Exotic families of embedded spheres

by

Joshua Drouin

B.S., Murray State University, 2017

M.S., Kansas State University, 2019

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

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$$\ker\left[\pi_k(\mathbf{Emb}_S^{C^\infty}(S^2, Z)) \to \pi_k(\mathbf{Emb}^{\mathrm{TOP}}(S^2, Z))\right]$$

may have an arbitrarily high-rank summand for a some 4-manifolds. Here, $\mathbf{Emb}_S^{C^\infty}(S^2, Z))$ represents the component to the embedding space containing a specific embedding S. This behavior is found for spheres of arbitrary self-intersection. We also establish an analogous result for homology groups.

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Approved by:

Major Professor
Dave Auckly

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Table of Contents

List of Figures

List of Tables

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Chapter 1

Introduction and Preliminaries

1.1 History

The difference between the smooth and topological categories has been a topic of extensive study for several decades. We seek to examine this difference via exotic $k-$dimensional families of embeddings of spheres. The study of 4-dimensional exotic objects began in the 1980s, via the work of Freedman and Donaldson. Freedman classified simply-connected topological 4-manifolds [13]. Donaldson proved that there are topological 4-manifolds that do not admit a smooth structure [7], and developed an invariant to distinguish smooth structures on the same topological 4-manifold [10].

Later, several authors constructed exotic embeddings of surfaces in 4-manifolds ([27], [11], [22], [5]). Recently, authors have studied exotic families of embeddings ([23], [2], [3]). Let $\mathrm{Emb}^{C^\infty}(S^2, X)$, and $\mathrm{Emb}^{\mathrm{Top}}(S^2, X)$, denote the space of smooth, and respectively topological, embeddings. Exotic embeddings are detected by the kernel of the map $H_0(\mathrm{Emb}^{C^\infty}(S^2, X))$ to $H_0(\mathrm{Emb}^{\mathrm{Top}}(S^2, X))$. This naturally leads to the question of the kernel on higher homology and homotopy groups. This kernel depends on the specific component of the space of embeddings. Auckly and Ruberman [2] have shown that the kernels for both homology and homotopy are large for components of spheres of self-intersection $(+1)$. We have generalized these results to show that the kernels have large rank summands in components with arbitrary self-intersection.

A spherical family that is homologically non-trivial will automatically be homotopically non-trivial. This means it is natural consider spherical families first. The homotopy group of a spaces is constructed from maps of $k-$dimensional spheres into the space, and a map between spaces induces a map between homotopy groups.

Exotic families are defined recursively, starting from an exotic embedding of surfaces. To pass from a $k-$dimensional family to a $(k+1)-$dimensional family, one uses a process called *stabilization*. Often, exotic objects become smoothly equivalent after stabilization. If a surface is connected, the one-handle surgery is equivalent to taking the connected sum with $S^1 \times S^1$. Following the work of Quinn and Perron, it was shown that closed surfaces with the same genus representing the same homology class in a simply-connected manifold always become smoothly isotopic after some number of two-handle surgeries ([28], [29]). Often, exotic surfaces in a 4-manifold become equivalent after one two-handle surgery. If a 4-manifold is simply-connected, two-handle surgery is equivalent to taking a connect-sum with a copy of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.

1.2 Our Results

Theorem 1.2.1. *Given any rank r and non-negative integers k, p, and n with $0 \leq k < p$ and $k \equiv p \mod 2$, there exists a manifold $Z_{r,p}$ and maps:*

$$j_{i,n}^k : S^k \to Emb^{C^\infty}(S^2, Z_{r,p} \setminus N(T))$$

where $N(T)$ is the neighborhood of an embedded torus and the following conditions hold:

i) $j_{i,n}^k(v)$ *has self-intersection n, for $v \in S^k$.*

ii) the homotopy classes of $[j_{i,n}^k]$ in $\pi_k(Emb^{TOP}(S^2, Z_{r,p} \setminus N(T)))$ is trivial.

iii) there is a map $\Phi : Emb^{C^\infty}(S^2, Z_{r,p} \setminus N(T)) \to Emb^{C^\infty}(S^2, Z_{r,p}\#(S^2 \times S^2))$ such that the homotopy class of $[\Phi \circ j_{i,n}^k]$ is trivial in $\pi_k(Emb^{C^\infty}(S^2, Z_{r,p}\#(S^2 \times S^2)))$.

iv) any pair of families of embeddings $j_{i,n}^k$ and $j_{\ell,n}^k$ are psuedoisotopic.

v) the homology classes of $[j_{i,n}^k]$ in $H_k(Emb^{C^\infty}(S^2, Z_{r,p}))$ are independent.

It follows that the kernel of $H_k(Emb^{C^\infty}(S^2, Z_{r,p})) \to H_k(Emb^{TOP}(S^2, Z_{r,p}))$ admits a rank r summand that is in the image of the Hurewicz map $\pi_k(Emb^{C^\infty}(S^2, Z_{r,p})) \to H_k(Emb^{C^\infty}(S^2, Z_{r,p}))$.

1.3 Conventions

There is a notable amount of notation throughout this paper. This section can be used as a reference to quickly recall the meanings and conventions we have used as well as establishing conditions that will be present in most constructions.

First, unless otherwise stated, we assume that all manifolds are smooth, oriented, and compact. Additionally, to work with gauge theory invariants on a manifold, we denote β as the choice of chosen homology orientation. While many manifolds are discussed, we use the letter Z to denote an arbitrary 4-manifold, or $\mathcal{Z}$ for a family of 4-manifolds, that we wish to examine. Additionally, inside our manifolds Z we will have an embedded torus T of self-intersection 0. This embedded torus allows to carefully perform submanifold sum of our 4-manifolds.

Second, we maintain the following convention to differentiate between families of diffeomorphisms, families of embeddings, families of spaces, and families of isotopies. Families of diffeomorphisms, denoted by α^k are elements in $\pi_k(\mathrm{Diff}(Z))$. This superscript k is consistently used to represent the level of homotopy that we are examining, whether for families of diffeomorphisms or families of embeddings. Families of embeddings, denoted by $j^k_{i,n}$ are elements in $\pi_k(\mathrm{Emb}(S^2, Z))$, and via an extension of the adjoint we will typically view them as:

$$\widehat{j}^k_{i,n} : S^k \times S^2 \to S^k \times Z$$

The hat notation is consistently used in families of embeddings to represent this extension of the adjoint. If $i = X$ or $i = Y$ it will refer to families created from the initial embeddings into the manifolds X and Y. We refer to this as the base case, and often will use the letter a to highlight this base case, instead of i. The letter i will be used when we refer to a collection of embeddings $i = 1, \ldots, r$. The subscript n refers to the self-intersection number of these embedded spheres.

We also maintain consistent subscripts and superscripts for specific 4-manifolds. In the statement of Theorem 1.2.1 we have the manifolds $Z_{r,p}$. The subscripts r and p are used to record that for the manifolds $Z_{r,p}$ the kernel of the map from the kth homotopy group of space of smooth embeddings into the kth homotopy group of the space of topological embeddings will have a rank r summand for all $k \leq p$ where k and p have the same parity. Contrast this subscript-only notation with the more prevalent $Z^{k,r}$ superscript-only notation. The r still represents the rank r summand in the kernel, and now the k represents the level of homotopy that we are examining. To help distinguish between manifolds that involve the parity condition or not, we will use the subscript-only notation and superscript-only notation, respectively.

Connect sum and submanifold sum are used throughout, typically over the torus T

mentioned earlier. We develop the following convention for examining maps of summed manifolds. Maps may be defined on manifolds using a certain name and then extended across the sum via identity. However, to simplify notation we will continue to refer to the extended map by its original name. For example, for manifolds X, Y, and Z, and some map $f : X \to Y$, we would continue denote the extension $f\#1_Z : X\#Z \to Y\#Z$ as just f.

Additionally, we perform submanifold sum recursively for our main constructions. A larger manifold may be created by summing together numerous smaller copies. Moreover, we may be interested in how maps into the smaller copies impact maps into the larger sum, notably how a map into the **last** summed copy influences the rest of the sum. To distinguish this **last** summed copy we will use the tilde notation. For example, let $j : S^2 \to Z$ be an embedding of sphere. Denote the manifold $Z^k = k\#_T Z$ as k copies of Z summed together. Into each summed portion there is a copy of the embedding j, and to emphasize the **last** summed copy, we denote this embedding by $\tilde{j}$.

Chapter 2

Background Information

This chapter outlines the preliminary material, including standard definitions as well as a quick summary of work by Auckly and Ruberman to create families of diffeomorphisms that are crucial for our construction. In order to carefully construct the desired families and invariants, a substantial amount of notation is required. To assist with this, table 2.1 contains the most important notation used throughout the entire paper.

2.1 Definitions

A basic invariant of embeddings is a family of manifolds created by submanifold sum. A submanifold sum is a generalization of a connected sum. Instead of joining two manifolds along a tubular neighborhood of points, we can join two manifolds along a tubular neighborhood of submanifolds.

Definition 2.1.1. Let X and Y be two $n-$dimensional manifolds each with an embedded $k-$dimensional submanifold F. Denote the normal bundles of F inside X and Y by $N_X(F)$ and $N_Y(F)$, respectively. Suppose τ is a fibre orientation reversing isomorphism between these normal bundles. Then we define the submanifold sum of X and Y over F by $X\#_F Y := ((X \setminus F) \sqcup (Y \setminus F)) / \sim$, under the equivalence relation that $v \sim \tau \left(v \, ||v||^{-2}\right)$.

An important operation we will utilize in the application of our invariant, is called blow-

Notation	Meaning	Page
$E(2)$	$K3$, the second elliptic surface	10
$N(T)$	Neighborhood of embedded self-intersection 0 torus T	10
N^1	First of three nuclei of $E(2)$. It contains $N(T)$	10
$\cdot\#_T\cdot$	Submanifold sum over the torus T	6
X	$E(2)\#\mathbb{CP}^2$, has non-zero Donaldson invariants	10
Y	$3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$, Cork twist of X	10
ψ	$Y \to X$, a homeomorphism	11
φ	$Y\#\mathbb{CP}^2 \to X\#\mathbb{CP}^2$, a diffeomorphism	11
W	Akbulut cork, used to create Y	10
j_a	Embeddings of S^2 into $a = X$ or $a = Y$	12
Z	$Y\#\mathbb{CP}^{\#}2\overline{\mathbb{CP}}^2$, with embedded $(+1)$ sphere	12
G^Z	Topological isotopy on Z that maps j_Y to j_X	12
F^Z	Stable-Smooth isotopy on $Z\#(S^2 \times S^2)$ that maps j_Y to j_X	12
K^Z	Pseudoisotopy on Z that maps j_Y to j_X	12
X^r	$X\#_T E(2)^{(\#_T)^{r+1}}\#\overline{\mathbb{CP}}^2$	14
N^2 and N^3	Second and third nuclei of $E(2)$, used for 2-log transforms	10
X_i^r	Manifolds created by 2-log transforms of nuclei in X^r	15
Y^r	$(4r+1)\mathbb{CP}^2\#(20r+34)\overline{\mathbb{CP}}^2$, Cork twist of X^r	15
ψ_i	$Y^r \to X_i^r$, collection of homeomorphisms	15
φ_i	$Y^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2 \to X_i^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, collection of diffeomorphisms	15
V	Special Manifold used for Parity Condition	16
$Z^{0,r}$	$Y^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, with embedded $(+1)$ sphere	15
$Z^{k,r}$	$Z^{k,r} := Z^{k-1,r}\#_T Z^{0,r}$	15
α_i^k	Families of Diffeomorphisms	16
G_i^k	Family of topological isotopies on $Z^{k,r}$	16
F_i^k	Family of stable-smooth isotopies on $Z^{k,r}\#(S^2 \times S^2)$	16
K_i^k	Family of pseudoisotopies on $Z^{k,r}$	16
$Z_{r,p}$	Created by summing $Z^{1,r}$ or $Z^{2,r}$ to copies of V, based upon parity.	16
$\widehat{j}_{i,n}^k$	Collection of Families of Embeddings	28
$\widetilde{j_{i,n}}$	The embedding of S^2 into X_i^r in the **last** $Z^{0,r}$ summed copy of $Z^{k+1,r}$	28
$\widetilde{G^Z}_i$	Collection of topological isotopies in the **last** summed $Z^{0,r}$	28
$\widetilde{F_i^Z}$	Collection of stable-smooth isotopies in the **last** summed $Z^{0,r}$	28
$\widetilde{K_i^Z}$	Collection of pseudoisotopies in the **last** summed $Z^{0,r}$	28
$\widehat{J}_{\overline{\mathbb{CP}}^2}$	$\Sigma^k \times S^2 \to \Sigma^k \times n\overline{\mathbb{CP}}^2$, Standard Family of Embeddings	31
$\mathcal{Z}_{\widehat{j}_{i,n}^k}$	Family of Spaces associated to Family of Embeddings	50
$D^{H_k(\mathrm{Emb})}$	Invariant for Families of Embeddings	50
$\mathbb{D}$	Konno's Invariant for Families of Spaces	51
$D^{H_k(\mathrm{Diff})}$	Auckly-Ruberman Invariant for Families of Diffeomorphisms	62

Table 2.1: Table of Notation

up surgery. First introduced by [3], it is a combination of blow-ups, blow-downs, and surgery.

Definition 2.1.2. For X, a $4-$manifold, to blow-up a point $p \in X$ is the operation where we remove a small neighborhood around p and glue in $\overline{\mathbb{CP}}^2$, replacing the point p with $\overline{\mathbb{CP}}^1$ the $2-$sphere of self-intersection 1. If this point p lies on a surface in X, this operation reduces the self-intersection of the surface by 1.

The blow-up operation is essentially the operation that glues on $X \rightarrow X\#\overline{\mathbb{CP}}^2$, assuming the point is chosen generically. Conversely to replacing a point with a $2-$sphere, the blow-down operation "crushes" such a $2-$sphere back down to a point.

Definition 2.1.3. For X, a $4-$manifold, we can blow-down a $2-$sphere of self-intersection ± 1 by collapsing the entire sphere to a point. We denote this operation as X/S.

The final component to address before we define blow-up surgery is surgery. The following definition is generalized to an $n-$manifold, but in this paper we will largely be interested in $4-$manifolds and $2-$spheres.

Definition 2.1.4. For X, an $n-$manifold, with embedded $k-$sphere, we can surger out this sphere by first cutting out the interior of $S^k \times D^{n-k}$ and gluing in a $D^{k+1} \times S^{n-k-1}$.

Finally, with the preliminary details defined, we can put them all together to define blow-up surgery. Since this focus of this paper pertains to $4-$manifolds and $2-$spheres, our definition will not be generalized to higher dimensions.

Definition 2.1.5. Given a $4-$manifold X with an embedded $2-$sphere S of some self-intersection n, we can perform blow-up surgery by first blowing-up n points on S and surgering out the resulting sphere of self-intersection 0.

Definition 2.1.6. A homology orientation β for a 4-manifold X is:

$$\beta \in \Lambda^{\text{Top}} H^0(X) \otimes \Lambda^{\text{Top}} H^2_+(X) \otimes \Lambda^{\text{Top}} (H^1(X))^* \setminus \{0\}$$

Where $H^2_+(X)$ is the space of self-dual harmonic 2-forms on X [8].

2.2 Spaces of Embeddings

With these important tools defined, let us recall that this paper focuses on embeddings spaces. The following section defines the important embedding spaces that we will work with. An *embedding* of a space F into a space X is a homeomorphism onto its image. An embedding $j : F \hookrightarrow X$ is *locally flat* if for every x_0, an element in F, there exists a parameterization $\psi : \mathbb{R}^k \to F$ with $\psi(0) = x_0$, and a chart $\varphi : U \subset X \to \mathbb{R}^n$ with $\varphi \circ j \circ \psi(x) = (x, 0)$. A *smooth embedding* is a topological embedding between manifolds so that the derivative is injective at each point.

For simplicity, we will assume all of these spaces are oriented. We denote the space of embeddings of F to X by $\mathrm{Emb}^{C^0}(F, X)$, the space of locally flat embeddings by $\mathrm{Emb}^{\mathrm{TOP}}(F, X)$, and the space of smooth embeddings by $\mathrm{Emb}^{C^\infty}(F, X)$. It is easy to see that these spaces are distinct. Indeed, let $W_K(n)$ denote the n-trace of a knot K, that is, $W_K(n)$ is result of attaching a $n-$framed $2-$handle along K in the boundary of a $4-$disc. For the $1-$trace of the right-handed trefoil, $W_{T_{2,3}}(1)$, there is an element of $\mathrm{Emb}^{C^0}(S^2, W_{T_{2,3}}(1))$ representing a generator of the second the homology: map one hemisphere to the core of the $2-$handle and map the other hemisphere to the cone on the knot. However, there is not an element of $\mathrm{Emb}^{\mathrm{TOP}}(S^2, W_{T_{2,3}}(1))$ representing a generator of the second the homology. Indeed, according to [26, Theorem 1.2] if a homology class x in a manifold N is represented by a locally flat sphere, then

$$KS(N, \partial N) + \mu(\partial N) = \frac{1}{8}[\sigma(N) - x \cdot x] \pmod 2$$

In this case, one sees that $KS(W_{T_{2,3}}(1), \partial W_{T_{2,3}}(1)) = 0$, $\mu(\partial W_{T_{2,3}}(1)) = 1$, $\sigma(W_{T_{2,3}}(1)) = 1$, and $x \cdot x = 1$. So it is clear that there is no e0lement of $\mathrm{Emb}^{\mathrm{TOP}}(S^2, W_{T_{2,3}}(1))$ representing a generator of the second the homology, showing that it is distinct from $\mathrm{Emb}^{C^0}(S^2, W_{T_{2,3}}(1))$.

To examine how $\mathrm{Emb}^{\mathrm{TOP}}(F, X)$ is distinct from $\mathrm{Emb}^{C^\infty}(F, X)$, consider the $0-$Whitehead double of the trefoil, $D_0(T_{2,3})$. According to [12, Theorem 1.14], $D_0(T_{2,3})$ has a generator in the $1-$trace represented by a locally flat sphere. However, according to [15, Theorem 2.1] the $0-$twisted Whitehead Double of the Trefoil has kinkiness $k = (0, 1)$, and, since this is nonzero, it cannot be smoothly slice. In fact, Gompf proves more, he shows that the $0-$Whitehead Double is not $1-$Shake Slice, i.e. there is no element of $\mathrm{Emb}^{C^\infty}(S^2, D_0(T_{2,3}))$ representing a generator of the second the homology. Clearly any smooth embedding is locally flat, by the tubular neighborhood theorem, and since we are only considering continuous embeddings: $\mathrm{Emb}^{C^\infty}(F,X) \subset \mathrm{Emb}^{\mathrm{TOP}}(F,X) \subset \mathrm{Emb}^{C^0}(F,X)$. Moreover, if we do not need to distinguish between these three embedding spaces we will use the notation $\mathrm{Emb}^{\cdot}(F,X)$ to represent any of these three. Finally, we can endow all of these embeddings spaces with the compact-open topology.

2.3 Families of Diffeomorphisms

The focus of this paper is on families of embedding, but in order to properly define them, we must first establish families of diffeomorphisms. We will now summarize the construction of families of diffeomorphisms by Auckly and Ruberman in [2] starting with a base case of 2 embedded spheres before generalizing to higher rank r.

