Direct and inverse electromagnetic scattering problems for bi-anisotropic periodic media

by

Trung T. Truong

B.S., Ho Chi Minh City University of Education, Vietnam, 2018

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2023

Abstract

This work focuses on electromagnetic direct and inverse scattering problems for bi-anisotropic periodic media in three dimensions. The scattering phenomenon of interest occurs when electromagnetic waves interact with a bi-anisotropic periodic scattering object and is governed by the time-harmonic Maxwell's equations with Rayleigh radiation condition. Bi-anisotropy means the object is characterized by three quantities: permittivity, permeability and magnetoelectric coupling coefficient. These are 3×3 matrix-valued functions of the spatial variable and make up two constitutive relations in addition to the Maxwell's equations. In the equations, the incident fields which we send towards the object are known. For the direct problem, we also know the three quantities characterizing the object, and the goal is to solve for the scattered fields. The inverse problem, on the other hand, aims to find the shape of the scattering object from the measurements of the scattered fields on two planes, one above and one below the object. Both problems have many applications in real life, especially in the study of photonic crystals.

Regarding the direct problem, we formulate the Maxwell's equations and constitutive relations with Rayleigh radiation condition into an equivalent integro-differential equation. We then show, under certain constraints on the material parameters of the scattering object, existence and uniqueness of solution. To numerically solve the problem, we utilize a periodization technique and fast Fourier Transform to switch to solving for the Fourier coefficients of the unknown. This helps avoid direct computation of the Green's function which has singularities. Then, we analyze convergence of a Galerkin scheme and implement it to obtain a numerical solution to the problem. The results regarding the direct problem are taken from the published paper[22] of myself and my advisor.

For the inverse problem, we rigorously justify the Factorization method to reconstruct the shape of the scattering medium from scattered field data. To be precise, we prove a unique

determination of the geometry of the scattering object. This determination can also be easily implemented numerically, thus, gives a fast imaging algorithm to find the shape of the scattering object. We also provide numerical results to show how the method performs with different types of bi-anisotropic periodic objects, from simple to more complex geometries. The results regarding the inverse problem are taken from the published paper[23] of myself and my advisor.

Direct and inverse electromagnetic scattering problems for bi-anisotropic periodic media

by

Trung T. Truong

B.S., Ho Chi Minh City University of Education, Vietnam, 2018

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2023

Approved by:

Major Professor
Dinh-Liem Nguyen

Copyright

Abstract

This work focuses on electromagnetic direct and inverse scattering problems for bi-anisotropic periodic media in three dimensions. The scattering phenomenon of interest occurs when electromagnetic waves interact with a bi-anisotropic periodic scattering object and is governed by the time-harmonic Maxwell's equations with Rayleigh radiation condition. Bi-anisotropy means the object is characterized by three quantities: permittivity, permeability and magnetoelectric coupling coefficient. These are 3×3 matrix-valued functions of the spatial variable and make up two constitutive relations in addition to the Maxwell's equations. In the equations, the incident fields which we send towards the object are known. For the direct problem, we also know the three quantities characterizing the object, and the goal is to solve for the scattered fields. The inverse problem, on the other hand, aims to find the shape of the scattering object from the measurements of the scattered fields on two planes, one above and one below the object. Both problems have many applications in real life, especially in the study of photonic crystals.

Regarding the direct problem, we formulate the Maxwell's equations and constitutive relations with Rayleigh radiation condition into an equivalent integro-differential equation. We then show, under certain constraints on the material parameters of the scattering object, existence and uniqueness of solution. To numerically solve the problem, we utilize a periodization technique and fast Fourier Transform to switch to solving for the Fourier coefficients of the unknown. This helps avoid direct computation of the Green's function which has singularities. Then, we analyze convergence of a Galerkin scheme and implement it to obtain a numerical solution to the problem. The results regarding the direct problem are taken from the published paper[22] of myself and my advisor.

For the inverse problem, we rigorously justify the Factorization method to reconstruct the shape of the scattering medium from scattered field data. To be precise, we prove a unique

determination of the geometry of the scattering object. This determination can also be easily implemented numerically, thus, gives a fast imaging algorithm to find the shape of the scattering object. We also provide numerical results to show how the method performs with different types of bi-anisotropic periodic objects, from simple to more complex geometries. The results regarding the inverse problem are taken from the published paper[23] of myself and my advisor.

Table of Contents

List of Figures

Acknowledgments

It goes without saying that this dissertation could not be done by myself alone. I would like to thank all the people who have been supporting me in completing this work and during my time in graduate school.

My biggest thank, without a doubt, goes to my advisor, Dr. Dinh-Liem Nguyen. He has helped me in much more ways than I could ever ask for. His knowledge in the field of inverse problems and professional advice are invaluable in my research. The opportunities he created for me and guided me to such as giving talks at seminar and conferences, mentoring undergraduate students, applying for awards, expanding my network, etc., have tremendously helped me gain more experience and confidence in academia. Last but not least, his words of encouragement and the discussions outside of work have helped me overcome many hard times. I also would like to thank my other committee members, Dr. Pietro Poggi-Corradini, Dr. Nathan Albin, Dr. Tuan Nguyen and Dr. Jingru Mu, for their time and effort regarding my exams, defense and for carefully reading this dissertation. I am thankful for all the professors and staff in the math department, who have been maintaining a warm, welcoming and productive environment for me and all graduate students.

As I live thousands of miles away from home, the friends around me are the main source of comfort that I have and without them, I would not have had such a wonderful time in graduate school. I am thankful for Thu, Huy, Thinh, My, Hien, Kamrul, Chanh and many other Vietnamese friends for hanging out and helping me a lot when I am in need. I thank Neeti and Bishnu for the time we spent taking classes together and preparing for exams. That time was stressful but also memorable.

I am extremely grateful for my mom, my dad and my brother for the spiritual support from afar. They have been keeping me company every day through video calls and ease my homesick. I am blessed to have such a lovely family who always think of me as a little son and a little brother, even though I feel like I am big now.

Finally, I am thankful for my girlfriend, Vi. She has been by my side the entire time of writing this dissertation, doing all the cooking and cleaning. I thank her for giving me a little hug and a little kiss when I am tired, and for being with me. Without her, I would have finished this dissertation much sooner, but I am not complaining, because the time we spend together is always time well spent.

Chapter 1

Introduction

1.1 Motivation

We study in this dissertation numerical methods for solving both direct and inverse electromagnetic scattering problems for bi-anisotropic periodic media. This work is motivated by applications of electromagnetic scattering from complex periodic media (e.g. bi-anisotropic media, chiral media) in optics and metamaterials; and nondestructive evaluations[1–4].

The direct problem involves finding the scattered electric and magnetic fields given the corresponding incident fields and the material parameters of the scattering medium. Results on well-posedness of the direct problem for both bounded and periodic scattering media can be found in[5–11]. To our knowledge results on numerical methods for direct and inverse scattering problems for electromagnetic complex media are limited. The authors in[12] studied numerical analysis of a finite element method for solving the direct problem for periodic chiral media. Spectral Galerkin methods for the integro-differential equation formulation of the direct scattering problem were studied in[13] for chiral bounded objects and in[14] for bi-anisotropic bounded objects. The advantages of the spectral Galerkin method studied in[14] are that this spectral method, which is based on a periodization technique, allows us to avoid the evaluation of the quasiperiodic Green's tensor and to use the fast Fourier transform in the numerical implementation of the method. In this dissertation, we study this spectral method to solve the direct problem for bi-anisotropic periodic media by extending the results in[14;15]. More precisely, we follow the periodization process of[15] for the integro-differential equation. The Garding estimates for the integro-differential equation can be done similarly as[14]. The uniqueness of solution, the mapping properties of the periodized integro-differential operators, and the numerical examples for the Galerkin method are the new ingredients of the study for the direct scattering problem in this dissertation. The results in this part are taken from the published paper[22] of myself and my advisor.

The inverse problem aims to reconstruct the geometry of the scattering object using measurement of the scattered electric or magnetic fields . We refer to[16–19] for some uniqueness results in the case of bounded scattering objects. To numerically solve the inverse problem, the factorization method was studied in[20] for the case of chiral bounded objects. This method was also studied in[21] for periodic chiral media. The analysis of the Factorization method for the inverse problem of interest in this dissertation is technically more complicated since it is concerned with a more complex type of materials. Under some assumptions (Assumption 1.2.1) corresponding to the absorbing material case and the smallness of one of the coefficients, we can prove coercivity of the middle operator in the Factorization method. This is also the key ingredient in the justification of the Factorization method. The justification remains open if the Assumption 1.2.1 does not hold true (e.g. the coefficients are all real-valued). The results in this part are taken from the published paper[23] of myself and my advisor.

1.2 The general setup, notation and assumptions

We are concerned with the scattering problem for a bi-anisotropic medium which is 2π-periodic in x_1, x_2 and bounded in x_3 (here x_1, x_2, x_3 are the three coordinates of a vector $\mathbf{x} = (x_1, x_2, x_3)^\top$ in $\mathbb{R}^3$). Let $D_0 \subset \mathbb{R}^3$ be the interior of the medium. The problem is governed by the time-harmonic Maxwell's equations with frequency $\omega > 0$

$$\operatorname{curl} \mathbf{E} + i\omega \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{H} - i\omega \mathbf{D} = 0, \quad \text{in } \mathbb{R}^3, \tag{1.2.1}$$

where the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, the electric flux density $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$ are three dimensional vector-valued functions, along with the constitutive relations

$$\mathbf{B} = \mu\mathbf{H} + \xi\sqrt{\varepsilon_0\mu_0}\,\mathbf{E}, \quad \mathbf{D} = \varepsilon\mathbf{E} + \overline{\xi}\sqrt{\varepsilon_0\mu_0}\,\mathbf{H} \tag{1.2.2}$$

where the permittivity ε and the permeability μ are 3×3 matrix-valued bounded functions. We assume that these functions are 2π-periodic in x_1 and x_2 and that they are respectively equal $\mu_0 I_3$ and $\varepsilon_0 I_3$ in $\mathbb{R}^3 \setminus \overline{D_0}$ (I_3 is the identity matrix). The function ξ is related to the magnetoelectric effect (see[24]) which is also a 3×3 matrix-valued bounded function. This function is also assumed to be 2π-periodic in x_1 and x_2 and to be zero in $\mathbb{R}^3 \setminus \overline{D_0}$.

We introduce the relative quantities

$$\varepsilon_r := \frac{\varepsilon}{\varepsilon_0}, \quad \mu_r := \frac{\mu}{\mu_0}$$

and the scaled quantities (with the same notations as the original ones)

$$\mathbf{E} := \sqrt{\varepsilon_0}\,\mathbf{E}, \quad \mathbf{H} := \sqrt{\mu_0}\,\mathbf{H}.$$

Using these new quantities and plugging (1.2.2) into (1.2.1) we obtain

$$\operatorname{curl}\mathbf{E} - ik(\mu_r\mathbf{H} + \xi\mathbf{E}) = 0, \quad \operatorname{curl}\mathbf{H} + ik(\varepsilon_r\mathbf{E} + \overline{\xi}\mathbf{H}) = 0 \tag{1.2.3}$$

where $k := \omega\sqrt{\varepsilon_0\mu_0}$ is the wave number.

Assume that μ_r is invertible almost everywhere in $\mathbb{R}^3$, by the first equation in (1.2.3) we can write

$$\mathbf{H} = -\frac{i}{k}\mu_r^{-1}\operatorname{curl}\mathbf{E} - \mu_r^{-1}\xi\mathbf{E}. \tag{1.2.4}$$

Plugging this into the second one and rearranging the resulting equation we obtain

$$\operatorname{curl}(\mu_r^{-1}\operatorname{curl}\mathbf{E}) + ik\left[\overline{\xi}\mu_r^{-1}\operatorname{curl}\mathbf{E} - \operatorname{curl}(\mu_r^{-1}\xi\mathbf{E})\right] - k^2(\varepsilon_r - \overline{\xi}\mu_r^{-1}\xi)\mathbf{E} = 0. \tag{1.2.5}$$

Let $\mathbf{d} = (d_1, d_2, d_3) \in \mathbb{R}^3$ be the wave vector satisfying $d_3 \neq 0$ and $|\mathbf{d}| = k$, we consider the plane electromagnetic wave given by

$$\mathbf{E}^{in} = \frac{1}{\omega\varepsilon_0} (\mathbf{s} \times \mathbf{d}) e^{i\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{H}^{in} = \mathbf{s} e^{i\mathbf{d}\cdot\mathbf{x}} \tag{1.2.6}$$

where $\mathbf{s}$ is the polarization vector satisfying $\mathbf{s} \cdot \mathbf{d} = 0$. The interaction between this plane wave and the medium gives rise to the scattered wave $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$, and the total wave $(\mathbf{E}, \mathbf{H})$ given by $\mathbf{E} = \mathbf{E}^{in} + \mathbf{E}^{sc}$, $\mathbf{H} = \mathbf{H}^{in} + \mathbf{H}^{sc}$ satisfies (1.2.4)−(1.2.5). From now on we will be working with $\mathbf{E}$ only since $\mathbf{E}$ and $\mathbf{H}$ are intimately related by (1.2.4).

For simplicity we denote $\mathbf{u} := \mathbf{E}^{sc}$. Note that by construction, $\mathbf{E}^{in}$ satisfies

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^{in} - k^2\mathbf{E}^{in} = 0,$$

thus (1.2.5) can be rewritten as

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u} = k^2 \left[(P - \overline{\xi}\mu_r^{-1}\xi)(\mathbf{E}^{in} + \mathbf{u}) + \frac{i}{k}\overline{\xi}\mu_r^{-1}(\operatorname{curl}\mathbf{E}^{in} + \operatorname{curl}\mathbf{u}) \right] \\ + \operatorname{curl}\left[Q(\operatorname{curl}\mathbf{E}^{in} + \operatorname{curl}\mathbf{u}) - ik\mu_r^{-1}\xi(\mathbf{E}^{in} + \mathbf{u}) \right] \end{aligned} \tag{1.2.7}$$

where $P := \varepsilon_r - I$ and $Q := I - \mu_r^{-1}$ are the contrasts. It is clear that outside of $\overline{D_0}$, ε_r and μ_r equal the identity matrix, so $P = Q = 0$. Thus we have

$$\operatorname{supp}(P) \cup \operatorname{supp}(Q) \cup \operatorname{supp}(\xi) \subset \overline{D_0}.$$

We now define that for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0) \in \mathbb{R}^3$, a function $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is called $\boldsymbol{\alpha}$-*quasiperiodic* if for any $n = (n_1, n_2, 0) \in \mathbb{Z}^3$

$$\mathbf{v}(x_1 + n_1 2\pi, x_2 + n_2 2\pi, x_3) = e^{2\pi i\alpha\cdot n}\mathbf{v}(x_1, x_2, x_3), \quad \mathbf{x} \in \mathbb{R}^3.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0) = (d_1, d_2, 0)$ and $\boldsymbol{\alpha}_m = (\alpha_1 + m_1, \alpha_2 + m_2, 0)$ for $m \in \mathbb{Z}^2$. Then it can be seen that the plane wave $\mathbf{E}^{in}$ is $\boldsymbol{\alpha}$-quasiperiodic. It is well-known that we will seek for the unique solution $\mathbf{u}$ of the direct problem as an $\boldsymbol{\alpha}$-quasiperiodic function.

Next we prescribe a radiation condition for $\mathbf{u}$. Recall that the medium D_0 is bounded in x_3, thus let $h > 0$ be such that

$$h > \sup\{|x_3| : \mathbf{x} \in D_0\} \tag{1.2.8}$$

and for $m \in \mathbb{Z}^2$ define

$$\beta_m = \begin{cases} \sqrt{k^2 - |\boldsymbol{\alpha}_m|^2}, & |\boldsymbol{\alpha}_m| \leq k \\ i\sqrt{|\boldsymbol{\alpha}_m|^2 - k^2}, & |\boldsymbol{\alpha}_m| > k \end{cases},$$

then $\mathbf{u}$ is called radiating if it can be expressed as the following Rayleigh expansion

$$\mathbf{u}(\mathbf{x}) = \sum_{m \in \mathbb{Z}^2} \widehat{\mathbf{u}}_m^{\pm} e^{i(\boldsymbol{\alpha}_m \cdot \mathbf{x} \pm \beta_m (x_3 \mp h))} \quad \text{for } x_3 \gtrless \pm h \tag{1.2.9}$$

where

$$\widehat{\mathbf{u}}_m^{\pm} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i\boldsymbol{\alpha}_m \cdot \mathbf{x}} \mathbf{u}(x_1, x_2, \pm h) \mathrm{d}x_1 \mathrm{d}x_2, \quad m \in \mathbb{Z}^2.$$

Note that all but finitely many terms in (1.2.9) are exponentially decaying, which helps us easily deduce absolute convergence of the series for every $\mathbf{x}$. Moreover, we can choose $\boldsymbol{\alpha}$ and k such that β_m are nonzero for all $m \in \mathbb{Z}^2$, and this will be the case for the remaining part of this dissertation.

