dots, 1, \dots) = (0, \dots, 0, \dots).$$

The map M is often called an odometer monodromy.

Theorem 2.2.1. *Σ is a compact, connected, but not locally path connected space. Each path component is homeomorphic to a real line which is dense in Σ.*

Proof. Since, by geometric definition, Σ is a nested intersection of compact, connected spaces, it is compact and connected.

Since locally Σ is a product of a Cantor set and an interval, each path component is homeomorphic to either a circle or a real line. Let us show that it is in fact a real line.

Let us consider a partial order on the space of sequences of 0's and 1's. We say $x > y$ if there exists k such that $x_n = y_n$ for all $n \geq k$, and the word $x_{k-1}x_{k-2}\cdots x_0$ is larger than $y_{k-1}y_{k-2}\cdots y_0$ when interpreted as dyadic integers.

Since $M(x) > x$, each path component is homeomorphic to a real line.

For any finite sequence $(y_0, \ldots, y_n)$, $y_i \in \{0, 1\}$, and any point $x \in C$, there exists $l \in \mathbb{Z}$ such that $(M^{\circ l}x)_i = y_i$ for $i = 0, \ldots, n$. This implies density of each path component in Σ.

Since each path component passes through countably many points in C, but C contains uncountably many points, Σ is not locally path connected.

$\square$

2.3 The Solenoidal Map τ_d

Hubbard and Oberste-Vorth[8] have shown that there are countably many conjugacy classes of unbraided solenoidal maps. The following solenoidal map is in the class relevant to Hénon maps[7]. Let $T = \mathbb{S}^1 \times \mathbb{D}$, where $\mathbb{S}^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 2\}$. Define

$$\tau_d(\zeta, z) = (\zeta^d, \zeta + \epsilon z \zeta^{1-d})$$

The first coordinate of the mapping τ_d tells us how many times $\mathbb{S}^1$ is wrapped in $\tau_d(T)$. The second coordinate of the mapping gives us a shrinking with ϵ, a translation with $\zeta + z$, and a rotation factor with ζ^{1-d}. Note that the rotation factor is important for extending the map to $\mathbb{R}^3$.

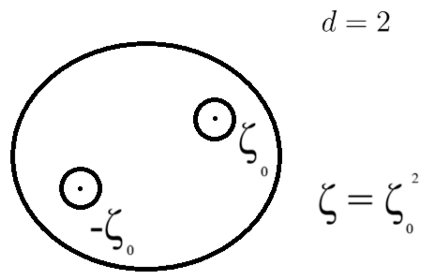

Figure 2.3: Slice of first iteration of τ_d for $d = 2$

Chapter 3

Orientation Preserving Homeomorphism

The main result of this theorem is due to Hubbard and Oberste-Vorth[7].

3.1 Inductive Limits

The inductive limit is defined as follows:

Definition 3.1.1. *Given a space X and a mapping $f : X \to X$, define the inductive limit $\varinjlim(X, f)$ to be*

$$\varinjlim(X, f) = X \times \mathbb{N}/\sim,$$

where the equivalence is generated by setting $(x, m) \sim (f(x), m + 1)$.

We will use the following notations:

- $[x, m]$ is the equivalence class of (x, m) in $\varinjlim(X, f)$.
- $q : X \times \mathbb{N} \to X \times \mathbb{N}/\sim$ is the quotient map.

For an arbitrary (X, f), the inductive limit can create a non-Hausdorff space. For one-to-one map, the inductive limit is used to create spaces on which the extension of the map is bijective.

3.2 Inductive Limit of τ_d

We present the proof of the following theorem from Hubbard and Oberste-Vorth[7].

Theorem 3.2.1.

The inductive limit $\varinjlim(T, \tau_d)$ *is homeomorphic to* $S^3 \setminus \Sigma_d$, *where* $\Sigma_d = \bigcap_m \tau_d^{\circ m}(T)$.

3.3 Extension of τ_d to $\mathbb{S}^3$

To construct the inductive limit of $\tau_d : T \to T$, we first extend it to an orientation preserving homeomorphism $h_d : \mathbb{S}^3 \to \mathbb{S}^3$.

Let us consider a decomposition of $\mathbb{S}^3$ into two complementary solid tori,

$$\mathbb{S}^3 = \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$$

$$\mathbb{S}^3 = T' \cup T''$$

with $T' = \{(u, v) : |u| \leq \frac{1}{\sqrt{2}}\}$, $T'' = \{(u, v) : |v| \leq \frac{1}{\sqrt{2}}\}$.

