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RESPONSE OF NONLINEAR NONSTATIONARY
VIBRATIONAL SYSTEMS WITH N DEGREES OF FREEDOM
SUBJECTED TO ARBITRARY PULSE EXCITATIONS

by

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I. INTRODUCTION

The study of nonlinear problems has assumed a great relevance and significance due to rapid advancement in science and engineering. Nonlinearities in systems are inherent and in most cases it becomes a necessity to incorporate them in the system equations. Many situations in engineering require the knowledge of nonlinear analysis to improve design and performance. The study helps in understanding certain special features of the system, to anticipate their existence and also account for it in design. In the analysis of nonlinear systems significant contributions have been made, however the methods and procedures are still inadequate and need a great deal of further investigation.

The demand and need for nonlinear analysis is important in quantum mechanics, solid mechanics, fluid mechanics, aeronautics and so on. In the theory of vibration, the class of problems involving nonlinear nonstationary systems has been receiving considerable attention. By nonstationary it is meant that the system components are time dependent. These components may be masses, restoring forces, amplitude of external excitation, material properties, frequency or damping present in the system.

The application of the theory of nonlinear nonstationary systems are numerous. Scalia and Torrisi [1], applied approximate solutions to study the behavior of a variable mass gyroscope, while Ilin and Zhigula [2], analysed the dynamics of viscoelastic cables in load lifting devices. Tordion and Gauvin

[3], carried out a stability analysis of gear trains with variable meshing stiffness. The classic example of a pendulum of variable length has been analysed by many authors in the past. Sinha and Chou [4] and Olberding and Sinha [17], have approached the problem by the use of the Krylov, Bogoliubov and Mitropolskii [KBM] method. Mitropolskii [5], has presented an analysis of the torsional oscillations of a crankshaft with variable stiffness.

The parametrically excited nonlinear systems treated as a subclass of the nonlinear nonstationary vibrational problems has been receiving considerable attention. Applications include the analysis by Tesak et al. [6], on parametrically excited beam columns with periodic loads. Tso [7], Ghoborah and Tso [8], Iwatsubo et al. [9] and Crespo da Silva [10] have also studied such systems. Mitropolskii [5], has presented the problem of a column subjected to periodic force of variable frequency. Evan Iwanowskii [11], has considered a beam column subjected to axial periodic loads. Nishikawa and Willems [12], have studied the satellite attitude stability. A study on helicopter blade vibration was carried out by Friedmann and Silverthorn [13]. Mitropolskii [5], has presented the analysis of monofrequency oscillation of a turbine blade. Paidoussis and Sunderarajan [14], studied the nonlinear oscillations of a pipe carrying fluid with variable flow rate and subjected to parametric excitation. Gorchakov [15], studied the torsional vibrations of a crankshaft during a transition through resonance. The list is by no means complete but definitely indicates the vast applications and the need for further investigation of nonlinear

nonstationary systems.

The analysis of nonlinear systems suffers from the fact that closed form solutions are not possible in majority of the situations. The class of nonlinear problems involving time dependent parameters has been studied through the use of approximate techniques. Numerical solutions commonly used to solve nonlinear differential equations do not provide adequate insight into the general behavior of the system. Excessive computation needs to be performed to obtain a general idea of the behavior of the system. In addition to being cumbersome for analysis, such solutions are not economical in terms of money and time spent on it. Hence any analytical approximate solutions obtained help in understanding the general behavior of the system in terms of the system parameters.

Nonlinear nonstationary vibrational systems have been studied both for the case of single degree and multiple degrees of freedom. The free, as well as the forced response of such systems have been analysed by Mitropolskii [5], Iwanowskii [11] and Bolotin [16]. Recently Sinha and Olberding [18], studied the response of nonlinear nonstationary systems with arbitrary pulse excitation. However, for the case of multiple degrees of freedom such analysis has been restricted to the special case of periodic excitations only. Also, in the case of parametrically excited nonlinear systems, the analysis has been restricted to the special case of system parameters varying periodically with time.

The purpose of the present study is to generalize the approach suggested by Sinha and Olberding [17] and Olberding [18] to nonlinear nonstationary systems with multiple degrees of freedom subjected to arbitrary pulse excitations. Such analyses are necessary when the systems are subjected to shock or impulse loading conditions. Approximate analytical solutions could be of considerable help for engineers and designers since it may offer a better insight into the general behavior of the system and ease in manipulation of parameters to optimize performance.

The method of averaging of Bogoliubov and Mitropolskii [19] is used as the main tool for the analysis. The study is restricted to a first order approximation.

In Chapter II, the general analysis of the system has been made. The KBM method is introduced and applied to a N degree of freedom, nonstationary vibrational system. Explicit expressions for amplitude and phase variations are derived.

In chapter III, the specific case of a two degrees of freedom system with nonstationary parameters is being considered. Following the procedure outlined in chapter I, expressions for amplitude and phase variations are obtained. Special cases of nonlinearities and specific examples of pulse excitations have been considered for purposes of illustration.

A beam column subjected to nonperiodic parametric excitation and arbitrary pulse load is considered in chapter IV. The amplitude of deflection for such a system is being presented. The first two mode shapes have been approximated using a two term Galerkin's approximation. The results obtained have been compared with a fourth order Runge-Kutta numerical scheme.

II. EQUATION OF MOTION AND GENERAL ANALYSIS OF NONLINEAR NONSTATIONARY SYSTEMS WITH N-DEGREES OF FREEDOM SUBJECTED TO ARBITRARY PULSE EXCITATION.

2.1 Introduction:

Nonstationary vibrational systems have been receiving considerable attention. For the case of a single degree of freedom system, analytical approximate solutions have been obtained for a general class of problems. Sinha and Chou [4] applied the Krylov, Bogoliubov and Mitropolskii method for a step excitation case. The monograph by Mitropolskii [5] contains a significant portion of the analysis of a single degree of freedom system. The book by Evan Iwanowskii [11] also contains significant amount of analysis and results. For the case of multiple degrees of freedom, Rangacharyulu, Srirangarajan and Dasarathy [20] have approached the problem via the KBM method, where the parameters of the system were stationary. Bauer [21] has applied approximate techniques to obtain solutions of stationary systems subjected to arbitrary pulse excitation. The books by Mitropolskii [5] and Evan Iwanowskii [11] have analysis of N degrees of freedom nonstationary system for the case of monofrequency oscillations only.

In this chapter the approach suggested by Sinha and Olberding [17] and Olberding [18] is being applied for the case of nonlinear nonstationary system with N degrees of freedom subjected to arbitrary pulse excitation. In such problems the presence of the forcing

term poses additional complication and is being overcome by considering a transformation of the dependent variable. The general procedure and expressions for amplitude variation and variation of instantaneous frequency are being obtained, by the use of the KBM method. The analysis is restricted to the case of a first order approximation.

2.2 Equation of Motion:

The equations of motion for nonlinear nonstationary systems having N degrees of freedom with slowly varying system parameters subjected to arbitrary pulse excitations can be reduced to the following mathematical form.

$$\frac{d}{dt} \left[m_{ij}(\tau) \frac{dq_j}{dt} \right] + b_{ij}(\tau) q_j + \epsilon f_i(q_j, \dot{q}_j) = g_i(\tau); \quad (2-1)$$

$$i, j = 1, \dots, N.$$

where the summation convention has been used.

The generalized coordinates are represented by q_j , the nonstationary inertia terms are represented by $m_{ij}(\tau)$ and the stiffness or restoring type terms are described by $b_{ij}(\tau)$. The nonlinearities which may be functions of displacement, velocity or both are represented by the function f . ϵ is a small parameter characterizing the smallness of nonlinearity. The nonstationary parameters $m_{ij}(\tau)$, $b_{ij}(\tau)$ and the excitation forces $g_i(\tau)$ are functions of the slowing time τ , which is defined as " $\tau = \epsilon t$ ". $m_{ij}(\tau)$, $b_{ij}(\tau)$ and $g_i(\tau)$ are considered to be smooth functions of τ . The inertia and stiffness terms are assumed to be positive definite. Without loss of generality, the initial conditions for the system may be chosen to be zero. Hence,

$$q_j = 0 \text{ at } t = 0 \quad (2-2a)$$

$$\dot{q}_j = 0 \text{ at } t = 0 \quad (2-2b)$$

Introducing the transformation,

$$q_j = y_j + p_j \quad (2-3a)$$

$$\dot{q}_j = \dot{y}_j + \dot{p}_j \quad (2-3b)$$

equation (2-1) takes the form,

$$\begin{aligned} \frac{d}{dt}[m_{ij}(\tau)\frac{dy_j}{dt}] + \frac{d}{dt}[m_{ij}(\tau)\frac{dp_j}{dt}] + b_{ij}(\tau)[y_j + p_j] + \\ + \epsilon f_i(y_j + p_j, \dot{y}_j + \dot{p}_j) = g_i(\tau) \end{aligned} \quad (2-4)$$

The initial conditions given by equation (2-2a) and (2-2b)

change to

$$y_j = -p_j, \quad \dot{y}_j = -\dot{p}_j \quad \text{at } t = 0. \quad (2-5a \text{ \& } 2-5b)$$

If p_j are chosen such that

$$\frac{d}{dt} \left[m_{ij}(\tau) \frac{dp_j}{dt} \right] + b_{ij}(\tau)p_j = g_i(\tau) \quad (2-6)$$

then eqn. (2-4) reduces to the form,

$$\frac{d}{dt} \left[m_{ij}(\tau) \frac{dy_j}{dt} \right] + b_{ij}(\tau)y_j + \epsilon f_i(y_j + p_j, \dot{y}_j + \dot{p}_j) = 0. \quad (2-7)$$

It may be observed from eqn. (2-7) that for a first order approximate solution for y_j , it is necessary to obtain the solution of p_j independent of ϵ . Hence it suffices to consider only the solution of eqn. (2-6) of the zeroeth order of ϵ obtained from

$$p_j(\tau) = b_{ji}^{-1} g_i(\tau) \quad (2-8)$$

This is evident if eqn. (2-6) is rewritten as

$$\epsilon^2 m_{ij}(\tau) \frac{d^2 p_j}{d\tau^2} + \epsilon^2 \frac{d}{d\tau} \left[m_{ij}(\tau) \right] \frac{dp_j}{d\tau} + b_{ij}(\tau)p_j = g_i(\tau). \quad (2-9)$$

Knowing $p_j(\tau)$, eqn. (2-7) can be written in the standard form, viz.,

$$\frac{d}{dt} \left[m_{ij}(\tau) \frac{dy_j}{dt} \right] + b_{ij}(\tau) y_j + \varepsilon f_i(y_j, \dot{y}_j, \tau) = 0. \quad (2-10)$$

This allows the application of the Krylov Bogoliubov and Mitropolskii's method which is explained in the following section.

2.3 Application of KBM Method:

This is one of the perturbation methods. For nonlinear systems with nonstationary parameters the solution of the differential equation gives rise to overtones. The amplitude of response is not stationary but increases or decreases depending on whether energy is being dissipated or absorbed by the system. These effects due to nonlinearities and nonstationary parameters are accounted for by constructing asymptotic approximations. These are periodic functions of the instantaneous frequency with a period 2π . The method of averaging of Krylov, Bogoliubov-Mitropolskii [22] serves as a tool to obtain such expansions.

Consider the unperturbed equation by setting $\varepsilon = 0$ and considering τ as a constant in eqn. (2-10), viz.,

$$m_{ij}(\tau) \frac{d^2 y_j}{dt^2} + b_{ij}(\tau) y_j = 0 \quad (2-11)$$

Note that $m_{ij}(\tau)$ and $b_{ij}(\tau)$ are constants. Assuming normal mode solution for y_j ,

$$y_j = \sum_{k=1}^N \alpha_{jk} a_k \cos \psi_k \quad (2-12)$$

and,

$$\dot{y}_j = -\sum_{k=1}^N \alpha_{jk} \omega_k a_k \sin \psi_k \quad (2-13)$$

$$j = 1, \dots, N.$$

where

a_k - are the amplitudes

ψ_k - are the instantaneous frequencies

ω_k - normal frequencies

The normal frequencies are obtained from the characteristic matrix and its determinant,

$$\det || b_{ij}(\tau) - m_{ij}(\tau)\omega^2 || = 0. \quad (2-14)$$

The quantities α_{jk} represent the amplitude ratios obtained for every eigenvalue ω_k .

As outlined, it is necessary to construct asymptotic approximations to account for the nonlinearities which give rise to overtones in the solution. We seek solutions of the form,

$$y_j = \sum_{k=1}^N \alpha_{jk} a_k \cos \psi_k + \varepsilon \sum_{k=1}^N U_k(a_1, a_2, \dots, a_N, \psi_1, \psi_2, \dots, \psi_N) \\ j = 1, \dots, N. \quad (2-15)$$

The instantaneous frequencies given by ψ_k are defined by the relations

$$\frac{d\psi_k}{dt} = \omega_k + \frac{d\theta_k}{dt} \quad (2-16)$$

where $\frac{d\theta_k}{dt}$ is the variation of phase.

The functions U_k are periodic functions of the angles ψ_k with a period 2π . The amplitudes a_k and the instantaneous frequencies ψ_k are defined by the following differential equations

$$\frac{da_k}{dt} = \sum_{m=1}^r \varepsilon^m A_{km}(a_1, a_2, \dots, a_N, \tau) \quad (2-17) \\ k = 1, \dots, N$$

and,

$$\frac{d\psi_k}{dt} = \omega_k + \sum_{m=1}^r \epsilon^m B_{km} (a_1, a_2, \dots, a_N, \tau) \quad (2-18)$$

where r is finite.

The construction of asymptotic approximation to the analytical solution of eqn. (2-7) reduces to simply finding the expressions A_{km} and B_{km} . However, for a first order approximate solution of equation (2-7), it is sufficient to find the expressions for amplitude variation and variation of the instantaneous frequencies of the form,

$$\frac{da_k}{dt} = \epsilon A_k (a_1, a_2, \dots, a_N, \tau) \quad (2-19a)$$

and

$$\frac{d\psi_k}{dt} = \omega_k + \epsilon B_k (a_1, a_2, \dots, a_N, \tau). \quad (2-19b)$$

Since the expressions for amplitude variation and variation of instantaneous frequencies contain terms of the first order in ϵ , the expression U_k in eqn. (2-15) would be of the order of ϵ . Inclusion of U_k in eqn. (2-15) would give rise to a second order term in ϵ , hence may be neglected. It is then sufficient for a first order approximate solution, to obtain

$$y_j = \sum_{k=1}^N \alpha_{jk}(\tau) a_k \cos \psi_k \quad (2-20)$$

In order to obtain the functions A_k and B_k , consider eqn. (2-20). The derivative of eqn. (2-20) is written as

$$\begin{aligned} \frac{dy_j}{d\tau} &= \sum_{k=1}^N \alpha_{jk}(\tau) \cos \psi_k \frac{da_k}{dt} + \sum_{k=1}^N a_k \cos \psi_k \frac{d\alpha_{jk}(\tau)}{dt} - \\ &\quad - \sum_{k=1}^N \alpha_{jk}(\tau) a_k \omega_k \sin \psi_k - \sum_{k=1}^N \alpha_{jk}(\tau) a_k \sin \psi_k \frac{d\omega_k}{dt} \end{aligned} \quad (2-21)$$

$$j = 1, \dots, N.$$

However $\frac{dy_j}{dt}$ is required to be of the form given by eqn. (2-13)

by assumption; hence we obtain a condition, viz.,

$$\begin{aligned} & \sum_{k=1}^N \alpha_{jk}(\tau) \cos \psi_k \frac{da_k}{dt} - \sum_{k=1}^N \alpha_{jk}(\tau) a_k \sin \psi_k \frac{d\theta_k}{dt} + \\ & + \sum_{k=1}^N \frac{d\alpha_{jk}(\tau)}{dt} a_k \cos \psi_k = 0, \end{aligned} \quad (2-22)$$

$$j = 1, \dots, N.$$

Substituting for $\frac{dy_j}{dt}$ in eqn. (2-7) yields,

$$\begin{aligned} & \frac{d}{dt} \left[\sum_{j,k=1}^N m_{ij}(\tau) (-\alpha_{jk}(\tau) a_k \omega_k \sin \psi_k) \right] + \\ & + \sum_{j,k=1}^N b_{ij}(\tau) (\alpha_{jk}(\tau) a_k \cos \psi_k) + \epsilon f_i(y_k, \dot{y}_k, \tau) = 0, \end{aligned} \quad (2-23)$$

$$i = 1, \dots, N;$$

and expansion of the above leads to,

$$\begin{aligned} & - \sum_{j,k=1}^N m_{ij}(\tau) \alpha_{jk}(\tau) \omega_k \sin \psi_k \frac{da_k}{dt} - \\ & - \sum_{j,k=1}^N m_{ij}(\tau) \alpha_{jk}(\tau) a_k \omega_k \cos \psi_k \frac{d\theta_k}{dt} = \\ & \sum_{j,k=1}^N m_{ij}(\tau) \alpha_{jk}(\tau) a_k \omega_k^2 \cos \psi_k + \sum_{j,k=1}^N \frac{d}{dt} \left[m_{ij}(\tau) \alpha_{jk}(\tau) \omega_k \right] \times \\ & \times a_k \sin \psi_k - \sum_{j,k=1}^N b_{ij}(\tau) \alpha_{jk}(\tau) a_k \cos \psi_k - \epsilon f_i(y_k, \dot{y}_k, \tau). \end{aligned} \quad (2-24)$$

Eqs. (2-22) and (2-24) may be arranged in a matrix form as,

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$$\begin{bmatrix}
\alpha_{11}(t) \cos \psi_1 & \dots & \alpha_{1k}(t) \cos \psi_k & \dots & -\alpha_{11}(t) a_1 \sin \psi_1 & \dots & -\alpha_{1k}(t) a_k \sin \psi_k \\
\vdots & & \vdots & & \vdots & & \vdots \\
\alpha_{11}(t) \cos \psi_1 & \dots & \alpha_{1k}(t) \cos \psi_k & \dots & -\alpha_{11}(t) a_1 \sin \psi_1 & \dots & -\alpha_{1k}(t) a_k \sin \psi_k \\
-\sum_{j=1}^N m_{1j}(t) \alpha_{j1}(t) \omega_1 \sin \psi_1 & \dots & -\sum_{j=1}^N m_{1j}(t) \alpha_{jk}(t) \omega_k \sin \psi_k & \dots & -\sum_{j=1}^N m_{1j}(t) \alpha_{j1}(t) a_1 \omega_1 \cos \psi_1 & \dots & -\sum_{j=1}^N m_{1j}(t) \alpha_{jk}(t) a_k \omega_k \cos \psi_k \\
\vdots & & \vdots & & \vdots & & \vdots \\
-\sum_{j=1}^N m_{ij}(t) \alpha_{j1}(t) \omega_1 \sin \psi_1 & \dots & -\sum_{j=1}^N m_{ij}(t) \alpha_{jk}(t) \omega_k \sin \psi_k & \dots & -\sum_{j=1}^N m_{ij}(t) \alpha_{j1}(t) a_1 \omega_1 \cos \psi_1 & \dots & -\sum_{j=1}^N m_{ij}(t) \alpha_{jk}(t) a_k \omega_k \cos \psi_k
\end{bmatrix}
\begin{bmatrix}
\frac{da_1}{dt} \\
\vdots \\
\frac{da_k}{dt} \\
\frac{d\theta_1}{dt} \\
\vdots \\
\frac{d\theta_k}{dt}
\end{bmatrix}$$

$$= \begin{bmatrix}
\sum_{k=1}^N \frac{d\alpha_{1k}(t)}{dt} a_k \cos \psi_k & \dots & \sum_{j,k=1}^N m_{1j}(t) \alpha_{jk}(t) a_k \omega_k^2 \cos \psi_k + \sum_{j,k=1}^N \frac{d}{dt} [m_{1j}(t) \alpha_{jk}(t) \omega_k] a_k \sin \psi_k - \\
-\sum_{j,k=1}^N b_{1j}(t) \alpha_{jk}(t) a_k \cos \psi_k - c f_1(y_k, \dot{y}_k, t) & \dots & \\
\vdots & & \vdots \\
-\sum_{j,k=1}^N m_{ij}(t) \alpha_{jk}(t) a_k \omega_k^2 \cos \psi_k + \sum_{j,k=1}^N \frac{d}{dt} [m_{1j}(t) \alpha_{jk}(t) \omega_k] a_k \sin \psi_k - \\
-\sum_{j,k=1}^N b_{ij}(t) \alpha_{jk}(t) a_k \cos \psi_k - c f_i(y_k, \dot{y}_k, t)
\end{bmatrix}$$

(2-25)

$i = 1, \dots, N$

Let,

M represent the $2N \times 2N$ matrix on the left side of equation (2-25),

Z be the column vector containing terms of amplitude and instantaneous frequency variation and,

S be the column vector on the right side of equation (2-25).

