RENORMALIZATIONS OF THE KONTSEVICH INTEGRAL AND THEIR BEHAVIOR UNDER BAND SUM MOVES

by

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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DOCTOR OF PHILOSOPHY

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Abstract

We generalize the definition of the framed Kontsevich integral initially presented in [LM1]. We study the behavior of the renormalized framed Kontsevich integral $\hat{Z}_f$ under band sum moves and show that it can be further renormalized into some invariant $\widetilde{Z}_f$ that is well-behaved under moves for which link components of interest are locally put on top of each other. Originally, Le, Murakami and Ohtsuki ([LM5], [LM6]) showed that another choice of normalization is better suited for moves for which link components involved in the band sum move are put side by side. We show the choice of renormalization leads to essentially the same invariant and that the use of one renormalization or the other is just a matter of preference depending on whether one decides to have a horizontal or a vertical band sum. Much of the work on $\widetilde{Z}_f$ relies on using the tangle chord diagrams version of $\hat{Z}_f$ ([ChDu]). This leads us to introducing a matrix representation of tangle chord diagrams, where each chord is represented by a matrix, and tangle chord diagrams of degree m are represented by stacks of m matrices, one for each chord making up the diagram. We show matrix congruences for some appropriately chosen matrices implement on the modified Kontsevich integral $\widetilde{Z}_f$ the band sum move on links. We show how $\widetilde{Z}_f$ in matrix notation behaves under the Reidemeister moves and under orientation changes. We show that for a link L in plat position, $Z_f(L)$ in book notation is enough to recover its expression in terms of chord diagrams. We elucidate the relation between $\check{Z}_f$ and $\widetilde{Z}_f$ and show the quotienting procedure to produce 3-manifold invariants from those as introduced in [LM5] is blind to the choice of normalization, and thus any choice of normalization leads to a 3-manifold invariant.

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Chapter 1

Introduction

This thesis has for aim to study 3-manifold invariants built from the framed Kontsevich integral Z_f ([LM1]). The unframed Kontsevich integral Z, first introduced in [K], is a functional on knots that becomes an isotopy invariant of knots once it is corrected in an appropriate manner. The normalized framed Kontsevich integral defined by Le, Murakami and Ohtsuki which they denote $\hat{Z}_f$ [LM1] is also an isotopy invariant and will be the starting point of our study. In [LM5], a 3-manifold invariant Ω is constructed by further renormalizing $\hat{Z}_f$ into some other invariant $\check{Z}_f$. One property that Ω must satisfy for it to be a true 3-manifold invariant is that it be invariant under band sum moves. Band sum moves between link components correspond to handle sliding on their respective 2 handles if a framed link is regarded as attaching data for 2-handles [GS]. Band sum moves on tangle chord diagrams ([ChDu]) however (tangles with chords ending on them) are not well-defined. One has to specify a window in which link components involved in the band sum are locally frozen. In [LM5], the invariance of Ω under band sum moves between link components locally put side by side relies on $\check{Z}_f$ behaving a certain way under such moves. In this paper, we start with $\hat{Z}_f$ and consider what normalization makes it well-behaved under band sum moves where link components of interest are locally on top of each other. We then show the equivalence between the normalization we get and that obtained in [LM5]. That two renormalizations come into play is not surprising since as Kontsevich initially pointed out ([K]) the Kontsevich integral depends very much on the choice of a time axis, and thus fixing two windows where link components are rotated will lead to different results locally.

One fundamental remark is the following: from [LM1] we only use the normalization of the Kontsevich integral at local extrema which we generalize by using an arbitrary parameter M. Apart from enabling one to identify whether a given tangle T has a local extremum by looking at the coefficients of $Z_f(T)$, this will allows us to locate those once $Z_f(T)$ is expressed in a "book notation" that we introduce in the present work. In [LM1] Le and Murakami defined a truncated invariant Z obtained from a limiting procedure on the original Kontsevich integral Z (same notation) on braids that enables one to compute the values of that invariant on q-tangles (or non-associative tangles). In this paper we work with geometric tangles only, that is tangles whose strands are separated from one another by a measurable distance. By the universality of $\hat{Z}_f$, the renormalized Kontsevich integral, the construction performed in [LM1] and the one done in the present work yield the same results. Whenever we present a result for geometric tangles initially proved by Le, Murakami and Ohtsuki for their invariant Z_f (or $\hat{Z}_f$, or even $\check{Z}_f$) using q-tangles, then we will mention the corresponding reference.

We then study the quotienting map ([LM5]) necessary for producing 3-manifold invariants from the Kontsevich integral and show that any normalization survives the quotienting process, and thus any choice of normalization yields a 3-manifold invariant. In particular this lifts the redundancy of having two normalizations $\check{Z}_f$ and $\widetilde{Z}_f$ describing the same things.

We formalize all these statements. We briefly recall the definition of knots and links, handle decompositions, the 4 dimensional 2-handlebodies determined by links, and that the diffeomorphism types of boundaries of such spaces are invariant under two operations, one of which is handle slide. Finally we will define the Kontsevich integral, its correction, and show what its expression is in the framed case. We show how it behaves under band sum moves.

1.1 Links and handlebodies

We define a framed e-component link L in S^3 [RK1] to be a smooth embedding $\gamma : \coprod_{1\leq i\leq e} S^1 \to S^3$, for e finite, along with integers $n_1, ..., n_e$, one for each embedding $\gamma_i := \gamma|_{S^1}$ of the i-th S^1 in $\coprod_{1\leq i\leq e} S^1$. Those integers define the framing for each embedded circle, and are defined as follows. We first consider the case of knots. A knot is obtained for $e = 1$. A knot K in S^3 bounds an embedded compact surface called a Seifert surface [S]. The zero framing of the normal bundle of K is obtained from the outward normal vector to the orientable Seifert surface F of K. A non-zero framing given by some positive integer n is obtained by twisting the zero framing n times in a clockwise direction. This corresponds to associating the integer n to the embedding γ defining the knot K. Doing this for all embeddings of circles γ_i, $1 \leq i \leq e$ for the link L one obtains the integer numbers $n_1, \cdots, n_e$ respectively. Links will be oriented unless specified otherwise.

Now a link L in S^3 determines a 4-manifold M_L obtained by adding 2-handles to the 4-ball D^4 along the circles in S^3 defining L, with the gluing performed using the framing along the circles. Before constructing M_L, one therefore needs to introduce the concept of handles and that of handle decompositions. We will mainly follow [GS].

Definition 1.1.1. Let X be an n-dimensional manifold. For $0 \leq k \leq n$, $D^k \times D^{n-k}$ is called an n-dimensional k-handle once it is attached to the boundary ∂X of X along $\partial D^k \times D^{n-k}$ via an embedding $\varphi : \partial D^k \times D^{n-k} \to \partial X$, called the attaching map.

If h is the handle thus specified, one writes $X \cup_\varphi h$ to denote the resulting n-dimensional manifold ([GS]). One can smooth corners during the attaching map in such a manner that one views $X \cup_\varphi h$ as a smooth n-dimensional manifold. In terms of M_L resulting from the gluing of a 2-handle along L using the framing on each circle, note that we have a deformation of $X \cup_\varphi h$ onto $X \cup_{\varphi|\partial D^k \times 0} D^k \times 0$. By the tubular neighborhood theorem, one can reconstruct $X \cup_\varphi h$ from the restriction of the attaching map φ to $D^k \times 0$ along with a normal framing of $\varphi|_{D^k\times 0}(\partial D^k)$. In particular, to construct $X \cup_\varphi h$, it suffices to have $\varphi_c : \partial D^k = S^{k-1} \to \partial X$ with a local trivialization of the normal bundle, $\varphi_c = \varphi|_{D^k \times 0}$ where the subscript c stands for core, $D^k \times 0$ is called the core of the k-handle, and one also needs a normal framing f of $\varphi_c(S^{k-1})$. This is giving a knot in ∂X along with a framing of that knot. One can generalize this to links, and apply it to the case $n = 4$, $k = 2$, $X = D^4$. The resulting space M_L is called a handle body since it is obtained from D^4 by attaching handles to it.

We have the following theorem:

Theorem 1.1.2 ([RK1])**.** $\partial M_L = \partial M_{L'}$ (orientation preserving diffeomorphism) if and only if one can pass from L to L' following a sequence of two operations, one being the blow up or down of a circle of framing ± 1, the other being the band sum move between two link components.

These two moves are commonly referred to as Kirby I and Kirby II moves respectively. Kirby II corresponds in M_L to doing a handle slide of one 4-dimensional 2-handle over another. There are 3-dimensional (resp. 4-dimensional) formalisms called 3-d Kirby Calculus (resp. 4-d Kirby calculus) [RK1] [RK2] [GS] meant to facilitate computations on handlebodies in 3 and 4 dimensions respectively. In both formalisms there are band sum moves between link components representing handle slides on handles, whence our interest in studying the behavior of the Kontsevich integral under such moves.

In what follows, we consider the second operation, the band sum move. It is defined as follows [RK1]: consider two link components K_i and K_j of L. Pick one knot, say K_j, push it off itself using its framing ϕ_j and in such a manner that it misses L. In this manner one obtains $\tilde{K}_j$, in addition to having K_j. Then one connect sums K_i with $\tilde{K}_j$ with the use of a band connecting K_i to $\tilde{K}_j$, and the connected sum is denoted:

$$K_i \underset{b}{\#} \tilde{K}_j := K_i' \tag{1.1}$$

where the b underneath the pound sign indicates that one has performed a band sum ([RK1]). One insists that the band b misses the rest of L. Formally what that means is that if γ_i and γ_j are two embeddings defining the two knots K_i and K_j in S^3 respectively, then if $b : I \times I \to S^3$ is an embedding such that $b(I \times I) \cap \gamma_i = b(\{0\} \times I)$ and $b(I \times I) \cap \gamma_j = b(\{1\} \times I)$, then one formally writes:

$$\gamma_i \underset{b}{\#} \gamma_j = \gamma_i \cup \gamma_j - b(\partial I \times I) \cup b(I \times \partial I) \tag{1.2}$$

Observe [RK2] that the band is allowed to have any number of half twists in it, either left or right twists, and since the circles are oriented this will lead to a notion of subtracting or adding a circle to another, respectively. In the terminology of handle bodies, for an n-dimensional k-handle, $\partial D^k \times 0$ is called the attaching region. In our situation one has $n = 4$, $k = 2$, and the 2-handle has $S^1 \times 0$ as the attaching sphere, which is nothing but a knot component of the link under consideration. The band sum of one knot over another corresponds to a certain move between 2-handles called a handle slide. Following [GS], for $0 \leq k \leq n$, given two n-dimensional k-handles h_1 and h_2 attached to some n-dimensional manifold X along its boundary ∂X as done above, one can isotope the attaching sphere $S^{k-1} \times 0$ of h_1 in $\partial(X \cup h_2)$ and slide it along a disk $D^k \times \{pt\} \subset h_2$ until it comes back down to ∂X. Observe that $\partial(D^k \times \{pt\}) = S^{k-1} \times \{pt\}$ is the attaching sphere of h_2 pushed away from itself using the framing on that link component.

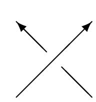
1.2 Isotopy invariants and Vassiliev invariants

One can define an equivalence relation on the category of knots called ambient isotopy [GS], denoted by $\sim$. An isotopy between embeddings $\gamma_1, \gamma_2 : X \to Y$ is a homotopy $\gamma : X \times I \to Y$

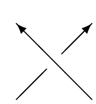
through embeddings. An ambient isotopy between two knots $\gamma_0 : S^1 \to S^3$ and $\gamma_1 : S^1 \to S^3$ is an isotopy $\gamma : S^1 \times I \to S^3$ through diffeomorphisms $\Gamma : S^3 \times I \to S^3$ such that $\Gamma_0 = id_{S^3}$ and $\gamma_t = \Gamma_t \circ \gamma_0$ for each t. The above definition for isotopy carries over to the case of links.

In a first time, one is interested in isotopy classes of knots since original definitions for the concepts that we introduce presently where made in the case of knots. To distinguish one class from another, one needs a function on knots that takes different values on different classes, but which must of course be constant on equivalence classes, and such an object is rightfully called an isotopy invariant or invariant for short. Typically an invariant is valued in some abelian group G, and if we denote by $\mathbb{Z}Knots$ the abelian group generated by oriented knots, we write $\Gamma : \mathbb{Z}Knots/\sim \to G$ for an invariant Γ, it is a G-valued functional on the set of equivalence classes of knots. One can generalize this definition to the case of links: if $\mathbb{Z}Links/\sim$ is the set of equivalence classes of oriented links, G is an abelian group, then a link invariant Γ will be a map $\Gamma : \mathbb{Z}Links/\sim \to G$.

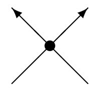
Of those invariants, Vassiliev invariants [V] are of particular interest. Such invariants are functionals on $\mathbb{Z}Knots/\sim$ that are extended to be defined on singular knots whose singularities are transversal self-intersections. One first defines a positive crossing in the image of a knot to be:



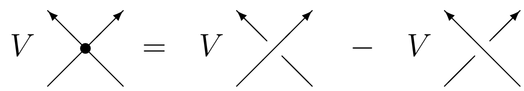
where the arrows indicate the orientation on the portion of the knot that is being displayed. A negative crossing is represented as:



One represents a transversal self-intersection as follows:

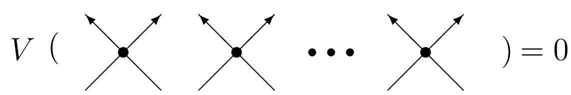


At this point one can extend invariants of knots to also be defined on singular knots, with the use of the relation:



(1.3)

where V is any knot invariant. By iterating this procedure, one can extend a knot invariant to be defined on knots with multiple double-crossings. If a knot invariant V has its extension vanishing on knots that have more than m self-intersections, one says that V is a Vassiliev invariant of type m:



where the argument of V above has more than m self-intersections. Due to the local nature of the extension of knot invariants to the case of singular knots, we can generalize the definition of Vassiliev invariant to the case of links. A Vassiliev invariant of links will be said to be of type m if it evaluates to zero on any singular link with more than m double-crossings (not necessarily same component intersections).

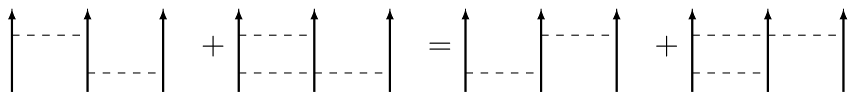
Among the Vassiliev invariants of type m, one is of particular interest: the degree m part of the Kontsevich integral first introduced in [K]. One should note that the Kontsevich integral as it was initially defined is not strictly speaking a knot invariant, but once it is corrected as done below, then it becomes an honest knot invariant, and its final corrected form is a universal Vassiliev invariant in the sense that every finite type Vassiliev invariant factors through it [DBN2]. Before introducing this integral, one defines the algebra $\mathcal{A}$ [K] in which it takes its values.

1.3 The algebra $\mathcal{A}$ of chord diagrams

For a singular oriented knot whose only singularities are transversal self-intersections, the preimage of each singular crossing under the embedding map defining the knot yields two distinct points on S^1. Each singular point in the image therefore yields a pair of points on S^1 that are conventionally connected by a chord for book keeping purposes. A knot with m singular points will yield m distinct chords on S^1. One refers to such a circle with m chords on it as a chord diagram of degree m, the degree being the number of chords. The support of the graph is an oriented S^1, and it is regarded up to orientation preserving diffeomorphisms of the circle. More generally, for a singular oriented link all of whose singularities are double-crossings, the preimage of each singular crossing under the embedding map defining the link yields a pair of distinct points on possibly different circles depending on whether the double crossing was on a same component or between different components of the link. One also connects points forming a pair by a chord. An e-component link with m singular points will yield m chords on $\coprod^e S^1$. One still calls such a graph a chord diagram. The support now is $\coprod^e S^1$ regarded up to orientation preserving diffeomorphism of each S^1. One denotes by $\mathcal{D}(\amalg^e S^1)$ the $\mathbb{C}$-vector space spanned by chord diagrams with support on $\amalg^e S^1$. One writes $\mathcal{D}$ for $\mathcal{D}(S^1)$. There is a grading on $\mathcal{D}(\amalg^e S^1)$ given by the number of chords featured in a diagram. If $\mathcal{D}^{(m)}(\amalg^e S^1)$ denotes the subspace of chord diagrams of degree m, then one writes:

$$\mathcal{D}(\amalg^e S^1) = \oplus_{m \geq 0} \mathcal{D}^{(m)}(\amalg^e S^1) \tag{1.4}$$

One quotients this space by the 4-T relation which locally looks like:



where solid lines are intervals on $\amalg^e S^1$ on which a chord foot rests, and arrows indicate the orientation of each strand. One further quotients this space by the framing independence relation: if a chord diagram has a chord forming an arc on S^1 with no other chord ending in between its feet, then the chord diagram is set to zero. The resulting quotient space is the $\mathbb{C}$-vector space generated by chord diagrams mod the 4-T relation and framing independence and is denoted by $\mathcal{A}(\amalg^e S^1)$. One writes $\mathcal{A}$ for $\mathcal{A}(S^1)$ ([BN]). The grading of $\mathcal{D}(\amalg^e S^1)$ is preserved by the quotient, inducing a grading on $\mathcal{A}(\amalg^e S^1)$:

$$\mathcal{A}(\amalg^e S^1) = \oplus_{m \geq 0} \mathcal{A}^{(m)}(\amalg^e S^1) \tag{1.5}$$

where $\mathcal{A}^{(m)}(\amalg^e S^1)$ is obtained from $\mathcal{D}^{(m)}(\amalg^e S^1)$ by modding out by the 4-T and the framing independence relations. The connected sum of circles can be extended to chorded circles, thereby defining a product on $\mathcal{A}$ that one denotes by $\cdot$ ([BN]), making it into an algebra that is associative and commutative [BN]. More generally $\mathcal{A}(\amalg^e S^1)$ is a module over $\otimes^e \mathcal{A}$. The Kontsevich integral will be valued in the graded completion $\overline{\mathcal{A}}(\amalg^e S^1)$ of the algebra $\mathcal{A}(\amalg^e S^1)$.

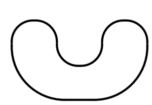
1.4 The original definition of the Kontsevich integral

As far as knots are concerned, we will work with Morse knots and geometric tangles, not q-tangles ([LM1], [LM2], [LM5], [LM6]), and for that purpose one considers the following decomposition of $\mathbb{R}^3$ as the product of the complex plane and the real line: $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \simeq \mathbb{C} \times \mathbb{R}$, with local coordinates z in the complex plane and t on the real line for time. A Morse knot K is such that $t \circ K$ is a Morse function on S^1. If one denotes by Z the Kontsevich integral functional on knots, if K is a Morse knot, one defines ([K], [BN], [CL2]):

$$Z(K) := \sum_{m \geq 0} \frac{1}{(2\pi i)^m} \int_{t_{min} < t_1 < ... < t_m < t_{max}} \sum_{P \; applicable} (-1)^{\varepsilon(P)} D_P \prod_{1 \leq i \leq m} \text{dlog} \, \triangle z(t_i)[P_i] \tag{1.6}$$

where t_{min} and t_{max} are the min and max values of t on K respectively, P is an m-vector each entry of which corresponds to a pair of points on the image of the knot K. We write $P = (P_1, ..., P_m)$, where the i-th entry P_i corresponds to a pair of points on the knot, and if we further situate these paired points at some height t_i, we can denote these two points by z_i and z_i', so that we can write $\triangle z(t_i)[P_i] := z_i - z_i'$. One refers to such P's as pairings. We denote by K_P the knot K with m pairs of points placed on it following the prescription given by P, and then connecting points at a same height by a chord. A pairing is said to be applicable if each entry corresponds to a pair of two distinct points on the knot, at the same height [BN]. For a pairing $P = (P_1, \cdots, P_m)$ giving the position of m pairs of points on K, one denotes by $\varepsilon(P)$ the number of those points ending on portions of K that are locally oriented down. For example if $P = (z(t), z'(t))$ and K is locally oriented down at $z(t)$, then $z(t)$ will contribute 1 to $\varepsilon(P)$. We also define the length of $P = (P_1, \cdots, P_m)$ to be $|P| = m$. If we denote by ι_K the embedding defining the knot then D_P is defined to be the chord diagram one obtains by taking the inverse image of K_P under ι_K: $D_P = \iota_K^{-1} K_P$. This generalizes immediately to the case of Morse links, and in this case the geometric coefficient will not be an element of $\overline{\mathcal{A}}$ but will be an element of $\overline{\mathcal{A}}(\coprod_e S^1)$ if the argument of Z is an e-component link.

Now if one wants to make this integral into a true knot invariant, then one corrects it as follows. Consider the embedding in S^3 of the trivial knot as:

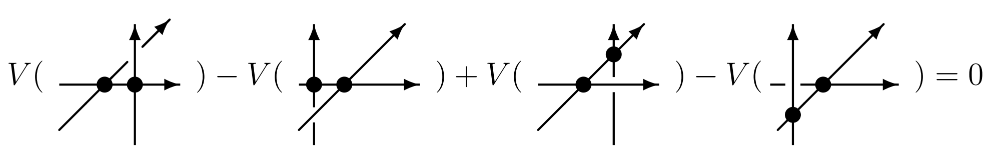
$U =$  (1.7)

Consider the following correction [K]:

$$\hat{Z} := Z(\cup\!\!\cup)^{-m}.Z \tag{1.8}$$

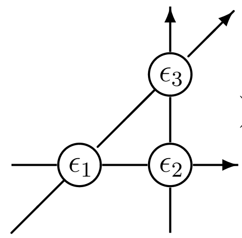
where the dot is the product on chord diagrams extended by linearity, and m is a function that captures the number of maximal points of any knot K that is used as an argument of Z. Defining $\nu := Z(U)^{-1}$, this reads $\hat{Z} = \nu^m \cdot Z$. Equivalently, we can define $\hat{Z}$ as being Z with the provision that ν multiplies each maximal point of a given knot K in the expression for $Z(K)$. As pointed out in [W], there are two possible corrections to the Kontsevich integral: the one we just presented, and the other one obtained by using $1-m$ as an exponent of the Kontsevich integral of the hump instead of just $-m$. In this manner the corrected version is multiplicative under connected sum, while using the above correction it behaves better under cabling operations. We will argue later that a modified version of $\hat{Z}$ using $1-m$ as an exponent of ν is a most convenient renormalization choice for band sum moves purposes and the construction of topological invariants of 3-manifolds. In the case of links, there will be one such correction for each component of the link, with a power m_i on the correction term for the i-th component, where m_i is the number of maximal points of the i-th link component. Equivalently, $\hat{Z}(L)$ is the same as $Z(L)$ save that every i-th link component in the expression for $Z(L)$ is multiplied by ν^{m_i}, $1 \leq i \leq e$.

Observe that it is not enough to have the Kontsevich integral being valued in $\overline{\mathcal{D}}$/fr.ind.. Indeed, by [DBN3], [ChDu], a Vassiliev invariant V satisfies the 4 terms relation:



Using (1.3) twice on each term, we can express this relation as a linear combination:

$$\sum_{\substack{1\leq i\leq 3\\ \epsilon_i=\pm}} \pm V\Big($$



$$\Big) = 0 \tag{1.9}$$

with $\epsilon_i = 1$ (resp. -1), $1 \leq i \leq 3$ for a positive crossing (resp. a negative crossing). In this relation all terms cancel except a collection of differences, each difference being an expression of the change of V under the third Reidemeister move. $\hat{Z}$ being a Vassiliev invariant, it also satisfies the 4 terms relation. Now, for a chord diagram D, contracting all chords of D one gets a knot K_D to use the notations of [BN]. Then $\hat{Z}(K_D) = D +$ higher order terms [BN], [DBN2], [ChDu] using the fact that $\hat{Z}$ is a Vassiliev invariant. This enables us to rewrite the 4 terms relation for $\hat{Z}$ as a relation involving only chord diagrams. This relation is exactly the 4-T relation. In other terms, the 4-T relation must be imposed for $\hat{Z}$ to be invariant under the third Reidemeister move.

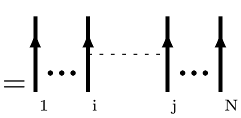
1.5 The Kontsevich Integral of tangles

One considers the generalization of the Kontsevich integral from knots to tangles as discussed in [BN], [LM1], [ChDu].

For this purpose, one considers a slightly more general algebra of chord diagrams as defined in [LM5]: For X a compact oriented 1-dimensional manifold with labeled components, a chord diagram with support on X is the manifold X together with a collection of chords with feet on X. One represents such chord diagrams by drawing the support X as solid lines, the graph consisting of dashed chords. One introduces an equivalence relation on the set of all chord diagrams: two chord diagrams D and D' with support on X are equivalent if there is a homeomorphism $f : D \to D'$ such that the restriction $f|_X$ of f to X is a homeomorphism of X that preserves components and orientation. One denotes by $\mathcal{A}(X)$ the complex vector space spanned by chord diagrams with support on X modulo the 4-T and framing independence relations. $\mathcal{A}(X)$ is still graded by the number of chords as:

$$\mathcal{A}(X) = \oplus_{m \geq 0} \mathcal{A}^{(m)}(X) \tag{1.10}$$

where $\mathcal{A}^{(m)}(X)$ is the complex vector space spanned by chord diagrams of degree m. Write $\overline{\mathcal{A}}(X)$ for the graded completion of $\mathcal{A}(X)$. We define a product on $\mathcal{A}(X)$ case by case. For example, if $X = \amalg^{e>1} S^1$, there is no well defined product defined on $\mathcal{A}(X)$. If $X = I^N$, the concatenation induces a well defined product on $\mathcal{A}(I^N)$. The product of two chord diagrams D_1 and D_2 in this case is defined by putting D_1 on top of D_2 and is denoted by $D_1 \times D_2$. More generally, for $D_i \in \mathcal{A}(X_i)$, $i = 1, 2$, $D_1 \times D_2$ is well-defined if X_1 and X_2 can be glued strand-wise. One chord diagram of degree 1 we will use repeatedly is the following:

$\Omega_{ij} =$  (1.11)

For T a tangle, one defines $Z(T) \in \overline{\mathcal{A}}(T)$ by:

$$Z(T) := \sum_{m \geq 0} \frac{1}{(2\pi i)^m} \int_{t_{min} < t_1 < \ldots < t_m < t_{max}} \sum_{P \ applicable} (-1)^{\varepsilon(P)} T_P \prod_{1 \leq i \leq m} \mathrm{dlog} \, \triangle z(t_i)[P_i] \tag{1.12}$$

exactly as $Z(K)$ was defined in (1.6) with the difference that T_P is the tangle T with m chords placed on it following the prescription given by P. Following [ChDu], one refers to T_P as a tangle chord diagram ([ChDu]). One defines the Kontsevich integral of a tangle T to be trivial if $Z(T) = T$. When working with links, we will sometimes omit the orientation on link components for convenience unless it is necessary to specify them. We will sometimes need the map S ([LM5]) on chord diagrams: suppose C is a component of X. If we reverse the orientation of C, one gets another oriented manifold from X that we will denote by X'

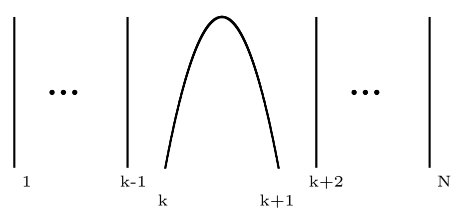
([LM5]). This induces a linear map:

$$S_{(C)} : \mathcal{A}(X) \to \mathcal{A}(X') \tag{1.13}$$

defined by associating to any chord diagram D in $\mathcal{A}(X)$ the element $S_{(C)}(D)$ obtained from D by reversing the orientation of C and multiplying the resulting chord diagram by $(-1)^m$ where m is the number of vertices of D ending on the component C. Suppose $Z(T)$ is known for some oriented tangle T, and T' is an oriented tangle with the same skeleton as T's, but with possible reversed orientations on some of its components. Then one can find $Z(T')$ by symply applying $S_C(Z(T))$ iteratively as many times as there are components C of T' that have an orientation different from that of T.

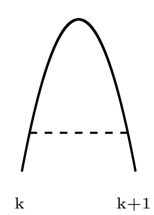
1.6 Integral of framed oriented links

In the framed case, in which the framing independence relation is no longer imposed, upon computing the Kontsevich integral $Z(T)$ of a tangle T with local extrema, one runs into computational problems. For instance, the Kontsevich integral of the following tangle:



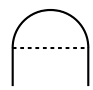
(1.14)

has possible integrands $\mathrm{dlog}(z_k - z_{k+1})$ in degree 1 corresponding to the chord diagram:



(1.15)

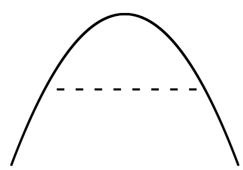
where z_k and z_{k+1} are local coordinates on the strands indexed by k and $k+1$ respectively, and such integrands are made to vanish by virtue of the framing independence in the unframed case:



$$= \quad 0 \tag{1.16}$$

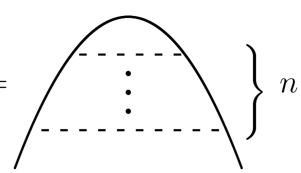
However, in the framed setting, without this relation, one has to somehow avoid the divergence as $z_k - z_{k+1} \to 0$. Le and Murakami ([LM1]) solved this problem as follows. If ω is the chord diagram defined by:

$\omega =$

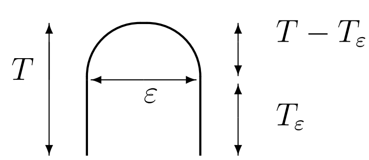


then Le and Murakami define $\varepsilon^{\pm\omega/(2\pi i)}$ for some $\varepsilon \in \mathbb{R}$ as a formal power series expansion of $\exp(\pm\frac{\omega}{2\pi i}\log\varepsilon)$ where they formally write:

$\omega^n =$



Then they consider the following type of tangle T ([LM1]):



where $T = (T - T_\varepsilon) \times T_\varepsilon$, and the diagram above defines the two tangles $T - T_\varepsilon$ and T_ε. We could equally have had this tangle upside down in the event that we had a minimum. One has the following result of Le and Murakami:

Proposition 1.6.1 ([LM1])**.** Let T be a tangle as above. Then:

$$Z_f(T) = \lim_{\varepsilon \to 0} \varepsilon^{\,\omega/(2\pi i)} Z(T_\varepsilon) \tag{1.17}$$

is well-defined, while if this tangle is turned upside down so that it is a local minimum, then:

$$Z_f(T) = \lim_{\varepsilon \to 0} Z(T_\varepsilon) \varepsilon^{-\omega/(2\pi i)} \tag{1.18}$$

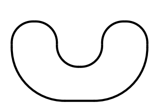
is well-defined as well.

Observe that it is no longer true that Z_f is an element of the completion of $\mathcal{A}(X) = \mathcal{D}(X)/(\text{4-T, fr. ind.})$ since one no longer imposes the framing independence relation. Rather Z_f becomes an element of the completion $\overline{\hat{\mathcal{A}}}(X) := \overline{\hat{\mathcal{A}}(X)}$ of the algebra $\hat{\mathcal{A}}(X) = \mathcal{D}(X)/\text{4-T}$ where we have adopted the original notation $\hat{\mathcal{A}}$ from [K]. In Chapter 2 we will fully develop the formalism of the framed Kontsevich integral.

Let L be an e-components framed oriented link in the blackboard framing represented by a link diagram $\mathcal{D}$. Recall that we consider Morse knots and links; one considers $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$ and one can arrange that our knots live in $\mathbb{C} \times I$. If t is the variable in I, then one says that a knot K is a Morse knot if $t(K)$ is a Morse function ([K]). Let m_i be the number of maximal points of the i-th component of $\mathcal{D}$ with respect to t. Then following [LM5], one defines:

$$\hat{Z}_f(L) = Z_f(\mathcal{D}) \cdot (\nu^{m_1} \otimes \cdots \otimes \nu^{m_e}) \in \overline{\hat{\mathcal{A}}}(\coprod^e S^1) \tag{1.19}$$

where $\nu = Z_f(U)^{-1}$ and:

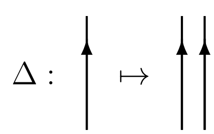
$U =$  (1.20)

Further, since we regard $\hat{\mathcal{A}}(\amalg^e S^1)$ as a $\otimes^e \mathcal{A}$-module, each ν^{m_i} acts only on the i-th component, and it does so by connected sum [LM2]. Strictly speaking, we should write:

$$\nu^{m_1} \otimes \cdots \otimes \nu^{m_e} \blacksquare = (\nu^{m_1} \blacksquare) \otimes \cdots \otimes (\nu^{m_e} \blacksquare) \tag{1.21}$$

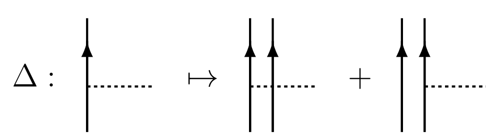
It is more economical to define $\hat{Z}_f$ as being Z_f with the provision that in the expression for $Z_f(L)$, ν multiplies each local max of each link component. Though as defined $\hat{Z}_f$ is already an isotopy invariant, we will use a certain renormalization of $\hat{Z}_f$ that is exceptionally well-behaved under band sum moves; we will define the modified $\widetilde{Z}_f$ to be $\hat{Z}_f$ with the provision that in the expression for $\hat{Z}_f(L)$, each link component is multiplied by ν^{-1}. If one uses ν instead as a renormalizing factor, one would get $\check{Z}_f$ as introduced in [LM3], [LM4], [LM5], [LM6]. It is important to note that $\check{Z}_f$ and $\widetilde{Z}_f$ are essentially the same invariant. What distinguishes them is their behavior under band sum moves.