We now introduce notation and assumptions related to the spatial domains and function spaces that we will be using throughout this dissertation. First, due to periodicity of all the introduced quantities and the condition (1.2.9), we can restrict our problem from $\mathbb{R}^3$ to one period

$$\Omega = (-\pi, \pi)^2 \times \mathbb{R}.$$

Also, for any $r > 0$, we define a truncation of Ω as

$$\Omega_r = (-\pi, \pi)^2 \times (-r, r).$$

Next we introduce some Sobolev spaces for vector-valued functions

$$
\begin{aligned}
L^2(\mathcal{O})^3 &= \{\mathbf{w} = (w_1, w_2, w_3)^T : w_1, w_2, w_3 \in L^2(\mathcal{O})\},\\
H(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in L^2(\mathcal{O}, \mathbb{C}^3) : \mathrm{curl}\,\mathbf{w} \in L^2(\mathcal{O}, \mathbb{C}^3)\},\\
H_{\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} : \mathbf{w}|_V \in H(\mathrm{curl}\,, V) \text{ for all } V \subset \mathcal{O} \text{ open, bounded}\},\\
H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in H(\mathrm{curl}\,, \mathcal{O}) : \mathbf{w} = \mathbf{W}|_{\mathcal{O}} \text{ for some } \boldsymbol{\alpha}\text{-quasiperiodic}\\
&\qquad \text{function } \mathbf{W} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathbb{R}^3)\},\\
H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) : \mathbf{w} = \mathbf{W}|_{\mathcal{O}} \text{ for some } \boldsymbol{\alpha}\text{-quasiperiodic}\\
&\qquad \text{function } \mathbf{W} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathbb{R}^3)\},
\end{aligned}
$$

along with $L^\infty(\mathbb{R}^3)^{3\times 3}$ which is the space of 3×3 matrix-valued functions whose components are functions in $L^\infty(\mathbb{R}^3)$. We define the following norm on this space

$$\|\gamma\|_\infty = \max_{i,j=\overline{1,3}} \|\gamma_{ij}\|_\infty$$

for $\gamma \in L^\infty(\mathbb{R}^3)^{3\times 3}$.

Let $D = D_0 \cap \Omega$. It is clear that D is open and bounded. Finally, the following assumption holds true throughout the theoretical analysis in this dissertation.

Assumption 1.2.1. Assume that D has Lipschitz boundary; ε_r, μ_r, μ_r^{-1}, ξ are in $L^\infty(\mathbb{R}^3)^{3\times 3}$, symmetric almost everywhere in $\mathbb{R}^3$ and ξ is real-valued. Moreover,

1. There exist $c_1, c_2 > 0$ such that for all $\mathbf{z} \in \mathbb{C}^3$

   $$\mathrm{Re}\,(\mu_r^{-1})\mathbf{z} \cdot \overline{\mathbf{z}} \geq c_1 |\mathbf{z}|^2, \quad \mathrm{Re}\,(\varepsilon_r - \xi\mu_r^{-1}\xi)\mathbf{z} \cdot \overline{\mathbf{z}} \geq c_2 |\mathbf{z}|^2$$

   a.e in D and that

   $$\||\mu_r^{-1}\xi|_F\|_\infty < c_1 c_2.$$

2. There exist $c_3, c_4 > 0$ such that for all $\mathbf{z} \in \mathbb{C}^3$

$$-\mathrm{Im}\,(\mu_r^{-1})\mathbf{z} \cdot \bar{\mathbf{z}} \geq c_3|\mathbf{z}|^2, \qquad \mathrm{Im}\,(\varepsilon_r - \xi\mu_r^{-1}\xi)\mathbf{z} \cdot \bar{\mathbf{z}} \geq c_4|\mathbf{z}|^2$$

a.e in D and that

$$\frac{1}{2}\left(\|\mu_r^{-1}\xi\|_\infty^2 + 1\right) \leq \min\{c_3, c_4\}\,.$$

Here the norm $\||\cdot|_F\|_\infty$ is defined as, for a 3×3 matrix-valued function $A = (a_{ij})$,

$$\||A|_F\|_\infty = \left\| \left(\sum_{i,j=1}^{3} |a_{ij}|^2 \right)^{1/2} \right\|_\infty .$$

An example of the parameters satisfying both Assumptions 1.2.1 is

$$\varepsilon_\mathrm{r} = \begin{bmatrix} 1+0.75i & 0 & 0 \\ 0 & 1+0.9i & 0 \\ 0 & 0 & 1+0.8i \end{bmatrix}, \quad \mu_\mathrm{r}^{-1} = \begin{bmatrix} 1-0.7i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1-0.9i \end{bmatrix}, \tag{1.2.10}$$

$$\xi = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.05 \end{bmatrix} \quad \text{in } D, \quad \text{and } \varepsilon_\mathrm{r} = \mu_\mathrm{r}^{-1} = I_3, \quad \xi = 0 \quad \text{in } \Omega \setminus \overline{D}. \tag{1.2.11}$$

The remaining part of this dissertation is organized as follows.

- In chapter 2, we introduce two equivalent formulations for the direct problem: as a variational problem and as an integro-differential equation. We show that the integro-differential equation satisfies a Garding-type estimate, thus obeys Fredholm alternative. Then, uniqueness is proved and existence follows. To find a numerical approximation for the solution, we study a spectral Galerkin method and prove its convergence as well as provide numerical results.

- In chapter 3, we state the inverse problem of interest and study the factorization method to solve it. We first prove a unique characterization of the scattering medium, then obtain a range identity of two important operators, and finally provide a way to determine the geometry of the medium as a Picard criterion.

- In chapter 4, we give a brief summary of the work in this dissertation as well as possible future work related to this topic.

Chapter 2

The Direct Problem

2.1 Integro-differential formulation and well-posedness of the direct problem

In this section, we will formulate the direct problem into an integro-differential equation. Define the operators

$$\mathcal{S}\mathbf{v} := (P - \xi\mu_r^{-1}\xi)\mathbf{v} + \frac{i}{k}\xi\mu_r^{-1}\mathrm{curl}\,\mathbf{v}, \quad \mathcal{T}\mathbf{v} := Q\mathrm{curl}\,\mathbf{v} - ik\mu_r^{-1}\xi\mathbf{v},$$
$$\tilde{\mathcal{S}}(\mathbf{v}_1, \mathbf{v}_2) := (P - \xi\mu_r^{-1}\xi)\mathbf{v}_2 + \frac{i}{k}\xi\mu_r^{-1}\mathbf{v}_1, \quad \tilde{\mathcal{T}}(\mathbf{v}_1, \mathbf{v}_2) := Q\mathbf{v}_1 - ik\mu_r^{-1}\xi\mathbf{v}_2,$$

then (1.2.7) can be written as

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u} = k^2(\mathcal{S}\mathbf{u} + \tilde{\mathcal{S}}(\mathbf{f}, \mathbf{g})) + \mathrm{curl}\,(\mathcal{T}\mathbf{u} + \tilde{\mathcal{T}}(\mathbf{f}, \mathbf{g})). \tag{2.1.1}$$

Note that (2.1.1) needs to be solved in the variational sense, therefore we will be seeking solutions in $H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,, \Omega)$ and $\mathbf{u}$ is called a solution of (2.1.1) in the variational sense if it satisfies

$$\int_\Omega (\operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\mathbf{v}} - k^2 \mathbf{u} \cdot \overline{\mathbf{v}})\, d\mathbf{x}$$
$$= k^2 \int_D (\mathcal{S}\mathbf{u} + \tilde{\mathcal{S}}(\mathbf{f}, \mathbf{g})) \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \int_D (\mathcal{T}\mathbf{u} + \tilde{\mathcal{T}}(\mathbf{f}, \mathbf{g})) \cdot \operatorname{curl} \overline{\mathbf{v}}\, d\mathbf{x} \quad (2.1.2)$$

for all $\mathbf{v} \in H_{\boldsymbol{\alpha}}(\operatorname{curl}, \Omega)$ with compact support.

First, we have some properties of the operators $\mathcal{S}$, $\mathcal{T}$, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$.

Lemma 2.1.1. *For $\mathbf{v} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\operatorname{curl}, \Omega)$, $\mathcal{S}\mathbf{v}$ and $\mathcal{T}\mathbf{v}$ are compactly supported in D. Moreover, $\mathcal{S}$ and $\mathcal{T}$ are bounded linear operators from $H_{\boldsymbol{\alpha}}(\operatorname{curl}, D)$ to $L^2(D)^3$.*

For $\mathbf{v}_1, \mathbf{v}_2 \in L^2(D)^3$, $\tilde{\mathcal{S}}(\mathbf{v}_1, \mathbf{v}_2)$ and $\tilde{\mathcal{T}}(\mathbf{v}_1, \mathbf{v}_2)$ are compactly support in D. Moreover, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ are bounded linear operators from $L^2(D)^3 \times L^2(D)^3$ to $L^2(D)^3$.

Proof. The proof can be done by using Cauchy-Schwarz inequality and boundedness of the material parameters. □

Let G_k be the $\boldsymbol{\alpha}$-quasiperiodic Green's function of the Helmholtz equation, then G_k has the Fourier expansion (see[25])

$$G_k(\mathbf{x}) = \frac{i}{8\pi^2} \sum_{m \in \mathbb{Z}^2} \frac{1}{\beta_m} e^{i(\boldsymbol{\alpha}_m \cdot \mathbf{x} + \beta_m |x_3|)} \quad (2.1.3)$$

for $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq (2\pi n_1, 2\pi n_2, 0)$, $n_1, n_2 \in \mathbb{Z}$.

The variational problem (2.1.2) along with the radiation condition (1.2.9) is equivalent to the following integro-differential equation

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = \mathcal{A}\tilde{\mathcal{S}}(\mathbf{f}, \mathbf{g}) + \mathcal{B}\tilde{\mathcal{T}}(\mathbf{f}, \mathbf{g}) \quad \text{in } \Omega \quad (2.1.4)$$

where

$$\mathcal{A}\mathbf{h}(\mathbf{x}) := (k^2 + \nabla \mathrm{div}) \int_D G_k(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y},$$
$$\mathcal{B}\mathbf{h}(\mathbf{x}) := \mathrm{curl} \int_D G_k(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}$$

in the sense that, if a radiating $\mathbf{u} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$ solves (2.1.2) then $\mathbf{u}|_D$ solves (2.1.4). Conversely, if $\mathbf{u} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, D)$ solves (2.1.4) in then $\mathbf{u}$ can be extended by the right-hand side of (2.1.4) to a radiating solution of (2.1.2) in $H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$. The proof for this equivalence is similar to that in[26].

The operator $I - \mathcal{A}\mathcal{S} - \mathcal{B}\mathcal{T}$ on the left-hand side of (2.1.4) satisfies a Garding-type estimate, and thus the Fredholm property. Note that $\overline{D} \subset \Omega_h$, so we only need to know $\mathbf{u}$ on Ω_h and then extend it to the whole Ω using integration. On the solution space $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$, we define the inner product

$$(\cdot,\cdot)_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} = k^2(\cdot,\cdot)_{L^2(\Omega_h)^3} + (\mathrm{curl}\,\cdot, \mathrm{curl}\,\cdot)_{L^2(\Omega_h)^3}$$

where $(\cdot,\cdot)_{L^2(\Omega_h)^3}$ is the usual inner product in $L^2(\Omega_h)^3$.

Under Assumption 1.2.1, the following Fredholm property can be proved in a similar manner as in[14].

Theorem 2.1.2. *There exist a constant $C > 0$ and a compact operator K on $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$ such that*

$$\mathrm{Re}\,(\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} \geq C\|\mathbf{u}\|^2_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} + \mathrm{Re}\,(K\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)}.$$

Since the Fredholm property holds, existence of solution to (2.1.4) is equivalent to uniqueness, therefore we show that it has a unique solution.

Theorem 2.1.3. *The integro-differential equation (2.1.4) has a unique solution in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$.*

Proof. Suppose (2.1.4) has two solutions $\mathbf{u}_1, \mathbf{u}_2$ in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$. Let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, we have

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = 0 \tag{2.1.5}$$

in Ω_h. In particular, $\mathbf{u}|_D$ solves the (2.1.5) in D. Therefore $\mathbf{u}|_D$ can be extended by the right-hand side of (2.1.5) to a radiating solution $\tilde{\mathbf{u}} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,, \Omega)$ of the variational problem

$$\begin{aligned}\int_\Omega (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\mathbf{v}} - k^2 \tilde{\mathbf{u}} \cdot \overline{\mathbf{v}})\; d\mathbf{x} = \int_D (k^2(P - \xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\mathbf{v}}\; d\mathbf{x} \\ + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}}) \cdot \mathrm{curl}\,\overline{\mathbf{v}}\; d\mathbf{x}\end{aligned} \tag{2.1.6}$$

for all $\mathbf{v} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega)$ with compact support. Let $\phi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that $\phi(t) = 1$ if $|t| \leq h$ and $\phi(t) = 0$ if $|t| \geq 2h$. Using the test function $\mathbf{v}(\mathbf{x}) = \phi(x_3)\tilde{\mathbf{u}}(\mathbf{x})$ in (2.1.6) yields

$$\begin{aligned}\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2 \tilde{\mathbf{u}} \cdot \overline{\phi\tilde{\mathbf{u}}})\; d\mathbf{x} = \int_D (k^2(P - \xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\tilde{\mathbf{u}}}\; d\mathbf{x} \\ + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}}) \cdot \mathrm{curl}\,\overline{\tilde{\mathbf{u}}}\; d\mathbf{x}.\end{aligned} \tag{2.1.7}$$

Since $\tilde{\mathbf{u}}$ is divergence-free and solves $\mathrm{curl}\,^2\tilde{\mathbf{u}} - k^2\tilde{\mathbf{u}} = 0$ on $\Omega_{2h} \setminus \Omega_h$, we have

$$\begin{aligned}&\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2 \tilde{\mathbf{u}} \cdot \overline{\phi\tilde{\mathbf{u}}})\; d\mathbf{x} \\ &= \int_{\Omega_h} (|\mathrm{curl}\,\tilde{\mathbf{u}}|^2 - k^2|\tilde{\mathbf{u}}|^2)\; d\mathbf{x} + \left(\int_{\{x_3=h\}} - \int_{\{x_3=-h\}}\right)(e_3 \times \mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\tilde{\mathbf{u}}}\; ds(\mathbf{x}) \\ &= \int_{\Omega_h} (|\mathrm{curl}\,\tilde{\mathbf{u}}|^2 - k^2|\tilde{\mathbf{u}}|^2)\; d\mathbf{x} + \left(\int_{\{x_3=h\}} - \int_{\{x_3=-h\}}\right)\left(\tilde{u}_3\frac{\partial\overline{\tilde{u}_3}}{\partial x_3} - \overline{\tilde{u}_1}\frac{\partial\tilde{u}_1}{\partial x_3} - \overline{\tilde{u}_2}\frac{\partial\tilde{u}_2}{\partial x_3}\right) ds(\mathbf{x}).\end{aligned}$$

Using radiation condition (1.2.9) on the boundary term then taking the imaginary part of both sides gives

$$\mathrm{Im}\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2 \tilde{\mathbf{u}} \cdot \overline{\phi\tilde{\mathbf{u}}})\; d\mathbf{x} = -4\pi^2 \sum_{m:\beta_m>0} \beta_m(|\hat{u}_m^+|^2 + |\hat{u}_m^-|^2) \leq 0.$$