Equation (2-25) may be then written in the form,

$$[M][Z] = [S] \quad (2-25a)$$

from which

$$[Z] = [M]^{-1}[S] \quad (2-25b)$$

provides the expressions for amplitude and instantaneous frequency variation. The structure of equation (2-25) reveals that these expressions would be functions of the amplitudes a_k and instantaneous frequencies ψ_k . In general, for a N degree of freedom system, the elements of the $2N \times 2N$ matrix and also the elements of the column vector on the right side of equation (2-25) are not simple. Hence, obtaining explicit expressions for amplitude and instantaneous frequency variation become cumbersome. However, such expressions may be represented in functional forms as,

$$\frac{da_k}{dt} = F_k(a_1, a_2, \dots, a_N, \psi_1, \psi_2, \dots, \psi_N, \tau) \quad (2-25c)$$

and,

$$\frac{d\psi_k}{dt} = G_k(a_1, a_2, \dots, a_N, \psi_1, \psi_2, \dots, \psi_N, \tau) \quad (2-25d)$$

$$k = 1, \dots, N$$

Where F_k and G_k are obtained using equation (2-25b).

From this, approximate expressions determining the relationship of amplitude and instantaneous frequency can be obtained by employing the averaging technique, viz.,

$$\frac{da_k}{dt} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} F_k(a_1, a_2, \dots, a_N, \psi_1, \psi_2, \dots, \psi_N, \tau) d\psi_1 d\psi_2 \dots d\psi_N$$

N times

$$k = 1, \dots, N. \quad (2-26)$$

and

$$\frac{d\theta_k}{dt} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} G_k(a_1, a_2, \dots, a_N, \psi_1, \psi_2, \dots, \psi_N, \tau) d\psi_1 d\psi_2 \dots d\psi_N$$

N times

$$k = 1, \dots, N. \quad (2-27)$$

Averaging over a period of 2π leaves $\frac{da_k}{dt}$ and $\frac{d\theta_k}{dt}$ as a function of the amplitudes and the slowing time τ , which can be written as

$$\frac{da_k}{dt} = F'_k(a_1, \dots, a_N, \tau) \quad (2-28)$$

$$\frac{d\theta_k}{dt} = G'_k(a_1, \dots, a_N, \tau) \quad (2-29)$$

$$k = 1, \dots, N$$

In general eqns. (2-28) and (2-29) form coupled $2N$ first order differential equations. General closed form solutions for such a system is however not possible. In practice certain special kinds of nonlinearities do arise and offer significant simplification in solving these equations. Using simple integration schemes the expressions for amplitudes may be solved for. This offers the possibility of using simple quadrature schemes to solve for the instantaneous frequency values by merely substituting the known values of amplitude. The functions F'_k and G'_k

represent A_k and B_k in eqns. (2-19a) and (2-19b). The total solution of the system may then be obtained from

$$q_j = y_j + p_j \quad (2-3a)$$

where y_j is given by eqn. (2-12)

and p_j is given by eqn. (2-8)

In the following chapter, a nonlinear nonstationary system with two degrees of freedom is being considered as a typical case for ease of illustration.

2.4 Discussion

The differential equations describing a nonlinear nonstationary system with N degrees of freedom can be reduced to a standard form through a suitable transformation of the dependent variable. The method of Krylov-Bogoliubov and Mitropolskii can be used to construct an analytical approximate solution to the equivalent system. This leads to $2N$ expressions of amplitude and instantaneous frequency variation.

The structure of the expressions for amplitude and frequency variation reveals that simple integration schemes may be applied. These may be further simplified under special circumstances. In the special case when the nonlinearity is dependent strictly on displacement type terms, the amplitude of response varies only due to the nonstationary nature of the parameters of the system. When the nonlinearity is dependent on velocity type terms only, the amplitude of response may have significant contribution in its variation due to such a nonlinearity. These interesting possibilities are being studied in the following chapter for a specific case of a two degrees of freedom system.

III. ANALYSIS OF TWO DEGREES OF FREEDOM SYSTEM

3.1 Introduction:

The analysis presented in chapter II provides the general idea of the nature of the problem and its solution. However in many cases problems often reduce to the case of two degrees of freedom. In this chapter, specific results for a two degrees of freedom system are derived and typical examples are presented. In the past, Bauer [20], has used a two mass vibrational system subjected to pulse excitation. Ranga-Charyulu, et al. have presented the approximate solution of two degrees of freedom system, subjected to step excitation. All these studies have been restricted to the case of systems with stationary parameters. For a single degree of freedom system with nonstationary parameters, Sinha and Olberding [17] and Olberding [18] have considered a variable length pendulum subjected to step and arbitrary pulse excitations. However analysis for a nonlinear nonstationary system with two degrees of freedom subjected to arbitrary pulse has not been considered in the past.

In this chapter, the response of a two degrees of freedom nonlinear nonstationary system subjected to arbitrary pulse is being considered in detail. A general analysis following the procedure outlined in chapter II is attempted and explicit expressions for variation of amplitudes and instantaneous

frequencies are derived. Special cases of nonlinearities and their effect on the behavior of the system is studied. The system is subjected to pulse excitation of the form of a step, an exponentially decaying pulse and an exponential step. The results are compared with a fourth order Runge-Kutta numerical scheme.

3.2 Equation of Motion:

Consider a nonlinear nonstationary system with two degrees of freedom subjected to arbitrary pulse excitation. The equations of motion can easily be obtained from equation (2-1) with $k, j = 1, 2$ as,

$$\begin{aligned} \frac{d}{dt} \left[m_{11}(\tau) \frac{dq_1}{dt} \right] + \frac{d}{dt} \left[m_{12}(\tau) \frac{dq_2}{dt} \right] + b_{11}(\tau)q_1 + b_{12}(\tau)q_2 + \\ + \epsilon f_1(q, \dot{q}) = U_1(\tau) \end{aligned} \quad (3-1)$$

and

$$\begin{aligned} \frac{d}{dt} \left[m_{21}(\tau) \frac{dq_1}{dt} \right] + \frac{d}{dt} \left[m_{22}(\tau) \frac{dq_2}{dt} \right] + b_{21}(\tau)q_1 + b_{22}(\tau)q_2 + \\ + \epsilon f_2(q, \dot{q}) = U_2(\tau) \end{aligned} \quad (3-2)$$

The notations carry the same meaning as in chapter II. $U_1(\tau)$ and $U_2(\tau)$ represents arbitrary pulse excitations. Without any loss of generality the initial conditions may be assumed to be zero, ie.,

$$q_1 = q_2 = 0 \text{ at } t = 0, \quad (3-3)$$

$$\dot{q}_1 = \dot{q}_2 = 0 \text{ at } t = 0. \quad (3-4)$$

Following the outline presented in chapter II, the generalized

coordinates are transformed to

$$q_1 = y_1 + p_1 \quad (3-5)$$

and,

$$q_2 = y_2 + p_2. \quad (3-6)$$

From equations (2-6) and (2-7) the solutions for p_1 and p_2 can be obtained from,

$$\begin{bmatrix} b_{11}(\tau) & b_{12}(\tau) \\ b_{21}(\tau) & b_{22}(\tau) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} U_1(\tau) \\ U_2(\tau) \end{bmatrix} \quad (3-7)$$

Using Cramer's rule equations (3-7) yields

$$p_1(\tau) = \frac{U_1(\tau) b_{22}(\tau) - U_2(\tau) b_{12}(\tau)}{b_{11}(\tau) b_{22}(\tau) - b_{12}(\tau) b_{21}(\tau)} \quad (3-8)$$

and

$$p_2(\tau) = \frac{U_2(\tau) b_{11}(\tau) - U_1(\tau) b_{21}(\tau)}{b_{11}(\tau) b_{22}(\tau) - b_{12}(\tau) b_{21}(\tau)}. \quad (3-9)$$

It is now easy to transform equations (3-1) and (3-2) to the standard form of Krylov, Bogoliubov and Mitropolskii, viz.,

$$\begin{aligned} & \frac{d}{dt} \left[m_{11}(\tau) \frac{dy_1}{dt} \right] + \frac{d}{dt} \left[m_{12}(\tau) \frac{dy_2}{dt} \right] + b_{11}(\tau)y_1 + b_{12}(\tau)y_2 + \\ & + \varepsilon f_1(y + p, \dot{y} + \dot{p}) = 0 \end{aligned} \quad (3-10)$$

and

$$\begin{aligned} & \frac{d}{dt} \left[m_{21}(\tau) \frac{dy_1}{dt} \right] + \frac{d}{dt} \left[m_{22}(\tau) \frac{dy_2}{dt} \right] + b_{21}(\tau)y_1 + b_{22}(\tau)y_2 + \\ & + \varepsilon f_2(y + p, \dot{y} + \dot{p}) = 0. \end{aligned} \quad (3-11)$$

The initial conditions are then written as

$$y_1 = -p_1 \text{ and } y_2 = -p_2 \text{ at } t = 0 \quad (3-12)$$

$$\dot{y}_1 = -\dot{p}_1 \text{ and } \dot{y}_2 = -\dot{p}_2 \text{ at } t = 0 \quad (3-13)$$

Following eqn. (2-17), solutions for eqns. (2-10) and (2-11) are assumed as,

$$y_1 = a_1 \cos \psi_1 + a_2 \cos \psi_2 \quad (3-14)$$

$$y_2 = \alpha_{21}(\tau) a_1 \cos \psi_1 + \alpha_{22}(\tau) a_2 \cos \psi_2 \quad (3-15)$$

where

$$\frac{d\psi_1}{dt} = \omega_1(\tau) + \frac{d\theta_1}{dt} \quad (3-16)$$

and

$$\frac{d\psi_2}{dt} = \omega_2(\tau) + \frac{d\theta_2}{dt} \quad (3-17)$$

$\omega_1(\tau)$ and $\omega_2(\tau)$ are nonstationary normal frequencies

a_1, a_2 are the amplitudes

θ_1, θ_2 represent the phase

$\alpha_{21}(\tau), \alpha_{22}(\tau)$ are the amplitude ratios obtained from the normal mode solution of the linear system.

For the first order approximation, the expressions for amplitude and instantaneous frequency variation from eqns. (2-17) and (2-18) are given by,

$$\frac{da_1}{dt} = \varepsilon A_{11}(a_1, a_2, \tau) \quad (3-18)$$

$$\frac{da_2}{dt} = \varepsilon A_{21}(a_1, a_2, \tau) \quad (3-19)$$

$$\frac{d\theta_1}{dt} = \varepsilon B_{11}(a_1, a_2, \tau) \quad (3-20)$$

and

$$\frac{d\theta}{dt} = \varepsilon B_{21}(a_1, a_2, \tau). \quad (3-21)$$

Following the analysis presented in sec. (2-3), explicit expressions for amplitude and instantaneous frequency variations are obtained as,

$$\begin{aligned} \frac{da_1}{dt} = & \left[-(m_{21}(\tau)\alpha_{12}(\tau) + m_{22}(\tau)\alpha_{22}(\tau)) \times \right. \\ & \times \left(-\frac{1}{2} \left[\alpha_{11}(\tau) \frac{d}{dt} (m_{11}(\tau)\omega_1(\tau)) + \alpha_{21}(\tau) \frac{d}{dt} (m_{12}(\tau)\omega_1(\tau)) \right] \times \right. \\ & \times \left. a_1 + \frac{\varepsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_1(y, \dot{y}, \tau) \sin \psi_1 d\psi_1 d\psi_2 \right) - \\ & - (m_{11}(\tau)\alpha_{12}(\tau) + m_{12}(\tau)\alpha_{22}(\tau)) \times \\ & \times \left(-\left[\frac{1}{2} \alpha_{12}(\tau) \frac{d}{dt} (m_{21}(\tau)\omega_1(\tau)) + \alpha_{21}(\tau) \frac{d}{dt} (m_{22}(\tau)\omega_1(\tau)) \right] \times \right. \\ & \times \left. a_1 + \frac{\varepsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_2(y, \dot{y}, \tau) \sin \psi_1 d\psi_1 d\psi_2 \right) + \\ & + \left. \alpha_{12}(\tau)\omega_1(\tau)(m_{11}(\tau)m_{22}(\tau) - m_{21}(\tau)m_{12}(\tau))a_1 \frac{d\alpha_{21}(\tau)}{dt} \right] \times \\ & \times \left[-\omega_1(\tau) (\alpha_{11}(\tau)\alpha_{22}(\tau) - \alpha_{21}(\tau)\alpha_{12}(\tau)) (m_{11}(\tau)m_{22}(\tau) - \right. \\ & \left. - m_{12}(\tau)m_{21}(\tau)) \right]^{-1} \quad (3-22) \end{aligned}$$

$$\begin{aligned} \frac{da_2}{dt} = & \left[-\alpha_{11}(\tau) (m_{21}(\tau)m_{12}(\tau) - m_{11}(\tau)m_{22}(\tau))\omega_2(\tau)a_2 \frac{d\alpha_{22}(\tau)}{dt} + \right. \\ & + (m_{21}(\tau)\alpha_{11}(\tau) + m_{22}(\tau)\alpha_{21}(\tau)) \cdot \frac{1}{2} \times \end{aligned}$$

$$\begin{aligned}
& \times \left(- \left[\alpha_{12}(\tau) \frac{d}{dt} (m_{11}(\tau)\omega_2(\tau)) + \alpha_{22}(\tau) \frac{d}{dt} (m_{12}(\tau)\omega_2(\tau)) \right] \times \right. \\
& \times \left. a_2 + \frac{\varepsilon}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_1 \sin \psi_2 d\psi_1 d\psi_2 \right) - (m_{11}(\tau)\alpha_{11}(\tau) + \\
& + m_{12}(\tau)\alpha_{21}(\tau)) \frac{1}{2} \left(- a_2 \left[\alpha_{12}(\tau) \frac{d}{dt} (m_{21}(\tau)\omega_2(\tau)) \right. \right. \\
& + \left. \left. \alpha_{22}(\tau) \frac{d}{dt} (m_{22}(\tau)\omega_2(\tau)) \right] + \frac{\varepsilon}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_2 \sin \psi_2 d\psi_1 d\psi_2 \right) \times \\
& \times \left[-\omega_2(\tau) (\alpha_{11}(\tau)\alpha_{22}(\tau) - \alpha_{21}(\tau)\alpha_{12}(\tau)) (m_{11}(\tau)m_{22}(\tau) - \right. \\
& \left. - m_{12}(\tau)m_{21}(\tau)) \right]^{-1} \quad (3-23)
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta_1}{dt} = & \left[- (m_{21}(\tau)\alpha_{12}(\tau) + m_{22}(\tau)\alpha_{22}(\tau)) \times \right. \\
& \times \left[- \frac{1}{2}(m_{11}(\tau)\alpha_{11}(\tau) + m_{12}(\tau)\alpha_{21}(\tau))\omega_1^2(\tau)a_1 + \right. \\
& + \frac{1}{2}(b_{11}(\tau)\alpha_{11}(\tau) + b_{12}(\tau)\alpha_{21}(\tau))a_1 + \\
& + \frac{\varepsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_1 \cos \psi_1 d\psi_1 d\psi_2 \left. \right] + \\
& + (m_{11}(\tau)\alpha_{12}(\tau) + m_{12}(\tau)\alpha_{22}(\tau)) \left[- \frac{1}{2}(m_{21}(\tau)\alpha_{11}(\tau) + \right. \\
& + m_{22}(\tau)\alpha_{21}(\tau))\omega_1^2(\tau)a_1 + \frac{1}{2}(b_{21}(\tau)\alpha_{11}(\tau) + b_{22}(\tau)\alpha_{21}(\tau))a_1 + \\
& + \frac{\varepsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_2 \cos \psi_1 d\psi_1 d\psi_2 \left. \right] \left[- (\alpha_{11}(\tau)\alpha_{22}(\tau) - \right.
\end{aligned}$$

$$- \alpha_{12}(\tau) \alpha_{21}(\tau)) (m_{11}(\tau) m_{22}(\tau) - m_{12}(\tau) m_{21}(\tau)) a_1 \omega_1(\tau) \Big]^{-1} \quad (3-24)$$

and,

$$\begin{aligned} \frac{d\theta}{dt} = & \left[(m_{21}(\tau) \alpha_{11}(\tau) + m_{22}(\tau) \alpha_{21}(\tau)) \times \right. \\ & \times \left[-\frac{1}{2} (m_{11}(\tau) \alpha_{12}(\tau) + m_{12}(\tau) \alpha_{22}(\tau)) a_2 \omega_2^2(\tau) + \right. \\ & + (b_{11}(\tau) \alpha_{12}(\tau) + b_{12}(\tau) \alpha_{22}(\tau)) \frac{a_2}{2} + \\ & + \frac{\epsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_1(y, \dot{y}, \tau) \cos \Psi_1 \cos \Psi_2 d\Psi_1 d\Psi_2 - \\ & - (m_{11}(\tau) \alpha_{11}(\tau) + m_{12}(\tau) \alpha_{21}(\tau)) \left[-\frac{1}{2} (m_{21}(\tau) \alpha_{12}(\tau) + \right. \\ & + m_{22}(\tau) \alpha_{22}(\tau)) a_2 \omega_2^2(\tau) + (b_{21}(\tau) \alpha_{12}(\tau) + b_{22}(\tau) \alpha_{22}(\tau)) \frac{a_2}{2} + \\ & + \left. \frac{\epsilon}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_2(y, \dot{y}, \tau) \cos \Psi_1 \cos \Psi_2 d\Psi_1 d\Psi_2 \right] \times \\ & \times \left[-a_2 \omega_2(\tau) (\alpha_{11}(\tau) \alpha_{22}(\tau) - \alpha_{12}(\tau) \alpha_{21}(\tau)) (m_{11}(\tau) m_{22}(\tau) - \right. \\ & \left. \left. - m_{12}(\tau) m_{21}(\tau)) \right]^{-1} \right] \quad (3-25) \end{aligned}$$

However the constant of integration needs the initial conditions for amplitude and instantaneous frequency. From equations (2-8), (2-9), (2-12), (2-13), (2-14), and (2-15) the initial conditions for $a_1(t)$, $a_2(t)$, $\Psi_1(t)$ and $\Psi_2(t)$ can be obtained as,

$$a_1(0) = \sqrt{\left[\frac{p_2(0) - \alpha_{22}(0)p_1(0)}{(\alpha_{22}(0) - \alpha_{21}(0))} \right]^2 + \left[\frac{\dot{p}_2(0) - \alpha_{22}(0)\dot{p}_1(0)}{\omega_1(0)(\alpha_{21}(0) - \alpha_{22}(0))} \right]^2} \quad (3-26)$$

$$a_2(0) = \sqrt{\left[\frac{p_2(0) - \alpha_{21}(0)p_1(0)}{(\alpha_{21}(0) - \alpha_{22}(0))} \right]^2 + \left[\frac{\dot{p}_2(0) - \alpha_{21}(0)\dot{p}_1(0)}{\omega_2(0)(\alpha_{22}(0) - \alpha_{21}(0))} \right]^2} \quad (3-27)$$

$$\psi_1(0) = \cos^{-1} \left[\frac{p_2(0) - \alpha_{22}(0)p_1(0)}{(\alpha_{22}(0) - \alpha_{21}(0))a_1(0)} \right] \quad (3-28)$$

$$\psi_2(0) = \cos^{-1} \left[\frac{p_2(0) - \alpha_{21}(0)p_1(0)}{(\alpha_{21}(0) - \alpha_{22}(0))a_2(0)} \right] \quad (3-29)$$

The amplitudes and instantaneous frequencies are obtained by integrating eqns. (3-22) through (3-25). It may be shown that under special cases, depending on the type of nonlinearities in the system, simplification of the expressions for amplitude and instantaneous frequencies are possible. The total solution of the system is then given by

$$q_1 = y_1 + p_1 \quad (3-30)$$

and

$$q_2 = y_2 + p_2 \quad (3-31)$$

In the following section special cases of nonlinearities are being considered.

3.3 Special Cases of Nonlinearities:

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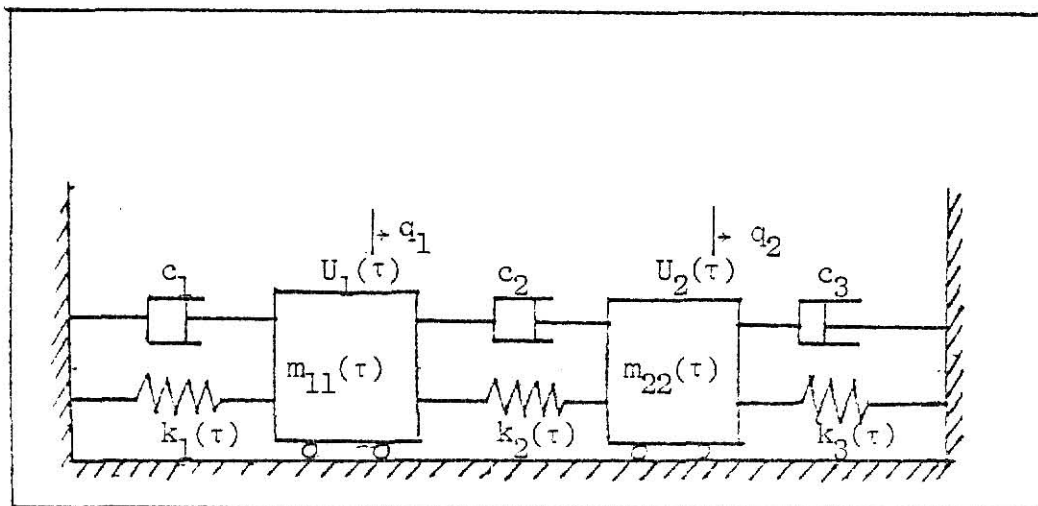


Fig 1. Typical dynamically decoupled nonlinear system with two degrees of freedom and slowly varying parameters, subjected to arbitrary pulse excitation.