The map $\rho : T' \to T''$ with $\rho : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} v \\ u \end{bmatrix}$ interchanges T' and T''.

Let

$$u = x_1 + ix_2, \qquad v = x_3 + ix_4.$$

We project stereographically from the north pole

$$N = (0, 0, 0, 1).$$

The stereographic projection $\sigma : \mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ is

$$\sigma(x_1, x_2, x_3, x_4) = \frac{(x_1, x_2, x_3)}{1 - x_4}.$$

Substitute

$$x_1 = \mathrm{Re}(u), \quad x_2 = \mathrm{Im}(u), \quad x_3 = \mathrm{Re}(v), \quad x_4 = \mathrm{Im}(v).$$

Thus

$$\sigma(u, v) = \left(\frac{\mathrm{Re}(u)}{1 - \mathrm{Im}(v)}, \frac{\mathrm{Im}(u)}{1 - \mathrm{Im}(v)}, \frac{\mathrm{Re}(v)}{1 - \mathrm{Im}(v)} \right) = (x, y, z).$$

We can write ρ in (x, y, z) coordinates as follows:

$$\rho_1 : \mathbb{S}^3 \to \mathbb{S}^3 \text{ as } \rho_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \frac{1}{x^2+(y-1)^2+z^2} \begin{bmatrix} 2z \\ x^2 + y^2 + z^2 - 1 \\ 2x \end{bmatrix}$$

Let r_y be the reflection with respect to the y-axis. Note that $r_y \circ \rho_1 = \rho_1 \circ r_y$. Consider, two unknotted solid tori T_1 and T_2 in $\mathbb{R}^3$ linked with linking number d (see fig 3.1).

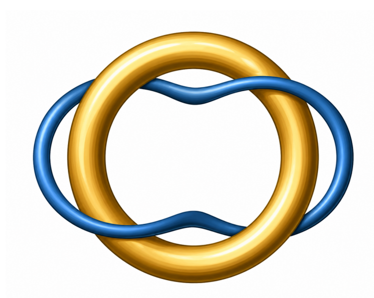

Figure 3.1: Two solid tori linked with linking number $d = 2$.[7]

Let $R_d : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation by $\frac{\pi}{d}$ around the z-axis. Tori T_1 and T_2 can be chosen so that $R_d(T_1) = T_2$.

Given a homeomorphism $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ we denote $\rho_\alpha = \alpha^{-1} \circ \rho_1 \circ \alpha$.

Proposition 3.3.1. *There exists an orientation-preserving homeomorphism*

$$\alpha : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$$

such that

(a) α maps T_1 to T';

(b) α commutes with reflection in the y-axis r_y;

(c) the restriction of $h_d := \rho_\alpha \circ R_d$ to T_1 is a solenoidal mapping conjugate to τ_d;

(d) h_d is conjugate to its inverse.

We first construct α on T_1, preserving symmetry with respect to r_y. Then extend to the rest of $\mathbb{R}^3$, preserving the same symmetry.

Since $R_d(T_1) = T_2$, the image $\alpha \circ R_d(T_1)$ is a torus in T'' wrapped around d times.

The image $\rho_1(\alpha(T_2))$ is a torus in T'. By finally applying α^{-1}, we get that the image $\rho_\alpha \circ R_d(T_1) \subset T_1$ is a torus in T_1 wrapped around d times. By adjusting α, we can ensure enough stretching and contracting, so $h_d : T_1 \to T_1$ is a solenoid map. One can show that it is conjugate to τ_d.

Since R_d and ρ_1 are homeomorphisms, then $h_d : \mathbb{S}^3 \to \mathbb{S}^3$ is also a homeomorphism.

$$(h_d)^{-1} = \rho_\alpha \circ R_d^{-1}.$$

We show that r_y conjugates h_d to h_d^{-1}. Note that r_y is an involution, so $r_y^{-1} = r_y$. Then

$$r_y \circ (h_d)^{-1} \circ r_y = r_y \circ \rho_\alpha \circ R_d^{-1} \circ r_y = \rho_\alpha \circ r_y \circ R_d^{-1} \circ r_y = \rho_\alpha \circ R_d.$$

Furthermore, we have two invariant solenoids $\Sigma_+ \subset T_1$ and $\Sigma_- \subset T_2$ corresponding to the images and pre-images of h_d respectively.

