An introduction to discrepancy theory

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Abstract

This paper introduces the basic elements of geometric discrepancy theory. After some background we discuss lower bounds for two problems, Schmidt's theorem giving a lower bound for convex sets and Roth's orthogonal method for the lower bound of the L_2 discrepancy of axis-parallel rectangles in the unit square. Then we introduce two sets with low worst-case discrepancy, the Van der Corput set for two dimensions and the Halton-Hammersley set for arbitrary dimension.

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Chapter 1

Introduction

Discrepancy Theory is a concerned with questions of evenness of distribution, historically arising from the study of uniform distributions. Given an infinite sequence x_i with $i \in \mathbb{N}$ and $x_i \in [0, 1]$ and letting $Z[n, \alpha] = |\{1 \leq i \leq n : x_i \in [0, \alpha)\}|$ for $\alpha \in [0, 1]$ and $n \in \mathbb{N}$, then the sequence is said to be uniformly distributed if

$$\lim_{n \to \infty} \frac{Z[n, \alpha]}{n} = \alpha$$

for all $\alpha \in [0, 1]$. This describes only the asymptotic qualities of the sequence, however. To some degree sequences can be more or less uniform than others, and this observation is the origin of discrepancy theory. From this idea comes our first definition of discrepancy,

$$D[n, \alpha] = Z[n, \alpha] - n\alpha$$

where $n\alpha$ is the number of points we would expect in the interval $[0, \alpha)$, and hence $D[n, \alpha]$ measures how much the actual number of points in the interval differs from what we expect, and hence the word discrepancy. From the above formula for discrepancy we can see that

$$\frac{D[n, \alpha]}{n} = \frac{Z[n, a]}{n} - \alpha,$$

and we know for a uniformly distributed sequence the second term goes to 0 as n goes to infinity, which implies that $D[n, \alpha] = o(n)$. We will be able to find much stronger results than this, but this is a start. From this result comes the first big question in discrepancy theory, whether there exists a function $f(n)$ with $\lim_{n \to \infty} f(n) \to \infty$ such that for all infinite sequences in the unit interval

$$\sup_{\alpha \in [0,1]} |D[n, \alpha]| \geq f(n)$$

for infinitely many $n \in \mathbb{N}$. Intuitively it would seem that the discrepancy must go to zero or at least reach some upper bound as n gets arbitrarily large; however, as we will see, this is not the case. In order to see this we will reformulate the problem into an equivalent geometric problem, as Roth did in his 1954 paper [1] that answered the above question in the positive. This reformulation looks at the unit square, and how we can minimize the discrepancy of n points placed in the plane. To begin we define the discrepancy of a rectangle $R \in [0, 1]^2$ as

$$D(P, R) = n \cdot vol(R) - |P \cap R|$$

where $vol(R)$ is in this case just the area of R and P is our chosen set of points $p_i \in [0, 1]^2$ with $i \in \{0, 1, \ldots, n-1\}$. This is fundamentally the same idea as the discrepancy for one dimension, the number of points we expect from the given area minus what we actually have. Going one step of generalization further to $\mathcal{R}_2$, the set of all axis parallel rectangles in the unit square, the discrepancy for any individual point set P is,

$$D(P, \mathcal{R}_2) = \sup_{R \in \mathcal{R}_2} |n \cdot vol(R) - |P \cap R||$$

which essentially finds the axis parallel rectangle with highest discrepancy in the unit square.

Finally, we will need to look at all point sets in order to find the best possible case for a given number of points n, which we can write as

$$D(n, \mathcal{R}_2) = \inf_{P \subset [0,1]^2, |P|=n} D(P, \mathcal{R}_2).$$

This form of discrepancy is often called the worst-case discrepancy, since it essentially finds the rectangle with the highest discrepancy. There are many other ways we can measure discrepancy however, one important alternative is the L_2 or average discrepancy.

Asking whether this last formulation is bounded above as $n \to \infty$ is equivalent to our prior question for infinite sequences, but it put the question in a more familiar setting and enabled the finding of an answer. Now we will move on to establishing some lower bounds.