The doubling map Δ on strands defined by:



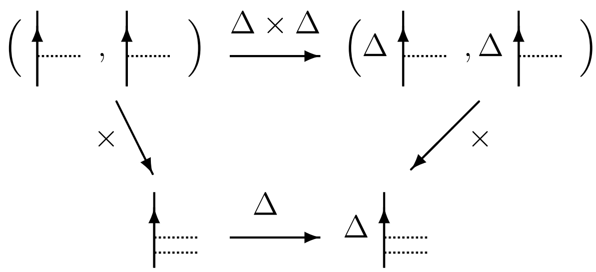
(1.22)

induces a map ([LM4]) $\Delta : \mathcal{A}(I) \to \mathcal{A}(I^2)$ on chord diagrams that is defined as follows on one chord:



(1.23)

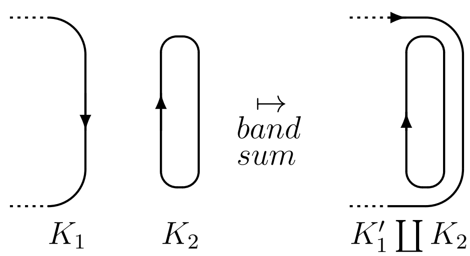
and for chord diagrams of degree greater than 1, we impose that the following square be commutative and use induction:



(1.24)

In [LM6], the behavior of $\check{Z}_f$ under band sum moves for which link components of interest are locally side by side is given by:

Theorem 1.6.2. ([LM6]) Let L be a framed oriented link. Suppose K_1 and K_2 are two link components of L, and K_1 is band summed over K_2, which one pictorially represents as:



(1.25)

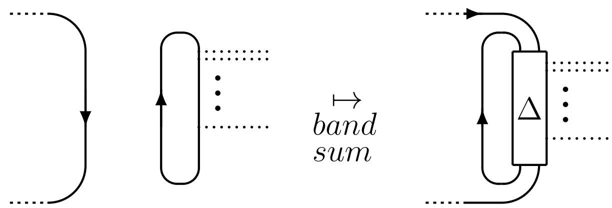
where K_1' is the result of doing a band sum move of K_1 over K_2, and one denotes by L' the link obtained from L after such an operation. In the above picture, we have only symbolically displayed K_1 and K_2, and not other components that may be linked to either or both components. If one writes:

$$\check{Z}_f(L) = \sum_{\substack{\text{chord} \\ \text{diagrams } X}} c_X X \tag{1.26}$$

then Le and Murakami find:

$$\check{Z}_f(L') = \sum_{\substack{\textit{chord}\\ \textit{diagrams } X}} c_X X' \tag{1.27}$$

where X and its corresponding chord diagram X' after the band sum move are given below:

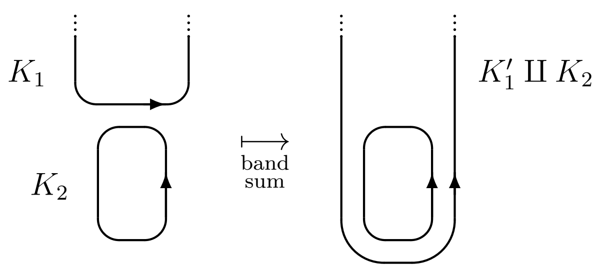


X $\qquad$ X' (1.28)

To be specific, the map Δ doubles strands, and the chords on those strands as well. Since one operates a band sum move here, the Δ enclosed in the box means by abuse of notation ([LM2], [LM5], [LM6]) that the doubling of strands coming with the band sum move proper has been performed and the only thing left to be done is to double chords accordingly.

For band sum moves for which link components of interest are locally on top of each other, we have the following theorem which is proved later in Chapter 4:

Theorem 1.6.3. Let L be a framed oriented link. Suppose K_1 and K_2 are two link components of L, and K_1 is band summed over K_2, which we pictorially represent as:



where K_1' is the result of doing a band sum move of K_1 over K_2, and one denotes by

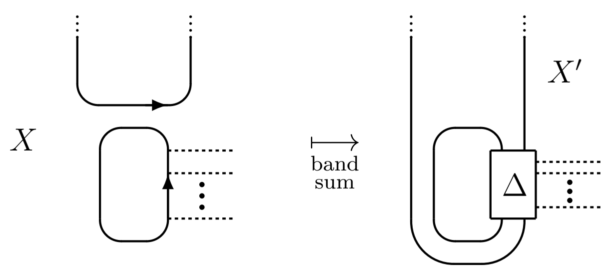
L' the link obtained from L after such an operation. In the above picture, we have only symbolically displayed K_1 and K_2, and not other components that may be linked to either or both components. If:

$$\widetilde{Z}_f(L) = \sum_{\substack{chord \\ diagrams\ X}} c_X X \tag{1.29}$$

for coefficients c_X, then we can write:

$$\widetilde{Z}_f(L') = \sum_{\substack{chord \\ diagrams\ X}} c_X X' \tag{1.30}$$

where X and its corresponding chord diagram X' after the band sum move are given below:

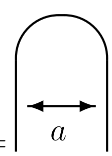


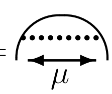
Chapter 2

The Framed Kontsevich integral

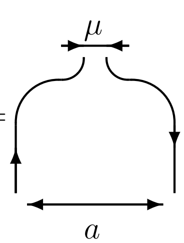
2.1 First definition and Preliminary lemmas

We now consider the problem of having to integrate over tangles that have local extrema in the framed case, in which one no longer imposes the framing independence relation. In that case one normalizes the integral at local extrema as was done in [LM1]. First define the following tangles and chord diagrams:

$T_a =$ 

$\omega =$ 

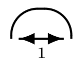
(2.1)

$T_a^\mu =$ 

(2.2)

It is convenient to define a formal tangle chord diagram consisting of a single chord stretching between two strands of opposite orientations, and to call such a graph Ω. This enables one to write T_a^μ above with m chords on it as $T_a^\mu \times \Omega^m$ or $\Omega^m \times T_a^\mu$. We would also have ω above written simply as $T_\mu \times \Omega$. Define $S : \mathcal{A} \to \hat{\mathcal{A}}$ to be the standard inclusion algebra map that maps elements of $\mathcal{A}$ to the subspace of $\hat{\mathcal{A}}$ in which all basis chord diagrams with an isolated chord have coefficient zero as in [BNGRT], [ChDu].

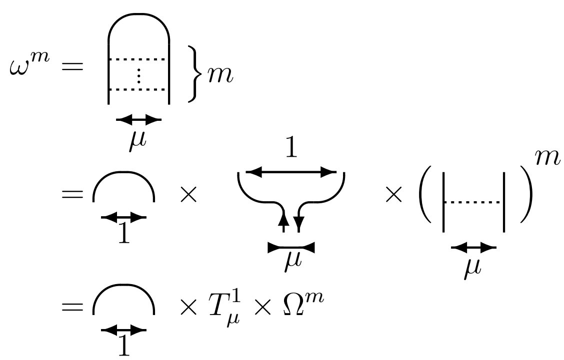
Definition 2.1.1 ([LM1])**.** We define Z_f of the above tangle T_a as:

$Z_f(T_a) = \lim_{\mu \to 0}$  $\times \mu^{\Omega/2\pi i} \times T^1_\mu \times SZ(T^\mu_a)$ (2.3)

which Le and Murakami write in a symbolic, compact form as:

$$Z_f(T_a) := \lim_{\mu\to 0} \mu^{\omega/(2\pi i)} \times SZ(T^\mu_a) \tag{2.4}$$

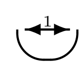
where in [LM1], ω^m is defined by:



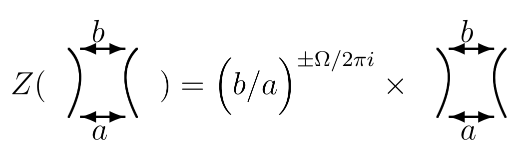
(2.5)

(2.6)

(2.7)

For a local minimum, if we take our tangle T^a to be a single local minimum, then one uses the normalization $Z_f(T^a) = \lim_{\mu\to 0} SZ(T^a_\mu) \times T^\mu_1 \times \mu^{-\Omega/2\pi i} \times$  also written $Z_f(T^a) = \lim_{\mu\to 0} SZ(T^a_\mu) \times \mu^{-\omega/(2\pi i)}$. In that expression ω and T^a_μ are our old ω and T^μ_a respectively, flipped upside down.

Lemma 2.1.2. For $a, b > 0$, we have:



with a plus sign for strands with same orientations, and a minus sign for opposite orientations.

Proof. The proof is a simple computation:

$$Z(T_a^b) = \sum_{n\geq 0} \frac{1}{(2\pi i)^n} \int_{0<t_1<\cdots<t_n<1} (\pm 1)^n \Omega^n \times T_a^b \times \text{dlog } \triangle z(t_1) \cdots \text{dlog } \triangle z(t_n) \tag{2.8}$$

$$= \sum_{n\geq 0} \frac{1}{(2\pi i)^n} (\pm 1)^n \Omega^n \times T_a^b \times I_n \tag{2.9}$$

with:

$$I_n = \int_{0<t_1<\cdots<t_n<1} \text{dlog } \triangle z(t_1) \cdots \text{dlog } \triangle z(t_n) \tag{2.10}$$

$$= \int_{0<t_2<\cdots<t_n<1} \text{dlog } \triangle z(t_2) \cdots \text{dlog } \triangle z(t_n) \cdot \log \frac{\triangle z(t_2)}{a} \tag{2.11}$$

$$= \int_{0<t_3<\cdots<t_n<1} \text{dlog } \triangle z(t_3) \cdots \text{dlog } \triangle z(t_n) \int_{t_2<t_3} \text{dlog } \triangle z(t_2) \log \frac{\triangle z(t_2)}{a} \tag{2.12}$$

$$= \int_{0<t_3<\cdots<t_n<1} \text{dlog } \triangle z(t_3) \cdots \text{dlog } \triangle z(t_n) \int_{t_2<t_3} \text{dlog} \frac{\triangle z(t_2)}{a} \log \frac{\triangle z(t_2)}{a} \tag{2.13}$$

$$= \int_{0<t_3<\cdots<t_n<1} \text{dlog } \triangle z(t_3) \cdots \text{dlog } \triangle z(t_n) \cdot \frac{1}{2} \log^2 \frac{\triangle z(t_3)}{a} \tag{2.14}$$

$$= \int_{0<t_4<\cdots<t_n<1} \text{dlog } \triangle z(t_4) \cdots \text{dlog } \triangle z(t_n) \int_{t_3<t_4} \text{dlog } \triangle z(t_3) \cdot \frac{1}{2} \log^2 \frac{\triangle z(t_3)}{a} \tag{2.15}$$

$$= \int_{0<t_4<\cdots<t_n<1} \text{dlog } \triangle z(t_4) \cdots \text{dlog } \triangle z(t_n) \cdot \frac{1}{3} \cdot \frac{1}{2} \log^3 \frac{\triangle z(t_4)}{a} \tag{2.16}$$

$$= \cdots = \frac{1}{n!} \log^n \frac{\triangle z(t_n = 1)}{a} = \frac{1}{n!} \log^n \frac{b}{a} \tag{2.17}$$

so that:

$$Z(T_a^b) = \sum_{n\geq 0} \frac{1}{(2\pi i)^n} (\pm 1)^n \Omega^n \times T_a^b \times I_n \tag{2.18}$$

$$= \sum_{n\geq 0} \frac{1}{(2\pi i)^n} (\pm 1)^n \Omega^n \times T_a^b \times \frac{1}{n!} \log^n \frac{b}{a} \tag{2.19}$$

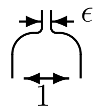
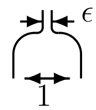
$$= \sum_{n\geq 0} \frac{1}{n!} \Big((\pm \frac{\Omega}{2\pi i}) \log \frac{b}{a} \Big)^n \times T_a^b \tag{2.20}$$

$$= \sum_{n\geq 0} \frac{1}{n!} \Big(\log (b/a)^{\pm \frac{\Omega}{2\pi i}} \Big)^n \times T_a^b \tag{2.21}$$

$$= e^{\log \left((b/a)^{\pm \Omega/2\pi i} \right)} \times T_a^b = (b/a)^{\pm \Omega/2\pi i} \times T_a^b \tag{2.22}$$

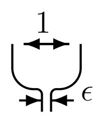
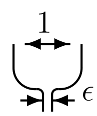
$\square$

Corollary 2.1.3. For $\epsilon > 0$:

$Z\Big($  $\Big) = \epsilon^{\pm\Omega/2\pi i} \times$  (2.23)

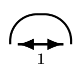
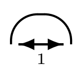
with a plus sign for same orientations, a minus sign for opposite orientations.

Corollary 2.1.4.

$Z($  $) = \epsilon^{\mp\Omega/2\pi i} \times$  (2.24)

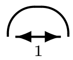
with a plus sign for same orientations and a minus sign for opposite orientations.

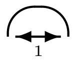
Lemma 2.1.5.

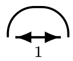
$Z_f(T_a) =$  $\times\, SZ(T_a^1) =:$  $\times\, SZ(T_a^{\text{1-resolved}})$ (2.25)

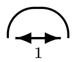
where $T_a^{\text{1-resolved}}$ is identical with the tangle T_a except that it is seen as being analytically probed by isolated chords near the local maximum that deform it into a spout of opening width 1, whence the name 1-resolved.

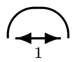
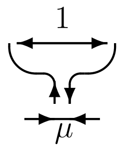
Proof. Using Definition 2.1.1 for the same tangle T_a:

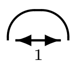
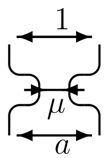
$Z_f(T_a) = \lim_{\mu\to 0}$  $\times\, \mu^{\Omega/2\pi i} \times T^1_\mu \times SZ(T^\mu_a)$ (2.26)

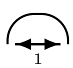
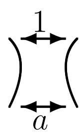
$= \lim_{\mu\to 0}$  $\times \Big(\sum_{m\geq 0} \frac{1}{m!}\frac{1}{(2\pi i)^m}\Omega^m \log^m \mu\Big) \times T^1_\mu \times SZ(T^\mu_a)$ (2.27)

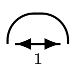
$= \lim_{\mu\to 0}$  $\times \sum_{m\geq 0} \Big(\frac{1}{m!}\frac{1}{(2\pi i)^m}\Omega^m \log^m \mu \times T^1_\mu\Big) \times SZ(T^\mu_a)$ (2.28)

$= \lim_{\mu\to 0}$  $\times \sum_{m\geq 0} \frac{1}{m!}\frac{1}{(2\pi i)^m} T^1_\mu \times \Omega^m \log^m \mu \times SZ(T^\mu_a)$ (2.29)

$= \lim_{\mu\to 0}$  $\times\, SZ($  $) \times SZ(T^\mu_a)$ (2.30)

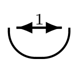
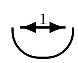
$= \lim_{\mu\to 0}$  $\times\, SZ($  $)$ (2.31)

$=$  $\times\, SZ($  $)$ (2.32)

$=$  $\times\, SZ(T^1_a)$ (2.33)

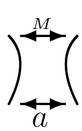
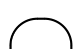
where in (2.30) we used Lemma 2.1.4, and in going from (2.31) to (2.32) we have used the invariance of the Kontsevich integral under horizontal deformations [BN], [ChDu]. □

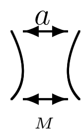
Lemma 2.1.6.

$Z_f(T^a) = SZ(T^a_1) \times$  $= SZ(T^a_{\text{1-resolved}}) \times$  (2.34)

Proof. Same as for the preceding lemma. □

Remark 2.1.7. From Lemmas 2.1.5 and 2.1.6 we might want to generalize the definition of Z_f near local extrema to:

$Z_f[\mathrm{M}](T_a) =$  $\times\, SZ($  $) =:$  $\times\, SZ(T_a^{M\text{-resolved}})$ (2.35) and:

$$Z_f[\mathrm{M}](T^a) = SZ(\ $$ 

$$\ ) \times \smile =: SZ(T^{a\ M\text{-resolved}}) \tag{2.36}$$

thereby defining a family of framed Kontsevich integrals $Z_f[\mathrm{M}]$ parametrized by $M > 0$. Without loss of generality, we can focus on a local maximum. An equivalent definition to (2.35) would be:

$$Z_f[\mathrm{M}](T_a) = \frown \times \lim_{(\mu/M)\to 0} \Big(\mu/M\Big)^{\Omega/2\pi i} \times T^M_\mu \times SZ(T^\mu_a) \tag{2.37}$$

$$= \frown \times \lim_{\xi\to 0} \xi^{\Omega/2\pi i} \times T^M_{M\xi} \times SZ(T^{M\xi}_a) \tag{2.38}$$

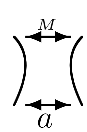
Definition 2.1.1 being a special case thereof for which $M = 1$. To see that (2.37) is well-defined, it suffices to write:

$$\begin{aligned} &\frown \times \lim_{(\mu/M)\to 0} \Big(\mu/M\Big)^{\Omega/2\pi i} \times T^M_\mu \times SZ(T^\mu_a) \\ &= \frown \times \lim_{(\mu/M)\to 0} \Big(M/\mu\Big)^{-\Omega/2\pi i} \times T^M_\mu \times SZ(T^\mu_a) \\ &= \frown \times \lim_{(\mu/M)\to 0} SZ(T^M_\mu) \times SZ(T^\mu_a) \end{aligned}$$

where in the last step we have used Lemma 2.1.2 with $a = \mu$, $b = M$ and mixed orientations on the strands since we work at a local maximum. We have a concatenation of tangles $T^M_\mu \times T^\mu_a = T^M_a$ leading to the equality $SZ(T^M_\mu) \times SZ(T^\mu_a) = SZ(T^M_a)$. Using it in the last equality above, we get:

$$\frown \times \lim_{(\mu/M)\to 0} \Big(\mu/M\Big)^{\Omega/2\pi i} \times T^M_\mu \times SZ(T^\mu_a)$$

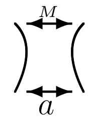
$$= \frown \times \lim_{(\mu/M)\to 0} SZ(T^M_a) \tag{2.39}$$

$$= \frown \times SZ(\ $$ 

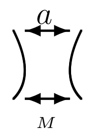
$$\ ) \tag{2.40}$$

This leads to generalizing the definition of the normalization for the value of Z_f of local extrema.

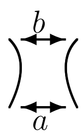
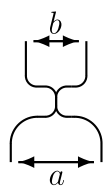
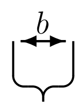
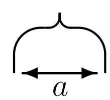
Definition 2.1.8. For $M > 0$:

$$Z_f[\mathrm{M}](T_a) = $$  $\times\, SZ($  $)$ (2.41)

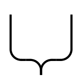
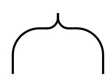
and:

$$Z_f[\mathrm{M}](T^a) = SZ($$  $) \times$  (2.42)

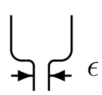
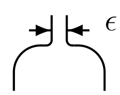
Remark 2.1.9. By invariance of Z under horizontal deformations:

$Z($  $) = Z($  $) = Z($  $) \times Z($  $)$ (2.43)

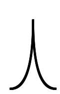
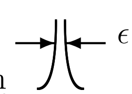
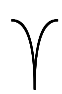
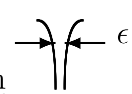
Since Z is defined for Morse knots, there is no ambiguity as to what:

 and 

mean; it is really

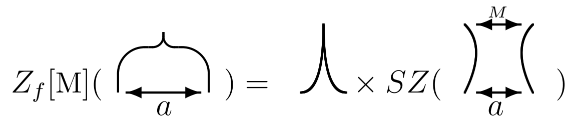
$\lim_{\epsilon \to 0}$  and $\lim_{\epsilon \to 0}$ 

respectively. The first equality in (2.43) is justified as $Z(T_a^b)$ is well-defined. To make sense of the second equality, one would have to make sense of the values of Z on those pinched tangles. The problem is the same as the one encountered for local extrema. If we formally define:

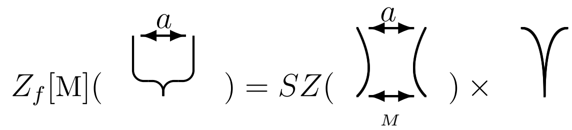
 $= \lim_{\epsilon \to 0}$  and  $= \lim_{\epsilon \to 0}$ 

then we can define:

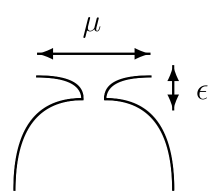
Definition 2.1.10.



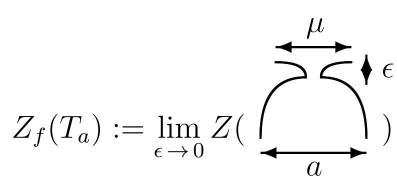
and:



Remark 2.1.11. Long after this work was finished, it was pointed to the author ([DM]) that a similar construction has already been performed ([ZD]). In the present work we just generalize the work of Le, Murakami et al ([LM1]) concerning the normalization of the Kontsevich integral at local extrema. In [ZD], the problem of divergences at local extrema is circumvented by invoking a QFT technique without naming it, but which from the looks of it should be multiplicative renormalization. A scale μ is fixed (the equivalent of M for us). Without loss of generality one works with a local maximum, which is opened to some width μ, the opening bottleneck having a length ϵ, as in:



Then:



(2.44)

While performing such a computation, only chords both of whose feet are on the bottleneck are allowed. This is imposed while in our picture this follows precisely because we have a bottleneck such that long chords vanish on it. Further it is understood that the bottleneck has a width ϵ. Thus in the limit the bottleneck is cut off. What we do however is retain the top of the tangle with a bottleneck and formally close it off.

One last remark is in order here. Though the following Lemma is true in the unframed case, it fails in the unframed case as the note that follows points out.

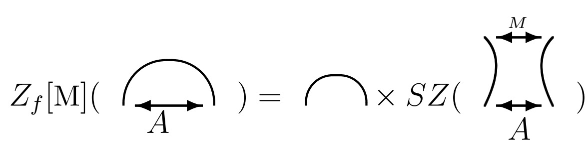
Lemma 2.1.12. ([ChDu]) For a tangle T, $Z(T)$ is invariant under a dilation of T by a factor of Λ with respect to the time axis.

Proof. We have:

$$
\begin{aligned}
Z(T) &= \sum_{m\geq 0} \frac{1}{(2\pi i)^m} \int_{0<t_1<\cdots<t_m<1} \sum_{|P|=m} (-1)^{\epsilon(P)} T_P \cdot \prod_{1\leq i\leq m} \text{dlog } \triangle z(t_i)[P_i] \\
&= \sum_{m\geq 0} \frac{1}{(2\pi i)^m} \sum_{|P|=m} (-1)^{\epsilon(P)} T_P \int_{0<t_1<\cdots<t_m<1} \prod_{1\leq i\leq m} \text{dlog } \triangle z(t_i)[P_i]
\end{aligned}
$$

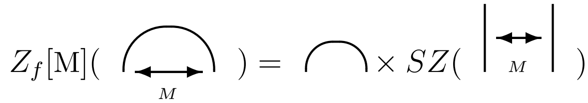
If we squeeze the tangle T uniformally with respect to the time axis by a factor of Λ then each coordinate z along T is being mapped to z/Λ. Thus $\text{dlog } \triangle z \mapsto \text{dlog}(\triangle z/\Lambda) = \text{dlog } \triangle z$ making the above multivariate integral invariant, as well as $Z(T)$ which is a sum of such integrals. $\square$

Note 2.1.13. Z_f[M] is not invariant under resizing by a factor of Λ the way Z was. For example:



(2.45)

is non trivial if $A \neq M$, while if we resize the tangle on the left hand side of the above equation by a factor of $\Lambda = M/A$, then we have:



(2.46)

which is trivial.

2.2 Properties of the framed Kontsevich integral

It is well-known that Z is invariant under horizontal deformations ([BN]), which was further generalized for the parameter-free, framed Kontsevich integral Z_f ([LM1]). We show this persists in the presence of a parameter:

Lemma 2.2.1. For $M > 0$, $Z_f[\mathrm{M}]$ is invariant under horizontal deformations.

Proof. $Z_f[\mathrm{M}]$ is defined locally. In particular, away from local extrema $Z_f[\mathrm{M}] = Z$, which is invariant under horizontal deformations. $\square$

The following is the framed generalization of a well-known result in the unframed case ([BN]), also proved in the framed case for Le and Murakami's parameter-free Z_f ([LM1]).

Proposition 2.2.2. For $M > 0$, $Z_f[\mathrm{M}]$ is multiplicative.

Proof. Multiplicativity means that if $T = T_1 \times \cdots \times T_n$ where local extrema and pinched extremities are located at times other than those at which a concatenation is performed, then $Z_f[\mathrm{M}](T) = \times_{1 \leq i \leq n} Z_f[\mathrm{M}](T_i)$. By definition of Z_f, away from those extrema $Z_f = SZ$, which is multiplicative. $\square$

The following Theorem was initially presented in [LM1] for the parameter-free Z_f defined on q-tangles, which we generalize as follows:

Theorem 2.2.3. $pZ_f[\mathrm{M}] = Z$ for all $M > 0$, where $p : \hat{\mathcal{A}} \to \mathcal{A}$.

Proof. Fix $M > 0$. By definition of $Z_f(L)$ for L a framed oriented link with possible pinched extremities, for a given tangle chord diagram and its corresponding coefficient, the contribution of long chords to that coefficient is computed from L punctured (c.f. (2.37) and (2.38)) and is thus independent of M. It reproduces the value one would obtain from $Z(L)$ in the unframed case. Isolated chords near an extremum are projected out by p. The result follows. $\square$

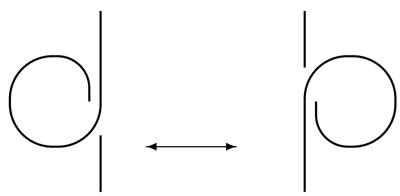
The following two Theorems were originally presented in [LM1] for the parameter-free $\hat{Z}_f$.

Theorem 2.2.4. For $M > 0$, $\hat{Z}_f[\mathrm{M}]$ is multiplicative.

Proof. This follows from $Z_f[\mathrm{M}]$ being multiplicative (Proposition 2.2.2) and the definition of $\hat{Z}_f[\mathrm{M}]$. □

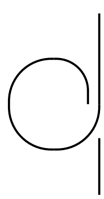
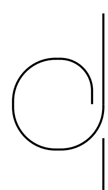
Theorem 2.2.5. For $M > 0$, $\hat{Z}_f[\mathrm{M}]$ is an isotopy invariant.

Proof. $\hat{Z}_f[\mathrm{M}]$ is invariant under all the Reidemeister moves since Z is, except the straightening of humps (henceforth referred to as the cancellation of critical points move) discussed in (2.51) and the following move:

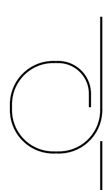


(2.47)

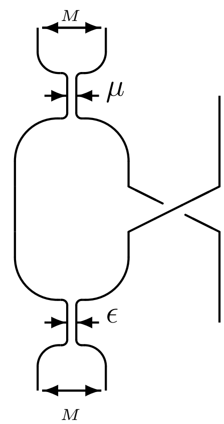
We have to show $Z_f[\mathrm{M}]$ is invariant under this move, and by definition of $\hat{Z}_f[\mathrm{M}]$ it will follow that so is this latter. By definition of $Z_f[\mathrm{M}]$:

$Z_f[\mathrm{M}]($  $) =$  $\times\, SZ($  $)^{M\text{-resolved}} \times$  (2.48)

As far as non isolated chords are concerned, this amounts to computing:

$SZ($  $)$ (2.49)

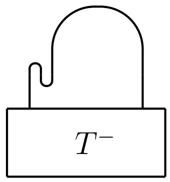
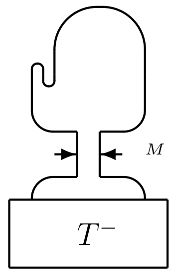
which is invariant under the unframed counterpart to the move depicted in (2.47) and is therefore invariant under that move as well. For isolated chords and mixed chords made up of isolated and long chords, we use:

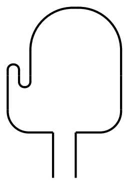
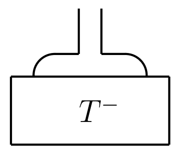
$\lim\limits_{\epsilon,\mu\to 0}$  $\times\, SZ($ 

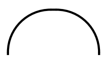
$) \times$  (2.50)

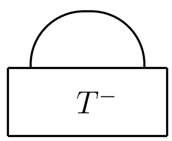
Under the move (2.47) the strands are rotated 180° with respect to the time axis, and complex coordinates are switched accordingly to leave the computation of the above limit invariant by virtue of the fact that $\mathrm{dlog}(z - z') = \mathrm{dlog}(z' - z)$.

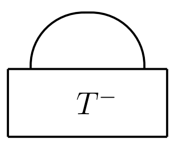
We now consider the cancellation of critical points move. Since $Z_f[\mathrm{M}]$ is invariant under all moves except this particular move, we can drag a hump along a component until it reaches a local max. Then squeezing this local max into a needle along with the hump on it, we essentially have to consider the following:

$Z_f[\mathrm{M}]($  $) = Z_f[\mathrm{M}]($  $)$ (2.51)

$= Z_f[\mathrm{M}]($  $) \times \Big($  $\#$  $\Big) \times Z_f[\mathrm{M}]($  $)$ (2.52)

$= Z_f[\mathrm{M}](U) \# Z_f[\mathrm{M}]($  $) \times Z_f[\mathrm{M}](T^-)$ (2.53)

$= Z_f[\mathrm{M}](U) \cdot Z_f[\mathrm{M}]($  $)$ (2.54)

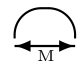
$= \nu^{-1} Z_f[\mathrm{M}]($  $)$ (2.55)

Thus multiplying by ν^2 on both sides, we have the invariance of $\hat{Z}_f$ under the cancellation of critical points move. $\square$

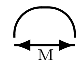
Proposition 2.2.6. For L a link, $Z_f[\mathrm{M}](L)$ is independent of $M > 0$ and we denote it by $Z_f(L)$.

Proof. A change in scale in $\mathbb{R}^3$ by a factor of $\lambda \neq 0$ induces a dilation map d_λ on tangles, which in turn induces a map d_λ^* on $Z_f[\mathrm{M}]$. We write:

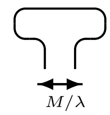
$$d_\lambda^* Z_f[\mathrm{M}](T_a) = Z_f[\mathrm{M}](d_\lambda T_a) = Z_f[\mathrm{M}](T_{\lambda a}) \tag{2.56}$$

$=$ 

$\times SZ(T^M_{\lambda a})$ (2.57)

$=$ 

$\times SZ(d_\lambda T_a^{M/\lambda})$ (2.58)

$=$ 

$\times SZ(T_a^{M/\lambda})$ (2.59)

$=$  $\times SZ(T_a^{M/\lambda})$ (2.60)

$$= Z_f[M/\lambda](T_a) \tag{2.61}$$

where in the next to last step we have used the invariance of the original Kontsevich integral under dilations. It is not difficult to see that for a tangle T with only one local maximum, the above computation generalizes to:

$$d_\lambda^* Z_f[\mathrm{M}](T) = Z_f[\mathrm{M}](d_\lambda T) \tag{2.62}$$

$=$  $\times SZ((d_\lambda T)^{\text{M-resolved}})$ (2.63)

$=$  $\times SZ(d_\lambda(T^{M/\lambda\text{-resolved}}))$ (2.64)

$=$  $\times SZ(T^{M/\lambda\text{-resolved}})$ (2.65)

$$= Z_f[M/\lambda](T) \tag{2.66}$$

Likewise, one would show that for a tangle T with a unique local minimum, one has $d_\lambda^* Z_f[\mathrm{M}](T) = Z_f[M/\lambda](T)$. The same computations essentially carry over to the case of tangles with pinched extremities. On associators and crossings, for any $M > 0$, $Z_f[\mathrm{M}] = Z$. Thus on elementary tangles, as well as for tangles with pinched extremities, for $M > 0$ we have:

$$d_\lambda^* Z_f[\mathrm{M}] = Z_f[M/\lambda] \tag{2.67}$$

Let K be a knot that we decompose as a concatenation of l elementary tangles as $K = \times_{1 \leq i \leq l} T_i$. Then:

$$d_\lambda^* Z_f[\mathrm{M}](K) = Z_f[\mathrm{M}](d_\lambda K) \tag{2.68}$$

$$= Z_f[\mathrm{M}](d_\lambda(\times_i T_i)) \tag{2.69}$$

$$= Z_f[\mathrm{M}](\times_i(d_\lambda T_i)) \tag{2.70}$$

$$= \times_{1 \leq i \leq l} Z_f[\mathrm{M}](d_\lambda T_i) \tag{2.71}$$

$$= \times_i Z_f[M/\lambda](T_i) \tag{2.72}$$

$$= Z_f[M/\lambda](\times_i T_i) \tag{2.73}$$

$$= Z_f[M/\lambda](K) \tag{2.74}$$

Thus $d_\lambda^* Z_f[\mathrm{M}] = Z_f[M/\lambda]$ on knots as well, and this generalizes easily to the case of links. Now observe that by isotopy invariance of $\hat{Z}_f[\mathrm{M}]$ we have:

$$\hat{Z}_f[\mathrm{M}](d_\lambda K) = \hat{Z}_f[\mathrm{M}](K) \tag{2.75}$$

from which it follows that:

$$Z_f[\mathrm{M}](d_\lambda K) = Z_f[\mathrm{M}](K) \tag{2.76}$$

and coupled with (2.74), we get $Z_f[\mathrm{M}](K) = Z_f[M/\lambda](K)$ for all $M > 0$, $\lambda > 0$, thus $Z_f[\mathrm{M}](K)$ is independent of M, and we denote it by $Z_f(K)$. Likewise, for a link L, one would show following the same line of reasoning that $Z_f[\mathrm{M}](L)$ is independent of M, and we denote it by $Z_f(L)$. $\square$

Corollary 2.2.7. For L a link, $\hat{Z}_f[\mathrm{M}](L)$ is independent of $M > 0$ and we denote it by $\hat{Z}_f(L)$, as it coincides with Le and Murakami's $\hat{Z}_f(L)$.

Proof. This follows readily from the definition of $\hat{Z}_f[\mathrm{M}]$ and the previous Proposition. $\square$

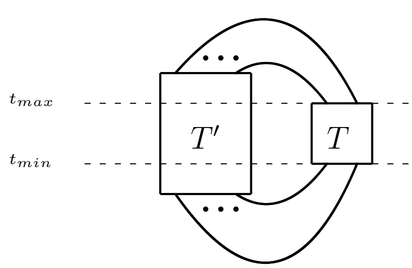
Chapter 3

Fundamental results

In this section we essentially discuss two very important results. The first, implicit in much of the work that has been done on the Kontsevich integral, is the long chords Lemma. A proof in the unframed case can be found in [ChDu]. We show it is also true in the framed case, and not only for knots but for links as well. The second result is the statement $\Delta\hat{Z}_f = \hat{Z}_f(\Delta)$ ([LM2]) which will require a few lemmas before we proceed to proving it.