On the other hand, by Assumption 1.2.1,

$$\operatorname{Im}\left(\int_D (k^2(P-\xi\mu_r^{-1}\xi)\tilde{\mathbf{u}}+ik\xi\mu_r^{-1}\operatorname{curl}\tilde{\mathbf{u}})\cdot\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}+\int_D (Q\operatorname{curl}\tilde{\mathbf{u}}-ik\mu_r^{-1}\xi\tilde{\mathbf{u}})\cdot\operatorname{curl}\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}\right)$$
$$\geq k^2c_2\|\tilde{\mathbf{u}}\|^2+c_1\|\operatorname{curl}\tilde{\mathbf{u}}\|^2+k\operatorname{Re}(\xi\mu_r^{-1}\operatorname{curl}\tilde{\mathbf{u}},\tilde{\mathbf{u}})-k\operatorname{Re}(\mu_r^{-1}\xi\tilde{\mathbf{u}},\operatorname{curl}\tilde{\mathbf{u}}),$$

here $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(D)$ norm and $L^2(D)$ inner product, respectively. Using Cauchy-Schwarz inequality and the fact that μ_r^{-1} and ξ are symmetric we have

$$k\operatorname{Re}(\xi\mu_r^{-1}\operatorname{curl}\tilde{\mathbf{u}},\tilde{\mathbf{u}})\geq-\|\xi\mu_r^{-1}\operatorname{curl}\tilde{\mathbf{u}}\|\|k\tilde{\mathbf{u}}\|\geq-\frac{1}{2}\left(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}\|\operatorname{curl}\tilde{\mathbf{u}}\|^2+k^2\|\tilde{\mathbf{u}}\|^2\right)$$

and

$$-k\operatorname{Re}(\mu_r^{-1}\xi\tilde{\mathbf{u}},\operatorname{curl}\tilde{\mathbf{u}})\geq-\|k\mu_r^{-1}\xi\tilde{\mathbf{u}}\|\|\operatorname{curl}\tilde{\mathbf{u}}\|\geq-\frac{1}{2}\left(k^2\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}\|\tilde{\mathbf{u}}\|^2+\|\operatorname{curl}\tilde{\mathbf{u}}\|^2\right).$$

Therefore

$$\operatorname{Im}\left(\int_D (k^2(P-\xi\mu_r^{-1}\xi)\tilde{\mathbf{u}}+ik\xi\mu_r^{-1}\operatorname{curl}\tilde{\mathbf{u}})\cdot\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}+\int_D (Q\operatorname{curl}\tilde{\mathbf{u}}-ik\mu_r^{-1}\xi\tilde{\mathbf{u}})\cdot\operatorname{curl}\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}\right)$$
$$\geq C_1\|\operatorname{curl}\tilde{\mathbf{u}}\|^2+k^2C_2\|\tilde{\mathbf{u}}\|^2,$$

where

$$C_1=c_1-\frac{1}{2}(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}+1)>0,$$
$$C_2=c_2-\frac{1}{2}(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}+1)>0.$$

Combining the estimates for the two sides of (2.1.7) we have

$$C_1\|\operatorname{curl}\tilde{\mathbf{u}}\|^2+k^2C_2\|\tilde{\mathbf{u}}\|^2\leq 0,$$

and thus $\tilde{\mathbf{u}}=0$ in D or $\mathbf{u}|_D=0$. Due to (2.1.5), $\mathbf{u}=0$ in $H_{\boldsymbol{\alpha}}(\operatorname{curl},\Omega_h)$, which means

$\mathbf{u}_1 = \mathbf{u}_2$ in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$. $\square$

2.2 Periodization of the integro-differential equation

We will be focusing on solving

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = \mathcal{A}\tilde{\mathcal{S}}(\mathbf{f}, \mathbf{g}) + \mathcal{B}\tilde{\mathcal{T}}(\mathbf{f}, \mathbf{g}), \quad \mathbf{u} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h). \tag{2.2.1}$$

The idea is to transform (2.2.1) into a periodic equation so that we can utilize all the tools of periodic integral equations, especially the fast Fourier transform (FFT), while ensuring that the solution to this periodic equation can be used to find $\mathbf{u}$ that solves (2.2.1).

For $R > 2h$ let

$$\mathcal{K}_R(\mathbf{x}) = G_k(\mathbf{x}), \quad \mathbf{x} \in \Omega_R \setminus \{0\}$$

and then extend $\mathcal{K}_R$ $2R$-periodically in x_3 and $\boldsymbol{\alpha}$-quasiperiodically in x_1 and x_2 to the entire $\mathbb{R}^3$. Since G is already $\boldsymbol{\alpha}$-quasiperiodic and smooth everywhere but at its singularities, extending along the x_1 and x_2 axes will not affect the smoothness of $\mathcal{K}_R$. However doing so along the x_3 axis does as G is not periodic in x_3. Thus next we smoothen $\mathcal{K}_R$ by defining

$$\mathcal{K}(\mathbf{x}) = \mathcal{K}_R(\mathbf{x})\chi(x_3), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \{(n_1 2\pi, n_2 2\pi, n_3 2R) : n_1, n_2, n_3 \in \mathbb{Z}\}$$

where $\chi : \mathbb{R} \to [0, 1]$ is a $2R$-periodic smooth function satisfying

$$\chi(t) = \begin{cases} 1, & -2h \leq t \leq 2h \\ 0, & t = \pm R \end{cases} \quad \text{and} \quad \chi'(\pm R) = \chi''(\pm R) = \chi'''(\pm R) = 0.$$

The explicit formula for the Fourier coefficients of $\mathcal{K}_R$ can be found in[15]. This helps us avoid the evaluation of $\mathcal{K}_R$ in the spectral domain. We also extend P, Q, ξ from Ω_R to $\mathbb{R}^3$ 2π-periodically in x_1, x_2 and $2R$-periodically in x_3, and extend $\mathbf{f}$, $\mathbf{g}$ from Ω_R to $\mathbb{R}^3$ $\boldsymbol{\alpha}$-quasiperiodically in x_1, x_2 and $2R$-periodically in x_3.

Next, for $\mathbf{h} \in H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$, we define

$$
\begin{aligned}
\mathcal{A}_p\mathbf{h}(\mathbf{x}) &:= (k^2 + \nabla \mathrm{div}) \int_D \mathcal{K}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}, \\
\mathcal{B}_p\mathbf{h}(\mathbf{x}) &:= \mathrm{curl} \int_D \mathcal{K}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}
\end{aligned}
$$

as periodized versions of $\mathcal{A}$ and $\mathcal{B}$ respectively.

We derive the following periodic integro-differential equation

$$
\mathbf{u} - \mathcal{A}_p\mathcal{S}\mathbf{u} - \mathcal{B}_p\mathcal{T}\mathbf{u} = \mathcal{A}_p\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g}) + \mathcal{B}_p\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g}), \quad \mathbf{u} \in H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R). \tag{2.2.2}
$$

Next we will analyze the periodic integral operators in the equation. To this end, for $\lambda \geq 0$, we consider periodic Sobolev spaces $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$ and $H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ as spaces consisting of functions in $L^2(\Omega_R)^3$ that satisfy

$$
\begin{aligned}
\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} &= \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^\lambda |\widehat{\mathbf{w}}(\mathbf{j})|^2 < \infty, \\
\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} &= \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^\lambda \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2 \right) < \infty,
\end{aligned}
$$

respectively, where $\tilde{\mathbf{j}} = \left(\alpha_1 + j_1, \alpha_2 + j_2, \frac{\pi j_3}{R}\right)$. Note that $H^0_{\boldsymbol{\alpha},p}(\Omega_R) = L^2(\Omega_R)^3$ and $H^0_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R) = H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$. Here the orthonormal basis for $L^2(\Omega_R)^3$ is defined as

$$
\phi_{\mathbf{j}}(\mathbf{x}) = \frac{1}{\sqrt{8R\pi^2}} \exp\left(i\alpha_{\tilde{j}} \cdot \mathbf{x} + i\frac{\pi j_3}{R} x_3 \right), \quad \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}^3 \tag{2.2.3}
$$

where $\tilde{j} = (j_1, j_2)$, and for $\mathbf{w} \in L^2(\Omega_R)^3$, the Fourier coefficients $\widehat{\mathbf{w}}(\mathbf{j})$ are given by

$$
\widehat{\mathbf{w}}(\mathbf{j}) = \int_{\Omega_r} \mathbf{w}(\mathbf{x})\overline{\phi_{\mathbf{j}}(\mathbf{x})}d\mathbf{x}. \tag{2.2.4}
$$

In fact

$$
\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = k^2\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \|\mathrm{curl}\,\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}.
$$

Note that the extension of a function $\mathbf{w}$ in $H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ to the entire $\mathbb{R}^3$ using the Fourier expansion

$$\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^3} \widehat{\mathbf{w}}(\mathbf{j})\phi_{\mathbf{j}}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3$$

is $\boldsymbol{\alpha}$-quasiperiodic in x_1, x_2 and $2R$-periodic in x_3.

Let $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times 3}$ be the space of 3×3 matrix-valued functions whose columns are functions in $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$. Now we prescribe an assumption on regularity of the parameters as well as the solution $\mathbf{u}$ of (2.2.2).

Assumption 2.2.1. Assume that $\varepsilon_r, \mu_r, \mu_r^{-1}, \xi \in H^{\eta+\frac{1}{2}+\epsilon}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times 3}$ for some $\epsilon > 0$ and $\eta > \frac{1}{2}$. Moreover, $\mathbf{u} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ for some λ such that $0 \le \lambda < \eta - \frac{1}{2}$.

Lemma 2.2.2. *$\mathcal{A}_p$ and $\mathcal{B}_p$ are bounded linear operators from $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$ to $H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$.*

Proof. The linearity is obvious. From the Fourier coefficients of $\mathcal{K}$ in[15] there exist $C_1, C_2 > 0$ such that $|\widehat{\mathcal{K}}(\mathbf{j})| \le C_1|\mathbf{j}|^{-2}$ and $|\tilde{\mathbf{j}}| \le C_2|\mathbf{j}|$ for all $\mathbf{j}\in\mathbb{Z}^3\setminus\{0\}$. Let $\mathbf{v}\in H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$,

$$\begin{aligned}
&\|\mathcal{A}_p\mathbf{v}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\lambda\left(k^2\left|\widehat{\mathcal{A}_p\mathbf{v}}(\mathbf{j})\right|^2 + \left|\tilde{\mathbf{j}}\times\widehat{\mathcal{A}_p\mathbf{v}}(\mathbf{j})\right|^2\right)\\
&= \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\lambda\left(k^2\left|k^2\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j}) + \tilde{\mathbf{j}}\left(\tilde{\mathbf{j}}\cdot\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right)\right|^2 + \left|\tilde{\mathbf{j}}\times k^2\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2\right)\\
&\le k^2(2k^4+2|\boldsymbol{\alpha}|^4+k^2|\boldsymbol{\alpha}|^2)|\widehat{\mathcal{K}}(0)|^2|\widehat{\mathbf{v}}(0)|^2 + C_1^2k^2\left(2k^4+2C_2^4+C_2^2k^2\right)\sum_{\mathbf{j}\in\mathbb{Z}^3\setminus\{0\}}(1+|\mathbf{j}|^2)^\lambda|\widehat{\mathbf{v}}(\mathbf{j})|^2.
\end{aligned}$$

Hence let $C^2 = k^2\max\left\{(2k^4+2|\boldsymbol{\alpha}|^4+k^2|\boldsymbol{\alpha}|^2)|\widehat{\mathcal{K}}(0)|^2, C_1^2\left(2k^4+2C_2^4+C_2^2k^2\right)\right\}$ we have

$$\|\mathcal{A}_p\mathbf{v}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \le C\|\mathbf{v}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

On the other hand

$$\begin{aligned}
\|\mathcal{B}_p\mathbf{v}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} &= \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\lambda\left(k^2\left|\widehat{\mathcal{B}_p\mathbf{v}}(\mathbf{j})\right|^2+\left|\tilde{\mathbf{j}}\times\widehat{\mathcal{B}_p\mathbf{v}}(\mathbf{j})\right|^2\right)\\
&= \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\lambda\left[k^2\left|\tilde{\mathbf{j}}\times\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2+\left|\tilde{\mathbf{j}}\times\left(\tilde{\mathbf{j}}\times\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2\right)\right]\\
&\le (k^2|\boldsymbol{\alpha}|^2+|\boldsymbol{\alpha}|^4)|\widehat{\mathcal{K}}(0)|^2|\widehat{\mathbf{v}}(0)|^2+C_1^2\left(C_2^2k^2+C_2^4\right)\sum_{\mathbf{j}\in\mathbb{Z}^3\setminus\{0\}}(1+|\mathbf{j}|^2)^\lambda|\widehat{\mathbf{v}}(\mathbf{j})|^2.
\end{aligned}$$

Let $C'^2=\max\left\{(k^2|\boldsymbol{\alpha}|^2+|\boldsymbol{\alpha}|^4)|\widehat{\mathcal{K}}(0)|^2, C_1^2\left(C_2^2k^2+C_2^4\right)\right\}$ we have

$$\|\mathcal{B}_p\mathbf{v}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}\le C'\|\mathbf{v}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

□

The operators $\mathcal{A}_p\mathcal{S}$ and $\mathcal{B}_p\mathcal{T}$ have some interesting mapping properties. First the following lemma will be handy.

Lemma 2.2.3. *If $A\in H^\gamma_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$ and $\mathbf{v}\in H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)$ where $\gamma>\frac{1}{2}$ and $0\le\nu<\gamma-\frac{1}{2}$ then there exists $C>0$ such that*

$$\|A\mathbf{v}\|_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)}\le C\|\mathbf{v}\|_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

Proof. Suppose $A=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$, $v=\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T$ then

$$\|A\mathbf{v}\|^2_\nu=\left\|\sum_{j=1}^3\mathbf{a}_jv_j\right\|^2_\nu\le 3\sum_{j=1}^3\|\mathbf{a}_jv_j\|^2_\nu. \tag{2.2.5}$$

For $j = 1, 2, 3$ we have

$$\begin{aligned}
\|\mathbf{a}_j v_j\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} &= \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\nu |\widehat{\mathbf{a}_j v_j}(\mathbf{j})|^2 = \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\nu \left|\sum_{\mathbf{l}\in\mathbb{Z}^3}\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})\widehat{v_j}(\mathbf{l})\right|^2 \\
&\le \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})||\widehat{v_j}(\mathbf{l})|\right)^2 \\
&\le 2^{2\nu}\sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}\left[(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}} + (1+|\mathbf{l}|^2)^{\frac{\nu}{2}}\right]|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})||\widehat{v_j}(\mathbf{l})|\right)^2 \\
&\le 2^{2\nu+1}(S_1+S_2)
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})||\widehat{v_j}(\mathbf{l})|\right)^2, \\
S_2 &= \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})||\widehat{v_j}(\mathbf{l})|\right)^2.
\end{aligned}$$

Applying the Cauchy - Schwarz inequality consecutively gives

$$\begin{aligned}
S_1 &\le \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})||\widehat{v_j}(\mathbf{l})|^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})|\right) \\
&= \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{l})|\right)\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{v_j}(\mathbf{l})|^2\sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{j}-\mathbf{l})| \\
&\le \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{v_j}(\mathbf{l})|^2
\end{aligned}$$

and

$$
\begin{aligned}
S_2 &\le \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})|\right)\\
&= \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|^2\\
&\le \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{v}_j(\mathbf{l})|^2.
\end{aligned}
$$

Thus

$$
\|\mathbf{a}_j v_j\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} \le 2^{2\nu+2}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{v}_j(\mathbf{l})|^2.
$$

Also by the Cauchy - Schwarz inequality

$$
\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)^2 \le \sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu-\gamma}\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\gamma}|\widehat{\mathbf{a}}_j(\mathbf{l})|^2 = C_1^2\|\mathbf{a}_j\|^2_{H^{\gamma}_{\boldsymbol{\alpha},p}(\Omega_R)},
$$

where $C_1^2 = \sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu-\gamma}$ (this series converges since $\nu-\gamma<-\frac{1}{2}$). Hence

$$
\|\mathbf{a}_j v_j\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} \le 2^{2\nu+2}C_1^2\|\mathbf{a}_j\|^2_{H^{\gamma}_{\boldsymbol{\alpha},p}(\Omega_R)}\|\mathbf{v}\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)}. \tag{2.2.6}
$$

By (2.2.5) and (2.2.6)

$$
\|A\mathbf{v}\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} = \sum_{j=1}^{3}\|\mathbf{a}_j v_j\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} \le 3.2^{2\nu+2}C_1^2\max_j\left\{\|\mathbf{a}_j\|^2_{H^{\gamma}_{\boldsymbol{\alpha},p}(\Omega_R)}\right\}\|\mathbf{v}\|^2_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)}.
$$

Let $C=\sqrt{3}.2^{\nu+1}C_1\max_j\left\{\|\mathbf{a}_j\|_{H^{\gamma}_{\boldsymbol{\alpha},p}(\Omega_R)}\right\}$, we have

$$
\|A\mathbf{v}\|_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} \le C\|\mathbf{v}\|_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)}.
$$

□

Lemma 2.2.4. *The matrix-valued functions $P,\ Q,\ \xi\mu_r^{-1},\ \xi\mu_r^{-1}\xi,\ \mu_r^{-1}\xi$ belong to $H^{\eta}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times 3}$.*

Moreover, the operators $\mathcal{S}$, $\mathcal{T}$ are bounded linear operators from $H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ to $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$.