3.4 Proof of Theorem 3.2.1

Proof. Let

$$q : T \times \mathbb{N} \to (T \times \mathbb{N})/\sim$$

be the quotient map on T and define

$$j_m : T \to T \times \mathbb{N}, \qquad j_m(x) = (x, m).$$

Define

$$\Phi : (T \times \mathbb{N})/\sim \to \mathbb{S}^3 \setminus \Sigma$$

by

$$\Phi([x, m]) = h_d^{\circ -m}(x).$$

Recall that $h_d|_T = \tau_d$.

Observe that

$$\Phi \circ q \circ j_m(x) = \Phi([x, m]) = h_d^{\circ -m}(x),$$

which is continuous since h_d is a homeomorphism. Hence Φ is continuous by the universal property of the quotient topology.

Next we construct the inverse. Let $z \in \mathbb{S}^3 \setminus \Sigma$. Since

$$\mathbb{S}^3 \setminus \Sigma = \bigcup_{m \geq 0} h_d^{\circ -m}(T),$$

there exists m such that $z \in h_d^{\circ -m}(T)$. Define

$$\Psi(z) = [h_d^{\circ m}(z), m].$$

We check that Ψ is well-defined. If $z \in h_d^{\circ -m}(T) \cap h_d^{\circ -n}(T)$ with $m \geq n$, it follows that

$$h_d^{\circ m}(z) = h_d^{\circ (m-n)}(h^{\circ n}(z)) = \tau_d^{\circ (m-n)}(h_d^{\circ n}(z)).$$

Thus

$$(h_d^{\circ n}(z), n) \sim (\tau_d^{\circ (m-n)}(h_d^{\circ n}(z)), m) = (h_d^{\circ m}(z), m),$$

so $[h_d^{\circ n}(z), n] = [h_d^{\circ m}(z), m]$.

Therefore Ψ is well-defined. On each set $h_d^{\circ -m}(T)$, Ψ is continuous since it is the composition of the continuous map h_d^m with the quotient map. Since $\mathbb{S}^3 \setminus \Sigma$ is the union of the sets $h_d^{\circ -m}(T)$, Ψ is continuous.

Finally,

$$\Psi(\Phi([x, m])) = [h_d^{\circ m}(h_d^{\circ -m}(x)), m] = [x, m]$$

and

$$\Phi(\Psi(z)) = h_d^{\circ -m}(h_d^{\circ m}(z)) = z.$$

Hence Φ is a homeomorphism. $\square$

Chapter 4

Dynamically significant sets

For use in the remainder of the discussion, let us consider the following constructions and definitions throughout. For any mapping F, let $F^{\circ n}$ denote the n-fold composition of F or $F^{\circ -n}$ for negative powers of n. Inspired by 1-dimensional polynomial dynamics, we define the following sets for the Hénon map F :

$$K_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \sup_n \left\| F^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| < \infty \right\} \qquad U_+ = \mathbb{C}^2 - K_+,$$

$$K_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \sup_n \left\| F^{\circ -n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| < \infty \right\} \qquad U_- = \mathbb{C}^2 - K_-.$$

Further, define

$$J_\pm = \partial K_\pm, \quad K = K_+ \cap K_-, \quad \text{and} \quad J = J_+ \cap J_-.$$

The sets J, J_+, and J_- are the analogs of the Julia set for 1-dimensional polynomials.

4.1 The Regions V_+, V_-, and W

Let us consider the generalized Hénon map of degree d

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix} = \begin{pmatrix} x^d + q(x) - ay \\ x \end{pmatrix},$$

where $p(x)$ is a monic polynomial of degree d. Below we define an immediate basin of infinity on which the first coordinate of F is dominant and behaves like x^d. We introduce the following constant α. For a generalized Hénon map of degree d, choose

$$\alpha \geq \max\{|x| : |x|^d - |q(x)| - (|a| + 2)|x| = 0\}$$

Note that the particular value for α is often not as important as its existence.

We can now define the regions V_+, V_-, and $W \subset \mathbb{C}^2$ by

$$V_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |y| \leq |x| \text{ and } |x| \geq \alpha \right\},$$

$$V_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |x| \leq |y| \text{ and } |x| \geq \alpha \right\},$$

and

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; |x| \leq \alpha \text{ and } |y| \leq \alpha \right\}.$$

We are interested in the region V_+ since all the points that belong to U_+ will eventually pass through V_+. That is $U_+ = \bigcup_{n\geq 0} F^{\circ -n}(V_+)$. Thus, the dynamics of the set U_+ can be viewed as the dynamics of the union of pre-images of the set V_+.