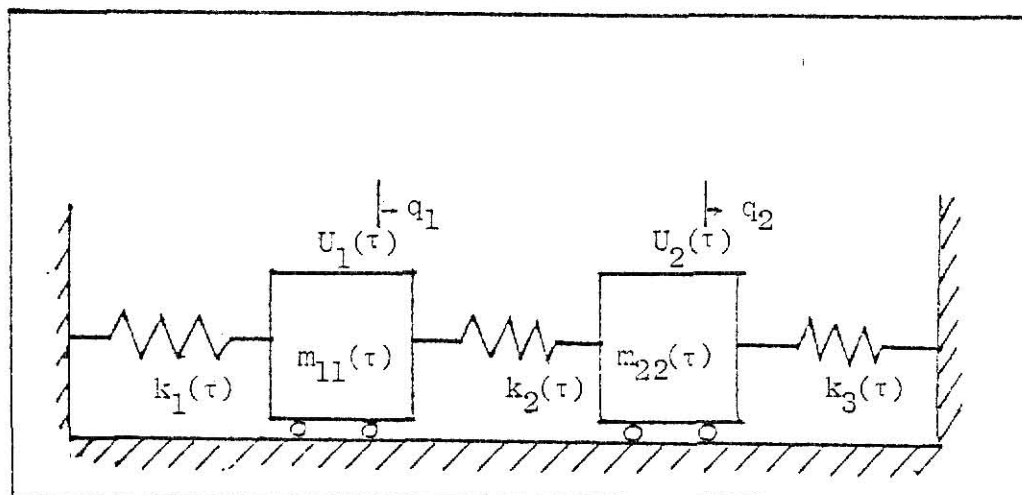


Fig 2. Nonlinear nonstationary system with two degrees of freedom having cubic restoring elements.

$$m_{11}(\tau) = (1 + 0.2\tau) ; \quad m_{22}(\tau) = (1 + 0.25\tau)$$

$$\text{Force } F_{k_1(\tau)} = (3 + 0.1\tau)z + (z)^3 ; \quad \text{Force } F_{k_3(\tau)} = (3 + 0.1\tau)z$$

$$\text{Force } F_{k_2(\tau)} = (3 + 0.1\tau)z + (z)^3$$

c_1, c_2 , & c_3 — viscous coefficient of friction.

z — relative displacement.

$U_1(\tau), U_2(\tau)$ = arbitrary pulse

3.3.1 Nonlinearities depending on displacement type terms only

These are typical of softening or hardening type springs in nonlinear vibrational systems. The nonlinearities are purely dependent on displacement type terms. Such nonlinearities can be represented mathematically as,

$$f_i(q_j, \dot{q}_j) = f_i(q_j). \quad (3-32)$$

Through the transformations of the dependent variable suggested by eqns. (3-5) and (3-6), eqns. (3-32) can be expressed as,

$$f_i(q_j, \dot{q}_j) = f_i(q_j) = f_i(y_j, \tau) \quad (3-33)$$

Substitution of eqn. (3-33) in the expressions for amplitude variation, viz., eqns. (3-22) and (3-23) yields,

$$\begin{aligned} \frac{1}{a_1} \frac{da_1}{dt} = & \left[\alpha_{12}(\tau) \omega_1(\tau) (m_{11}(\tau) m_{22}(\tau) - m_{21}(\tau) m_{12}(\tau)) \right] \times \\ & \times \left[\frac{d\alpha_{21}(\tau)}{dt} + (m_{21}(\tau) \alpha_{12}(\tau) + m_{22}(\tau) \alpha_{22}(\tau)) \right] \times \\ & \times \frac{1}{2} \left[\alpha_{11}(\tau) \frac{d}{dt} (m_{11}(\tau) \omega_1(\tau)) + \alpha_{21}(\tau) \frac{d}{dt} (m_{12}(\tau) \omega_1(\tau)) \right] + \\ & + (m_{11}(\tau) \alpha_{12}(\tau) + m_{12}(\tau) \alpha_{22}(\tau)) \frac{1}{2} \left[\alpha_{12}(\tau) \frac{d}{dt} (m_{21}(\tau) \omega_1(\tau)) + \right. \\ & \left. + \alpha_{21}(\tau) \frac{d}{dt} (m_{22}(\tau) \omega_1(\tau)) \right] \times \left[-\omega_1(\tau) (\alpha_{11}(\tau) \alpha_{22}(\tau) - \right. \\ & \left. - \alpha_{21}(\tau) \alpha_{12}(\tau)) (m_{11}(\tau) m_{22}(\tau) - m_{12}(\tau) m_{21}(\tau)) \right]^{-1} \end{aligned} \quad (3-34)$$

and,

$$\begin{aligned} \frac{1}{a_2} \frac{da_2}{dt} = & \left[-\alpha_{11}(\tau) (m_{21}(\tau) m_{12}(\tau) - m_{11}(\tau) m_{22}(\tau)) \omega_2(\tau) \right] \times \\ & \times \frac{d\alpha_{22}(\tau)}{dt} - \frac{1}{2} (m_{21}(\tau) \alpha_{11}(\tau) + m_{22}(\tau) \alpha_{21}(\tau)) \times \end{aligned}$$

$$\begin{aligned}
& \times \left[\alpha_{12}(\tau) \frac{d}{dt} (m_{11}(\tau) \omega_2(\tau)) + \alpha_{22}(\tau) \frac{d}{dt} (m_{12}(\tau) \omega_2(\tau)) \right] + \\
& + (m_{11}(\tau) \alpha_{11}(\tau) + m_{12}(\tau) \alpha_{21}(\tau)) \frac{1}{2} \left[\alpha_{12}(\tau) \frac{d}{dt} (m_{21}(\tau) \omega_2(\tau)) + \right. \\
& + \left. \alpha_{22}(\tau) \frac{d}{dt} (m_{22}(\tau) \omega_2(\tau)) \right] \times \left[-\omega_2(\tau) (\alpha_{11}(\tau) \alpha_{22}(\tau) - \right. \\
& - \left. \alpha_{21}(\tau) \alpha_{12}(\tau)) (m_{11}(\tau) m_{22}(\tau) - m_{12}(\tau) m_{21}(\tau)) \right]^{-1} \quad (3-35)
\end{aligned}$$

Denoting the right hand sides of eqn. (3-34) and (3-35) by $h_1(\tau)$ and $h_2(\tau)$, it is possible to express the amplitudes as,

$$a_1(t) = a_1' e^{\int h_1(\tau) d\tau} \quad (3-36)$$

and

$$a_2(t) = a_2' e^{\int h_2(\tau) d\tau} \quad (3-37)$$

where a_1' and a_2' are the constants obtained from the initial conditions. Observe that in some cases a simple numerical quadrature scheme may be needed to integrate $h_1(\tau)$ and $h_2(\tau)$. The expressions $a_1(t)$ and $a_2(t)$ from equation (3-36) and (3-37) can now be substituted in eqn. (3-24) and eqn. (3-25) to obtain $\frac{d\psi_1}{dt}$ and $\frac{d\psi_2}{dt}$ as explicit functions of time t . Subsequently a

simple numerical quadrature scheme can be used to integrate these expressions. The variation of amplitude is strictly due to the nonstationary parameters of the system. However there is a significant contribution to the change in phase due to the presence of the nonlinearities having displacement type terms only. A typical example is being presented for this case.

Example - 1:

Consider the system shown in fig. [2] having a cubic nonlinear spring. For simplicity, a dynamically decoupled nonstationary system with nonstationary parameters varying linearly with respect to the slowing time τ is being considered. The nonlinearities arising in the differential equation of this system are expressed as,

$$f_1(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = [(y_1 - y_2 + p_1 - p_2)^3 + (y_1 + p_1)^3] \quad (3-39)$$

$$f_2(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = -[(y_1 - y_2 + p_1 - p_2)^3] \quad (3-40)$$

The effect of such nonlinearities on the system response may be studied by solving for the values of amplitudes and instantaneous frequencies in the system. Substituting the quantities in eqns. (3-22) and (3-23) yields,

$$\begin{aligned} \frac{da_1}{dt} = & \left[a_1 \alpha_{12}(\tau) \omega_1(\tau) m_{11}(\tau) m_{22}(\tau) \frac{d\alpha_{21}(\tau)}{dt} - \right. \\ & \left. - m_{22}(\tau) \alpha_{22}(\tau) - \frac{1}{2} \frac{d}{dt} (m_{11}(\tau) \omega_1(\tau)) + \frac{1}{2} m_{11}(\tau) \alpha_{21}(\tau) \frac{d}{dt} (m_{22}(\tau) \omega_1(\tau)) \right] \\ & \times \left[-\omega_1(\tau) \alpha_{11}(\tau) \alpha_{22}(\tau) m_{11}(\tau) m_{22}(\tau) \right]^{-1} \end{aligned} \quad (3-41)$$

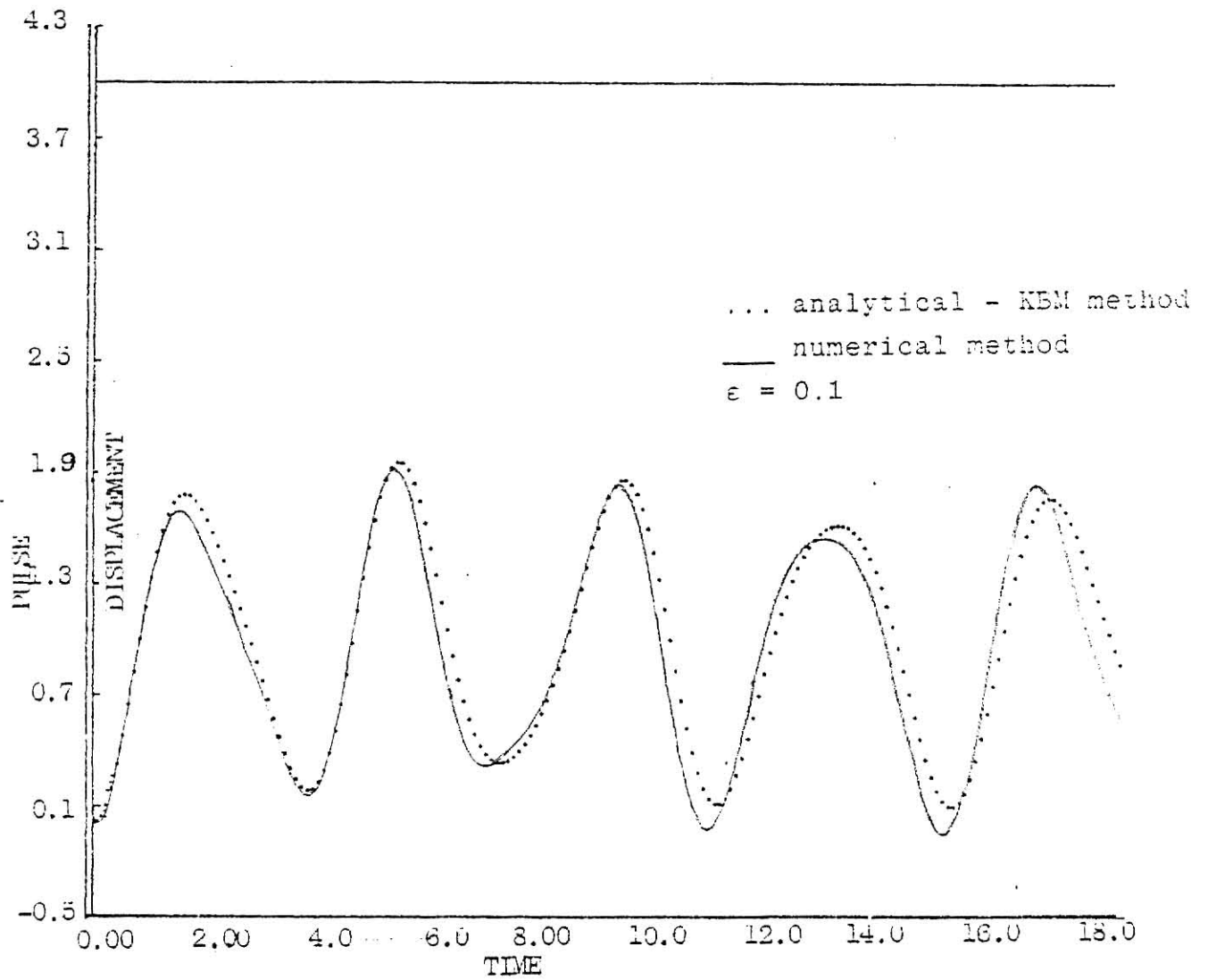
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$$\begin{aligned} \frac{da_2}{dt} = & \left[\alpha_{11}(\tau) m_{11}(\tau) m_{22}(\tau) a_2 \omega_2(\tau) \frac{d\alpha_{22}(\tau)}{dt} - \right. \\ & - \frac{1}{2} m_{22}(\tau) \alpha_{21}(\tau) \frac{d}{dt} (m_{11}(\tau) \omega_2(\tau)) + \\ & \left. + \frac{1}{2} m_{11}(\tau) \alpha_{11}(\tau) \frac{a_2}{2} \frac{d}{dt} (m_{22}(\tau) \omega_2(\tau) \alpha_{22}(\tau)) \right] \times \\ & \times \left[-\omega_2(\tau) \alpha_{11}(\tau) \alpha_{22}(\tau) m_{11}(\tau) m_{22}(\tau) \right]^{-1} \end{aligned} \quad (3-42)$$

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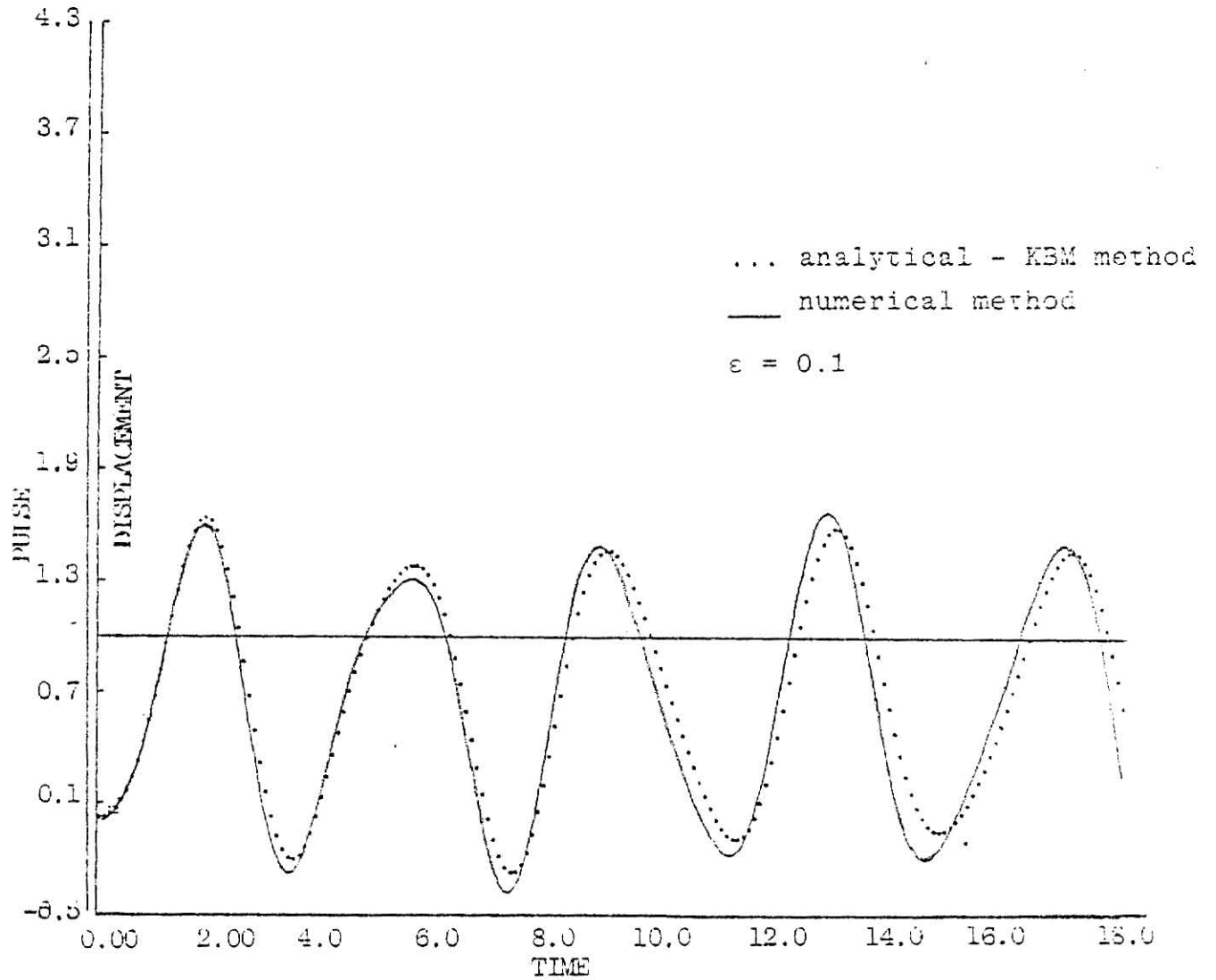
Step function excitation

Fig.3. Time-displacement q_1 result for the system;

$$\frac{d}{dt} \left[(1 + 0.2\tau) \frac{dq_1}{dt} \right] + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon(q_1^3 + (q_1 - q_2)^3) = 4$$

$$\frac{d}{dt} \left[(1 + 0.25\tau) \frac{dq_2}{dt} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \varepsilon(q_2 - q_1)^3 = 1$$

Step function excitation

Fig.4. Time-displacement q_2 result for the system.

$$\frac{d}{dt} (1 + 0.2\tau) \frac{dq_1}{dt} + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon(q_1^3 + (q_1 - q_2)^3) = 4$$

$$\frac{d}{dt} (1 + 0.25\tau) \frac{dq_2}{dt} - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \varepsilon(q_2 - q_1)^3 = 1$$

Denoting the right hand side expressions of eqn. (3-41) and (3-42) by $h_1(\tau)$ and $h_2(\tau)$ respectively the amplitudes may be expressed in the form

$$a_1 = a_1' e^{\int h_1(\tau) d\tau} \quad (3-43)$$

$$a_2 = a_2' e^{\int h_2(\tau) d\tau} \quad (3-44)$$

where a_1' and a_2' are constants obtained from the initial conditions. Using the known values of amplitudes the values of instantaneous frequencies maybe computed by substituting the known values of amplitude and the expressions for nonlinearities in the system given by eqn. (3-39) and (3-40), in eqn. (3-24) and (3-25). This yields,

$$\begin{aligned} \frac{d\theta_1}{dt} = & \left[-m_{22}(\tau)\alpha_{22}(\tau) \left[-\frac{1}{2} m_{11}(\tau)\omega_1^2(\tau) a_1 + \right. \right. \\ & + \frac{1}{2}(b_{11}(\tau) + b_{12}(\tau)\alpha_{21}(\tau))a_1 + F_{11}(a_1, a_2, \tau) \left. \right] + \\ & + m_{11}(\tau) \left[-\frac{1}{2} m_{22}(\tau)\alpha_{21}(\tau)\omega_1^2(\tau)a_1 + \right. \\ & + \frac{1}{2} a_1(b_{21}(\tau) + b_{22}(\tau)\alpha_{21}(\tau)) + F_{12}(a_1, a_2, \tau) \left. \right] \times \\ & \times \left[-\alpha_{22}(\tau)\alpha_{11}(\tau)m_{11}(\tau)m_{22}(\tau)a_1\omega_1(\tau) \right]^{-1} \end{aligned} \quad (3-45)$$

and

$$\begin{aligned} \frac{d\theta_2}{dt} = & \left[m_{22}(\tau)\alpha_{21}(\tau) \left[-\frac{1}{2} a_2 m_{11}(\tau)\alpha_{21}(\tau)\omega_2^2(\tau) + \right. \right. \\ & + (b_{11}(\tau) + b_{12}(\tau)\alpha_{22}(\tau)) \frac{a_2}{2} + F_{21}(a_1, a_2, \tau) \left. \right] - \\ & - m_{11}(\tau)\alpha_{11}(\tau) \left[\frac{1}{2} a_2 m_{22}(\tau)\alpha_{22}(\tau)\omega_2^2(\tau) + \right. \\ & + (b_{21}(\tau) + b_{22}(\tau)\alpha_{22}(\tau)) \frac{a_2}{2} + F_{22}(a_1, a_2, \tau) \left. \right] \times \\ & \times (a_2\omega_2(\tau)\alpha_{22}(\tau)\alpha_{11}(\tau)m_{11}(\tau)m_{22}(\tau)) \end{aligned} \quad (3-46)$$

where,

$$\begin{aligned}
 F_{11}(a_1, a_2, \tau) &= 0.375a_1^3 + 0.75a_1a_2^2 + 1.5a_1p_1^2 \\
 &\quad + 0.375(1 - \alpha_{21}(\tau))^3a_1^3 + \\
 &\quad + 1.5a_1a_2^2(1 - \alpha_{21}(\tau))(1 - \alpha_{22}(\tau))^2 \\
 F_{12}(a_1, a_2, \tau) &= - \left[0.375(1 - \alpha_{21}(\tau))^3a_1^3 + 1.5(1 - \alpha_{21}(\tau)) \right. \\
 &\quad \left. (1 - \alpha_{22}(\tau))^3a_1a_2^2 + 1.5(1 - \alpha_{21}(\tau))(p_1 - p_2)^2a_1 \right] \\
 F_{21}(a_1, a_2, \tau) &= 0.375a_2^3 + 0.75a_1^3a_2 + 1.5p_1^2a_2 \\
 &\quad + 0.375(1 - \alpha_{22}(\tau))^3a_2^3 + 0.75(1 - \alpha_{21}(\tau))^2a_1^2 \\
 &\quad (1 - \alpha_{22}(\tau))a_2 + 1.5(1 - \alpha_{22}(\tau))a_2p_2^2 \\
 F_{22}(a_1, a_2, \tau) &= - \left[0.375(1 - \alpha_{22}(\tau))^3a_2^3 + 0.75(1 - \alpha_{21}(\tau))^2a_1^2 \right. \\
 &\quad \left. (1 - \alpha_{22}(\tau))a_2 + 1.5(1 - \alpha_{22}(\tau))a_2p_2^2 \right] \quad (3-47)
 \end{aligned}$$

The constants may be computed using the initial conditions given by equations (3-26) through (3-29). From the form of eqns. (3-41) and (3-42) it is observed that the variation of amplitude is strictly due to the nonstationary values of the parameters of the system. However the shift in phase is significantly increased or decreased due to nonlinearities of the type under discussion. This is evident from eqns. (3-43) and (3-45). The results presented in fig.(3)&(4) indicate a small shift in phase. This increases the time period of the system and could be of importance in design.