Proof. For the first part, we only prove for $\xi\mu_r^{-1}\xi$, the other cases are either similar or straightforward. Suppose $\xi = \begin{bmatrix}\boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\end{bmatrix}$, then $\mu_r^{-1}\xi = \begin{bmatrix}\mu_r^{-1}\boldsymbol{\xi}_1 & \mu_r^{-1}\boldsymbol{\xi}_2 & \mu_r^{-1}\boldsymbol{\xi}_3\end{bmatrix}$. For $j = 1,2,3$, since $\mu_r^{-1} \in H^{\eta+\frac{1}{2}+\epsilon}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times 3}$ and $\boldsymbol{\xi}_j \in H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)$, by Lemma 2.2.3 there exists $c > 0$ such that

$$\|\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} \le c\|\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} < \infty.$$

Similarly we can show that $\|\bar{\xi}\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} \le c'\|\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} < \infty$ for some $c' > 0$. Thus $\xi\mu_r^{-1}\xi \in H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times 3}$.

Linearity of $\mathcal{S}$ and $\mathcal{T}$ is clear, we only prove their boundedness. For $\mathbf{u} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$, $\mathrm{curl}\,\mathbf{u} \in H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$. Therefore by lemma 2.2.3

$$\begin{aligned}\|\mathcal{S}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} &\le \|(P - \xi\mu_r^{-1}\xi)\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \left\|\frac{i}{k}\xi\mu_r^{-1}\mathrm{curl}\,\mathbf{u}\right\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C_1 k\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + C_2\|\mathrm{curl}\,\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} &\le \|Q\mathrm{curl}\,\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \|ik\mu_r^{-1}\xi\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C_1'\|\mathrm{curl}\,\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + C_2' k\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C'\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}\end{aligned}$$

where $C_1, C_1', C_2, C_2', C, C'$ are positive numbers. □

The following theorem gives us the the relation between the periodic equation (2.2.2) and the original one as well as its Fredholm property.

Theorem 2.2.5. *If $\mathbf{u} \in H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ is a solution to (2.2.2) then $\mathbf{u}|_{\Omega_h} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$ solves (2.2.1). Moreover, there exist a constant $C > 0$ and a compact operator K on*

$H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ *such that*

$$\mathrm{Re}\,(\mathbf{u} - \mathcal{A}_p\mathcal{S}\mathbf{u} - \mathcal{B}_p\mathcal{T}\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \geq C\|\mathbf{u}\|^2_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} + \mathrm{Re}\,(K\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

The proof can be done using the same reasoning as in Theorem 5.2 and Theorem 5.3 in[15].

2.3 A spectral Galerkin method

For $N \in \mathbb{N}$, define

$$\mathbb{Z}^3_N = \left\{ \mathbf{j} = \begin{bmatrix} j_1 & j_2 & j_3 \end{bmatrix}^T \in \mathbb{Z}^3 : -\frac{N}{2} < j_1, j_2, j_3 \leq \frac{N}{2} \right\}.$$

Now we consider

$$\mathcal{T}_N = \mathrm{span}\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^3_N\right\}$$

which is a finite dimensional subspace of $H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ and let $P_N : H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R) \to \mathcal{T}_N$ be the orthogonal projection onto $\mathcal{T}_N$, i.e

$$P_N\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^3_N} \widehat{\mathbf{u}}(\mathbf{j})\phi_{\mathbf{j}}(\mathbf{x}).$$

We approximate (2.2.2) by the following Galerkin equation

$$\begin{aligned}(\mathbf{u}_N - \mathcal{A}_p\mathcal{S}\mathbf{u}_N - \mathcal{B}_p\mathcal{T}\mathbf{u}_N, \mathbf{w}_N)_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \\ = (\mathcal{A}_p\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g}) + \mathcal{B}_p\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g}), \mathbf{w}_N)_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}, \quad \mathbf{u}_N \in \mathcal{T}_N \quad (2.3.1)\end{aligned}$$

for all $\mathbf{w}_N \in \mathcal{T}_N$, which is equivalent to

$$\mathbf{u}_N - P_N\mathcal{A}_p\mathcal{S}\mathbf{u}_N - P_N\mathcal{B}_p\mathcal{T}\mathbf{u}_N = P_N\mathcal{A}_p\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g}) + P_N\mathcal{B}_p\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g}), \quad \mathbf{u}_N \in \mathcal{T}_N. \qquad (2.3.2)$$

We will see that convergence is achieved without smoothness of the parameters, but the smoother they are the higher order of convergence we will get.

Lemma 2.3.1. *There exists $C_\lambda > 0$ such that for all $\mathbf{w} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$*

$$\|P_N\mathbf{w} - \mathbf{w}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C_\lambda N^{-\lambda}\|\mathbf{w}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

Proof. For $\mathbf{w} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$

$$\begin{aligned}
\|P_N\mathbf{w} - \mathbf{w}\|^2_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} &= \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&= \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} \frac{1}{(1+|\mathbf{j}|^2)^\lambda}(1+|\mathbf{j}|^2)^\lambda \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&\leq \frac{1}{\left(1+\frac{3N^2}{4}\right)^\lambda} \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} (1+|\mathbf{j}|^2)^\lambda \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&\leq \left(\frac{4}{7}\right)^\lambda N^{-2\lambda}\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.
\end{aligned}$$

□

Recall that the equation (2.2.2) satisfies a Garding estimate, we have the following convergence theorem for the Galerkin method.

Theorem 2.3.2. *There exists $N_0 > 0$ such that for all $N \geq N_0$, the equation (2.3.2) has a unique solution $\mathbf{u}_N$ and the following error estimate holds*

$$\|\mathbf{u}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C_\lambda N^{-\lambda}\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}$$

where $C_\lambda > 0$ is a constant depending on λ and $\mathbf{u}$ is the solution to (2.2.2).

Proof. Since (2.2.2) satisfies the Garding estimate in Theorem 2.2.5, by Theorem 4.2.9 in [27], there exists N_0 and a positive constant C independent of N such that for all $N \geq N_0$,

$$\|\mathbf{u}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C \inf_{\mathbf{w}_N\in\mathcal{T}_N} \|\mathbf{w}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

Moreover, according to Lemma 2.3.1,

$$\inf_{\mathbf{w}_N\in\mathcal{T}_N}\|\mathbf{w}_N-\mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}\le\|P_N\mathbf{u}-\mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}\le C_\lambda N^{-\lambda}\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)},$$

therefore the desired estimate holds. □

By this theorem, we can conclude that the spectral Galerkin method converges with arbitrarily high order provided $\mathbf{u}$ is sufficiently smooth. The discretization of equation (2.3.2) as well as the process of calculating the Fourier coefficients of the involved quantities can be done similarly as in[15], therefore are omitted here.

2.4 Numerical examples

We now present numerical results for the Galerkin method. The computational domain is the rectangular box $(-\pi,\pi)^2\times(-1,1)$ $(h=1)$ which is partitioned uniformly in each dimension into N^3 grid points for some $N\in\mathbb{N}$. The wave number $k=3\pi/2$. We choose the following $\boldsymbol{\alpha}$-quasiperiodic plane wave as the incident electric field

$$\mathbf{E}^{in}(\mathbf{x})=\begin{bmatrix}1\\1\\0\end{bmatrix}e^{ikx_3}.$$

We ran the simulation in MATLAB and observed the scattered electric field $\mathbf{u}$ at $\mathbf{x}=(0,0,1.5)$ when N varies from 20 to 100 with step size 10. For each value of N, we calculated the relative error between $\mathbf{u}$ of the current step and that of the previous one, that means

$$\mathrm{error}_j=\frac{|\mathbf{u}_{N_j}(\mathbf{x})-\mathbf{u}_{N_{j-1}}(\mathbf{x})|}{|\mathbf{u}_{N_{j-1}}(\mathbf{x})|},\quad j=1,2,\dots,8.$$

Here $N_0=20$ and $N_j=N_{j-1}+10$. We consider three test cases in the numerical simulations.

Small smooth spheres

The first case is a medium consisting of a small sphere in each period with the geometry given by

$$D = \left\{ \mathbf{x} \in \Omega_h : x_1^2 + x_2^2 + x_3^2 < 0.2^2 \right\}$$

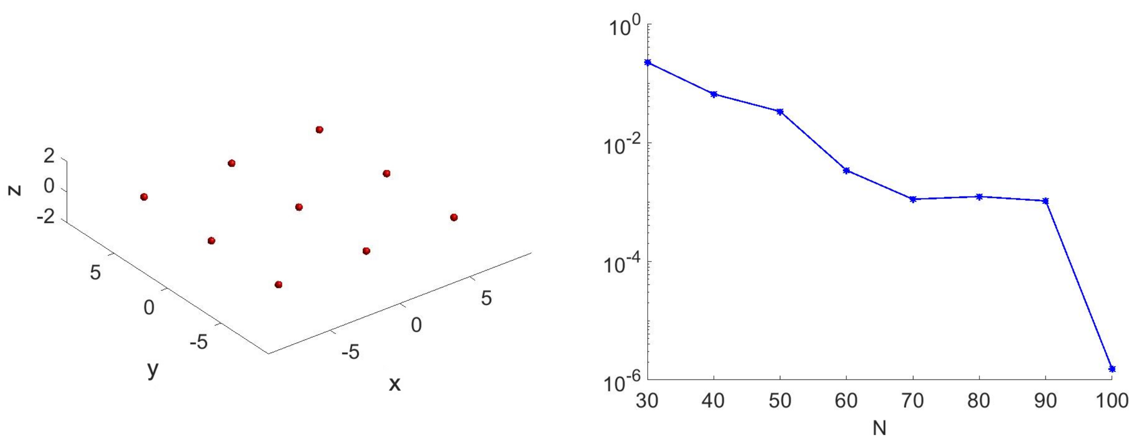
and, for $\mathbf{x} \in \Omega_h$, the material parameters are

$$\begin{aligned} \varepsilon_r(\mathbf{x}) &= I + \gamma_0(\mathbf{x})\text{diag}\left\{0.2, 0.4, 0.1\right\}, \\ \mu_r^{-1}(\mathbf{x}) &= I + \gamma_0(\mathbf{x})\text{diag}\left\{0.3, 0.1, 0.2\right\}, \\ \xi(\mathbf{x}) &= \gamma_0(\mathbf{x})\text{diag}\left\{0.01, 0.02, 0.05)\right\}, \end{aligned}$$

where $\gamma_0 : \mathbb{R}^3 \to [0, 1]$ is a smooth function that equals 1 at the center of the sphere and 0 outside it. More specifically, for $\mathbf{x} \in D$,

$$\gamma_0(\mathbf{x}) = \exp\left(1 - \frac{0.2^2}{0.2^2 - x_1^2 - x_2^2 - x_3^2}\right).$$

The three-dimensional image for the geometry as well as the relative errors for increasing values of N are shown in Figure 2.1. The error decreases as N increases and the difference between $N = 90$ and $N = 100$ is very slim.



(a) Periodic structure (9 periods) (b) Relative errors of $\mathbf{u}$ at $(0, 0, 1.5)$

Figure 2.1: Solving the direct scattering problem for periodically aligned spheres.

Smooth rectangular boxes

Next we consider a medium that consists of one rectangular box in each period. The geometry is given by

$$D = \{\mathbf{x} \in \Omega_h : |x_1| < 0.5, |x_2| < 0.5, |x_3| < 0.2\}$$

and, for $\mathbf{x} \in \Omega_h$, the material parameters are

$$\begin{aligned}
\varepsilon_r(\mathbf{x}) &= I + \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\{0.2, 0.4, 0.1\}, \\
\mu_r^{-1}(\mathbf{x}) &= I + \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\{0.3, 0.1, 0.2\}, \\
\xi(\mathbf{x}) &= \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\{0.01, 0.02, 0.05)\},
\end{aligned}$$

where $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R} \to [0, 1]$ are smooth functions such that their product equals 1 at the center of the box and 0 outside it. For $\mathbf{x} \in D$, they are given by

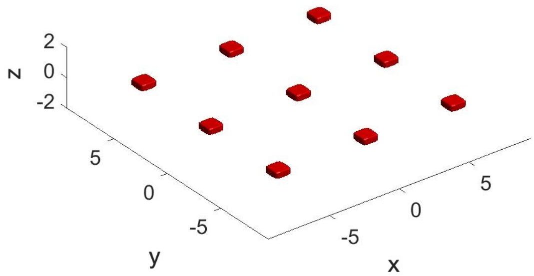
$$\begin{aligned}
\gamma_1(x_1) &= \exp\left(1 - \frac{0.5^2}{0.5^2 - x_1^2}\right) \\
\gamma_2(x_2) &= \exp\left(1 - \frac{0.5^2}{0.5^2 - x_2^2}\right) \\
\gamma_3(x_3) &= \exp\left(1 - \frac{0.2^2}{0.2^2 - x_3^2}\right).
\end{aligned}$$

Figure 2.2 shows the true geometry and relative errors for the rectangular box. Overall, the error decreases from $N = 30$ to $N = 100$. There is an abnormal rise in error from $N = 60$ to $N = 70$, but the error is already very small at this point so this might be due to roundoff errors.

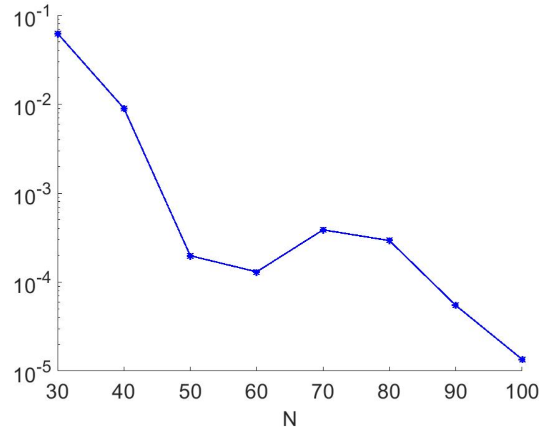
Smooth flat strip

The last scattering medium is a strip that spans the entire period horizontally. The geometry is

$$D = \{\mathbf{x} \in \Omega_h : |x_3| < 0.4\}$$



(a) Periodic structure (9 periods)



(b) Relative errors of $\mathbf{u}$ at $(0, 0, 1.5)$

Figure 2.2: Solving the direct scattering problem for periodically aligned boxes.

and, for $\mathbf{x} \in \Omega_h$, the material parameters are

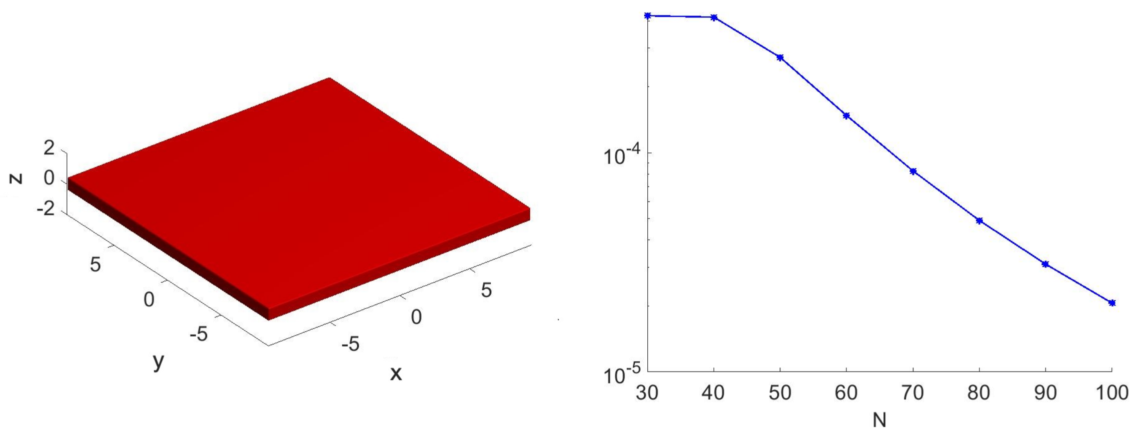
$$
\begin{aligned}
\varepsilon_r(\mathbf{x}) &= I + \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.2, 0.4, 0.1\}\,, \\
\mu_r^{-1}(\mathbf{x}) &= I + \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.3, 0.1, 0.2\}\,, \\
\xi(\mathbf{x}) &= \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.01, 0.02, 0.05)\}\,,
\end{aligned}
$$

where

$$
\gamma_4(\mathbf{x}) = \begin{cases} \exp\left(1 - \frac{0.4^2}{0.4^2 - x_3^2}\right) & \text{if } \mathbf{x} \in D, \\ 0 & \text{if } \mathbf{x} \in \Omega_h \setminus D. \end{cases}
$$

In this case, the error gradually decreases as N increases, see Figure 2.3.

In summary, the spectral method converges numerically for all examples. The periodization of the integro-differential equation along with the use of the fast Fourier transform helps significantly improve computational time and accuracy since we can avoid direct computation of the $\boldsymbol{\alpha}$-quasiperiodic Green's function.