3.1 The long chords lemma

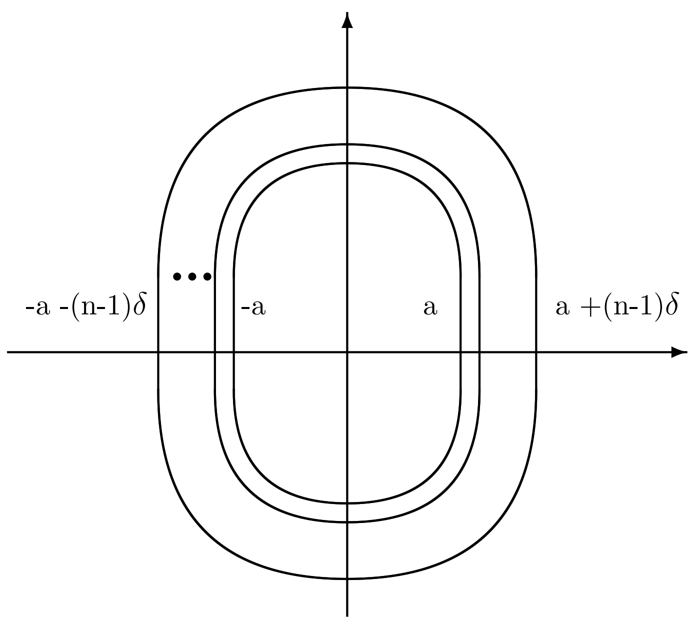
To discuss the long chord lemma, we will follow the notations of [ChDu]. Suppose L is a link, with a distinguished tangle T as in:



located between times t_{min} and t_{max}. The idea is that if one shrinks the tangle T, in the limit when it is shrunk to zero size the chords between it and the rest of the link L in the expression for $Z_f(L)$ go away. The exact statement is that if one denotes by $Z_{f,T}(L)$ that part of $Z_f(L)$ where in the horizontal strip between the times t_{min} and t_{max} one only has chords both feet of which are either on T or not at all, then $Z_f(L) = Z_{f,T}(L)$.

3.1.1 The Kontsevich integral of n unlinked circles

The case of n unlinked circles is a simple instance where computations can be done by hand. In this case, the long chords Lemma is true because all tangle chord diagrams have zero coefficients. The case of n unlinked circles corresponds to the simple case where both T and T' are trivial. By invariance of Z under horizontal deformations, we essentially have to show that n unlinked circles as in:

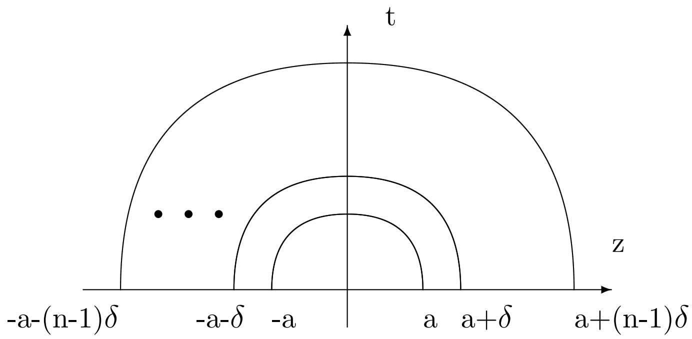


(3.1)

have trivial Kontsevich integral. We first show the result in the unframed case and then prove it in the framed case.

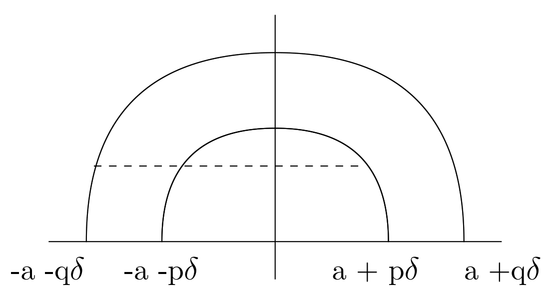
Lemma 3.1.1.1. The Kontsevich integral of n unlinked circles as above is trivial.

Proof. We first consider the types of chords we can have and their corresponding log differentials. It is convenient to present the n strands as circles equally spaced by a distance δ as in:



(3.2)

with a similar setting for bottom strands, which number n as well. For chords with support on those strands, we have long chords stretching from a q-th strand on one side to a p-th strand on the other side of the knot, $0 \leq p, q \leq n-1$, short chords stretching between the q-th and p-th strands, $0 \leq q, p \leq n-1$, both on a same side of the knot, and self chords. We first consider a long chord stretching from the q-th strand on the left to the p-th one on the right as in:

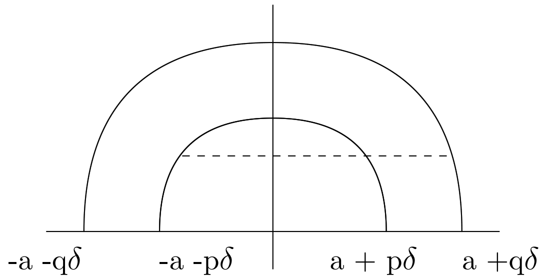


(3.3)

Such a chord diagram we denote by $C^{(n)+}_{(q,p)}$, where the plus sign indicates that we consider the upper portion of n unlinked circles, and (q,p) indicates that we consider one chord only stretching from the q-th strand on the left to the p-th strand on the right. At height t, the separation from one chord foot to the other is $\triangle z(t) = \sqrt{(a+p\delta)^2 - t^2} + \sqrt{(a+q\delta)^2 - t^2}$ from which we get:

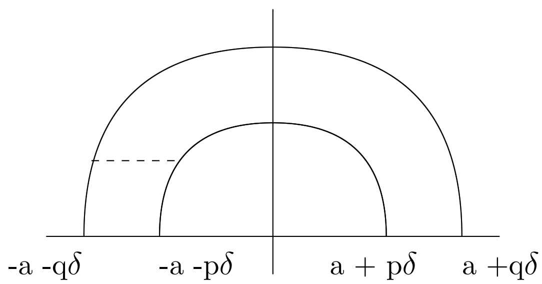
$$\text{dlog}\, \triangle z(t) = -\frac{tdt}{\sqrt{(a+q\delta)^2 - t^2} \cdot \sqrt{(a+p\delta)^2 - t^2}} \tag{3.4}$$

For a chord stretching from the p-th strand to the q-th strand on the right, that is for $C^{(n)+}_{(p,q)}$:



(3.5)

we get the same log differential. We also have short chords between strands on the left as in the following tangle chord diagram that we denote by $C^{(n)+}_{(qp,-)}$:

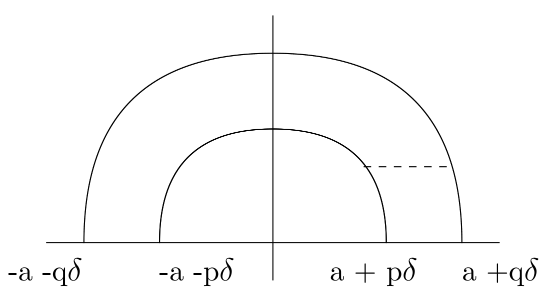


(3.6)

with chords of length $\sqrt{(a+q\delta)^2 - t^2} - \sqrt{(a+p\delta)^2 - t^2}$, leading to a log differential:

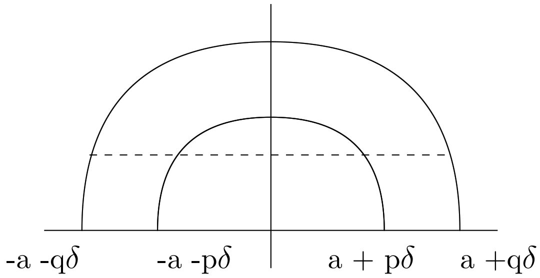
$$\text{dlog } \triangle z(t) = \frac{tdt}{\sqrt{(a+q\delta)^2 - t^2} \cdot \sqrt{(a+p\delta)^2 - t^2}} \tag{3.7}$$

and short chords on strands on the right side of the knot as in the chord diagram $C^{(n)+}_{(-,pq)}$:



(3.8)

with the same log differential. We also have to consider self-chords that do not cancel if they are not isolated, such as in the chord diagram $C^{(n)+}_{(q,q)}$:



(3.9)

These chords have length $2\sqrt{(a+q\delta)^2 - t^2}$ and thus yield a log differential:

$$\text{dlog } \triangle z(t) = -\frac{tdt}{(a+q\delta)^2 - t^2} \tag{3.10}$$

We obtain similar chord lengths and corresponding log differentials for chords stretching between strands at the bottom of the knot. What we have at this point is that the log differentials for each chord are odd functions times the differential dt.

We see that we essentially have three possibilities for chords: long chords, short chords, and non-isolated self-chords. For long chords and short chords we assume that $q > p$ for computational purposes. Chord diagrams are graded by their number of chords, which are numbered from the bottom up. The k-th chord at time t_k is stretching between the q_k-th strand and the p_k-th one. The Kontsevich integral of n unlinked trivial knots is a sum over $m \geq 0$ of terms proportional to integrals of the form:

$$\int_{\substack{-a-p_m\delta<t_m<a+p_m\delta_m \\ -a-p_{m-1}\delta<t_{m-1}<\min(a+p_{m-1}\delta,t_m) \\ \cdots \\ -a-p_1\delta<t_1<\min(a+p_1\delta,t_2)}} \prod_{1\leq i\leq m} \text{dlog } \triangle z(t_i) \tag{3.11}$$

times some tangle chord diagrams. For convenience, we write this integral as:

$$\int_{-a-p_m\delta<t_m<a+p_m\delta_m} \text{dlog } \triangle z(t_m)\Psi(t_m) \tag{3.12}$$

where $\Psi(t_m)$ comes from the integration of $m-1$ log differentials. As we have shown above, the log differential is odd, the region of integration is symmetric with respect to zero, so if we can show that $\Psi(t_m)$ is an even function of t_m then we would have this m-th term in the Kontsevich integral to be vanishing, and this for all such terms and for all $m \geq 0$, so this would show the result. Thus it suffices to show that Ψ is an even function. We have:

$$\Psi(t_2) = \int_{-a-p_1\delta<t_1<\min(a+p_1\delta,t_2)} \operatorname{dlog} \triangle z(t_1) \tag{3.13}$$

If $t_2 > a+p_1\delta$, then $\Psi(t_2) = \int_{-a-p_1\delta}^{a+p_1\delta} \operatorname{dlog} \triangle z(t_1) = 0 = \Psi(-t_2)$. If $t_2 < a+p_1\delta$ then $\Psi(t_2) = \log \triangle z(t_2) - \log \triangle z(-a-p_1\delta)$ while:

$$\Psi(-t_2) = \int_{-a-p_1\delta<t_1<-t_2} \operatorname{dlog} \triangle z(t_1) \tag{3.14}$$

$$= \log \triangle z(-t_2) - \log \triangle z(-a-p_1\delta) \tag{3.15}$$

$$= \log \triangle z(t_2) - \log \triangle z(-a-p_1\delta) = \Psi(t_2) \tag{3.16}$$

Thus $\Psi(t_2)$ is an even function of t_2. Suppose $\Psi(t_{m-1})$ is an even function of t_{m-1}. Then we have:

$$\Psi(t_m) = \int_{-a-p_{m-1}\delta<t_{m-1}<\min(a+p_{m-1}\delta,t_m)} \operatorname{dlog} \triangle z(t_{m-1})\Psi(t_{m-1}) \tag{3.17}$$

an integral of an odd integrand, which we denote by Γ. Let $A = a + p_{m-1}\delta$. If $t_m > A$ then $\Psi(t_m) = \int_{-A}^{A} \Gamma(t)dt = 0 = \Psi(-t_m)$. If $t_m < A$ then $\Psi(-t_m) = \int_{-A}^{-t_m} \Gamma(t)dt = \int_{A}^{t_m} \Gamma(u)du$, while:

$$\Psi(t_m) = \int_{-A<t<t_m} \Gamma(t)dt \tag{3.18}$$

$$= \int_{A}^{-t_m} \Gamma(u)du \tag{3.19}$$

$$= \int_{A}^{t_m} \Gamma(u)du + \int_{t_m}^{-t_m} \Gamma(u)du \tag{3.20}$$

$$= \int_{A}^{t_m} \Gamma(u)du = \Psi(-t_m) \tag{3.21}$$

and $\Psi(t_m)$ is therefore an even function. This being true for any $m \geq 0$ it is true for all $m \geq 0$, so Ψ is an even function and we have shown the result. This proof is using basic properties of the Kontsevich integral. We can give a second proof that uses some invariance properties of the Kontsevich integral per se. It suffices to invoke the invariance of Z under moves other than the cancellation of critical points move as well as its multiplicative property to write:

$$Z(\underbrace{S^1 + \cdots + S^1}_{n}) = nZ(S^1) \tag{3.22}$$

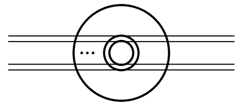
$$= nZ(\frown) \times Z(\smile) \tag{3.23}$$

$$= n \frown \times \smile = nS^1 \tag{3.24}$$

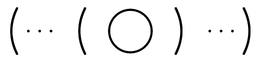
□

Lemma 3.1.1.2. The framed Kontsevich integral of n unlinked circles is trivial.

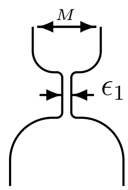
Proof. As for the preceding lemma, we will give two proofs. The first one uses the definition of the framed Kontsevich integral, the second is much shorter and just uses some invariance properties of the framed Kontsevich integral. For the first proof, we consider a segmentation of n circles at each local extremum and use the multiplicativity of the framed Kontsevich integral.

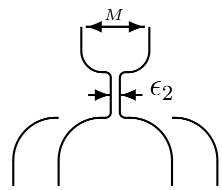
$Z_f(nS^1) = Z_f($  $)$ (3.25)

$= Z_f[\mathrm{M}]($  $) \times Z_f[\mathrm{M}]($  $)\times$

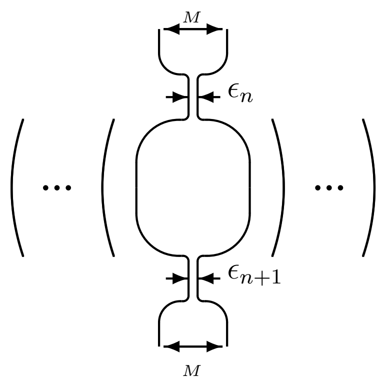
$\cdots \times Z_f[\mathrm{M}]($  $) \times \cdots$

$\times Z_f[\mathrm{M}]($  $) \times Z_f[\mathrm{M}]($  $)$ (3.26)

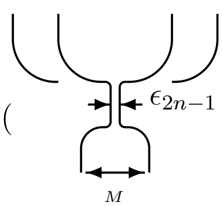
$=$  $^n \times \lim_{\epsilon_1,\cdots,\epsilon_{2n}\to 0} SZ($ 

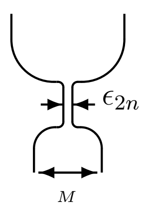
$) \times SZ($ 

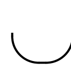
$)\times$

$\cdots \times SZ($ 

$) \times \cdots$

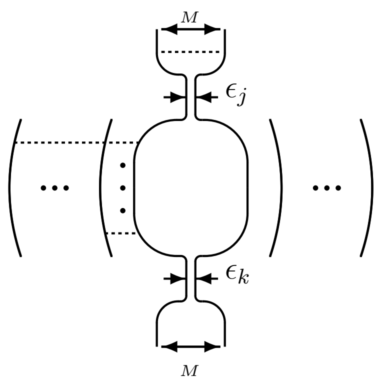
$SZ($ 

$) \times SZ($ 

$) \times$  ^{n} (3.27)

$$= SZ(nS^1) + \text{mixed chords} \tag{3.28}$$

The first term is equal to $nSZ(S^1) = nS^1$ by the preceding lemma, and mixed chords here means that this is the part of $Z_f(nS^1)$ with at least one integrated chord that would have been isolated in the unframed case. One new problem arises however from the fact that those chords can now escape through a spout. Supposing we consider such a chord at a local maximum with other long chords below that will prevent the isolated chord to go all the way down through the spout at the local minimum, in this instance we can no longer invoke a symmetry argument. A paradigm example is the following:

$SZ($  $)$ (3.29)

whose coefficient modulo sign and a power of $1/2\pi i$ is:

$$\int_{\substack{-a-p_{m+1}\delta<t_{m+1}<a+p_{m+1}\delta+A \\ -a-p_{m+1}\delta<t_m<\min(a+p_{m+1}\delta,t_{m+1}) \\ \cdots \\ -a-p_{m+1}\delta<t_1<\min(a+p_{m+1}\delta,t_2)}} \prod_{1\leq i\leq m+1} \mathrm{dlog}\ \triangle z(t_i) \tag{3.30}$$

where $\triangle z(t_{m+1})$ corresponds to the top chord which is escaping through the puncture, until it reaches the top of the open tangle at the top in a time A. All the other chords stop at $a+p_{m+1}\delta$. Using the notations of the previous proof, we consider:

$$\int_{\substack{-a-p_{m+1}\delta<t_{m+1}<a+p_{m+1}\delta+A \\ -a-p_{m+1}\delta<t_m<\min(a+p_{m+1}\delta,t_{m+1})}} \mathrm{dlog}\ \triangle z'(t_{m+1})\mathrm{dlog}\ \triangle z(t_m)\Psi(t_m) \tag{3.31}$$

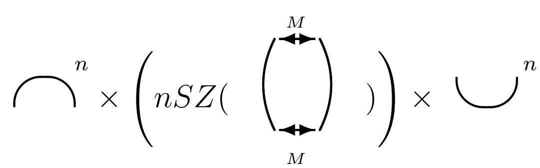
with $\triangle z'(t_{m+1})$ the length of the isolated chord. We write this integral as:

$$\begin{aligned}\int_{-a-p_{m+1}\delta}^{a+p_{m+1}\delta} \mathrm{dlog}\ \triangle z'(t_{m+1}) \int_{-a-p_{m+1}\delta}^{t_{m+1}} \mathrm{dlog}\ \triangle z(t_m)\Psi(t_m) \\ + \int_{a+p_{m+1}\delta}^{a+p_{m+1}\delta+A} \mathrm{dlog}\ \triangle z'(t_{m+1}) \int_{-a-p_{m+1}\delta}^{a+p_{m+1}\delta} \mathrm{dlog}\ \triangle z(t_m)\Psi(t_m)\end{aligned} \tag{3.32}$$

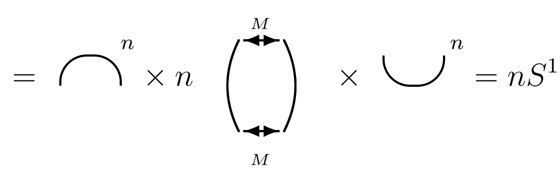
The second integral vanishes because the region of integration in t_m is symmetric, dlog $\triangle z(t_m)$ is odd, and $\Psi(t_m)$ is an even function from the preceding proof. The first integral can be rewritten as:

$$\int_{-a-p_{m+1}\delta}^{a+p_{m+1}\delta} \mathrm{dlog}\ \triangle z'(t_{m+1})\Psi(t_{m+1}) \tag{3.33}$$

with $\Psi(t_{m+1})$ even and dlog $\triangle z(t_{m+1})$ odd since the domain of integration is the circle. The region of integration being symmetric, the integral vanishes. For the more general case of having more than one isolated chord on top, say n of them, we would get $2+n-1$ integrals of either of the forms discussed above, which have been showed to vanish, therefore giving a vanishing contribution to such more complicated mixed tangle chord diagrams. Thus any mixed tangle chord diagram with what would have been isolated chords in the unframed case on top has a vanishing coefficient. We would similarly show that the same result holds for isolated chords at the bottom. Thus all tangle chord diagrams with mixed chords vanish. We are left with $Z_f(nS^1) = nS^1$. The second proof uses the invariance of Z_f under moves other than the cancellation of critical points move. Using this property, we can write the framed Kontsevich integral of n unlinked circles as the framed Kontsevich integral of n parallel circles of same radius $a > M$ separated by a distance $\Lambda \gg a$. For Λ very large, chords stretching from one circle to another yield a negligible contribution to the Kontsevich integral. Equivalently, one could have put all n circles on top of each other and we would not have had chords between them in the expression for $Z_f(nS^1)$, thereby yielding the same result. We are left with:

$$Z_f(nS^1) = Z_f[\mathrm{M}](nS^1) =$$ 

(3.34)



(3.35)

□

3.1.2 $Z_f(L)$: General case

We now tackle the general case: T is non trivial.

Theorem 3.1.2.1 ([ChDu], [CL])**.** For a knot K and a distinguished tangle T: $Z(K) = Z_T(K)$

Proposition 3.1.2.2. For L a link and T a distinguished tangle, then $Z_f(L) = Z_{f,T}(L)$.

Proof. The preceding Theorem is valid for links as well. Indeed, the proof consists in squeezing the tangle T in a window of width ϵ and to let $\epsilon \to 0$. In the limit of very small values of ϵ, chords between T and the rest of L have a negligible contribution to $Z_f(L)$. More generally, tangle chord diagrams with at least one such long chord have a negligible contribution. In the framed case, for L with $2n$ local extrema, it suffices to write:

$$Z_f(L) = Z_f[\mathrm{M}](L) = \frown^{n} \times SZ(L^{\,M\text{-resolved}}) \times \smile^{n} \tag{3.36}$$

If we consider $L^{\,M\text{-resolved}}$, then the part of L that constitutes T is also M-resolved. Thus by Theorem 3.1.2.1, we have:

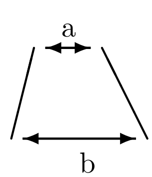
$$Z_f(L) = \frown^{n} \times SZ_{T^{\mathrm{M\text{-}resolved}}}(L^{\,M\text{-resolved}}) \times \smile^{n} \tag{3.37}$$

which is equal to $Z_{f,T}(L)$ by definition of this latter. □

3.2 The statement $\Delta\hat{Z}_f = \hat{Z}_f\Delta$

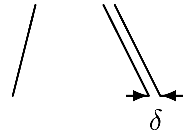
3.2.1 The doubling operator Δ

Before setting to prove the relation $\hat{Z}_f\Delta = \Delta\hat{Z}_f$ ([LM2], [LM3], [LM4], [LM5], [LM6]), we remind the reader what Δ means. We first recall the well-known result $Z(\Delta T) = \Delta Z(T)$ if T is a tangle without local extrema [LM2]. We illustrate this equality with the simple tangle T:

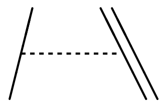
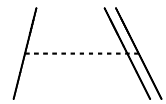


(3.38)

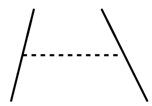
We now double the right strand of T to get another tangle ΔT whose Kontsevich integral we would like to compute. Consider:

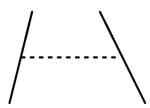
$\Delta T =$  (3.39)

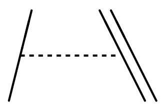
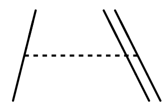
We will display only the degree 1 terms of $Z(\Delta T)$:

$Z(\Delta T) = \cdots \pm \frac{1}{2\pi i} \log \frac{a}{b}$  $\pm \frac{1}{2\pi i} \log \frac{a+\delta}{b+\delta}$  $+ \cdots$

while:

$\Delta Z(T) = \cdots + \Delta \Big(\pm \frac{1}{2\pi i} \log \frac{a}{b}$  $\Big) + \cdots$ (3.40)

$= \cdots \pm \frac{1}{2\pi i} \log \frac{a}{b} \cdot \Delta$  $+ \cdots$ (3.41)

$= \cdots \pm \frac{1}{2\pi i} \log \frac{a}{b}$  $\pm \frac{1}{2\pi i} \log \frac{a}{b}$  $+ \cdots$ (3.42)

which shows that $Z(\Delta T) = \Delta Z(T)$ is true only in the limit $\delta \to 0$. Thus in general writing ΔT means the replica of the tangle T is geometrically infinitesimally close to T itself and thus can be considered to be analytically coincident with the domain of T. We adopt the notation $\Delta(\delta)$ to denote the map on tangles that creates a replica of a tangle a distance δ off of it. Some ambiguity arises concerning the position of the replica, whether it should be to the right or to the left of an original tangle without local extrema, and what convention is to be used when there are local extrema. Typically it will be clear from the context what $\Delta(\delta)$ does. Irrespective of the particular $\Delta(\delta)$, one has:

$$Z(\Delta T) = Z(\lim_{\delta \to 0} \Delta(\delta) T) \tag{3.43}$$

which is independent of the distance between the original tangle and its copy. We will call ΔT a doubled tangle and $\Delta(\delta)T$ a δ-doubled tangle for $\delta \neq 0$.

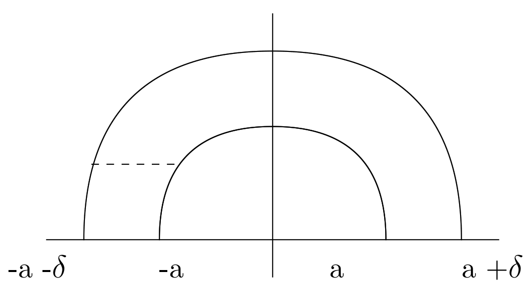
Lemma 3.2.1.1. ([LM2]) $Z(\Delta T) = \Delta Z(T)$ if T has no local extrema, nor does it have pinched extremities.

Proof. It suffices to write:

$$Z(\Delta T) = Z(\lim_{\delta \to 0} \Delta(\delta) T) = \Delta Z(T) \tag{3.44}$$

□

Warning 3.2.1.2. The statement $Z(\Delta T) \neq \Delta Z(T)$ if T has a local extremum is properly written $Z(\Delta(\delta)T) \neq \Delta Z(T)$ if $\delta \neq 0$. Pinched extremities have not been considered in the literature so far, and a same remark would hold in that case. In words, the vanishing of the coefficient of chords with feet on parallel δ-doubled strands is no longer true in case we double a local extremum or a pinched extremity. Without loss of generality, consider a tangle consisting of a single local maximum, its feet being separated by a distance $2a$. By invariance of Z under horizontal deformations including tweaking local extrema into needles, it suffices to consider Z of a semi-circle of radius a. At the first order already, the coefficient of:

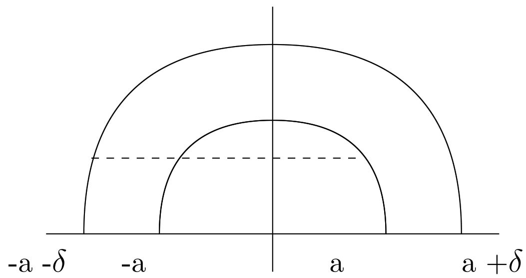


(3.45)

is:

$$\begin{aligned} &\frac{1}{(2\pi i)} \log \frac{\sqrt{(a+\delta)^2 - a^2} - \sqrt{a^2 - a^2}}{\delta} \\ &= \frac{1}{(4\pi i)} \log \frac{2a+\delta}{\delta} \end{aligned} \tag{3.46}$$

The chord diagram with a same short chord on the right hand side has the same coefficient. For a δ-doubled local max, we also have long chords of the form:

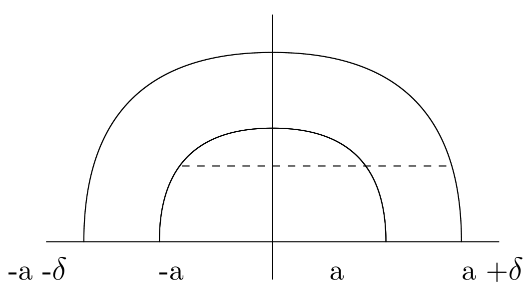


(3.47)

with the following coefficient in the Kontsevich integral, using the same orientations as above for consistency:

$$
\begin{aligned}
-\frac{1}{(2\pi i)}\log\frac{\sqrt{(a+\delta)^2-a^2}+\sqrt{a^2-a^2}}{2a+\delta} \\
&= -\frac{1}{(2\pi i)}\log\frac{\sqrt{\delta}\sqrt{2a+\delta}}{2a+\delta} \\
&= -\frac{1}{(4\pi i)}\log\frac{\delta}{2a+\delta} \\
&= \frac{1}{(4\pi i)}\log\frac{2a+\delta}{\delta} \qquad (3.48)
\end{aligned}
$$

The chord diagram with a long chord:



(3.49)

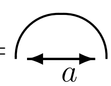
has the same coefficient. What we have so far is that long chords and short chords have the same coefficient, and thus the first order term of the Kontsevich integral of a δ-doubled local max is equal to $(1/(\pi i))\log(2a+\delta)/\delta$, which is non zero. What we have shown is that the Kontsevich integral of a δ-doubled local max is non trivial, and likewise the Kontsevich integral of a δ-doubled local min is non trivial either. This means in particular that $\Delta Z(T) = Z(\Delta(\delta)T)$ is not true of tangles T with local extrema since the statement is not even true in first order.

3.2.2 The relation between $\Delta\hat{Z}_f$ and $\hat{Z}_f\Delta$

By definition of Δ, we can now make sense of $Z_f(\Delta K)$. The following Lemma is a statement about $Z_f[\mathrm{M}](\Delta(\delta)T)$. The Proposition that follows proves that $Z_f(\Delta K) = \Delta Z_f(K)$.

Lemma 3.2.2.1. ([LM2]) If T has a local extremum, $M > 0$, $\delta > 0$, then $Z_f[\mathrm{M}](\Delta(\delta)T) \neq \Delta Z_f[\mathrm{M}](T)$.

Proof. It suffices to consider the tangle:

$T_a =$ 

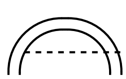
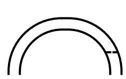
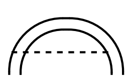
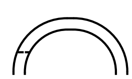
(3.50)

a semi-circle of radius a. We have:

$Z_f[\mathrm{M}](\Delta(\delta)T_a) =$  $^2 \times SZ((\Delta(\delta)T_a)^{M\text{-resolved}})$ (3.51)

$$= SZ(\Delta(\delta)T_a) + \text{other terms} \tag{3.52}$$

where "other terms" are tangle chord diagrams with at least one chord that would have been isolated on $\Delta(\delta)T_a$ in the unframed case. Thus in degree 1, the only tangle chord diagram with a chord between T_a and its replica is coming from $SZ(\Delta(\delta)T_a)$ where we have 4 occurrences of such a term:

 ,  ,  and  (3.53)

with overall coefficient

$$\frac{1}{\pi i}\log((2a+\delta)/\delta) \tag{3.54}$$

On the other hand:

$$\Delta Z_f[\mathrm{M}](T_a) = \Delta\Big(\frown \times SZ(T_a{}^{M\text{-resolved}})\Big) \tag{3.55}$$

$$= \Delta\Big(\frown \times SZ(\overset{M}{\underset{a}{)(}})\Big) \tag{3.56}$$

$$= \Delta \frown \times \Delta SZ(\overset{M}{\underset{a}{)(}}) \tag{3.57}$$

$$= \Delta \frown \times SZ(\Delta \overset{M}{\underset{a}{)(}}) \tag{3.58}$$

$$= \Delta \frown \times \Big(\alpha \lim_{\epsilon\to 0} \overset{\epsilon}{))}\cdots((\ + \alpha\ ))\cdots((\ + \cdots\Big) \tag{3.59}$$

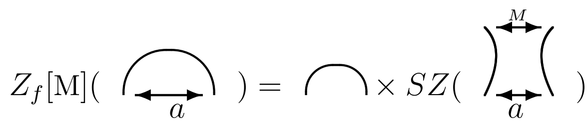
with $\alpha = \lim_{\epsilon\to 0} -1/2\pi i \log((M+\epsilon)/(a+\epsilon)) = -1/2\pi i \log M/a$ (note that this is one operation we wouldn't have been able to do at local extrema), yielding an overall coefficient of $-1/\pi i \log(M/a)$ for the degree one tangle chord diagram with one chord between T_a and its replica. This is finite, whereas (3.54) is large for small values of δ. Not only are the coefficients of $Z_f(\Delta(\delta)T_a)$ and $\Delta Z_f(T_a)$ different, they have unlike behaviors as δ becomes very small. $\square$

In the preceding Lemma we studied the limiting behavior of $Z_f[\mathrm{M}](\Delta(\delta)T)$ as $\delta \to 0$ and saw that it was different from $\Delta Z_f[\mathrm{M}](T)$. However this does not say anything about $Z_f[\mathrm{M}](\Delta T)$ as it equals $Z_f[\mathrm{M}](\lim_{\delta\to 0}\Delta(\delta)T)$.

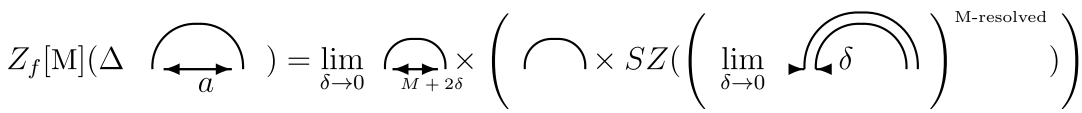
Note that the result that follows was initially proved in [LM2] for the parameter-free $\hat{Z}_f$ and what we call the δ-doubling map in this work.

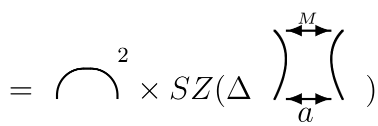
Proposition 3.2.2.2. $\Delta Z_f(K) = Z_f(\Delta K)$.