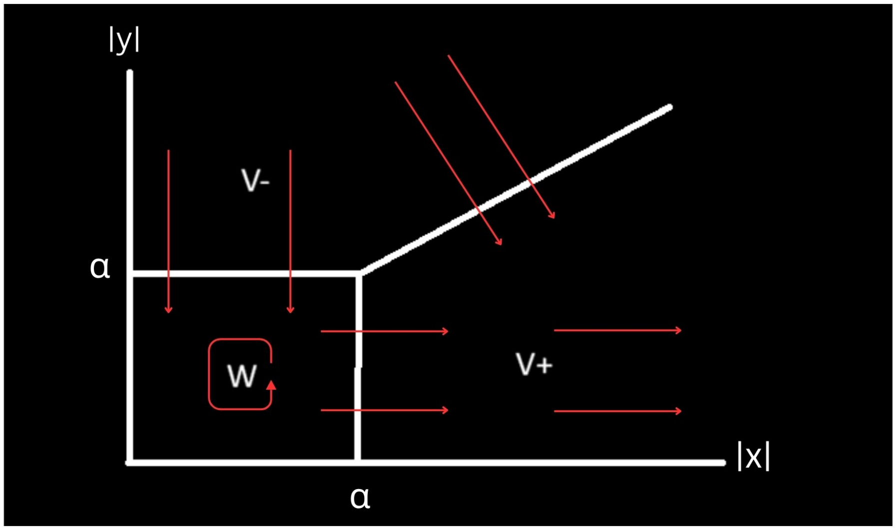

Figure 4.1: $V+, V_{-}, W$

For large $|x|$, we estimate

$$|x^d + q(x) - ay| \geq |x^d| - |q(x)| - |ay|.$$

In V_+, we have $|y| \leq |x|$, hence

$$|x^d + q(x) - ay| \geq |x|^d - |q(x)| - |a|\,|x|.$$

Thus, for sufficiently large $|x|$, $(x, y) \in V_+$, the Hénon map behaves approximately like

$$F\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} x^d \\ x \end{pmatrix}.$$

To understand the topology on V_+, we consider

$$|x| \colon V_+ \to \mathbb{R}_+,$$

whose level sets are

$$|x|^{-1}(r) = \{(x, y) \in V_+ : |x| = r,\ |y| \leq r\}\,.$$

This function produces level sets which are geometrically solid tori, however $|x|$ does not respect dynamics. We consider instead φ_+ which behaves approximately like x but respects dynamics. To build φ_+, we will need to recall Böttcher's Theorem.

Chapter 5

Böttcher's Theorem

Let us recall Böttcher's Theorem from single variable complex dynamics[11].

Let $f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$ $d \geq 2$ and $a_d \neq 0$. There exist $\mathcal{N}(\infty)$ and $\mathcal{N}(0)$ neighborhoods of ∞ and 0, respectively, and a corresponding local change of coordinates

$$\phi : \mathcal{N}(\infty) \to \mathcal{N}(0), \qquad \phi(\infty) = 0,$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{N}(\infty) \arrow[r, "F"] \arrow[d, "\phi"'] & \mathcal{N}(\infty) \arrow[d, "\phi"] \\
\mathcal{N}(0) \arrow[r, "z \mapsto z^d"'] & \mathcal{N}(0)
\end{tikzcd}$$

Furthermore, ϕ is unique up to multiplication by an $(d-1)$st root of unity.

The map ϕ is defined as a limit

$$\phi = \lim_{n \to \infty} \phi_n(z),$$

where,

$$\phi_n(z) = \sqrt[d^n]{f^{\circ n}(z)}$$

with an appropriate choice of root.

We then want to look at the analogue of $\phi(z)$ in $\mathbb{C}^2$.

For a map $G : \mathbb{C}^2 \to \mathbb{C}^2$, we write $G(x, y) = (G_1(x, y), G_2(x, y))$.

$$\varphi_{\pm}\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty}\left[\left(F^{\circ\pm n}\right)_1\begin{pmatrix} x \\ y \end{pmatrix}\right]^{1/d^n}$$

Below we explain how to choose the branch of d^n -root in the definition of ϕ_+ for $(x, y) \in V_+$.

Let $F(x, y) = (x^d + q(x) - ay, x)$ and $x_0 = x$ and $y_0 = y$ then $(x_n, y_n) = F(x_{n-1}, y_{n-1}) = (x_{n-1}^d + q(x) - ay_{n-1}, x_{n-1})$.