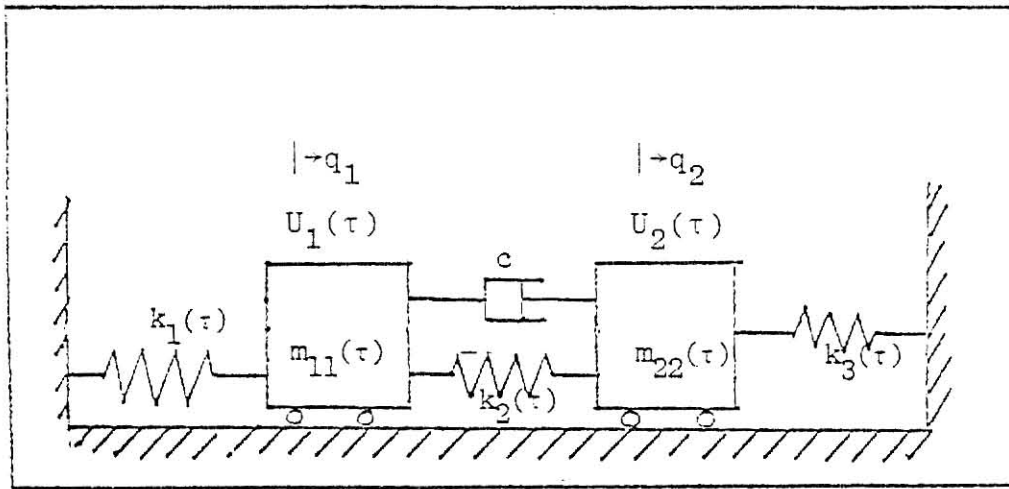


Fig.4a. Nonlinear nonstationary system with two degrees of freedom and cubic damping elements.

$$m_{11}(\tau) = (1 + 0.2\tau); \quad m_{22}(\tau) = (1 + 0.25\tau)$$

$$k_1(\tau) = k_2(\tau) = k_3(\tau) = (3 + 0.1\tau)$$

$$\text{Force } F_c = (\dot{z})^3$$

z = relative displacement

q_1, q_2 = generalized co-ordinates

$U_1(\tau), U_2(\tau)$ = arbitrary pulse

3.3.2. Systems With Nonlinearities Due to Velocity Type Terms Only

Nonlinearities in systems due to velocity type terms alone may be represented in a general form as,

$$f_i(q_j, \dot{q}_j) = f_i(\dot{q}_j) \quad (3-48)$$

Further, through the transformation of the dependent variable suggested by eqns. (2-3) and (2-4) it may be rewritten in the form,

$$f_i(\dot{q}_j) = f_i(\dot{y}_j, \tau) \quad (3-49)$$

For this case the expressions of amplitudes and instantaneous frequencies are given by equations (3-22) through (3-25). The expressions for instantaneous frequency variation further reduce to

$$\begin{aligned} \frac{d\theta_1}{dt} = a_1 & \left[-m_{21}(\tau)\alpha_{12}(\tau) + m_{22}(\tau)\alpha_{22}(\tau) \left[-\frac{1}{2} (m_{11}(\tau)\alpha_{11}(\tau) + \right. \right. \\ & + m_{12}(\tau)\alpha_{21}(\tau))\omega_1^2(\tau) + \frac{1}{2} (b_{11}(\tau)\alpha_{11}(\tau) + b_{12}(\tau)\alpha_{21}(\tau)) \Big] + \\ & + (m_{11}(\tau)\alpha_{12}(\tau) + m_{12}(\tau)\alpha_{22}(\tau)) \left[-\frac{1}{2} (m_{21}(\tau)\alpha_{11}(\tau) + \right. \\ & + m_{22}(\tau)\alpha_{21}(\tau))\omega_1^2(\tau) + \frac{1}{2} (b_{21}(\tau)\alpha_{11}(\tau) + b_{22}(\tau)\alpha_{21}(\tau)) \Big] \Big] \times \\ & \times \left[(\alpha_{11}(\tau)\alpha_{22}(\tau) - \alpha_{21}(\tau)\alpha_{12}(\tau))(m_{11}(\tau)m_{22}(\tau) - m_{12}(\tau)m_{21}(\tau))\omega_1(\tau) \right]^{-1} \end{aligned} \quad (3-50)$$

and

$$\begin{aligned} \frac{d\theta_2}{dt} = a_2 & \left[(m_{21}(\tau)\alpha_{11}(\tau) + m_{22}(\tau)\alpha_{21}(\tau)) \left[-\frac{1}{2} (m_{11}(\tau)\alpha_{12}(\tau) + \right. \right. \\ & + m_{12}(\tau)\alpha_{22}(\tau))\omega_2^2(\tau) + \frac{1}{2} (b_{11}(\tau)\alpha_{12}(\tau) + b_{12}(\tau)\alpha_{22}(\tau)) \Big] - \\ & - (m_{11}(\tau)\alpha_{11}(\tau) + m_{12}(\tau)\alpha_{21}(\tau)) \left[-\frac{1}{2} (m_{21}(\tau)\alpha_{12}(\tau) + \right. \end{aligned}$$

$$\begin{aligned}
& + m_{22}(\tau)\alpha_{22}(\tau)\omega_2^2(\tau) + \frac{1}{2}(b_{21}(\tau)\alpha_{12}(\tau) + b_{22}(\tau)\alpha_{22}(\tau)) \Big] \times \\
& \times \left[-(\alpha_{11}(\tau)\alpha_{22}(\tau) - \alpha_{21}(\tau)\alpha_{12}(\tau))(m_{11}(\tau)m_{22}(\tau) - m_{12}(\tau)m_{21}(\tau))\omega_1(\tau) \right]^{-1}
\end{aligned}
\tag{3-51}$$

Note that once the values of amplitudes are known as explicit functions of time, these expressions may be computed by a simple quadrature scheme.

It is observed that a shift in phase occurs due to the presence of nonstationary parameters in the system. Such nonlinearities offer no contribution to the phase. On the other hand, amplitude variations are affected by the presence of nonlinear dampers in the system. This contribution is in addition to the variation of amplitude caused due to nonstationary parameters. An example where nonlinearities in the system are due to velocity type terms alone, is being presented for typical values of system parameters.

Example 2

For the system represented by fig.[4], the nonlinear dampers give rise to nonlinearities of the form:

$$f_1(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = (\dot{y}_1 - \dot{y}_2 + \dot{p}_1 - \dot{p}_2)^3 \tag{3-52}$$

and

$$f_2(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = -(\dot{y}_1 - \dot{y}_2 + \dot{p}_1 - \dot{p}_2)^3. \tag{3-53}$$

As outlined in the previous example, it is necessary to first find the expressions for amplitude and instantaneous frequencies.

These may be obtained by using eqns. (3-22) through (3-25).

These are given by,

$$\begin{aligned} \frac{da_1}{dt} = & \left[\omega_1(\tau) m_{11}(\tau) m_{22}(\tau) a_1 \frac{d\alpha_{21}(\tau)}{dt} - \right. \\ & - m_{22}(\tau) \alpha_{22}(\tau) \left[-\frac{a_1}{2} \frac{d}{dt} (m_{11}(\tau) \omega_1(\tau)) + F_{11}(a_1, a_2, \tau) \right] - \\ & - m_{11}(\tau) \left[-\frac{a_1}{2} \alpha_{21}(\tau) \frac{d}{dt} (m_{22}(\tau) \omega_1(\tau)) + F_{12}(a_1, a_2, \tau) \right] \times \\ & \times \left[-\omega_1(\tau) m_{11}(\tau) m_{22}(\tau) (\alpha_{11}(\tau) \alpha_{22}(\tau) - \alpha_{21}(\tau) \alpha_{12}(\tau)) \right]^{-1} \end{aligned} \quad (3-54)$$

and,

$$\begin{aligned} \frac{da_2}{dt} = & \left[m_{11}(\tau) m_{22}(\tau) \omega_2(\tau) a_2 \frac{d\alpha_{22}(\tau)}{dt} + m_{22}(\tau) \alpha_{21}(\tau) \right. \\ & \times \left[-\frac{a_2}{2} \frac{d}{dt} (m_{11}(\tau) \omega_2(\tau)) + F_{21}(a_1, a_2, \tau) \right] - \\ & - m_{11}(\tau) \alpha_{11}(\tau) \left[-\frac{a_2}{2} \alpha_{22}(\tau) \frac{d}{dt} (m_{22}(\tau) \omega_2(\tau)) + \right. \\ & + F_{22}(a_1, a_2, \tau) \left. \right] \times \left[-\omega_2(\tau) m_{11}(\tau) m_{22}(\tau) (\alpha_{11}(\tau) \alpha_{22}(\tau) - \right. \\ & - \alpha_{21}(\tau) \alpha_{12}(\tau)) \left. \right]^{-1} \end{aligned} \quad (3-55)$$

where,

$$\begin{aligned} F_{11}(a_1, a_2, \tau) = & (\alpha_{21}(\tau) - 1)^3 \omega_1(\tau)^3 a_1^{0.375} + 0.25 \times \\ & \times a_2^2 a_1 \omega_2(\tau)^2 \omega_1(\tau) (\alpha_{22}(\tau) - 1)^2 (\alpha_{21}(\tau) - 1) + \\ & + 1.5(\alpha_{21}(\tau) - 1) a_1 \omega_1(\tau) \times 0.0005063, \end{aligned}$$

$$F_{12}(a_1, a_2, \tau) = -F_{11}(a_1, a_2, \tau),$$

$$F_{21}(a_1, a_2, \tau) = 0.375 a_2^3 (\alpha_{22}(\tau) - 1)^3 \omega_2(\tau)^3 + 0.75 a_1^2 \times$$

$$\begin{aligned} & \times \omega_1^2(\tau)(\alpha_{21}(\tau) - 1)^2(\alpha_{22}(\tau) - 1) a_2 \omega_2(\tau) + \\ & + 3a_2 \omega_2(\tau) \times 0.0005063(\alpha_{22}(\tau) - 1), \end{aligned}$$

and

$$F_{22}(a_1, a_2, \tau) = -F_{21}(a_1, a_2, \tau) \quad (3-56)$$

Contrary to the case discussed in example No. 1, nonlinearities of the type under discussion simplifies significantly the expressions for instantaneous frequency. This may be observed from the following expressions.

$$\begin{aligned} \frac{d\theta_1}{dt} = & \left[-\frac{1}{2} m_{22}(\tau) \alpha_{22}(\tau) m_{11}(\tau) \omega_1^2(\tau) + \frac{1}{2} m_{22}(\tau) \alpha_{22}(\tau) \times \right. \\ & \times (b_{11}(\tau) + b_{12}(\tau) \alpha_{21}(\tau)) - \frac{1}{2} m_{11}(\tau) m_{22}(\tau) \alpha_{21}(\tau) \omega_1^2(\tau) + \\ & + \left. \frac{1}{2} m_{11}(\tau) (b_{21}(\tau) + \alpha_{21}(\tau) b_{22}(\tau)) \right] \times \left[-m_{11}(\tau) m_{22}(\tau) \times \right. \\ & \times \left. \omega_1(\tau) (\alpha_{22}(\tau) - \alpha_{21}(\tau)) \right]^{-1} \quad (3-57) \end{aligned}$$

and,

$$\begin{aligned} \frac{d\theta_2}{dt} = & \left[m_{22}(\tau) \alpha_{21}(\tau) \left[-\frac{1}{2} m_{11}(\tau) \omega_2^2(\tau) + \frac{1}{2} (b_{11}(\tau) + b_{12}(\tau) \times \right. \right. \\ & \times \left. \left. \alpha_{22}(\tau)) \right] - m_{11}(\tau) \left(\frac{-m_{22}(\tau) \alpha_{22}(\tau) \omega_2^2(\tau)}{2} \right) + \right. \\ & + \left. \frac{1}{2} (b_{21}(\tau) + b_{22}(\tau) \alpha_{22}(\tau)) \right] \times \left[-m_{11}(\tau) m_{22}(\tau) \omega_2(\tau) \right. \\ & \times \left. \left. (\alpha_{22}(\tau) - \alpha_{21}(\tau)) \right]^{-1} \quad (3-58) \end{aligned}$$

The instantaneous frequencies may be obtained through the application of a simple quadrature scheme, once the amplitude values a_1 and a_2 are known as functions of t . It is evident from eqn. (3-54) and eqn. (3-55) that a significant contribution

Exponential step pulse.

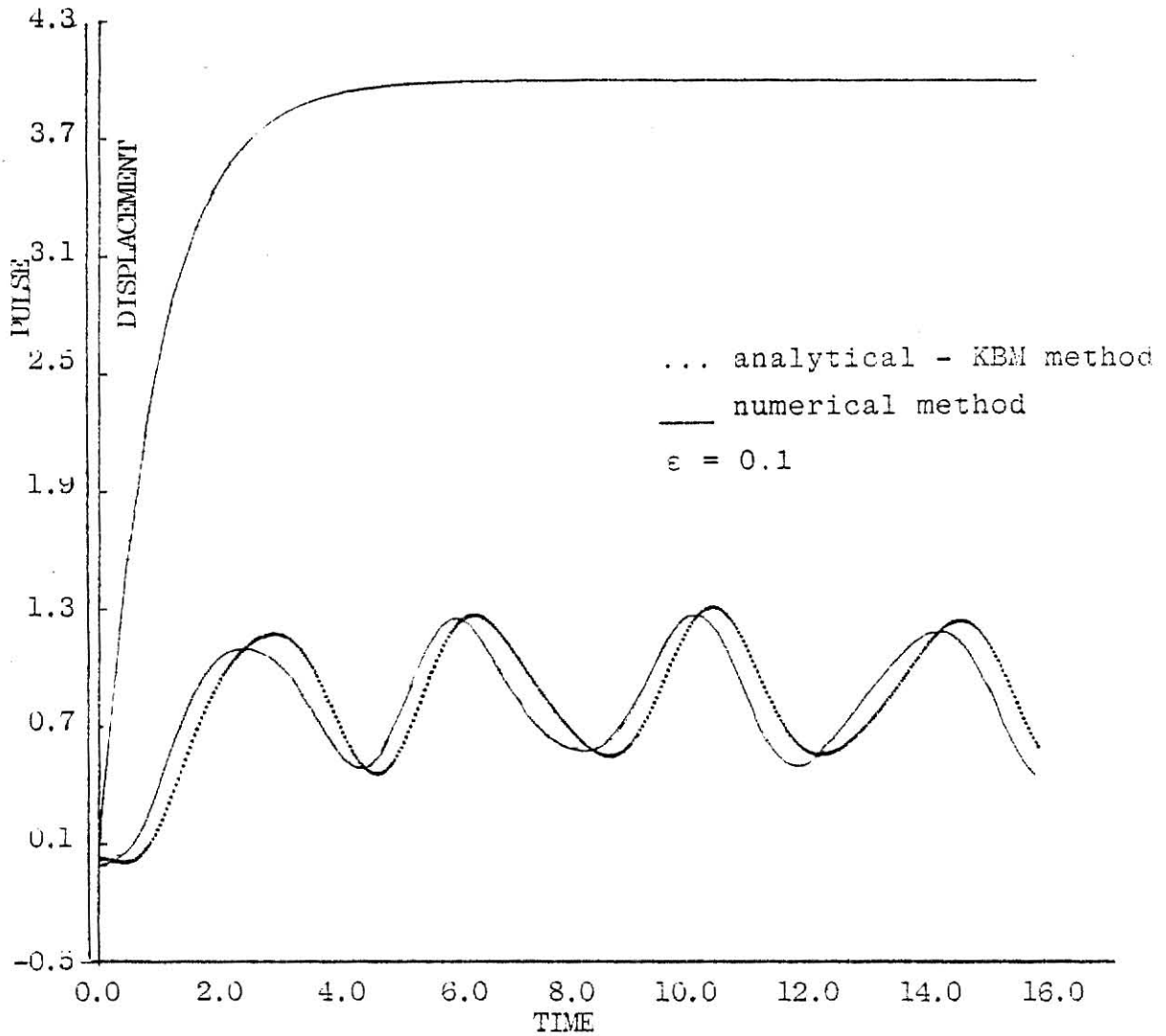
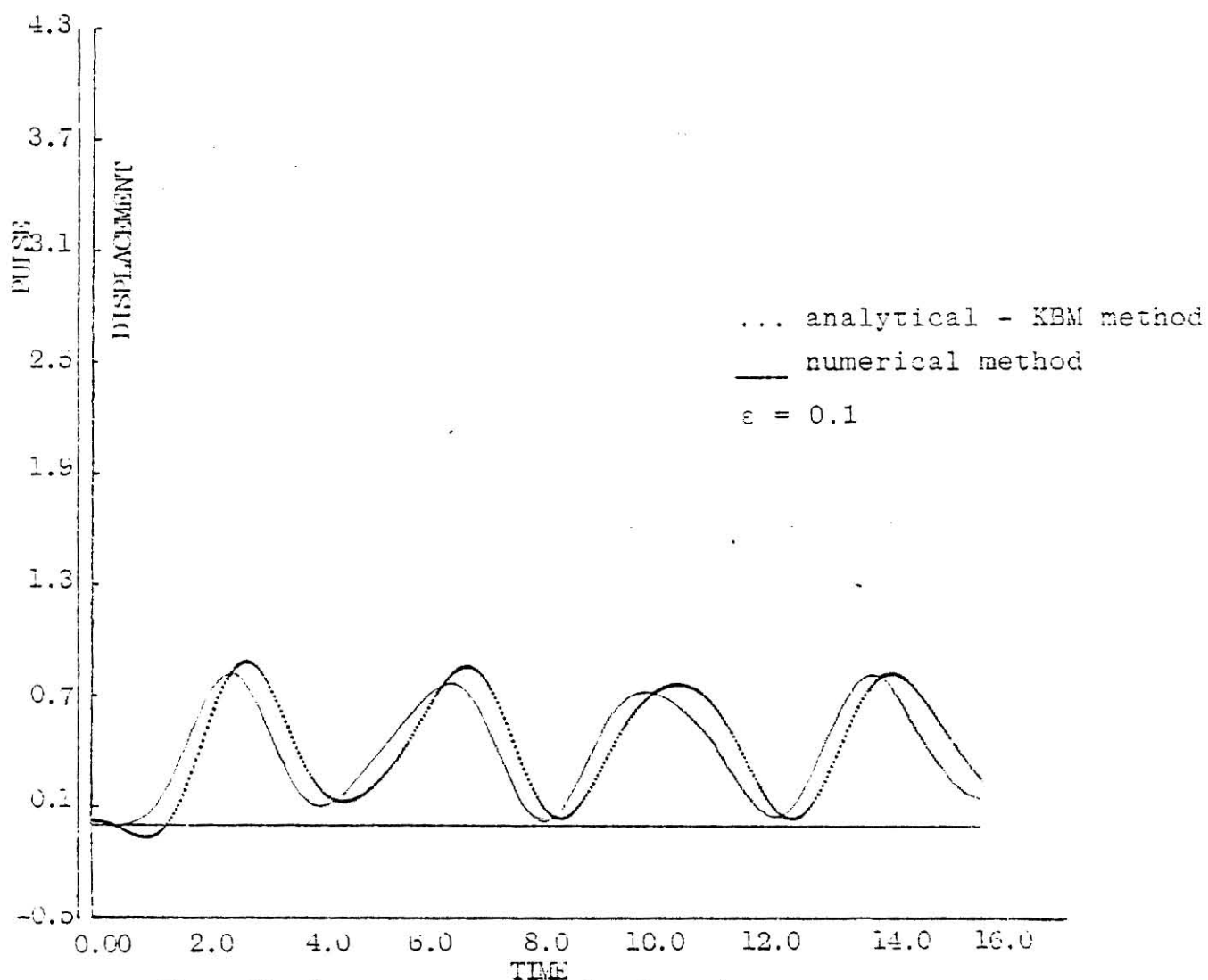


Fig.5. Time-displacement q_1 results for the system.

$$\frac{d}{d\tau} \left[(1 + 0.2\tau) \frac{dq_1}{d\tau} \right] + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon (\dot{q}_1 - \dot{q}_2)^3 = 4(1 - e^{-1.0\tau})$$

$$\frac{d}{d\tau} \left[(1 + 0.25\tau) \frac{dq_2}{d\tau} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 - \varepsilon (\dot{q}_1 - \dot{q}_2)^3 = 0$$

Exponential step pulse

Fig.6. Time-displacement q_2 result for the system

$$\frac{d}{dt} \left[(1 + 0.2\tau) \frac{dq_1}{dt} \right] + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon (\dot{q}_1 - \dot{q}_2)^3 = 4(1 - e^{-1.0\tau})$$

$$\frac{d}{dt} \left[(1 + 0.25\tau) \frac{dq_2}{dt} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 - \varepsilon (\dot{q}_1 - \dot{q}_2)^3 = 0$$

to the variation of amplitudes is due to the nonlinearity in the system. The expressions offer scope for manipulation of the nonlinearity and the parametric variation in the system, to obtain a desired response. A shift in phase of the system is caused strictly due to the change of the parameters in the system.