By the work of Akbulut [1], Gompf and Stipicz [16], and many others, we know that there are two smooth 4-manifolds $X := E(2)\#\overline{\mathbb{CP}}^2$, an embedded Akbulut cork W, and a cork-twist of X along W we denote as Y. Where $E(2) \cong K3$ is the second elliptic surface [17]. An important detail we utilize throughout is that $E(2)$ can be fibred in three distinct ways, such that the fibres and nuclei are disjoint [19]. Let us denote these nuclei by N^1, N^2, and N^3. Embedded in the first nucleus is the neighborhood $N(T)$ of our essential torus T. The latter two nuclei will be used later to perform 2-log transforms to further generalize our manifolds X and Y. A final important condition that we will utilize in generalizing these manifolds to higher rank, is that the Akbulut cork W is disjoint from the nucleus N^1 [4].

These manifolds X and Y have a few notable properties, X has non-trivial Donaldson invariants and Y has trivial Donaldson invariants. Moreover, X and Y are topologically equivalent via a homeomorphism ψ, and are smoothly equivalent after one external stabilization via a diffeomorphism φ. That is:

$$Y \xrightarrow[C^0]{\psi} X \qquad \text{and} \qquad Y^+ \xrightarrow[C^\infty]{\varphi} X^+$$

where $X^+ = X\#\mathbb{CP}^2$ and $Y^+ = Y\#\mathbb{CP}^2$. One important detail to note is that we assume in these maps, as well as any maps related to X and Y going forward, is that they are identity when restricted to $N(T)$, a neighborhood around our essential torus T. Among other operations, this notably allows us to perform connect-sums carefully without worrying about the influence of the gluing operation.

Remark. An important point about our morphisms ψ and φ is that they behave similarly on the homology of the manifolds Y and X. Precisely, they should induce the same map on homology when we sum the identity map of $\mathbb{CP}^2$ onto ψ, that is:

$$H_*(\varphi) = H_*(\psi\#1_{\mathbb{CP}^2})$$

This can be seen in [2, Proposition 3.3 and 3.6] which is a generalization of Wall's work [32]. Wall's results are particularly for closed manifolds, but also apply to 4-manifolds with the boundary of a homology sphere.

Since we are interested in embeddings of spheres into these 4-manifolds, let us consider the two following ways to embed an S^2 into Y^+. The $S^2 \cong \mathbb{CP}^1$ is viewed as the hyperplane section of $\mathbb{CP}^2$. One embedding, j_Y, into Y^+ is a modification standard embedding of the hyperplane section into the $\mathbb{CP}^2$ summand of Y^+ that drags a "finger" into $N(T)$, but still disjoint from T. The second embedding, j_X, is defined by $j_X = \varphi^{-1}(\psi\#1_{\mathbb{CP}^2}) \circ j_Y$.

$$\begin{tikzcd}
 & S^2 \arrow[dl, bend right, "j_Y"'] \arrow[drr, bend left, "j_X"] \arrow[d, hook] & & \\
Y^+ \arrow[rr, bend right, "\psi\#1_{\mathbb{CP}^2}"'] & (\mathbb{CP}^2)^\times \arrow[l, hook'] \arrow[r, hook] & X^+ \arrow[r, "\varphi^{-1}"'] & Y^+
\end{tikzcd}$$

Where $(\mathbb{CP}^2)^\times$ means a punctured $\mathbb{CP}^2$. Examining the different ways these spheres can embedded is the basic idea going forward. To better examine the embedding classes, let us define the manifold that will serve as the base of our recursive process. Let $Z := Y\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. By [4], we know that there is a topological isotopy G^Z, a stable-smooth isotopy F^Z, and a pseudoisotopy K^Z that are identity at time 0, and at time 1 map the spheres j_Y to j_X. Isotopies G^Z and F^Z are illustrated here, and K^Z is discussed later.

$$\begin{tikzcd}
G^Z : I \times Z \arrow[rr, "C^\infty"] \arrow[dr] & & I \times Z \arrow[dl] \\
& I & \\
F^Z : I \times Z\#(S^2 \times S^2) \arrow[rr, "C^\infty"] \arrow[ur] & & I \times Z\#(S^2 \times S^2) \arrow[ul]
\end{tikzcd}$$

These isotopies have one important condition. Denote the exceptional divisors of $2\overline{\mathbb{CP}}^2$ as E_i for $i = 1, 2$ and the exceptional divisor of $\mathbb{CP}^2$ as H. When G^Z and F^Z are restricted to $I \times E_i$ we want this to also be identity. Using these exceptional divisors, we can create diffeomorphisms that are complex conjugation in a neighborhood of $H + E_1 + E_2$ and in a neighborhood of E_1, we denote these as $\rho^{H+E_1+E_2}$ and ρ^{E_1}. Composing these diffeomorphisms, we create a diffeomorphism:

$$f = \rho^{E_1}\rho^{H+E_1+E_2}$$

This diffeomorphism f is now combined with the diffeomorphism φ from before, as well as identity maps on X and Y to create the the fundamental building block for our recursive construction, denoted as α'.

$$\alpha' := (1_X\#f)^\varphi(1_Y\#f^{-1})$$

Where the notation h^φ is conjugation by φ, that is $h^\varphi = \varphi^- h\varphi$. Note that under the G^Z map we can see that α' is topologically isotopic to the identity, and under the F^Z map it is smoothly isotopic to the identity after a single stabilization. Thus, we may finally define the base case diffeomorphism for the recursion process: denote $\alpha^0 : S^0 \to \mathrm{Diff}(Z)$ where $\alpha^0(1) = 1_Z$ and $\alpha^0(-1) = \alpha'$ where we view $S^0 = \{-1, 1\}$. In addition, by [4, Theorem C], after stabilization the diffeomorphism $\alpha^0 \# 1_{S^2 \times S^2}$ is smoothly isotopic to identity via a map we denote as F^0. This F^0 serves as the base case of our families of isotopies. From these base cases, we can define our tower of manifolds Z^{k+1} by: $Z^{k+1} := Z^k \#_T Z^0$ where $Z^0 := Z$; as well as the families of diffeomorphims α, families of topological isotopies G, and families of stable-smooth isotopies F by using the commutators:

$$\alpha^{k+1} := \left[F^k, 1_{Z^k} \#_T \alpha^0\right]$$

$$G^{k+1} := \left[F^k, 1_{Z^k} \#_T G^Z\right]$$

$$F^{k+1} := \left[F^k, 1_{Z^k} \#_T F^Z\right]$$

Where the notation $\#_T$ means we perform submanifold sum over the neighborhood of the embedded torus. With these primary families defined, we move on to a fourth family that is slightly different. Instead of a topological or smooth isotopy, this fourth family involves psuedoisotopy. A psuedoisotopy, originally attributed to Cerf ([6]) can be thought of as a concordance.

Definition 2.3.1. Given an n-dimensional smooth manifold X, a psuedoisotopy K, of X, is a diffeomorphism $K : X \times I \to X \times I$ such that $K_{X \times 0}$ and $K_{(\partial X) \times I}$ are both the identity.

Above, we were interested in how diffeomorphisms, or families of diffeomorphisms, may be isotopic. The concept of pseudoisotopy plays quite nicely with these diffeomorphisms. Two diffeormorphisms $\alpha_0, \alpha_1 \in \mathrm{Diff}(Z)$ are said to be pseudoisotopic if there is a pseudoisotopy $K : I \times Z \to I \times Z$ such that $K(0, x) = \alpha_0(X)$ and $K(1, x) = \alpha_1(X)$.

Definition 2.3.2. An element $\alpha \in \pi_k(\mathrm{Diff}(Z))$ is pseudoisotopic to identity, that is, it pseudocontracts if there is diffeomorphism $K : I \times S^k \times Z \to I \times S^k \times Z$ such that when restricted to $S^k \times Z$ maps $(\theta, x) \to (\theta, \alpha(\theta)(x))$.

Thus, when we examine families of diffeomorphisms, we may define a new family of diffeomorphisms similar to how we defined G^Z and F^Z above. Much like before, Auckly and

Ruberman utilize the commutator, but emphasize that need to be more particular here. So, let us define a diffeomorphism:

$$K^{k+1} : S^0 \times I \times S^{k+1} \times Z_{k+1} \to \left(S^0 \times I \times S^{k+1}\right) \tilde{\times}_{\alpha^{k+1}} Z_{k+1}$$

by $K^{k+1}(\epsilon, r, \theta, z) = \left(K_D^{k+1}(\epsilon, r, \theta, z), K_Z^{k+1}(\epsilon, r, \theta, z)\right)$ where the maps:

$$K_D^{k+1}(\epsilon, r, \theta, z) = \begin{cases} (\epsilon, \frac{1}{2}[F^k, 1_{Z^k} \#_T K]_T(2r-1, \theta, z) + \frac{1}{2}, \theta), & r \geq \frac{1}{2} \text{ and } \epsilon = -1 \\ (\epsilon, r, \theta), & \text{else} \end{cases}$$

$$K_Z^{k+1}(\epsilon, r, \theta, z) = \begin{cases} \frac{1}{2}[F^k, 1_{Z^k} \#_Z K]_T(2r-1, \theta, z), & r \geq \frac{1}{2} \text{ and } \epsilon = -1 \\ z, & \text{else} \end{cases}$$

Where the map K is a pseudoisotopy that induces a self-diffeomorphism on $I \times S^k \times Z$. For consistency sake, we will shorten the notation of K^{k+1} as: $K^{k+1} = \left[F^k, 1_{Z^k} \#_T K^Z\right]$. With this collection of four families of maps, α, G, F, and K defined, we refer to these four families, as well as the family of spaces Z^k, as the rank 1 case.

Recall that we are interested in families of any rank r. Utilizing the above constructions, which gives us the rank one case, we now seek to generalize to higher ranks. The full construction and properties are given in [2, Proposition 3.6], but the process is summarized here. The first change is to the underlying manifolds X and Y. Recall, these were the topologically, but not smoothly, equivalent manifolds that became diffeomorphic after one stabilization. For an integer r, we create the manifold X^r by starting with the base manifold $X := E(2) \# \overline{\mathbb{CP}}^2$ and, similar to the recursive construction of Z^k, fibre summing over the embedded self-intersection 0 torus inside $E(2)$ with additional copies of $E(2)$.

$$X^r := X \#_T E(2)^{(\#_T)^{r+1}} \# \overline{\mathbb{CP}}^2$$

Similar to the rank 1 case constructed above, we now seek to have a Y^r that are homeomorphic to X^r and diffeomorphic after a stabilization. Recall that Y was defined to be obtained by cork twisting of X along W. We will adapt this cork twisting construction of Y to the generalized version of Y^r. However, we must be careful in this construction.

In each $E(2)$ summand of X^r there are three nuclei $N^{i,1}$, $N^{i,2}$, and $N^{i,3}$ for $i = 0, \ldots, r$ ([17]). Using these nuclei, we perform log transforms of multiplicity 2 on X^r along fibres in the $N^{i,2}$ and $N^{i,3}$. A log transforms of multiplicity 2 means that in each nucleus we replace a $D^2 \times T^2$ with a second copy of $D^2 \times T^2$ such that the new meridian wraps twice around the original meridian. We denote the collection of manifolds resulting from these transforms as X^r_i, again for $i = 0, \ldots, r$. By [2, Proposition 3.6], we know there is a homeomorphism from $X^r \to X^r_i$ and a diffeomorphism from $X^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2 \to X^r_i \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ that preserve the embedded torus T, which is the generic fibre of the $N^{i,1}$ nucleus, and preserve the embedded cork W. Thus, we define Y^r to be the result of cork twisting X^r along W. This construction allows us to extend to desired homeomorphism $\psi_i : Y^r \to X^r_i$ and diffeomorphism from $\varphi_i : Y^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2 \to X^r_i \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$. Similar to how we constructed α' in the rank 1 case, this extension of the morphisms allows us to define an $(\alpha^i)'$ in the general rank r case.

$$\alpha'_i := (1_{X^r_i} \# f)^{\varphi_i} (1_{Y^r} \# f^{-1})$$

Finally, we can define the generalized spaces and families of diffeomorphisms for these higher ranks as follows. First, denote $Z^{0,r} := Y^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$, and, similar to before, denote $Z^{k+1,r} := Z^{k,r} \#_T Z^{0,r}$. For the base case of α we define $\alpha^0_i(0) = \alpha'_i$ and $\alpha^0_i(1) = 1$ allowing us to define the rest of the tower in the analogous way:

$$\alpha^{k+1}_i := \left[F^k_i, 1_{Z^{k,r}} \#_T \alpha^0_i\right]$$

$$G^{k+1}_i := \left[F^k_i, 1_{Z^{k,r}} \#_T G^Z\right]$$

$$F^{k+1}_i := \left[F^k_i, 1_{Z^{k,r}} \#_T F^Z\right]$$

$$K^{k+1}_i := \left[F^k_i, 1_{Z^{k,r}} \#_T K^Z\right]$$

In addition to higher ranks, an interesting fact we can also examine is that these families of spaces $Z^{k,r}$ can be modified to have rank r summands in the kernels of other homotopy groups, not just the kth group. Stated another way, for $k \leq p$ and $k \equiv p \mod 2$, the kernels of the homotopy groups π_k will all contain rank r summands.

As before, the full details for families of diffeomorphims are present in [2, Proposition 3.4], but are summarized here before we generalize the process to families of embeddings. The main ingredient is a special 4-manifold V defined by Auckly and Ruberman [2] which is a slight alteration of a construction by Gompf [18]. Notably, V has $b_2^+ = 9$, $b_2^- = 45$, and contains two disjoint copies of the $N(2)$ nucleus that each contain a copy of a torus T. Denote these tori as T_1 and T_2. If $\mathbf{c}_V \in \widehat{C}_V$ is Poincaré dual to $\sigma_1 + \sigma_2$ where σ_1 and σ_2 are sections in the nuclues $N(2)$, then $\mathbf{c}_V[T_1] = \mathbf{c}_V[T_2] \equiv 1 \mod 2$, and by [2, Proposition 3.5 (7)] V has non-zero Donaldson invariants, specifically $q_{\mathbf{c}_V,V} = -8$. Furthermore, for, let $\mathbf{c}_V \in \widehat{C}_V$ be as above, let $\mathbf{c}_Z \in \widehat{C}^k_Z$, and define $\mathbf{c}_{V,Z} \in \hat{C}^k_{Z\#_T V}$ such that when restricted to $Z \setminus T$, respectively $V \setminus T$, the class $\mathbf{c}_{V,Z}$ agrees with $\mathbf{c}_j$, respectively $\mathbf{c}_V$. We have by [2, Proposition 3.5 (6)] that the Donaldson invariants of the sum $0Z\#_T V$ are the product of the Donaldson invariants of Z and V, that is:

$$q_{\mathbf{c}_{V,Z},Z\#_T V} = q_{\mathbf{c}_V,V} q_{\mathbf{c}_Z,Z} = -8 q_{\mathbf{c}_Z,Z}$$

Definition 2.3.3. Using V, we are interested in constructing a manifold $Z_{r,p}$ from the established manifold $Z^{k,r}$. The argument depends upon whether p is even or odd, but they are very similar. When p is odd we set $q_o := (p-1)/2$ and when p is even we set $q_e := (p-2)/2$. We then define our manifolds in the odd and even cases as follows:

$$\underset{\text{Odd Case}}{Z_{r,p} := Z^{1,r}\#_T V^{(\#_T)^{q_o}}} \qquad\qquad \underset{\text{Even Case}}{Z_{r,p} := Z^{2,r}\#_T V^{(\#_T)^{q_e}}}$$

This summing allows us to utilize a final interesting property of V. By [33] and [2] we have that there is an r_0 such that for any $r \geq r_0$ that sum $V\#_T E(2r)$ is *almost completely* *decomposable.*

Definition 2.3.4. A manifold is completely decomposable if it is diffeomorphic to a connected sum of copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

Definition 2.3.5. A manifold X is almost completely decomposable if $X\#\mathbb{CP}^2$ is completely decomposable.

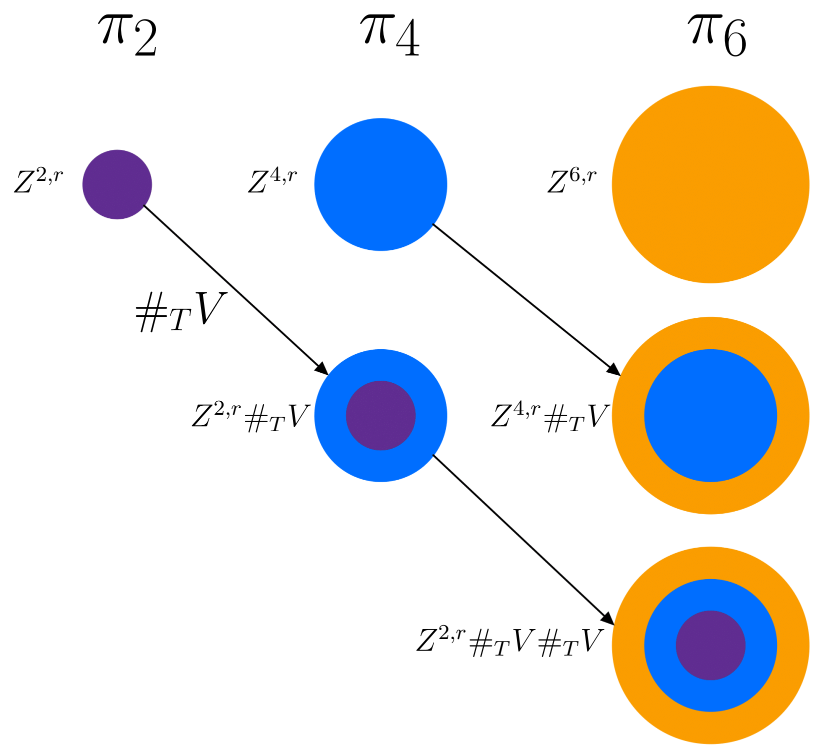


Figure 2.1: The Summing of V Yields Appropriate Kernel in Higher Homotopy.

In either the even or odd case, for sufficiently large r, this property of $V\#_T E(2r)$ and our construction of $Z^{k,r}$ allows us to define a sequence of diffeomorphisms. In the odd case we have that:

$$Z_{r,p} := Z^{1,r}\#_T V^{(\#_T)^{q_o}} \cong Z^{3,r}\#_T V^{(\#_T)^{q_o-1}} \cong \cdots = Z^{2\ell+1,r}\#_T V^{(\#_T)^{q_o-\ell}}$$

and we have the analogous equality in the even case. Formally explained in [2, Proof of Theorem 1.1], the intuitive idea is as follows. Start with a manifold that has a $\mathbb{Z}^r$ summand in the kernel of π_k, summing to this manifold a copy of V results in manifold that still has a $\mathbb{Z}^r$ summand in the kernel of π_k but now is diffeomorphic to the manifolds that has a $\mathbb{Z}^r$ summand in the kernel of π_{k+2}.

This fibre sum of copies of V can be visualized in Figure 2.1. Each column represents different levels of homotopy groups, and the top rows represent the kernel in homotopy of the base manifold $Z^{k,r}$, and rows represent summing on copies of V. Each concentric ring represents an additional summation. For example, in row 1 column 1, we have a black circle that represents the kernel of the manifold $Z^{2,r}$, and in row 1 column 2, we have a blue circle that represents the kernel of manifold $Z^{4,r}$. If we sum a copy of V to $Z^{2,r}$, we now have the picture in row 2 column 2. We have augmented the manifold $Z^{2,r}$ now that its π_4 kernel also contains a $\mathbb{Z}^r$ summand, just like the kernel of π_4 for $Z^{4,r}$. We can sum an additional copy of V to $Z^{2,r}$ to receive the picture in row 3 column 3. We have augmented the manifold $Z^{2,r}$ now that its π_6 kernel also contains a $\mathbb{Z}^r$ summand, just like the kernel of π_6 for $Z^{6,r}$.