We define φ_+ by the following telescoping product

$\varphi_+(x, y) = x_0 \cdot \frac{\sqrt[d]{x_1}}{x_0} \cdot \dots \cdot \frac{\sqrt[d^n]{x_n}}{\sqrt[d^{n-1}]{x_{n-1}}} \cdot \dots$.

Now consider the nth iteration, $\frac{\sqrt[d]{x_n}}{x_{n-1}} = \sqrt[d]{\frac{x_n}{x_{n-1}^d}} = \sqrt[d]{\frac{x_{n-1}^d + q(x_{n-1}) - ay_{n-1}}{x_{n-1}^d}} = \sqrt[d]{1 + \frac{q(x_{n-1}) - ay_{n-1}}{x_{n-1}^d}}$

If $(x_0, y_0) \in V_+$, then $(x_{n-1}, y_{n-1}) \in V_+$. Hence, $|y_{n-1}| \le |x_{n-1}|$ and $|x_{n-1}| \to \infty$.

$$\left|\frac{q(x_{n-1}) - a(y_{n-1})}{x_{n-1}^d}\right| \le \left|\frac{q(x_{n-1})}{x_{n-1}^d}\right| + \left|\frac{a}{x_{n-1}^{d-1}}\right| \to 0.$$

The principal branch of $\sqrt[d]{z}$ is well-defined in a neighborhood of the point $z = 1$.

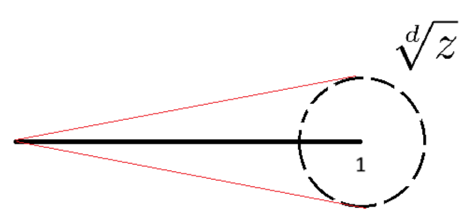

Figure 5.1: Branch of $\sqrt[d]{z}$

This calculation also shows that

$$\varphi_+(x,y) \sim x \quad \text{when } (x,y) \to \infty \text{ in } V^+.$$

Chapter 6

Main results

6.1 Level Sets of Green's Function

The function

$$G_{\pm}\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{d^n} \log_{+} \left\| F^{\circ \pm n}(x) \right\| ,$$

is a generalization of the Green's function for polynomial maps. Note that $G_+(x, y) = \ln|\varphi_+(x, y)|$.

Theorem 6.1.1. *$G_+ : U_+ \to \mathbb{R}_+$ is a trivial fibration whose fibers are homeomorphic to $\mathbb{S}^3 \setminus \Sigma_d$.*

Proof. Represent the set

$$U_+(r) = G_+^{-1}(\log r)$$

as the increasing union

$$U_+(r) = V_+(r) \ \cup \ F^{-1}\big(V_+(r^2)\big) \ \cup \ F^{-2}\big(V_+(r^4)\big) \ \cup \ \cdots ,$$

where

$$V_+(r) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in V_+ \;\middle|\; G_+ \begin{pmatrix} x \\ y \end{pmatrix} = \log r \right\}.$$

Below we list the main propositions in the proof of this theorem from Hubbard and Oberste-Vorth[7].

The first proposition is a direct consequence of the fact that for $(x, y) \in V^+$ with $|x|, |y|$ large, $\varphi_+(x, y) \sim x$.

Proposition 6.1.1. *(a) For large r, $V_+(r)$ is homeomorphic to a solid torus, and*

$$\varphi_+ : V_+(r) \to \{z \mid |z| = r\}$$

is a fibration with fibers homeomorphic to closed disks.

(b) There exists a sufficiently large R such that,

$$G_+ : V_+ \to \mathbb{R}_+$$

is a trivial bundle over (R, ∞) with fibers homeomorphic to solid tori.

Let $z = \frac{y}{x}$, $\zeta = \varphi_+(x, xz)$, and $\eta = z\zeta$

To show that F is conjugate to τ_d we need the following expansions:

Proposition 6.1.2. *The following asymptotic expansion holds:*

$$x = \zeta + o(|\zeta|) + \left(\frac{a}{d\,\zeta^{d-2}} + o\left(\frac{1}{|\zeta|^{d-2}} \right) \right) z + o(|z|).$$

This leads us to the following result:

$$F\begin{pmatrix}\zeta\\ z\end{pmatrix} = \left(\zeta^d,\ \frac{1}{\zeta^{d-1}} + o\!\left(\frac{1}{|\zeta|^{d-1}}\right) + \left(\frac{a}{d\,\zeta^{2d-2}} + o\!\left(\frac{1}{|\zeta|^{2d-2}}\right)\right) z + o(|z|)\right).$$

In these coordinates, the following expression holds:

$$F\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \left(\zeta^d,\ \zeta + o(|\zeta|) + \left(\frac{a}{d\,\zeta^{d-1}} + o\!\left(\frac{1}{|\zeta|^{d-1}}\right)\right) \eta + o(|\eta|)\right).$$

It follows that, F is conjugate to τ_d.