It is worth noting here that throughout most of this paper we will prove results for axis-parallel boxes using axis-parallel corners, which are just boxes that have to touch the origin, and can be written

$$C_x = [0, x_1) \times [0, x_2) \times \ldots \times [0, x_d),$$

for some $x = (x_1, \ldots, x_d)$. Using corners will give us the same asymptotic discrepancy up to some constant while simplifying the proofs. To show this is equivalent consider for the two dimensional case the inequality for any point set P,

$$D(P, \mathcal{C}_2) \leq D(P, \mathcal{R}_2) \leq 4D(P, \mathcal{C}_2)$$

where $\mathcal{C}_2$ is the set of all corners in the plane. The first inequality is clear since every corner is an axis parallel rectangle and so the set of corners is a subset of the set of rectangles. The second inequality follows from the fact that we can construct every rectangle using 4 corners and the relative compliment operation. This inequality holds in d dimensions as

$$D(P, \mathcal{C}_d) \leq D(P, \mathcal{R}_d) \leq 2^d(P, \mathcal{C}_d).$$

Since the corner case is always at most a constant multiple greater than the rectangular case, we can use it without disturbing any asymptotic qualities while simplifying the work.

Chapter 2

Discrepancy Lower Bounds

In the introduction we were introduced to the discrepancy of axis-parallel boxes. However, the study of discrepancy covers much more than just this, and has been applied to a wide range of geometric objects. While the main focus of this report is axis-parallel boxes, there are some very surprising results for other classes of objects that we will briefly discuss.

Firstly, there is a somewhat surprising division into two classes of discrepancy among geometric objects. On the one hand we have the class of polytopes with at least 4 sides restricted to the scaling and translation operations (i.e. no rotation). These objects have upper and lower bounds that tend to be in $O(\log^c n)$ for some fixed constant c. On the other hand we have the class of polytopes that allow rotation, along with objects with curved boundaries, which tend to have upper and lower bounds that grow as fractional powers of n. This is quite surprising, since it implies that allowing the rotation of our boxes, or simply halving them to create triangles, makes our discrepancy grow at a much worse rate. As an example we will consider a fun result due to Wolfgang Schmidt [2] providing a lower bound for the collection of convex sets in the unit square. This proof is based on Chen's write up [3].

Theorem 1. *The collection of all convex sets $\mathcal{A} \subset [0,1]^2$ has discrepancy*

$$\sup_{A \in \mathcal{A}} |D(P,A)| \gg n^{1/3}$$

for every point set P with n points in the unit square.

Proof. This proof works by chopping up the circle of diameter 1 inscribed in the unit square in chunks with area $\frac{1}{2n}$. Then using the fact that each chunk should have half a point, which is not possible, we can guarantee that we have either enough of the chunks with too many points (one or greater) or no points at all to prove our lower bound. We begin by taking segments with area $\frac{1}{2n}$ out of the inscribed circle which we will call A. This operation guarantees that the shape formed from the circle minus the segments will remain convex. By a geometric argument we will not go into here we know that we can remove $\gg n^{1/3}$ disjoint segments of area $\frac{1}{2n}$. Since each of these segments, which we will call S has either no points from P or at least one point, we know they will have $D(P,S) = -1/2$ and $D(P,S) \geq 1/2$ respectively. We will now separate these segments into $S_1, ..., S_k$ which contain no points of P, and $T_1, ..., T_m$ segments which contain at least one point of P. Then since the removal of any of these segments will still give us a convex set we can look at the convex sets $A \setminus (S_1 \cup ... \cup S_k)$ and $A \setminus (T_1 \cup ... \cup T_m)$. Then since all segments are disjoint we can write the discrepancy of these set differences as $D(A \setminus (S_1 \cup ... \cup S_k)) = D(P,A) - \sum_{i=1}^{k} D(P,S_i)$ and $D(A \setminus (T_1 \cup ... \cup T_m)) = D(P,A) - \sum_{i=1}^{m} D(P,T_i)$ respectively. Here we get to a crucial step that is indicative of a larger trend in discrepancy proofs, namely, finding and adding together collections of small discrepancy areas without canceling out our gains. Now we subtract

$$\begin{aligned} D(P, A \setminus (S_1 \cup ... \cup S_k)) - D(P, A \setminus (T_1 \cup ... \cup T_m)) &= \sum_{i=1}^{m} D(P,T_i) - \sum_{i=1}^{k} D(P,S_i) \\ &= |\sum_{i=1}^{k} D(P,S_i)| + |\sum_{i=1}^{m} D(P,T_i)| \\ &\geq \frac{m+k}{2} \end{aligned}$$

which cancels out the discrepancies of the original circle while adding the discrepancies of our chunks. Since the number of disjoint segments we can cut the circle into is $\gg n^{1/3}$ this implies $\frac{m+k}{2} \gg n^{1/3}$. This in turn implies

$$\max\{|D(A \setminus (S_1 \cup ... \cup S_k))|, |D(A \setminus (T_1 \cup ... \cup T_m))| \geq \frac{m+k}{4} \gg n^{1/3}.$$