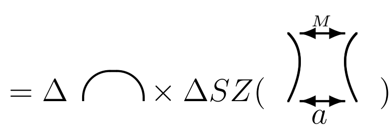
Proof. On knots, $Z_f = Z_f[\mathrm{M}]$ for all $M > 0$. Fix $M > 0$. By multiplicativity of $Z_f[\mathrm{M}]$, it suffices to consider the doubling operation on one local extremum and one pinched extremity. Without loss of generality we can consider a local maximum. The proof for a roof-shaped extremity is exactly the same. We have:

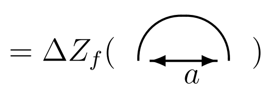

(3.60)

Thus if we double the local max, we get:


(3.61)


(3.62)


(3.63)


(3.64)

$\square$

Corollary 3.2.2.3. If L is an e-components framed oriented link, K_i a component of L, $1 \le i \le e$, Δ_i the doubling map on K_i, then

$$\Delta_i Z_f(L) = Z_f(\Delta_i L) \tag{3.65}$$

Proof. There are three types of chords to consider: chords none of whose feet are on K_i, for which the contribution to $Z_f(L)$ make (3.65) true, chords that have only one foot on K_i, with a corresponding contribution to $Z_f(L)$ that make (3.65) hold by definition of Δ_i, and chords both of whose feet are on K_i, for which we invoke Proposition 3.2.2.2 above. $\square$

Corollary 3.2.2.4.

$$\Delta\nu^{-1} = \nu^{-1} \otimes \nu^{-1} \tag{3.66}$$

Proof. It follows from the previous proposition applied to $K = U$ and the fact that $\nu = Z_f(U)^{-1}$ that:

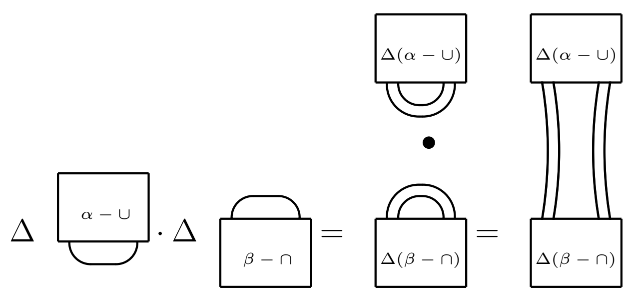
$$\nu^{-1} \otimes \nu^{-1} = Z_f(\Delta U) = \Delta Z_f(U) = \Delta\nu^{-1} \tag{3.67}$$

□

Proposition 3.2.2.5. For K and K' two knots, $\alpha = \sum_P c_P K_P$ and $\beta = \sum_Q d_Q K'_Q$ in $\hat{\mathcal{A}}(K)$ and $\hat{\mathcal{A}}(K')$ respectively, we have:

$$\Delta\alpha \cdot \Delta\beta = \Delta(\alpha \cdot \beta) \tag{3.68}$$

where the product $\Delta\alpha \cdot \Delta\beta$ is defined by:



(3.69)

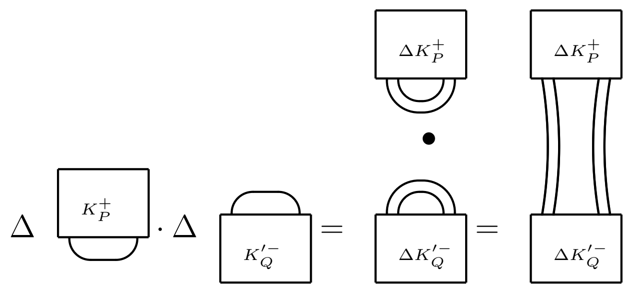
Proof. On the one hand:

$$\Delta\alpha \cdot \Delta\beta = \sum c_P d_Q \Delta K_P \cdot \Delta K'_Q \tag{3.70}$$

On the other hand:

$$\Delta(\alpha \cdot \beta) = \sum c_P d_Q \Delta(K_P \cdot K'_Q) \tag{3.71}$$

We show $\Delta K_P \cdot \Delta K'_Q = \Delta(K_P \cdot K'_Q)$ from which the result will follow. If we isolate the local minimum of K that will be used in the product, removing this local minimum the remaining tangle is denoted by K^+. Likewise if we isolate the local maximum of K' that will be used in the product, removing this local maximum the remaining tangle is denoted by K'^-. Then we consider:



(3.72)

$$= \Delta(K_P \cdot K_Q) \tag{3.73}$$

□

Lemma 3.2.2.6. $\left(\Delta\nu^{-1}\right)^{-1} = \Delta\nu$

Proof. It follows from the previous Proposition that:

$$\Delta\nu^{-1} \cdot \Delta\nu = \Delta(\nu^{-1} \cdot \nu) = \Delta 1 = 1 \tag{3.74}$$

□

Corollary 3.2.2.7. ([LM4]) $\Delta\nu = \nu \otimes \nu$.

Proof. From the previous Lemma $\Delta\nu = (\Delta\nu^{-1})^{-1}$. By Corollary 3.2.2.4 this equals $(\nu^{-1} \otimes \nu^{-1})^{-1}$, which equals $\nu \otimes \nu$. □

Corollary 3.2.2.8. $\Delta(\nu^p) = \nu^p \otimes \nu^p$ for all $p \in \mathbb{Z}$.

Proof. Fix $p \in \mathbb{Z}$. By Proposition 3.2.2.5, we have $\Delta(\nu^p) = (\Delta\nu)^p$. By Lemmas 3.2.2.4 and 3.2.2.7, we have that this equals $(\nu \otimes \nu)^p$, which is further equal to $\nu^p \otimes \nu^p$. □

Theorem 3.2.2.9. ([LM2]) For an e-component link L, Δ_i the doubling map on the i-th component, $1 \leq i \leq e$ we have $\hat{Z}_f(\Delta_i L) = \Delta_i \hat{Z}_f(L)$.

Proof. By definition, one has:

$$\hat{Z}_f(L) = \nu^{m_1} \otimes \cdots \otimes \nu^{m_e} \cdot Z_f(L) \tag{3.75}$$

Thus if one doubles the i-th component, $1 \leq i \leq e$:

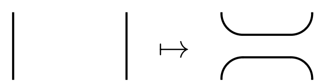
$$\begin{aligned}
\Delta_i \hat{Z}_f(L) &= \Delta_i(\nu^{m_1} \otimes \cdots \otimes \nu^{m_e} \cdot Z_f(L)) && (3.76)\\
&= \nu^{m_1} \otimes \cdots \otimes \Delta_i(\nu^{m_i}) \otimes \cdots \otimes \nu^{m_e} \cdot \Delta_i Z_f(L) && (3.77)\\
&= \nu^{m_1} \otimes \cdots \otimes (\Delta_i \nu)^{m_i} \otimes \cdots \otimes \nu^{m_e} \cdot \Delta_i Z_f(L) && (3.78)\\
&= \nu^{m_1} \otimes \cdots \otimes 1 \otimes \cdots \otimes \nu^{m_e} \cdot \Big((\Delta_i \nu)^{m_i} \cdot \Delta_i Z_f(L) \Big) && (3.79)\\
&= \nu^{m_1} \otimes \cdots \otimes 1 \otimes \cdots \otimes \nu^{m_e} \cdot \Big((\nu \otimes \nu)^{m_i} \cdot \Delta_i Z_f(L) \Big) && (3.80)\\
&= \nu^{m_1} \otimes \cdots \otimes (\nu \otimes \nu)^{m_i} \otimes \cdots \otimes \nu^{m_e} \cdot \Delta_i Z_f(L) && (3.81)\\
&= \nu^{m_1} \otimes \cdots \otimes (\nu^{m_i} \otimes \nu^{m_i}) \otimes \cdots \otimes \nu^{m_e} \cdot \Delta_i Z_f(L) && (3.82)\\
&= \nu^{m_1} \otimes \cdots \otimes (\nu^{m_i} \otimes \nu^{m_i}) \otimes \cdots \otimes \nu^{m_e} \cdot Z_f(\Delta_i L) && (3.83)\\
&= \hat{Z}_f(\Delta_i L) && (3.84)
\end{aligned}$$

where in (3.78) we have used Lemma 3.2.2.7, and in (3.82) we used $\Delta Z_f = Z_f \Delta$ (Proposition 3.2.2.2). This being true for any $1 \leq i \leq e$, the result follows: $\hat{Z}_f(\Delta) \equiv \Delta \hat{Z}_f$. $\square$

Chapter 4

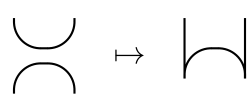
Behavior of the Kontsevich integral under band sum moves

In [LM3], [LM5], [LM6], Le, Murakami and Ohtsuki considered the band sum move of one link component over another by locally considering the two link components of interest to be side by side. The band sum thus looks horizontal as in:



(4.1)

Kontsevich introduced a modified Kontsevich integral in his seminal paper [K] that turned out to be multiplicative under connected sum. The band sum move being somewhat similar in nature to a connected sum, one could investigate whether this modified integral is also well-behaved under band sum moves for which link components of interest are locally on top of each other. In this case the band sum looks like:

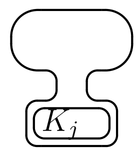


(4.2)

We claim that once we generalize Kontsevich's modified integral for knots to links this is indeed the case. A band sum move of some component K_i over some other component K_j of a link L is effected by doubling K_j into itself and a copy that we will denote by $K_j^{(2)}$ and then by connect summing K_i with this copy $K_j^{(2)}$.

4.1 Unlinked case

We assume both K_i and K_j have orientations such that the band sum move of K_i over K_j results in the addition of K_i to K_j in the terminology of [RK1]. The case of reverse orientations will be treated in Section 8. We want to compute the Kontsevich integral of:

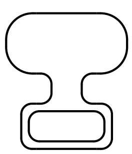
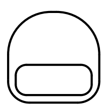
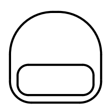
$K_i \# \Delta K_j \left\{ \right.$  $K_i \# K_j^{(2)}$ (4.3)

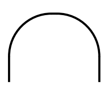
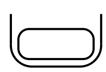
where $K_i \# \Delta K_j$ is a short-hand for $(K_i \# K_j^{(2)}) \amalg K_j$. We have:

Proposition 4.1.1. If K_i and K_j are unlinked:

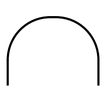
$$\hat{Z}_f(K_i \# \Delta K_j) = \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \Delta \hat{Z}_f(K_j) \tag{4.4}$$

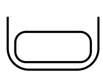
Proof. We compute:

$\hat{Z}_f($  $) = \hat{Z}_f($  $) = \hat{Z}_f[\mathrm{M}]($  $)$ (4.5)

$= \hat{Z}_f[\mathrm{M}]($  $) \times \hat{Z}_f[\mathrm{M}]($  $)$ (4.6)

where in the second line:

$\hat{Z}_f[\mathrm{M}]($  $) := \hat{Z}_f[\mathrm{M}](K_i^+)$ (4.7)

$\hat{Z}_f[\mathrm{M}]($  $) := \hat{Z}_f[\mathrm{M}](K_j \amalg K_j^{(2)-})$ (4.8)

We seek to rewrite (4.6). It suffices to consider:

$$
\begin{aligned}
\hat{Z}_f(K_i) &= \hat{Z}_f[\mathrm{M}](K_i) \\
&= \hat{Z}_f[\mathrm{M}](K_i^+) \times \hat{Z}_f[\mathrm{M}](\quad \cup \quad) && (4.9) \\
&= \hat{Z}_f[\mathrm{M}](K_i^+) \times Z_f[\mathrm{M}](\quad \cup \quad) && (4.10)
\end{aligned}
$$

as well as:

$$
\begin{aligned}
\hat{Z}_f(\Delta K_j) = \hat{Z}_f[\mathrm{M}](\Delta K_j) &= \hat{Z}_f[\mathrm{M}](\quad \cap \quad) \times \hat{Z}_f[\mathrm{M}](K_j \amalg K_j^{(2)-}) && (4.11) \\
&= \nu \otimes 1 \cdot Z_f[\mathrm{M}](\quad \cap \quad) \times \hat{Z}_f[\mathrm{M}](K_j \amalg K_j^{(2)-}) && (4.12)
\end{aligned}
$$

where the first component in the tensor product refers to $K_j^{(2)}$ and the second to K_j proper. Then:

$$
\begin{aligned}
Z_f[\mathrm{M}](\quad \cap \quad) \times \hat{Z}_f[\mathrm{M}](K_j \amalg K_j^{(2)-}) &= \nu^{-1} \otimes 1 \cdot \hat{Z}_f(\Delta K_j) && (4.13) \\
&= \nu^{-1} \otimes 1 \cdot \Delta \hat{Z}_f(K_j) && (4.14)
\end{aligned}
$$

by Theorem 3.2.2.9 which states that $\Delta\hat{Z}_f = \hat{Z}_f(\Delta)$. Then we can rewrite (4.6) as:

$$\hat{Z}_f[\mathrm{M}](\bigcap) \times \hat{Z}_f[\mathrm{M}](\quad)$$

$$=\hat{Z}_f[\mathrm{M}](\bigcap) \times \hat{Z}_f[\mathrm{M}](\mid \quad \mid) \times \hat{Z}_f[\mathrm{M}](\quad) \tag{4.15}$$

$$=\hat{Z}_f[\mathrm{M}](\bigcap) \times SZ(\mid \quad \mid) \times \hat{Z}_f[\mathrm{M}](\quad) \tag{4.16}$$

$$=\hat{Z}_f[\mathrm{M}](\bigcap) \times \Big(SZ(\quad) \times \smile \# \frown \times SZ(\quad)\Big) \times \hat{Z}_f[\mathrm{M}](\quad) \tag{4.17}$$

$$= \Big(\hat{Z}_f[\mathrm{M}](K_i^+) \times Z_f[\mathrm{M}](\smile)\Big) \# \Big(Z_f[\mathrm{M}](\frown) \times \hat{Z}_f[\mathrm{M}](K_j \amalg K_j^{(2)-})\Big) \tag{4.18}$$

$$= \Big(\hat{Z}_f[\mathrm{M}](K_i^+) \times Z_f[\mathrm{M}](\smile)\Big) \# \Big(\nu^{-1} \otimes 1 \cdot \Delta\hat{Z}_f(K_j)\Big) \tag{4.19}$$

$$= \hat{Z}_f(K_i) \cdot (\nu^{-1} \otimes 1 \cdot \Delta\hat{Z}_f(K_j)) \tag{4.20}$$

$$= \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \Delta\hat{Z}_f(K_j) \tag{4.21}$$

so we have essentially proved that we have:

$$\hat{Z}_f(K_i \# \Delta K_j) = \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \Delta\hat{Z}_f(K_j) \tag{4.22}$$

□

Remark 4.1.2. Observe that (4.22) written in the form:

$$\hat{Z}_f(K_i \# \Delta K_j) = \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \hat{Z}_f(\Delta K_j) \tag{4.23}$$

is surprisingly similar to the statement [K]:

$$\hat{Z}_f(K_i \# K_j) = \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \hat{Z}_f(K_j) \tag{4.24}$$

One now defines, as Kontsevich did, a modified version of the hatted Kontsevich integral.

Definition 4.1.3. We define $\widetilde{Z}_f$ to be $\hat{Z}_f$ with the provision that every link component in its argument has a factor of ν^{-1} multiplying it. Equivalently:

$$\widetilde{Z}_f(L) = \nu^{-1} \otimes \cdots \otimes \nu^{-1} \cdot \hat{Z}_f(L) \tag{4.25}$$

Proposition 4.1.4. ([K]) If K_i and K_j are unlinked:

$$\widetilde{Z}_f(K_i \# \Delta K_j) = \widetilde{Z}_f(K_i) \cdot \Delta \widetilde{Z}_f(K_j) \tag{4.26}$$

Proof. Multiplying both sides of (4.22) by $\nu^{-1} \otimes \nu^{-1}$ where the first ν^{-1} factor acts on $K_i \# K_j^{(2)}$, the second on K_j, we get:

$$\nu^{-1} \otimes \nu^{-1} \cdot \hat{Z}_f(K_i \# \Delta K_j) = \nu^{-1} \otimes 1 \cdot \hat{Z}_f(K_i) \cdot \nu^{-1} \otimes \nu^{-1} \cdot \Delta \hat{Z}_f(K_j) \tag{4.27}$$

Now using:

$$\nu^{-1} \otimes \nu^{-1} \cdot \Delta \hat{Z}_f(K_j) = \Delta(\nu^{-1}) \cdot \Delta \hat{Z}_f(K_j) \tag{4.28}$$

$$= \Delta(\nu^{-1} \cdot \hat{Z}_f(K_j)) \tag{4.29}$$

$$= \Delta \widetilde{Z}_f(K_j) \tag{4.30}$$

we get:

$$\widetilde{Z}_f(K_i \# \Delta K_j) = \widetilde{Z}_f(K_i) \cdot \Delta \widetilde{Z}_f(K_j) \tag{4.31}$$

□

On the other hand for two unlinked knots:

Proposition 4.1.5. If K_i and K_j are unlinked:

$$\widetilde{Z}_f(K_i \amalg K_j) = \widetilde{Z}_f(K_i) + \widetilde{Z}_f(K_j) \tag{4.32}$$

Proof. For two unlinked knots, we have:

$$\hat{Z}_f(K_i \amalg K_j) = \hat{Z}_f(K_i) + \hat{Z}_f(K_j) \tag{4.33}$$

That this is true is immediate. It suffices to compute $\hat{Z}_f(K_i \amalg K_j)$ in two ways. The first is performed by putting the two knots side by side, so that:

$$\hat{Z}_f(K_i \amalg K_j) = \hat{Z}_f(K_i) + \hat{Z}_f(K_j) + \text{terms with chords between } K_i \text{ and } K_j \tag{4.34}$$

The second way to perform this computation is by putting one knot above the other, in which case we do not get chords between one component and the other, and by isotopy invariance of $\hat{Z}_f$ it follows that in the above sum the terms with chords between the two components vanish and we are left with (4.33). From there, by modifying the hatted Kontsevich integral for each component, we get:

$$\begin{aligned}
&\nu^{-1} \otimes \nu^{-1} \cdot \hat{Z}_f(K_i \amalg K_j) \\
&\qquad = (\nu^{-1} \cdot) \otimes (\nu^{-1} \cdot) \hat{Z}_f(K_i \amalg K_j) && (4.35) \\
&\qquad = (\nu^{-1} \cdot) \otimes (\nu^{-1} \cdot) \Big(\hat{Z}_f(K_i) + \hat{Z}_f(K_j) \Big) && (4.36) \\
&\qquad = \nu^{-1} \cdot \hat{Z}_f(K_i) + \nu^{-1} \cdot \hat{Z}_f(K_j) && (4.37)
\end{aligned}$$

or:

$$\widetilde{Z}_f(K_i \amalg K_j) = \widetilde{Z}_f(K_i) + \widetilde{Z}_f(K_j) \tag{4.38}$$

□

Remark 4.1.6. Before the band sum move, we have:

$$\widetilde{Z}_f(K_i \amalg K_j) = \widetilde{Z}_f(K_i) + \widetilde{Z}_f(K_j) \tag{4.39}$$

and after the band sum move:

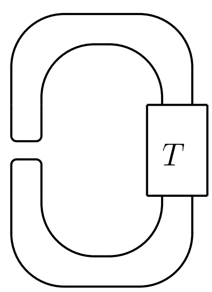
$$\widetilde{Z}_f(K_i \# \Delta K_j) = \widetilde{Z}_f(K_i) \cdot \Delta \widetilde{Z}_f(K_j) \tag{4.40}$$

The essence of these two statements is that whatever chords we have on the j-th component in the expression for $\widetilde{Z}_f$ before band sum move are doubled after the move.

4.2 Linked case

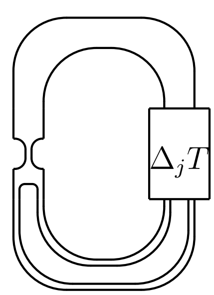
We now treat the general case wherein K_i and K_j are linked to other components of the given link they are part of, and are linked between themselves. In this case the desired map on $\hat{Z}_f$ that should be induced by the band sum move on link components turns out to be non-existent. A way around this difficulty amounts to fixing a small window in which only strands from the link components involved in the band sum show, and are disconnected from the rest of the link as it were, or "frozen", and then a map on $\hat{Z}_f$ induced by the band sum on link components with a fixed window can be worked out ([DM]). In [O], [LM3], [LM5], [LM6] the window is fixed so that the two link components involved in the band sum are side by side. We chose a window in which they are locally on top of each other.

If L is the link under consideration, we can arrange that by isotopy invariance of $\hat{Z}_f$ we just have to compute:

$\hat{Z}_f(L) = \hat{Z}_f($ 

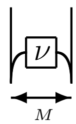
$)$ (4.41)

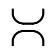
where the tangle T represents the link L minus one local max from K_i and one local min from K_j, both local extrema having been moved to a side using the isotopy invariance of $\hat{Z}_f$. Using the same trick, it suffices to compute:

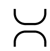
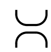
$\hat{Z}_f(L') = \hat{Z}_f[\mathrm{M}](L') = \hat{Z}_f[\mathrm{M}]($ 

$)$ (4.42)

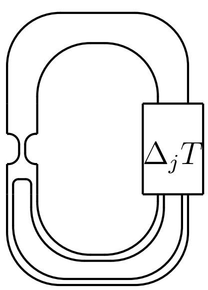
Proposition 4.2.1. If L is an e-components framed oriented link, L' is the link resulting from the band sum of a link component K_i over some other link component K_j, $1 \leq i, j \leq e$, then:

$\hat{Z}_f(L') =$ 

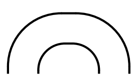
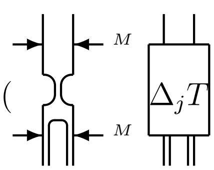
$\cdot\, \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.43)

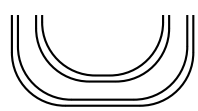
where $\cdot$ is the product on Z induced by the long chords lemma ([ChDu]) and $L \setminus$  is the link L from which its portion involving K_i and K_j to the left that looks like  has been severed.

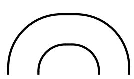
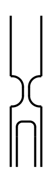
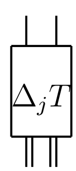
Proof.

$\hat{Z}_f[\mathrm{M}]($ 

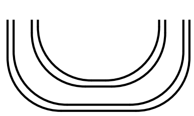
$)$ (4.44)

$= \hat{Z}_f[\mathrm{M}]($  $) \times \hat{Z}_f[\mathrm{M}]($ 

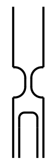
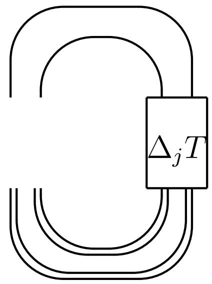
$) \times \hat{Z}_f[\mathrm{M}]($  $)$

$= \hat{Z}_f[\mathrm{M}]($  $) \times \Big(\hat{Z}_f[\mathrm{M}]($  $) + \hat{Z}_f[\mathrm{M}]($ 

$) \Big)$

$\times\, \hat{Z}_f[\mathrm{M}]($  $)$

where in the second step we have used the long chords lemma ([ChDu]). In the same paper it is shown that the long chords lemma induces a product on tangle chord diagrams written $\cdot$, that in this case would read:

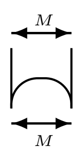
$\hat{Z}_f[\mathrm{M}]($  $) \cdot \hat{Z}_f[\mathrm{M}]($ 

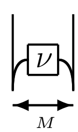
$)$ (4.45)

Note that this product is not the product on $\hat{\mathcal{A}}$ induced by the connected sum of S^1's, nor is it a concatenation per se. All that is left is to compute the first term in the product as the second exactly equals:

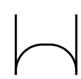
$\hat{Z}_f[\mathrm{M}](\Delta_j\Big(L \setminus$  $\Big)) = \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.46)

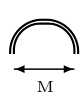
Lemma 4.2.2.

$\hat{Z}_f[\mathrm{M}]($ 

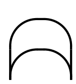
$) =$ 

(4.47)

Proof. We present two proofs. The first proof consists in computing $Z_f($  $)$ exactly. It suffices to write:



$= Z_f[\mathrm{M}]($  $)$ (4.48)

$= Z_f[\mathrm{M}]($  $)$ (4.49)

$= Z_f[\mathrm{M}]($  $) \times Z_f[\mathrm{M}]($  $)$ (4.50)

$=$  $\times Z_f[\mathrm{M}]($  $)$ (4.51) where in the first step we used the fact that the two strands are analytically coincident and thus are M-resolved together. Moreover, since the base is chosen to be M, $Z_f[\mathrm{M}](\;)$ is trivial. In the second step we used the invariance of $Z_f[\mathrm{M}]$ under the stretching move thus displayed. It follows that:

$$Z_f(\quad) = \qquad (4.52)$$

and:

$$\hat{Z}_f[\mathrm{M}](\quad) = 1 \otimes \nu \cdot Z_f[\mathrm{M}](\quad) = \boxed{\nu} \qquad (4.53)$$

Second proof:

$$\hat{Z}_f[\mathrm{M}](\quad) = \qquad \times \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.54)$$

$$= \quad \# \quad \times \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.55)$$

$$= \left(\hat{Z}_f[\mathrm{M}](\quad) \# Z_f[\mathrm{M}](\quad) \right) \times \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.56)$$

$$= \hat{Z}_f[\mathrm{M}](\quad) \# Z_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.57)$$

$$= Z_f[\mathrm{M}](\quad) \# \nu^{-1} \otimes 1 \cdot \hat{Z}_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.58)$$

$$= \hat{Z}_f[\mathrm{M}](\quad) \# \nu^{-1} \otimes 1 \cdot \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.59)$$

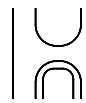
$$= \#^{K_i}_{K_j^{(2)}} \nu^{-1} \otimes 1 \cdot \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.60)$$

where in the last relation we denoted by $\#^{K_i}_{K_j^{(2)}}$ a connected sum between K_i and $K_j^{(2)}$ and we used the following lemma:

Lemma 4.2.3.

$$\hat{Z}_f[\mathrm{M}](\quad) \# \hat{Z}_f[\mathrm{M}](\quad) = \#^{K_i}_{K_j^{(2)}} \hat{Z}_f[\mathrm{M}](\quad) \qquad (4.61)$$

Proof. Consider the following tangle:



(4.62)

Then:

$$\hat{Z}_f[\mathrm{M}](\quad) = \hat{Z}_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad) \tag{4.63}$$

so that:

$$\#^{K_i}_{K_j^{(2)}} \hat{Z}_f[\mathrm{M}](\quad) = \#^{K_i}_{K_j^{(2)}} \Big(\hat{Z}_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad)\Big) \tag{4.64}$$

If in either expression we consider only those tangle chord diagrams none of whose chords have any foot on the straight strand, then:

$$\#^{K_i}_{K_j^{(2)}} \hat{Z}_f[\mathrm{M}](\quad) = \hat{Z}_f[\mathrm{M}](\quad) \# \hat{Z}_f[\mathrm{M}](\quad) \tag{4.65}$$

$\square$

So far we have:

$$\hat{Z}_f[\mathrm{M}](\quad) = \#^{K_i}_{K_j^{(2)}} \nu^{-1} \otimes 1 \cdot \hat{Z}_f[\mathrm{M}](\quad) \tag{4.66}$$

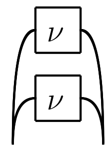
We have:

$$\hat{Z}_f[\mathrm{M}](\quad) = \nu \otimes \nu \cdot Z_f[\mathrm{M}](\Delta \quad) \tag{4.67}$$

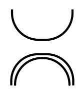
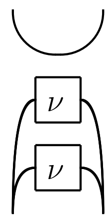
In $Z_f(\Delta \quad)$ there are no non-self chords between one local max and its replica as they would be integrated from M at the top down to the base which is of width M. Thus:

$$Z_f[\mathrm{M}](\Delta \quad) = \tag{4.68}$$

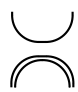
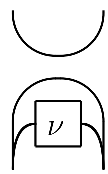
From which it follows that:

$\hat{Z}_f[\mathrm{M}]($  $) =$  (4.69)

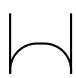
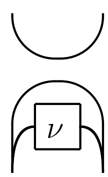
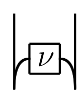
and:

$\hat{Z}_f[\mathrm{M}]($  $) =$  (4.70)

so that:

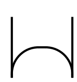
$\nu^{-1} \otimes 1 \cdot \hat{Z}_f[\mathrm{M}]($  $) =$  (4.71)

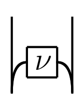
From which we get:

$\hat{Z}_f[\mathrm{M}]($  $) = \#^{K_i}_{K_j^{(2)}}$  $=$  (4.72)

This completes the proof of the Lemma. $\square$

Back to the proof of the Proposition:

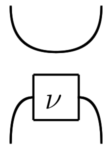
$\hat{Z}_f[\mathrm{M}](L') = \hat{Z}_f[\mathrm{M}]($  $) \cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.73)

$=$  $\cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.74)

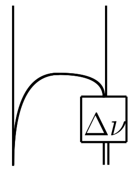
$\square$

Proposition 4.2.4. For L an e-components framed oriented link, L' the link resulting from the band sum move of one component K_i over some other component K_j, $1 \leq i, j \leq e$, then the resulting induced map on $\widetilde{Z}_f$ is given by:

$$\widetilde{Z}_f(L) = \widetilde{Z}_f[\mathrm{M}](L) = \quad \cdot\, \widetilde{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{4.75}$$

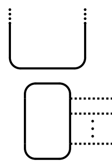


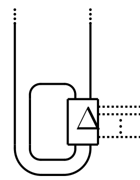
$$\longmapsto \widetilde{Z}_f(L') = \widetilde{Z}_f[\mathrm{M}](L') = \quad \cdot\, \Delta_j \widetilde{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{4.76}$$



which we can symbolically write as:

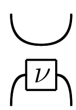
$$\widetilde{Z}_f(L) = \sum \lambda \quad \longmapsto \widetilde{Z}_f(L') = \sum \lambda \tag{4.77}$$



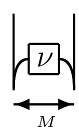


Proof. The previous proposition showed that:

$$\hat{Z}_f(L) = \hat{Z}_f[\mathrm{M}](L) = \quad \cdot\, \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{4.78}$$

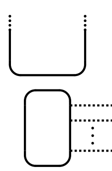


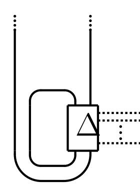
$$\longmapsto \hat{Z}_f(L') = \hat{Z}_f[\mathrm{M}](L') = \quad \cdot\, \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{4.79}$$



We will use this statement and determine what normalization $\widetilde{Z}_f$ leads to a statement of the form:

$$\widetilde{Z}_f(L) = \sum \lambda \quad \longmapsto \widetilde{Z}_f(L') = \sum \lambda$$

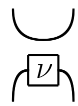


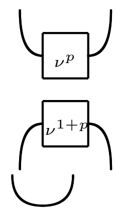


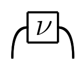
Let $\widetilde{Z}_f$ be $\hat{Z}_f$ with the provision that every link component in the expression of $\hat{Z}_f(L)$

is multiplied by ν^p for some p yet to be determined. Without loss of generality we can work with a two component link. The proof for other link components in addition to those involved in the band sum move is the same but is just a clutter of tensor products. Before band sum move:

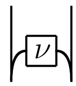
$$\widetilde{Z}_f[M](L) = \nu^p \otimes \nu^p \cdot \hat{Z}_f[\mathrm{M}](L) \qquad (4.80)$$

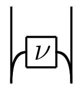
$= \nu^p \otimes \nu^p \cdot$  $\cdot \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.81)

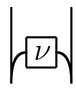
$=$  $\cdot \hat{Z}_f[\mathrm{M}](L \setminus$  $)$ (4.82)

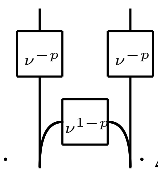
$=$  $\cdot \widetilde{Z}_f[M](L \setminus$  $)$ (4.83)

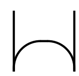
After the band sum move:

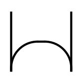
$\widetilde{Z}_f[M](L') = \nu^p \otimes \nu^p \cdot$  $\cdot\, \Delta \hat{Z}_f[\mathrm{M}](L \setminus \asymp)$ (4.84)

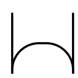
$= \nu^p \otimes \nu^p \cdot$  $\cdot\, \Delta\Big(\nu^{-p} \otimes \nu^{-p} \cdot \widetilde{Z}_f[M](L \setminus \asymp)\Big)$ (4.85)

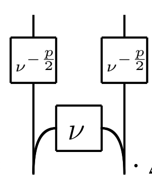
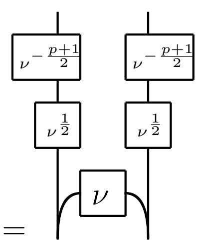
$= \nu^p \otimes \nu^p \cdot$  $\cdot\, \nu^{-p} \otimes \nu^{-p} \otimes \nu^{-p} \cdot \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.86)

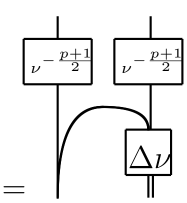
$= \nu^p \otimes \nu^p \cdot$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.87)

$= \nu^p \otimes \nu^p \cdot \nu^{-2p} \otimes \nu^{1-p} \cdot$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.88)

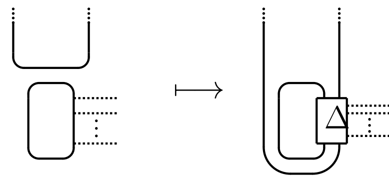
$= \nu^{-p} \otimes \nu \cdot$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.89)

$= \nu^{-p/2} \cdot \nu^{-p/2} \otimes \nu \cdot$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.90)

$=$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp) =$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.91)

$=$  $\cdot\, \Delta \widetilde{Z}_f[M](L \setminus \asymp)$ (4.92)

Thus if we want a statement of the form:



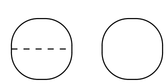
then using (4.83) it follows that we must have $p = -1$, whence the result. $\square$

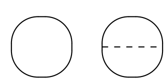
Chapter 5

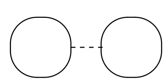
The book notation

5.1 Linking matrix and degree 1 Kontsevich integral

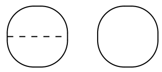
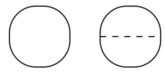
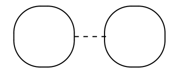
It was originally Kontsevich's idea ([K]) to generalize Gauss' definition of the linking number via an integral, to the algebraic completion of the sum of generalized such integrals valued in tangle chord diagrams. The old observation of Yetter [Y2] that the degree 1 part of the framed Kontsevich integral of a link with labeled components behaves like the linking matrix under band sum move is thus not very surprising, though it has been generally overlooked in most of the literature that tends to focus on knots rather than links. He sees that as a motivation for introducing the book notation that we will cover in the next section. For now we show his claim in the simple case of a two component link $L = K_i \cup K_j$ where the two knots K_i and K_j are trivial and unlinked for the simplicity of exposition. We take the following basis for $\mathcal{A}(S^1 \amalg S^1)$:

$\frac{1}{2}$  (5.1)

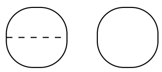
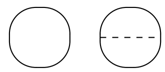
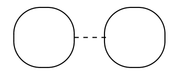
$\frac{1}{2}$  (5.2)

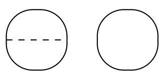
 (5.3)

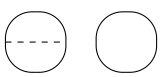
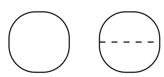
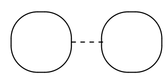
We write:

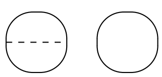
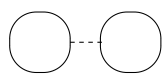
$Z_1(L) = c_i\Big(\frac{1}{2}$  $\Big) + c_j\Big(\frac{1}{2}$  $\Big) + c_{ij}\Big($  $\Big)$ (5.4)

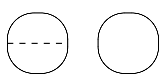
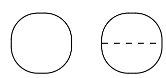
After the band sum move of K_i over K_j, the resulting link is denoted by L' and we have:

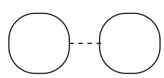
$Z_1(L') = c_i\Big(\frac{1}{2}$  $\Big)' + c_j\Big(\frac{1}{2}$  $\Big)' + c_{ij}\Big($  $\Big)'$ (5.5)

$= c_i\Big(\frac{1}{2}$  $\Big)$

$+ c_j\Big(\frac{1}{2}$  $+\frac{1}{2}$  $\pm$  $\Big)$

$+ c_{ij}\Big(\pm$  $+$  $\Big)$ (5.6)

$= [c_i + c_j \pm 2c_{ij}]\Big(\frac{1}{2}$  $\Big) + c_j\Big(\frac{1}{2}$  $\Big)$

$+ [\pm c_j + c_{ij}]\Big($  $\Big)$ (5.7)

Now observe that in the basis for $\mathcal{A}(S^1 \amalg S^1)$ we chose, the coefficients of Z_1 transform like the linking matrix entries under band sum move:

$$L_i \mapsto L_i' = L_i + L_j \pm 2L_{ij} \tag{5.8}$$

$$L_j \mapsto L_j' = L_j \tag{5.9}$$

$$L_{ij} \mapsto L_{ij}' = L_{ij} \pm L_j \tag{5.10}$$

The matrix congruence that implements the band sum move on both the coefficients of $Z_1(L)$ and the linking matrix is given by:

$$M = \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix} \tag{5.11}$$

We check that if:

$$A = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \tag{5.12}$$

stands for either of

$$\begin{bmatrix} L_i & L_{ij} \\ L_{ji} & L_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c_i & c_{ij} \\ c_{ji} & c_j \end{bmatrix} \tag{5.13}$$

with $L_{ij} = L_{ji}$ and $c_{ij} = c_{ji}$, then:

$$\begin{aligned} M^T A M &= \begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{ii} + a_{jj} \pm 2a_{ij} & a_{ij} \pm a_{jj} \\ a_{ji} \pm a_{jj} & a_{jj} \end{bmatrix} \end{aligned} \tag{5.14}$$

In the following, we seek to generalize such a transformation.