Chapter 3

Construction of Examples

3.1 Embellishing the Spheres

In Section 2.3, we briefly outline the construction for interesting families of diffeomorphisms. These diffeomorphisms form the foundation, as now we will define interesting families of embeddings using the diffeomorphisms. To make this construction more clear, we will mirror the construction above for families of diffeomorphisms. We will start with the base case when there are two embedded spheres, before generalizing to a collection.

Recall in the base case we are examining two spheres being embedded into the manifold $Z = Y\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ with self-intersection $(+1)$. Going forward, we are not only interested in $(+1)$ spheres, but in spheres of non-negative self-intersection n. These (n) spheres are carefully constructed by using the $(+1)$ spheres and a process called *Embellishments* developed by Auckly, Kim, Melvin, and Ruberman in [3]. For any non-negative self-intersection (n), they constructed infinite collections of spheres that were topologically equivalent, but not smoothly equivalent, as detected by an operation called *blow-up/blow-down*. They refer to the construction as embellishing a sphere. Additionally, they noticed that by changing the orientation of the ambient manifold, they could construct spheres of negative self-intersection as well. We are adapting this construction to families of spheres. The full process is outlined in [3], and we will summarize it here.

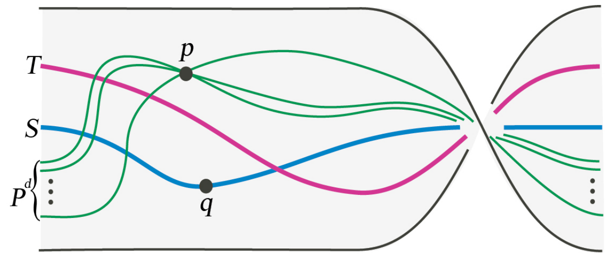


Figure 3.1: Configuration of projective lines in $(\mathbb{CP}^2)^\circ$ in Z, taken from [3]

In our manifold Z, fix a pair of base points, p and q, and isotope an embedded $(+1)$ sphere S so that q is on S but p is not. Take a tubular neighborhood U of S that contains both p and q. Within U choose a line T that is away from p and q as well as a pencil P^d of $d \geq 1$ number of lines in U that are disjoint from q and disjoint from $T \cap S$. Figure 3.1 is an illustration of this construction taken from [3]. To construct a sphere of self-intersection $n > 1$, denoted by S_n, we simply need to blow-up the points p and q, glue, and then smooth the configuration. More formally, to create a sphere of self-intersection $(2d)$, we take $S \cup P^d$, blow-up at p and q, and then smooth. To create a sphere of self-intersection $(2d+1)$, we take $T \cup P^d$, blow-up at p and q, and then smooth. As an example, when $d = 2$, the picture, also taken from [3], would appear as in figure 3.2.

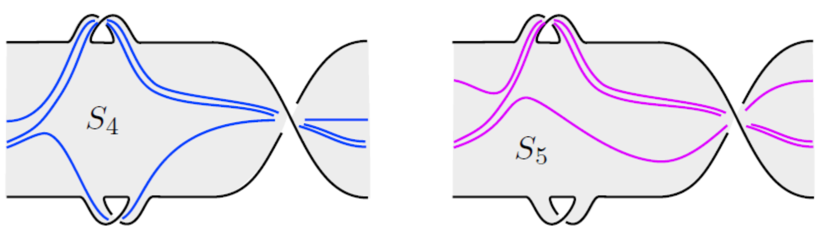


Figure 3.2: Example of embellishment, taken from [3]

Proposition 3.1.1. Let $[S_n]$ denote the homology class of the embellished sphere as constructed above for a postive integer n. If $\tilde{S}$ and $\tilde{P}$ denote the blow-up of S at q, and P at p, respectively, then $[S_n]^2 = n$. That is, S_n has self-intersection n.

Proof. When $n = 2d$, we have that $[S_{2d}] = [\tilde{S} \cup \tilde{P}]$. Therefore, if we denote E_p as the exceptional divisor for the blow-up at p, and E_q as the exceptional divisor for the blow-up at q, we can see that

$$[S_{2d}] = [\tilde{S} \cup \tilde{P}] = [S + E_q] + d[S + E_p] = (d+1)[S] + d[E_p] + [E_q]$$

Meaning that when we examine the self-intersection $[S_{2d}]^2$, we have that:

$$((d+1)[S] + d[E_p] + [E_q])^2 = (d+1)^2 - d^2 - 1 = 2d$$

Since $[S]^2 = 1$, $[E_p]^2 = [E_q]^2 = -1$, and all of the cross-terms are zero. Similarly, when $n = 2d+1$, we have $[S_{2d+1}] = [T \cup \tilde{P}] = [S] + d[S] + d[E_p]$. Thus, it is easy to see that $[S_{2d+1}] = 2d+1$.

$\square$

By [3, Proposition 3.5], using the tubular neighborhood theorem, isotopic (+1) spheres in our manifold Z, will have isotopic embellishments. Therefore, our original isotopies on the (+1)-sphere, G^Z, F^Z, and K^Z, are also isotopies of our embellished sphere. Stated more formally, if $j_{Y,n}$ and $j_{X,n}$ represent the embeddings of the embellished spheres into Y^+ and X^+ respectively, then, at time 1, G^Z and K^Z map $j_{Y,n}$ to $j_{X,n}$ in the manifold Z, and F^Z, at time 1, maps $j_{Y,n}$ to $j_{X,n}$ in the manifold $Z\#(S^2 \times S^2)$.

One particularly helpful way to view these spheres is as sections of an appropriate manifold. Recall, when the sphere had self-intersection (+1) we viewed it as the section $\mathbb{CP}^1$ inside of $\mathbb{CP}^2$, which is how we originally embedded it. Now, we interpret this embellished sphere S_n as the section of the nth Hirzebruch surface, denoted by $\mathbb{F}_n$. Since $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ is diffeomorphic to $\mathbb{F}_n\#\overline{\mathbb{CP}}^2$ by handle slides, this allows us to view our manifold

$$Z = Y\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2 \cong Y\#\mathbb{F}_n\#\overline{\mathbb{CP}}^2$$

3.2 Interesting Families of Embeddings

With the spheres embellished, we now will create a new tower of manifolds, embeddings, and isotopies, similar to the diffeomorphisms before. We will begin with the base case of only two embedded spheres, before generalizing to collections of embeddings.

First, denote $Z^0 := Z$ and recursively construct $Z^k := Z^{k-1} \#_T Z^0$. Second, let us view $S^0 = \{-1, 1\}$ and define the two base cases of our tower of embeddings $\widehat{j}^0_{Y,n}$ and $\widehat{j}^0_{X,n}$ by:

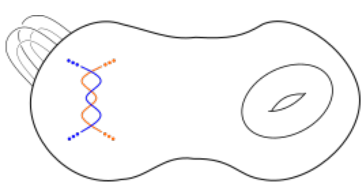
$$\begin{aligned}
\widehat{j}^0_{Y,n} &: S^0 \times S^2 \to S^0 \times Z^0 \\
\widehat{j}^0_{Y,n}(\varepsilon, w) &= (\varepsilon, j_{Y,n}(w)) \\
\widehat{j}^0_{X,n} &: S^0 \times S^2 \to S^0 \times Z^0 \\
\widehat{j}^0_{X,n}(\varepsilon, w) &= \begin{cases} \widehat{j}^0_{Y,n}(\varepsilon, w) & \varepsilon = 1 \\ (-1, j_{X,n}(w)) & \varepsilon = -1 \end{cases}
\end{aligned}$$

Third, we now define the base case of our tower of stable-smooth isotopy using the original stable-smooth isotopy F^Z. Recall in our original definition in that $F^Z : I \times (Z\#(S^2 \times S^2)) \to I \times (Z\#(S^2 \times S^2))$ such that F^Z_0 is identity and F^Z_1 maps $j_{Y,n}$ to $j_{X,n}$. We will denote these isotopies as the same letter from before to emphasize the similarity. We define the base family of stable-smooth ambient isotopy by:

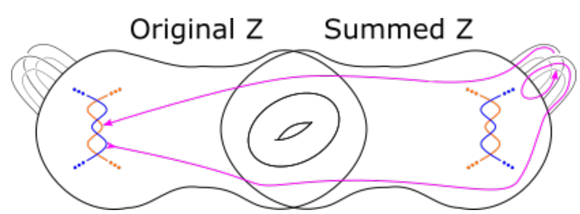
$$\begin{aligned}
F^0 : I \times S^0 \times \left(Z^0\#(S^2 \times S^2)\right) &\to I \times S^0 \times \left(Z^0\#(S^2 \times S^2)\right) \\
F^0(t, \varepsilon, z) &= (t, \varepsilon, p_z \circ F^Z(1/2(1-\varepsilon)t, z))
\end{aligned}$$

Where $p_z \circ F^Z$ is projection onto the $(Z\#(S^2 \times S^2))$ component of F^Z.

Remark. Before we define the rest of the tower, let us highlight a few important details. Intuitively, the recursive construction of the manifold can be visualized as spokes of the manifold Z being summed over the axle of T. Inside Z there are embedded spheres that are topologically equivalent, smoothly distinct, but can be made smoothly equivalent after a stabilization. Moreover, recall that $Z^0 = Z = Y\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2 \cong Y\#(S^2 \times S^2)\#\overline{\mathbb{CP}}^2$. This means that when we proceed up the tower of manifolds to Z^{k+1} by fibre summing, we are summing on a copy of $(S^2 \times S^2)$, which means we are stabilizing the Z^k manifold. Once the manifold has been stabilized, the families of embedded spheres become smoothly equivalent under the isotopy F. This smooth isotopy, however, must interact with the new summed copy of Z^0. Since Z^{k+1} has many copies of $Z \setminus T$ it has many embeddings of the sphere into any of these copies. The embedding we care about is the embedding into the **last** copy, which we denote by the tilde notation $\tilde{j}$. The interaction of the isotopy F with the embedded sphere $\tilde{j}$ in the new summed copy results in the creation of the paramterized families of embeddings and isotopies. In Figure 3.3b we see the case when perform a single sum. The blue and orange are embedded spheres and the pink path represents the isotopy F. The isotopy maps one sphere to the other in the original manifold, but creates two one-parameter families of spheres because of the two embedded spheres in the summed Z component. The recursion process occurs as we sum on more spokes of Z, and continue to perform the stabilization of the previous family of spheres, thereby creating higher paramterized families with each new spoke.



(a) A single copy of Z



(b) $Z\#_T Z$ with isotopy.

Figure 3.3: Submanifold Sum allows for a Smooth Isotopy

Next, for ease of definition, we will decompose the $(k+1)$-sphere so that it is viewed as $S^{k+1} \cong (I \times S^k)/\big((0 \times S^k) \cup (1 \times S^k)\big)$. Finally, we can recursively define the family of embeddings $\widehat{j}^{k+1}_{a,n}$ and the family of stable-smooth isotopies F^{k+1} by:

$$\begin{aligned}
\widehat{j}^{k+1}_{a,n} &: S^{k+1} \times S^2 \to S^{k+1} \times Z^{k+1} \\
&\widehat{j}^{k+1}_{a,n}(t, v, w) = F^k(t, v, \widetilde{j_{a,n}}(w)) \\
F^{k+1} &: I \times S^{k+1} \times \left(Z^{k+1}\#(S^2 \times S^2)\right) \to I \times S^{k+1} \times \left(Z^{k+1}\#(S^2 \times S^2)\right) \\
&F^{k+1}(s, t, v, z) = (s, t, v, p_z \circ F^k(t, v, \widetilde{F^Z}(s, z)))
\end{aligned}$$

Where a represents either X or Y, $s \in I$, $[t, v] \in S^{k+1}$, and $p_z \circ F^k$ is projection onto the $\left(Z^{k+1}\#(S^2 \times S^2)\right)$ component of F^k. Additionally, the embedding $\widetilde{j_{a,n}}$ and the isotopy $\widetilde{F^Z}$ are happening in the Z^0 summand (or "newest spoke") in $Z^{k+1} = Z^k\#_T Z^0$.

Remark. Regarding the constructions above, we quickly clarify the map F^k is not defined on all of $Z^{k+1} = Z^k\#_T Y\#(S^2 \times S^2)\#\overline{\mathbb{CP}}^2$, however it is only defined on $Z^k\#(S^2 \times S^2) \setminus N(t)$. Thus, we can simply extend F^k by defining it to be identity on the other portion. Due to the already substantial amount of notation, we did not wish further complicate things.

Proposition 3.2.1. The family of embeddings $\widehat{j}^k_{a,n}$ and the family of stable-smooth isotopies F^k, as defined above, are well-defined. Furthermore, $F^k(1, \widehat{j}^k_{Y,n}) = \widehat{j}^k_{x,n}$, that is, F^k is indeed an isotopy between the families of embeddings.

Proof. We begin with the base case of $k = 0$ before continuing by induction. In our construction we have implicitly stated the basepoint of $S^0 = \{-1, 1\}$ is 1; meaning that F^0 should be identity at the basepoint, independent of time, and the usual isotopy otherwise. Indeed when $\varepsilon = 1$ we have that:

$$F^0(t, 1, z) = (t, 1, p_z \circ F^Z(0, z)) = (t, 1, z)$$

since F^Z was defined to be identity at time 0. When $\varepsilon = -1$ and when $t = 0$ we have that for $w \in S^2$ the map F^0 is identity everywhere, and notably on the embedding $\widehat{j}^0_{Y,n}$:

$$F^0(0, -1, \widehat{j}^0_{Y,n}(-1, w)) = (0, -1, p_z \circ F^Z(0, j_{Y,n}(w))) = (0, -1, j_{Y,n}(w)) = \widehat{j}^0_{Y,n}(-1, w)$$

When $t = 1$ we have that F^0 maps $\widehat{j}^0_{Y,n}$ to $\widehat{j}^0_{X,n}$:

$$F^0(1, -1, \widehat{j}^0_{Y,n}(-1, w)) = (1, -1, p_z \circ F^Z(1, j_{Y,n}(w))) = (1, -1, j_{X,n}(w)) = \widehat{j}^0_{X,n}(-1, w)$$

Moreover, F^0 at $t = 1$ is identity away from these spheres, since the isotopy F^Z was defined to be as well. Thus, these maps work as intended in the base case. Now, let us consider the higher order families. For arbitrary $(k + 1)$, we first need to ensure that they are well-defined for our decomposition of S^{k+1}. Recall that we decomposed $S^{k+1} \cong (I \times S^k)/\left((0 \times S^k) \cup (1 \times S^k)\right)$. Let us first examine

$$F^{k+1}(s, t, v, z) = (s, t, v, p_z \circ F^k(t, v, \widetilde{F^Z}(s, z)))$$

We need to ensure when $t = 0$, respectively when $t = 1$, that F^{k+1} is independent of the choice of v. When $t = 0$ we have that:

$$F^{k+1}(s, 0, v, z) = (s, 0, v, p_z \circ F^k(0, v, \widetilde{F^Z}(s, z)))$$

Note that $F^k(0, v, \widetilde{F^Z}(s, z)))$ is the lower order isotopy at time zero, which by induction we see is identity, regardless of v. Therefore, at $t = 0$ our map F^k is also independent of v. When $t = 1$ we have that:

$$F^{k+1}(s, 1, v, z) = (s, 1, v, p_z \circ F^k(1, v, \widetilde{F^Z}(s, z)))$$

Similar to the $t = 0$ case, we note that $F^k(0, v, \widetilde{F^Z}(s, z)))$ is the lower order isotopy at time one, which, by induction, we see is identity outside a neighborhood of our embeddings of spheres. Hence, we have that F^{k+1} is well-defined. Before we show that F^{k+1} is an isotopy between $\widehat{j}^{k+1}_{Y,n}$ and $\widehat{j}^{k+1}_{X,n}$, let us ensure that $\widehat{j}^{k+1}_{Y,n}$ and $\widehat{j}^{k+1}_{X,n}$ are well-defined too.

Once again, we need to ensure when $t = 0$, respectively when $t = 1$, that $\widehat{j}^{k+1}_{a,n}$ is independent of the choice of v, for $a = X, Y$. Recall that for $[t, v] \in S^{k+1}$ and for $w \in S^2$ we have:

$$\widehat{j}^{k+1}_{a,n}(t,v,w) = F^k(t,v,\widetilde{j_{a,n}}(w))$$

Thus, these maps are wholly determined by the lower order F^k, which, as shown above, are independent of choice of v, by induction. All that remains is to ensure that at $s=0$ our map F^{k+1} is identity, and at $s=1$ our map F^{k+1} maps $\widehat{j}^{k+1}_{Y,n}$ to $\widehat{j}^{k+1}_{X,n}$. Indeed this is true, since when $s=0$ we have that $F^z(0,z)$ is identity, which determines the rest of F^{k+1}, and when $s=1$ we have that for $w \in S^2$:

$$\begin{aligned}
F^{k+1}(1,\widehat{j}^{k+1}_{Y,n}(t,v,w)) &= (1,t,v,p_z \circ F^k(t,v,\widetilde{F^Z}(1,j_{Y,n}(w)))) \\
&= (1,t,v,p_z \circ F^k(t,v,\widehat{j}^{k}_{X,n}(w))) && \text{By Induction} \\
&= \widehat{j}^{k+1}_{X,n}(w)
\end{aligned}$$

Hence, we have that the map F^{k+1} behaves as desired. $\square$

With the family of embeddings $\widehat{j}^{k+1}_{a,n}$ and the family of stable-smooth isotopies F^{k+1}, as defined above, we can now define the rest of our families. We define the families of topological isotopies G^{k+1} and families of pseudoisotopies K^{k+1} on embeddings by:

$$\begin{aligned}
&G^{k+1} : I \times S^{k+1} \times Z^{k+1} \to I \times S^{k+1} \times Z^{k+1} \\
&\qquad G^{k+1}(s,t,v,z) = (s,t,v,p_z \circ F^k(t,v,\widetilde{G^Z}(s,z))) \\
&K^{k+1} : I \times S^{k+1} \times Z^{k+1} \to I \times S^{k+1} \times Z^{k+1} \\
&\qquad K^{k+1}(s,t,v,z) = (s,t,v,p_z \circ F^k(t,v,\widetilde{K^Z}(s,z)))
\end{aligned}$$

where $s \in I$, $[t,v] \in S^{k+1}$, and $p_z \circ F^k$ is projection onto the $\big(Z^{k+1}\#(S^2 \times S^2)\big)$ component of F^k. Additionally, the isotopy $\widetilde{G^Z}$ and the isotopy $\widetilde{K^Z}$ are happening in the Z^0 summand (or "newest spoke") in $Z^{k+1} = Z^k \#_T Z^0$. The family G^{k+1}, respectively K^{k+1}, can be easily seen to be well-defined since it is determined by F^k and G^Z, respectively K^Z, which behave similar to F^Z. That is, they were constructed over the base manifold Z and are identity at time 0, and at time 1 map the sphere j_Y to the sphere j_X.

With these base families defined, much like in section 2.3, we now seek to generalize this construction to higher ranks. We begin by first altering the underlying space, Z. Recall in the earlier section, we used the base manifold X and fibred on additional copies of the $E(2)$ surface to create X^r. We then perform log-transforms on nuclei to create $\{X_i^r\}$, performed a cork twist to define Y^r, and finally augmented the manifold Z^k to $Z^{k,r}$ using these operations. We will follow this construction again to define our manifold $Z^r := Y^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$. Like before we do not simply have two embedded spheres, but one for each X_i^r, $i = 1, \ldots, r$, and one for Y^r. We define these embeddings as $j_i = \varphi_i^{-1}(\psi_i \# 1_{\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2}) j_{Y^r}$ for $i = 1, \ldots, r$. By [2, Proposition 3.6] and [4, Theorem C], our isotopies G^Z, F^Z, and K^Z can be extended to collections of isotopies G_i^Z, F_i^Z, and K_i^Z. That is for each $i = 1, \ldots, r$, we have that j_i is topologically isotopic to j_{Y^r} via G_i^Z, that j_i is stable-smooth isotopic to j_{Y^r} via F_i^Z, and that j_i is pseudoisotopic to j_{Y^r} via K_i^Z.