Hence, at least one of our two collections will be big enough to satisfy our bound, and we are done. $\square$

Now we will move on to Roth's Orthogonal Function method [1] for finding a lower bound for axis parallel rectangles. This result was very important because it proved that the discrepancy of axis-parallel boxes in $[0,1]^2$ is unbounded as $n \to \infty$ and since Roth also showed this problem is equivalent to the infinite sequences in $[0,1]$ problem that this is also unbounded. The following proof, proceeding based on Matousek's [4] sketch of the original proof, is for the L_2 or average discrepancy, but we will show afterwards that a corollary gives a similar lower bound for the worst-case discrepancy. The average (L_2) discrepancy for a point set $P \subset [0,1]^2$ over all corners in the unit square is defined as

$$D_2(P, \mathcal{C}_2) = \sqrt{\int_{[0,1]^2} D(P, C_{\vec{x}})^2 d\vec{x}}.$$

Theorem 2. *For every point set P of n points $p_i \in [0,1]^2$ we have,*

$$D_2(n, \mathcal{C}) = \Omega(\log^{1/2} n).$$

Proof. We will write $D(\vec{x})$ as shorthand for $D(P, C_{\vec{x}})$. Then we fix $P \subset [0,1]^2$ with n points. Since we wish to find a lower bound for

$$\sqrt{\int_{[0,1]^2} D(\vec{x}) d\vec{x}},$$

we will use the Cauchy-Schwarz inequality with a function F to be definied later so that

$$\int_{[0,1]^2} FD d\vec{x} \leq \sqrt{\int_{[0,1]^2} F^2 d\vec{x}} \sqrt{\int_{[0,1]^2} D^2 d\vec{x}}$$

and thus

$$\sqrt{\int_{[0,1]^2} D^2 d\vec{x}} \geq \frac{\int_{[0,1]^2} FD d\vec{x}}{\sqrt{\int_{[0,1]^2} F^2 d\vec{x}}},$$

where the integral of F^2 is non-zero. Thus we need to show the right half of the inequality equals $\Omega(\log^{1/2} n)$. To do this we will first show that

$$\sqrt{\int_{[0,1]^2} F^2 d\vec{x}} = O(\log^{1/2} n).$$

In order to do this we first fix m such that $2n \leq 2^m < 4n$. Now we will define our functions f_j that will make up F. The trick here is that the way we define f_j will lead to their being mutually orthogonal, and hence canceling out and simplifying our calculations. Let $f_j : [0,1]^2 \to \{-1, 0, 1\}$ for $j = 0, 1, ..., m$. Then we break the unit square up into smaller rectangles with dimensions based on j as 2^j rectangles horizontally and 2^{m-j} rectangles vertically. If a point of P lies in any of these smaller rectangles, f_j takes the value zero on that rectangle; otherwise these rectangles are broken down into 4 equally sized rectangles arranged 2×2. In this case, f_j takes on the values -1 in the top left and lower right rectangles and 1 in the top right and lower left rectangles. This leads to

$$\int_{[0,1]^2} f_i f_j d\vec{x} = 0$$

when $i \neq j$, since $f_i f_j$ creates a similar function with small rectangles with 1's and -1's in the same or inverted positions, or zero everywhere. This orthogonality allows us to find our estimate to $\sqrt{\int_{[0,1]^2} F^2 d\vec{x}}$. First we define $F = f_0 + ... + f_m$ then (we will stop writing integrating variable for now for readability)

$$\int_{[0,1]^2} F^2 = \sum_{i,j=0}^{m} \int_{[0,1]^2} f_i f_j$$

which by orthogonality gives

$$\sum_{i,j=0}^{m} \int_{[0,1]^2} f_i f_j = \sum_{i=0}^{m} \int_{[0,1]^2} f_i^2,$$

but since $f_i^2 = 1$ everywhere it is not 0 and since $m \leq \log_2 n + 2$, we have

$$\sum_{i=0}^{m} \int_{[0,1]^2} f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log_2 n + 3.$$

This shows $\sqrt{\int_{[0,1]^2} F^2 d\vec{x}} = O(\log^{1/2} n)$.