Proposition 6.1.3. *If $g : V_+(r) \to T$ is a homeomorphism with $g(\gamma_r)$ a curve on T bounding a disk in $\mathbb{S}^3 \setminus T$, then there exists a homeomorphism $g' : V_+(r^d) \to T$ such that the diagram*

$$\begin{tikzcd}
V_+(r) \arrow[r, "g"] \arrow[d, "F"'] & T \arrow[d, "\tau_d"] \\
V_+(r^d) \arrow[r, "g'"'] & T
\end{tikzcd}$$

commutes. Moreover, $g'(\gamma_{r^d})$ is again a curve which bounds in $\mathbb{S}^3 \setminus T$.

Then, by induction, the following diagram commutes:

$$\begin{tikzcd}
V_+(r) \arrow[rr] \arrow[dr, equal] \arrow[dd, "G"'] & & F^{\circ -1}(V_+(r^d)) \arrow[rr] \arrow[dr, "F"] \arrow[dd, "G'"'] & & F^{\circ -2}(V_+(r^{d^2})) \arrow[rr] \arrow[dr, "F^{\circ 2}"] \arrow[dd, "G''"'] & & F^{\circ -3}(V_+(r^{d^3})) \arrow[rr] \arrow[dr, "F^{\circ 3}"] \arrow[dd, "G'''"'] & & \cdots \\
& V_+(r) \arrow[rr, "F"] \arrow[dd, "g"'] & & V_+(r^d) \arrow[rr, "F"] \arrow[dd, "g'"'] & & V_+(r^{d^2}) \arrow[rr, "F"] \arrow[dd, "g''"'] & & V_+(r^{d^3}) \arrow[r] \arrow[dd, "g'''"'] & \cdots \\
T \arrow[rr] \arrow[dr, equal] & & \tau_d^{\circ -1}(T) \arrow[rr] \arrow[dr, "\tau_d"] & & \tau_d^{\circ -2}(T) \arrow[rr] \arrow[dr, "\tau_d^{\circ 2}"] & & \tau_d^{\circ -3}(T) \arrow[rr] \arrow[dr, "\tau_d^{\circ 3}"] & & \cdots \\
& T \arrow[rr, "\tau_d"'] & & T \arrow[rr, "\tau_d"'] & & T \arrow[rr, "\tau_d"'] & & T \arrow[r] & \cdots
\end{tikzcd}$$

Figure 6.1: Commutative diagram by induction[7]

In the end,

$$U_+(r) = \bigcup_{k=0}^{\infty} F^{\circ -k}\big(V_+(r^{d^k})\big)$$

is homeomorphic to

$$\bigcup_{k=0}^{\infty} \tau_d^{\circ -k}(T) = \mathbb{S}^3 \setminus \Sigma_d.$$

for large enough r. For any $r > 1$, there exists n_0 such that for all $k \geq n_0$, $U_+(r)$ can be represented as

$$U_+(r) = \bigcup_k F^{-\circ k}\left(V_+\left(r^{d^{k+n_0}}\right)\right).$$

Hence, the theorem holds for all level sets of G^+.

$\square$

Chapter 7

The Foliation of U_+

7.1 Description of the foliation

Since G^+ is a pluriharmonic submersion on U^+, its level sets $U^+(r)$ are foliated by Riemann surfaces. We denote this foliation by $\mathcal{F}^+$. Since we are in V_+, $G^+ = \ln|\varphi_+|$, the leaves of the foliation $\mathcal{F}^+$ are level sets of the function φ_+. This foliation can be propagated to the dynamics to all of U^+.