(a) Periodic structure (9 periods) (b) Relative errors of $\mathbf{u}$ at $(0, 0, 1.5)$

Figure 2.3: Solving the direct scattering problem for a flat strip

Chapter 3

The Inverse Problem

3.1 Formulation of the inverse problem

In this section, we formulate the inverse problem that we are interested in solving. First, let us recall that $\mathbf{u} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$ is called a radiating variational solution to the direct problem if it satisfies the Rayleigh radiation condition

$$\mathbf{u}(\mathbf{x}) = \sum_{m\in\mathbb{Z}^2} \widehat{\mathbf{u}}_m^{\pm} e^{i(\boldsymbol{\alpha}_m\cdot\mathbf{x}\pm\beta_m(x_3\mp h))} \quad \text{for } x_3 \gtrless \pm h \tag{3.1.1}$$

and

$$\begin{aligned}\int_\Omega (\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - k^2\mathbf{u}\cdot\overline{\mathbf{v}})\,d\mathbf{x}& \\ = k^2\int_D(\mathcal{S}\mathbf{u}+\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g}))\cdot\overline{\mathbf{v}}\,d\mathbf{x} &+ \int_D(\mathcal{T}\mathbf{u}+\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g}))\cdot\mathrm{curl}\,\overline{\mathbf{v}}\,d\mathbf{x}\end{aligned} \tag{3.1.2}$$

for all $\mathbf{v} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega)$ with compact support. The operators $\mathcal{S}$, $\mathcal{T}$, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ are defined by

$$\mathcal{S}\mathbf{v} := (P-\xi\mu_r^{-1}\xi)\mathbf{v} + \frac{i}{k}\xi\mu_r^{-1}\mathrm{curl}\,\mathbf{v}, \quad \mathcal{T}\mathbf{v} := Q\mathrm{curl}\,\mathbf{v} - ik\mu_r^{-1}\xi\mathbf{v},$$
$$\tilde{\mathcal{S}}(\mathbf{v}_1,\mathbf{v}_2) := (P-\xi\mu_r^{-1}\xi)\mathbf{v}_2 + \frac{i}{k}\xi\mu_r^{-1}\mathbf{v}_1, \quad \tilde{\mathcal{T}}(\mathbf{v}_1,\mathbf{v}_2) := Q\mathbf{v}_1 - ik\mu_r^{-1}\xi\mathbf{v}_2.$$

An equivalent integro-differential equation to the above variational problem is

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = \mathcal{A}\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g}) + \mathcal{B}\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g}) \quad \text{in } \Omega \tag{3.1.3}$$

where

$$\mathcal{A}\mathbf{h}(\mathbf{x}) := (k^2 + \nabla\mathrm{div}) \int_D G_k(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y},$$
$$\mathcal{B}\mathbf{h}(\mathbf{x}) := \mathrm{curl} \int_D G_k(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}.$$

We have shown that the direct problem is well-posed, thus we can define the solution operator $S : L^2(D,\mathbb{C}^3) \times L^2(D,\mathbb{C}^3) \to \ell^2(\mathbb{Z}^2,\mathbb{C}^4)$

$$S(\mathbf{f},\mathbf{g}) = (\widehat{u}^+_{m,1}, \widehat{u}^-_{m,1}, \widehat{u}^+_{m,2}, \widehat{u}^-_{m,2})_{m\in\mathbb{Z}^2}$$

where $(\widehat{u}^+_{m,1}, \widehat{u}^-_{m,1}, \widehat{u}^+_{m,2}, \widehat{u}^-_{m,2})_{m\in\mathbb{Z}^2}$ are the Rayleigh sequences of the first two components $(\widehat{\mathbf{u}}^\pm_m)_{m\in\mathbb{Z}^2}$ of the radiating solution $\mathbf{u}$ of (3.1.2).

For $\mathbf{x} = (x_1, x_2, x_3)^\top$, we denote $\tilde{\mathbf{x}} = (x_1, x_2, -x_3)^\top$. For the inverse problem, we consider many incident plane waves as follows

$$\varphi^{(l)\pm}_m = p^{(l)}_m e^{i(\alpha_m\cdot\mathbf{x}+\beta_m x_3)} \pm \tilde{p}^{(l)}_m e^{i(\alpha_m\cdot\mathbf{x}-\beta_m x_3)}, \quad l = 1,2,\ m \in \mathbb{Z}^2, \tag{3.1.4}$$

where the polarizations $p^{(1)}_m$, $p^{(2)}_m$ are linearly independent vectors such that $|p^{(1)}_m| = |p^{(2)}_m| = 1$ and $\varphi^{(l)\pm}_m$ are divergence-free. One possible choice could be

$$p^{(1)}_m = \frac{(0, \beta_m, -\alpha_{m,2})^\top}{\sqrt{|\alpha_{m,2}|^2 + |\beta_m|^2}}, \quad p^{(2)}_m = \frac{(-\beta_m, 0, \alpha_{m,1})^\top}{\sqrt{|\alpha_{m,1}|^2 + |\beta_m|^2}}. \tag{3.1.5}$$

Note that these incident plane waves were proposed in [28] for the analysis of the Factorization method for the Maxwell's equations in non-magnetic periodic structures. We now define the Herglotz operator $H : \ell^2(\mathbb{Z}^2,\mathbb{C}^4) \to L^2(D,\mathbb{C}^3) \times L^2(D,\mathbb{C}^3)$ that maps a sequence of coefficients $(a_m) = (a^+_{m,1}, a^-_{m,1}, a^+_{m,2}, a^-_{m,2})_{m\in\mathbb{Z}^2}$ into

$$H(a_m) = \begin{bmatrix} \sum\limits_{m\in\mathbb{Z}^2} \left(\frac{a^+_{m,1}}{\beta_m w^+_m} \operatorname{curl} \varphi^{(1)+}_m + \frac{a^+_{m,2}}{\beta_m w^+_m} \operatorname{curl} \varphi^{(2)+}_m + \frac{a^-_{m,1}}{\beta_m w^-_m} \operatorname{curl} \varphi^{(1)-}_m + \frac{a^-_{m,2}}{\beta_m w^-_m} \operatorname{curl} \varphi^{(2)-}_m \right) \\ \sum\limits_{m\in\mathbb{Z}^2} \left(\frac{a^+_{m,1}}{\beta_m w^+_m} \varphi^{(1)+}_m + \frac{a^+_{m,2}}{\beta_m w^+_m} \varphi^{(2)+}_m + \frac{a^-_{m,1}}{\beta_m w^-_m} \varphi^{(1)-}_m + \frac{a^-_{m,2}}{\beta_m w^-_m} \varphi^{(2)-}_m \right) \end{bmatrix} \tag{3.1.6}$$

where the weights

$$w^+_m = \begin{cases} i, & |\alpha_m| \le k \\ e^{-i\beta_m h}, & |\alpha_m| > k \end{cases}, \quad w^-_m = \begin{cases} 1, & |\alpha_m| \le k \\ e^{-i\beta_m h}, & |\alpha_m| > k \end{cases}, \qquad m \in \mathbb{Z}^2$$

are to help simplify the calculations in the proofs of some analytical properties of H, see[21]. Here these weights are kept in the definition of H for the convenience of the presentation. From[21] we know that H is a compact and injective operator.

We consider a near field measurement which is motivated by uniqueness results in[29] and applications in near field optics. Let $N : \ell^2(\mathbb{Z}^2, \mathbb{C}^4) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ be the near field operator which maps a sequence $(a_m) \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ to the Rayleigh sequences of the first two components of the radiating variational solution $\mathbf{u}$ of (3.1.2) with $(\mathbf{f}, \mathbf{g}) = H(a_m)$, that means

$$N(a_m) = (\widehat{u}^+_{m,1}, \widehat{u}^-_{m,1}, \widehat{u}^+_{m,2}, \widehat{u}^-_{m,2})_{m\in\mathbb{Z}^2}.$$

Now we are ready to state the inverse problem of interest.

Inverse problem. Given the near-field operator N, find the shape D of the scatterer in Ω.

3.2 A characterization of the scatterer

In this section we will show that D can be characterized by the range of the adjoint operator H^* of the Herglotz operator H defined in (3.1.6). Let us introduce two auxiliary operators which are $E : L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ and $W : \ell^2(\mathbb{Z}^2, \mathbb{C}^4) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ defined by

$$E(\mathbf{f}, \mathbf{g}) = (\widehat{u}^+_{m,1}, \widehat{u}^-_{m,1}, \widehat{u}^+_{m,2}, \widehat{u}^-_{m,2})_{m\in\mathbb{Z}^2}$$

where $\mathbf{u}$ is the radiating variational solution to

$$\operatorname{curl}^2\mathbf{u} - k^2\mathbf{u} = \operatorname{curl}\mathbf{f} + \mathbf{g} \tag{3.2.1}$$

and

$$W(a_m) = 8\pi^2 \begin{bmatrix} w^{*+}_m\overline{p^{(1)}_{m,1}} & w^{*+}_m\overline{p^{(1)}_{m,2}} & w^{*+}_m\overline{p^{(1)}_{m,1}} & w^{*+}_m\overline{p^{(1)}_{m,2}} \\ w^{*+}_m\overline{p^{(2)}_{m,1}} & w^{*+}_m\overline{p^{(2)}_{m,2}} & w^{*+}_m\overline{p^{(2)}_{m,1}} & w^{*+}_m\overline{p^{(2)}_{m,2}} \\ w^{*-}_m\overline{p^{(1)}_{m,1}} & w^{*-}_m\overline{p^{(1)}_{m,2}} & -w^{*-}_m\overline{p^{(1)}_{m,1}} & -w^{*-}_m\overline{p^{(1)}_{m,2}} \\ w^{*-}_m\overline{p^{(2)}_{m,1}} & w^{*-}_m\overline{p^{(2)}_{m,2}} & -w^{*-}_m\overline{p^{(2)}_{m,1}} & -w^{*-}_m\overline{p^{(2)}_{m,2}} \end{bmatrix} \begin{bmatrix} a^+_{m,1} \\ a^-_{m,1} \\ a^+_{m,2} \\ a^-_{m,2} \end{bmatrix}$$

where

$$w^{*+}_m = \begin{cases} e^{-i\beta_m h}, & |\alpha_m| \le k \\ i, & |\alpha_m| > k \end{cases}, \quad w^{*-}_m = \begin{cases} ie^{-i\beta_m h}, & |\alpha_m| \le k \\ i, & |\alpha_m| > k \end{cases}, \qquad m \in \mathbb{Z}^2$$

and $p^{(1)}_m$, $p^{(2)}_m$ are given by (3.1.5). Note that E is well-defined thanks to well-posedness of (3.2.1) for all frequencies $k > 0$ (see[30]). In addition, we will extend $\mathbf{f}$ and $\mathbf{g}$ in (3.2.1) by zero outside of D if needed.

The proof of the following lemma can be done as in[21].

Lemma 3.2.1. *The adjoint operator $H^* : L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ satisfies*

$$H^* = WE.$$

Next we introduce the α-quasiperiodic Green's tensor

$$\mathbb{G}_k(\mathbf{x}, \mathbf{z}) = G_k(\mathbf{x} - \mathbf{z})I_3 + \frac{1}{k^2}\nabla_{\mathbf{x}}\operatorname{div}_{\mathbf{x}}(G_k(\mathbf{x} - \mathbf{z})I_3), \quad \mathbf{x}, \mathbf{z} \in \Omega,\ x_3 \ne z_3$$

where G_k is given by (2.1.3) and $\nabla_{\mathbf{x}}$, $\operatorname{div}_{\mathbf{x}}$ are taken componentwise and columnwise respectively. Note that for a fixed $\mathbf{z} \in \Omega$, $\mathbb{G}_k(\mathbf{x}, \mathbf{z})$ solves

$$\operatorname{curl}\operatorname{curl}\mathbb{G}_k(\mathbf{x}, \mathbf{z}) - k^2\mathbb{G}_k(\mathbf{x}, \mathbf{z}) = \delta_{\mathbf{z}}(\mathbf{x})I_3 \tag{3.2.2}$$

in the sense of distribution and it satisfies the Rayleigh expansion radiation condition (the curl is taken columnwise).

Let $\mathbf{p} = (p_1, p_2, p_3)^\top \in \mathbb{R}^3$ and $\Psi_{\mathbf{z}}(\mathbf{x}) = k^2\mathbb{G}_k(\mathbf{x}, \mathbf{z})\mathbf{p}$. Denote by

$$(\widehat{\Psi}_{\mathbf{z},m}) = (\widehat{\Psi}^+_{m,1}(\mathbf{z}), \widehat{\Psi}^-_{m,1}(\mathbf{z}), \widehat{\Psi}^+_{m,2}(\mathbf{z}), \widehat{\Psi}^-_{m,2}(\mathbf{z}))_{m\in\mathbb{Z}^2}$$

the Rayleigh sequences of the first two components of $\Psi_{\mathbf{z}}$. Then $(\widehat{\Psi}_{\mathbf{z},m})$ can be explicitly given by

$$(\widehat{\Psi}_{\mathbf{z},m}) = \begin{bmatrix} (k^2 - \alpha^2_{m,1})\widehat{G}^+_{k,m}(\mathbf{z})p_1 - \alpha_{m,1}\alpha_{m,2}\widehat{G}^+_{k,m}(\mathbf{z})p_2 - \alpha_{m,1}\beta_m\widehat{G}^+_{k,m}(\mathbf{z})p_3 \\ (k^2 - \alpha^2_{m,1})\widehat{G}^-_{k,m}(\mathbf{z})p_1 - \alpha_{m,1}\alpha_{m,2}\widehat{G}^-_{k,m}(\mathbf{z})p_2 + \alpha_{m,1}\beta_m\widehat{G}^-_{k,m}(\mathbf{z})p_3 \\ -\alpha_{m,1}\alpha_{m,2}\widehat{G}^+_{k,m}(\mathbf{z})p_1 + (k^2 - \alpha^2_{m,2})\widehat{G}^+_{k,m}(\mathbf{z})p_2 - \alpha_{m,2}\beta_m\widehat{G}^+_{k,m}(\mathbf{z})p_3 \\ -\alpha_{m,1}\alpha_{m,2}\widehat{G}^-_{k,m}(\mathbf{z})p_1 + (k^2 - \alpha^2_{m,2})\widehat{G}^-_{k,m}(\mathbf{z})p_2 + \alpha_{m,2}\beta_m\widehat{G}^-_{k,m}(\mathbf{z})p_3 \end{bmatrix}_{m\in\mathbb{Z}^2}$$

where $(\widehat{G}^\pm_{k,m}(\mathbf{z}))_{m\in\mathbb{Z}^2}$ are the Rayleigh sequences of $G_k(\cdot - \mathbf{z})$.

Denote by $\mathcal{R}(A)$ the range of some operator A. The domain D can be characterized by the $\mathcal{R}(H^*)$ as follows.

Theorem 3.2.2. *A point $\mathbf{z} \in \Omega$ belongs to D if and only if $W(\widehat{\Psi}_{\mathbf{z},m}) \in \mathcal{R}(H^*)$.*

Proof. The proof can be done similarly as that of Lemma 7 in [28]. □

Note that since the definition of H^* involves D, we can't compute D using this characterization. However, this characterization will serve as an intermediate step to connect D to the near field operator N that is given. This is the goal of the Factorization method that is studied in the next section.

3.3 The Factorization method

This section is dedicated to the analysis of the Factorization method. Let $T : L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3) \to L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)$ be defined by

$$T(\mathbf{f},\mathbf{g}) = \begin{bmatrix} Q & -ik\mu_{\mathrm{r}}^{-1}\xi \\ ik\xi\mu_{\mathrm{r}}^{-1} & k^2(P-\xi\mu_{\mathrm{r}}^{-1}\xi) \end{bmatrix} \begin{bmatrix} \mathbf{f}+\operatorname{curl}\mathbf{u} \\ \mathbf{g}+\mathbf{u} \end{bmatrix}$$

where $\mathbf{u}$ is the radiating solution to (3.1.3).