5.2 Books of matrices

Note that in the previous section, as is customary whenever we compute the Kontsevich integral of tangles, the diagrams are tangle chord diagrams and not chord diagrams as elements of $\hat{\mathcal{A}}(\coprod S^1)$ as initially defined by Kontsevich. Indeed, for a link L sliced into n horizontal strips in each of which we have a tangle T_i, $1 \leq i \leq n$ such that $L = T_1 \times \cdots \times T_n$, it is generally understood that when we compute $\hat{Z}_f(L)$ by using the multiplicativity property of the Kontsevich integral $\hat{Z}_f$:

$$\hat{Z}_f(L) = \hat{Z}_f(T_1) \times \cdots \times \hat{Z}_f(T_n) \tag{5.15}$$

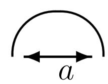
the resulting object $\hat{Z}_f(L)$ is a sum of tangle chord diagrams with coefficients in front of each diagram being obtained from the Kontsevich integral itself. By definition, $\widetilde{Z}_f$ is also multiplicative and $\widetilde{Z}_f(L)$ is also a sum of tangle chord diagrams with complex coefficients. The Kontsevich integral $\widetilde{Z}_f$ of a link L can be written:

$$\widetilde{Z}_f(L) = \sum_{\text{chord diagr. X}} c_X X = \sum_{m \geq 0} \sum_{|X|=m} c_X X \tag{5.16}$$

where by $|X| = m$ we mean that the tangle chord diagram X has chord degree m, and we sum over all such tangle chord diagrams, for all $m \geq 0$. In what follows, we fix $m \geq 1$. We will be working with an e-component link $L = \coprod_{1 \leq l \leq e} K_l$.

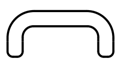
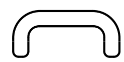
5.2.1 Vertical slicing of tangles

Before slicing links, we have to fully determine where local extrema will be located on any given link. The definitions of $\hat{Z}_f$ and $\widetilde{Z}_f$ each introduce factors of $\nu = Z_f(U)^{-1}$ which will yield additional local extrema on the link L upon acting on $Z_f(L)$. Since those factors of ν are acting on local maxima, it suffices to consider products of the form:

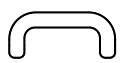
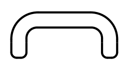
$\nu \cdot Z_f[\mathrm{Q}]($ 

$)$

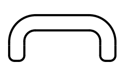
for $Q > 0$. Observe that if we write:

$Z_f($  $) = \sum_{|P| \geq 0} c_P \cdot \Big($  $\Big)_P$ (5.17)

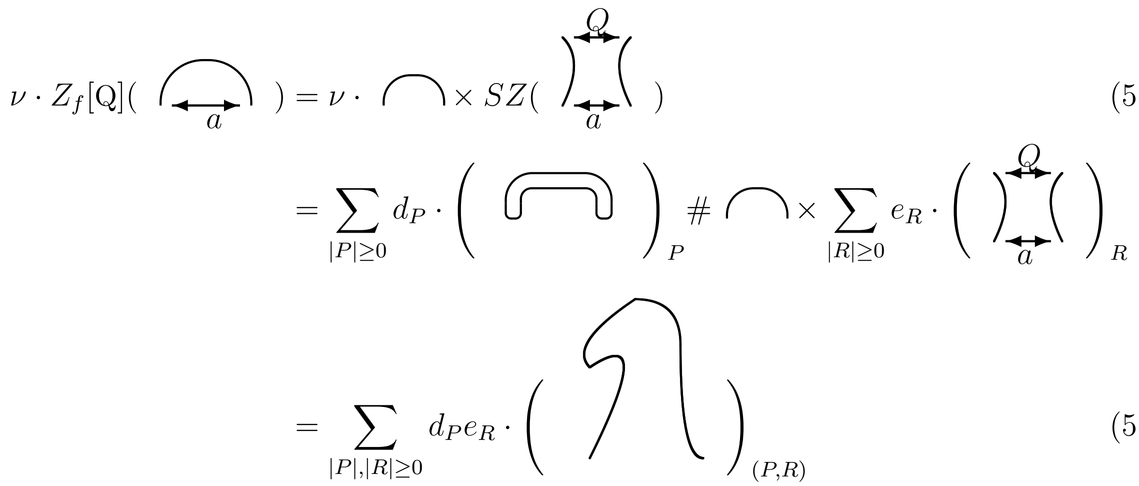
then:

$\nu = Z_f($  $)^{-1} = \sum_{|P| \geq 0} d_P \cdot \Big($  $\Big)_P$ (5.18)

with coefficients d_P such that

$\nu \cdot Z_f($  $) =$  (5.19)

It follows that:



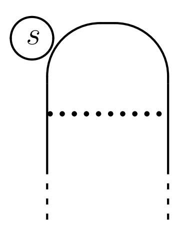
(5.20)

(5.21)

This simple computation shows that the skeleton L will be modified in the expression for $Z_f(L)$ by the introduction of a hook of the above form at each local max but for one local max on each component by definition of $\widetilde{Z}_f$.

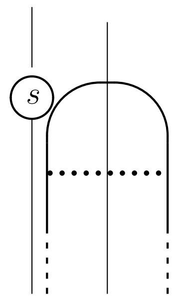
Definition 5.2.1.1. Let $\widetilde{L}$ be the link L with all but one local max on each component being tweaked into a left pointing hook as above.

We consider the band sum move of K_i over K_j. The result of such a band sum move is denoted by L'. However we work with $\widetilde{L}$ and $\widetilde{L}'$ as the rest of the section will make clear. We slice $\widetilde{L}$ into N vertical strips as follows. For a local maximum on the s-th component K_s such as:



(5.22)

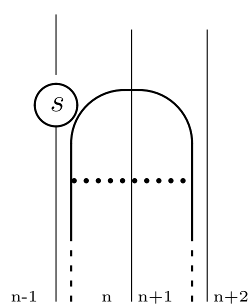
the slicing is performed as follows:



(5.23)

We want each local max to be enclosed within two vertical slices to distinguish neighboring local extrema of a same component. If we call the vertical slices on either side of a local max dividing slices, it follows in practice that consecutive local extrema share a dividing slice is sufficient as we will see later. We do this at each local max of each component K_s of the link $\widetilde{L}$. We slice each local min of each component in like manner, keeping in mind that consecutive local extrema can share dividing slices. We number those vertical strips formed from this slicing procedure starting from the left.

In the above situation, we would have the strips with the following labels:

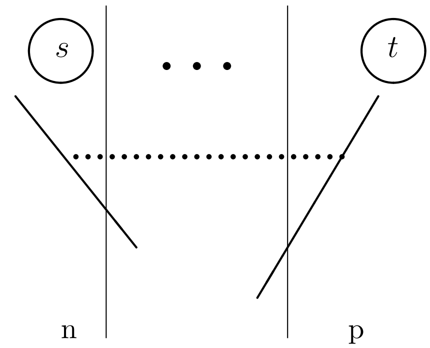


(5.24)

We now discuss the labeling of the chords. For each time, we have a chord. Thus it is natural to number the chords from the bottom up. If $t_1 < \cdots t_m$ are the different times corresponding to m different chords, then those corresponding chords will be labeled $1, \cdots, m$ respectively.

5.2.2 Matrix representation of tangle chord diagrams

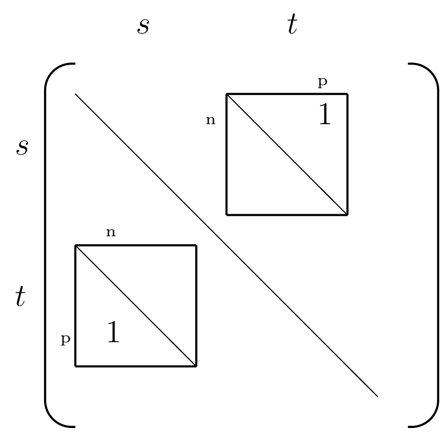
Consider the generic situation of one chord stretching between 2 components of L indexed by s and t:



(5.25)

where we have displayed only portions of the s and t components on which the a-th chord is ending, $1 \leq a \leq m$ and $1 \leq n, p \leq N$ are strip indices. We will index such a chord by (s, t) and (n, p). Chords can be moved up or down along a link and thus their feet may end up in vertical strips with different indices. To avoid such ambiguity, we require that chords be moved up along the link until at least one of their feet reaches a local max.

We represent each such chord by a $eN \times eN$ matrix in the basis given by the ordering of the components, and of the strips, as $(1, 2, \cdots, N, 1, 2, \cdots, N, \cdots, N)$ the first N vectors $1, 2, \cdots, N$ corresponding to the first component of L, followed by those for the second component, and so on, until the e-th component. The st block of that matrix will carry information about chords between the s and t components of the link. In the above situation, the st block has all its entries zero, except the np entry which is one. Note that the matrix is symmetric, so all blocks are empty for that particular chord, except the ts-th one, whose pn entry is one as well. We represent such a matrix as follows where without loss of generality we have chosen $s < t$ and $n < p$:



$$(5.26)$$

We will refer to such a matrix as a page, and we denote it by $A_{s,t,n,p}$ with obvious notations. We do this for all chords of a given tangle chord diagram X of degree m. The information about its chords will therefore be given by m ordered pages from the bottom up, the collection of which will be referred to as a book. We denote a book as follows:

$$A_{I,J,U,V} := \underset{1 \leq a \leq m}{\times} A_{i_a, j_a, u_a, v_a} \tag{5.27}$$

where I, J, U and V are multi-indices defined by:

$$I = (i_1, \cdots, i_m) \tag{5.28}$$

$$J = (j_1, \cdots, j_m) \tag{5.29}$$

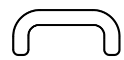
$$U = (u_1, \cdots, u_m) \tag{5.30}$$

$$V = (v_1, \cdots, v_m) \tag{5.31}$$

with $1 \leq i_l, j_l \leq e$ are component indices, $1 \leq u_l, v_l \leq N$ are strip indices for $1 \leq l \leq m$. We denote the size of such multi-indices by $|I| = |J| = |U| = |V| = m$. In the above example, for the a-th chord, we have $i_a = s$, $j_a = t$, $u_a = n$ and $v_a = p$. In the product of pages defining a book, pages are ordered from the bottom up. Now instead of using the notation X for a tangle chord diagram, we use the book notation $A_{I,J,U,V}$ which incorporates the information about the chords on the tangle. We have:

$$Z_f(L) = \sum_{\substack{\text{chord} \\ \text{diagr. X}}} a_X X = \sum_{m \geq 0} \sum_{\substack{|I|=|J|=m \\ |U|=|V|=m}} a_{I,J,U,V} A_{I,J,U,V}(L) \tag{5.32}$$

However $\widetilde{Z}_f(L)$ uses powers of

$\nu = Z_f(U)^{-1} = Z_f($  $)$ (5.33)

and with the above slicing performed on $\widetilde{L}$ we are able to use books $A_{IJUV}(\widetilde{L}) = A_{IJUV}$. We can therefore write:

$$\widetilde{Z}_f(L) = \sum_{\substack{\text{chord} \\ \text{diagr. X}}} c_X X = \sum_{m \geq 0} \sum_{\substack{|I|=|J|=m \\ |U|=|V|=m}} c_{I,J,U,V} A_{I,J,U,V} \tag{5.34}$$

where we have $c_{I,J,U,V} := c_{A_{IJUV}}$.

Chapter 6

Behavior of the Kontsevich integral $\widetilde{Z}_f$ under band sum moves using the book notation

What we have in (1.30) is the following: if h_{ij} denotes the band sum move map on links corresponding to the band sum move of the i-th component of L over its j-th component, then we can write $L' = h_{ij}(L)$, so that $\widetilde{Z}_f(L') = \widetilde{Z}_f(h_{ij}L)$. What we would like however is to find a map $\mathbf{h}_{ij}$ induced from h_{ij} that acts on the Kontsevich integral expressed in book notation to yield its corresponding values after band sum moves. In so doing, we use a different notation for the Kontsevich integral in book notation as once it is written as a linear combination of books, the resulting object is no longer a link invariant. We write ${}^bZ$ for Z written in book notation. We seek a map $\mathbf{h}_{ij}$ that implements the band sum move on ${}^b\widetilde{Z}_f$. We claim that there is such a map, and that moreover for any link L, we have $\mathbf{h}_{ij}({}^b\widetilde{Z}_f(L)) = {}^b\widetilde{Z}_f(h_{ij}L)$. In other terms, the following diagram is commutative:

$$
\begin{tikzcd}
L \arrow[r, "h_{ij}"] \arrow[d, "{}^b\widetilde{Z}_f"'] & h_{ij}L \arrow[d, "{}^b\widetilde{Z}_f"] \\
{}^b\widetilde{Z}_f(L) \arrow[r, "\mathbf{h_{ij}}"'] & \mathbf{h}_{ij}\,{}^b\widetilde{Z}_f(L) = {}^b\widetilde{Z}_f(h_{ij}L)
\end{tikzcd}
\tag{6.1}
$$

It is important to remember that using the isotopy invariance of $\widetilde{Z}_f$, we can arrange that the band sum does not introduce new local extrema, and is far away enough from the rest of the link L that we can use the Long Chords Lemma.

We have the following fact about books; since no two chords can be positioned at the same height t on a tangle, pages, which represent chords, can allow for the possibility to hold many other chords by virtue of the non-simultaneity of chords. In this manner there is no ambiguity as to what chord in a page is represented by which entry. Further if a page holds information about more than one chord, we can split the matrix in as many matrices as there are chords represented in the original page. We illustrate this situation presently:

$$\begin{array}{cc} & \begin{matrix} g & h & k & l \\ \scriptstyle \mathrm{n} & \scriptstyle \mathrm{p} & \scriptstyle \mathrm{u} & \scriptstyle \mathrm{v} \end{matrix} \\ \begin{matrix} g \ \scriptstyle \mathrm{n} \\ h \ \scriptstyle \mathrm{p} \\ k \ \scriptstyle \mathrm{u} \\ l \ \scriptstyle \mathrm{v} \end{matrix} & \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \end{array} \tag{6.2}$$

This represents a page carrying information about two chords, one between the g-th and k-th components, the other between the h-th and l-th components. For the first chord, the foot on K_g is in the n-th strip, and the foot on K_k is in the u-th strip. For the second chord, the foot on K_h is in the p-th strip, and the foot on K_l is in the v-th strip. We have indicated to the left and above the matrix the block indices g, h, k and l, and in small letters the strips within the blocks where a foot is ending, and those are n, p, u and v. Such a matrix splits as follows:

$$\begin{array}{cc} & \begin{array}{cccc} \underset{\mathrm{n}}{g} & \underset{\mathrm{p}}{h} & \underset{\mathrm{u}}{k} & \underset{\mathrm{v}}{l} \end{array} \\ \begin{array}{cc} g & \mathrm{n} \\ h & \mathrm{p} \\ k & \mathrm{u} \\ l & \mathrm{v} \end{array} & \begin{pmatrix} \ddots & & 1 & \\ & \ddots & & \\ 1 & & \ddots & \\ & & & \ddots \end{pmatrix} \end{array} + \begin{array}{cc} & \begin{array}{cccc} \underset{\mathrm{n}}{g} & \underset{\mathrm{p}}{h} & \underset{\mathrm{u}}{k} & \underset{\mathrm{v}}{l} \end{array} \\ \begin{array}{cc} g & \mathrm{n} \\ h & \mathrm{p} \\ k & \mathrm{u} \\ l & \mathrm{v} \end{array} & \begin{pmatrix} \ddots & & & \\ & \ddots & & 1 \\ & & \ddots & \\ & 1 & & \ddots \end{pmatrix} \end{array} \tag{6.3}$$

Since a page holds some information about one chord only by non-simultaneity of chords, if a page displays the information about more than one chord, we can isolate the information about each chord as a direct sum of pages each of which carries information about a unique chord. If we call the original matrix A and the two spin-off matrices B and C, then inserting A in a book of m pages as follows:

$$A^- \times A \times A^+ \tag{6.4}$$

where A^- are the first m^- pages of the book, and A^+ are the last m^+ pages, with $m^- + 1 + m^+ = m$, then we can write:

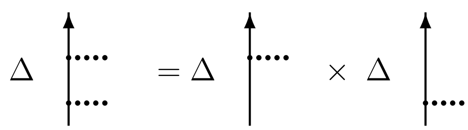
$$A^- \times A \times A^+ = A^- \times B \times A^+ + A^- \times C \times A^+ \tag{6.5}$$

representing two chord diagrams, one of which has its $(m^- + 1)$-st chord represented by the page B, the other has its $(m^- + 1)$-st chord represented by C. We have done this for a page A containing the information about two chords. We generalize (6.4) by iterating this process for pages that contain information about more than 2 chords and generalize (6.5) by iteration for books with two or more pages written as a sum of matrices. All this can be easily summarized by saying that

$$\bigoplus_{m\geq 0} \times_{1\leq i\leq m} \mathbf{Symm}_{\,eN\times eN}$$

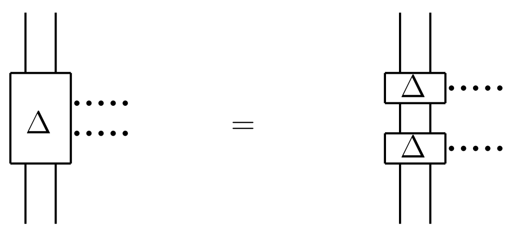
forms a tensor algebra.

We now discuss the band sum move proper. By virtue of the fact that we have:



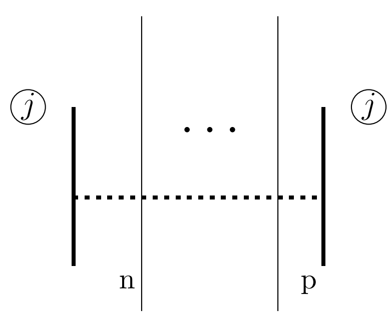
(6.6)

or by abuse of notation:



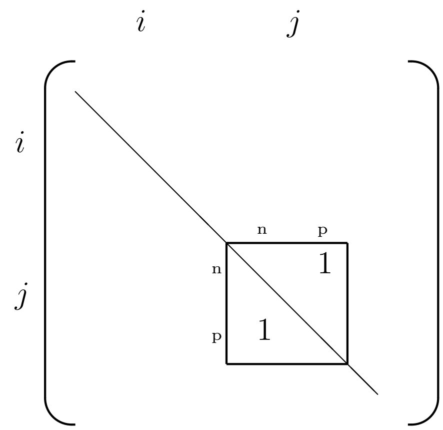
(6.7)

then studying the doubling of chords during the band sum move can be done one chord after another. It suffices to work with one page at a time. During the band sum move of the handle corresponding to the i-th component K_i over the j-th component K_j of the link L, we encounter two different situations:



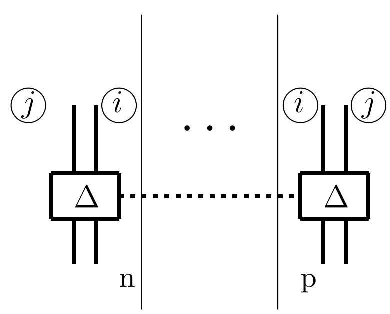
(6.8)

where $n < p$. In this first case, a given chord starts and ends on the j-th component, with matrix representation given by:



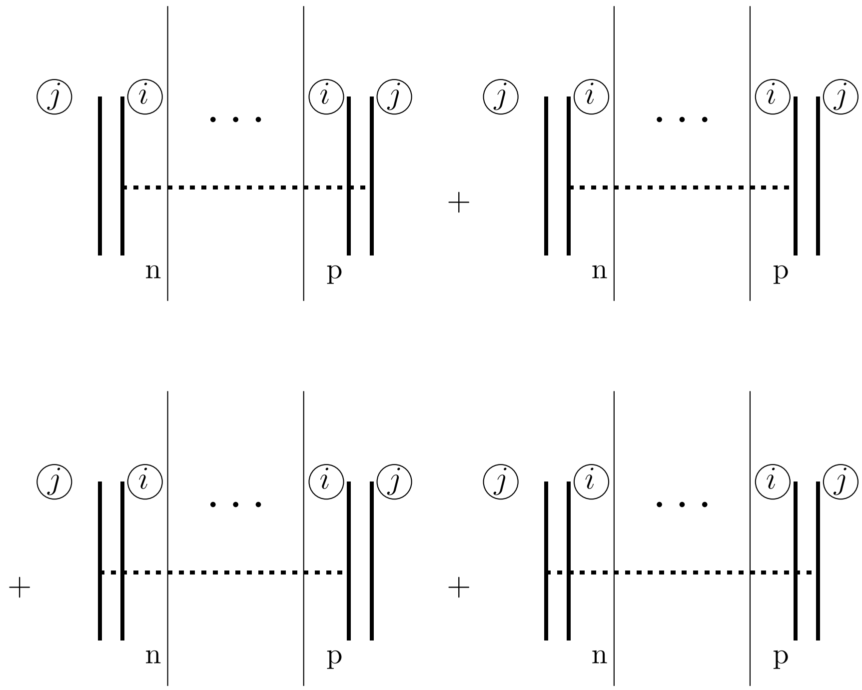
(6.9)

where without loss of generality we have chosen $i < j$ and the case $i > j$ is dealt with by a simple change of basis. Under a band sum move we obtain:



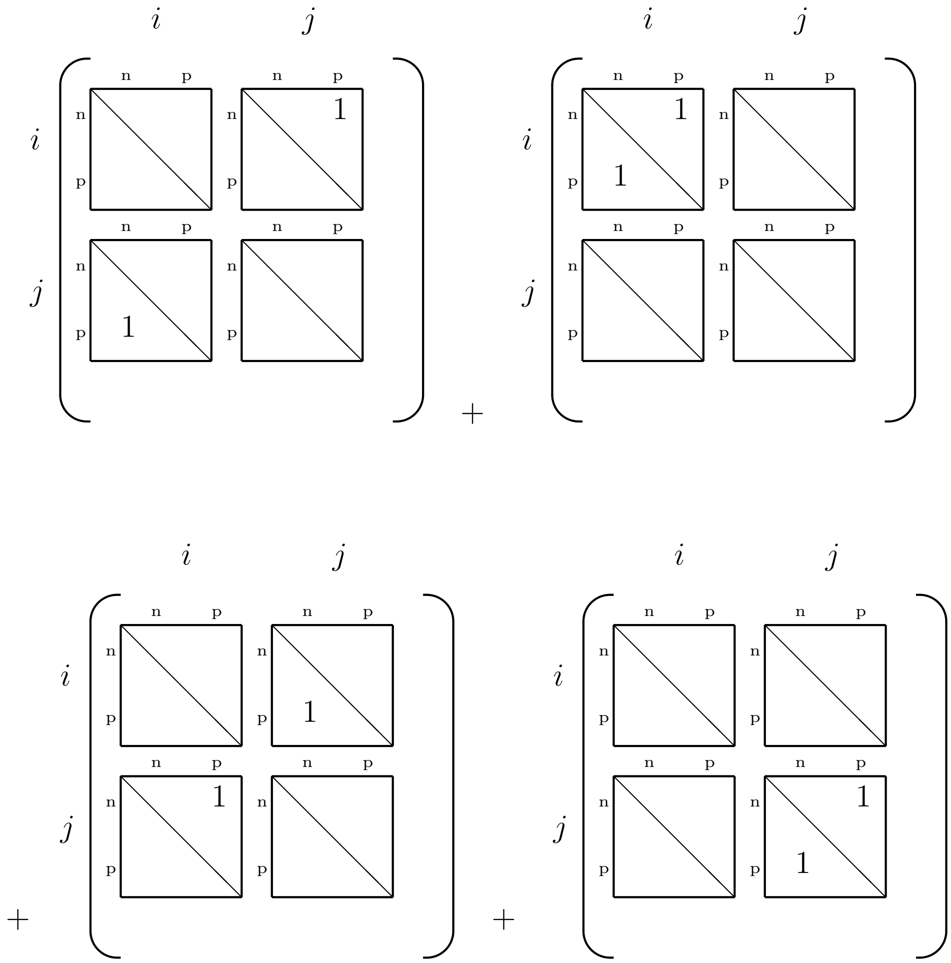
(6.10)

which equals:

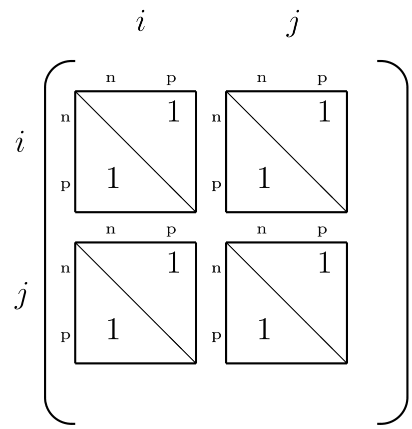


(6.11)

a sum of chord diagrams that correspond, in this order, to the following sum of matrices:

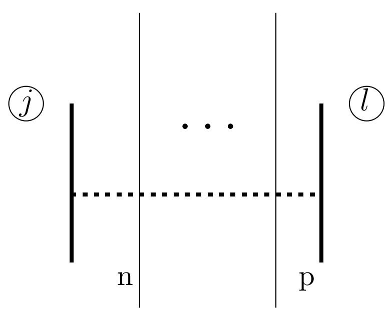


which combines into:



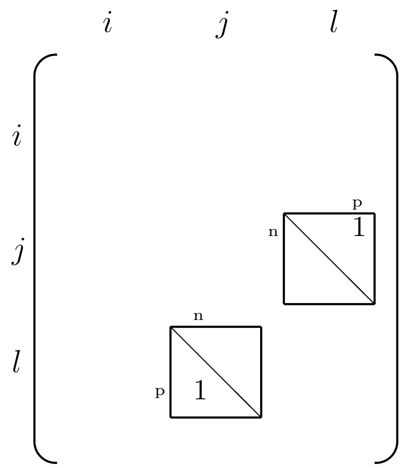
(6.12)

The second situation we can have is the case where the chord starts on the j-th component but ends on some other l-th component. Without loss of generality we can pick $i < j < l$. For other arrangements of these indices we modify the basis for our matrices accordingly. We discuss the case $l = i$ right after since it doesn't consist in a basis change. Pictorially we have:



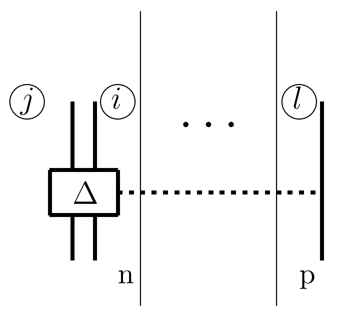
(6.13)

with the following matrix representation:



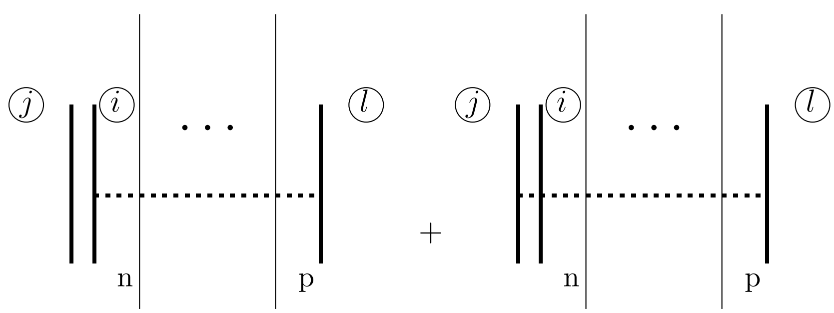
(6.14)

After a band sum move we get the following chord diagram:

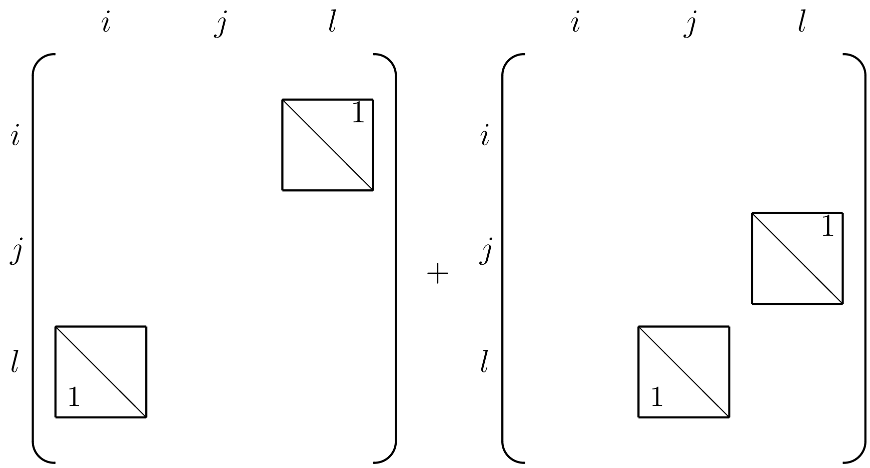


(6.15)

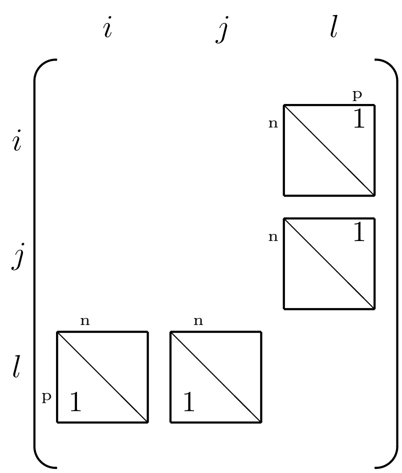
which equals:



represented, in this order, by the sum of matrices:

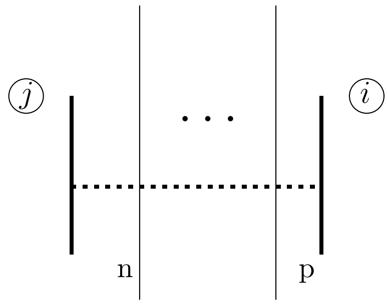


combining into:



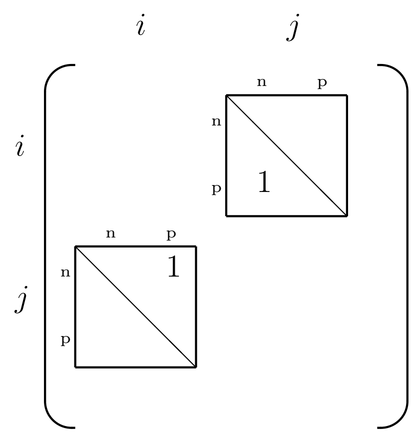
(6.16)

For the case $l = i$, we pictorially have:



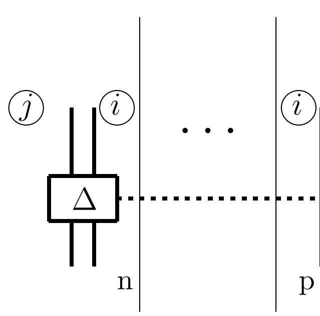
(6.17)

with the following matrix representation:



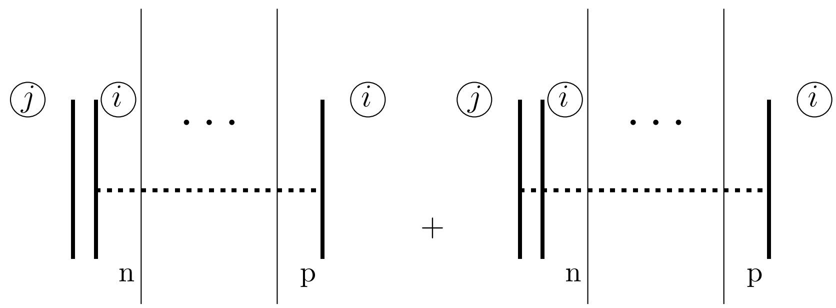
(6.18)