The response of such a system due to an exponential step pulse is presented in fig.[5] and fig.[6]. Note that one of the masses alone was subjected to an exponential step and the response is in fair agreement with the numerical solution. The rapid increase in step seems to increase discrepancies between numerical and analytical methods. This needs further investigation.

Example:- 3

Consider the system represented in fig [7], having nonstationary parameters and a mixed type nonlinearity. The nonlinearities appear in the form,

$$f_1(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = 0.113(\dot{y}_1 - \dot{y}_2 + \dot{p}_1 - \dot{p}_2)^3 + (y_1 + p_1)^3 + 0.2(\dot{y}_1 + \dot{p}_1) + 0.1(y_1 - y_2 + p_1 - p_2)^3 \quad (3-59)$$

and

$$f_2(y_1, y_2, \dot{y}_1, \dot{y}_2, \tau) = -0.113(\dot{y}_1 - \dot{y}_2 + \dot{p}_1 - \dot{p}_2)^3 - (y_1 + p_1)^3 + 0.2(\dot{y}_1 + \dot{p}_1) - 0.1(y_1 - y_2 + p_1 - p_2)^3 \quad (3-60).$$

The system is being subjected to

- i) a step excitation
- ii) exponentially decaying pulse

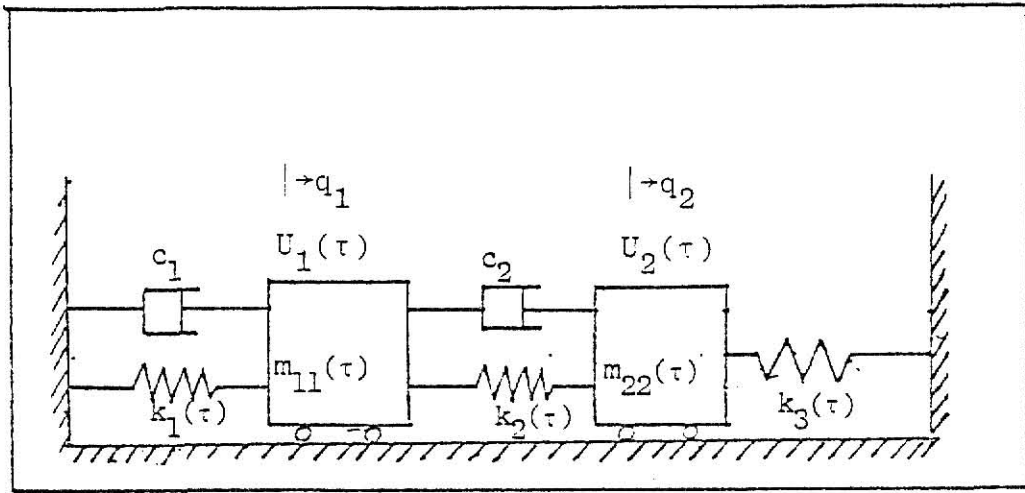


Fig.7. System with mixed type nonlinearity

$$m_{11}(\tau) = (1 + 0.2\tau)$$

$$m_{22}(\tau) = (1 + 0.25\tau)$$

$$\text{Force } F_{k_1}(\tau) = (3 + 0.1\tau)z + (z)^3$$

$$\text{Force } F_{k_2}(\tau) = (3 + 0.1\tau) + z + 0.1(z)^3$$

$$\text{Force } F_{k_3}(\tau) = (3 + 0.1\tau)z$$

c_1, c_2 = co-efficient of viscous friction

q_1, q_2 = generalized co-ordinates ; z -Relative displacement.

$U_1(\tau), U_2(\tau)$ = arbitrary pulse

It is required to calculate the particular solutions given by eqns. (3-8) and (3-9). The procedure to obtain the approximate solution is outlined in the earlier example.

i) Particular solutions for a step excitation:

The system in fig. (7) is subjected to a step excitation of the form.

$$\begin{aligned} u_1(\tau) &= 0 \left\{ \begin{array}{l} t \leq 0- \\ 4 \left\{ \begin{array}{l} t \geq 0+ \end{array} \right. \end{array} \right. \\ u_2(\tau) &= 0 \left\{ \begin{array}{l} t \leq 0- \\ 1 \left\{ \begin{array}{l} t \geq 0+ \end{array} \right. \end{array} \right. \end{aligned}$$

for which, using eqns. (3-8) and (3-9)

$$p_1(\tau) = (27 + 0.9\tau) \times (0.03\tau^2 + 1.8\tau + 27)^{-1}$$

and

$$p_2(\tau) = (18 + 0.6\tau) \times (0.03\tau^2 + 1.8\tau + 27)^{-1}$$

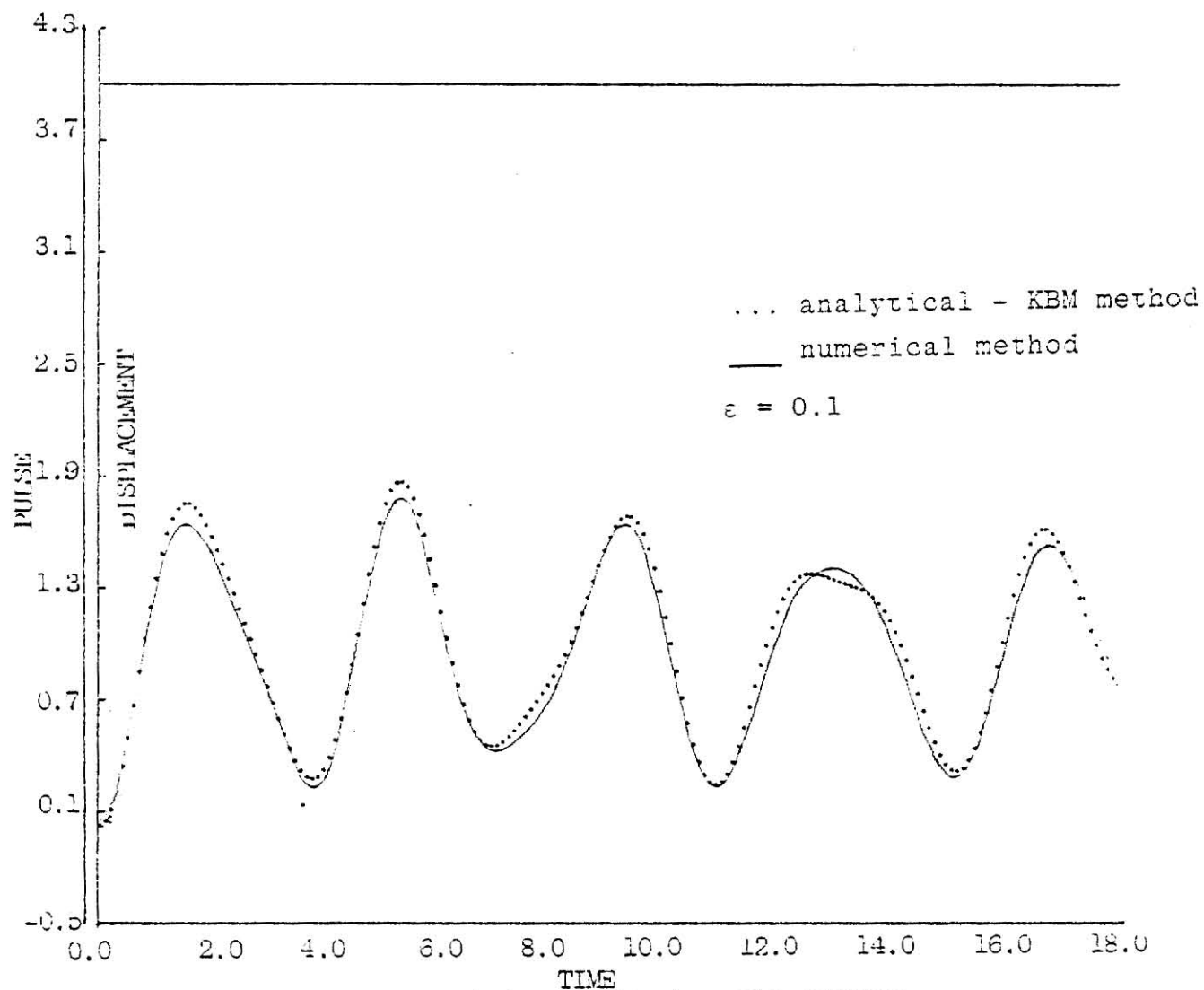
It then becomes convenient to perform the subsequent computation following the outline presented earlier through the use of eqns. (3-22) through (3-25). The constants needed are also computed using eqns. (3-26) through (3-29). Figure [8], Fig. [9], Fig.[12], Fig.[13], Fig. [14] and Fig. [15] provide the response of systems with step excitation and compared to numerical solution using a fourth order Runge-Kutta method.

ii) Exponentially decaying pulse:

The following exponentially decaying pulse are considered

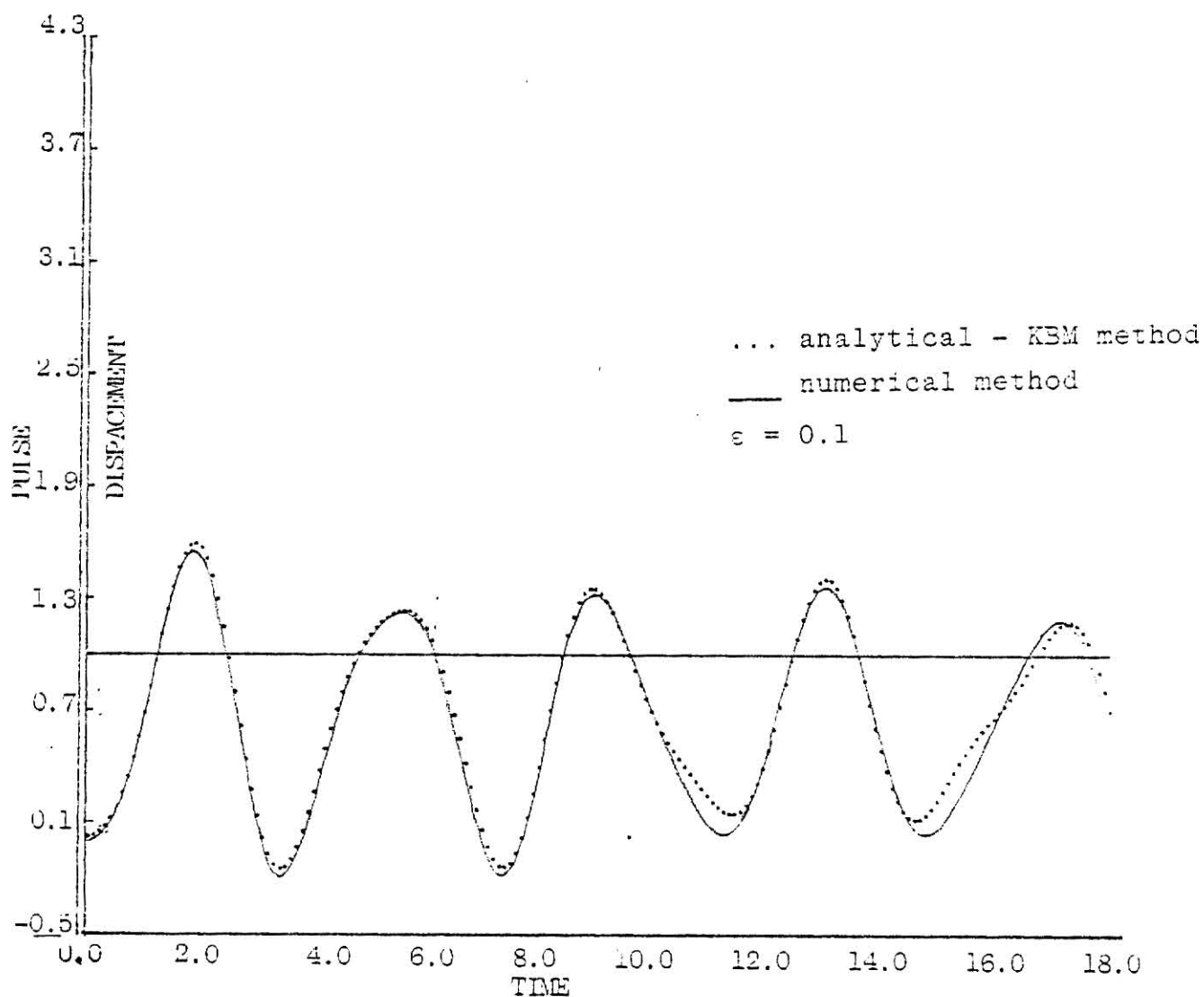
$$\begin{aligned} u_1(\tau) &= 1.0 e^{-1.5\tau} \quad t \geq 0 \\ u_2(\tau) &= 2.0 e^{-2\tau} \quad t \geq 0. \end{aligned}$$

Step function excitation

Fig.8. Time-displacement (q_1), result for the system.

$$\begin{aligned}
 & \frac{d}{dt} \left((1 + 0.2\tau) \frac{dq_1}{dt} \right) + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \\
 & + \epsilon(q_1^3 + 0.1(q_1 - q_2)^3 + 0.313\dot{q}_1 - .113\dot{q}_2) = 4 \\
 & \frac{d}{dt} \left((1 + 0.2\tau) \frac{dq_1}{dt} \right) - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \epsilon(0.1(q_2 - q_1)^3 + \\
 & + 0.113(\dot{q}_2 - \dot{q}_1)) = 1
 \end{aligned}$$

Step function excitation.

Fig. 9. Time-displacement (q_2), results for the system.

$$\begin{aligned} \frac{d}{dt} \left[1 + 0.2\tau \frac{dq_1}{dt} \right] - (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon(q_1^3 + 0.1(q_1 - q_2)^3 - \\ + (0.313\dot{q}_1 - 0.113\dot{q}_2)) = 4 \\ \frac{d}{dt} \left[1 + 0.2\tau \frac{dq_2}{dt} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \varepsilon(0.2(q_2 - q_1)^3 - \\ + 0.113(\dot{q}_2 - \dot{q}_1)) = 1 \end{aligned}$$

Response - Exponentially decaying pulse.

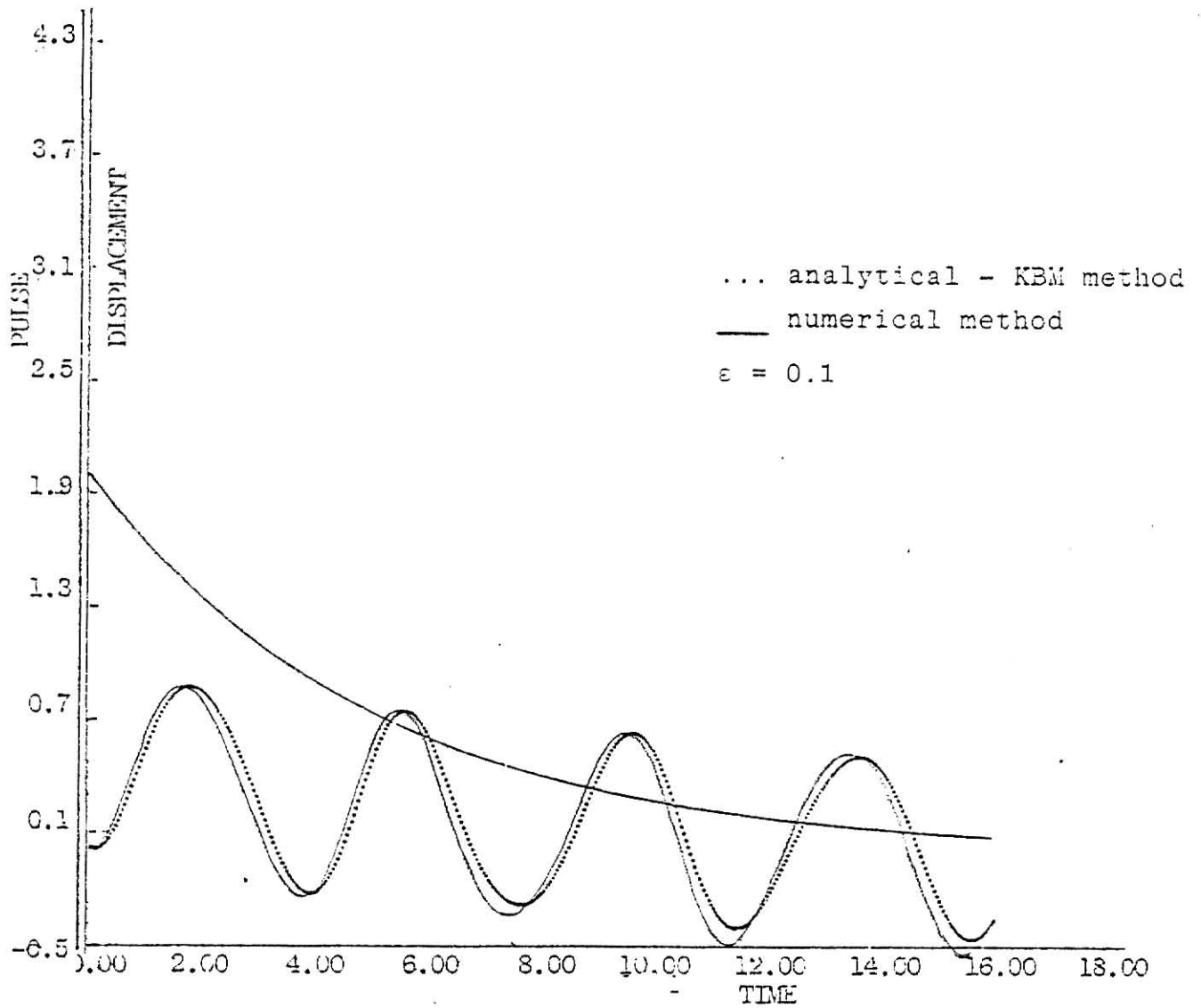


Fig. 10. Time displacement (q_2) result for,

$$\frac{d}{dt}((1+0.2\tau)\frac{dq_1}{dt}) + (6+0.2\tau)q_1 - (3+0.1\tau)q_2 + \epsilon(-133(\dot{q}_1 - \dot{q}_2)) = 1.0e^{-1.5\tau}$$

$$\frac{d}{dt}((1+0.25\tau)\frac{dq_2}{dt}) - (3+0.1\tau)q_1 + (6+0.2\tau)q_2 + \epsilon(213)(\dot{q}_2 - \dot{q}_1) = 2.0e^{-2\tau}$$

Response- Exponentially decaying pulse.