Lemma 3.3.1. *T is a bounded linear operator on $L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)$.*

Proof. Linearity of T follows from linearity and well-posedness of (3.1.2). We will show its boundedness. For $(\mathbf{f},\mathbf{g})\in L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)$

$$\|T(\mathbf{f},\mathbf{g})\|^2_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)} = \|\mathcal{S}\mathbf{u}+\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g})\|^2_{L^2(D,\mathbb{C}^3)} + \|\mathcal{T}\mathbf{u}+\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g})\|^2_{L^2(D,\mathbb{C}^3)}.$$

Let $\mathbf{u}\in H_{\alpha,\mathrm{loc}}(\operatorname{curl},\Omega)$ be the solution of (3.1.3). Thanks to well-posedness of the equation, the operator $I-\mathcal{A}\mathcal{S}-\mathcal{B}\mathcal{T}$ is boundedly invertible and

$$\mathbf{u} = (I-\mathcal{A}\mathcal{S}-B\mathcal{T})^{-1}[\mathcal{A}\tilde{\mathcal{S}}(\mathbf{f},\mathbf{g})+\mathcal{B}\tilde{\mathcal{T}}(\mathbf{f},\mathbf{g})].$$

Since $\mathcal{A},\mathcal{B}$ are bounded and the scatterer's parameters are in $L^\infty(\mathbb{R}^3,\mathbb{C}^{3\times3})$, it follows that there exists $c>0$ such that

$$\|\mathbf{u}\|_{H(\operatorname{curl},D)} \le c\|(\mathbf{f},\mathbf{g})\|_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)}.$$

This implies

$$\|\mathbf{g}+\mathbf{u}\|_{L^2(D,\mathbb{C}^3)} \le (c+1)\|(\mathbf{f},\mathbf{g})\|_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)},$$

$$\|\mathbf{f}+\operatorname{curl}\mathbf{u}\|_{L^2(D,\mathbb{C}^3)} \le (c+1)\|(\mathbf{f},\mathbf{g})\|_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)}.$$

Therefore by the definition of $\mathcal{S}$ and $\mathcal{T}$ there exist $c_1, c_2 > 0$ such that

$$\|$$

$$mathcalS\mathbf{u} + \tilde{\mathcal{S}}(\mathbf{f}, \mathbf{g})\|_{L^2(D,\mathbb{C}^3)} \leq c_1 \|(\mathbf{f}, \mathbf{g})\|_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)},$$

$$\|\mathcal{T}\mathbf{u} + \tilde{\mathcal{T}}(\mathbf{f}, \mathbf{g})\|_{L^2(D,\mathbb{C}^3)} \leq c_2 \|(\mathbf{f}, \mathbf{g})\|_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)},$$

and the boundedness of T follows. $\square$

Let $\operatorname{Im} T$ be the imaginary part of T defined by

$$\operatorname{Im} T = \frac{1}{2i}(T - T^*).$$

In the next lemma we prove the coercivity of $\operatorname{Im} T$ which is the key ingredient in the analysis of the Factorization method.

Lemma 3.3.2. *There exists $c > 0$ such that for all $(\mathbf{f}, \mathbf{g}) \in L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$*

$$(\operatorname{Im} T(\mathbf{f}, \mathbf{g}), (\mathbf{f}, \mathbf{g}))_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)} \geq c\|(\mathbf{f}, \mathbf{g})\|^2_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)}. \tag{3.3.1}$$

Proof. For the convenience of the presentation of this proof we will use $(\cdot, \cdot)$ and $\|\cdot\|$ indistinctively for the inner product and norm of $L^2(D, \mathbb{C}^3)$ and $L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$. Let $\mathbf{h}_1 = \mathbf{f} + \operatorname{curl} \mathbf{u}$, $\mathbf{h}_2 = \mathbf{g} + \mathbf{u}$ we have

$$\begin{aligned}
(T(\mathbf{f}, \mathbf{g}), (\mathbf{f}, \mathbf{g})) &= \int_D (Q\mathbf{h}_1 - ik\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2) \cdot (\overline{\mathbf{h}}_1 - \operatorname{curl} \overline{\mathbf{u}})\, \mathrm{d}\mathbf{x} \\
&\quad + \int_D (ik\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1 + k^2(P - \xi\mu_{\mathrm{r}}^{-1}\xi)\mathbf{h}_2) \cdot (\overline{\mathbf{h}}_2 - \overline{\mathbf{u}})\, \mathrm{d}\mathbf{x} \\
&= S_1 - S_2,
\end{aligned}$$

where

$$S_1 = (Q\mathbf{h}_1, \mathbf{h}_1) + k^2((P - \xi\mu_{\mathrm{r}}^{-1}\xi)\mathbf{h}_2, \mathbf{h}_2) + ik(\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1, \mathbf{h}_2) - ik(\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2, \mathbf{h}_1),$$
$$S_2 = \int_D (Q\mathbf{h}_1 - ik\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2) \cdot \operatorname{curl}\overline{\mathbf{u}}\ \mathrm{d}\mathbf{x} + \int_D (ik\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1 + k^2(P - \xi\mu_{\mathrm{r}}^{-1}\xi)\mathbf{h}_2) \cdot \overline{\mathbf{u}}\ \mathrm{d}\mathbf{x}.$$

Therefore, with the fact that

$$(\operatorname{Im} T(\mathbf{f}, \mathbf{g}), (\mathbf{f}, \mathbf{g})) = \operatorname{Im}\,(T(\mathbf{f}, \mathbf{g}), (\mathbf{f}, \mathbf{g}))$$

we have

$$(\operatorname{Im} T(\mathbf{f}, \mathbf{g}), (\mathbf{f}, \mathbf{g})) = \operatorname{Im} S_1 - \operatorname{Im} S_2.$$

Let us first consider $\operatorname{Im} S_1$. From Assumption 1.2.1 we have

$$\operatorname{Im} S_1 \geq C_1\|\mathbf{h}_1\|^2 + k^2 C_2\|\mathbf{h}_2\|^2 + k\mathrm{Re}(\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1, \mathbf{h}_2) - k\mathrm{Re}(\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2, \mathbf{h}_1).$$

Recall that μ_{r}^{-1} and ξ are symmetric. We estimate

$$\begin{aligned}
k\mathrm{Re}(\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1, \mathbf{h}_2) &= \mathrm{Re}(\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1, k\mathbf{h}_2) \\
&= \frac{1}{2}\left(\|\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1 + k\mathbf{h}_2\|^2 - \|\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1\|^2 - k^2\|\mathbf{h}_2\|^2\right) \\
&\geq \frac{1}{2}\left(\left(\|\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1\| - \|k\mathbf{h}_2\|\right)^2 - \|\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1\|^2 - k^2\|\mathbf{h}_2\|^2\right) \\
&= -\|\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1\|\|k\mathbf{h}_2\| \\
&\geq -\frac{1}{2}\left(\||\xi\mu_{\mathrm{r}}^{-1}|_F\|_{L^\infty}^2\|\mathbf{h}_1\|^2 + k^2\|\mathbf{h}_2\|^2\right) \\
&= -\frac{1}{2}\left(\||\mu_{\mathrm{r}}^{-1}\xi|_F\|_{L^\infty}^2\|\mathbf{h}_1\|^2 + k^2\|\mathbf{h}_2\|^2\right),
\end{aligned}$$

and similarly

$$
\begin{aligned}
-k\mathrm{Re}(\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2,\mathbf{h}_1) &= -\mathrm{Re}(k\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2,\mathbf{h}_1)\\
&= \frac{1}{2}\left(\|k\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2-\mathbf{h}_1\|^2 - k^2\|\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2\|^2 - \|\mathbf{h}_1\|^2\right)\\
&\geq -\frac{1}{2}\left(k^2\||\mu_{\mathrm{r}}^{-1}\xi|_F\|^2_{L^\infty}\|\mathbf{h}_2\|^2 + \|\mathbf{h}_1\|^2\right).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathrm{Im}\,S_1 &\geq \left[C_1 - \frac{1}{2}\left(\||\mu_{\mathrm{r}}^{-1}\xi|_F\|^2_{L^\infty}+1\right)\right]\|\mathbf{h}_1\|^2 + k^2\left[C_2 - \frac{1}{2}\left(\||\mu_{\mathrm{r}}^{-1}\xi|_F\|^2_{L^\infty}+1\right)\right]\|\mathbf{h}_2\|^2\\
&= c_1\|\mathbf{h}_1\|^2 + k^2 c_2\|\mathbf{h}_2\|^2
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= C_1 - \frac{1}{2}\left(\||\mu_{\mathrm{r}}^{-1}\xi|_F\|^2_{L^\infty}+1\right) > 0,\\
c_2 &= C_2 - \frac{1}{2}\left(\||\mu_{\mathrm{r}}^{-1}\xi|_F\|^2_{L^\infty}+1\right) > 0.
\end{aligned}
$$

In order to estimate $\mathrm{Im}\,S_2$ we note that since $\mathbf{u}$ is the radiating solution to (3.1.2), for all $\mathbf{v}\in H_\alpha(\mathrm{curl}\,,\Omega)$ with compact support we have

$$
\begin{aligned}
\int_\Omega (\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - k^2\mathbf{u}\cdot\overline{\mathbf{v}})\,\mathrm{d}\mathbf{x} &= \int_D (Q\mathbf{h}_1 - ik\mu_{\mathrm{r}}^{-1}\xi\mathbf{h}_2)\cdot\mathrm{curl}\,\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x}\\
&\quad + \int_D (ik\xi\mu_{\mathrm{r}}^{-1}\mathbf{h}_1 + k^2(P-\xi\mu_{\mathrm{r}}^{-1}\xi)\mathbf{h}_2)\cdot\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Let $\Omega_r = \{\mathbf{x}\in\Omega : |x_3| < r\}$ and consider $r>0$ such that $\overline{D}\subset\Omega_r$. Consider a scalar cut-off function $\varphi\in C^\infty(\mathbb{R})$ such that $\varphi(t)=1$ for $|t|<r$ and $\varphi=0$ for $|t|>2r$. Then setting $\mathbf{v}(\mathbf{x}) = \varphi(x_3)\mathbf{u}(\mathbf{x})$ it belongs to $H_\alpha(\mathrm{curl}\,,\Omega)$ with compact support in Ω_{2r}. Substituting $\mathbf{v}$ in the above variational form yields

$$
\int_\Omega (\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,(\varphi\overline{\mathbf{u}}) - k^2\mathbf{u}\cdot(\varphi\overline{\mathbf{u}}))\,\mathrm{d}\mathbf{x} = S_2.
$$

Therefore, using Green's identities and the fact that $\mathbf{u}$ solves $\operatorname{curl}^2\mathbf{u} - k^2\mathbf{u} = 0$ in $\Omega_{2r} \setminus \Omega_r$ we have

$$\begin{aligned}
S_2 &= \int_{\Omega_r} (|\operatorname{curl}\mathbf{u}|^2 - k^2|\mathbf{u}|^2)\mathrm{d}\mathbf{x} + \int_{\Omega_{2r}\setminus\Omega_r} (\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}(\varphi\overline{\mathbf{u}}) - k^2\mathbf{u}\cdot(\varphi\overline{\mathbf{u}}))\mathrm{d}\mathbf{x} \\
&= \int_{\Omega_r} (|\operatorname{curl}\mathbf{u}|^2 - k^2|\mathbf{u}|^2)\mathrm{d}\mathbf{x} + \int_{\Omega_{2r}\setminus\Omega_r} (\operatorname{curl}^2\mathbf{u} - k^2\mathbf{u})\cdot(\varphi\mathbf{u})\mathrm{d}\mathbf{x} \\
&\quad + \left(\int_{\{x_3=r\}\cap\Omega} - \int_{\{x_3=-r\}\cap\Omega}\right)(e_3\times\operatorname{curl}\mathbf{u})\cdot\overline{\mathbf{u}}\,\mathrm{d}\mathbf{x} \\
&= \int_{\Omega_r} (|\operatorname{curl}\mathbf{u}|^2 - k^2|\mathbf{u}|^2)\mathrm{d}\mathbf{x} + \left(\int_{\{x_3=r\}\cap\Omega} - \int_{\{x_3=-r\}\cap\Omega}\right)(e_3\times\operatorname{curl}\mathbf{u})\cdot\overline{\mathbf{u}}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Therefore, taking the imaginary part of both sides we have

$$\operatorname{Im} S_2 = \operatorname{Im}\left(\int_{\{x_3=r\}\cap\Omega} - \int_{\{x_3=-r\}\cap\Omega}\right)(e_3\times\operatorname{curl}\mathbf{u})\cdot\overline{\mathbf{u}}\,\mathrm{d}\mathbf{x}.$$

Using the Rayleigh expansion radiation condition for $\mathbf{u}$ and a straightforward calculation give

$$\lim_{r\to\infty}\left(\int_{\{x_3=r\}\cap\Omega} - \int_{\{x_3=-r\}\cap\Omega}\right)(e_3\times\operatorname{curl}\mathbf{u})\cdot\overline{\mathbf{u}}\,\mathrm{d}\mathbf{x} = -i4\pi^2\sum_{m:\beta_m>0}\beta_m(|\widehat{u}_m^+|^2 + |\widehat{u}_m^-|^2).$$

We thus obtain

$$\operatorname{Im} S_2 = -4\pi^2\sum_{m:\beta_m>0}\beta_m(|\widehat{u}_m^+|^2 + |\widehat{u}_m^-|^2) \le 0.$$

From the estimates for $\operatorname{Im} S_1$ and $\operatorname{Im} S_2$ and the fact that $(\operatorname{Im} T(\mathbf{f},\mathbf{g}),(\mathbf{f},\mathbf{g})) = \operatorname{Im} S_1 - \operatorname{Im} S_2$, we obtain

$$(\operatorname{Im} T(\mathbf{f},\mathbf{g}),(\mathbf{f},\mathbf{g})) \ge c_1\|\mathbf{h}_1\|^2 + k^2c_2\|\mathbf{h}_2\|^2. \tag{3.3.2}$$

Now suppose that there is no $c > 0$ such that (3.3.1) holds. Then there exists a sequence $\{(\mathbf{f}_j,\mathbf{g}_j)\}_j \subset L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)$ such that $\|(\mathbf{f}_j,\mathbf{g}_j)\| = 1$ and

$$(\operatorname{Im} T(\mathbf{f}_j,\mathbf{g}_j),(\mathbf{f}_j,\mathbf{g}_j)) \xrightarrow{j\to\infty} 0.$$

By (3.3.2) we have $\mathbf{h}_1^j = \mathbf{f}_j + \operatorname{curl} \mathbf{u}_j \xrightarrow{j\to\infty} 0$ and $\mathbf{h}_2^j = \mathbf{g}_j + \mathbf{u}_j \xrightarrow{j\to\infty} 0$ in $L^2(D, \mathbb{C}^3)$ where $\mathbf{u}_j$ is the radiating variational solution to

$$\operatorname{curl}^2 \mathbf{u}_j - k^2 \mathbf{u}_j = k^2(\mathcal{S}\mathbf{u}_j + \tilde{\mathcal{S}}(\mathbf{f}_j, \mathbf{g}_j)) + \operatorname{curl}(\mathcal{T}\mathbf{u}_j + \tilde{\mathcal{T}}(\mathbf{f}_j, \mathbf{g}_j)).$$

Then $\mathbf{u}_j$ also satisfies

$$\begin{aligned}
\mathbf{u}_j &= \mathcal{A}(\mathcal{S}\mathbf{u}_j + \tilde{\mathcal{S}}(\mathbf{f}_j, \mathbf{g}_j)) + \mathcal{B}(\mathcal{S}\mathbf{u}_j + \tilde{\mathcal{S}}(\mathbf{f}_j, \mathbf{g}_j)) \\
&= \mathcal{A}\left[\frac{i}{k}\xi\mu_{\mathrm{r}}^{-1}(\mathbf{f}_j + \operatorname{curl}\mathbf{u}_j) + (P - \xi\mu_{\mathrm{r}}^{-1}\xi)(\mathbf{g}_j + \mathbf{u}_j)\right] \\
&\quad + \mathcal{B}\left[Q(\mathbf{f}_j + \operatorname{curl}\mathbf{u}_j) - ik\mu_{\mathrm{r}}^{-1}\xi(\mathbf{g}_j + \mathbf{u}_j)\right].
\end{aligned}$$

Hence, there exist $c_3, c_4 > 0$ such that

$$\begin{aligned}
\|\mathbf{u}_j\|_{H(\operatorname{curl}, D)} \leq c_3 &\left(\frac{1}{k}\||\xi\mu_{\mathrm{r}}^{-1}|_F\|_{L^\infty}\|\mathbf{f}_j + \operatorname{curl}\mathbf{u}_j\| + \||P - \xi\mu_{\mathrm{r}}^{-1}\xi|_F\|_{L^\infty}\|\mathbf{g}_j + \mathbf{u}_j\|\right) \\
&+ c_4\left(\||Q|_F\|_{L^\infty}\|\mathbf{f}_j + \operatorname{curl}\mathbf{u}_j\| + k\||\mu_{\mathrm{r}}^{-1}\xi|_F\|_{L^\infty}\|\mathbf{g}_j + \mathbf{u}_j\|\right).
\end{aligned}$$

Thus $\mathbf{u}_j \xrightarrow{j\to\infty} 0$ in $H(\operatorname{curl}, D)$ and therefore $(\mathbf{f}_j, \mathbf{g}_j) \xrightarrow{j\to\infty} 0$ in $L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$ which contradicts $\|(\mathbf{f}_j, \mathbf{g}_j)\| = 1$. □

Lemma 3.3.3. *We have the following factorization*

$$WN = H^*TH.$$

Proof. Together with $H^* = WE$ from Lemma 3.2.1, the proof follows from the factorizations $N = SH$ and $S = ET$ that can be easily verified. □

From the above lemma we obtain

$$\operatorname{Im}(WN) = H^*(\operatorname{Im} T)H,$$

and more importantly, we can deduce the following result about the connection between the ranges of H^* and $\mathrm{Im}\,(WN)$.