Once again, since we are not only interested in the self-intersection $(+1)$ spheres, we will need to perform embellishments on the spheres in this space like before. Since the collection represented by j_i spheres are all embedded into different components of X^r, and the embellishment operation is done in a neighborhood of each sphere, we may simply embellish each sphere one at a time which we denote by $j_{i,n}$. With this collection of embedded spheres embellished, denote $Z^{0,r} := Z^r$ and define the base families of embeddings and stable-smooth isotopies by:

$$\widehat{j}^0_{Y^r,n} : S^0 \times S^2 \to S^0 \times Z^0$$

$$\widehat{j}^0_{Y^r,n}(\varepsilon, w) = (\varepsilon, j_{Y^r,n}(w))$$

$$\widehat{j}^0_{i,n} : S^0 \times S^2 \to S^0 \times Z^0$$

$$\widehat{j}^0_{i,n}(\varepsilon, w) = \begin{cases} \widehat{j}^0_{Y^r,n}(\varepsilon, w) & \varepsilon = 1 \\ (-1, j_{i,n}(w)) & \varepsilon = -1 \end{cases}$$

$$F^0_i : I \times S^0 \times \big(Z^{0,r}\#(S^2 \times S^2)\big) \to I \times S^0 \times \big(Z^{0,r}\#(S^2 \times S^2)\big)$$

$$F^0_i(t, \varepsilon, z) = (t, \varepsilon, p_z \circ F^Z_i(1/2(1-\varepsilon)t, z))$$

The rest of the recursive tower construction proceeds as one expects.

Definition 3.2.2. We define $Z^{k+1,r} := Z^{k,r}\#_T Z^{0,r}$ and the families of embeddings and isotopies by:

$$\widehat{j}^{k+1}_{i,n} : S^{k+1} \times S^2 \to S^{k+1} \times Z^{k+1,r}$$

$$\widehat{j}^{k+1}_{i,n}(t, v, w) = F^k_i(t, v, \widetilde{j_{i,n}}(w))$$

$$F^{k+1}_i : I \times S^{k+1} \times \big(Z^{k+1,r}\#(S^2 \times S^2)\big) \to I \times S^{k+1} \times \big(Z^{k+1,r}\#(S^2 \times S^2)\big)$$

$$F^{k+1}_i(s, t, v, z) = (s, t, v, p_z \circ F^k_i(t, v, \widetilde{F^Z_i}(s, z)))$$

$$G^{k+1}_i : I \times S^{k+1} \times Z^{k+1,r} \to I \times S^{k+1} \times Z^{k+1,r}$$

$$G^{k+1}_i(s, t, v, z) = (s, t, v, p_z \circ F^k_i(t, v, \widetilde{G^Z_i}(s, z)))$$

$$K^{k+1}_i : I \times S^{k+1} \times Z^{k+1,r} \to I \times S^{k+1} \times Z^{k+1,r}$$

$$K^{k+1}_i(s, t, v, z) = (s, t, v, p_z \circ F^k_i(t, v, \widetilde{K^Z_i}(s, z)))$$

again where $s \in I$, $[t, v] \in S^{k+1}$, and $p_z \circ F^k$ is projection onto the $\big(Z^{k+1}\#(S^2 \times S^2)\big)$ component of F^k. Additionally, the isotopies $\widetilde{G^Z_i}$, $\widetilde{F^Z_i}$, and $\widetilde{K^Z_i}$ are happening in the Z^0 summand (or "newest spoke") in $Z^{k+1} = Z^k\#_T Z^0$.

Remark. Our families of embeddings, as defined above, are actually constructed as an extension of the adjoint. These embeddings $\widehat{j}_{i,n}^{k+1}$ should first be viewed as:

$$j_{i,n}^{k+1} : S^k \to \mathrm{Emb}(S^2, Z^{k+1,r})$$

Next, applying the adjoint we have a map $S^k \times S^2 \to Z$, and extending the adjoint to be identity on the first component, we have our embeddings:

$$\widehat{j}_{i,n}^{k+1} : S^{k+1} \times S^2 \to S^{k+1} \times Z^{k+1,r}$$

To maintain convention with later constructions, the hat notation refers to the extension of the adjoint.

Chapter 4

The Invariants

This chapter outlines the creation of the Family Invariant used to detect the exotic families of embeddings. We first outline the basic construction of our invariant using submanifold sum. However, the usage of submanifold sum on general families may not be possible. Recall, in order to perform a submanifold sum we require an orientation reversing isomorphism of the normal bundles. In general, this may not be possible. Since these are paramterized embeddings, the parameterization may result in "twisting." This twisting of the parameter component results in non-isomorphic normal bundles. To address this, we next define a Twisting Invariant to determine if submanifold sum is possible. Once this problem of twisting is addressed, we define our Family Invariant on families of embeddings by using Konno's invariant [25] on the associated family of manifolds created by the submanifold sum. Finally, we relate Konno's invariant on a family of manifolds to the Auckly-Ruberman invariant, which utilizes degree 0 Donaldson invariants [2], on the associated family of diffeomorphisms.

4.1 Invariant Setup

We are interested in the topology of the embedding spaces, specifically we want to examine $\pi_k(\mathrm{Emb}^{\cdot}(F, X))$ and $H_k(\mathrm{Emb}^{\cdot}(F, X))$. A useful way to examine these groups is to begin with the homotopy classes and to recall that an element of a such a homotopy group is a equivalence class of a sphere into the space of embeddings. Similarly, an element of the homology is an equivalence class of singular chains which may be viewed as a map of a $k-$cycle into the space of embeddings.

$$J : \Sigma^k \to \mathrm{Emb}^{\cdot}(F, X)$$

A natural way to view this family is via the adjoint, which gives us a map: $\tilde{J} : \Sigma^k \times F \to X$. We will actually extend this map $\tilde{J}$ into a slightly more useful form, which is simply identity on the first component.:

$$\widehat{J} : \Sigma^k \times F \to \Sigma^k \times X$$

Let us endow $\mathrm{Emb}^{\cdot}(F, X)$ with the compact-open topology. We first highlight that, in general, these spaces are not path connected. Two elements in the same component of $\mathrm{Emb}^{C^0}(F, X)$ represent the same homology class; thus, if two embeddings represent different homology classes, they cannot lie in the same path component. Indeed, if there was a path between them, that path would be a homotopy of the two maps, and maps that are homotopic represent the same homology class.

To create the desired invariant, we first note that for the purposes of this paper we will restrict F to the 2-sphere S^2, meaning we are examining the embedding space $\mathrm{Emb}^{\cdot}(S^2, X)$. Any component of this space will represent a well-defined homology class. By changing orientation of the ambient manifold, we may assume that the self-intersection number is non-negatived. Let us denote this self-intersection number by n. To define an invariant for a homology calss of embeddings, we will take a family submanifold sum with a standard model to obtain a family of 4-manifolds and then apply a family invariant to this family of 4-manifolds. In principle we could use any family invariant. In this case, we will use the Donaldson characteristic class as developed by Konno [25].

For a family of embeddings $J : \Sigma^k \to \mathrm{Emb}^{\cdot}(S^2, X)$ we use the adjoint to extended this to the analogous embedding $\widehat{J} : \Sigma^k \times S^2 \to \Sigma^k \times X$. In the case of $n > 0$, the standard model family of embeddings is $\widehat{J_{\overline{\mathbb{CP}}^2}} : \Sigma^k \times S^2 \to \Sigma^k \times n\overline{\mathbb{CP}}^2$. This map is simply identity on the first component, and the second component maps S^2 to the sphere of self-intersection $(-n)$ inside $n\overline{\mathbb{CP}}^2$ created by summing each of the exceptional divisors in $n\overline{\mathbb{CP}}^2$. In the case of $n = 0$, denote $0\overline{\mathbb{CP}}^2$ as S^4 and we will consider a map where $\Sigma^k \times S^2$ embeds into $\Sigma^k \times S^4$, in the usual way.

We now describe how to create families of spaces from the families of embeddings using the sub-manifold sum construction. Recall that in the definition of submanifold sum, we needed a fibre orientation reversing isomorphism of the normal bundles of the submanifold. So for our families, when the normal bundle of $\Sigma^k \times S^2$ in $\Sigma^k \times X$ is fiber orientation reversing to the normal bundle of $\Sigma^k \times S^2$ in $\Sigma^k \times n\overline{\mathbb{CP}}^2$ we say the family of embeddings J has a <u>non-negative standard normal bundle</u>. With this, we finally can define our associated k-dimensional family of spaces to be:

Definition 4.1.1. For $J : \Sigma^k \to \mathrm{Emb}(S^2, X)$ with non-negative standard normal bundle we define the associated family of spaces to be

$$\mathcal{Z}_J := (\Sigma^k \times X)\#_{\Sigma^k \times S^2}(\Sigma^k \times n\overline{\mathbb{CP}}^2)$$

Where $\Sigma^k \times S^2$ embeds into the left component by $\widehat{J}$ and into the right component by $\widehat{J_{\overline{\mathbb{CP}}^2}}$.

However, are we guaranteed to have a non-negative standard normal bundle? In short: no. We need a fibre orientation reversing isomorphism between the normal bundles, and Since $\Sigma^k \times S^2$ is a codimension 2 submanifold, in both $\Sigma^k \times X$ and $\Sigma^k \times n\overline{\mathbb{CP}}^2$, its normal bundles are classified by the Euler Class. Moreover, the Euler class is an element of the second cohomology of $\Sigma^k \times S^2$. This poses a potential problem in case when Σ^k has non-zero second cohomology. Consider the example above when $\Sigma^k = S^2$.

Example 4.1.2. Let $X := S^2 \times S^2$ and consider two spherical families in the class $[b \times S^2]$, where b is a fixed point in S^2. We define these spherical families as:

$$S^2 \times S^2 \xrightarrow{\widehat{J_0}} S^2 \times X \quad \text{by} \quad \widehat{J}_0(x, y) = (x, (b, y))$$

$$S^2 \times S^2 \xrightarrow{\widehat{J_1}} S^2 \times X \quad \text{by} \quad \widehat{J}_1(x, y) = (x, (x, y))$$

We will now show that these two families have distinct normal bundles. Since these are codimension two submanifolds, we must examine the second homology of our domain: $H_2(S^2 \times S^2) = \mathbb{Z} \oplus \mathbb{Z}$. Therefore, we have two generators $A = b_0 \times S^2$ and $B = S^2 \times b_0$, where b_0 is the basepoint. Next, restrict the families $\widehat{J}_1$ and $\widehat{J}_0$ to these generators. Beginning with $\widehat{J}_1$, we can observe that:

$$
\begin{aligned}
\widehat{J}_1 \circ A(x) &= \widehat{J}_1(b_0, y) = (b_0, (b_0, y)) \\
\widehat{J}_1 \circ B(x) &= \widehat{J}_1(x, b_0) = (x, (x, b_0))
\end{aligned}
\tag{4.1}
$$

The map (4.1) simply returns the diagonal of $S^2 \times S^2 \times b_0$. Next, let us consider the following diagram:

$$
\begin{tikzcd}
 & (x,y) \arrow[r, mapsto] & x & \\
(x,y) \arrow[d, mapsto] & S^2 \times S^2 \arrow[r, "\varphi"] \arrow[d, "\widehat{J}_1"] & S^2 \arrow[d, "j_1"] & x \arrow[d, mapsto] \\
(x,x,y) & S^2 \times S^2 \times S^2 \arrow[r] & S^2 \times S^2 & (x,x) \\
 & (w,x,y) \arrow[r, mapsto] & (w,x) &
\end{tikzcd}
$$

We now claim that the normal bundle of the $\widehat{J}_1$ is the pull-back of the normal bundle of j_1. Noting that $\widehat{J}_1(x_t, y_t) = (j_1(x_t), y_t)$, for any t, we can observe that that derivative of this map is: $(\widehat{J}_1)_*(\dot{x}_0, \dot{y}_0) = (j_*(\dot{x}_0), \dot{y}_0)$. Therefore, as we examine the normal bundle, we have that:

$$
\begin{aligned}
T(S^2 \times S^2 \times S^2)/(\widehat{J}_1)_*(\dot{x}_0, \dot{y}_0) &= \left(T(S^2) \times T(S^2) \times T(S^2)\right)/(j_*(\dot{x}_0), \dot{y}_0) \\
&\simeq \left(T(S^2) \times T(S^2)\right)/j_*(\dot{x}_0)
\end{aligned}
$$

Combining this observation with the fact that j_1 is simply the diagonal map on its image, we note that j_1 can be represented by the generators of the second homology in the form: $j_1 = B + A$. Meaning that $j_1^2 = 2$, and, more importantly, that the Euler class is $e(N(j_1))(S^2) = 2$.

Since characteristic classes are natural with respect to the pull back, we may also note that:

$$e(N(\widehat{J}_1)) = e(\varphi^*(N(j))) = \varphi^*(e(N(j)))$$

Examining this with respect to the generators, we can observe the following:

$$\begin{aligned} e(N(\widehat{J}_1))(A) &= e(\varphi^*(N(j)))(A) \\ &= e(N(j))(\varphi_*(A)) \\ &= e(N(j))(0) \\ &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} e(N(\widehat{J}_1))(B) &= e(\varphi^*(N(j)))(B) \\ &= e(N(j))(\varphi_*(B)) \\ &= e(N(j))(S^2) \\ &= 2 \end{aligned}$$

However, if we repeat the above process with $\widehat{J}_0$ we arrive at a slightly different outcome. The first observation is $\widehat{J}_0$ applied to the generators,

$$\begin{aligned} \widehat{J}_0 \circ A(x) &= \widehat{J}_0(b_0, y) = (b_0, (b, y)) \\ \widehat{J}_0 \circ B(x) &= \widehat{J}_0(x, b_0) = (x, (b, b_0)) \end{aligned} \tag{4.2}$$

Similar to the $\widehat{J}_1$ case, if we observe the map 4.2 above, the map returns simply the first S^2. Let us observe a familiar diagram:

$$\begin{tikzcd} & (x,y) \arrow[r, mapsto] & x & \\ (x,y) \arrow[d, mapsto] & S^2 \times S^2 \arrow[r, "\varphi"] \arrow[d, "\widehat{J}_0"] & S^2 \arrow[d, "j_0"] & x \arrow[d, mapsto] \\ (x,b,y) & S^2 \times S^2 \times S^2 \arrow[r] & S^2 \times S^2 & (x,b) \\ & (w,x,y) \arrow[r, mapsto] & (w,x) & \end{tikzcd}$$

Like before, the normal bundle of the $\widehat{J}_0$ is the pull-back of the normal bundle of j_0, and following the analogous steps as above, we can conclude that:

$$\begin{aligned} T(S^2 \times S^2 \times S^2)/(\widehat{J}_0)_*(\dot{x}_0, \dot{y}_0) &= \left(T(S^2) \times T(S^2) \times T(S^2)\right)/((j_0)_*(\dot{x}_0), \dot{y}_0) \\ &\simeq \left(T(S^2) \times T(S^2)\right)/(j_0)_*(\dot{x}_0) \end{aligned}$$

In this case, however, we note that j_0 can be represented by the generators of the second homology in the form: $j_0 = B$. Meaning that $j_0^2 = 0$, and, more importantly, that the Euler class is also 0. Like before, since characteristic classes are natural with respect to the pull back, we can finally observe that:

$$\begin{aligned} e(N(\widehat{J}_0))(A) &= e(\varphi^*(N(j_0)))(A) \\ &= e(N(j_0))(\varphi_*(A)) \\ &= e(N(j))(0) \\ &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} e(N(\widehat{J}_0))(B) &= e(\varphi^*(N(j_0)))(B) \\ &= e(N(j_0))(\varphi_*(B)) \\ &= e(N(j))(S^2) \\ &= 0 \end{aligned}$$

We can now easily observe that the normal bundles of $\widehat{J}_0$ and $\widehat{J}_1$ are distinct and not isomorphic. When restricted to generator A, both bundles had the same Euler number, but when we restricted to generator B had different numbers. The B generator corresponded to the first S^2 component, that is: the parameter component. So, in this example, the problem only existed in the parameter component and not in the embedding component, i.e. the second S^2. In general, our constructed families $\widehat{J}$ may also have this problem with the parameter component, and for this reason we refer to this as "parameter twisting", or just "twisting".

4.2 The Twisting Invariants

To address this problem of parameter twisting, we will create an invariant on our families of embeddings that will detect if a submanifold sum is possible. To create this invariant we utilize the universal normal bundle.

Lemma 4.2.1. *Consider the commutative diagram below where $J \in H_k(Emb(S^2, X))$ and $\widehat{J}$ is the induced adjoint; along with the map $u(j, v) = (j, j(v))$.*

$$\begin{tikzcd} Emb(S^2, X) \times S^2 \arrow[r, hook, "u"] & Emb(S^2, X) \times X \\ \Sigma^k \times S^2 \arrow[u, "J \times 1_{S^2}"] \arrow[r, hook, "\widehat{J}"] & \Sigma^k \times X \arrow[u, "J \times 1_X"'] \end{tikzcd}$$

If $J \times 1_X$ is transverse to $Emb(S^2, X) \times S^2$, then the normal bundle of $\widehat{J}$, denoted by $N_{w,J_w(v)}\Sigma^k \times S^2$ is isomorphic to $(J \times 1_{S^2})^*\mathcal{N}$, the pullback of the normal bundle of u

Definition 4.2.2. Given a smooth map $f : X \to Y$ where Z is a submanifold of Y via an embedding i, we say that f is transverse to Z if for every $z \in Z$

$$T_z i\,(TZ) \oplus (T_m(f))\,T_m X = T_{i(z)}Y$$

Lemma 4.2.3. *The map $J \times 1_X$ is transverse to the submanifold $Emb(S^2, X) \times S^2$.*

Proof. To show that $J \times 1_X$ is transverse to the submanifold $\mathrm{Emb}(S^2, X) \times S^2$, we need to find a $(Di, \dot{v}) \in T(\mathrm{Emb}(S^2, X) \times S^2)$ and a $(\dot{w}, \dot{y}) \in T(\Sigma^k \times X)$ such that for any $(Dj, \dot{x}) \in T(\mathrm{Emb}(S^2, X) \times X)$ the following equivalence is true:

$$(Dj, \dot{x}) = \left(T_{(J_w,v)}u\right)(Di, \dot{v}) + \left(T_{(w,J_w(v))})S^k \times X\right)(\dot{w}, \dot{y})$$

To find such elements, let us examine the components of the right-hand side above. By definition, we have that:

$$\begin{aligned} \left(T_{(J_w,v)}u\right)(Di, \dot{v}) &= \frac{d}{dt}u(j_t, v_t)|_{t=0} = (Di, Di(v_0) + j_*\dot{v}) \\ \left(T_{(w,J_w(v))})\Sigma^k \times X\right)(\dot{w}, \dot{y}) &= \frac{d}{dt}(J \times 1_X)(w_t, y_t)|_{t=0} = (J_*(\dot{w}), \dot{y}) \end{aligned}$$

Therefore, by setting $i = j$, $\dot{w} = 0$, $\dot{v} = 0$, and $\dot{y} = \dot{x} - Dj(v_0)$ we have found elements $(Di, \dot{v})$ and $(\dot{w}, \dot{y})$ showing that $J \times 1_X$ is transverse to the submanifold $\mathrm{Emb}(S^2, X) \times S^2$. $\square$

Utilizing this arugment about transversality, we can now can examine the normal bundles of these spaces in the commutative diagram above.