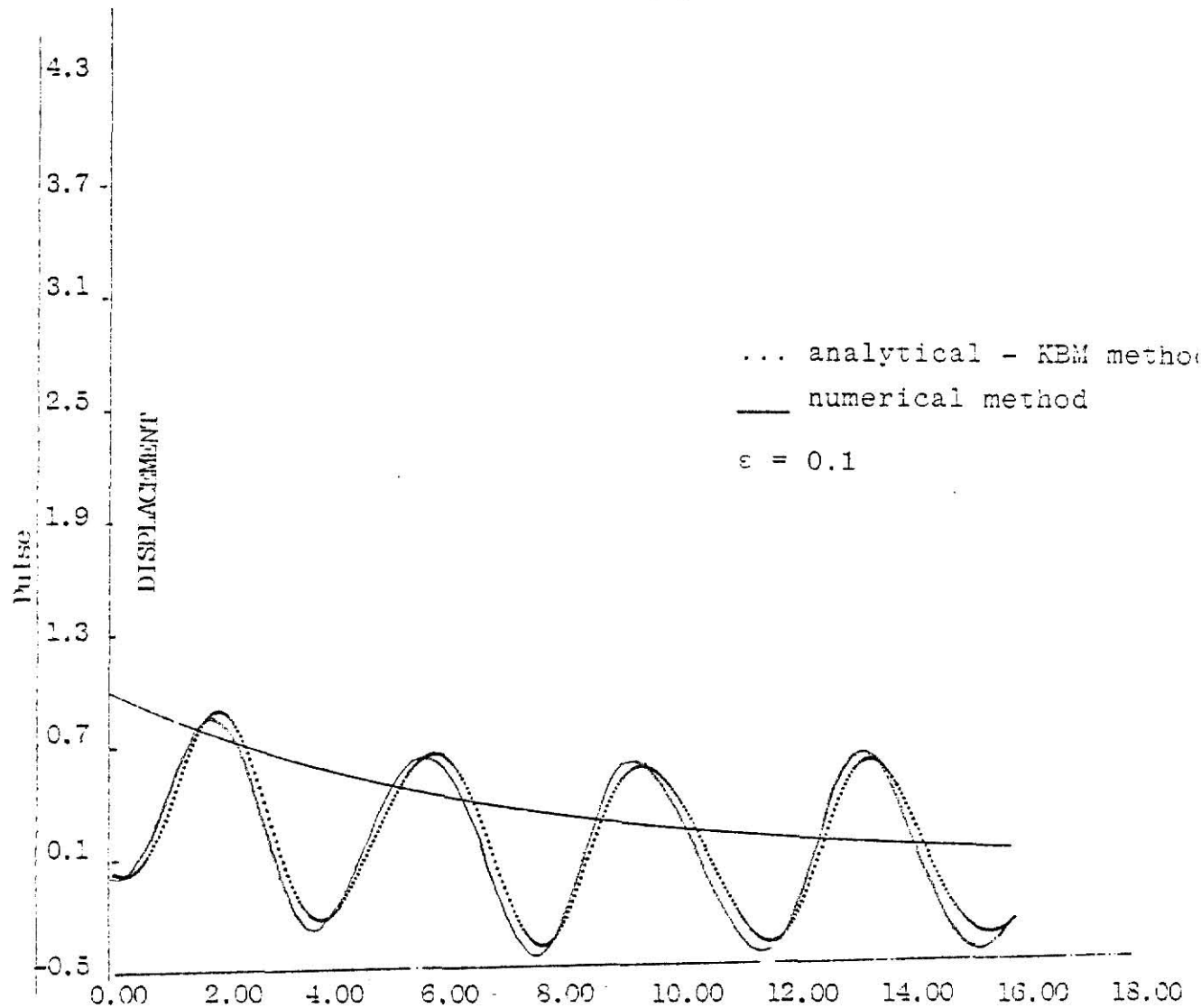


Fig.11. Time displacement (q_1) result for,

$$\frac{d}{dt}((1+0.2\tau)\frac{dq_1}{dt})+(6+0.2\tau)q_1-(3+0.1\tau)q_2+\varepsilon(0.133(\dot{q}_1-\dot{q}_2))=1.0e^{-1.5\tau}.$$

$$\frac{d}{dt}((1+0.25\tau)\frac{dq_2}{dt})-(3+0.1\tau)q_1+(6+0.2\tau)q_2+\varepsilon(.213(\dot{q}_2-\dot{q}_1))=2.0e^{-2\tau}.$$

System subjected to step excitation.

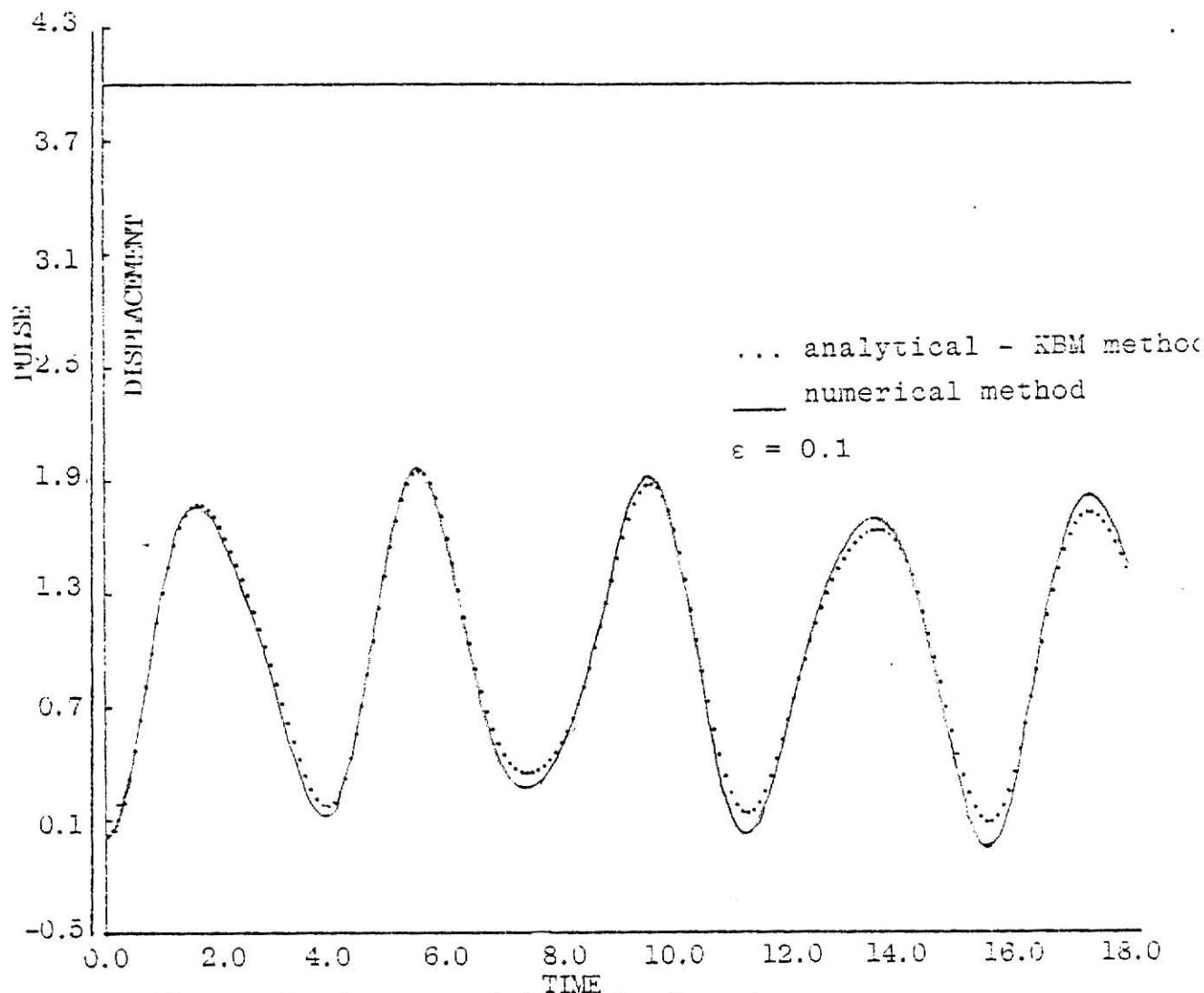


Fig.12 Time-displacement (q_1) result for the system.

$$\frac{d}{dt} \left[(1 + 0.2\tau) \frac{dq_1}{dt} \right] + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \epsilon 0.133 (\dot{q}_1 - \dot{q}_2) = 4$$

$$\frac{d}{dt} \left[(1 + 0.25\tau) \frac{dq_2}{dt} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \epsilon 0.213 (\dot{q}_2 - \dot{q}_1) = 1$$

System subjected to step excitation.

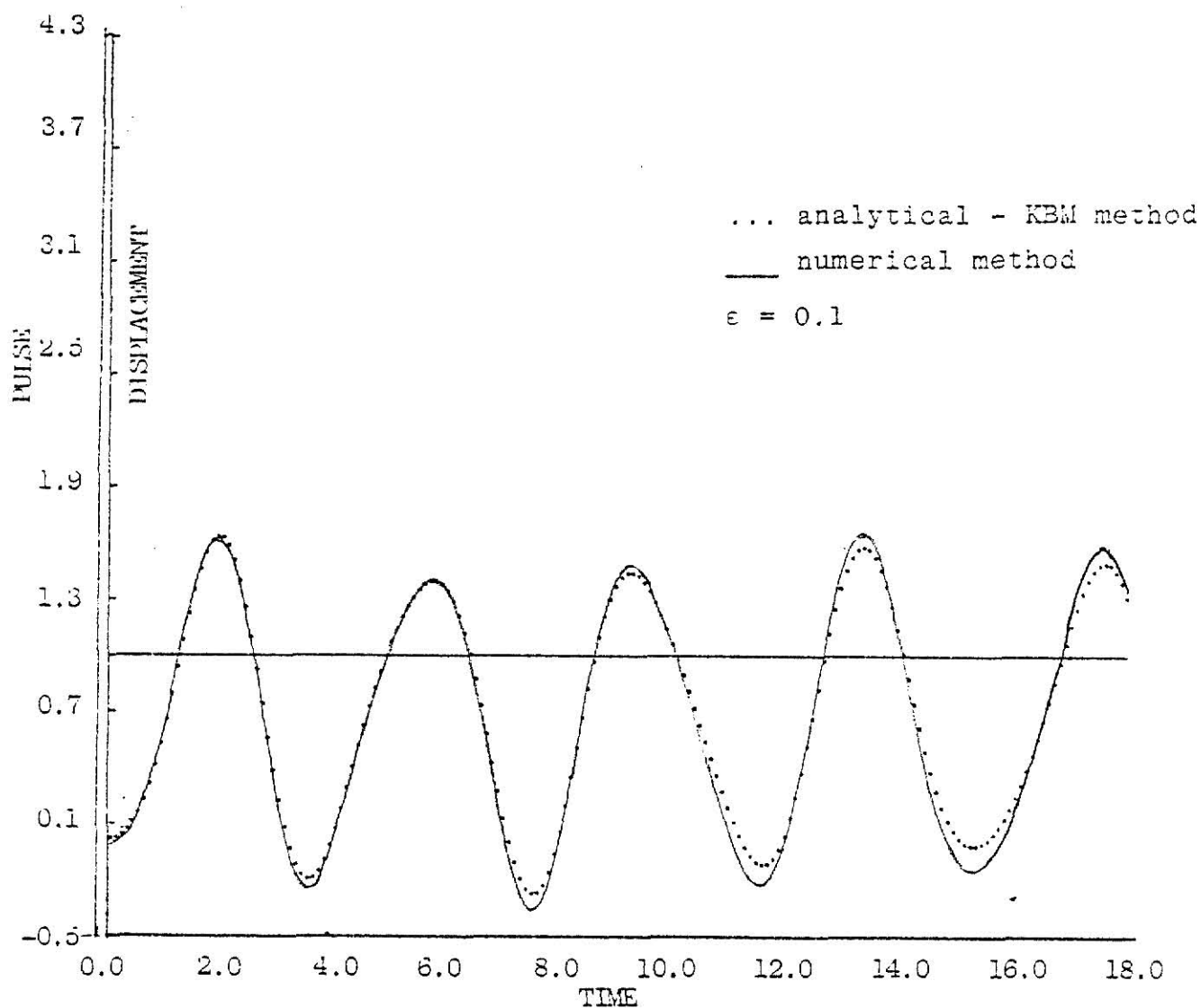


Fig.13. Time displacement (q_2) results for the system.

$$\frac{d}{dt} \left[(1 + 0.2\tau) \frac{dq_1}{dt} \right] + (6 + 0.2\tau)q_1 - (3 + 0.1\tau)q_2 + \varepsilon 0.133(\dot{q}_1 - \dot{q}_2) = 4$$

$$\frac{d}{dt} \left[(1 + 0.25\tau) \frac{dq_2}{dt} \right] - (3 + 0.1\tau)q_1 + (6 + 0.2\tau)q_2 + \varepsilon 0.213(\dot{q}_2 - \dot{q}_1) = 1$$

The particular solutions are given by

$$\begin{aligned}
 p_1(\tau) &= [e^{-1.5\tau}(6 + 0.2\tau) + 2e^{-2\tau}(3 + 0.1\tau)] \times \\
 &\quad \times (0.03\tau^2 + 1.8\tau + 27)^{-1} \\
 p_2(\tau) &= [2e^{-2\tau}(6 + 0.2\tau) - e^{-1.5\tau}(3 + 0.1\tau)] \times \\
 &\quad \times (0.03\tau^2 + 1.8\tau + 27)^{-1}
 \end{aligned}$$

Further analysis follows the procedure adopted for the case of step excitation. Fig. [10] and Fig. [11] present the response of the system.

Discussion:

In this chapter approximate expressions for amplitude and instantaneous frequency variations are obtained for a two degrees of freedom system. The expressions for amplitude and instantaneous frequency variations are given by eqns. (3-22) through (3-25). Both the nonstationary parameters and the nonlinearities of the system are present in these expressions. By setting τ equal to a constant, the expressions (3-22) through (3-25) represent the case of a stationary system, and these are consistent with the expressions provided by Rangacharyulu, Srirangarajan and Dasarathy [20]. By doing so it is observed that the amplitude and instantaneous frequencies are affected only by the nonlinearities of the system; whereas, when τ represents the slowing time, the parameters of the system contribute to the variation of amplitude and instantaneous frequencies of the system.

The special case where the nonlinearities of the system

Step Excitation- System with parameters decreasing slowly.

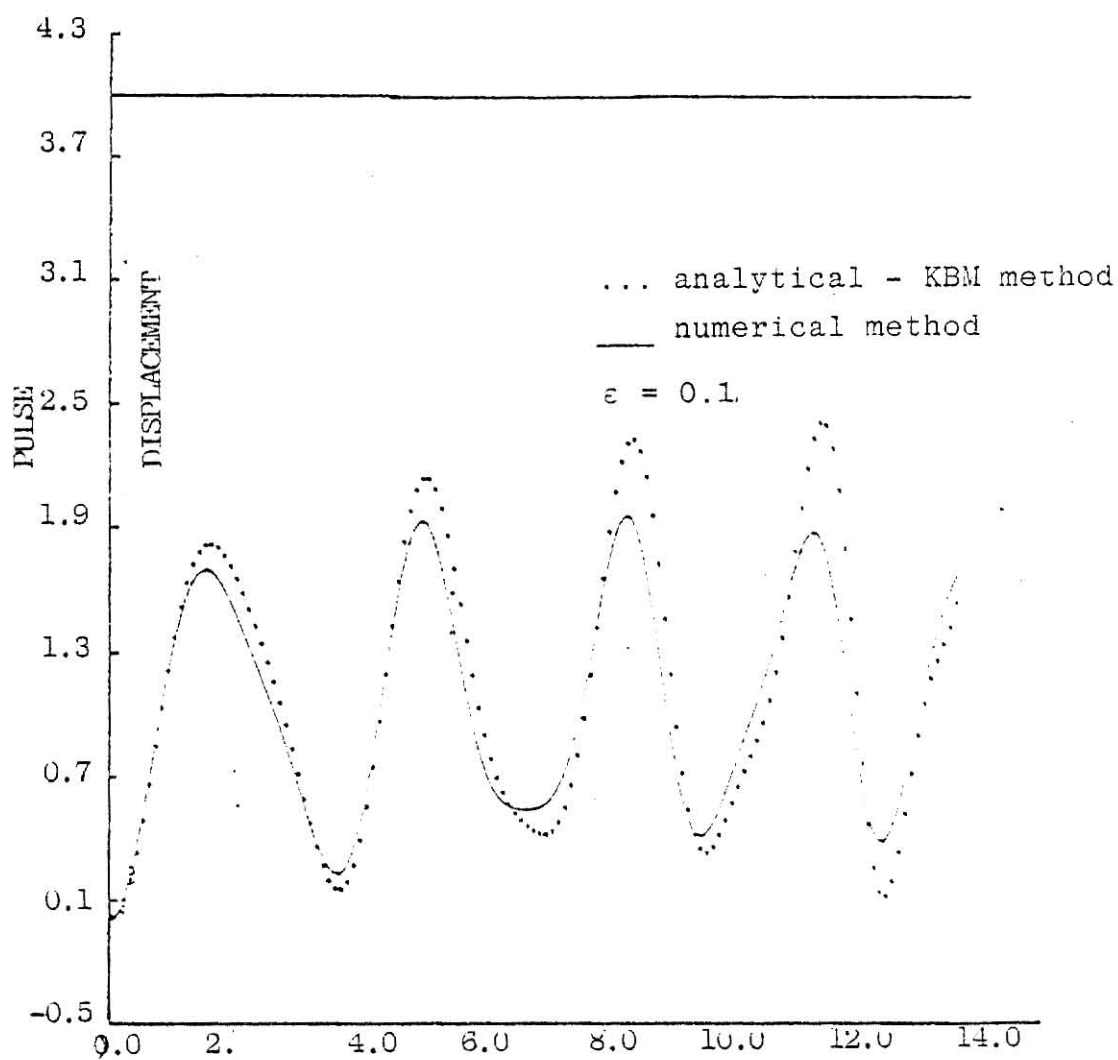


Fig.14. Time-displacement^{TIME}(q_1) result for the system

$$\frac{d}{dt}((1-0.4\tau)\frac{dq_1}{dt}) + (6-0.4\tau)q_1 - (3-0.1\tau)q_2 + \varepsilon [q_1^3 + (q_1 - q_2)^3] = 4$$

$$\frac{d}{dt}((1-0.45\tau)\frac{dq_2}{dt}) - (3-0.1\tau)q_1 + (6-0.4\tau)q_2 + \varepsilon [(q_2 - q_1)^3] = 1$$

Step Excitation- System with parameters decreasing slowly.

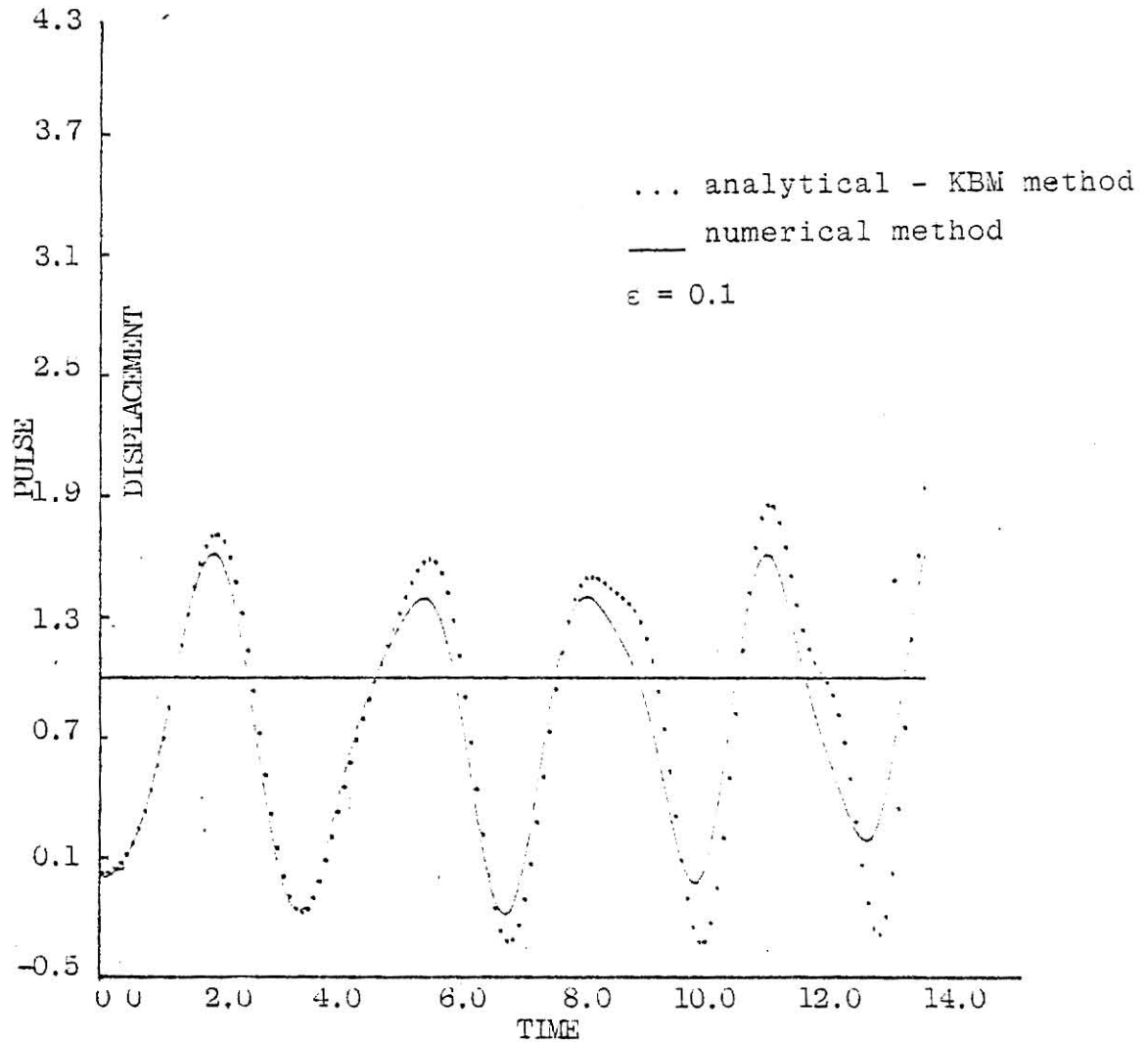


Fig.15. Time-displacement(q_2)result for the system

$$\frac{d}{dt}((1-0.4\tau)\frac{dq_1}{dt}) + (6-0.4\tau)q_1 - (3-0.1\tau)q_2 + \varepsilon [q_1^3 + (q_1 - q_2)^3] = 4$$

$$\frac{d}{dt}((1-0.45\tau)\frac{dq_1}{dt}) - (3-0.1\tau)q_1 + (6-0.4\tau)q_2 + \varepsilon [(q_2 - q_1)^3] = 1$$

contain displacement type terms alone, presented in Sec. 3.3.1, results in a greater change of the instantaneous frequencies of the system. An increase or decrease of the instantaneous frequency depends, however, on the nature of the nonlinear elements of the system. For the case of a hard cubic spring, for which the response is presented in Fig. (3) and (4), the instantaneous frequency was found to decrease. The amplitude of response also changes, but such a change is due to the nonstationary nature of the parameters only.