After a band sum move, we get the following diagram:

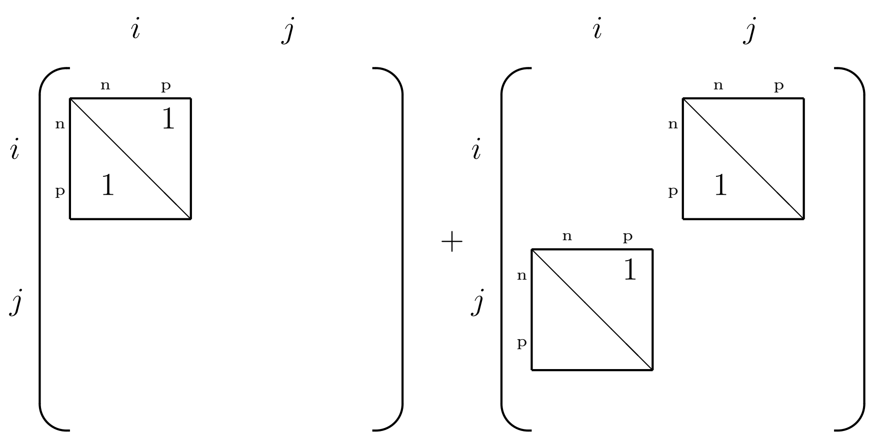


(6.19)

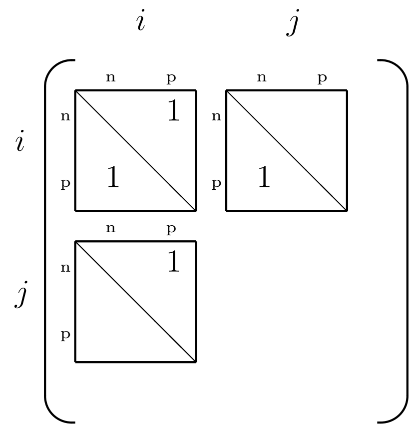
which equals:



represented in this order by the sum of matrices:



combining into:



(6.20)

We present now our main result:

Theorem 6.1. For an e-component link L for which the i-th component K_i is being band summed over the j-th component K_j, the induced map on ${}^{b}\widetilde{Z}_f$ is denoted by $\mathbf{h}_{ij}$ and is defined by:

$$\mathbf{h}_{ij}\,{}^{b}\widetilde{Z}_f(L) = \sum_{m\geq 0} \sum_{\substack{|I|=|J|=m \\ |U|=|V|=m}} c_{IJUV} \underset{1\leq a\leq m}{\times} M_{ij}^T A_{i_a,j_a,u_a,v_a} M_{ij} \tag{6.21}$$

where M_{ij} is an $eN \times eN$ matrix with ones on its diagonal and the ji block is the $N \times N$ identity matrix I_N. For $i < j$, we write such a matrix as:

$$\begin{array}{cc}
 & \begin{array}{cc} i & \qquad j \end{array} \\
\begin{array}{c} i \\ \\ \\ j \end{array} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ \boxed{\mathrm{I}_N} & & & 1 \end{pmatrix}
\end{array} \tag{6.22}$$

If we define:

$$M_{ij}^m := \mathop{\times}_{1\leq a\leq m} M_{ij} \tag{6.23}$$

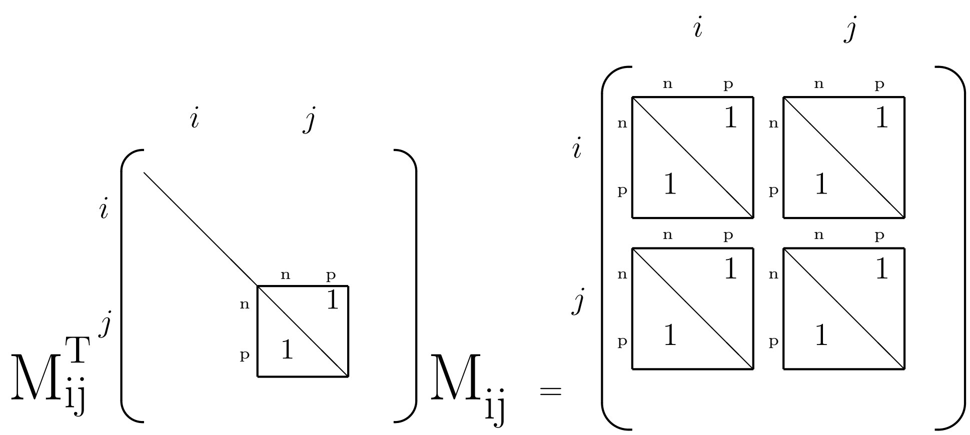
acting on books of m pages chord-wise, then we can rewrite the above formula in compact form as:

$$\mathbf{h}_{ij}\, {}^b\widetilde{Z}_f(L) = \sum_{m\geq 0} \sum_{\substack{|I|=|J|=m \\ |U|=|V|=m}} c_{IJUV} M_{ij}^{m\,T} A_{IJUV} M_{ij}^m \tag{6.24}$$

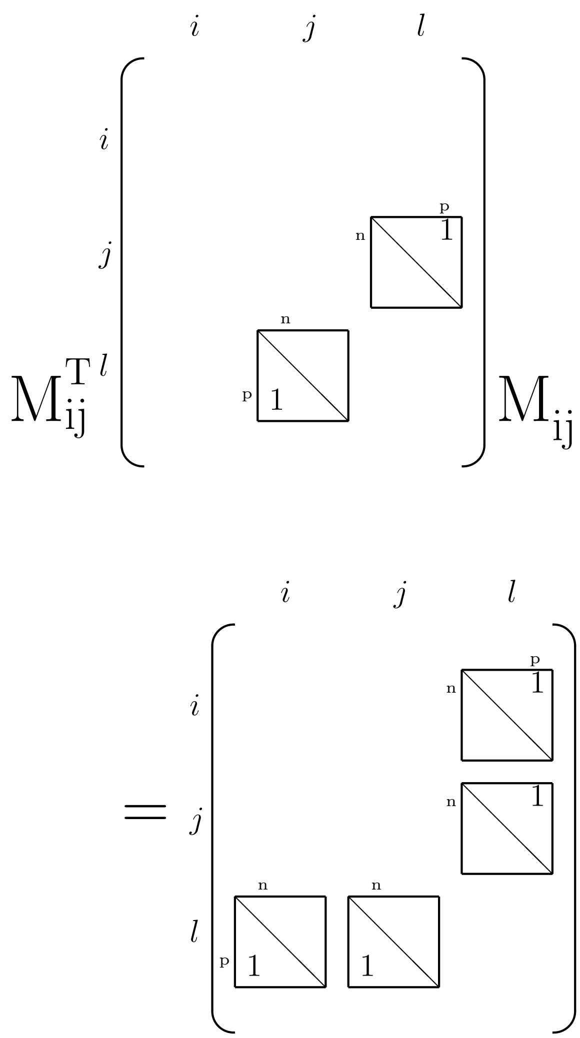
We may even generalize this further by defining $M_{ij}^{\cdot}$ to be the product of as many matrices M_{ij} as the book they operate on have pages, which leads to having the even simpler formula:

$$\mathbf{h}_{ij}\, {}^b\widetilde{Z}_f(L) = M^{\cdot\,T} \left({}^b\widetilde{Z}_f(L) \right) M^{\cdot} \tag{6.25}$$

Proof. The only cases that need to be studied are those where a chord has a foot on the j-th component, since all other tangle chord diagrams are left invariant by $\mathbf{h}_{ij}$, and of those there are only two kinds that we presented just before the statement of the Theorem. By elementary matrix multiplication, we have in the first case, for one page:

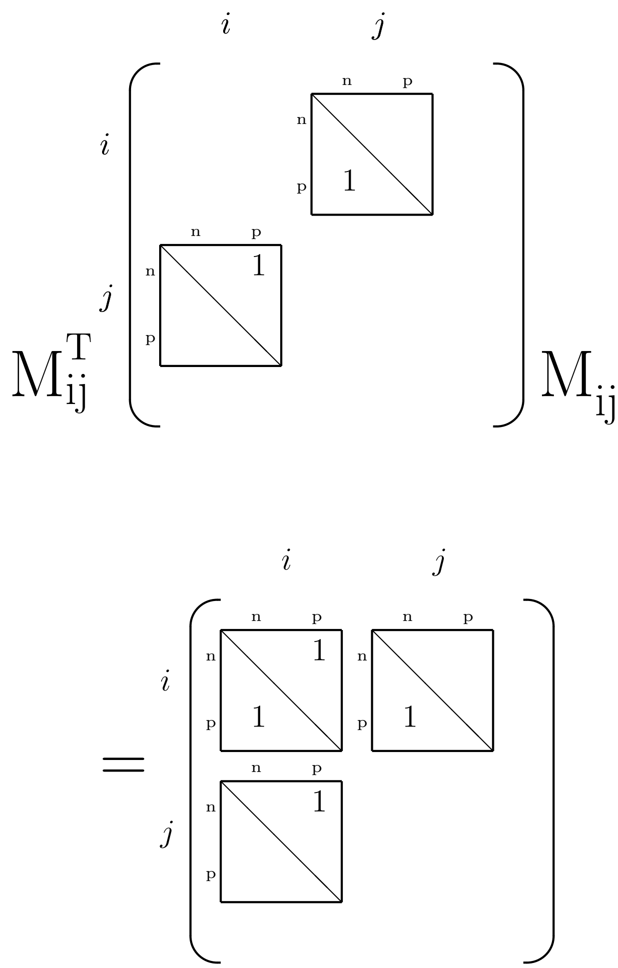


which is what we expected from the above considerations. In the second case of interest, if $l \neq i$:



(6.26)

which is what was expected as well. In case $l = i$:

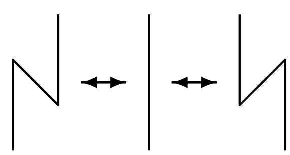


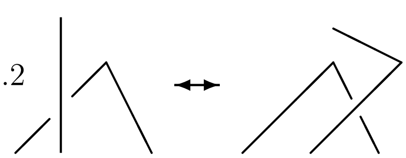
This completes the proof. $\square$

Chapter 7

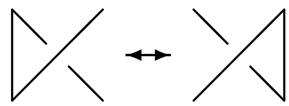
Isotopy invariance of $\widetilde{Z}_f$ in book notation

From Theorem 2.2.5 $\hat{Z}_f$ is an isotopy invariant. By definition, so is $\widetilde{Z}_f$. If we consider $\widetilde{Z}_f(L)$ as just a formal sum of books with complex coefficients however, we have a representation of tangle chord diagrams by books that is no longer isotopy invariant, whence the introduction in the previous chapter of the notation ${}^bZ$. We use the fact that ambient isotopic links are in the same class if and only if they are related by the Reidemeister moves [Rei] [Y]. Instead of studying the behavior of ${}^b\widetilde{Z}_f(L)$ under any arbitrary isotopy, we study Reidemeister moves of tangles. They are pictured as follows:

$\Delta.\pi.1$  (7.1)

$\Delta.\pi.2$  (7.2)

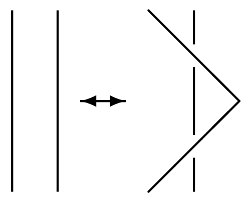
$\Omega.1.f$



(7.3)

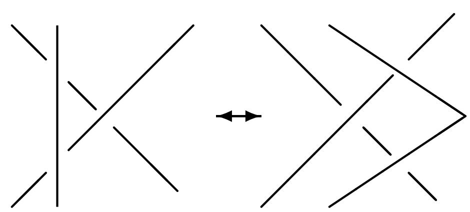
which is the framed version of the original $\Omega.1$ Reidemeister move.

$\Omega.2$



(7.4)

$\Omega.3$



(7.5)

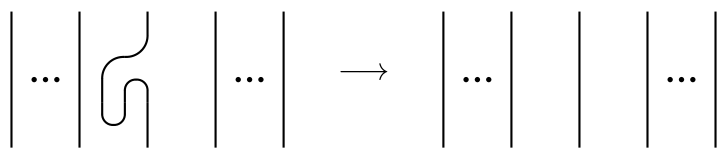
Proposition 7.1. The book representation of $\widetilde{Z}_f(L)$ of any link L is invariant under the Reidemeister moves $\Omega.2$ and $\Omega.3$.

Proof. In both cases we have an equivalence of two tangle diagrams which yield the same page for any chord ending on them after possibly moving the chord up the tangle. $\square$

Those are the only moves that leave ${}^b\widetilde{Z}_f(L)$ invariant. For the other Reidemeister moves, there are a few changes to be performed on each page.

7.1 Behavior of ${}^b\widetilde{Z}_f$ under $\Delta.\pi.1$

In a slice presentation of knots, and using the isotopy invariance of $\widetilde{Z}_f$, it suffices to consider strands other than those involved in the move $\Delta.\pi.1$ to be vertical, and the straightened out strand involved in $\Delta.\pi.1$ to be straight as well. For instance, one of the moves involved in $\Delta.\pi.1$ would look like:



(7.6)

In the expression for $\hat{Z}_f$, powers of ν cancel self-chords from the Kontsevich integral of the hump. Chords with one foot on the hump however are not canceled, whereas in the expression for $\hat{Z}_f$ of the right hand side of (7.6), all of whose strands are vertical, there are no such chords. This is in particular true of $\widetilde{Z}_f$. Moreover mixed chord diagrams involving self chords on the hump and long chords with one foot on the hump are not canceled by ν. They are canceled by squeezing the hump into a window of vanishing width. For each long chord with a foot on an ascending (resp. descending) part of the hump, there is a long chord with a foot on a descending (resp. ascending) part of the hump, each tangle chord diagram with log differentials that differ in sign and therefore cancel each other off. This is how we have all long chords ending on the hump go. Thus we seek a transformation on the expression for ${}^b\widetilde{Z}_f$ of the left hand side of (7.6) that effectively gets rid of chords with a foot on the hump.

Proposition 7.1.1. The $eN \times e(N-8)$ matrix $\mathrm{diag}(M_{\text{hump}\mapsto|})$ with:

$$\mathrm{M}_{\text{hump}\mapsto|} = \left(\begin{array}{c|c} I_n & \\ \hline \multicolumn{2}{c}{O_{8\times(N-8)}} \\ \hline & I_{N-n-8} \end{array}\right) \tag{7.7}$$

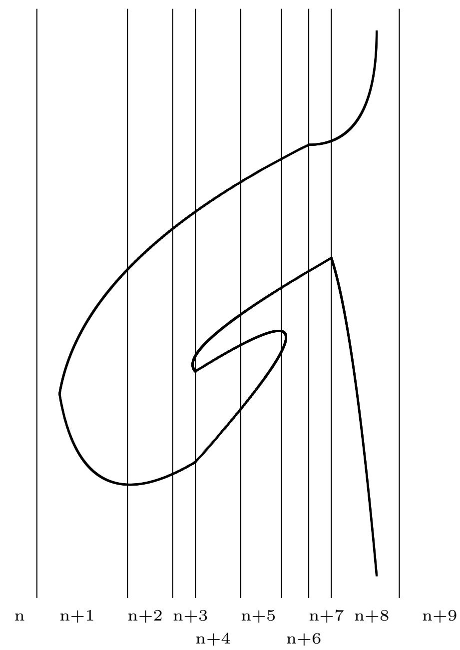
implements the move (7.6). In case the hump is on the first strand from the left:

$$\mathrm{M}_{\text{hump}\mapsto|} = \left(\begin{array}{c} O_{8\times(N-8)} \\ \hline I_{N-8} \end{array}\right) \tag{7.8}$$

If we let I_0 be the empty matrix then (7.7) implements the move (7.6) for all $n \geq 0$.

Proof. First observe that since we have a local max on the hump, the skeleton for both $\hat{Z}_f$

and $\widetilde{Z}_f$ is as follows, with added vertical slices for numbering purposes:



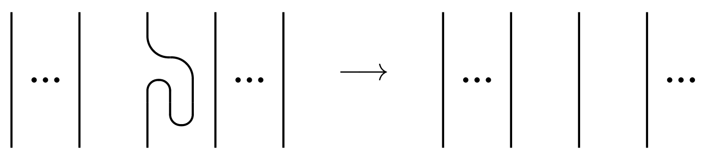
(7.9)

The straightened out strand on the right hand side of (7.6) is in the $(n+1)$-st strip. The matrix that takes care of relabeling all the strips under the map depicted in (7.6) and kills all the chords on the hump is, for the block corresponding to the component on which the hump is located:

$$\mathrm{M}_{\text{hump}\mapsto|} = \left(\begin{array}{c|c} I_n & \\ \hline \multicolumn{2}{c}{O_{8\times(N-8)}} \\ \hline & I_{N-n-8} \end{array}\right) \tag{7.10}$$

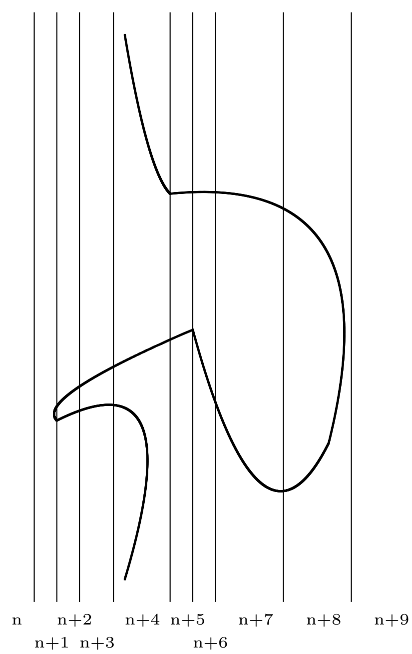
It is a $N \times (N-8)$ matrix. In other horizontal slices, chords ending on strands other than the one with the hump are moved up to be either in strips indexed 1 through n, or $n+9$ through N. Therefore the above matrix also takes care of the transformation of those chords under the move (7.6). We conclude that the $eN \times e(N-8)$ matrix $\mathrm{diag}(M_{\text{hump}\mapsto|})$ with $M_{\text{hump}\mapsto|}$ given above implements the move (7.6). $\square$

One would similarly show that the move:



(7.11)

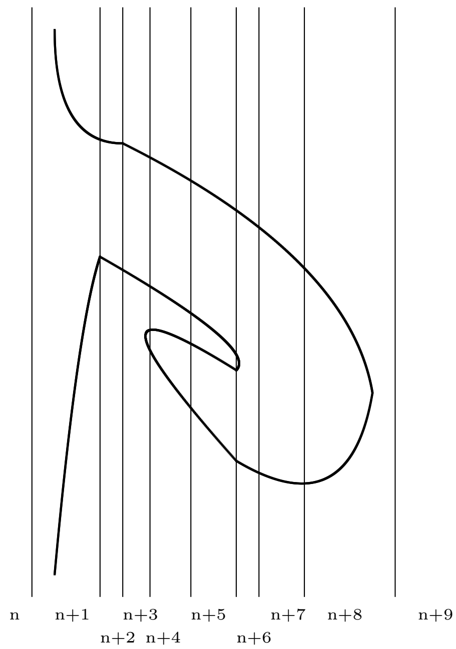
with a local slicing of the hump given by:



(7.12)

is taken care of by the same $eN \times e(N-8)$ matrix $\mathrm{diag}(M_{\mathrm{hump}\mapsto|})$ defined above.

Remark 7.1.2. We could equally have taken the following slicing and skeleton:



(7.13)

and the matrix would have been the same.

We have shown:

Proposition 7.1.3. The map $\mathbf{h}_{\Delta.\pi.1,\ \text{hump}\mapsto|}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect two of the $\Delta.\pi.1$ Reidemeister moves as depicted in (7.6) and (7.11) is given on a page by:

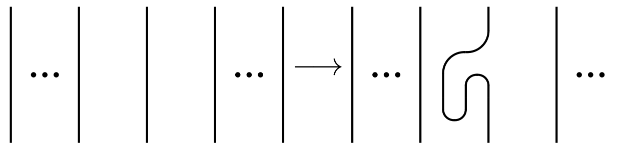
$$\mathbf{h}_{\substack{\Delta.\pi.1\\ \text{hump}\mapsto|}} A := M^T_{\substack{\Delta.\pi.1\\ \text{hump}\mapsto|}} A M_{\substack{\Delta.\pi.1\\ \text{hump}\mapsto|}} \tag{7.14}$$

where $M_{\Delta.\pi.1,\ \text{hump}\mapsto|}$ is the $eN \times e(N-8)$ matrix $\text{diag}(M_{\text{hump}\mapsto|})$ with $M_{\text{hump}\mapsto|}$ given by:

$$\mathrm{M}_{\text{hump}\mapsto|} = \left(\begin{array}{c|c} I_n & \\ \hline \multicolumn{2}{c}{O_{8\times(N-8)}} \\ \hline & I_{N-n-8} \end{array}\right) \tag{7.15}$$

for $n \geq 0$.

Proposition 7.1.4. The matrix that implements the move:



(7.16)

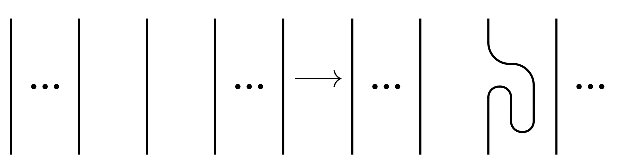
with the same slicing convention as in (7.9) is given by the $N \times (N+8)$ matrix:

$$\mathrm{M}_{|\mapsto \text{hump}} = \left(\begin{array}{c||c||c} I_n & & \\ \hline & O_{N\times 8} & \\ \hline & & I_{N-n} \end{array}\right) \tag{7.17}$$

for $n \geq 0$.

Proof. In other horizontal slices, chords ending on strands other than the one on which the hump is located are moved up to end in strips indexed 1 through n or $n+9$ through N. Therefore the above matrix also takes care of the transformation of those chords under the move (7.16). We conclude that the $eN \times e(N+8)$ matrix $\mathrm{diag}(M_{|\mapsto \text{hump}})$ with $M_{|\mapsto \text{hump}}$ given above implements the move (7.16). $\square$

Likewise, one would show that the same matrix implements the move:



(7.18)

with the slicing on the hump given as in (7.12). We have shown:

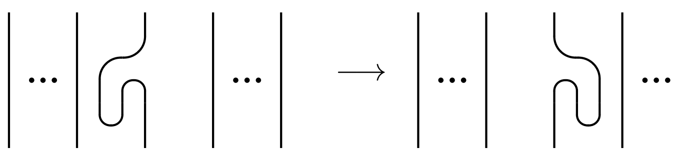
Proposition 7.1.5. The map $\mathbf{h}_{\Delta.\pi.1,\ |\mapsto\text{hump}}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect two of the $\Delta.\pi.1$ Reidemeister moves as depicted in (7.16) and (7.18) is given on a page by:

$$\mathbf{h}_{\substack{\Delta.\pi.1\\ |\mapsto\text{hump}}} A := M^T_{\substack{\Delta.\pi.1\\ |\mapsto\text{hump}}} A M_{\substack{\Delta.\pi.1\\ |\mapsto\text{hump}}} \tag{7.19}$$

where $M_{\Delta.\pi.1,\ |\mapsto\text{hump}}$ is the $eN \times e(N+8)$ $\text{diag}(M_{|\mapsto\text{hump}})$ matrix with $M_{|\mapsto\text{hump}}$ given by:

$$\mathrm{M}_{|\mapsto\text{hump}} = \left(\begin{array}{c|c||c} I_n & & \\ \hline & O_{N\times 8} & \\ \cline{3-3} & & I_{N-n} \end{array}\right) \tag{7.20}$$

Finally we consider the map that implements the following move on strands:



(7.21)

With slicing conventions as in (7.9) and (7.12), the corresponding transformation matrix is given by the $N \times N$ matrix:

$$\mathrm{M}_{\text{hump}\mapsto\text{hump}} = \begin{pmatrix} I_n & & \\ & O_{8\times 8} & \\ & & I_{N-n-8} \end{pmatrix} = \mathrm{M}_{\text{hump}\mapsto|} \cdot \mathrm{M}_{|\mapsto\text{hump}} \tag{7.22}$$

In other horizontal slices, chords ending on strands other than the one on which the hump is located are moved up to end in strips indexed 1 through n or $n+9$ through N. Therefore the above matrix also takes care of the transformation of those chords under the move (7.21). We conclude that the $eN \times eN$ matrix $\mathrm{diag}(M_{\mathrm{hump}\mapsto\mathrm{hump}})$ with $M_{\mathrm{hump}\mapsto\mathrm{hump}}$ given above implements the move (7.21). We have shown:

Proposition 7.1.6. The map $\mathbf{h}_{\Delta.\pi.1,\ \mathrm{hump}\mapsto\mathrm{hump}}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect the $\Delta.\pi.1$ Reidemeister move as depicted in (7.21) is given on a page by:

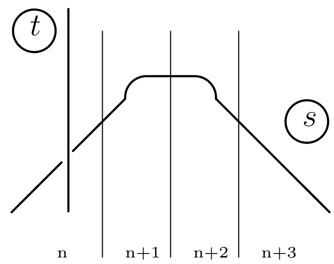
$$\mathbf{h}_{\substack{\Delta.\pi.1\\ \mathrm{hump}\mapsto\mathrm{hump}}} A := M^T_{\substack{\Delta.\pi.1\\ \mathrm{hump}\mapsto\mathrm{hump}}} A M_{\substack{\Delta.\pi.1\\ \mathrm{hump}\mapsto\mathrm{hump}}} \tag{7.23}$$

where $M_{\Delta.\pi.1,\ \mathrm{hump}\mapsto\mathrm{hump}}$ is the $eN \times eN$ matrix $\mathrm{diag}(M_{\mathrm{hump}\mapsto\mathrm{hump}})$ with $M_{\mathrm{hump}\mapsto\mathrm{hump}}$ given by:

$$\mathrm{M}_{\mathrm{hump}\mapsto\mathrm{hump}} = \begin{pmatrix} I_n & & \\ & O_{8\times 8} & \\ & & I_{N-n-8} \end{pmatrix} \tag{7.24}$$

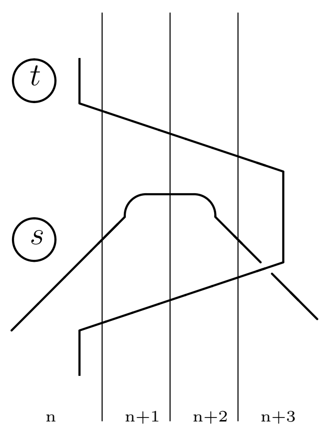
7.2 Behavior of ${}^b\widetilde{Z}_f$ under $\Delta.\pi.2$

The two tangle diagrams involved in the statement for the $\Delta.\pi.2$ Reidemeister move have the following presentation with vertical strips for book purposes:



(7.25)

and:



(7.26)

Proposition 7.2.1. With tangle diagrams as above, the map $\mathbf{h}_{\Delta.\pi.2}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect the Reidemeister move $\Delta.\pi.2$, is given on a page by:

$$\mathbf{h}_{\Delta.\pi.2}A := M^T_{\Delta.\pi.2}AM_{\Delta.\pi.2} \tag{7.27}$$

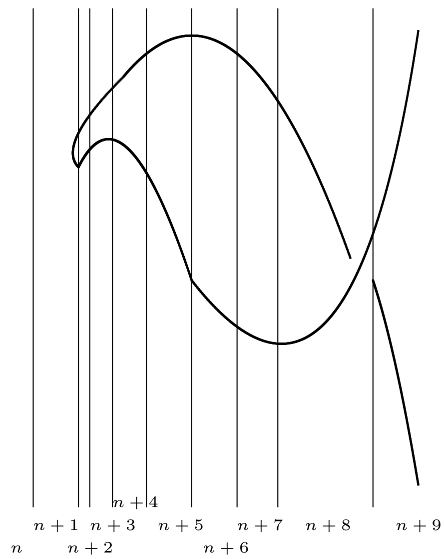
where $M_{\Delta.\pi.2}$ is the $eN \times eN$ identity matrix save for the tt block which is of the form:

$$\begin{array}{cc} & \begin{array}{cccccc} \phantom{I_{n-1}} & n & n+1 & n+2 & n+3 & \phantom{I_{N-n-3}} \end{array} \\ \begin{array}{c} \\ n \\ n+1 \\ n+2 \\ n+3 \\ \\ \end{array} & \begin{pmatrix} I_{n-1} & & & & & \\ & 0 & 0 & 0 & 1 & \\ & 0 & 0 & 1 & 0 & \\ & 0 & 1 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & \\ & & & & & I_{N-n-3} \end{pmatrix} \end{array} \tag{7.28}$$

Proof. Recall that chords on a given tangle are moved until they reach a local max. Keeping this in mind, under the move $\Delta.\pi.2$, only the t-th component moves, and thus only chords with a foot on this component need be considered. Suppose a chord is stretching between that component and the s-th component. Moving the chord up the s-th component so that it's localized near its local max, the foot of that chord on the s-th component is either in the $n+1$-st or $n+2$-nd strip, and its other foot on the t-th component is then in the n-th or $n+3$-rd strip depending on which tangle we are looking at. Under $\Delta.\pi.2$, chords with one foot in the n-th strip (resp. the $n+3$-rd strip) are moved to end up in the $n+3$-rd strip (resp. the n-th strip). Suppose now a chord is stretching between the t-th component and another l-th component, $l \neq s$. For a needle shaped local max on the s-th component, the corresponding vertical strips indexed by $n+1$ and $n+2$ are sufficiently narrow that chords from some l-th component, $l \neq s$ end on the t-th component in some strip other than the $n+1$-st or $n+2$-nd strips, that is in the n-th or $n+3$-rd strips, which are interchanged under $\Delta.\pi.2$. We conclude that $\Delta.\pi.2$ is implemented by a $eN \times eN$ matrix with a tt block of the form above which effectively switches the n-th and $n+3$-rd strips for $n \geq 1$, taking I_0 to be the empty matrix in the event that $n = 1$ indexes the first vertical strip. $\square$

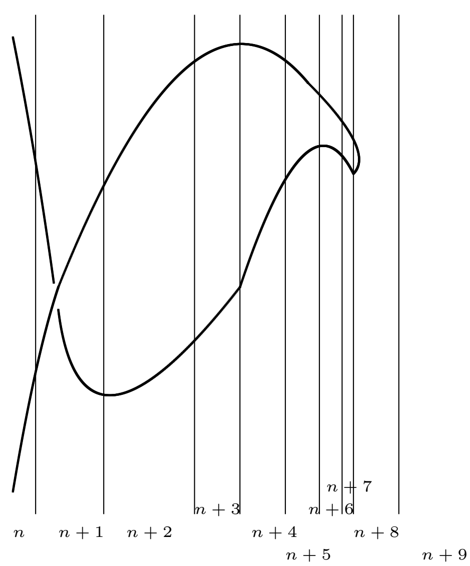
7.3 Behavior of ${}^{b}\widetilde{Z}_f$ under $\Omega.1.f$

The two equivalent tangle diagrams under this move are represented as follows along with the strips for book representation purposes:



(7.29)

and:



(7.30)

Proposition 7.3.1. With tangle diagrams as above, the map $\mathbf{h}_{\Omega.1.f}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect the Reidemeister move $\Omega.1.f$ on the l-th component of L, is given on a page by:

$$\mathbf{h}_{\Omega.1.f}A := M^T_{\Omega.1.f}AM_{\Omega.1.f} \tag{7.31}$$

where $M_{\Omega.1.f}$ is the $eN \times eN$ identity matrix save for the ll block which is of the form:

$$\begin{array}{cc}
 & \begin{array}{ccc} \qquad\qquad & \mathrm{n} \;\cdots\; \mathrm{n+9} & \qquad\qquad \end{array} \\
\begin{array}{c} \mathrm{n} \\ \vdots \\ \mathrm{n+9} \end{array} & \begin{pmatrix} I_{n-1} & & \\ & \begin{matrix} & & 1 \\ & {\cdot^{\cdot^{\cdot}}} & \\ 1 & & \end{matrix} & \\ & & I_{N-n-9} \end{pmatrix}
\end{array} \tag{7.32}$$

with ones on the transverse diagonal.