Proof of Lemma 4.2.1. We first view the bundles as quotient bundles:

$$\begin{aligned}
\mathcal{N} &= \left(T_{(j,j(v))}\left(\mathrm{Emb}(S^2,X)\times X\right)\right) / \left(T\left(\mathrm{Emb}(S^2,X)\times S^2\right)\right) \\
&\cong \left(T_j\mathrm{Emb}(S^2,X)\times T_{j(v)}X\right) / \left(T\mathrm{Emb}(S^2,X)\times TS^2\right) \\
&\cong \left(\left(T_j\mathrm{Emb}(S^2,X)\right) / \left(T\mathrm{Emb}(S^2,X)\right)\right)\times\left(\left(T_{j(v)}X\right) / \left(TS^2\right)\right) \\
N_{w,\alpha_w(v)}\Sigma^k\times S^2 &= \left(T_{w,\alpha_w(v)}\left(\Sigma^k\times X\right)\right) / \left(T\left(\Sigma^k\times S^2\right)\right) \\
&\cong \left(T_w\Sigma^k/T\Sigma^k\right)\times\left(T_{\alpha_w(v)}X/TS^2\right)
\end{aligned}$$

Next, let $\varphi:\mathcal{N}_{j,v}\to N_{j(v)}S^2$ by $\varphi([D_j,\dot{x}])=[\dot{x}]$. To show this is well-defined, first suppose $[D_{j_1},\dot{x}_1]\sim[D_{j_2},\dot{x}_2]$, which more precisely means that there is a $[D_{j_0},\dot{v}]\in\mathrm{Emb}(S^2,X)\times S^2$ such that $[D_{j_1},\dot{x}_1]=[D_{j_2},\dot{x}_2]+[D_{j_0},\dot{v}]$. With this, we have that:

$$\varphi([D_{j_0},\dot{v}])=[\dot{x}_2+\dot{v}]$$

However, in $N_{j(v)}S^2$ the class of $[\dot{x}_2+\dot{v}]$ is equivalent to the class $[\dot{x}_2]$. Hence, it is clear that φ is well-defined. To show this is an isomorphism, let us first note that the left component of $\mathcal{N}_{j,v}$ is can be defined by the short exact sequence:

$$0\longrightarrow T\left(\mathrm{Emb}(S^2,X)\right)\longrightarrow T\left(\mathrm{Emb}(S^2,X)\right)|_{\mathrm{Id}}\longrightarrow\frac{T_j\mathrm{Emb}(S^2,X)}{T\mathrm{Emb}(S^2,X)}\longrightarrow 0$$

However, we clearly note an isomorphism between $T\left(\mathrm{Emb}(S^2,X)\right)$ and $T\left(\mathrm{Emb}(S^2,X)\right)|_{\mathrm{Id}}$, meaning that $T_j\mathrm{Emb}(S^2,X)/T\mathrm{Emb}(S^2,X)\cong 0$. A similar argument shows that $T_w\Sigma^k/T\Sigma^k\cong 0$. Thus, we need only examine:

$$\left(T_{j(v)}X\right) / \left(TS^2\right)\quad\text{and}\quad\left(T_{J_w(v)}X\right) / \left(TS^2\right)$$

By Guillemin and Pollack [20], we know that if a smooth map $f : X \to Y$ is transverse to a submanifold Z, the preimage of that submanifold under the map is a submanifold, i.e. $f^{-1}(Z) \subset X$ is a submanifold. Therefore, since $(J \times 1_X)$ is transverse to $\mathrm{Emb}(S^2, X) \times S^2$, we have that $(J \times 1_X)^{-1}(\mathrm{Emb}(S^2, X) \times S^2)$ is a submanifold in $\Sigma^k \times X$. This means that whether under the map $\widehat{J}$ or under the composition $(J \times 1_X) \circ (J \times S^2)$ the output in $\Sigma^k \times X$ should be the same. Chiefly, this means that $J_w(v)$, the induced embedding J_w created from $w \in \Sigma^k$ is equivalent to $j(v)$ when viewed via the composition. That is:

$$\begin{tikzcd}
(J_w, v) \arrow[rrr, mapsto] & & & (J_w, J_w(v)) \\
& \mathrm{Emb}(S^2, X) \times S^2 \arrow[r, hook, "u"] \arrow[dr, "(J \times 1_X)^{-1}"] & \mathrm{Emb}(S^2, X) \times X & \\
& \Sigma^k \times S^2 \arrow[u, "J \times 1_{S^2}"] \arrow[r, hook, "\widehat{J}"] & \Sigma^k \times X \arrow[u, "J \times 1_X"'] & \\
(w, v) \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & (w, J_w(v)) \arrow[uuu, mapsto]
\end{tikzcd}$$

Thus, we have that $T_{j(v)}X/TS^2 \cong T_{J_w(v)}X/TS^2$, which finally implies that $N_{w,J_w(v)}\Sigma^k \times S^2$ is isomorphic to $\mathcal{N}$ via the pullback. $\square$

Utilizing this isomorphism, let us now examine the Euler class of the universal normal bundle. Since the bundles are isomorphic via the pullback, we have the following:

$$(J \times 1_{S^2})^* e(\mathcal{N}) = e(N_{\Sigma^k \times X}(\Sigma^k \times S^2))$$

The right-hand side of this equality is an element of $H^2(\Sigma^k \times S^2)$. Set $i^v_\Sigma : \Sigma^k \to \Sigma^k \times S^2$ where $i^v_\Sigma(\theta) = (\theta, v)$ and set $i^\theta_S : S^2 \to \Sigma^k \times S^2$ where $i^\theta_S(v) = (\theta, v)$. By the Künneth theorem, the map:

$$\big((i^v_\Sigma)^*, (i^\theta_S)^*\big) : H^2(\Sigma^k \times S^2) \to H^2(\Sigma^k) \times H^2(S^2)$$

is an isormorphism. Thus, we consider the two elements

$$(i_{\Sigma}^{v})^{*}(J \times 1_{S^2})^{*}e(\mathcal{N}) = (J \times \underline{v})^{*}e(\mathcal{N}) \in H^2(\Sigma^k)$$
$$(i_{S}^{\theta})^{*}(J \times 1_{S^2})^{*}e(\mathcal{N}) = (\underline{\theta} \times 1_{S^2})^{*}e(\mathcal{N}) \in H^2(S^2)$$

Where $\underline{v}$ and $\underline{\theta}$ are constant maps. Hence, the Euler class of the neighborhood of the embedding $e(N_{\Sigma^k \times X}(\Sigma^k \times S^2))$ is completely determined by: $(J \times \underline{v})^{*}e(\mathcal{N})$ and $(\underline{\theta} \times 1_{S^2})e(\mathcal{N})$. Since we embedded the original S^2 into our manifold X with self-intersection n, we know that the latter $(\underline{\theta} \times 1_{S^2})^{*}e(\mathcal{N}) = n$. The former highlights the concern we have discussed in that the parameter component may incur some "twisting". It is this twisting that we seek to avoid. So, to define the twisting invariant, let us now define a sub-set, in fact normal subgroup, Γ of $H_k(\mathrm{Emb}(S^2, X))$ by:

$$\Gamma := \left\{\gamma \in H_k(\mathrm{Emb}(S^2, X)) \mid \gamma = \left[J : \Sigma^k \to \mathrm{Emb}(S^2, X)\right],\ (J \times \underline{v})^{*}e(\mathcal{N}) = 0\right\}$$

Proposition 4.2.4. Γ is a normal subgroup of $H_k(\mathrm{Emb}(S^2, X))$.

Proof. Since $H_k(\mathrm{Emb}(S^2, X))$ is abelian if Γ is a subgroup, it is a normal subgroup. First, the identity element in $0 \in H_k(\mathrm{Emb}(S^2, X))$ can be represented by a map J_0 that maps the Σ^k simplex to the trivial embedding of S^2. Meaning that $(J_0 \times \underline{v})^{*}e(\mathcal{N}) = 0$. Second, for $\gamma \in \Gamma$, it can be represented by a map $J_\gamma : \Sigma^k \to \mathrm{Emb}(S^2, X)$. This simplex $\Sigma^k := \Sigma a_k \sigma_k$, where each $\sigma_k : \Delta^k \to \mathrm{Emb}(S^2, X)$, can be thought of as a polytope created by gluing together standard simplices σ_k with a weight $a_k = \pm 1$. Meaning that the element γ^{-1} can be simply represented as a map $J_{\gamma^{-1}} : -\Sigma^k \to \mathrm{Emb}(S^2, X)$, where $-\Sigma^k$ is created from Σ^k by simply multiplying the weights by -1. Finally, all that remains to be shown is that for any $\gamma_1, \gamma_2 \in \Gamma$ that $\gamma_1 + \gamma_2 \in \Gamma$. Since $\gamma_1, \gamma_2 \in \Gamma$, they can be represented by maps J_{γ_1} on Σ_1 and J_{γ_2} on Σ_2. Thus, let us define a map $J_{\gamma_1+\gamma_2}$ on $\Sigma_1 \sqcup \Sigma_2$ that is simply J_{γ_1} on Σ_1 and J_{γ_2} on Σ_2. Hence, since cohomology distributes across disjoint union we have that:

$$(J_{\gamma_1+\gamma_2} \times \underline{v})^{*}\, e(\mathcal{N}) = 0$$

$\square$

Definition 4.2.5 (Twisting Invariant)**.** Given a family of embeddings $J : \Sigma^k \to \mathrm{Emb}(S^2, X)$, we define an invariant to detect parameter twisting called η, where:

$$\eta : H_k(\mathrm{Emb}(S^2, X)) \to H_k(\mathrm{Emb}(S^2, X))/\Gamma$$

by $\eta(J) = [J : \Sigma^k \to \mathrm{Emb}(S^2, X)]$

Remark. In the definition of the twisting invariant, we did not specify if the embedding space was of smooth or topological (locally flat) embeddings. In reality, it is defined in the locally flat case. In the smooth case, there is no issue with the existence of a normal bundle, but one may worry in the locally flat case. By Hatcher ([21]), we know that over a locally flat space the fibres of normal bundles on charts can all be made parallel. Despite now being parallel, the fibres may not be sufficiently compatible to be called a normal bundle. In our case, these fibres are real 2-dimensional, and standardizing the fibres is a map in $\mathrm{Emb}((\mathbb{R}^2, 0), (\mathbb{R}^2, 0))$, that is, a map on $\mathbb{R}^2$ that fixes the origin. Kister showed in [24] that $\mathrm{Emb}((\mathbb{R}^2, 0), (\mathbb{R}^2, 0))$ deformation retracts onto $\mathrm{Homeo}(\mathbb{R}^2, 0)$. Moreover, Friberg shows in [14] that $\mathrm{SO}(2)$ deformation retracts onto $\mathrm{Homeo}(\mathbb{R}^2, 0)$. A bundle is classified by maps into its classifying space, up to homotopy. Therefore, our bundle is classified by maps into $B\mathrm{SO}(2)$. The classifying space of $\mathrm{SO}(2) \simeq S^1$ is $S^\infty/S^1 \simeq \mathbb{CP}^\infty$. By examining the long exact sequence of homotopy groups, the Hurewicz morphism, and Poincaré duality, we have that $H^2(\mathbb{CP}^\infty) \simeq \mathbb{Z}$ is generated by the Euler class, the same that we use for the definition of the twisting invariant. Therefore, our twisting invariant is properly defined on locally flat embeddings and smooth embeddings.

This invariant allows us to detect twisting in the parameter component. Informally, if a family of embeddings is zero under η there is no twisting in the normal bundles that will prevent a creation of submanifold sum, but if the family is non-zero, then there is some obstruction to the creation of a submanifold sum. Formally, we have that:

Theorem 4.2.6. *Given a family of embeddings $J : \Sigma^k \to Emb(S^2, X)$, if $\eta[J] = 0$, there exists an orientation reversing isomorphism of the normal bundles of $N_{\Sigma^k \times X}(J)$ and $N_{\Sigma^k \times n\overline{\mathbb{CP}}^2}(\widehat{J_{\overline{\mathbb{CP}}^2}})$.*

Proof. It is enough to show that the Euler classes of the two bundles sum to zero. By Lemma 4.2.1, since $\eta[J] = 0$, we know that:

$$(J \times 1_{S^2})^* : H^2(\mathrm{Emb}(S^2, X) \times S^2) \to H^2(\Sigma^k) \oplus H^2(S^2)$$

when restricted to $\Sigma^k \times \{\mathrm{pt}\}$ maps only to the first component, and specifically to zero in the first component. Thus, the map restricted to $\{\mathrm{pt}\} \times S^2$ completely determines the Euler class. More precisely, the S^2 is embedded into X with self-intersection n. Hence, $e(N_{\Sigma^k \times X}(J)) = n$. Similarly, when we examine $e(N_{\Sigma^k \times n\overline{\mathbb{CP}}^2}(\widehat{J_{\overline{\mathbb{CP}}^2}}))$, let us recall that

$$\widehat{J_{\overline{\mathbb{CP}}^2}} : \Sigma^k \times S^2 \to \Sigma^k \times n\overline{\mathbb{CP}}^2$$

is identity on the first component and embeds S^2 into $n\overline{\mathbb{CP}}^2$ as the sum the of the exceptional divisors. Once again, the second component wholly determines the Euler class. Since the sum of the exceptional divisors in $n\overline{\mathbb{CP}}^2$ has self-intersection $-n$, the Euler class of $e(N_{\Sigma^k \times n\overline{\mathbb{CP}}^2}(\widehat{J_{\overline{\mathbb{CP}}^2}})) = -n$.

Note, the Euler classes are the same number, but with opposite orientation. Thus, we can conclude that there is an orientation reversing isomorphism of the normal bundles. □

Using Theorem 4.2.6 and the twisting invariant we can now detect when a submanifold sum is possible or not. This means that for any family of embeddings J such that $\eta[J] = 0$, we are guaranteed a family of spaces $\mathcal{Z}_J$, as in definition 4.1.1.

Recall, we aim to define a family invariant, to detect exotic smooth structures, using these families of spaces. If the conditions of Theorem 4.2.6 are satisfied, we know that an isomorphism to create such a family of spaces does exist, but is this isomorphism unique? Thankfully, it is unique. The following lemmas are used to prove this in Theorem 4.2.9.

Lemma 4.2.7. *If K_{-n} denotes the sphere of self-intersection $(-n)$ in $n\overline{\mathbb{CP}}^2$ created by summing together the exceptional divisors, then there is an S^1 action on $n\overline{\mathbb{CP}}^2$ such that K_{-n} is in the fixed point set of this circle action.*

Proof. We will proceed by induction on n. In the base case of $n = 0$, we assume our embedded sphere has no self-intersection and is embedded into the usual 4-sphere. Viewing the 4-sphere as:

$$S^4 = \left\{(w, z) \in \mathbb{R}^3 \times \mathbb{C} \mid \ |w|^2 + |z|^2 = 1\right\}$$

we can easily see the circle action on S^4 by $\lambda[w, z] = [w, \lambda z]$. The fixed point of this action is when $z = 0$, which means that the set is all vectors in $\mathbb{R}^3$ with norm one, i.e. the equatorial 2-sphere with zero self-intersection.

When we consider $n = 1$, we select a point on the equatorial 2-sphere and blow-up. This blow-up changes the underlying 4-manifold to $\overline{\mathbb{CP}}^2$. We may view the circle action on $\overline{\mathbb{CP}}^2$ similar to the $n = 0$ case:

$$\lambda[z_0 : z_1 : z_2] = [z_0 : z_1 : \lambda z_2]$$

To examine the fixed-point set, we have two cases: either $z_2 = 0$, or both $z_0 = z_1 = 0$. Thus, our fixed point set is the embedded $\mathbb{CP}^1 \cong S^2$ and a single point $[0, 0, 1]$. To transition to higher n, we continue to select a point on the embedded $\mathbb{CP}^1$ and blow-up. It is sufficient to examine the $n = 2$ case, as all higher cases rely on a similar argument. In the $n = 2$ case, we are examining a self-intersection -2 sphere embedded into $2\overline{\mathbb{CP}}^2$. To examine the circle action, we will analyze the blow-up in greater detail:

$$\mathrm{BL}\overline{\mathbb{CP}}^2 = L_{-1} = \left\{ ([z_0 : z_1], w_0, w_1) \in \overline{\mathbb{CP}}^1 \times \mathbb{C}^2 \quad \middle| \quad \begin{vmatrix} w_0 & w_1 \\ z_0 & z_1 \end{vmatrix} = 0 \right\}$$

We may view the circle action, similar to before, as multiplication by λ to the right-factors:

$$\lambda([z_0 : z_1], w_0, w_1) = ([z_0 : \lambda z_1], w_0, \lambda w_1)$$

Therefore, the fixed point set requires that $w_1 = 0$, implying two cases we need to examine: either w_0 is also zero, or w_0 is nonzero. If $w_0 \neq 0$, then z_1 must be zero, and z_0 can be turned into 1. If $w_0 = 0$, then either z_0 or z_1 may be zero, but not both of them. In summary, we have the following components of our fixed-point set:

$$([1:0], w_0, 0) \qquad ([1:0], 0, 0) \qquad ([0:1], 0, 0)$$

where $w_0 \neq 0$. Thus, similar to the $n = 1$ case, the fixed-point set is comprised of our section with self-intersection -2 and two points. Proceeding by induction shows that the circle action preserves the underlying submanifold: K_{-n}. $\square$

Lemma 4.2.8. *Given a family of embeddings $J : \Sigma^k \to Emb(S^2, X)$ with $\eta[J] = 0$, suppose φ_1 and φ_2 are orientation reversing isomorphisms of the normal bundles:*

$$\begin{tikzcd} N_{\Sigma^k \times S^2}(J) \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_2"'] & -N_{\Sigma^k \times n\overline{\mathbb{CP}}^2}(J) \end{tikzcd}$$

Since these bundles are isomorphic, denote them as N. Then, the automorphism group of N is isomorphic to the set of maps from $\Sigma^k \times S^2$ to S^1, that is:

$$Aut(N) \cong Maps(\Sigma^k \times S^2, S^1)$$

Proof. First, let us recall for a principal G-bundle P over X, that the automorphisms of P are isomorphic to sections of the adjoint bundle $\mathrm{Ad}(P)$. Second, let us note that our normal bundle N is an oriented $\mathbb{R}^2$ bundle:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & N \arrow[d] \\ & \Sigma^k \times S^2 \end{tikzcd}$$

Via the Borel construction, we observe that $N = \mathrm{SO}(N) \times_{\mathrm{SO}(2)} \mathbb{R}^2$, and from this we see that

$\mathrm{Aut}(N) = \mathrm{SO}(N) \times_{\mathrm{Ad},\mathrm{SO}(2)} \mathrm{SO}(2)$. Moreover, note that $S^1 \cong \mathrm{SO}(2)$ is abelian, meaning that it is trivial under the adjoint action and so $\mathrm{Ad}(SO(N)) = \Sigma^k \times S^2 \times \mathrm{SO}(2)$, and we conclude that sections of $\mathrm{Ad}(SO(N))$ are identified with $\mathrm{Maps}(\Sigma^k \times S^2, S^1)$. $\square$

Theorem 4.2.9. *Given a family of embeddings $J : \Sigma^k \to Emb(S^2, X)$ with $\eta[J] = 0$, if there are two orientation reversing isomorphisms of the normal bundles φ_1 and φ_2:*

$$\begin{tikzcd} N_{\Sigma^k \times X}(J) \arrow[rr, bend left, "\varphi_1"] \arrow[rr, bend right, "\varphi_2"'] & & -N_{\Sigma^k \times n\overline{\mathbb{CP}}^2}(J) \end{tikzcd}$$

then the associated families of spaces, denoted by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, created via submanifold sum using φ_1 and φ_2, respectively, are isomorphic.