As presented in Sec. 3.2.2, the nonlinearities of the system containing velocity type terms only affect the amplitude of response significantly. The nonstationary parameters contribute to the variation of amplitude and also to the variation of the instantaneous frequency of the system. However, changes in instantaneous frequency in this case are relatively less. Fig. (5) and (6) present the response of the system with velocity type nonlinearities, subjected to an exponential step excitation.

The structure of equation (3-22) and (3-23) also offers an interesting possibility of adjusting the amplitude of response by judicious choice of the nonstationary parameters and the nonlinearities of the system. The numerator of these expressions have derivative terms involving the nonstationary mass, instantaneous frequency and the nonlinearities. It seems possible that the change in amplitude due to the nonlinearities of the system can be compensated by adjusting the rate of change of the

nonstationary parameters.

For the case where the system was subjected to exponentially decaying pulse, Fig (8) and Fig. (9) indicate that the analytical and numerical results are quite close. However, in the case of an exponential step pulse which rises quite rapidly, the discrepancies between analytical and numerical results are observed to be greater. Such an observation was also made in the case of a single degree of freedom system by Olberding [13]. Fig. (14) and Fig. (15) presents the results for a system with cubic nonlinearity where the nonstationary mass and stiffness are reducing functions of time. Upon comparing the results with that of example 1, it is found that the time period increases. It is also found that after significant reduction of the value of the parameters, the discrepancies between numerical and analytical solutions are quite high. A quantitative analysis for such behavior needs further investigation.

The results of most cases indicate that the discrepancies between numerical and analytical results are quite less.

IV: NONLINEAR RESPONSE OF A PARAMETRICALLY EXCITED BEAM-COLUMN SUBJECTED TO NONPERIODIC LOADS.

4.1 Introduction

In the conventional theory of beam vibration, it is assumed that the bar is free to move in the axial direction. However, immovable end hinges and hinges supported in a manner by which as they approach each other a tensile force is produced in the bar, needs to be considered. The effect of axial forces becomes important in such cases. Also, external axial forces, give rise to an additional parametric component in the differential equation of the system.

In the past a few authors have considered parametrically excited systems. Woinowsky and Krieger [22], investigated the effect of an axial force on the vibration of hinged bars. Tso and Asmis [23] reduced such problems to a two degrees of freedom system with multiple parametric resonance. Mitropolskii [5], has considered columns under the action of a longitudinal sinusoidal force of variable frequency. Evan Iwanowskii [11] has analyzed the action of an axial load on a column and has studied the effect of the geometric nonlinearity arising due to such loads. Bolotin [16], in his excellent text, "The Dynamic Stability of Elastic Systems", has presented a detailed analysis of columns subjected to parametric excitation. Recently Tesak, et al., [6], have presented the analysis of the response of a beam subjected to parametric excitation with periodic loads. In all such cases, it needs to be emphasized that the parametric excitations have been limited to periodic loads.

The response of beams subjected to parametric excitation with nonperiodic loads has not been considered in the past. In many industrial applications, nonperiodic axial forces on beams commonly occur due to shock or impulse forces and, hence, the response of such systems becomes important for purposes of design.

In this chapter the response of beams subjected to nonperiodic axial forces are being analyzed. An attempt is being made to approach the problem using the method outlined in Chapter III. The first two modes are approximated using a two term Galerkin approximation. This leads to two coupled nonlinear nonstationary ordinary differential equations. The subsequent analysis may then be carried out using the detailed procedure discussed in Chapter III.

4.2 Equation of Motion

Consider a beam column hinged at both ends and subjected to arbitrary transverse and axial loads. Relatively large deflections tend to stretch the neutral axis since the end displacements are restrained, giving rise to tensile forces. Following Tesak, et. al., [6], the governing differential equation for transverse deflection of the beam is written as

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = \left[P(t) + \frac{EA}{2l} \int_0^2 \left[\frac{\partial w}{\partial z} \right]^2 dz \right] \frac{\partial^2 w}{\partial x^2} + 2c \frac{\partial w}{\partial t} + g(x)f(t) \quad (4-1)$$

where,

E - the elastic modulus

I - the moment of inertia of the cross section of the beam

ρ - mass per unit length of the beam

A - the area of cross section

$P(t)$ - the longitudinal load

l - length of the beam

w - the transverse deflection

$g(x)f(t)$ - the transverse load

c - coefficient of friction

Note that for simplicity the transverse load is considered to be a separable function of time and space. Using the following nondimensional quantities,

$$\begin{aligned} x &= l x^* & w &= \frac{r^2}{l} w^* & t &= \frac{l}{r} \left(\frac{\rho}{E} \right) t^* \\ P(t) &= EA \left(\frac{r}{l} \right)^4 P^*(t^*) & c &= \frac{2Ar^3}{l^4} (\rho E) c^* \end{aligned} \quad (4-2)$$

Equation (4-1) is reduced to the nondimensional form,

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \varepsilon \left\{ \left[P(t) + \frac{1}{2} \int_0^2 \left(\frac{\partial w}{\partial z}(z, t) \right)^2 \partial z \right] \frac{\partial^2 w}{\partial x^2} - \right. \\ \left. - 2c \frac{\partial w}{\partial t} \right\} + \Delta f(t)g(x) \end{aligned} \quad (4-3)$$

where,

$$\Delta = \frac{EA r^4}{l^5}$$

ε is a small parameter defined by $\varepsilon = \frac{r^2}{l^2}$

Note that the asterisks have been dropped. Expressing the deflection as an expansion of the linear free oscillation modes,

$$w(x,t) = \sum_{m=1}^{\infty} U_m(t;\varepsilon) \phi_m(x) \quad (4-4)$$

where,

ϕ_m are the orthonormal solutions of the differential equations

$$\phi_m'''' - \omega_m^2 \phi_m'' = 0 \quad (4-5)$$

satisfying the boundary conditions

$$\phi_m = 0 \text{ and } \phi_m'' = 0 \text{ at } x = 0, x = 1 \quad (4-6)$$

where the primes denote derivatives with respect to x .

$$\omega_m = (m\pi)^2 \quad m = 1, \dots, n$$

are the orthonormal-frequencies. Substituting for $w(x,t)$ in the normalized equation (4-3) and applying Galerkin's method yields

$$\begin{aligned} \ddot{U}_n + \omega_n^2 U_n = \varepsilon \left[-P(t) \sum_{m=1}^n f_{nm} U_m + \sum_{m=1}^n \sum_{p=1}^n \sum_{q=1}^n \gamma_{nmpq} U_m U_p U_q - \right. \\ \left. - 2c \dot{U}_n' \right] + \Delta g(t) f_a \end{aligned} \quad (4-7)$$

where

$$f_{nm} = \int_0^1 \phi_n' \phi_m' dx$$

$$\gamma_{nmpq} = -\frac{1}{4} \int_0^1 \phi_n' \phi_m' dx \int_0^1 \phi_p' \phi_q' dx$$

$$f_a = \int_0^1 f(x)\phi_n(x)dx \quad (4-8)$$

c = nondimensional coefficient of friction

A two term approximation may be used to obtain the response of the system.

Considering

$$\phi_1(x) = \sin\pi x \quad \text{and} \quad \phi_2(x) = \sin 3\pi x$$

With $n = 1, 2$, equation (4-7) yields,

$$\begin{aligned} \ddot{U}_1 + (\pi^4 + \varepsilon\pi^2 P(t))U_1 + \varepsilon \left[\frac{1}{4}(U_1^3\pi^4 + U_2^2 U_1 9\pi^2) + 2c\dot{U}_1 \right] = \\ = 2 \int_0^1 \Delta f(x)g(t)\sin\pi x dx = Q_1(t) \end{aligned} \quad (4-9)$$

and

$$\begin{aligned} \ddot{U}_2 + (81\pi^4 + \varepsilon P(t)9\pi^2)U_2 + \varepsilon \left[\frac{1}{4}(9U_1^2 U_2 \pi^2 + 81U_2^3\pi^4) + 2c\dot{U}_2 \right] = \\ = 2 \int_0^1 \Delta f(x)g(t)\sin 3\pi x dx = Q_2(t) \end{aligned} \quad (4-10)$$

4.3 Application of KBM Method

It is seen that eqns. (4-9) and (4-10) are of the form of eqns. (3-1) and (3-2) respectively. The coefficients of U_1 and U_2 in these equations vary slowly with respect to time and hence, the procedure for a two degrees of freedom system outlined in Chapter III can be used to find approximate solutions of these equations. The quantities $p_1(t)$ and $p_2(t)$ given by eqns. (3-8) and (3-9) can be found from the matrix,

$$\begin{bmatrix} \pi^4 + \varepsilon \pi^2 P(t) & 0 \\ 0 & 81\pi^4 + \varepsilon P(t) 9\pi^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 \int_0^1 \Delta f(x) g(t) \sin \pi x dx \\ 2 \int_0^1 \Delta f(x) g(t) \sin 3\pi x dx \end{bmatrix} \quad (4-11)$$

therefore

$$p_1(t) = \frac{2 \int_0^1 \Delta f(x) g(t) \sin \pi x dx}{\pi^4 + \varepsilon \pi^2 P(t)} \quad (4-12)$$

$$p_2(t) = \frac{2 \int_0^1 \Delta f(x) g(t) \sin 3\pi x dx}{81\pi^4 + \varepsilon P(t) 9\pi^2} \quad (4-13)$$

Knowing p_1 and p_2 , the transformation of the dependent variable given by eqn. (3-5) and (3-6) can be applied to eqns. (4-9) and (4-10). The eqns. (4-9) and (4-10) then reduce to the form.

$$\ddot{y}_1 + (\pi^4 + \varepsilon \pi^2 P(t)) y_1 + \varepsilon \left[\frac{1}{4} \left[\pi^4 (y_1 + p_1)^3 + (y_2 + p_2)^2 (y_1 + p_1) 9\pi^2 \right] + 2c(\dot{y}_1 + \dot{p}_1) \right] = 0 \quad (4-14)$$

and

$$\ddot{y}_2 + (81\pi^4 + \varepsilon 9\pi^2 P(t)) y_2 + \varepsilon \left[\frac{1}{4} (9(y_1 + p_1)^2 (y_2 + p_2) \pi^2 + 81(y_2 + p_2)^3 \pi^4) + 2c(\dot{y}_2 + \dot{p}_2) \right] = 0 \quad (4-15)$$

This is in the form of eqns. (3-10) and (3-11) for which the solutions are given by equations (3-22) through (3-25). The deflection of the beam is then obtained by substituting for U_1 and U_2 in eqn. (4-4). In the following section examples using nonperiodic loads are considered for purposes of illustration.

An Example:

Let the beam be subjected to an axial force $P(t)$ and transverse load $f(x)g(t)$ of the following form

$$P(t) = b_0 + b_1 e^{-\sigma\tau} \quad (4-16)$$

and

$$f(x)g(t) = 2b_2 \cdot x \cdot e^{-\delta\tau} \quad (4-17)$$

where b_0 , b_1 , b_2 , σ , and δ are constants.

The quantities p_1 and p_2 for this choice of loading, are obtained from equations (4-12) and (4-13) as

$$p_1 = \frac{2b_2 e^{-\delta\tau}}{\pi [\pi^4 + \varepsilon \pi^2 (b_0 + b_1 e^{-\sigma\tau})]} \quad (4-17a)$$

and

$$p_2 = \frac{2b_2 e^{-\delta\tau}}{3\pi [81\pi^4 + \varepsilon 9\pi^2 (b_0 + b_1 e^{-\sigma\tau})]} \quad (4-17b)$$

With these values of p_1 and p_2 the approximate solutions for equations (4-14) and (4-15) can be obtained by the use of equations (3-22) through (3-25). The equations for amplitude and instantaneous frequencies take the form,

$$\frac{da_1}{dt} = \frac{1}{2\Omega_1(\tau)\alpha_{22}(\tau)m_{11}m_{22}} \left[m_{11}\alpha_{22}(\tau) \left[-a_1m_{22} \frac{d}{dt} \Omega_1(\tau) - \right. \right. \\ \left. \left. - 2\epsilon ca_1\Omega_1(\tau) \right] \right] , \quad (4-18a)$$

$$\frac{da_2}{dt} = \frac{-1}{\Omega_2(\tau)\alpha_{22}(\tau)m_{11}m_{22}} \left[m_{11}m_{22}\alpha_{22}(\tau) \frac{a_2}{2} \frac{d}{dt} \Omega_2(\tau) + \right. \\ \left. + \epsilon m_{11}a_2c\alpha_{22}(\tau)\Omega_2(\tau) \right] , \quad (4-18b)$$

$$\frac{d\theta_1}{dt} = \left[-m_{22}\alpha_{22}(\tau) \left[-m_{11}\Omega_1^2 \frac{a_1}{2} + \frac{b_{11}a_1}{2} + \right. \right. \\ \left. + \epsilon(a_1^3 \frac{3}{8} + 0.75a_1a_2^2 + 3a_1p_1^2 + \right. \\ \left. + \alpha_{22}^2a_2^2a_1 \times 0.75 + p_2^2 \times 0.5) \right] \times \\ \times \left[-\alpha_{22}m_{11}m_{22}a_1\Omega_1(\tau) \right]^{-1} \quad (4-19a)$$

and

$$\frac{d\theta_2}{dt} = \frac{-1}{a_2\Omega_2(\tau)\alpha_{22}m_{11}m_{22}} \left[-m_{11} - \frac{1}{2} a_2\Omega_2^2(\tau)m_{22}\alpha_{22}(\tau) + \right. \\ \left. + b_{22}(\tau)\alpha_{22}(\tau)a_2 \times 0.5 + \left\{ \frac{1}{4} \left[9\pi^2(\alpha_{22}(\tau)a_2^3 \frac{3}{8} + \right. \right. \right. \\ \left. \left. + \frac{1}{2}p_1^2\alpha_{22}(\tau)a_2 + p_1p_2a_2) + \right. \right. \\ \left. \left. + 81\pi^4 \left(\alpha_{22}(\tau)^3a_2^3 \frac{3}{8} + \alpha_{22}(\tau)1.5a_2p_2^2 \right) \right] \right\} \right] \quad (4-19b)$$

where

$$\Omega_1(\tau) = [\pi^4 + \epsilon\pi^2P(\tau)]^{1/2}$$

$$\Omega_2(\tau) = (81\pi^4 + \varepsilon 9\pi^2 P(t))^{1/2}$$

Note that the natural frequencies are slowly varying functions since $P(t)$ is a time varying load. The amplitude a_1 , a_2 are obtained by a direct integration as,

$$a_1 = a_{1c} \frac{1}{\sqrt{\Omega_1(\tau)}} e^{-\varepsilon c t} \quad (4-20)$$

and

$$a_2 = a_{2c} \frac{e^{-\varepsilon c t}}{\alpha_{22}(\tau) \sqrt{\Omega_2(\tau)}} \quad (4-21)$$

where a_{1c} and a_{2c} are obtained using the initial conditions.

With the explicit expressions for the amplitudes given by eqns. (4-20) and (4-21) the instantaneous frequencies can be obtained by using a simple quadrature scheme using eqns. (4-19a) & (4-19b).

Then $y_1(t)$ and $y_2(t)$ are obtained as,

$$y_1 = a_1 \cos(\Omega_1 t + \theta_1) \quad (4-22)$$

$$y_2 = a_2 \cos(\Omega_2 t + \theta_2) \quad (4-23)$$

and hence

$$U_1(t) = y_1 + p_1 \quad (4-24)$$

$$U_2(t) = y_2 + p_2 \quad (4-25)$$

The deflection of the beam can now be calculated from equation

$$w(x,t) = U_1(t)\phi_1(x) + U_2(t)\phi_2(x) \quad (4-26)$$

which is written as

$$w(x,t) = U_1(t)\sin\pi x + U_2(t)\sin 3\pi x \quad (4-26a)$$

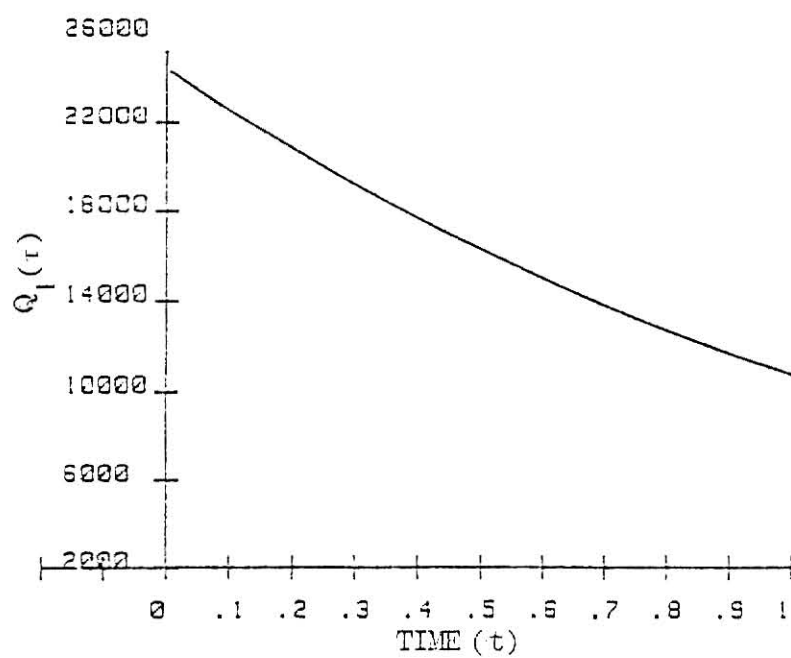


Fig.16. An exponentially decaying function; $24506e^{-82\tau}$

$$\tau = \varepsilon t$$

$$\varepsilon = 0.01$$

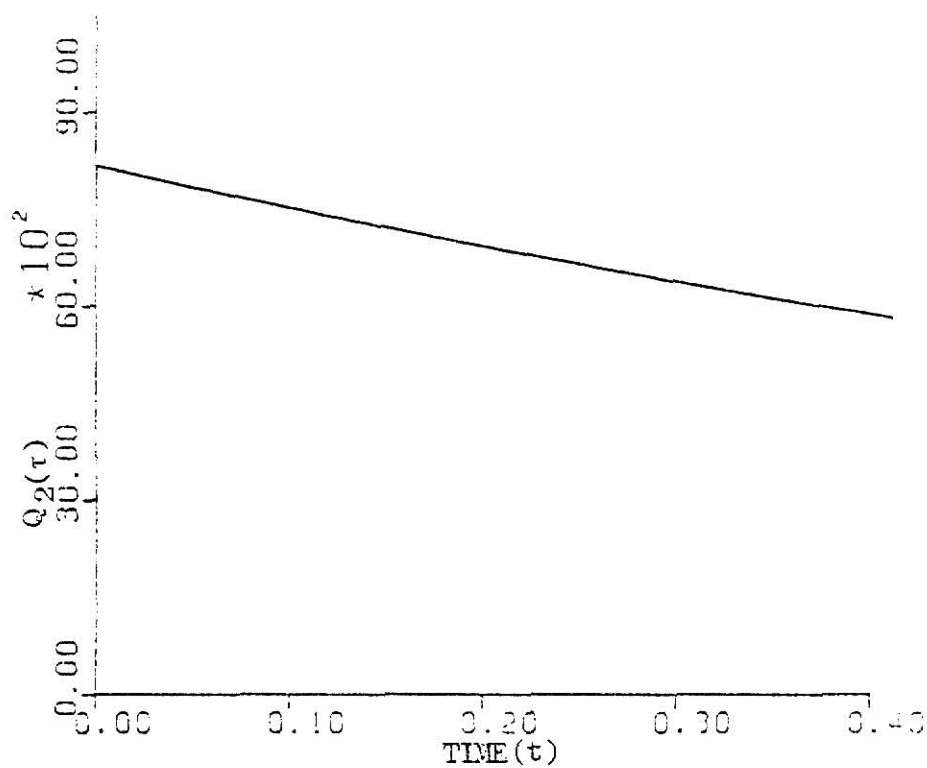


Fig. 17. An exponentially decaying function; $8169 e^{-82\tau}$

$$\tau = \varepsilon t$$

$$\varepsilon = 0.01$$

Time-displacement results for a parametrically excited beam column with exponentially decaying load.