Theorem 3.3.4. $\mathrm{Im}\,(WN)$ *is a positive definite, compact and self-adjoint operator on* $\ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ *with an eigensystem* $(\lambda_j, (\phi^j_m))_{j\in\mathbb{N}}$. *Thus its square root*

$$(\mathrm{Im}\,(WN))^{1/2}(\psi_m) = \sum_{j=0}^{\infty} \sqrt{\lambda_j}\ (\psi_m, \phi^j_m)_{\ell^2(\mathbb{Z}^2,\mathbb{C}^4)}\phi^j_m, \quad (\psi_m) \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4),$$

is well-defined and

$$\mathcal{R}(H^*) = \mathcal{R}((\mathrm{Im}\,(WN))^{1/2}).$$

Proof. $\mathrm{Im}\,(WN)$ is clearly self-adjoint and its compactness follows from the compactness of H. Moreover, for $(a_m) \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ that is a nonzero sequence we have

$$\begin{aligned}(\mathrm{Im}\,(WN)(a_m), (a_m))_{\ell^2(\mathbb{Z}^2,\mathbb{C}^4)} &= (H^*(\mathrm{Im}\,T)H(a_m), (a_m))_{\ell^2(\mathbb{Z}^2,\mathbb{C}^4)} \\ &= ((\mathrm{Im}\,T)H(a_m), H(a_m))_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)} \\ &\geq c\|H(a_m)\|^2_{L^2(D,\mathbb{C}^3)\times L^2(D,\mathbb{C}^3)} > 0\end{aligned}$$

for some $c > 0$ thanks to the coercivity of $\mathrm{Im}\,T$ and injectivity of H. Hence all the eigenvalues of $\mathrm{Im}\,(WN)$ are positive. The range identity follows directly from the Corollary 1.22 in [31]. $\square$

Combining Theorem 3.2.2 and Theorem 3.3.4 we have the following characterization of D.

Theorem 3.3.5. *A point* $\mathbf{z} \in \Omega$ *belongs to* D *if and only if* $W(\widehat{\Psi}_{\mathbf{z},m}) \in \mathcal{R}((\mathrm{Im}\,(WN))^{1/2})$. *If we denote by* $(\lambda_j, (\phi^j_m))_{j\in\mathbb{N}}$ *the eigensystem of* $\mathrm{Im}\,(WN)$ *then the above criterion is equivalent to*

$$\sum_{j=0}^{\infty} \frac{\left|\left(W(\widehat{\Psi}_{\mathbf{z},m}), (\phi^j_m)\right)_{\ell^2(\mathbb{Z}^2,\mathbb{C}^4)}\right|^2}{\lambda_j} < \infty. \tag{3.3.3}$$

This is a necessary and sufficient characterization for D from the range of $(\mathrm{Im}\,(WN))^{1/2}$ which gives us the unique determination of D. It also provides a fast way to reconstruct D

by plotting the reciprocal value of the series (3.3.3) for many points $\mathbf{z}$ sampling some domain that contains D.

3.4 Numerical examples

We present in this section some numerical examples for imaging of bi-anisotropic periodic structures via the Picard criterion (3.3.3). For $M \in \mathbb{N}$, we set

$$\mathbb{Z}_M^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 : -M/2 + 1 \leq j_1, j_2 \leq M/2\}.$$

To generate the synthetic near-field data for the numerical examples we solve the direct problem using the spectral solver studied in[15]. More precisely, the direct problem is solved for the incident plane waves $\varphi_j^{(l)\pm}$ for $j \in \mathbb{Z}_M^2$, and the Rayleigh coefficients of the scattered fields are computed on $x_3 = \pm 1$, again for all indices in $\mathbb{Z}_M^2$. Let $\boldsymbol{N}_M$ be the block matrix of the discretized near-field operator N

$$\boldsymbol{N}_M = \begin{pmatrix} \left((\hat{u}_{1,n}^{+})_j^{(1)+}\right)_{j,n} & \left((\hat{u}_{1,n}^{+})_j^{(1)-}\right)_{j,n} & \left((\hat{u}_{1,n}^{+})_j^{(2)+}\right)_{j,n} & \left((\hat{u}_{1,n}^{+})_j^{(2)-}\right)_{j,n} \\ \left((\hat{u}_{2,n}^{+})_j^{(1)+}\right)_{j,n} & \left((\hat{u}_{2,n}^{+})_j^{(1)-}\right)_{j,n} & \left((\hat{u}_{2,n}^{+})_j^{(2)+}\right)_{j,n} & \left((\hat{u}_{2,n}^{+})_j^{(2)-}\right)_{j,n} \\ \left((\hat{u}_{1,n}^{-})_j^{(1)+}\right)_{j,n} & \left((\hat{u}_{1,n}^{-})_j^{(1)-}\right)_{j,n} & \left((\hat{u}_{1,n}^{-})_j^{(2)+}\right)_{j,n} & \left((\hat{u}_{1,n}^{-})_j^{(2)-}\right)_{j,n} \\ \left((\hat{u}_{2,n}^{-})_j^{(1)+}\right)_{j,n} & \left((\hat{u}_{2,n}^{-})_j^{(1)-}\right)_{j,n} & \left((\hat{u}_{2,n}^{-})_j^{(2)+}\right)_{j,n} & \left((\hat{u}_{2,n}^{-})_j^{(2)-}\right)_{j,n} \end{pmatrix}, \qquad (3.4.1)$$

where the indices j, n in each subblock belong both to $\mathbb{Z}_M^2$, and $\hat{u}_{(1,2),n}^{\pm}$ are the first two components of the Rayleigh coefficients of the scattered field in (3.1.1). The notation $(\,\cdot\,)_j^{(l)\pm}$ for $l = 1, 2$ indicates the dependence of these coefficients on the corresponding incident wave $\varphi_j^{(l)\pm}$.

Let $\boldsymbol{W}\boldsymbol{N}_M$ be the discretization of WN. The Hermitian matrix $\mathrm{Im}\,(\boldsymbol{W}\boldsymbol{N}_M)$ has an eigendecomposition $\mathrm{Im}\,(\boldsymbol{W}\boldsymbol{N}_M) = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$, where $\boldsymbol{D}$ is the diagonal matrix containing $4M^2$ eigenvalues $\boldsymbol{\lambda}_n$ of $\mathrm{Im}\,(\boldsymbol{W}\boldsymbol{N}_M)$ and $\boldsymbol{V}$ is an orthogonal matrix containing the eigenvectors

$(\boldsymbol{\varphi}_{j,n})_{j=1}^{4M^2}$. Then

$$(\mathrm{Im}\,(\boldsymbol{W}\boldsymbol{N}_M))^{1/2} = \boldsymbol{V}\,|\boldsymbol{D}|^{1/2}\,\boldsymbol{V}^{-1}. \tag{3.4.2}$$

Then the criterion (3.3.3) is numerically exploited for imaging by plotting the function

$$\mathbf{z} \mapsto P_M(\mathbf{z}) = \left[\sum_{n=1}^{4M^2} \frac{|A_n(\mathbf{z})|^2}{\boldsymbol{\lambda}_n}\right]^{-1}, \tag{3.4.3}$$

where $A_n(\mathbf{z}) = \sum_{j=1}^{4M^2} \boldsymbol{W}(\widehat{\Psi}_{\mathbf{z},j})\overline{\boldsymbol{\phi}_{j,n}}$. If the series in (3.4.3) approximates the true value of the exact Picard series in (3.3.3), then P_M should be very small outside of D and considerably larger inside D.

To consider noise in the scattering data we add a complex-valued noise matrix $\boldsymbol{X}$ containing random numbers whose real and imaginary parts are uniformly distributed on $(-1,1)$ to the data matrix $\boldsymbol{N}_M$. Denoting by δ the noise level, the noisy data matrix $(\boldsymbol{N}_M)_\delta$ is then given by

$$(\boldsymbol{N}_M)_\delta = \boldsymbol{N}_M + \delta\frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}\,\|\boldsymbol{N}_M\|\,,$$

where the matrix norm $\|\cdot\|$ is the Frobenius norm. For such noisy data, the eigenvalue decomposition in (3.4.2) has to be replaced by a singular value decomposition, that will not be detailed here. We truncate the singular values to regularize the Factorization method. For all of the examples below we only keep the singular values that are greater than or equal to 10^{-2} for the regularization.

We consider four numerical examples for which the periodic structures are motivated by two-dimensional photonic crystals. Here are the detailed information of the periodic structures we consider in this section.

a) We consider the structure of periodically aligned balls. The reconstruction result for this example is presented in Figure 3.1. Recall that $\overline{D} = [\mathrm{supp}(Q) \cup \mathrm{supp}(P) \cup \mathrm{supp}(\xi)] \cap \Omega$.

In this example D is given by

$$\begin{aligned} D = &\left\{(x_1, x_2, x_3)^\top : \left(x_1 - \frac{\pi}{2}\right)^2 + \left(x_2 - \frac{\pi}{2}\right)^2 + x_3^2 < 0.6^2\right\} \\ &\cup \left\{(x_1, x_2, x_3)^\top : \left(x_1 + \frac{\pi}{2}\right)^2 + \left(x_2 - \frac{\pi}{2}\right)^2 + x_3^2 < 0.6^2\right\} \\ &\cup \left\{(x_1, x_2, x_3)^\top : \left(x_1 - \frac{\pi}{2}\right)^2 + \left(x_2 + \frac{\pi}{2}\right)^2 + x_3^2 < 0.6^2\right\} \\ &\cup \left\{(x_1, x_2, x_3)^\top : \left(x_1 + \frac{\pi}{2}\right)^2 + \left(x_2 + \frac{\pi}{2}\right)^2 + x_3^2 < 0.6^2\right\}. \end{aligned}$$

b) We consider the structure of periodically aligned bars. The reconstruction result for this example is presented in Figure 3.2. In this example D is given by

$$\begin{aligned} D = &\left\{(x_1, x_2, x_3)^\top : x_1^2 + x_3^2 < \left(\frac{\pi}{6}\right)^2\right\} \\ &\cup \left\{(x_1, x_2, x_3)^\top : (x_1 - \pi)^2 + x_3^2 < \left(\frac{\pi}{6}\right)^2\right\} \\ &\cup \left\{(x_1, x_2, x_3)^\top : (x_1 + \pi)^2 + x_3^2 < \left(\frac{\pi}{6}\right)^2\right\}. \end{aligned}$$

c) We consider the structure of periodically aligned cubes. The reconstruction result for this example is presented in Figure 3.3. In this example D is given by

$$D = \left\{(x_1, x_2, x_3)^\top : |x_1| < \frac{\pi}{2},\ |x_2| < \frac{\pi}{2},\ |x_3| < 0.3\right\}.$$

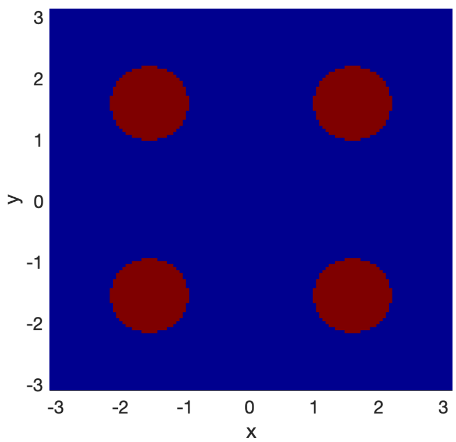
d) We consider a strip with periodically aligned holes. The reconstruction result for this example is presented in Figure 3.4. In this example D is given by

$$D = \left\{(x_1, x_2, x_3)^\top : x_1^2 + x_2^2 > \left(\frac{\pi}{2}\right)^2,\ |x_3| < 0.3\right\}.$$

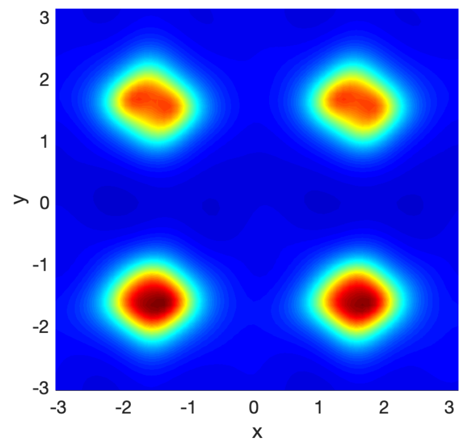
The numerical implementation is done using Matlab. In all of the examples in this section we use the following parameters.

$$\text{sampling domain} = (-\pi,\pi)^2 \times (-1,1), \quad k = \pi,$$
$$M = 20 \text{ (i.e. 1600 incident plane waves)},$$
$$\delta = 2\% \text{ (noise level)}, \quad \alpha = (\pi/2, \pi/2, 0).$$

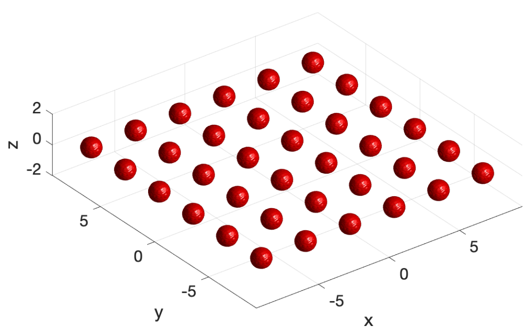
The sampling domain is probed by 32^3 sampling points. Recall that the data are measured at $\{x_3 = \pm 1\}$. The matrix-valued coefficients $\varepsilon_{\mathrm{r}}, \mu_{\mathrm{r}}^{-1}, \xi$ are given by (1.2.10)–(1.2.11) in all of the examples. The Rayleigh expansion in the radiation condition for $\mathbf{u}$ is truncated in $\mathbb{Z}_M^2$. Hence, for $M = 20$, we have 400 Rayleigh coefficients in each block of the data matrix (3.4.1). There are 32 coefficients corresponding to propagating modes in these 400 Rayleigh coefficients, and the rest corresponds to evanescent modes which are necessarily important for the quality of the reconstructions. The important role of evanescent modes in the numerical implementation has also been observed in previous works, see for example[28;32;33]. We can see in the Figures 3.1,3.2,3.3, and 3.4 that the Factorization method is able to provide reasonable reconstructions for different types of bi-anisotropic periodic structures.



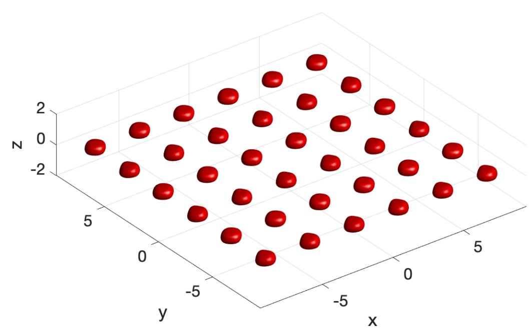
(a) Exact geometry viewed at $\Omega \cap \{z = 0\}$



(b) Reconstruction viewed at $\Omega \cap \{z = 0\}$

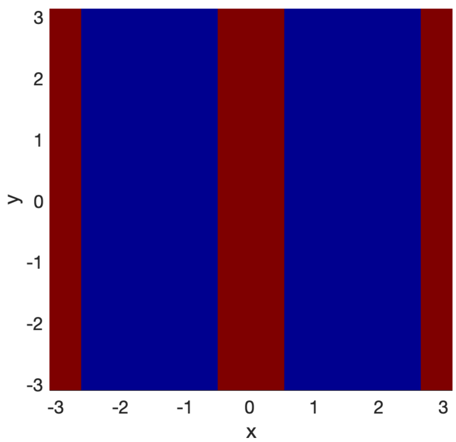


(c) Exact geometry in $(-3\pi, 3\pi)^2 \times (-2, 2)$

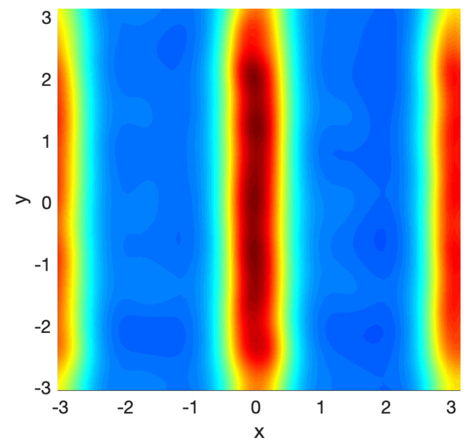


(d) Reconstruction

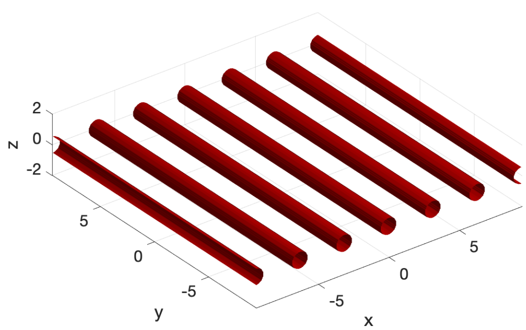
Figure 3.1: Shape reconstruction of periodically aligned balls. There is 2% artificial noise in the data. The isovalue for the isosurface plotting is chosen to be one third of the maximal value of the computed image.