Proof. By Lemma 4.2.8, we know that the connected components of $\mathrm{Aut}(N)$ are in bijection with the homotopy classes of maps from $\Sigma^k \times S^2$ to S^1, that is: $\pi_0(\mathrm{Aut}(N)) \cong \left[\Sigma^k \times S^2, S^1\right]$. This use of homotopy classes of maps allows us to utilize a nice property of Eilenberg-MacLane spaces. Since $S^1 \cong K(\mathbb{Z}, 1)$, we may relate these automorphisms to the first cohomology:

$$\begin{aligned} \pi_0(\mathrm{Aut}(N)) &\cong \left[\Sigma^k \times S^2, S^1\right] \\ &\cong \left[\Sigma^k \times S^2, K(\mathbb{Z}, 1)\right] \\ &\cong H^1(\Sigma^k \times S^2; \mathbb{Z}) \end{aligned}$$

Now, we utilize the universal coefficient theorem followed by the Künneth theorem, to reduce this cohomology of the product, to simply the cohomology of Σ^k.

$$\begin{aligned}
H^1(\Sigma^k \times S^2; \mathbb{Z}) &\cong \mathrm{Hom}(H_1(\Sigma^k \times S^2), \mathbb{Z}) && \text{U.C.T.} \\
&\cong \mathrm{Hom}\big((H_0(\Sigma^k) \otimes H_1(S^2)) \oplus (H_1(\Sigma^k) \otimes H_0(S^2)), \mathbb{Z}\big) && \text{Künneth} \\
&\cong \mathrm{Hom}\big(H_1(\Sigma^k), \mathbb{Z}\big) \\
&\cong H^1(\Sigma^k; \mathbb{Z}) && \text{U.C.T.}
\end{aligned}$$

Therefore, by one final use of the nice property of Eilenberg-MacLane spaces, we have that: $\pi_0(\mathrm{Aut}(N)) \cong \big[\Sigma^k, S^1\big]$. Thus, for the automorphism class $[\lambda] := [\varphi_2^{-1} \circ \varphi_1]$, we can represent $[\lambda]$ as a map $\lambda : \Sigma^k \to S^1$. From this, we define a family diffeomorphism Λ:

$$\begin{tikzcd}
\Lambda : \Sigma^k \times n\overline{\mathbb{CP}}^2 \arrow[rr] \arrow[dr] & & \Sigma^k \times n\overline{\mathbb{CP}}^2 \arrow[dl] \\
& \Sigma^k &
\end{tikzcd}$$

by $\Lambda(s, x) = (s, \lambda(s) \cdot x)$, where $\lambda(s) \cdot x$ is the circle action defined in Lemma 4.2.7. Finally, we can construct an isomorphism Φ between the families of spaces:

$$\begin{tikzcd}
\Phi : \mathcal{Z}_1 \arrow[rr] \arrow[dr] & & \mathcal{Z}_2 \arrow[dl] \\
& \Sigma^k &
\end{tikzcd}$$

To create Φ, first recall that

$$\mathcal{Z}_1 := \big((\Sigma^k \times X) \setminus (\Sigma^k \times S^2)\big) \bigcup_{\varphi_1} \Big((\Sigma^k \times n\overline{\mathbb{CP}}^2) \setminus (\Sigma^k \times S^2)\Big)$$

and $\mathcal{Z}_2$ is given by the analogous construction using φ_2. Given this presentation, we define our map Φ by:

$$\Phi = \begin{cases} \mathrm{Id} & \text{, on } (\Sigma^k \times X) \setminus (\Sigma^k \times S^2) \\ \Lambda & \text{, on } (\Sigma^k \times n\overline{\mathbb{CP}}^2) \setminus (\Sigma^k \times S^2) \end{cases}$$

By Lemma 4.2.7 we know that the circle action of Λ extends over the entire manifold, therefore, by the pasting lemma, we have that Φ is an isomorphism.

□

4.2.1 The Twisting Invariant and our Families

The twisting invariant we have defined here applies to any family. Now, let us emphasize how it applies to our constructed families. Recall that in the construction of our families of embeddings, we defined $\widehat{j}^k_{i,n} : S^k \times S^2 \to S^2 \times Z^{k,r}$ which is an extension of the adjoint from:

$$j^k_{i,n} : S^k \to \mathrm{Emb}(S^2, Z^{k,r})$$

This family is parameterized by S^k, which makes these calculations markedly easier. Recall that the twisting invariant is utilized to detect obstructions to a submanifold sum between a given family and a special family $\widehat{J_{\overline{\mathbb{CP}}^2}}$. In general there can be an obstruction, but when our families are paramterized by k-spheres, $k \neq 2$, we do not have this obstruction.

Indeed, in order to perform a submanifold sum in our case, we need to examine the Euler class of the normal bundles of $S^k \times S^2$ inside $S^k \times Z^k$ and inside $S^k \times n\overline{\mathbb{CP}}^2$. By construction of

$$\widehat{J_{\overline{\mathbb{CP}}^2}} : S^k \times S^2 \to S^k \times n\overline{\mathbb{CP}}^2$$

we have that it is identity on the first component, and maps to the sum of the exceptional divisors in the second component. This means that the Euler class is completely determined by the second component. So, the Euler class of this normal bundle is $(-n)PD[S^k \times \underline{v}]$, where PD is the Poincaré dual.

The potential obstruction appears when we examine the families $\widehat{j}^k_{i,n}$. For our constructions, since the Euler class is an element of the second cohomology, we have that:

$$e(N_{S^k \times X}(\widehat{j}^k_{i,n})) \in H^2(S^k \times S^2) \cong H^2(S^2)$$

Thus, the second component once again determines the Euler class. Meaning that the Euler class of this normal bundle is also n, since the S^2 is embedded into Z with self-intersection n.

One may worry that when $k = 2$, the existence of cohomology in the parameter component may prove an obstruction. However, due to how the families are created, we do not have an obstruction.

Proposition 4.2.10. The twisting invariant applied to our families of embeddings satisfies

$$\eta\left(\left[j^2_{i,n}\right]\right) = 0$$

Proof. By definition of η, we know that $\eta\left(\left[j^2_{i,n}\right]\right) = 0$ if and only if there is a representative J such that:

$$\left[j^2_{i,n}\right] = [J : \Sigma^2 \to \mathrm{Emb}(S^2, Z^{2,r})]$$

with $(J \times \underline{v})^* e(\mathcal{N}) = 0$. Thus, let us take $J = j^2_{i,n}$ and $\Sigma^2 = S^2$, and we simply need to show that: $(j^2_{i,n} \times \underline{v})^* e(\mathcal{N}) = 0$. By Lemma 4.2.1 we know that the normal bundle of $\widehat{j}^2_{i,n}$ is isomorphic to the pullback of $\mathcal{N}$ under $(j^2_{i,n} \times 1_{S^2})$:

$$N_{S^2 \times Z^{2,r}}(S^2 \times S^2) \cong (j^2_{i,n} \times 1_{S^2})^* \mathcal{N}$$

Recall in our original construction of j_Y, respectively j_X, in section 2.3, that the embedding of S^2 in Y^+ was a modification standard embedding of the hyperplane section into the $\mathbb{CP}^2$ summand of Y^+ where we push a "finger" of the sphere into $N(T)$, but still disjoint from T. If we select our point v such that $j^2_{i,n}(\theta, v) \in N(T) \setminus T$ we observe that all maps and constructions have been trivial on $N(T)$. Let B_{z_0} be a standard ball around $j^2_{i,n}(\theta, v) = z_0$, and denote the ball B_v as a ball in S^2 around v via the inverse of $j^2_{i,n}$.

Let us now define several maps and normal bundles guided by the diagram. Let ι be simple inclusion maps and let $i^v_S(\theta) = (\theta, v)$ and $i^B_S(\theta, w) = (\theta, w)$. The normal bundles, listed in the third column, are defined as follows. Denote the bundle $\mathcal{N}$ as the universal normal bundle, the bundle $N_{S^2\times Z^{2,r}}(S^2\times S^2)$ as the bundle defined over $\widehat{j}^2_{i,n}$, and the bundle $N_{S^2\times B_{z_0}}(S^2\times B_v)$ as the bundle defined over $\widehat{j}^2_{i,n}|_{S^2\times B_v}$.

$$
\begin{tikzcd}
\mathcal{N} \arrow[dr] & & \\
& \mathrm{Emb}(S^2, Z^{2,r})\times S^2 \arrow[r, "u"] & \mathrm{Emb}(S^2, Z^{2,r})\times Z^{2,r} \\
N_{S^2\times Z^{2,r}}(S^2\times S^2) \arrow[dr] & & \\
& S^2\times S^2 \arrow[uu, "j^2_{i,n}\times 1_{S^2}"] \arrow[r, "\widehat{j}^2_{i,n}"] & S^2\times Z^{2,r} \arrow[uu, "j^2_{i,n}\times 1_{Z^{2,r}}"'] \\
N_{S^2\times B_{z_0}}(S^2\times B_v) \arrow[dr] & & \\
& S^2\times B_v \arrow[uu, "i^B_S"] \arrow[r, "\widehat{j}^2_{i,n}|_{S^2\times B_v}"] & S^2\times B_{z_0} \arrow[uu, hook, "\iota"] \\
& S^2\times\{v\} \arrow[u, "i^v_S"] &
\end{tikzcd}
$$

By repeated use of Lemma 4.2.1, we have the following chain of equalities when evaluated on the fundamental class of the parameter S^2:

$$
\begin{aligned}
(j^2_{i,n}\times \underline{v})^* e(\mathcal{N})[S^2] &= (i^v_S)^*\left(i^B_S\right)^*(j^2_{i,n}\times 1_{S^2})^* e(\mathcal{N})[S^2] \\
&= (i^v_S)^*\left(i^B_S\right)^* e((j^2_{i,n}\times 1_{S^2})^*\mathcal{N})[S^2] \\
&= (i^v_S)^*\left(i^B_S\right)^* e(N_{S^2\times Z^{2,r}}(S^2\times S^2))[S^2] \\
&= (i^v_S)^* e(N_{S^2\times B_{z_0}}(S^2\times B_v))[S^2] \\
&\cong e((i^v_S)^* N_{S^2\times B_{z_0}}(S^2\times B_v))[S^2]
\end{aligned}
$$

To finally show that the parameter S^2 does not influence the calculation, let us define one more normal bundle $N_{B_{z_0}}(B_v)$ guided by the following diagram:

$$\begin{tikzcd}
N_{S^2\times B_{z_0}}(S^2\times B_v) \arrow[dr] & & \\
& S^2\times B_v \arrow[r, "\widehat{j}^2_{i,n}|_{S^2\times B_v}"] \arrow[d, "P_2"] & S^2\times B_{z_0} \arrow[d, "P_2"] \\
N_{B_{z_0}}(B_v) \arrow[r] & B_v \arrow[r, "\widehat{j}^2_{i,n}|_{B_v}"] & B_{z_0}
\end{tikzcd}$$

Where P_2 is simply projection onto the second factor. By one final use of Lemma 4.2.1 we note that $N_{S^2\times B_{z_0}}(S^2\times B_v)\cong P_2^*N_{B_{z_0}}(B_v)$. Thus we have that:

$$\begin{aligned}
(j^2_{i,n}\times \underline{v})^*e(\mathcal{N})[S^2] &= e((i^v_S)^*\,N_{S^2\times B_{z_0}}(S^2\times B_v))[S^2]\\
&= e((i^v_S)^*\,P_2^*N_{B_{z_0}}(B_v))[S^2]\\
&= e((P_2\circ i^v_S)^*N_{B_{z_0}}(B_v))[S^2]\\
&= e(N_{B_{z_0}}(B_v))[(P_2\circ i^v_S)_*(S^2)]\\
&= 0
\end{aligned}$$

Since $(P_2\circ i^v_S)_*(S^2)$ is trivial, since the maps P_2 and i^v_S are independent of the parameter S^2. $\square$

Hence, we see that there is no obstruction to the creation of families of manifolds via submanifold sum using our families of embeddings $\widehat{j}^k_{i,n}$ and the standard family $\widehat{J}_{\overline{\mathbb{CP}}^2}$ for any k.

Definition 4.2.11. We define these families of manifolds created by submanifold sum between our constructed families of embeddings $\widehat{j}^k_{i,n}$ and the standard family $\widehat{J}_{\overline{\mathbb{CP}}^2}$ by:

$$\mathcal{Z}_{\widehat{j}^k_{i,n}} := (S^k\times Z^{k,r})\#_{S^k\times S^2}(S^k\times n\overline{\mathbb{CP}}^2)$$

4.3 The Family Invariant

We have shown that for nice families of embeddings the isomorphism required for submanifold sum both exists and is unique. Finally, we can define our family invariant in order to show the exotic nature of our families of embeddings. This family invariant is based upon Konno's invariant from [25]. Konno's invariant associates to a family of 4-manifolds, with auxilary data, a cohomology class in the base. Let X be a closed 4-manifold and let $\mathfrak{P}$ be an equivalence class of an SO(3) bundle where $w_2(\mathfrak{P}) \neq 0$. Let $\mathcal{O}$ be data to specify an orientation on the Yang-Mills moduli space of X. Denote

$$\mathrm{Diff}(X, \mathfrak{P}, \mathcal{O}) := \left\{ f \in \mathrm{Diff}^+(X) \mid f^*\mathfrak{P} = \mathfrak{P}, f^*\mathcal{O} = \mathcal{O} \right\}$$

Konno defines a characteristic class $\mathbb{D}(X, \mathfrak{P}, \mathcal{O}) \in H^n(\mathrm{BDiff}(X, \mathfrak{P}, \mathcal{O}); \mathbb{Z})$

Remark. In [8], Donaldson uses an equivalence class of a spin^c lift of the SO(3) bundle together with a homology orientation to define an orientation on the Yang-Mills moduli space. We are denoting the spin^c lift as $\mathbf{c} \in \widehat{C}^k_X$ and the homology orientation by β. Auckly and Ruberman use this data to define their orientation on the Yang-Mills moduli space. In the proof of Proposition 3.25 in [8], Donaldson notices that after taking connect-sums with sufficiently many copies of $S^2 \times S^2$, any 4-manifold will admit an almost complex structure. This leads to an alternate way one could define orientation on the Yang-Mills moduli space. Notice that the element $\mathbf{c} \in \widehat{C}^k_X$ determines the isomorphism class $\mathfrak{P}$, and one could view the orientation data $\mathcal{O}$ as the data $(\mathbf{c}, \beta)$. Thus, we will denote Konno's characteristic class by $\mathbb{D}(X, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$.

Let $X \hookrightarrow \mathcal{X} \to \Xi$ be a continuous family with structure group $\mathbb{D}(X, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$, where Ξ is a $(k+1)$-cycle. This bundle is determined by a classifying map $f_{\mathcal{X}} : \Xi \to \mathrm{BDiff}(X, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$. In this case, one obtains a number $\mathbb{D}(X, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))[f^{\mathcal{X}}_*[\Xi]]$.

Definition 4.3.1 (Family Invariant)**.** For a k-dimensional family of embeddings J such that $\eta[J] = 0$, we define the Family Invariant $D^{H_k(\mathrm{Emb})}_{\mathbf{c},\beta}$ to be:

$$D^{H_k(\mathrm{Emb})}_{\mathbf{c},\beta}(J) := \mathbb{D}(Z, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))[f^{\mathcal{Z}_J}_*[\Xi]]$$

That is, the invariant on a family of embeddings is Konno's invariant applied to the associated family of spaces.

We now show that Konno's invariant ([25]) on a family of spaces is equivalent to the Auckly-Ruberman invariant ([2]) on the associated family of diffeomorphisms. Auckly and Ruberman use parameterized Donaldson invariants. These invariants are created by extending the degree zero Donaldson invariants to families of manifolds. The degree zero invariants are defined on an SO(3) bundle, $P_{\mathbf{c}}$, over a specific manifold Z. This bundle is related to a class $\mathbf{c} \in \widehat{C}^k_Z$ where $\widehat{C}^k_Z$ is the set of integral lifts of cohomology classes on Z, modulo some equivalence relations fully outlined in [2, Section 4]. For a $(k+1)$-dimensional-chain of metrics, $\{g_\theta\}$, created from a k-dimensional family of diffeomorphisms on our manifold on Z, the invariant is a signed count of a finite moduli space of anti-self-dual connections, $\#\mathcal{M}(\{g_\theta\})$.

A priori these two invariants are not equivalent, but given the specific construction of families of diffeomorphisms by Auckly and Ruberman, they are equivalent.

Theorem 4.3.2. *Suppose Ξ is a closed, smooth, oriented manifold of dimension $k+1$ and $c \in \widehat{C}^k_Z$. Let $\mathcal{Z} \to \Xi$ be a smooth family supporting a bundle $\mathcal{P}_c$ whose restriction to each fibre Z is isomorphic to P_c. This bundle is classified by a map $f^{\mathcal{Z}} : \Xi \to \mathit{BDiff}(C, \mathfrak{P}_{\boldsymbol{c}}, (\boldsymbol{c}, \beta))$. Let $\mathfrak{P}_{\boldsymbol{c}}$ be the isomorphism class of P_c and β be the homology orientation on Z. If $\{g_\theta\}_{\theta\in\Xi}$ is a good family of metrics then $\mathbb{D}(Z, \mathfrak{P}_{\boldsymbol{c}}, (\boldsymbol{c}, \beta))[f^{\mathcal{Z}}_*[\Xi]] = \#\mathcal{M}(\{g_\theta\}_{\theta\in\Xi})$.*

Proof. The key idea is that, for a *good* family of metrics, the virtual neighborhood, $\mathcal{U}$, used in Konno's invariant, is equivalent to the moduli space, $\mathcal{M}$, used in the Auckly-Ruberman invariant. The notation used is quite different in each paper, but the ideas are similar. Konno uses cellular homology and Auckly-Ruberman use a variant of singular homology. Auckly and Ruberman use generic metrics to get a signed count of points, while Konno uses virtual neighborhoods to get proper counts, even when the metrics are not generic.

We review both definitions to fix notation. Let Ξ, $\mathcal{Z}$, and $\mathbf{c}$ be as in the theorem statement. Represent Ξ as a sum of transverse singular simplices, see [2, Definition 4.1]. Consider now the parameterized configuration space

$$\mathcal{B}^*(\Delta) := \bigsqcup_{\theta \in \Delta} \mathcal{B}^*(\mathcal{Z}_\theta)$$

Where $\mathcal{B}^*(\mathcal{Z}_\theta)$ are irreducible connections, modulo gauge, on $\mathcal{Z}_\theta$. Konno uses $\mathscr{X}$ to denote this space. One distinction is that Auckly-Ruberman use L^3_1 connections with curvature in L^2 to define their configuration space. Konno uses L^2_k connections for large k to define his configuration space, ensuring that it is a Hilbert manifold. The fact that these constructions give equivalent moduli spaces will follow from elliptic regularity theory see, for example, Donaldson and Kronheimer [9, Section 7.2.3] for a comparison and discussion the various functions spaces. Notice there is a natural projection $\mathscr{X} = \mathcal{B}^*(\Delta) \to \Delta$.

Konno first represents cycles in $\mathsf{BDiff}(C, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$ via CW complexes, and then extends to arbitrary spaces representing cycles via a result from algebraic topology [25, Lemma 6.8]. Any simplicial complex is a CW complex; therefore, the Auckly-Ruberman setting is covered Konno's construction.

To define the value of Konno's characteristic class on a CW complex representing a cycle in $\mathsf{BDiff}(C, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$, Konno defines an inductive section of the bundle of fibre-metrics over the base space. These will be fibre-metrics so that the paramterized moduli space over the k-skeleta vanish. The good family of metrics $\{g_\theta\}_{\theta \in \Xi}$ defines such an inductive section.