Theorem 7.1.1. *The leaves of the foliation $\mathcal{F}^+$ of $U_+(r)$ are isomorphic to $\mathbb{C}$ and dense in $U_+(r)$.*

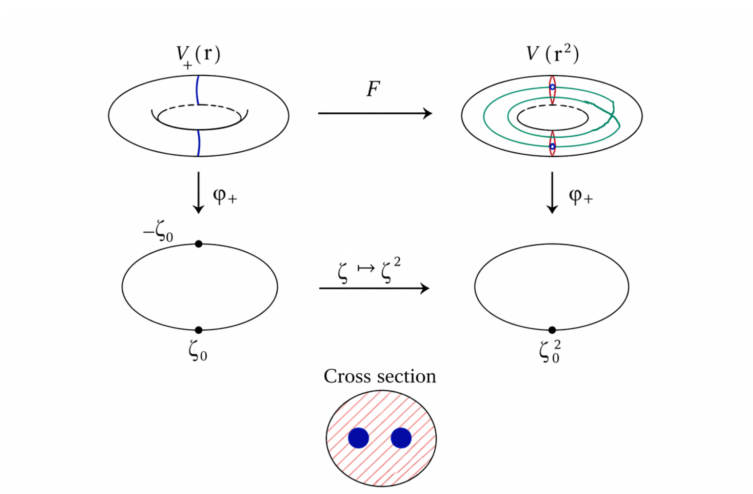

Figure 7.1: Map $F : V_+(r) \to V_+(r^2)$ for $d = 2$.

The diagram above illustrates the map $F : V_+(r) \to V_+(r^2)$ for the case $d = 2$.

The leaf $\mathcal{L}$ going through $\varphi_+^{-1}(\zeta)$ can be written as

$$\mathcal{L} = \varphi_+^{-1}(\zeta) \cup F^{-1}(\varphi_+^{-1}(\zeta^d)) \cup F^{-2}(\varphi_+^{-1}(\zeta^{d^2})) \cup \cdots ,$$

where each $F^{-n}(\varphi_+^{-n}(\zeta^{d^n}))$ is a disk. Since

$$\varphi_+^{-1}(\zeta) \subset F^{-1}(\varphi_+^{-1}(\zeta^d)) \subset F^{-2}(\varphi_+^{-2}(\zeta^{d^2})) \subset \cdots ,$$

the leaf $\mathcal{L}$ is an increasing union of disks.

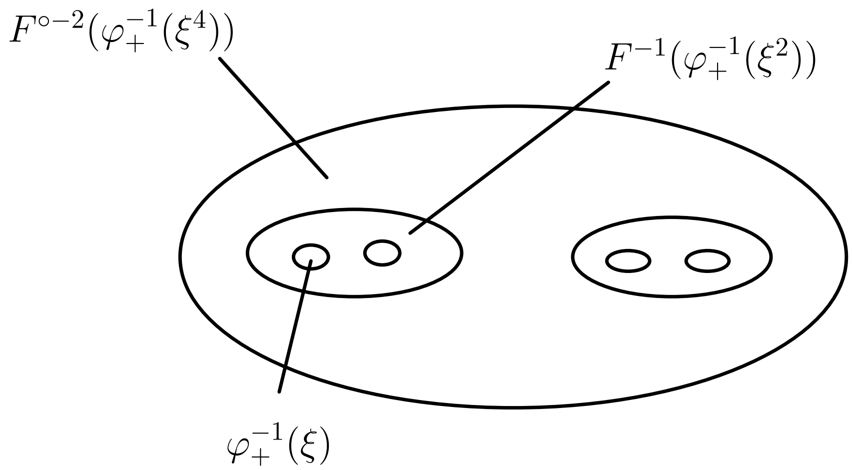

Figure 7.2: Leaf $\mathcal{L}$ of foliation $\mathcal{F}^+$

Note that

$$F^{\circ -k}\varphi_+^{-1}(\zeta^{d^k}) \bigcap V_+ = \bigcup_{\omega^{d^k}=1} \varphi_+^{-1}(\omega\zeta)$$

.

Since $F^{\circ -k}\varphi_+^{-1}(\xi^{d^k}) \subset \mathcal{L}$ and the d^{th} roots of unity are dense on the unit circle, we can conclude that the leaf $\mathcal{L}$ is dense in $V_+(r)$, where $r = |\varphi_+(\xi)|$. It follows that it is dense in $U_+(r)$.

To find the conformal type of the leaf, we use the following proposition.

Proposition 7.1.1. *Let X be a simply connected Riemann surface, and $K \subset X$ a compact connected simply connected subset not reduced to a point. Suppose $A_0, A_1, \dots$ is a sequence of disjoint annuli in $X \setminus K$ such that the inclusion of A_i into $X \setminus K$ is of degree 1. If*

$$\sum_{i=0}^{\infty} \operatorname{mod}(A_i) = \infty,$$

then the surface X is isomorphic to $\mathbb{C}$.