Now we just need to show $\int_{[0,1]^2} FDd\vec{x} = \Omega(\log n)$. To attain this, we will show $\int_{[0,1]^2} f_j D$ is bounded away from zero for each j, since

$$\int_{[0,1]^2} FD = \sum_{j=0}^{m} \int_{[0,1]^2} f_j D$$

and $2^m \geq 2n$. In turn since $2^m \geq 2n$ we are guaranteed at least half of the 2^m rectangles for each f_j will have no points of P, and the other rectangles have $f_j = 0$ everywhere. We will focus on these rectangles that do not have points. For any of these rectangles R we only need to show $\int_R f_j D = \Omega(\frac{1}{n})$ since there are at least n of these rectangles, and thus we will have our lower constant bound. Since each R is in turn broken down into four smaller rectangles as defined by f_j we can write

$$\int_R f_j D = \int_{LL} D(\vec{x}) - \int_{LR} D(\vec{x}) - \int_{UL} D(\vec{x}) + \int_{UR} D(\vec{x})$$

where LL is the lower left smaller rectangle, UL is the upper left rectangle, etc., and the sign of the integral depends on the value f_j takes on that rectangle. Since we know the height and width of each smaller rectangle by knowing j, we can define the vectors $\|\vec{a}\| = 2^{-j-1}$ in the direction of the x-axis and $\|\vec{b}\| = 2^{-m+j-1}$ in the direction of the y-axis. This will allow us to rewrite the discrepancy for the four smaller rectangles in terms of just the lower left rectangle as

$$\int_{LL} D(\vec{x}) - D(\vec{x}+\vec{a}) - D(\vec{x}+\vec{b}) + D(\vec{x}+\vec{a}+\vec{b})$$

which from the definition of discrepancy is

$$\int_{LL} n \cdot [vol(C_x) - vol(C_{x+\vec{a}}) - vol(C_{x+\vec{b}}) + vol(C_{x+\vec{a}+\vec{b}})]d\vec{x}$$

$$-\int_{LL} [|P \cap C_x| - |P \cap C_{x+\vec{a}}| - |P \cap C_{x+\vec{b}}| + |P \cap C_{x+\vec{a}+\vec{b}}|]d\vec{x}$$

The combinations of volumes (areas since this is $d = 2$) in the first term will end up by cancellation being the area of one of the small rectangles, or $\|\vec{a}\|\|\vec{b}\| = 2^{-m-2} = vol(R_{LL})$. The second term will be zero since the corners involved will overlap and cancel everywhere except the upper right rectangle, and since we know the whole rectangle R has no points neither does the smaller upper right rectangle. Then we have arrived at

$$\int_R f_j D = \int_{R_{LL}} n \cdot vol(R_{LL}).$$

Integrating, since neither term depends on $\vec{x}$, gives

$$= n \cdot vol(R_{LL})^2 = \frac{n}{2^{2m+4}}.$$

Since $m \geq \log_2 n + 1$, this is $= \Omega(\frac{1}{n})$, and thus our $\int_{[0,1]^2} f_j D$ will each be greater than some non-zero constant, which implies $\int_{[0,1]^2} FD = \Omega(\log n)$ so we are done. □

Knowing that the L_2 discrepancy is in $\Omega(\log^{1/2} n)$ will guarantee us that the worst-case discrepancy is also, since the worst case is always at least as bad as the average. This is clear from

$$\int_{[0,1]^2} D(P, C_{\vec{x}})^2 d\vec{x} \leq \int_{[0,1]^2} \sup_{C_{\vec{x}} \in \mathcal{C}_2} |D(P, C_{\vec{x}})|^2 d\vec{x}.$$

Chapter 3

Constructions of Logarithmic Worst Case Discrepancy Sets

In the previous section we have shown that the lower bound discrepancy for axis parallel rectangles is $\sup_{R\in\mathcal{R}} |(P,R)| \geq \log^{1/2} n$ which can be sharpened to $\log n$ [5] for any point set P of n points. In this section we will present two constructions of sets that attain close to their lower worst-case discrepancy bounds. First, we will discuss the Van der Corput set, which gives discrepancy equal to $\mathrm{O}(\log N)$ for axis parallel rectangles in the unit square. Then we will generalize the Van der Corput set to any finite dimension d with the Halton-Hammersley set, which gives worst case discrepancy $O(\log^{d-1} n)$ for axis parallel boxes of dimension d in the d dimensional unit cube.