The paramterized Yang-Mills may be encoded in a section of a Hilbert manifold over $\mathcal{B}^*(\Delta)$. Let $\{g_\theta\}_{\theta \in \Xi}$ be a good family of metrics, as in the theorem statement. Now, define

$$\mathcal{E}^*(\{g_\theta\}_{\theta \in \Delta}) := \bigsqcup_{\theta \in \Delta} \mathcal{A}^*(P_\theta) \times_{\mathcal{G}_\theta} \Omega^2_+(\mathrm{Ad}_* P_\theta)$$

Where P_θ is the restriction of P_c to the fibre $\mathcal{Z}_\theta$, $\mathcal{A}^*(P_\theta)$ is the space of irreducible connections on P_θ, $\mathcal{G}_\theta$ is the gauge group on P_θ, and $\Omega^2_+(\mathrm{Ad}_* P_\theta)$ is the space of 2-forms that are self-dual with respect to g_θ. Konno denotes this bundle by $\mathscr{E}$.

By varying θ, one could define a vector bundle with fibres $\Omega^2_+(\mathrm{Ad}_* P_\theta)$ over Ξ. Since the space of metrics is contractible, this bundle is trivial. Donaldson and Kronheimer [9] introduced a specific trivialization based conformal geometry and a reference metric h. Auckly and Ruberman use this formulation and define a map:

$$\mathcal{F}_h : \Xi \times \mathcal{A} \to \Omega^2_+(h)$$

Such that $\mathcal{F}_h(\theta, \mathcal{A}) = 0$ if and only if $\mathcal{A}$ is anti-self-dual with respect to g_θ [2, Definition 6.5]. Via a change of coordinates, this generates a section of the bundle $\mathcal{E}^* \to \mathcal{B}^*$, which Konno denotes by $s : \mathscr{E} \to \mathscr{X}$. We define this section by:

$$s(\theta, [A]) = (\theta, [A, F_A^{+g_\theta}])$$

Where $F_A^{+g_\theta}$ is the g_θ-self-dual part of the curvature.

Since $\{g_\theta\}_{\theta\in\Xi}$ is a good family of metrics, the derivative of the section, ds, is surjective [2, Definition 6.6]. To define Konno's invariant, Konno uses a family version of Ruan's ([30]) virtual neighborhood technique requiring the addition of a real vector space and a collection linear function in order to create a finite dimensional perturbation of the section, he denotes by $\tilde{s}$. Since the derivative of our section is already surjective we take the real vector space to be the zero space. Thus, the finite dimensional perturbation of the section is simply the section s itself.

This means that the family virtual neighborhood, which Konno denotes by $\mathcal{U}$, is equivalent to the moduli space $M(\{g_\theta\})$. Moreover, the projection map $\mathcal{U} \to \mathscr{X}$ is simply the inclusion map $M(\{g_\theta\}) \to \mathcal{B}^*$. The value of Konno's characteristic class on the cell represented by the simplex Δ is the fibre integral of the cap product of the Euler class of the trivial bundle $\mathcal{U} \times \mathbb{R}^N \to \mathcal{U}$ with the pullback of $[1] \in H^0(\mathscr{X})$. In our case, $N = 0$, and $\mathcal{U}$ is

$M(\{g_\theta\})$, and the pullback of [1] is just [1]; thus,

$$\mathbb{D}(Z, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))[\Delta] = \#\mathcal{M}(\{g_\theta\}_{\theta \in \Delta})$$

We are writing $\Xi = \Sigma a_\tau \tau$ where $\tau : \Delta^{k+1} \to \mathsf{BDiff}(C, \mathfrak{P}_{\mathbf{c}}, (\mathbf{c}, \beta))$. Summing over the transverse simplices, τ gives the desired result. $\square$

Chapter 5

Computing the Invariant

5.1 Associated Families

With the invariant defined, we now examine the families of spaces $\mathcal{Z}_{\widehat{j}^k_{i,n}}$ associated to our constructed family of embeddings $\widehat{j}^k_{i,n}$. Recall in section 4.2.11 we used submanifold sum to define:

$$\mathcal{Z}_{\widehat{j}^k_{i,n}} := (S^k \times Z^{k,r})\#_{S^k \times S^2}(S^k \times n\overline{\mathbb{CP}}^2)$$

This is a fibre bundle over S^k:

$$\begin{array}{ccc} Z^{k,r}\#_{S^2}n\overline{\mathbb{CP}}^2 & \hookrightarrow & \mathcal{Z}_{\widehat{j}^k_{i,n}} = (S^k \times Z^{k,r})\#_{S^k \times S^2}(S^k \times n\overline{\mathbb{CP}}^2) \\ & & \downarrow \\ & & S^k \end{array}$$

Any such family of spaces is classified by the $(k-1)$-homotopy group of diffeomorphisms of its fiber. This correspondence may be nicely understand via something called the *clutching construction.* Given $\gamma : S^{k-1} \to \mathrm{Diff}(Z)$, let us define a standardized bundle $\widehat{\mathcal{Z}}_\gamma$, similar to $\mathcal{Z}_{\widehat{j}^k_{i,n}}$, but where we view the sphere as two hemispheres glued to a thickened equator (or collar). Let us decompose the k-dimensional sphere by:

$$S^k = (D^k_+ \sqcup D^k_- \sqcup (I \times S^{k-1}))/\sim$$

Where $(0, \theta, z) \sim (-, \theta, z)$ and $(1, \theta, z) \sim (+, \theta, z)$. Using this decomposition, we define the standardized bundle by

$$\begin{array}{c} \widehat{\mathcal{Z}}_\gamma := \left((D^k_+ \sqcup D^k_- \sqcup (I \times S^{k-1})) \times Z\right) / \sim \\ \big\downarrow \\ S^k \end{array}$$

Where the submanifold $S^k \times S^2$ is embedded into $(D^k_+ \sqcup D^k_- \sqcup (I \times S^{k-1})) \times Z$ such that it is the trivial embedding in the lower hemisphere, and by γ in the upper hemisphere. In other words, the upper hemisphere is attached to the collar by: $(1, \theta, z) \sim (+, \theta, \gamma(\theta, z))$ and the lower hemisphere is attached to the collar by $(0, \theta, z) \sim (-, \theta, z)$. Using this standardized bundle, we will next define two family isomorphisms.

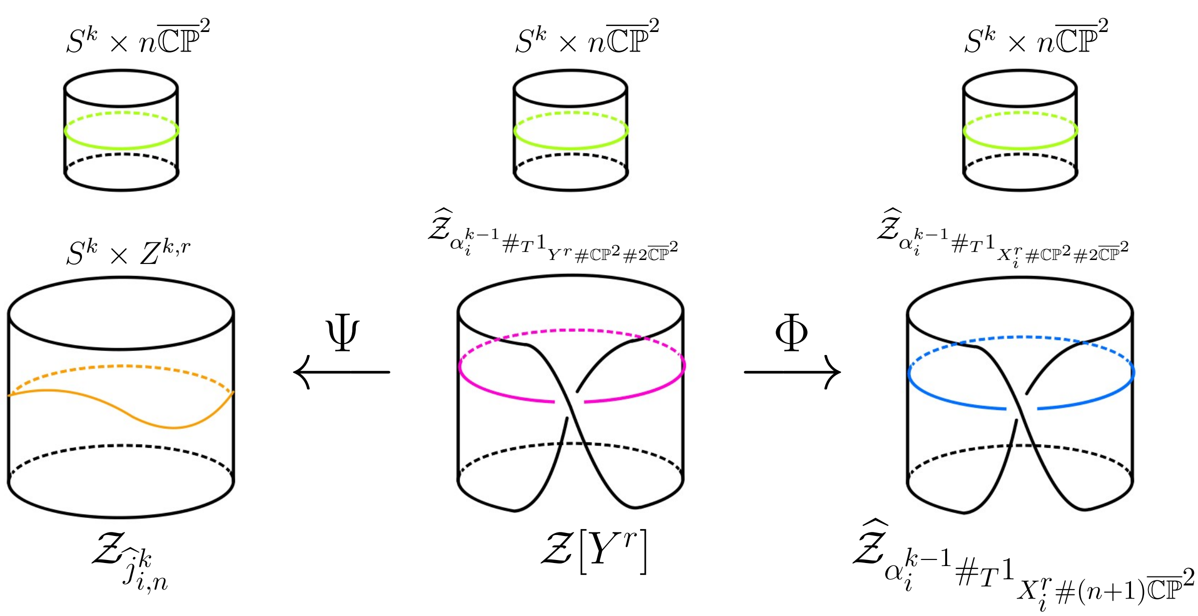


Figure 5.1: Isomorphisms between families

To define these isomorphisms, we refer to Figure 5.1 as a schematic. The top row of cylinders represents the standard family of embeddings $\widehat{J}_{\overline{\mathbb{CP}}^2} : S^k \times S^2 \to S^k \times n\overline{\mathbb{CP}}^2$ from section 4.1. Visualized in lime-green is this family $\widehat{J}_{\overline{\mathbb{CP}}^2}$, embedded in a standard way. The bottom-left cylinder represents the family of embeddings $\widehat{j}^k_{i,n}$ that we constructed in section

3.2.2. Visualized in orange is this family $\widehat{j}^k_{i,n}$, pictured more "curvy" as we hope to understand it better. The submanifold sum between the top-left and bottom-left cylinders over the families $\widehat{J}_{\overline{\mathbb{CP}}^2}$ and $\widehat{j}^k_{i,n}$ yields the family of manifolds $\mathcal{Z}_{\widehat{j}^k_{i,n}}$.

The second column of cylinders represents the submanifold sum between the families $\widehat{J}_{\overline{\mathbb{CP}}^2}$ and $\underline{\widetilde{j_{i,n}}} : S^k \times S^2 \to S^k \times \widehat{\mathcal{Z}}_{\alpha_i^{k-1} \#_T 1_{Y^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2}}$, where the latter are visualized in pink. Recall $j_{i,n}$ was the collection of embeddings defined in section 3.2 by embellishing the collection of embeddings $j_i = \varphi_i^{-1}(\psi_i \# 1_{\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2}) j_{Y^r}$. Additionally, recall that the tilde notation $\widetilde{j_{i,n}}$ means we restrict to the family $j_{i,n}$ in the **last** summand $Z^{0,r}$ in $Z^{k,r} = Z^{k-1,r} \#_T Z^{0,r}$. Finally, the underline notation means that this family is constant in the S^k component. This submanifold sum in the second column results in the family of manifolds

$$\mathcal{Z}[Y^r] := \widehat{\mathcal{Z}}_{\alpha_i^{k-1} \#_T 1_{Y^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2}} \#_{S^k \times S^2} (S^k \times n\overline{\mathbb{CP}}^2)$$

Similarly, the third column of cylinders represents the submanifold sum between the families $\widehat{J}_{\overline{\mathbb{CP}}^2}$ and $i_{X_i^r,n}$, where the latter are visualized in blue. The family $i_{X_i^r,n}$ is the embellished version of $i_{X_i^r} : S^2 \hookrightarrow \mathbb{CP}^2$, where this codomain is the $\mathbb{CP}^2$ summand of $X_i^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ from section 2.3. More formally, $i_{X^r_{i,n}} := 1 \#_T (\psi_i \# 1_{\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2}) j_{Y^r,n}))$, where 1 represents identity on all other portions of $X_i^r \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ except $\mathbb{CP}^2$. This submanifold sum in the third column results in the family of manifolds

$$\widehat{\mathcal{Z}}_{\alpha_i^{k-1} \#_T 1_{X_i^r \# (n+1)\overline{\mathbb{CP}}^2}}$$

We seek to better understand our bundle $\mathcal{Z}_{\widehat{j}^k_{i,n}}$ with a bundle created via the clutching construction. To do this, we will define family isomorphisms Ψ and Φ in the following ways. Define the map:

$$\Psi : \mathcal{Z}[Y^r] \to \mathcal{Z}_{\widehat{j}^k_{i,n}}$$

by

$$(-,\theta,z) \mapsto (-,\theta,z)$$
$$(t,\theta,z) \mapsto (t,\theta,F_i^{k-1}(t,\theta,z))$$
$$(+,\theta,z) \mapsto (+,\theta,z)$$

To define the map Φ let us first recall from section 2.3 that the diffeomorphisms $\varphi_i : Y^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2 \to X_i^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ and can be thought of as extensions of the original map $\phi : Y^+ \to X^+$ after we perform the log-transforms. Since φ_i are defined on $Y^r\#\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ we can extend them via the identity to $1\#_T\varphi_i$ so that $1\#_T\varphi_i$ is now defined on the whole of $Z^{k,r}$. With this, we define the map:

$$\Phi : \mathcal{Z}[Y^r] \to \widehat{\mathcal{Z}}_{\alpha_i^{k-1}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}}$$

by

$$(-,\theta,z) \mapsto (-,(1\#_T\varphi_i)(\theta,z))$$
$$(t,\theta,z) \mapsto (t,(1\#_T\varphi_i)(\theta,z))$$
$$(+,\theta,z) \mapsto (+,(1\#_T\varphi_i)(\theta,z))$$

Lemma 5.1.1. *The maps Ψ and Φ, as defined above, are well-defined family isomorphisms. Moreover, we have that:*

i) $\Psi \circ \widetilde{\underline{j_{i,n}}} = \widehat{j}_{i,n}^k$

ii) $\Phi \circ \widetilde{\underline{j_{i,n}}} = i_{X_i^r,n}$

Proof. We begin by examining Ψ. When $t = 0$ we have that: $\Psi(0,\theta,z) = (0,\theta,F_i^{k-1}(0,\theta,z)) = (-,\theta,z)$ since F_i^{k-1} is identity at time zero. When $t = 1$ we have that: $\Psi(1,\theta,z) = (1,\theta,F_i^{k-1}(1,\theta,z)) = (1,\theta,\alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta,z)) = (+,\theta,z)$ by our standard clutching construction. Thus Ψ is well-defined and since Ψ is either identity, or is determined by

F_i^{k-1}, which is a smooth isotopy, Ψ is a family isomorphism. Hence, let us now observe:

$$\begin{aligned}
\Psi \circ \underline{\widetilde{j_{i,n}}} &= \Psi(t, \theta, \underline{\widetilde{j_{i,n}}}(z)) \\
&= \Psi(t, \theta, 1\#_T\varphi_i^{-1}(\psi_i\#1_{\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2})j_{Y^r,n}) \\
&= (t, \theta, F_i^{k-1}\left(t, \theta, 1\#_T\varphi_i^{-1}(\psi_i\#1_{\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2})j_{Y^r,n}\right)) \\
&= (t, \theta, F_i^{k-1}\left(t, \theta, \widetilde{j_{i,n}}\right)) \\
&= \widehat{j}_{i,n}^k
\end{aligned}$$

Next, we examine Φ. When $t = 0$ we have that: $\Phi(0, \theta, z) = (0, (1\#_T\varphi_i)(\theta, z)) = (-, (1\#_T\varphi_i)(\theta, z))$. When $t = 1$, recall in construction of clutching that we had:

$$(1, \theta, z) \sim (+, \theta, \alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta, z))$$

Thus, we need to show that the following bottom-row equivalency is true:

$$\begin{tikzcd}
(1\theta, z) \arrow[r, dash, "\sim"] \arrow[d, mapsto, "\Phi"] & (+, \theta, \alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta, z)) \arrow[d, mapsto, "\Phi"] \\
(1, (1\#_T\varphi_i)(\theta, z)) \arrow[r, dash, "\overset{\text{NTS}}{\sim}"] & (+, (1\#_T\varphi_i)\alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta, z))
\end{tikzcd}$$

If we analyze the right column we see that:

$$\begin{aligned}
\Phi(+, \theta, \alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta, z))) &= (1, (1\#_T\varphi_i)(\alpha_i^{k-1}\#_T 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2}(\theta, z)))) \\
&= (1, (1 \circ \alpha_i^{k-1})\#_T(\varphi_i \circ 1_{Y^r\#(n+1)\overline{\mathbb{CP}}^2})(\theta, z)) \\
&= (1, (\alpha_i^{k-1} \circ 1)\#_T(1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2} \circ \varphi_i)(\theta, z)) \\
&= (1, (\alpha_i^{k-1}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2})(1\#_T\varphi_i)(\theta, z)) \\
&\sim (1, (1\#_T\varphi_i)(\theta, z))
\end{aligned}$$

Moreover, since Φ is well-defined and completely determined by the diffeomorphisms φ_i, it is a family isomorphism. Thus, all that remains to be shown is that $\Phi \circ \underline{\widetilde{j_{i,n}}} = i_{X_i^r,n}$. Consider

$$\begin{aligned}\Phi \circ \underline{\widetilde{j_{i,n}}} &= (t, (1\#_T\varphi_i)(\underline{\widetilde{j_{i,n}}})(\theta, z)) \\ &= (t, 1\#_T(\varphi_i\varphi_i^{-1}(\psi_i\#1_{\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2})j_{Y^r,n})) \\ &= (t, 1\#_T(\psi_i\#1_{\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2})j_{Y^r,n})) \\ &= i_{X_i^r,n}\end{aligned}$$

□

Therefore, our constructed families of spaces $\mathcal{Z}_{\widehat{j}^k_{i,n}}$ are diffeomorphic to the families of spaces $\widehat{\mathcal{Z}}_{\alpha_i^{k-1}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}}$ created via the clutching construction. Recall that via the clutching construction, the families $\widehat{\mathcal{Z}}_{\alpha_i^{k-1}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}}$ are classified by the $(k-1)-$homotopy group of diffeomorphisms of its fibre. Thus, let us examine the fibre, namely: $Z^{k,r}\#_{S^2}n\overline{\mathbb{CP}}^2$. Recall that $Z^{k,r} = (Z^{k-1,r}\#_T(X_i^r\#\mathbb{F}_n\#\overline{\mathbb{CP}}^2))$, so our fibre is:

$$Z^{k,r}\#_{S^2}n\overline{\mathbb{CP}}^2 = \left(Z^{k-1,r}\#_T X_i^r\right)\#\overline{\mathbb{CP}}^2\#\underbrace{\mathbb{F}_n\#_{S^2}n\overline{\mathbb{CP}}^2}_{\star}$$

This starred sum is the interesting part of the fibre. To analyze $\mathbb{F}_n\#_{S^2}\ n\overline{\mathbb{CP}}^2$ we utilize an operation called blowup-surgery. First introduced in [3, Definition 3.1], this operation was used in [3] to demonstrate that collections of spheres constructed by embellishments are smoothly distinct. We adapt this operation to families and observe that blowup-surgery on $\mathbb{F}_n$ is $\mathbb{F}_n/S^2 \cong n\overline{\mathbb{CP}}^2$. This blowup-surgery, can be viewed of as the following four steps: first a usual connect sum of $\mathbb{F}_n$ and $n\overline{\mathbb{CP}}^2$, then perform handle slides, surger out one of the spheres of self-intersection 0, and finally a handle cancellation. The Kirby diagram of this operation is illustrated in Figure 5.2. Thus, the starred submanifold sum is simply reduced to $n\overline{\mathbb{CP}}^2$. This reduction allows us to view the entire fibre of $\widehat{\mathcal{Z}}_\gamma$ as:

$$Z^{k-1,r}\#_T(X_i^r\#(n+1)\overline{\mathbb{CP}}^2)$$

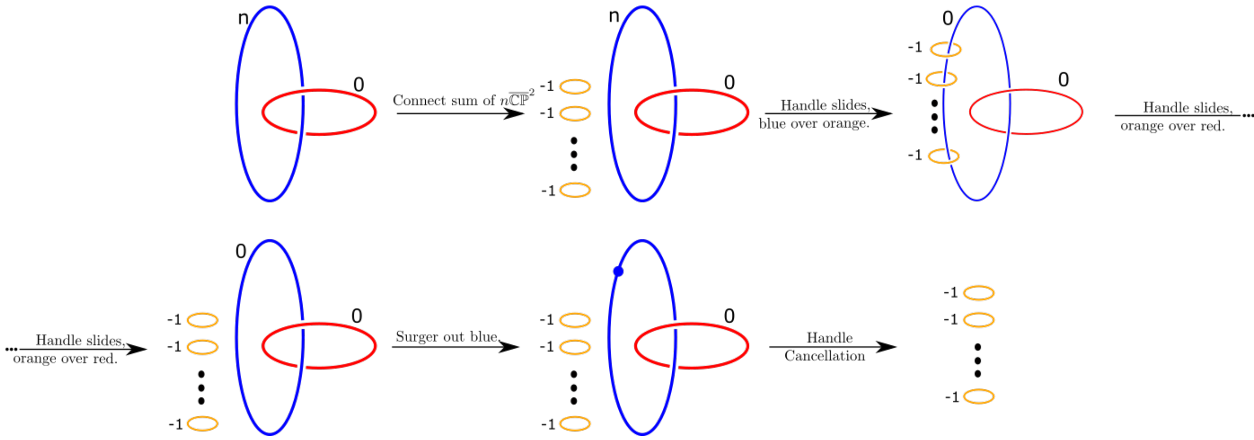


Figure 5.2: Blowup-surgery on $\mathbb{F}_n \#_{S^2} n\overline{\mathbb{CP}}^2$

Hence, using our diffeomorphisms Φ and Ψ, we can see that our constructed families of spaces $\mathcal{Z}_{\widehat{j}^k_{i,n}}$ are classified by the families of diffeomorphisms $\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}$.