Consider the following annulus (see Fig. 7.3):

$$A_\zeta = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in V_+ \;\middle|\; \phi_+ \begin{pmatrix} x \\ y \end{pmatrix} = \zeta, \; \left|\frac{x}{y}\right| \geq \frac{1}{2} \right\}.$$

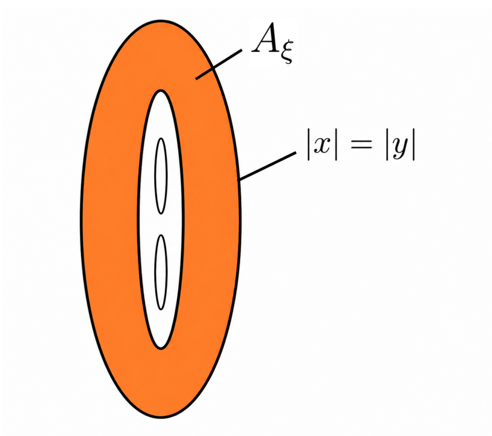

Figure 7.3: An annulus A_ξ

Recall that $\mathcal{L}$ be the leaf of foliation $\mathcal{F}^+$ going through $\varphi_+^{-1}(\xi)$. The annuli A_ξ, $F^{-1}(A_{\xi^d})$, $F^{-2}(A_{\xi^{d^2}}), \dots \subset \mathcal{L}$ and are disjoint. They all have the same modulus

$$\frac{\log 2}{2\pi}.$$

Hence, applying the proposition, we get that $\mathcal{L}$ is isomorphic to $\mathbb{C}$.

Bibliography

[1] Daniel S. Alexander and Robert L. Devaney. A century of complex dynamics. https://math.bu.edu/people/bob/papers/cent.pdf, 2014. Accessed: 2026-04-22.

[2] Michael Benedicks and Lennart Carleson. The dynamics of the hénon map. *Annals of Mathematics*, 133(1):73–169, 1991. doi: 10.2307/2944330.

[3] Pierre Fatou. Sur les substitutions rationnelles. *Bulletin de la Société Mathématique de France*, 47:161–271, 1919.

[4] Federico Aracne, Mikhail Lyubich, Curtis T. McMullen, and Mikhail Yampolsky, editors. *Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday*, volume 51 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2014. ISBN 978-0-691-14707-5.

[5] Michel Hénon. Numerical study of quadratic area-preserving mappings. *Journal of Applied Mathematics and Physics (ZAMP)*, 27:291–312, 1969. doi: 10.1007/BF01594969.

[6] Michel Hénon. A two-dimensional mapping with a strange attractor. *Communications in Mathematical Physics*, 50:69–77, 1976. doi: 10.1007/BF01608556.

[7] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain I: the global topology of dynamical space. *Publications Mathématiques de l'IHÉS*, 79: 5–46, 1994. doi: 10.1007/BF02698886. URL https://www.numdam.org/articles/10.1007/BF02698886/.

[8] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain II: projective and inductive limits of polynomials. *Publications Mathématiques de l'IHÉS*, 80:5–46, 1995. doi: 10.1007/BF02698887. URL https://www.numdam.org/articles/10.1007/BF02698887/.

[9] Gaston Julia. Mémoire sur l'itération des fonctions rationnelles. *Journal de Mathématiques Pures et Appliquées*, 1:47–245, 1918.

[10] Wikipedia contributors. Lorenz system. https://en.wikipedia.org/wiki/Lorenz_system, 2026. Accessed: 2026-04-23.

[11] John Milnor. *Dynamics in One Complex Variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 3 edition, 2006. ISBN 978-0-691-12488-5.

[12] Stephen Smale. Differentiable dynamical systems. *Bulletin of the American Mathematical Society*, 73(6):747–817, 1967. doi: 10.1090/S0002-9904-1967-11798-1.

[13] Wikipedia contributors. Solenoid (mathematics). https://en.wikipedia.org/wiki/Solenoid.

[14] D. van Dantzig. über topologisch homogene kontinua. *Fundamenta Mathematicae*, 15: 102–125, 1930.

[15] Leopold Vietoris. Über den höheren zusammenhang kompakter räume und eine klasse von zusammenhangstreuen abbildungen. *Mathematische Annalen*, 97(1):454–472, 1927. ISSN 0025-5831. doi: 10.1007/bf01447877.