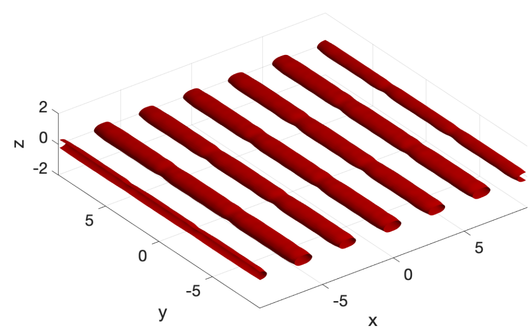
(a) Exact geometry viewed at $\Omega \cap \{z = 0\}$



(b) Reconstruction viewed at $\Omega \cap \{z = 0\}$

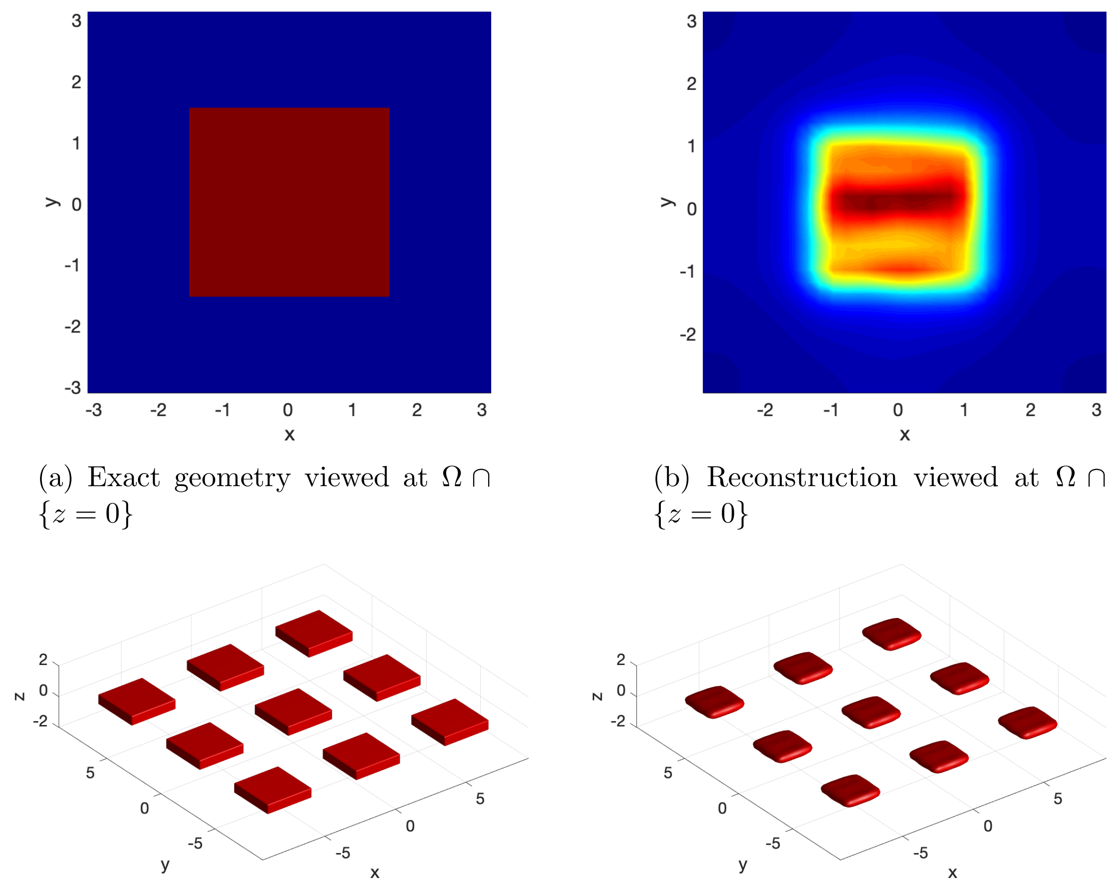


(c) Exact geometry in $(-3\pi, 3\pi)^2 \times (-2, 2)$



(d) Reconstruction

Figure 3.2: Shape reconstruction of periodically aligned bars. There is 2% artificial noise in the data. The isovalue for the isosurface plotting is chosen to be one third of the maximal value of the computed image.



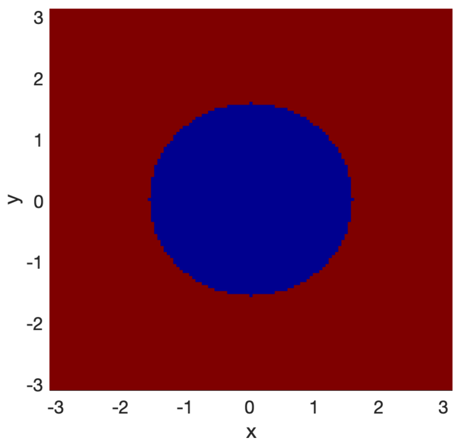
(a) Exact geometry viewed at $\Omega \cap \{z = 0\}$

(b) Reconstruction viewed at $\Omega \cap \{z = 0\}$

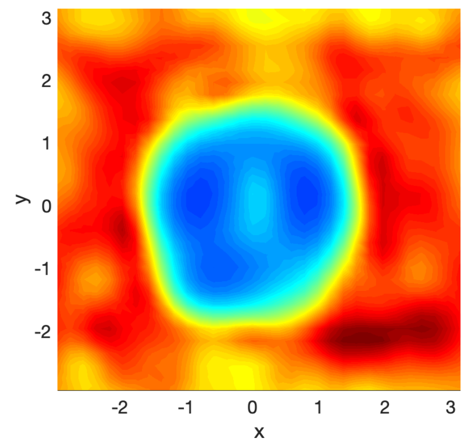
(c) Exact geometry in $(-3\pi, 3\pi)^2 \times (-2, 2)$

(d) Reconstruction

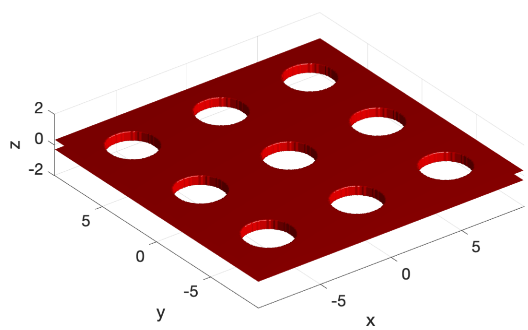
Figure 3.3: Shape reconstruction of periodically aligned cubes. There is 2% artificial noise in the data. The isovalue for the isosurface plotting is chosen to be one third of the maximal value of the computed image.



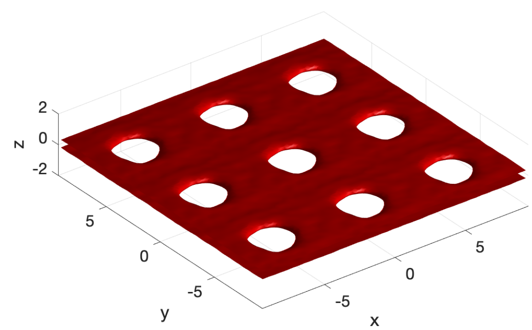
(a) Exact geometry viewed at $\Omega \cap \{z = 0\}$



(b) Reconstruction viewed at $\Omega \cap \{z = 0\}$



(c) Exact geometry in $(-3\pi, 3\pi)^2 \times (-2, 2)$



(d) Reconstruction

Figure 3.4: Shape reconstruction of a strip with periodically aligned holes. There is 2% artificial noise in the data. The isovalue for the isosurface plotting is chosen to be one third of the maximal value of the computed image.

Chapter 4

Conclusion

In conclusion, we have presented results for both direct and inverse scattering problems for bi-anisotropic periodic scattering media in the electromagnetic setting.

For the direct problem, we introduced a variational formulation of the problem and an equivalent integro-differential equation. We then showed that this equation satisfies a Garding-type estimate under some positivity assumptions on the real parts of permittivity, permeability and smallness assumptions on the magnetoelectric coupling coefficient. This leads to the fact that Fredholm alternative applies and we only needed to show uniqueness of solutions to establish well-posedness of the problem. Uniqueness was proved under some extra assumptions on positivity of the imaginary parts of the permittivity and permeability, as well as smallness of the magnetoelectric coupling coefficient. In order to find a good numerical solution, we periodized the integro-differential equation and looked for the Fourier coefficients of the unknown with respect to a Fourier-like basis. This technique helps avoid direct calculation of the α-quasiperiodic Green's function which has singularities and utilize fast Fourier Transform to reduce computational time. Convergence of the method can be obtained without any additional smoothness assumptions on the true solution, but we also showed that the more regular the true solution is the faster the convergence. Numerical results for three different bi-anisotropic periodic media further confirm convergence of the method.

In the future, we wish to show well-posedness of the problem with more relaxed conditions. For example, we would like to study the case where the physical parameters of the object are real-valued or have not too large imaginary parts. We can also attempt to extend the technique to more difficult cases, such as scattering problems for materials in which the constitutive relations are more complex.

Regarding the inverse problem, we reconstructed the geometry of the scattering object from near-field measurement of the scattered electric fields. The factorization method was the tool of interest to solve this problem. We first proved a characterization for the geometry of the scattering medium which involves the range of H^* (where H is the Herglotz operator). However, the range of H^* is still unknown since it depends on the object. We then showed a factorization of the near-field operator in terms H, H^* and a middle operator T. After that, we were able to connect the range of H^* with the range of a known operator. The key ingredient is coercivity of $\operatorname{Im} T$ which we fully justified. Finally, a unique determination of the geometry of the medium was established and we were able to solve the problem. Numerical results showed that the factorization method performs well for different bi-anisotropic periodic scattering media.

In order to prove that $\operatorname{Im} T$ is coercive, we once again needed to use the positivity and smallness assumptions similar to the direct problem. Therefore, future work may involve attempts to relax these conditions. Moreover, we are interested in studying and developing other shape reconstruction methods for bi-anisotropic periodic media that can handle scattering data with high levels of noise, or with flexible types of incident fields.

Bibliography

[1] D. L. Jaggard, N. Engheta, M. W. Kowarz, P. Pelet, J. C. Liu, and Y. Kim. Periodic chiral structures. *IEEE Trans. Antennas Propagat.*, 37:1447–1452, 1989.

[2] C. E. Kriegler, M. S. Rill, S. Linden, and M. Wegener. Bianisotropic photonic metamaterials. *IEEE J. Sel. Topics Quantum Electron.*, 16:367–375, 2010.

[3] K. M. Flood and D. L. Jaggard. Band-gap structure for periodic chiral media. *J. Opt. Soc. Am. A*, 13:1395, 1996.

[4] J. Pendry. A Chiral Route to Negative Refraction. *Science*, 19:1353–1355, 2004.

[5] C. Athanasiadis, G. Costakis, and I. G. Stratis. Electromagnetic scattering by a homogeneous chiral obstacle in a chiral environment. *SIAM J. Appl. Math.*, 64:245–258, 2000.

[6] H. Ammari and G. Bao. Maxwell's equations in periodic chiral structures. *Math. Nachr.*, 251:3–18, 2003.

[7] R. Potthast and I. G. Stratis. On the domain derivative for scattering by impenetrable obstacles in chiral media. *IMA J. Appl. Math.*, 68:621–635, 2003.

[8] P. Ciarlet Jr. and G. Legendre. Well-posedness of the Drude–Born–Fedorov model for chiral media. *Math. Models Methods Appl. Sci.*, 17:461, 2007.

[9] P.-H. Cocquet, P.-A. Mazet, and V. Mouysset. On well-posedness of some homogenized Drude–Born–Fedorov systems on a bounded domain and applications to metamaterials. *C. R. Math.*, 349:99–103, 2011.

[10] G. F. Roach, I. G. Stratis, and A. N. Yannacopoulos. *Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics*. Princeton Series in Applied Mathematics. Princeton University Press, 2012.

[11] S. Nicaise. Time domain study of the Drude–Born–Fedorov model for a class of heterogeneous chiral materials. *Math. Meth. Appl. Sci.*, 36:794–813, 2013.

[12] H. Ammari and G. Bao. Coupling of finite element and boundary element methods for the electromagnetic diffraction problem by a periodic chiral structure. *J. Comput. Math.*, 26:261–283, 2008.

[13] D.-L. Nguyen, T.-N. Nguyen, and M.-P. Tran. A Galerkin approximation for integro-differential equations in electromagnetic scattering from a chiral medium. *Appl. Anal.*, 96:159–172, 2017.

[14] D.-L. Nguyen. Direct and inverse electromagnetic scattering problems for bi-anisotropic media. *Inverse Problems*, 35:124001, 2019.

[15] D.-L Nguyen. A volume integral equation method for periodic scattering problems for anisotropic maxwell's equations. *Applied Numerical Mathematics*, 98:59–78, 2015. ISSN 0168-9274. doi: https://doi.org/10.1016/j.apnum.2015.08.005. URL https://www.sciencedirect.com/science/article/pii/S0168927415001178.

[16] S. R. McDowall. Boundary determination of material parameters from electromagnetic boundary information. *Inverse Problems*, 13:153–163, 1997.

[17] S. R. McDowall. An electromagnetic inverse problem in chiral media. *T. Am. Math. Soc.*, 352:2993–3013, 2000.

[18] S. Li. An inverse problem for Maxwell's equations in bi-isotropic media. *SIAM J. Math. Anal.*, 37:1027–1043, 2005.

[19] C. Athanasiadis and E. Kardasi. Inverse electromagnetic scattering by a perfect conductor in a chiral environment. *J. Inv. Ill-Posed Problems*, 16:1–18, 2008.

[20] S. Heumann. *The Factorization Method for Inverse Scattering from Chiral Media*. PhD thesis, Karlsruher Institut für Technologie, 2012.

[21] D.-L. Nguyen. The Factorization method for the Drude-Born-Fedorov model for periodic chiral structures. *Inverse Probl. Imaging*, 10:519–547, 2016.

[22] D.-L. Nguyen and T. Truong. Imaging of bi-anisotropic periodic structures from electromagnetic near field data. *J. Inverse Ill-Posed Probl.*, 30:205–219, 2022.

[23] D.-L. Nguyen and T. Truong. Fast numerical solutions to direct and inverse scattering for bi-anisotropic periodic maxwell's equations,. *AMS Contemporary Mathematics*, 2022.

[24] T. G. Mackay and A. Lakhtakia. *Electromagnetic Anisotropy and Bi-Anisotropy: A Field Guide*. World Scientific, Singapore, 2010.

[25] Tilo Arens. Scattering by biperiodic layered media: The integral equation approach. Habilitation Thesis, Universität Karlsruhe, 2010.

[26] A. Kirsch. An integral equation approach and the interior transmission problem for Maxwell's equations. *Inverse Probl. Imaging*, 1:159–180, 2007.

[27] S. Sauter and C. Schwab. *Boundary Element Methods*. Springer, 2007. ISBN 978-3540680925.

[28] A. Lechleiter and D.-L. Nguyen. Factorization method for electromagnetic inverse scattering from biperiodic structures. *SIAM J. Imaging Sci.*, 6:1111–1139, 2013.

[29] A. Kirsch. Uniqueness theorems in inverse scattering theory for periodic structures. *Inverse Problems*, 10:145–152, 1994.

[30] G. Schmidt. On the diffraction by biperiodic anisotropic structures. *Appl. Anal.*, 82: 75–92, 2003.

[31] A. Kirsch and N.I. Grinberg. *The Factorization Method for Inverse Problems*. Oxford Lecture Series in Mathematics and its Applications 36. Oxford University Press, 2008.

[32] T. Arens and N.I. Grinberg. A complete factorization method for scattering by periodic structures. *Computing*, 75:111–132, 2005.

[33] X. Jiang and P. Li. Inverse electromagnetic diffraction by biperiodic dielectric gratings. *Inverse Problems*, 33:085004, 2017.