Remark. There is a notable amount of construction throughout this paper, and is it illustrated in Figure 5.3. Intuitively, we began with families of embeddings, we used submanifold sum to create associated families of spaces from these families of embeddings, and now classify these families of spaces by families of diffeomorphisms. This construction allows us to utilize Konno's invariant [25] and the Auckly-Ruberman invariant [2] to prove the exotic nature of our families of embeddings.

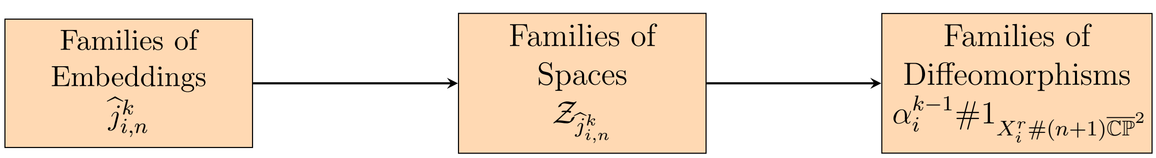


Figure 5.3: Families of Embeddings to Families of Diffeomorphisms

5.2 Computing the Invariant

Stated by Auckly and Ruberman in [2, Proposition 4.12] and proven by us in Theorem 4.3.2, for a good family of metrics, Konno's invariant $\mathbb{D}$ applied to the family of spaces is equivalent to the Auckly-Ruberman $D^{H_{k-1}(\mathrm{Diff})}$ invariant applied to the associated family of diffeomorphisms. That is, for the parameterizing space $\Xi = S^k$, $\mathbf{c}_j \in \widehat{C}^k_Z$, homology orientation β we have that

$$\mathbb{D}(Z^{k,r}, \mathfrak{P}_{\mathbf{c}_j}, (\mathbf{c}_j, \beta))[f_*^{\mathcal{Z}_{\widehat{j}^k_{i,n}}}[S^2]] = D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2})$$

Since the family of space $\mathcal{Z}_{\widehat{j}^k_{i,n}}$ is classified by the family of diffeomorphisms $\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}$, as discussed in section 5.1. Diagrammatically, our constructions can be viewed as below. Where the maps h are all Hurewicz morphisms, the map γ is the isomorphism evidenced by clutching and the fibration sequence, and the map s represents the submanifold sum construction from an associated family of embeddings.

$$\begin{tikzcd}
 & & \pi_{k-1}(\mathrm{Diff}(Z)) \arrow[r, "h"] \arrow[d, "\gamma"] & H_{k-1}(\mathrm{Diff}(Z)) \arrow[dd, "D^{H_{k-1}(\mathrm{Diff})}"] \\
\pi_k(\mathrm{Emb}(S^2, Z)) \arrow[rr, "s"] \arrow[d, "\eta_E"] & & \pi_k(\mathsf{BDiff}(Z)) \arrow[d, "h"] & \\
H_k(\mathrm{Emb}(S^2, Z)) \arrow[d, "\eta"] & \ker(\eta) \arrow[l, hook'] \arrow[r, "s"] & H_k(\mathsf{BDiff}(Z)) \arrow[r, "\mathbb{D}"] & \mathrm{Maps}(\hat{C}_{k-1}, \mathbb{Z}) \\
H_{k-2}(\mathrm{Emb}(S^2, Z)) & & &
\end{tikzcd}$$

With the construction of all of these components completed, all that remains to be shown is that the families of embeddings are indeed smoothly distinct.

Theorem 5.2.1. *Given families of embeddings $j^k_{i,n}$, as constructed in section 3.2, we can create an associated family of spaces, as shown in section 4.2.11, such that when we apply our family invariant we have that:*

$$\begin{aligned}
D^{H_k(Emb)}_{\boldsymbol{c}_j,\beta}(\widehat{j}^k_{i,n}) &= \mathbb{D}(Z^{k,r}, \mathfrak{P}_{\boldsymbol{c}_j}, (\boldsymbol{c}_j,\beta))[f_*^{\widehat{\mathcal{Z}}_{\widehat{j}^k_{i,n}}}[S^2]] \\
&= \mathbb{D}(Z^{k,r}, \mathfrak{P}_{\boldsymbol{c}_j}, (\boldsymbol{c}_j,\beta))[f_*^{\widehat{\mathcal{Z}}_{\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}}}[S^2]] \\
&= D^{H_{k-1}(Diff)}_{c_j,\beta}(\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}) \\
&= 2^{k-1}\delta_{ij}
\end{aligned}$$

where δ_{ij} is the Kronecker delta.

Proof. The first equality is by definition of our invariant in section 4.3.1, the second equality is shown by our family diffeomorphism Ψ in section 5.1, and the third equality is by Theorem 4.3.2. Thus, all the remains to be shown is that

$$D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}) = 2^{k-2}\delta_{ij}$$

So, we simply need to examine the Auckly-Ruberman invariant on our associated families of diffeomorphisms. Recall from section 2.3 that α_i^{k-1} is defined via the commutator, and so we have:

$$D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(\alpha_i^{k-1}\#1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}) = D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(\left[F_i^{k-2}, 1_{Z^{k-2,r}}\#_T\alpha_i^0\right]\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2})$$

Expanding the commutator yields

$$= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\Big(\left(\left(F_i^{k-2}\right)\left(1_{Z^{k-2,r}}\#_T\alpha_i^0\right)\left(F_i^{k-2}\right)^{-1}\left(1_{Z^{k-2,r}}\#_T\alpha_i^0\right)^{-1}\right)\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\Big)$$

Since $D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}$ is a homomorphism [2, Proposition 4.7], we can distribute the summed identity map and then view the right-most inverse as subtraction, giving us the following:

$$= D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\left(F_i^{k-2}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right)\left(1_{Z^{k-2,r}}\#_T\alpha_i^0\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right)\left(F_i^{k-2}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right)^{-1}\right)$$
$$- D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\left(1_{Z^{k-2,r}}\#_T\alpha_i^0\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right)\right)$$

Note, that $1_{Z^{k-2,r}}\#_T\alpha_i^0\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}$ is independent of our parameter component $\theta \in S^k$. Thus, it represents zero in $H_{k-1}(\mathrm{Diff})$, and so the invariant applied to it is zero. That is, the right-most subtracted component above is simply zero. To further simplify notation, let us denote

$$\tilde{F} := \left(F_i^{k-2}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right) \qquad \text{and} \qquad (1\#_T\alpha_i^0) := \left(1_{Z^{k-2,r}}\#_T\alpha_i^0\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2}\right)$$

Combining this notation with the definition of α_i^0 we have:

$$\begin{aligned}
&= D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\tilde{F}\left(1\#_T\alpha_i^0\right)\tilde{F}^{-1}\right)\\
&= D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\tilde{F}\left(1\#_T(1_{X_i^r}\#f)^{\varphi_i}(1_{Y^r}\#f^{-1})\right)\tilde{F}^{-1}\right)
\end{aligned}$$

Since the superscript φ_i notation represents conjugation by φ_i we have that $(1_{X_i^r}\#f)^{\varphi_i} = (\varphi_i)^{-1}(1_{X^r}\#f)\varphi_i$. Thus, we can distribute the sum with the identity map across this product to get:

$$= D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1\#_T 1_{X^r}\#f)(1\#_T\varphi_i)(1\#_T 1_{Y^r}\#f)^{-1}\tilde{F}^{-1}\right)$$

Similar to before, to simplify notation, sums between identities will be combined and list only the important component. That is, let us denote: $1\#_T 1_{X^r} = 1_X$ and $1\#_T 1_{Y^r} = 1_Y$, and this yields

$$= D_{c_j,\beta}^{H_{k-1}(\mathrm{Diff})}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)(1_Y\#_T f)^{-1}\tilde{F}^{-1}\right)$$

Inserting $\tilde{F}^{-1}\tilde{F}$ in the middle of this expression and using the homomorphism property of the invariant, we have the following:

$$
\begin{aligned}
&= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)\tilde{F}^{-1}\tilde{F}(1_Y\#_T f)^{-1}\tilde{F}^{-1}\right)\\
&= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)\tilde{F}^{-1}\left(\tilde{F}(1_Y\#_T f)\tilde{F}^{-1}\right)^{-1}\right)\\
&= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)\tilde{F}^{-1}\right) - D^{H_{k-1}(\mathrm{Diff})}\left(\tilde{F}(1_Y\#_T f)\tilde{F}^{-1}\right)
\end{aligned}
$$

Similar to before this subtracted component is simply zero, this time because the manifold Y was defined to have zero Donaldson invariants. Next, we may add and subtract $D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(1\#_T\varphi_i)$ and, once again using the homomorphism property of the invariant, we have the following:

$$
\begin{aligned}
&= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(1\#_T\varphi_i) + D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left(\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)\tilde{F}^{-1}\right) - D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}(1\#_T\varphi_i)\\
&= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left((1\#_T\varphi_i)\tilde{F}(1\#_T\varphi_i)^{-1}(1_X\#_T f)(1\#_T\varphi_i)\tilde{F}^{-1}(1\#_T\varphi_i)^{-1}\right)
\end{aligned}
$$

If we repeat the conjugation superscript notation where $\tilde{F}^{(1\#_T\varphi_i)^{-1}} := \tilde{F}(1\#_T\varphi_i)^{-1}$, then our expression becomes:

$$
= D^{H_{k-1}(\mathrm{Diff})}_{c_j,\beta}\left(\tilde{F}^{(1\#_T\varphi_i)^{-1}}(1_X\#_T f)((\tilde{F}^{-1})^{(1\#_T\varphi_i)^{-1}}\right)
$$

Hence, by [2, Theorem 4.10], we have the vital recursive step in calculating this invariant. The above expression at homology level $(k-1)$ can be reduced to homology level $(k-2)$ by:

$$
= 2D^{H_{k-2}(\mathrm{Diff})}_{c_j}\left((\alpha_i^{k-2}\#_T 1_{X_i^r\#(n+1)\overline{\mathbb{CP}}^2})\right)
$$

Utilizing this recursion step, by [31] and [2], we can repeat this process until we have reduced this expression to the Donaldson invariant q_{c_j} corresponding to our cohomology class c_j, applied to the underlying manifold.

$$= 2^{k-1} D_{c_j}^{H_0(\text{Diff})}\left((\alpha^0 \#_T 1_{X_i^r \#(n+1)\overline{\mathbb{CP}}^2})\right)$$
$$= 2^{k-1} q_{c_j, X_i^r \#(n+1)\overline{\mathbb{CP}}^2}$$

Moreover, by [19], we have that the Donaldson invariants of X_i^r and the invariants of $X_i^r \# \overline{\mathbb{CP}}^2$ coincide. Thus, our expression is simply:

$$= 2^{k-1} q_{c_j, X_i^r}$$

The cohomology class $\mathbf{c}_j$ were chosen so that the Donaldson invariant of $q_{c_j, X_i^r} = 1$ for $i = j$ and zero elsewhere. Therefore, our expression is simply the Kronecker delta δ_{ij}, yielding that:

$$D_{\mathbf{c}_j, \beta}^{H_k(\text{Emb})}(\widehat{j}_{i,n}^k) = 2^{k-1} \delta_{ij}$$

□

5.3 Proofs of Main Results

Finally, using Theorem 5.2.1, we may show that our families of embeddings create the desired rank r summand in the kernel and allow us to prove the result of our main theorem.

Proof of Theorem 1.2.1. The first several conditions in our theorem are proven in earlier sections.

i) By the embellishments process in section 3.1, proposition 3.1.1, and our definition of $\widehat{j}_{i,n}^k$ on page 28, our embedded spheres have self-intersection n.

ii) The families of topological isotopies G_i^k defined on page 28 show that the classes are trivial.

iii) The map Φ is defined first by inclusion, and then the contraction is given by the families of stable-smooth isotopies F_i^k defined on page 28.

iv) The families of psuedoisotopies K_i^k defined on page 28 show that the families are psuedoisotopic.

For part (v) and conclusion, recall in section 2.3, the special manifold V was constructed and used to define our manifolds $Z_{r,p}$ based upon the parity of p. When p is odd we set $q_o := (p-1)/2$ and when p is even we set $q_e := (p-2)/2$. We then define our manifolds in the odd and even cases as follows:

$$\underset{\text{Odd Case}}{Z_{r,p} := Z^{1,r} \#_T V^{(\#_T)^{q_o}}} \qquad \underset{\text{Even Case}}{Z_{r,p} := Z^{2,r} \#_T V^{(\#_T)^{q_e}}}$$

Moreover, recall that V contains two disjoint copies of the $N(2)$ nucleus that each contain a torus T. Denote these tori as T_1 and T_2. If $\mathbf{c}_V \in \widehat{C}_V$ is Poincaré dual to $\sigma_1 + \sigma_2$ where σ_1 and σ_2 are sections in the nuclei $N(2)$, then $\mathbf{c}_V[T_1] = \mathbf{c}_V[T_2] \equiv 1 \mod 2$, and by [2, Proposition 3.5 (7)] the Donaldson invariants of V are

$$q_{c_V,V} = -8$$

Notably, this is non-zero. Therefore, let $\mathbf{c}_V \in \widehat{C}_V$ be as above and $\mathbf{c}_j \in \widehat{C}_Z^k$ be as in Theorem 5.2.1. Notice that $c_j[T] = 1$, and define $\mathbf{c}_{V,j} \in \hat{C}^k_{Z\#_T V}$ such that when restricted to $Z \setminus T$, respectively $V \setminus T$, the class $\mathbf{c}_{V,j}$ agrees with $\mathbf{c}_j$, respectively $\mathbf{c}_V$. Thus, depending on the parity of p, we replace X_i^r in all of our constructions with:

$$\underset{\text{Odd Case}}{X_i^r \#_T V^{(\#_T)^{q_o}}} \qquad \text{or} \qquad \underset{\text{Even Case}}{X_i^r \#_T V^{(\#_T)^{q_e}}}$$

and then extend all maps onto the summed V portions by identity. Also, extend $\mathbf{c}_{V,j}$ to $\mathbf{c}_{V\#^{q_o},j}$ and to $\mathbf{c}_{V\#^{q_e},j}$, in the odd and even cases, respectively. Note that the Donaldson invariants will split nicely over these sums [2, Proposition 3.5 (6)]. In particular, if $c_u[T] = c_w[T] = 1$

mod 2, then $q_{c_u+c_w,U\#_T W} = q_{c_u,U} q_{c_w,W}$. Thus, we have:

$$q_{\mathbf{c}_{V\#q_o,j},X_i^r\#_T V} = (-8)^{q_o} q_{c_j,X_i^r} = (-8)^{q_o}\delta_{ij} \quad \text{and} \quad q_{\mathbf{c}_{V\#q_e,j},X_i^r\#_T V} = (-8)^{q_e} q_{c_j,X_i^r} = (-8)^{q_e}\delta_{ij}$$

Note, replacing X_i^r as we have done above, does not greatly change the rest of the construction, since the copies of V are summed over the essential torus T. The only change is altering the Donaldson invariants by a nonzero multiple. Therefore, we will examine the case where we do not sum on copies of V, for simplicity.

To show that the classes $[j_{i,n}^k]$ are independent, suppose $\Sigma a_i [j_{i,n}^k] = 0$, we need to show that $a_i = 0$. In theorem 5.2.1 we had that $D_{\mathbf{c}_j,\beta}^{H_k(\text{Emb})}(\widehat{j}_{i,n}^k) = 2^{k-1}\delta_{ij}$. Note that $\widehat{j}_{i,n}^k$ is simply the extension of the adjoint of $j_{i,n}^k$ as discussed on page 29. In theorem 5.2.1 we were not summing-on copies of V. Therefore, let us denote γ_p^k as the coefficients of δ_{ij}, in the general case when we sum on copies of V. Note, γ_p^k is essentially 2 to some power, the fact that it is non-zero is what is important. Combining these facts, for any ℓ, we have that:

$$0 = D_{\mathbf{c}_\ell,\beta}^{H_k(\text{Emb})}(\Sigma a_i [\widehat{j}_{i,n}^k]) = a_\ell D_{\mathbf{c}_\ell,\beta}^{H_k(\text{Emb})}([\widehat{j}_{\ell,n}^k]) = \gamma_p^k a_\ell$$

Then, we have that $\gamma_p^k = D_{\mathbf{c}_\ell,\beta}^{H_k(\text{Emb})}([\widehat{j}_{\ell,n}^k]) \neq 0$, which implies that $a_\ell = 0$. So, the classes $[j_{i,n}^k]$ are independent. Futhermore, recalling the construction of the twisting invariant η in section 4.2, we may view our invariant as a map:

$$D_{-,\beta}^{H_k(\text{Emb})} : \ker(\eta) \to \text{Maps}(\widehat{C}_Z^k, \mathbb{Z})$$

The independence of the classes above shows us that the collection $\left\{D_{-,\beta}^{H_k(\text{Emb})}([j_{i,n}^k])\right\}$ is independent in the image of $D_{-,\beta}^{H_k(\text{Emb})}$. This image, denoted $\text{Im}(D_{-,\beta}^{H_k(\text{Emb})})$, is a subset of $\text{Maps}(\widehat{C}_Z^k, \mathbb{Z})$, which is free, and so $\text{Im}(D_{-,\beta}^{H_k(\text{Emb})})$ is also free. This means that there is a map:

$$s : \text{Im}(D_{-,\beta}^{H_k(\text{Emb})}) \to \ker(\eta) \subset H_k(\text{Emb}^{C^\infty}(S^2, Z_{r,p}))$$

Such that $\left(D_{-,\beta}^{H_k(\text{Emb})} \circ s\right)$ is identity on $\text{Im}(D_{-,\beta}^{H_k(\text{Emb})})$. Finally, $s(\text{Im}(D_{-,\beta}^{H_k(\text{Emb})})$ is the desired subgroup of $H_k(\mathrm{Emb}^{C^\infty}(S^2, Z_{r,p}))$ that will be of rank at least r. $\square$

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