Proof. With the vertical slicing as in (7.29) and (7.30), it is immediate that the above matrix implements the move $\Omega.1.f$ on the l-th component. Chords with feet on other strands are in strips indexed 1 through n or $n + 10$ through N and stay put under $\Omega.1.f$, so for those we just use an identity matrix. We conclude that the above matrix implements the move $\Omega.1.f$. □

7.4 Adding and deleting strips

If M_n^+ denotes the matrix that inserts a strip between the n-th and $n+1$-st strip, then the transformation on pages that effects such a change is the $eN \times e(N+1)$ matrix $\mathrm{diag}(M_n^+)$ with:

$$\mathrm{M}_n^+ = \begin{array}{cc} & \begin{array}{ccc} & \scriptstyle \mathrm{n+1} & \phantom{I_{N-n}} \end{array} \\ \scriptstyle \mathrm{n+1} & \begin{pmatrix} I_n & 0 & \\ & \vdots & \\ & \vdots & I_{N-n} \\ & 0 & \end{pmatrix} \end{array} \tag{7.33}$$

and if M_n^- denotes the matrix that deletes the n-th strip, then the transformation on pages that effects such a change is the $e(N+1) \times eN$ matrix $\mathrm{diag}(M_n^-)$ with:

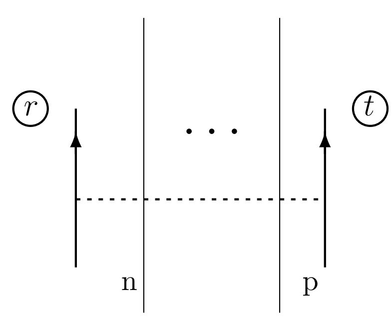
$$\mathrm{M}_n^- = \begin{array}{c} \begin{array}{cc} \phantom{I_{n-1}} \scriptstyle \mathrm{n\text{-}1} & \phantom{I_{N-n+2}} \end{array} \\ \begin{pmatrix} I_{n-1} & \\ & I_{N-n+2} \end{pmatrix} \end{array} \tag{7.34}$$

Chapter 8

Behavior of $\widetilde{Z}_f(L)$ in book notation under a change of orientation

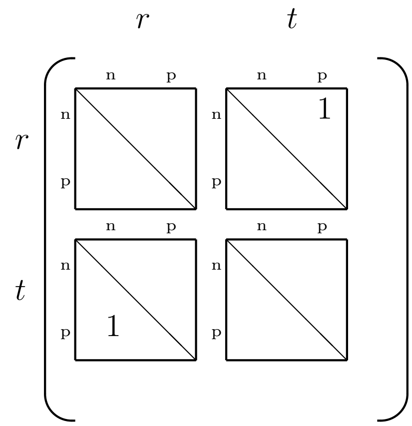
8.1 Behavior of ${}^b\widetilde{Z}_f$ under orientation change

Recall that the links we deal with are oriented. Under a change of orientation on a component of a link L, any chord diagram with a foot on that component has its coefficient being multiplied by -1. The map on chord diagrams in $\hat{\mathcal{A}}(\coprod S^1)$ that effects this change is denoted by S_r for the r-th component of the link L whose orientation is being changed, $1 \leq r \leq e$ [LM3]. We generalize this to tangle chord diagrams: S_r is the map $\hat{A}(\coprod_{1\leq l\leq e} K_l) \to \hat{A}(\coprod_{1\leq l\neq r\leq e} K_l + S_r K_r)$ induced from a change of orientation $S_r K_r$ on K_r. The book notation should reflect this change. Pictorially, the following local tangle chord diagram:



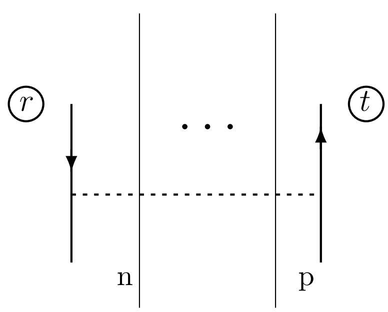
(8.1)

is represented in book notation by:



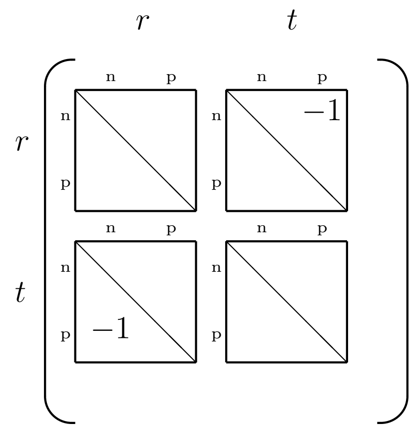
(8.2)

where without loss of generality we have chosen $r < t$. The same tangle chord diagram with the reverse orientation on the r-th component:



(8.3)

has book representation:



(8.4)

Proposition 8.1. With tangle diagrams as above, the map $\mathbf{h}_{S,r}$ that acts on $\widetilde{Z}_f(L)$ in book notation to effect the orientation change S_r on the r-th component of L, is given on a page by:

$$\mathbf{h}_{S,r}A := M_{S,r}^T A M_{S,r} \tag{8.5}$$

where $M_{S,r}$ is the $eN \times eN$ identity matrix save for the rr block which is the negative of the $N \times N$ identity matrix, $-I_N$:

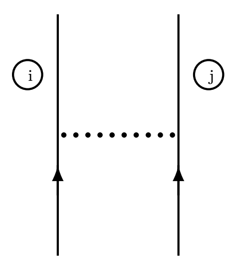
$$\begin{array}{c c} & \begin{array}{cc} r & \quad t \end{array} \\ \begin{array}{c} r \\ \\ t \end{array} & \begin{pmatrix} \boxed{-\mathrm{I}_N} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array} \tag{8.6}$$

There are as many modified blocks such as the rr-th above as there are components whose orientation is changed. For a chord with feet on the r-th and t-th components both of whose orientations are reversed, the matrix $M_{S,r,t}$ with two blocks rr and tt as $-I_N$ will effect the orientation change while leaving the coefficient in front of corresponding chord diagrams unchanged since $(-1) \times (-1) = 1$.

Proof. Matrix multiplication. $\square$

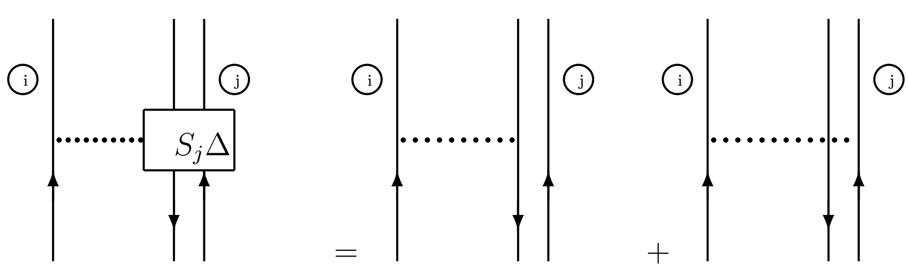
8.2 Behavior of $\widetilde{Z}_f(L)$ as K_i is being subtracted from K_j

We now consider the effect of having a subtraction of K_i from K_j as a result of operating a band sum move of K_i over K_j. As defined in [RK1], this corresponds to having a band such that upon doing the band sum move the components K_i and K_j end up having opposite orientations. If we locally represent the pieces of those two components with a chord stretching between them as:



(8.7)

then this chord behaves as follows under a band sum move of K_i over K_j:



(8.8)

If the latter tangle chord diagram has a coefficient of c_X in the expression for $\widetilde{Z}_f$, the former has a coefficient $-c_X$ however due to the orientation change resulting in the Kontsevich integral picking up an overall minus sign. It follows that the statement of Theorem 1.6.3 is no longer true in this case. We can nevertheless remedy this as follows.

Proposition 8.2.1. ([LM4], [LM5]) $Z_f(S_{K_i}L) = S_{K_i}Z_f(L)$ for $1 \leq i \leq e$.

Proof. This follows from the definition of Z_f and the fact that $Z(S_{K_i}L) = S_{K_i}Z(L)$. □

Theorem 8.2.2. ([LM4], [LM5]) $\hat{Z}_f(S_{K_i}L) = S_{K_i}\hat{Z}_f(L)$ for $1 \leq i \leq e$.

Proof. It suffices to write:

$$\begin{aligned}
\hat{Z}_f(S_{K_i}L) &= \nu^{m_1} \otimes \cdots \otimes \nu^{m_i} \otimes \cdots \otimes \nu^{m_e} \cdot Z_f(S_{K_i}L) && (8.9)\\
&= \nu^{m_1} \otimes \cdots \otimes S_{K_i}S_{K_i}\nu^{m_i} \otimes \cdots \otimes \nu^{m_e} \cdot S_{K_i}Z_f(L) && (8.10)\\
&= S_{K_i}\Big(\nu^{m_1} \otimes \cdots \otimes (S_{K_i}\nu)^{m_i} \otimes \cdots \otimes \nu^{m_e} \cdot Z_f(L)\Big) && (8.11)\\
&= S_{K_i}\hat{Z}_f(L) && (8.12)
\end{aligned}$$

□

Theorem 8.2.3. $\widetilde{Z}_f(S_{K_i}L) = S_{K_i}\widetilde{Z}_f(L)$ for $1 \leq i \leq e$.

Proof. Same as for the previous Theorem. □

Then we consider the following diagram, where without loss of generality we have reversed the orientation of the i-th component:

$$\begin{tikzcd}
\widetilde{Z}_f(L) \arrow[r, dashed] \arrow[d, "S_{K_i}"'] & \widetilde{Z}_f(L') = \widetilde{Z}_f(S_{K_i}[S_{K_i}L]') \\
\widetilde{Z}_f(S_{K_i}L) = \sum c_X \cdot X \arrow[r, "\text{band sum move}"'] & \sum c_X \cdot X' = \widetilde{Z}_f([S_{K_i}L]') \arrow[u, "S_{K_i}"']
\end{tikzcd}$$

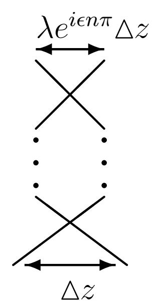
First applying the map S_{K_i} to $\widetilde{Z}_f(L)$ we obtain $\widetilde{Z}_f(S_{K_i}L)$ by Theorem 8.2.3. We write this quantity as $\sum c_X \cdot X$. We can apply Theorem 1.6.3 for $S_{K_i}L$ since then the band sum move results in K_i being added to K_j. We obtain $\sum c_X \cdot X'$ under the band sum move. This equals $\widetilde{Z}_f([S_{K_i}L]')$. By further reversing the orientation of K_i, we get, using Theorem 8.2.3 again, $\widetilde{Z}_f(S_{K_i}[S_{K_i}L]')$. Since $S_{K_i}[S_{K_i}L]' = L'$, this last quantity is $\widetilde{Z}_f(L')$. We have that the band sum move in the event of a subtraction is given by closing the above diagram, and this corresponds to the composition of S_{K_i}, a band sum, and S_{K_i}. Each of those maps can be represented in book notation by three respective matrices whose action is by congruence. This composition is then easily represented by a congruence with a matrix obtained by multiplying those three matrices.

Chapter 9

Recovering $Z_f(L) \in \overline{\mathcal{A}}(\amalg S^1)$ from ${}^bZ_f(L)$

Proposition 9.1. For an unknown e-component link L represented as a plat, from ${}^bZ_f(L)$ we can recover its expression in $\hat{\mathcal{A}}(\amalg^e S^1)$, without knowing the skeleton on which tangle chord diagrams represented by the books are based.

Proof. It suffices to know the algebraic number of crossings between strands. Indeed, chord diagrams are blind to the winding of the skeleta of tangle chord diagrams. Now, the degree one coefficient of the following tangle (where a crossing is either positive or negative, $\lambda > 0$, $\epsilon = \pm 1$, n the number of half-twists):



(9.1)

in $Z_{f,1}(L)$ is:

$$\pm\frac{1}{2\pi i}\log\frac{\lambda e^{i\epsilon n\pi}\ \Delta z}{\Delta z} = \pm\Big(\frac{1}{2\pi i}\log\lambda\Big)\ \pm\epsilon\frac{n}{2} \tag{9.2}$$

Thus if we have an odd number of half-twists between 2 strands, $n/2$ is fractional, an integer otherwise. It follows that from ${}^bZ_{f,1}(L)$ we can determine for a fixed strand the collection of strands to its right near the top of L it has an odd number of half-twists with, or equivalently what are those strands to its right near the top of L it passes on the right at the bottom of L. Doing this for all strands we can determine the permutation that to any strand near the top of L associates a strand near the bottom of the link. This is sufficient to associate to any book the element of $\mathcal{A}(\amalg^e S^1)$ it corresponds to. $\square$

Chapter 10

Normalizations in LMO Invariant Theory

Le, Murakami and Ohtsuki [LM5], [LM6] defined a quotienting map that enables one to construct a 3-manifold invariant from a renormalized version $\check{Z}_f$ of the hatted framed Kontsevich integral $\hat{Z}_f$. In Section 4.2, we showed using only the long chords lemma [ChDu], the multiplicativity of the framed Kontsevich integral (Proposition 2.2.2), as well as the fact that $\hat{Z}_f$ is an isotopy invariant (Theorem 2.2.5), that the normalization $\widetilde{Z}_f = \nu^{-1}\hat{Z}_f$ leads to a link invariant that is well-behaved under band sum moves for which in a small window where such moves are performed, link components of interest are locally on top of each other, and the resulting band is vertical. Independently, Le, Murakami and Ohtsuki showed ([LM3], [LM5], [LM6]) that if one uses the formalism of q-tangle diagrams [LM1] and carefully considers associators, one finds that another normalization $\check{Z}_f = \nu\hat{Z}_f$ is well-behaved under band sum moves for which in a small window where such moves are performed, link components of interest are locally put side by side, and the resulting band looks horizontal. In what follows we discuss these normalizations and show how they relate to each other. We then discuss the quotienting map [LM5] necessary for producing 3-manifold invariants from the Kontsevich integral and show that any normalization survives the quotienting process, and thus any choice of normalization yields a 3-manifold invariant.

10.1 Relation between $\widetilde{Z}_f$ and LMO's $\check{Z}_f$

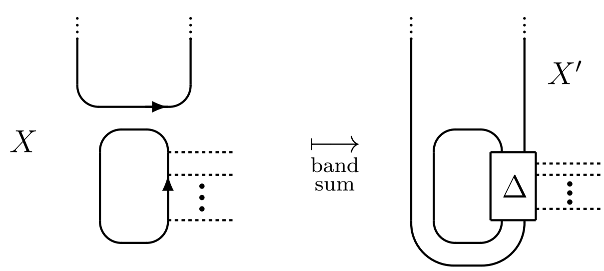
In Chapter 4 we showed that under band sum moves,

$$\widetilde{Z}_f(L) = \sum_{\substack{chord \\ diagrams\ X}} c_X X \tag{10.1}$$

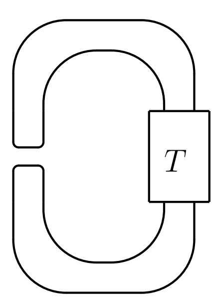
maps to:

$$\widetilde{Z}_f(L') = \sum_{\substack{chord \\ diagrams\ X}} c_X X' \tag{10.2}$$

where L is a given link, L' is the same link except that one of its components is band summed over another of its components by a vertical band that is a band whose midline is parallel to the time axis, and c_X are coefficients arising from computing the renormalized framed Kontsevich integrals of the appropriate framed links. In those expressions, X and X' are related as follows:

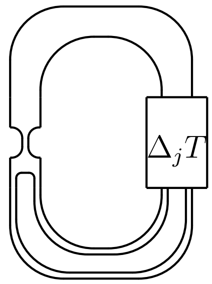


We proved such a statement by stretching one strand from each of the two components involved in the band sum to one side as in:

$$\hat{Z}_f(L) = \hat{Z}_f($$


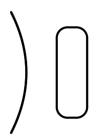
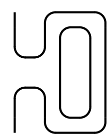
$$) \tag{10.3}$$

and by performing a band sum:

$\hat{Z}_f[\mathrm{M}]($ 

$)$ (10.4)

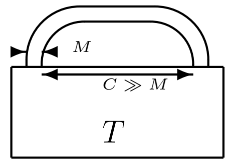
where in this particular instance, the j-th component is being band summed over. In [LM2], [LM3], [LM5], [LM6], [O] however, computations are performed by considering strands involved in the band sum side by side, resulting in a horizontal band that is a band whose midline is transverse to the time axis:

$\hat{Z}_f($  $) \quad \mapsto \quad \hat{Z}_f($  $)$ (10.5)

where the open strand is the portion of the link component operating the band sum and the closed component represents the link component being band summed over. This pictorial representation is symbolic and does not display how complex the link components are and whether they are linked to other link components. In order to compare both computations, we will use the equivalent normalization procedure whereby each local extremum is multiplied by $\nu^{1/2}$ instead of having only local maxima being multiplied by ν. This we do only in this section.

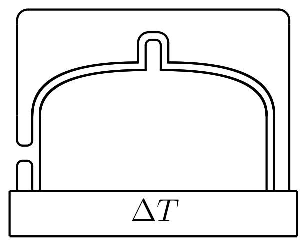
We consider an e-component link L. We focus on two of its components K_i and K_j, $1 \leq i, j \leq e$ and consider the band sum of K_i over K_j. The resulting link after band sum we call L'. Using the isotopy invariance of $\hat{Z}_f$ we can pull up one local max from each of K_i

and K_j so that we compute:

$\hat{Z}_f(L) = \hat{Z}_f($ 

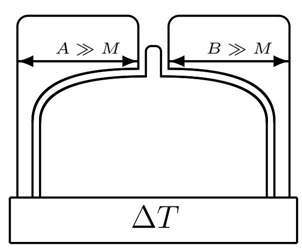
$)$ (10.6)

where T is what remains of L, and we have displayed the local maximum from K_i as the outer strand. Le Murakami et al ([LM3], [LM5], [LM6]) would compute $\hat{Z}_f$ after band sum move as:

$\hat{Z}_f($ 

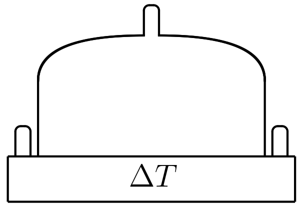
$)$ (10.7)

The protruding local maxima at the top are not necessary for Le and Murakami's computation, but are convenient to emphasize the relation with the computation that we suggest in this work:

$\hat{Z}_f($ 

$)$ (10.8)

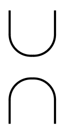
An intermediate stage both computations should be equal to by isotopy invariance of $\hat{Z}_f$ is the following:

$$\hat{Z}_f(\quad$$  $$\quad) \tag{10.9}$$

In the next section, we show (10.7) evaluates to (10.9). In the section that follows, we show (10.8) evaluates to this intermediate value, and this will show that whether we compute $\hat{Z}_f(L')$ using LMO's procedure of locally putting link components involved in the band sum side by side or putting them on top of each other as done in the present work, one really gets the same thing. In the last section, we take stock and draw some conclusions.

10.1.1 LMO's computation

In [LM3], [LM5], [LM6], [O], a window is fixed inside of which one has the following tangle:

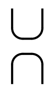
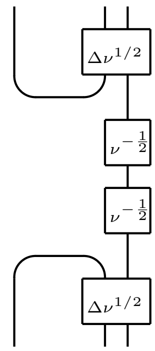


$$\tag{10.10}$$

Then one has:

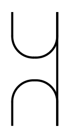
$$\hat{Z}_f(L') = \hat{Z}_f(\quad$$  $$\quad) \cdot \Delta_j \hat{Z}_f(L \setminus \mid \ \mid) \tag{10.11}$$

In [O], using q-tangles, it is shown that:

$$\hat{Z}_f(\ $$  $$\ ) = $$ 

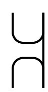
(10.12)

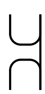
Outside of the window one has $\Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \vert\ \vert\ )$. This comes from computing $\hat{Z}_f[\mathrm{M}]\left(\Delta_j \left(L \setminus \vert\ \vert\ \right)\right)$. We have equality of these two quantities in the case of geometric tangles only when the duplicate $K_j^{(2)}$ of K_j is analytically coincident with K_j, that is in the window one really works with:



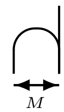
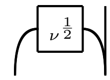
(10.13)

Then:

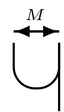
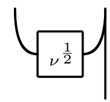
$$\hat{Z}_f(L') = \hat{Z}_f[\mathrm{M}](L') = \hat{Z}_f[\mathrm{M}](\ $$  $$\ ) \cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \vert\ \vert\ ) \qquad (10.14)$$

where the value of $\hat{Z}_f[\mathrm{M}](\ $  $\ )$ is given by the following lemma:

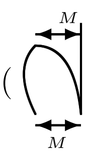
Lemma 10.1.1.1. For $M > 0$,

$\hat{Z}_f[\mathrm{M}]($  $) =$  (10.15)

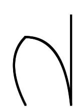
and:

$\hat{Z}_f[\mathrm{M}]($  $) =$  (10.16)

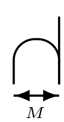
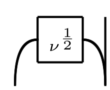
Proof. It suffices to consider the case of a local maximum, the local minimum computation being the same. We have, by isotopy invariance of $\hat{Z}_f$ that:

$\hat{Z}_f[\mathrm{M}]($  $) = \hat{Z}_f[\mathrm{M}]($  $) = \nu^{1/2} \otimes 1 \cdot Z_f[\mathrm{M}]($  $)$ (10.17)

where the first factor in the tensor product multiplies the local max and the identity multiplies the straight strand. We have:

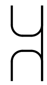
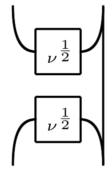
$Z_f[\mathrm{M}]($  $) =$  (10.18)

from which we get:

$\hat{Z}_f[\mathrm{M}]($  $) =$  (10.19)

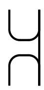
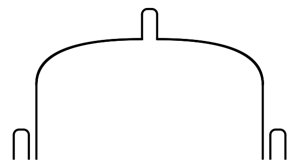
$\square$

Using this lemma it follows:

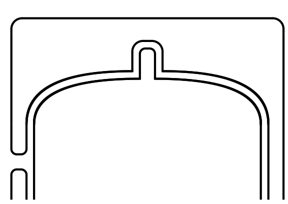
$\hat{Z}_f[\mathrm{M}]($  $) =$  (10.20)

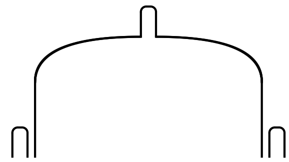
$\hat{Z}_f(L')$ as computed by Le, Murakami et al and rewritten for geometric tangles as done above does equal (10.9) as the following Proposition shows:

Proposition 10.1.1.2.

$\hat{Z}_f[\mathrm{M}]($  $) \cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \mid \; \mid) = \hat{Z}_f($  $)$ (10.21)

Proof. We slice $L \setminus \mid \; \mid$ into three horizontal strips, the middle one containing the window that was initially frozen. To the far right of that strip, strands see those in the window as straight strands as $C \gg M$. It follows that we can replace in the above expression $\hat{Z}_f[\mathrm{M}]($  $)$ by $\hat{Z}_f[\mathrm{M}]$ of the middle strip. By multiplicativity of the Kontsevich integral this equals

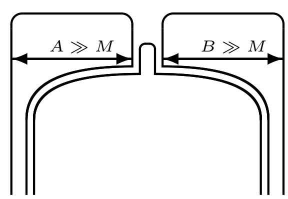
$\hat{Z}_f($  $)$ (10.22) which by isotopy invariance of $\hat{Z}_f$ is equal to (10.9) that we reproduce below for convenience:

$\hat{Z}_f($  $)$ (10.23)

$\square$

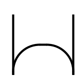
10.1.2 Alternate computation

In the present paper we considered doing:

$\hat{Z}_f($ 

$)$ (10.24)

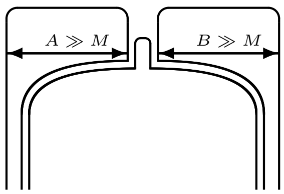
Slice L' into three horizontal slices so that the window containing the tangle:

 (10.25)

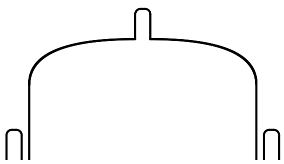
is exactly enclosed within the middle strip. Since $A \gg M$ and $B \gg M$ strands other than those in the window see those from the window as being straight, and thus we can write:

$$\hat{Z}_f[\mathrm{M}](\quad) \cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \quad) \tag{10.26}$$

$$= \hat{Z}_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad) \times \hat{Z}_f[\mathrm{M}](\quad) \tag{10.27}$$



$$= \hat{Z}_f[\mathrm{M}](\quad) \tag{10.28}$$



$$= \hat{Z}_f[\mathrm{M}](\quad) \tag{10.29}$$

where in the last step we have used the isotopy invariance of $\hat{Z}_f$.

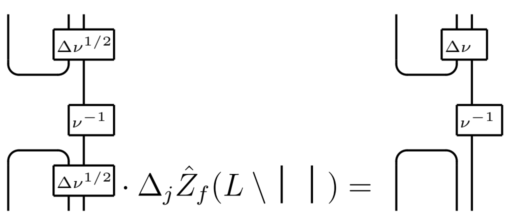
10.1.3 Taking stock: conclusions

In [O], [LM3], [LM5], [LM6], the behavior of $\hat{Z}_f$ under band sum moves on link components should be more properly called map on $\hat{Z}_f$ induced by band sum moves on link components with windows. The following statement holds at the level of q-tangles:

$$\hat{Z}_f(L) = \hat{Z}_f(\mid \; \mid) \cdot \hat{Z}_f(L \setminus \mid \; \mid) \tag{10.30}$$

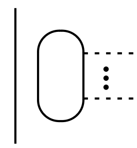
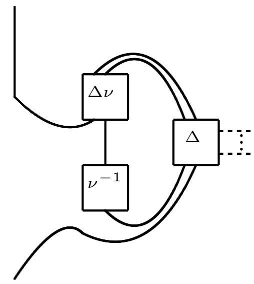
$$= \mid \; \mid \cdot \hat{Z}_f(L \setminus \mid \; \mid) \tag{10.31}$$

$$\longmapsto \hat{Z}_f(L') = \hat{Z}_f(\quad) \cdot \Delta_j \hat{Z}_f(L \setminus \mid \; \mid) \tag{10.32}$$

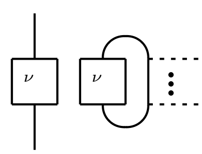


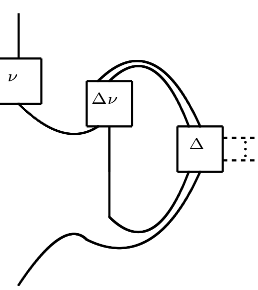
$$= \quad \cdot \Delta_j \hat{Z}_f(L \setminus \mid \; \mid) = \quad \cdot \Delta_j \hat{Z}_f(L \setminus \mid \; \mid) \tag{10.33}$$

In other terms:

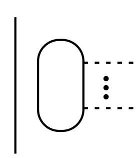
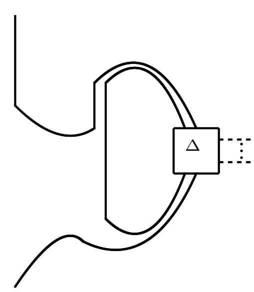
$$\hat{Z}_f(L) = \sum \lambda$$  $$\underset{\substack{band\\sum}}{\mapsto} \sum \lambda$$  (10.34)

where the pack of chords displayed above comes from computing $\hat{Z}_f(L \setminus |\;\; |\,)$. In other terms those chords are very much dependent on the window we have chosen. Multiplying both expressions by a power of ν on each link component we obtain the following statement about $\check{Z}_f$:

$$\check{Z}_f(L) = \sum \lambda$$ 

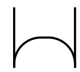
$$\underset{\substack{band\\sum}}{\mapsto} \sum \lambda$$  (10.35)

That is for $\check{Z}_f$ one has the simple statement:

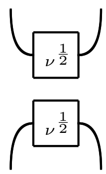
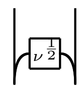
$$\check{Z}_f(L) = \sum \mu$$  $$\underset{\substack{band\\sum}}{\mapsto} \sum \mu$$  (10.36)

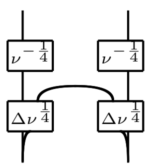
In the present paper we prefer to work with geometric tangles, for which a computation such as the one above can make use of windows, provided we deal with analytically coincident tangles. In this case we obtain:

$$\hat{Z}_f(L) = \hat{Z}_f[\mathrm{M}](\asymp) \cdot \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{10.37}$$

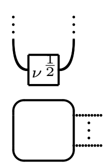
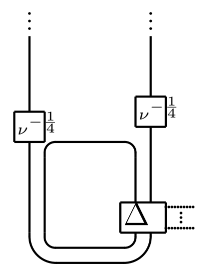
$$\longmapsto \hat{Z}_f[\mathrm{M}](L') = \hat{Z}_f[\mathrm{M}](\quad$$  $$\quad) \cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{10.38}$$

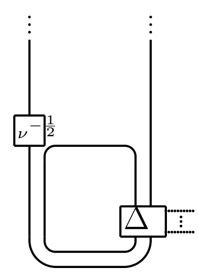
which we can explicity write as:

$$=$$  $$\cdot \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \mapsto$$  $$\cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{10.39}$$

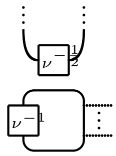
$$=$$  $$\cdot \Delta_j \hat{Z}_f[\mathrm{M}](L \setminus \asymp) \tag{10.40}$$

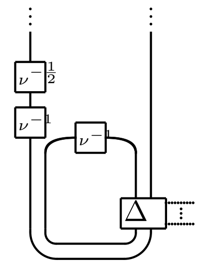
In other terms:

$$\hat{Z}_f(L) = \sum \lambda$$  $$\mapsto \hat{Z}_f(L') = \sum \lambda$$  $$\tag{10.41}$$

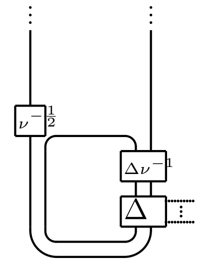
$$= \sum \lambda$$  $$\tag{10.42}$$

where the pack of chords come from $\hat{Z}_f[\mathrm{M}](L \setminus \asymp)$ as well as $\nu^{1/2}$ and is thus dependent on the choice of window. Multiplying this mapping by a factor of ν^{-1} on each link component we get:

$$\widetilde{Z}_f(L) = \sum \lambda$$ 

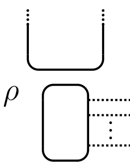
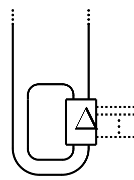
$$\mapsto \widetilde{Z}_f(L') = \sum \lambda$$ 

(10.43)

$$= \sum \lambda$$ 

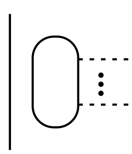
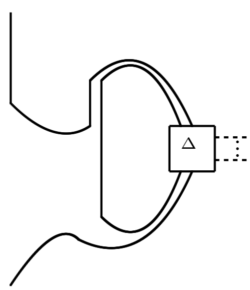
(10.44)

that is we have a statement of the form:

$$\widetilde{Z}_f(L) = \sum \rho$$ 
$$\longmapsto \widetilde{Z}_f(L') = \sum \rho$$ 

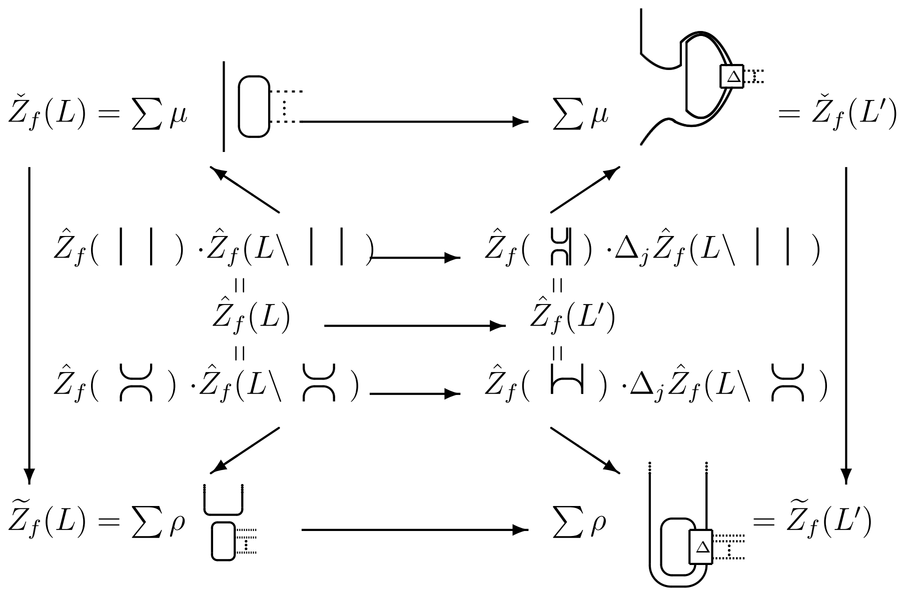
(10.45)

Thus the above statement, as well as:

$$\check{Z}_f(L) = \sum \mu$$ 
$$\underset{\substack{band\\ sum}}{\mapsto} \sum \mu$$ 

(10.46)

are true in their own right, but are not related by a simple renormalization as the packs of chords displayed in each statement are very much dependent on the window selected in $\hat{Z}_f(L)$ and $\hat{Z}_f(L')$, from which one normalization or the other led to symbolic, window dependent statements about $\check{Z}_f$ or $\widetilde{Z}_f$. However if one keeps track of what those chords really represent and one concatenates elements properly, one truly gets the following commutative diagram:



(10.47)

where horizontal maps denote a band sum move, and both diagonal and vertical maps are renormalizations. To be specific, ascending diagonal maps are multiplications of every link component in the expressions for $\hat{Z}_f(L)$ or $\hat{Z}_f(L')$ by ν, thus defining the corresponding $\check{Z}_f$ values, and descending diagonal maps are multiplications by ν^{-1} as prescribed by the definition of $\widetilde{Z}_f$. The vertical maps are multiplications of each link component of $\check{Z}_f(L)$ and $\check{Z}_f(L')$ by ν^{-2} to get the corresponding $\widetilde{Z}_f$ values.

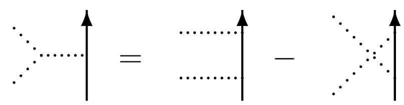
10.2 3-Manifold Invariants constructed from the Kontsevich Integral

In [LM5] a quotienting map is defined that allows one to construct a 3-manifold invariant from $\check{Z}_f$. The proof of invariance of the equivalence class of such an object under Kirby move II leaves room for other renormalizations to also be invariant by inserting a fixed power of ν

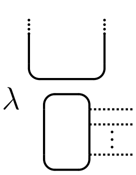
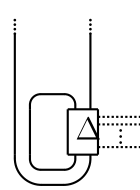
on each link component. We first show that this is so and introduce a family of 3-manifold invariants built from renormalizations of $\hat{Z}_f$. In the section that follows we generalize this further by arguing that renormalization by elements of $\hat{\mathcal{A}}(\amalg^e S^1)$ that verify certain more general conditions also lead to a family of 3-manifold invariants.