3.1 The Van der Corput Set

The Van der Corput set[6] is generally the introductory low discrepancy set for axis parallel rectangles. It is constructed using the so-called bit reversal function, which flips the binary (or n-nary) expression of a given number. This creates a sort of pseudo-random scattering effect caused by the least relevant digits becoming the most relevant and vice versa. To show the construction of this bit reversal sequence we first describe the dyadic expansion for any integer i as

$$i = \sum_{v=1}^{\infty} a_v 2^{v-1}$$

where a_v is either 0 or 1 depending on i, and so for a given integer the sequence of a_v's is equivalent to that number's binary expansion. Then the bit reversal function would flip the digits of each integer and then place them after the decimal to create a fraction. For instance 11 expressed in binary as 1011 becomes .1101 or $1/2 + 1/4 + 1/16 = 13/16$. We can express this by inverting the dyadic expansion as

$$r(i) = \sum_{v=1}^{\infty} a_v 2^{-v}.$$

Combining these two ideas we can arrive at our point set P. For P containing n points in $[0, 1]^2$ we have $p_i = (i/n, r(i))$ for $i = 0, 1, ..., n-1$. For example, when n=10 will look like

$$(0, 0), (1/10, 1/2), (2/10, 1/4), (3/10, 3/4), (4/10, 1/8),$$

$$(5/10, 5/8), (6/10, 3/8), (7/10, 7/8), (8/10, 1/16), (9/10, 9/16)$$

where the x coordinates advance incrementally but the y coordinates scatter between 0 and 1 as a result of the bit reversal. Now that we understand how to construct the Van der Corput set we will demonstrate its low discrepancy property following Matousek's argument [4]. We will do this using axis parallel corners for simplicity.

Proposition 3. *Over the set of all axis parallel rectangles, a given n point set $P = \{(i, r(i)) : i \in \{0, ..., n-1\}\}$ has*

$$|D(P, \mathcal{R}_2)| = O(\log(n)).$$

Proof. This proof works in the same as way many lower bound proofs in Discrepancy Theory. We divide our space into chunks that we know have discrepancy zero plus some small remainder, and then we add up how many of these chunks there are. To do this we begin with some notation. We will call a canonical interval any interval of the form $[\frac{k}{2^q}, \frac{k+1}{2^q})$ where $0 \leq k < 2^q$.

Lemma 4. *For rectangles R of the form $[0, a) \times I$, where $a \in [0, 1]$ and I is a canonical interval, we have $|D(P, R)| \leq 1$.*

To prove this we first fix some canonical interval I and some a and set $R = [0, a) \times I$. We know that any point p_i from P that falls in our rectangle R has its y-coordinate $r(i) \in I$, and therefore the first q terms of $\sum_{v=1}^{\infty} a_v 2^{-v}$ uniquely determined. This in turn, since we used the bit reversal operation to get $r(i)$, implies the first q terms of i's dyadic expansion (or the q least significant terms of its binary expansion) are fixed. This necessarily means all of the points with $r(i) \in I$ have x-coordinates spaced $\frac{2^q}{n}$ apart. This means we can divide $[0, 1] \times I$ into boxes with area $\frac{1}{2^q} \times \frac{2^q}{n} = \frac{1}{n}$ where the first term comes from the height of the canonical interval and the second from the width of the boxes of x-coordinates of points p_i falling in I. For an arbitrary box B with the characteristics just defined we have discrepancy $D(P, B) = n \cdot vol(B) - |P \cap B|$, and since we know one point of B will fall in each of these boxes $|P \cap B| = 1$ which will give us discrepancy 0 for each box. Then since our rectangle R falls in $[0, 1] \times I$ we know it can be made into a number of boxes of area $\frac{1}{n}$ with discrepancy 0 plus some remainder term that has discrepancy at most 1 in absolute value, since it is a piece of one of our boxes of discrepancy 0.

Lemma 5. *For any corner $C_{(x,y)} \in [0, 1]^2$ we can find at most $\lceil \log_2 n \rceil$ disjoint rectangles with canonical interval height and a remainder rectangle M with discrepancy 1, whose union is equivalent to $C_{(x,y)}$*

To show this we first fix $m \in \mathbb{N}$ with $2^m \geq n > 2^{m-1}$. Using this m we find the greatest integer multiple of $\frac{1}{2^m}$, which we call y_0 such that $y_0 \leq y$. Then the gap between y_0 and y is where our remainder rectangle $M = [0, x) \times [y_0, y)$ exists. However, we know that M cannot have area greater than $y - y_0 < \frac{1}{2^m} \leq \frac{1}{n}$, and also that there can be at most one point in M since we found m such that $\frac{1}{2^m}$ is the minimum possible distance between any of our points'

y-coordinates. This shows that the discrepancy of our remainder rectangle M is at most 1. Next we need to show that the corner less M can be broken into at most m rectangles with canonical interval height. To do this we notice that once we remove M we are left with a corner $C_{(x,y_0)}$ with height $[0,y_0)$. Then, if necessary, we can remove a rectangle of height $\frac{1}{2^m}$ to get a new corner $C_{(x,y_1)}$ where y_1 is some integer multiple of $\frac{1}{2^{m-1}}$. Then we can remove a rectangle of height $\frac{1}{2^{m-1}}$ if needed, and so on recursively, until $[0, y_0)$ has been chopped into at most m canonical intervals. Since m was chosen such that $2^m \geq n > 2^{m-1}$, we know that $m \leq \lceil \log_2 n \rceil$, and we are done with Lemma 5.

Then to finish the proof we just need to realize that $\lceil \log_2 n \rceil$ rectangles each with discrepancy at most 1 in absolute value plus our remainder rectangle M gives us

$$|D(P, \mathcal{C}_2)| = O(\log(n))$$

and thus

$$|D(P, \mathcal{R}_2)| = O(\log(n)).$$

□

Now we introduce the Halton-Hammersley set [7], which is essentially just an extension of the Van der Corput set to d dimensions. In order to form this set we fix $d-1$ distinct primes $p_1, p_2, ..., p_{d-1}$, where d is the number of dimensions of the space $[0,1]^d$ where we will be working. Then the Halton-Hammersley set with n points is,

$$P = \{\left(\frac{i}{n}, r_{p_1}(i), r_{p_2}(i), ..., r_{p_{d-1}}(i)\right) : i \in \{0, 1, ..., n-1\}\},$$

where r_{p_i} is the bit reversal function for base i.

Theorem 6. *[7][8]Given dimension d and a set of $d-1$ distinct primes, the Halton-Hammersley set with n points has discrepancy $O(log^{d-1}n)$ for axis-parallel boxes of dimension d .*

The proof of this theorem follows very closely to the prior proof of the discrepancy of the Van der Corput set; we break the corner into boxes of a certain size with 1 point in each box plus some remainder and then add up the number of these boxes plus some remainder. The difference for multiple dimensions is the use of the Chinese Remainder Theorem to assure that only one point will fall in each of the evenly spaced boxes.

Bibliography

[1] K. F. Roth. On irregularities of distribution. *Mathematika*, 1:73–79, 1954.

[2] W. Schmidt. Irregularities of distribution 9. *Acta Arith.*, 27:385–396, 1975.

[3] W. Chen. Lectures on discrepancy theory. http://www.williamchen-mathematics.info/researchfolder/dt12.pdf, 2012.

[4] Jiri Matoušek. *Geometric Discrepancy. An Illustrated Guide*, volume :Algorithms Combin. 18. Springer,Berlin, 2010.

[5] W. Schmidt. Irregularities of distribution 7. *Acta Arith.*, 21:45–50, 1972.

[6] J. G. van der Corput. Verteilungsfunktionen 1. *Akad. Wetensch. Amsterdam Proc.*, 38: 813–821, 1935.

[7] J. M. Hammersley. Monte carlo methods for solving multivariable problems. *Ann. of the New York Acad. Sci.*, 86:844–874, 1960.

[8] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, 2:84–90, 1960.