10.2.1 Renormalizations of $\hat{Z}_f$ and the 3-manifold invariants they generate

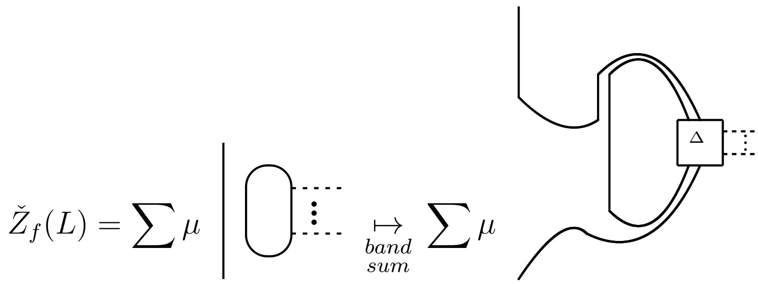
We first review the requisite formalism and definitions as covered in [LM5]. If $\mathcal{A}(X)$ denotes the algebra of chord diagrams with support on X mod the STU relation ([BN]):

 (10.48)

then one defines $\mathring{\mathcal{A}}(X)$ to be obtained by including in $\mathcal{A}(X)$ dashed trivial circles as well, with a graded completion denoted by $\mathring{\mathcal{A}}(X)^\wedge$. For $2n$ points arranged in a circle one denotes by P_n the relation equating the sum of all pairings of those $2n$ points by dotted arcs to zero. Denote by $L_{<2n}$ the equivalence relation between elements of $\mathring{\mathcal{A}}(\amalg S^1)$ defined as follows: for D_1, $D_2 \in \mathring{\mathcal{A}}(\amalg S^1)$, $D_1 \sim D_2$ if and only if $D_1 - D_2$ is a sum of chord diagrams with less than $2n$ chord feet on each summand. O_n is the equivalence relation such that a dashed circle is equivalent to $-2n$. In this section we use tangle chord diagrams that explicitly display chords ending on them. We build our reasoning on the statement (4.77):

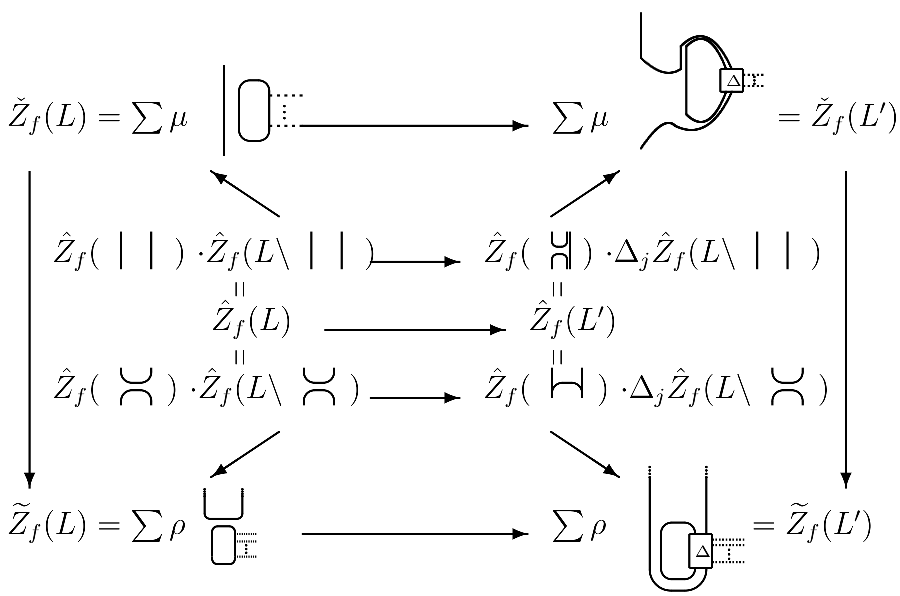
$\widetilde{Z}_f(L) = \sum \lambda$  $\longmapsto \widetilde{Z}_f(L') = \sum \lambda$  (10.49)

We could equally have started from:



(10.50)

as both renormalizations are equivalent by the following commutative diagram:



(10.51)

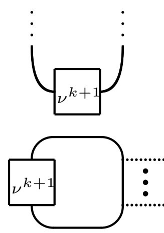
However chords displayed in (10.49) and (10.50) are different as this diagram shows. To avoid confusion, we fix chords that are shown to be complementary to the choice of window for which a convenient normalization is $\widetilde{Z}_f$. Thus we work with (10.49).

We now define a renormalization of such an invariant indexed by the number $k+1$ of powers of ν by which it is renormalized as:

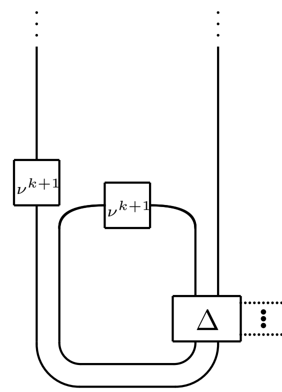
$$\overset{\vee(k)}{\mathrm{Z_f}} = \nu^{k+1} \otimes \cdots \otimes \nu^{k+1} \cdot \widetilde{Z}_f \tag{10.52}$$

in such a manner that $\mathrm{Z_f}^{\vee(0)} = \hat{Z}_f$, $\mathrm{Z_f}^{\vee(-1)} = \widetilde{Z}_f$, and $\mathrm{Z_f}^{\vee(1)} = \check{Z}_f$ as defined in [LM5]. If we multiply each link component of (10.49) by a $k+1$-st power of ν, then we get:

$$\overset{\vee(k)}{\mathrm{Z_f}}(L) = \sum \lambda$$



$$\mapsto \overset{\vee(k)}{\mathrm{Z_f}}(L') = \sum \lambda \tag{10.53}$$



The following result was originally given in [LM5] in the case $k = 1$.

Proposition 10.2.1.1. ([LM5]) For $k \in \mathbb{Z}$, L an e-components framed oriented link and n an integer, the class $[\mathrm{Z_f}^{\vee(k)}(L)]$ in $\mathring{\mathcal{A}}(\coprod^e S^1)^\wedge / L_{<2n}, P_{n+1}, O_n$ is independent of the orientation of the link L and is invariant under band sum moves.

Proof. We first show the invariance of $[\mathrm{Z_f}^{\vee(k)}(L)]$ under orientation change. For K a link component, $S_{(K)}$ the map on links that implements the orientation reversal on K, we have:

$$\overset{\vee(k)}{\mathrm{Z_f}}(S_{(K)}L) = \nu^k \otimes \cdots \otimes \nu^k \cdot \hat{Z}_f(S_{(K)}L) \tag{10.54}$$

$$= \nu^k \otimes \cdots \otimes \nu^k \cdot S_{(K)}\hat{Z}_f(L) \tag{10.55}$$

$$= S_{(K)}\Big(\nu^k \otimes \cdots \otimes S_{(K)}\nu^k \otimes \cdots \otimes \nu^k \cdot \hat{Z}_f(L)\Big) \tag{10.56}$$

$$= S_{(K)} \overset{\vee(k)}{\mathrm{Z_f}}(L) \tag{10.57}$$

Thus if we write $\mathrm{Z_f}^{\vee(k)}(L) = \sum c_D \cdot D$ where D represent tangle chord diagrams supported on L and c_D are the corresponding coefficients in the expansion of $\mathrm{Z_f}^{\vee(k)}(L)$, then we want to show $[D] = [S_{(K)}D]$. This was proved in [LM5]. Thus $[\mathrm{Z_f}^{\vee(k)}(L)] = [\mathrm{Z_f}^{\vee(k)}(S_{(K)}L)]$, and this for any link component K.

We now show the invariance under band sum moves. We have to prove that:

$$[\overset{\vee(k)}{\mathrm{Z_f}}(L)] = [\overset{\vee(k)}{\mathrm{Z_f}}(L')] \tag{10.58}$$

or equivalently that:

$$[\otimes^e \nu^{k+1} \cdot \widetilde{Z}_f(L)] = [\otimes^e \nu^{k+1} \cdot \widetilde{Z}_f(L')] \tag{10.59}$$

We follow exactly the same line of reasoning as in [LM5]. We write $\widetilde{Z}_f(L) = \sum c_D \cdot D$, so that:

$$\overset{\vee(k)}{\mathrm{Z_f}}(L) = \otimes^e \nu^{k+1} \cdot \sum c_D \cdot D \tag{10.60}$$

$$= \sum c_D \otimes^e \nu^{k+1} \cdot D \tag{10.61}$$

$$= \sum c_{D',D} D' \cdot D \tag{10.62}$$

where the chord diagrams D' come from powers of ν^{k+1} and the coefficients $c_{D',D}$ take account of the expansion of such powers of ν. We have $\widetilde{Z}_f(L') = \sum c_D \cdot \Delta D$, so that:

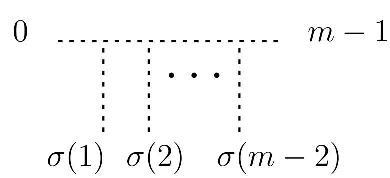
$$\overset{\vee(k)}{\mathrm{Z_f}}(L') = \otimes^e \nu^{k+1} \cdot \sum c_D \cdot \Delta D \tag{10.63}$$

$$= \sum c_D \otimes^e \nu^{k+1} \cdot \Delta D \tag{10.64}$$

$$= \sum c_{D',D} D' \cdot \Delta D \tag{10.65}$$

We want to show that $[D' \cdot D] = [D' \cdot \Delta D]$. If there are less than $2n$ chord feet on K_2, then both classes vanish by $L_{<2n}$. If there are exactly $2n$ chord feet on K_2, then all terms but one in $[D' \cdot \Delta D]$ have less than $2n$ chord feet on K_2, and therefore die by $L_{<2n}$. The surviving term is exactly $[D' \cdot D]$. If there are more than $2n$ chord feet on K_2, we use Lemma 3.1 of [LM5] that allows us to write a chord diagram with more than $2n$ chord feet on K_2 as a linear combination of chord diagrams with less chord feet on it. Applying this lemma repeatedly, we can reduce the number of chord feet on K_2. The proof involves only the use of the STU relation and P_{n+1}. If we denote by $\cdot\Delta$ the map $D' \cdot D \mapsto D' \cdot \Delta D$, then we want to show that it commutes with the STU relation and P_{n+1}. That this commutes with P_{n+1} is immediate since this relation does not involve the support of chord diagrams. Thus we turn to the commutation with the STU relation. In decreasing the degree of $D' \cdot D$, we can decrease the number of chords coming from D' or the number of chords from D. We start by considering the case where we decrease the number of chords from D. If D' were trivial, then we would just use the commutative diagram showing the commutativity of Δ and STU given in [LM5]. Adding D' decomposed as D'_1 on one strand and D'_2 on the second to the previous commutative diagram and replacing Δ by $\cdot\Delta$, we still have commutativity of the resulting diagram. If we now decrease the number of chords in D', then we wish to apply the STU relation to chords from D', and this commutes with applying the map $\cdot\Delta$ as well. Thus $[D' \cdot D] = [D' \cdot \Delta D]$, and this for any chord diagram D' coming from a $k+1$-st power of ν, and for all summands D of $\widetilde{Z}_f(L)$. Thus $[\otimes^e \nu^{k+1} \cdot \widetilde{Z}_f(L)] = [\otimes^e \nu^{k+1} \cdot \widetilde{Z}_f(L')]$, or equivalently $[\mathrm{Z_f}^{\vee(k)}(L)] = [\mathrm{Z_f}^{\vee(k)}(L')]$. $\square$

In order to have an invariant under Kirby move I, one moves the above equivalence class into another quotient space. Before doing so we make a few definitions, all of which once again can be found in the original paper [LM5]. If X is a set with m ordered elements labeled $0, 1, \cdots, m-1$, write $\mathcal{A}(m) = \mathcal{A}(X)$. For $\sigma \in S_{m-2}$, define T_σ to be the graph:



(10.66)

Define:

$$T_m = \sum_{\sigma \in S_{m-2}} \frac{(-1)^{r(\sigma)}}{(m-1) \cdot \frac{(m-2)!}{r(\sigma)!(m-2-r(\sigma))!}} T_\sigma \tag{10.67}$$

where $r(\sigma) = \mathrm{Card}\{i \mid \sigma(i) > \sigma(i+1)\}$. For $m \geq 2$, define $T^n_m \in \mathcal{A}(m)$ by:

$$T^n_m = \sum_{\substack{m_1+m_2+\cdots+m_n=m \\ m_1\geq m_2\geq\cdots\geq m_n\geq 2}} \coprod_{1\leq i\leq n} T_{m_i} \tag{10.68}$$

where for each choice of a configuration $(m_1, \cdots, m_n)$ the cyclic order of the T_{m_i}'s is preserved. If $m < 2m$, we set $T^n_m = 0$. Define:

$$\iota_n : \mathcal{A}(\amalg^e S^1) \to \mathcal{A}(\varnothing) \tag{10.69}$$

by sending a solid circle with m chords ending on it to T^n_m with those m chords grafted onto the m points defining the support of T^n_m. We now consider $[\mathrm{Z_f}^{\vee(k)}(L)] \in \mathring{\mathcal{A}}(\amalg^e S^1)^\wedge/L_{<2n}, P_{n+1}, O_n$, map it into $\mathring{\mathcal{A}}(\varnothing)^\wedge/P_{n+1}, O_n$, which we further project to $\mathring{\mathcal{A}}(\varnothing)/D_{>n}, P_{n+1}, O_n$. The resulting element we write $[\iota_n(\mathrm{Z_f}^{\vee(k)}(L))]$. The relation $D_{>n}$ sets to zero chord diagrams of degree greater than n. This last quotient space is isomorphic to $\mathcal{A}(\varnothing)/D_{>n}$ by Lemma 3.4 of [LM5].

The following result was originally given in [LM5] in the case $k = 1$.

Proposition 10.2.1.2. $[\iota_n(\mathrm{Z_f}^{\vee(k)}(L))] \in \mathcal{A}(\phi)/D_{>n}$ is independent of the orientation of L and is invariant under band sum moves.

Proof. This follows from the previous Proposition and by following the proof of the same statement for $\check{Z}_f$, Lemma 3.5 of [LM5]. $\square$

Recall the following definitions originally introduced in [LM5]: U_+ (resp. U_-) is the trivial knot with $+1$ (resp. -1) framing, and one puts an algebra structure on $\mathcal{A}(\phi)/D_{>n}$ whereby the disjoint union in the argument of the Kontsevich integral results in a product of equivalence classes. One lets σ_+ (resp.σ_-) be the number of positive (resp. negative) eigenvalues of the linking matrix for L. If M denotes the 3-manifold obtained from a surgery on the link L, we would like to define a family of invariants:

$$\Omega_{k,n}(M) = [\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))]^{-\sigma_+}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_-))]^{-\sigma_-}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L))] \in \mathcal{A}(\varnothing)/D_{>n} \tag{10.70}$$

with $\Omega_{1,n}(M) = \Omega_n(M)$ in [LM5]. If $\alpha^{(d)}$ denotes the degree d part of a sum of chord diagrams α as in [LM5], then we define:

$$\Omega_k(M) = 1 + \sum_{d \geq 1} \Omega_{k,d}(M)^{(d)} \tag{10.71}$$

with $\Omega_1(M) = \Omega(M)$ in [LM5]. The rest of this paper is devoted to showing that $[\iota_n({\mathrm{Z_f}}^{\vee(k)}(U_{\pm}))]$ are invertible, and we will mainly follow the argument given in [LM5] to emphasize that their proof can be easily generalized to the present case.

One first defines a linear map ([LM5]):

$$p : \mathcal{A}^{\wedge}(X \amalg X) \to \mathcal{A}^{\wedge}(X) \otimes \mathcal{A}^{\wedge}(X) \tag{10.72}$$

that sends any chord diagram D with support on $X \amalg X$ with at least one chord stretching between one X and the other copy of X to zero, $D\Big|_{X_L} \otimes D\Big|_{X_R}$ otherwise, where L and R refer to left and right copies respectively. If $X = \cup_{1 \leq i \leq e} X_i$, then by successively applying the doubling map Δ on each of the e components of X, one defines a map ([LM5]):

$$\Delta_{(X_1, \cdots, X_e)} : \mathcal{A}^{\wedge}(X) \to \mathcal{A}^{\wedge}(X \amalg X) \tag{10.73}$$

Define ([LM5]):

$$\hat{\Delta} := p \circ \Delta_{(X_1, \cdots, X_e)} \tag{10.74}$$

In other terms, $\hat{\Delta}$ doubles all components and kills those chord diagrams that have at least one chord stretching between one copy of X and the other. We have the following fact:

$$\hat{\Delta}(\overset{\vee(k)}{\mathrm{Z_f}}(L)) = \overset{\vee(k)}{\mathrm{Z_f}}(L) \otimes \overset{\vee(k)}{\mathrm{Z_f}}(L) \tag{10.75}$$

Indeed:

$$\hat{\Delta}(\overset{\vee(k)}{\mathrm{Z_f}}(L)) = p \circ \Delta_{(X_1,\cdots,X_e)}(\overset{\vee(k)}{\mathrm{Z_f}}(L)) \tag{10.76}$$

$$= p \circ \Big(\Delta_{(X_1)}(\nu^{k+1}) \otimes \cdots \otimes \Delta_{(X_e)}(\nu^{k+1}) \cdot \Delta_{(X_1,\cdots,X_e)}\widetilde{Z}_f(L)\Big) \tag{10.77}$$

$$= p \circ \Big((\Delta\nu)^{k+1} \otimes \cdots \otimes (\Delta\nu)^{k+1} \cdot \widetilde{Z}_f(\Delta L)\Big) \tag{10.78}$$

$$= p \circ \Big(\underbrace{\nu^{k+1} \otimes \cdots \otimes \nu^{k+1}}_{2e} \cdot \widetilde{Z}_f(\Delta L)\Big) \tag{10.79}$$

$$= p \overset{\vee(k)}{\mathrm{Z_f}}(\Delta L) \tag{10.80}$$

$$= \overset{\vee(k)}{\mathrm{Z_f}}(L) \otimes \overset{\vee(k)}{\mathrm{Z_f}}(L) \tag{10.81}$$

The following result was originally given in [LM5] in the case $k = 1$.

Lemma 10.2.1.3. $[\iota_n(\mathrm{Z_f}^{\vee(k)}(U_+))]$ and $[\iota_n(\mathrm{Z_f}^{\vee(k)}(U_-))]$ are invertible in $\mathcal{A}(\varnothing)/D_{>n}$.

Proof. The proof is the same as in [LM5]. Successive applications of $\hat{\Delta}$ lead to defining:

$$\hat{\Delta}^{(1)} = \hat{\Delta} \tag{10.82}$$

$$\hat{\Delta}^{(2)} = (\hat{\Delta} \otimes 1) \circ \hat{\Delta} \tag{10.83}$$

$$\hat{\Delta}^{(3)} = (\hat{\Delta} \otimes 1 \otimes 1) \circ \hat{\Delta}^{(2)} \tag{10.84}$$

$$\vdots$$

$$\hat{\Delta}^{(k)} = (\hat{\Delta} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1}) \circ \hat{\Delta}^{(k-1)} \tag{10.85}$$

Using the fact that $\hat{\Delta}(\mathrm{Z_f}^{\vee(k)}(L)) = \mathrm{Z_f}^{\vee(k)}(L) \otimes \mathrm{Z_f}^{\vee(k)}(L)$, we readily find:

$$\hat{\Delta}^{(n-1)} \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm) = \Big(\overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big)^{\otimes n} \tag{10.86}$$

One defines two maps induced by $\hat{\Delta}$. The first is denoted $\hat{\Delta}_{n,m}$ ([LM5]):

$$\begin{aligned} \hat{\Delta}_{n,m} : \mathring{\mathcal{A}}(\amalg^e S^1)\wedge)/L_{<2(n+m)}, P_{n+m+1}, O_{n+m} \\ \to \mathring{\mathcal{A}}(\amalg^e S^1)^\wedge/L_{<2n}, P_{n+1}, O_n \otimes \mathring{\mathcal{A}}(\amalg^e S^1)^\wedge/L_{<2m}, P_{m+1}, O_m \end{aligned} \tag{10.87}$$

One uses in what follows the less cumbersome notation PO_n for $P_{n+1}O_n$ as employed in [LM5]. The second map is denoted $\hat{\Delta}^{(l)}_{1,\cdots,1}$:

$$\hat{\Delta}^{(l)}_{1,\cdots,1} : \mathcal{A}(\varnothing)/D_{>l+1} \to \Big(\mathcal{A}(\varnothing)/D_{>1}\Big)^{\otimes l+1} \tag{10.88}$$

Now we invoke Lemma 4.2 of [LM5] which states that the following diagram is commutative (shown for three integers $n = n_1 + n_2$):

$$\begin{tikzcd}
\mathring{\mathcal{A}}(\amalg^e S^1)/L_{<2n}, PO_n \arrow[r, "\iota_n"'] \arrow[d, "\hat{\Delta}_{n_1,n_2}"'] & \mathring{\mathcal{A}}(\varnothing)/PO_n \arrow[d, "\hat{\Delta}"] \\
\mathring{\mathcal{A}}(\amalg^e S^1)/L_{<2n_1}, PO_{n_1} \otimes \mathring{\mathcal{A}}(\amalg^e S^1)/L_{<2n_2}, PO_{n_2} \arrow[r, "\iota_{n_1}\otimes\iota_{n_2}"] & \mathring{\mathcal{A}}(\varnothing)/PO_{n_1} \otimes \mathring{\mathcal{A}}(\varnothing)/PO_{n_2}
\end{tikzcd} \tag{10.89}$$

We apply this lemma $n-1$ times. The composition of the $n-1$ maps on the left is:

$$\Big(\hat{\Delta}_{1,1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-2}\Big) \circ (\hat{\Delta}_{2,1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-3}\Big) \circ \cdots \circ \Big(\hat{\Delta}_{n-2,1} \otimes 1\Big) \circ \hat{\Delta}_{n-1,1} \tag{10.90}$$

We denote by $\Big(\hat{\Delta}^{(n-1)}\, \mathrm{Z_f}^{\vee(k)}(U_\pm)\Big)_{1,\cdots,1}$ the image of $\mathrm{Z_f}^{\vee(k)}(U_\pm)$ under this map. The commutativity of the $n-1$ squares yields:

$$\hat{\Delta}^{(n-1)}_{1,\cdots,1}\Big([\iota_n \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)]\Big) = \Big[\iota_1^{\otimes n}\Big(\hat{\Delta}^{(n-1)} \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big)_{1,\cdots,1}\Big] \tag{10.91}$$

Now observe that:

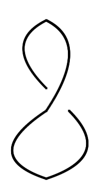
$$\Big(\hat{\Delta}^{(n-1)} \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big)_{1,\cdots,1} = \bigg(\overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big|_{\mathring{\mathcal{A}}(\amalg^e S^1)^\wedge/L_{<2}, PO_2}\bigg)^{\otimes n} \tag{10.92}$$

From which it follows that:

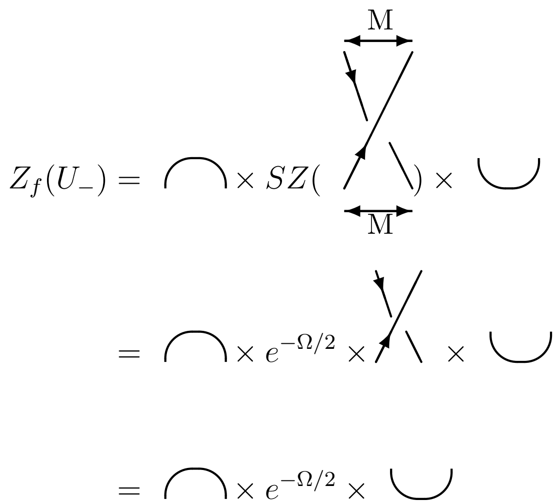
$$\Big[\iota_1^{\otimes n}\Big(\hat{\Delta}^{(n-1)} \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big)_{1,\cdots,1}\Big] = \Big[\iota_1 \overset{\vee(k)}{\mathrm{Z_f}}(U_\pm)\Big]^{\otimes n} \tag{10.93}$$

We wish to show that $[\iota_1\, \mathrm{Z_f}^{\vee(k)}(U_\pm)]$ is non-trivial. This will show that the constant part of $[\iota_n\, \mathrm{Z_f}^{\vee(k)}(U_\pm)]$ is non-zero, implying that it is invertible. Without loss of generality, we can work with U_-:

$$U_- = \tag{10.94}$$



We have, using the fact that $Z_f(K) = Z_f[\mathrm{M}](K)$ for any $M > 0$:



(10.95)

(10.96)

(10.97)

$$= \bigcirc - \frac{1}{2} \ominus + \cdots \tag{10.98}$$

On the other hand:

$$\nu = \bigcirc + \text{terms of even order} \tag{10.99}$$

Thus:

$$\overset{\vee(k)}{\mathrm{Z_f}}(U_-) = \nu^{k+1} \cdot \nu^{-1} \cdot \nu^{1} \cdot Z_f(U_-) = \bigcirc - \frac{1}{2} \ominus + \text{higher order terms} \tag{10.100}$$

In $\mathring{\mathcal{A}}(\amalg^e S^1)^\wedge / L_{<2}, P_2, O_1$, the relation $L_{<2}$ gets rid of the first term. Under the mapping ι_1, the second term maps to T_2^1 joined to an isolated chord, which is a trivial dashed circle, which according to the relation O_1 is equivalent to $-2 \cdot 1$. Thus $[\iota_1 \, \mathrm{Z_f}^{\vee(k)}(U_-)] = -1/2 \cdot (-2) +$ higher order terms $= 1 +$ higher order terms, and therefore $[\iota_n \, \mathrm{Z_f}^{\vee(k)}(U_-)]$ is invertible. One can show that the same result holds for $[\iota_n \, \mathrm{Z_f}^{\vee(k)}(U_+)]$. $\square$

This result shows that $\Omega_{k,n}(M)$ is well-defined. We have the following result originally given in [LM5] in the case $k = 1$.

Theorem 10.2.1.4. For $k \in \mathbb{Z}$, L a framed oriented link, M the 3-manifold obtained from a surgery on S^3 along L, the element $\Omega_{k,n}(M)$ defined by:

$$[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))]^{-\sigma_+}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_-))]^{-\sigma_-}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L))] \in \mathcal{A}(\phi)/D_{>n} \tag{10.101}$$

is a topological invariant of M for any integer n.

Proof. $\Omega_{k,n}$ is well-defined by the previous Lemma. Invariance under orientation change and band sum moves follows from Proposition 10.2.1.2. We work out the invariance under the first Kirby move, which consists of adding to or subtracting from a link a circle with ± 1 framing. If one adds a circle with framing $+1$, the framing of the component it is added to increases by 1, and likewise if one adds a circle with framing -1, the framing of the component it is added to decreases by 1. Then it suffices to write:

$$[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L \amalg U_+))] = [\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L))][\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))] \tag{10.102}$$

or equivalently:

$$[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))]^{-1}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L \amalg U_+))] = [\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L))] \tag{10.103}$$

so that:

$$\begin{aligned}&[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))]^{-(\sigma_++1)}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_-))]^{-\sigma_-}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L \amalg U_+))] = \\ &[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_+))]^{-\sigma_+}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(U_-))]^{-\sigma_-}[\iota_n(\overset{\vee(k)}{\mathrm{Z_f}}(L))]\end{aligned} \tag{10.104}$$

where on the left hand side of that equation $\sigma_+ + 1 = \sigma_+(L \amalg U_+)$. We have a similar computation for $L \amalg U_-$. $\square$

This shows that the following quantity:

$$\Omega_k(M) = 1 + \sum_{d \geq 1} \Omega_{k,d}(M)^{(d)} \tag{10.105}$$

is well-behaved and defines a family of topological invariants of 3-manifolds M, indexed by a natural number k.

10.2.2 General renormalizations of $\hat{Z}_f$ and the invariants they generate

In this section, we consider more general renormalizations of $\widetilde{Z}_f$ and study the conditions put on them to yield 3-manifold invariants. We consider an element $\gamma \in \hat{\mathcal{A}}(S^1)$ whose expansion we write $\sum_D c_D \cdot D$ for chord diagrams D with support on S^1 and coefficients c_D. We consider the general renormalization:

$$\widetilde{\mathcal{Z}}_f[\gamma](L) = \underbrace{\gamma \otimes \cdots \otimes \gamma}_{e} \cdot \widetilde{Z}_f(L) \tag{10.106}$$

for L an e-component link. We have:

$$\widetilde{Z}_f(U_\pm) = \nu^{-1} \cdot \nu \cdot Z_f(U_\pm) \tag{10.107}$$

with:

$$Z_f(U_\pm) = \frown \times e^{\pm\Omega/2} \times \smile \tag{10.108}$$

We use the notation θ to denote the unique chord diagram of degree one on the circle, and θ^p the solid circle with p parallel chords on it. Then we can write:

$$\frown \times \Omega^p \times \smile = \theta^p \tag{10.109}$$

so that:

$$\widetilde{Z}_f(U_\pm) = Z_f(U_\pm) = \sum_{p \geq 0} \frac{(\pm 1)^p}{p!} \frac{1}{2^p} \theta^p \tag{10.110}$$

We consider:

$$\begin{aligned} \widetilde{\mathcal{Z}}_f[\gamma](U_\pm) &= \gamma \cdot \widetilde{Z}_f(U_\pm) = \gamma \cdot Z_f(U_\pm) && (10.111) \\ &= \sum_D c_D \cdot D \cdot \sum_{p \geq 0} \frac{(\pm 1)^p}{p!} \frac{1}{2^p} \theta^p && (10.112) \\ &= \sum_{D, p \geq 0} c_D \frac{(\pm 1)^p}{p!} \frac{1}{2^p} D \cdot \theta^p && (10.113) \end{aligned}$$

We simplify this expression by letting:

$$\lambda_{p,\epsilon,D} = c_D \frac{\epsilon^p}{p!} \frac{1}{2^p} \tag{10.114}$$

with $\epsilon = \pm 1$, so that:

$$\widetilde{\mathcal{Z}}_f[\gamma](U_\pm) = \sum_{D,p\geq 0} \lambda_{p,\epsilon,D} D \cdot \theta^p \tag{10.115}$$

Recall that the chord degree is half the number of vertices, trivalent vertices included. Thus for D a chord diagram, if $|D|$ denotes the chord degree, we decompose that as $|D|^e + |D|^i$, where $|D|^e$ is half the number of external (or univalent) vertices, on solid lines, and $|D|^i$ is half the number of internal (or trivalent) vertices. We first regard $\widetilde{\mathcal{Z}}_f[\gamma](U_\pm)$ as an element of $\mathring{\mathcal{A}}(S^1)^\wedge/L_{<2n}, PO_n$ that we map by ι_n into $\mathring{\mathcal{A}}(\varnothing)^\wedge/PO_n$. We write:

$$\begin{aligned}
\iota_n\Big(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm)\Big) &= \iota_n\Big(\sum_{D,p\geq 0} \lambda_{p,\epsilon,D} D \cdot \theta^p\Big) && (10.116)\\
&= \sum_{D,p\geq 0} \lambda_{p,\epsilon,D}\ \iota_n(D \cdot \theta^p) && (10.117)\\
&= \sum_{m\geq 2n} \sum_{|D|^e+p=m/2} \lambda_{p,\epsilon,D}\ \iota_n(D \cdot \theta^p) && (10.118)
\end{aligned}$$

One can see $\iota_n(D \cdot \theta^p)$ in the above sum by pulling all the m chords from $D \cdot \theta^p$ out of the solid circle, and replacing this latter with T^n_m where its support of m points is grafted onto the m feet of chords from $D \cdot \theta^p$. We write the resulting object $D \cdot \theta^p(T^n_m)$. We will be interested in the constant part of such objects. Depending on m, n, and the choice of D and p, and using P_{n+1} in conjunction with O_n, we can write:

$$\iota_n(D \cdot \theta^p) = D \cdot \theta^p(T^n_m) = \alpha_{n,m,D} + \text{chord diagrams} \tag{10.119}$$

for some numbers $\alpha_{n,m,D}$, so that:

$$\begin{aligned}
\iota_n\Big(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm)\Big) &= \sum_{m\geq 2n} \sum_{|D|^e+p=m/2} \lambda_{p,\epsilon,D} D \cdot \theta^p(T^n_m) && (10.120)\\
&= \sum_{m\geq 2n} \sum_{|D|^e+p=m/2} \lambda_{p,\epsilon,D}\alpha_{n,m,D} + \text{chord diagrams of degree} \geq 1 && (10.121)
\end{aligned}$$

Once we have this quantity, we project it to $\mathring{\mathcal{A}}(S^1)/D_{>n}, PO_n \simeq \mathcal{A}(\varnothing)/D_{>n}$. In this latter space we define:

$$\Omega_{\gamma,n}(M) = [\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_+))]^{-\sigma_+}[\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_-))]^{-\sigma_-}[\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](L))] \tag{10.122}$$

and:

$$\Omega_\gamma(M) = 1 + \sum_{d\geq 1} \Omega_{\gamma,d}(M)^{(d)} \tag{10.123}$$

We have the following result:

Theorem 10.2.2.1. For $\gamma = \sum_D c_D \cdot D \in \hat{\mathcal{A}}(S^1)$, L a framed oriented link, M the 3-manifold obtained from a surgery on S^3 along L, then $\Omega_\gamma(M)$ is a topological invariant of M provided the coefficients of γ satisfy for all integers n:

$$\sum_{m\geq 2n} \sum_{|D|^e+p=m/2} \lambda_{p,\epsilon,D}\alpha_{n,m,D} \neq 0 \tag{10.124}$$

where $\epsilon = \pm 1$ in:

$$\lambda_{p,\epsilon,D} = c_D \frac{\epsilon^p}{p!}\frac{1}{2^p} \tag{10.125}$$

and:

$$D \cdot \theta^p(T^n_m) = \alpha_{n,m,D} + \text{chord diagrams of degree} \geq 1 \tag{10.126}$$

Proof. The proof of Proposition 10.2.1.1 still holds for $\widetilde{\mathcal{Z}}_f[\gamma](L) \in \mathring{\mathcal{A}}(\amalg^e S^1)^\wedge/L_{2n}, PO_n$ and shows that this quantity is independent of orientation change and band sum moves. In order to have $\Omega_{\gamma,n}(M)$ well-defined we need that $[\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm))]$ be invertible. In the case where $\gamma = \nu^{k+1}$ we used the fact that $\Delta\nu = \nu \otimes \nu$ which we can no longer assume is true for a general γ. Thus instead of following the same proof as for ν^{k+1}, we show directly that for all n, the elements $[\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm))]$ are invertible. We have computed in (10.121) that:

$$\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm)) = \sum_{m\geq 2n} \sum_{|D|^e+p=m/2} \lambda_{p,\epsilon,D}\alpha_{n,m,D} + \text{chord diagrams of degree} \geq 1 \tag{10.127}$$

showing that the constant part of $\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm))$ is:

$$\sum_{m\geq 2n} \sum_{2(|D|^e+p)=m} \lambda_{p,\epsilon,D}\alpha_{n,m,D} \tag{10.128}$$

and thus we need this to be non-zero for $[\iota_n(\widetilde{\mathcal{Z}}_f[\gamma](U_\pm))]$ to be invertible. The proof of invariance of $\Omega_{\gamma,n}(M)$ under Kirby move I is the same as that in Theorem 10.2.1.4 and carries over in our case to show that $\Omega_{\gamma,n}$ is invariant under such a move. This makes this quantity a topological invariant for any integer n, and thus $\Omega_\gamma(M)$ becomes a topological invariant of M. $\square$

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