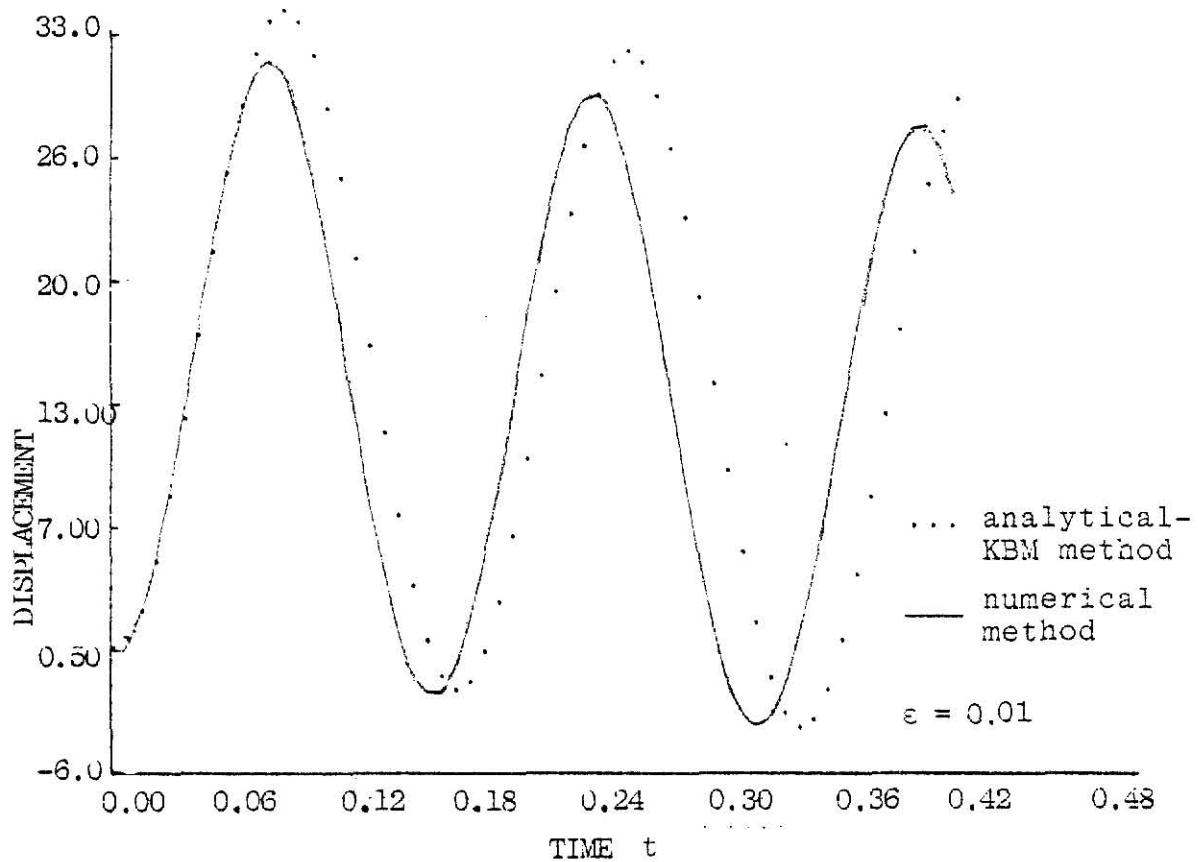


Fig.18 Time-response U_1 of the system

$$\begin{aligned} \frac{d^2 U_1}{dt^2} + (97.4 + 9.86\epsilon (12000 + 1000e^{-5\tau})) U_1 + \epsilon (24U_1^3 + 219U_1^2 U_2 + 6.0\dot{U}_1) &= \\ &= 24506e^{-82\tau} \\ \frac{d^2 U_2}{dt^2} + (7890 + 87.6\epsilon (12000 + 1000e^{-5\tau})) U_2 + \epsilon (219U_1^2 U_2 + 1972U_2^3 + 6.0\dot{U}_2) &= \\ &= 8169e^{-82\tau} \end{aligned}$$

Time displacement result for a parametrically excited beam column with exponentially decaying load.

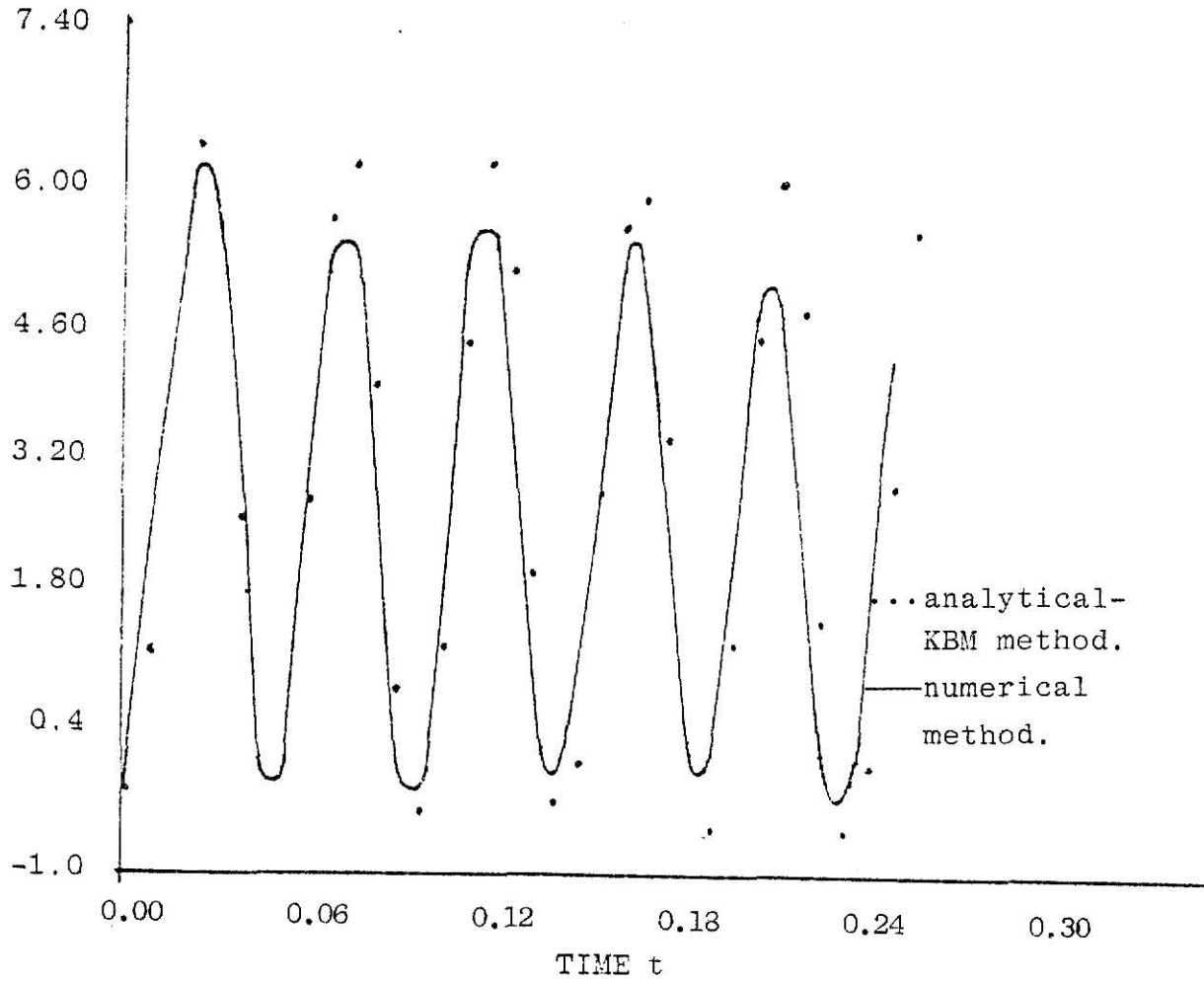


Fig. 19. Time-response U_2 of the system.

$$\begin{aligned} \frac{d^2 U_1}{dt^2} + (97.4 + \varepsilon 9.86(12000 + 1000e^{-5\tau})) U_1 + \varepsilon 24U_1^3 + 219U_1 U_2^2 + \\ + 6.0\dot{U}_1) = 24506e^{-82} \\ \frac{d^2 U_2}{dt^2} + (7890 + \varepsilon 87.6(12000 + 1000e^{-5\tau})) U_2 + \varepsilon(219U_1^2 U_2 + \\ + 1972U_2^3 + 6.0\dot{U}_2) = 8169e^{-82\tau} \end{aligned}$$

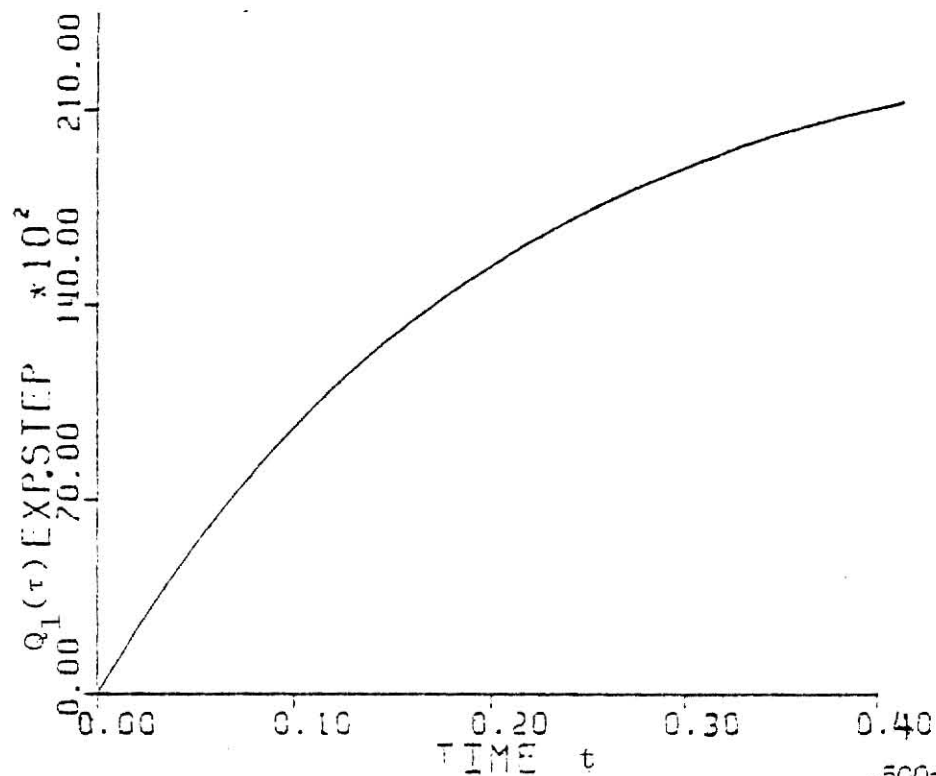
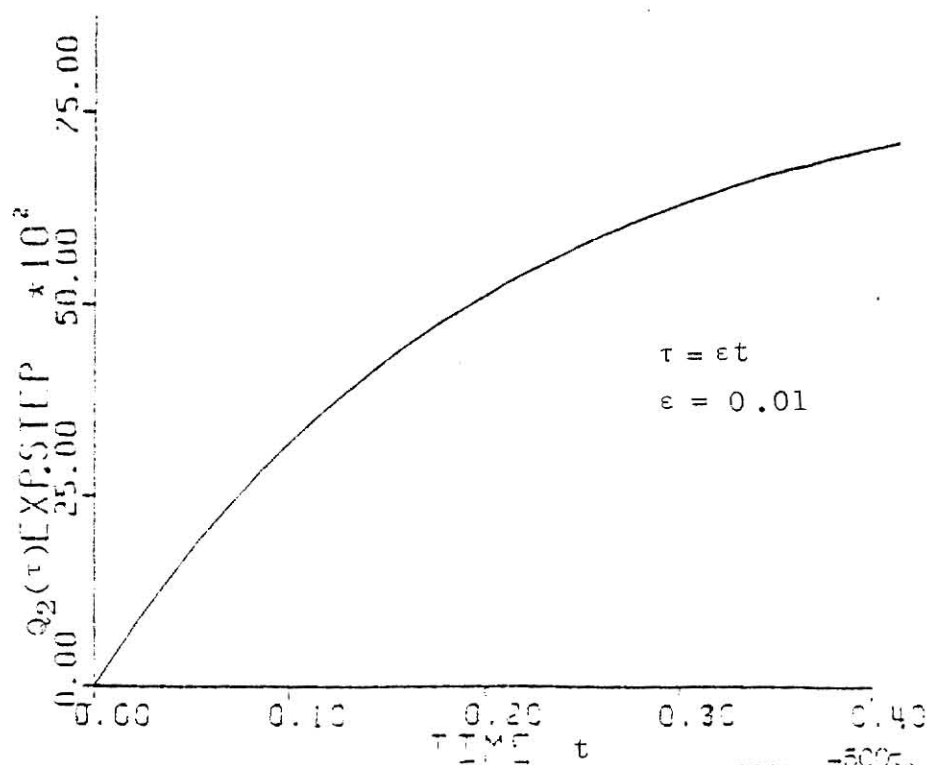


Fig.20. An exponential step function. $24506(1-e^{-500\tau})$

$$\tau = \epsilon t$$

$$\epsilon = 0.01$$



$$\tau = \epsilon t$$

$$\epsilon = 0.01$$

Fig.21. An exponential step function. $8169(1-e^{-500\tau})$

Time displacement results for a parametrically excited beam column with exponential step load.

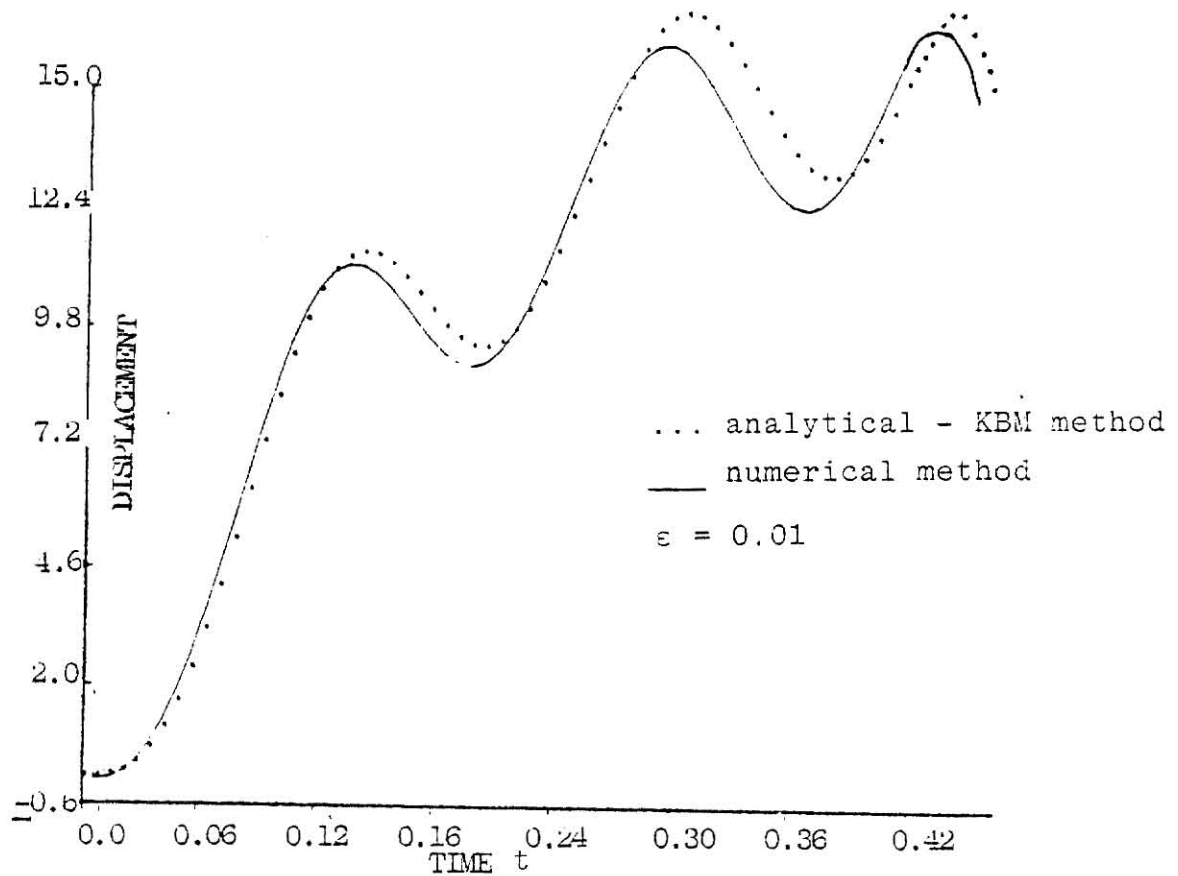


Fig.22. Time displacement U_1 results for

$$\frac{d^2 U_1}{dt^2} + (97.4 + \varepsilon \cdot 9.86(12000 + 1000e^{-5\tau})) U_1 + \varepsilon (24U_1^3 + 219U_1 U_2^2 + 6.0\dot{U}_1) = 24506(1 - e^{-500\tau})$$

$$\frac{d^2 U_2}{dt^2} + (7890 + \varepsilon \cdot 87.6(12000 + 1000e^{-5\tau})) U_2 + \varepsilon (219U_1^2 U_2 + 1972U_2^3 + 6\dot{U}_2) = 8169(1 - e^{-500\tau})$$

Time-displacement result for a parametrically excited beam column subjected to exponential step loads.

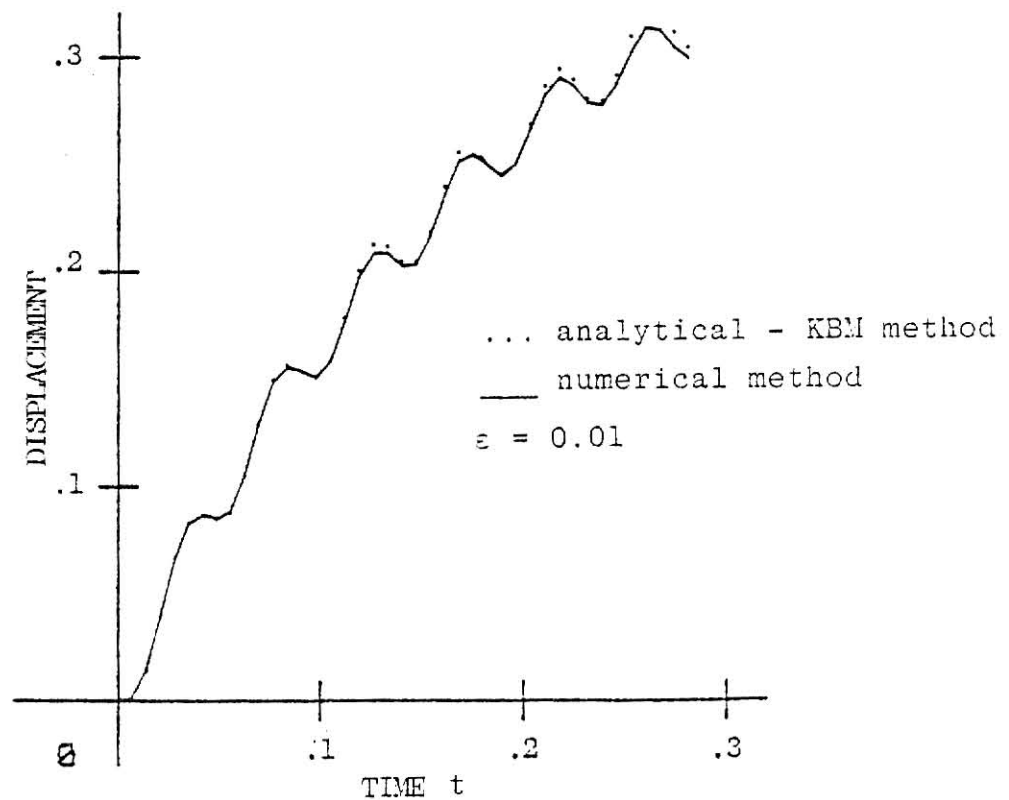


Fig.23. Time-displacement U_2 result for

$$\begin{aligned} \frac{d^2 U_1}{dt^2} + (97.4 + \varepsilon 9.86(12000 + 1000e^{-5\tau}))U_1 + \varepsilon(24U_1^3 + 219U_1U_2^2 + \\ + 6.0\dot{U}_1) = 24506(1 - e^{-500\tau}) \\ \frac{d^2 U_2}{dt^2} + (7890 + \varepsilon 87.6(12000 + 1000e^{-5\tau}))U_2 + \varepsilon(219U_1^2U_2 + 1972U_2^3 + 6\dot{U}_2) = \\ = 8169(1 - e^{-500\tau}) \end{aligned}$$

4.4 Discussion:

The governing differential equation for the deflection of a beam column subjected to arbitrary loads reduces to a partial nonlinear differential equation with a small parameter. It is possible, using a two term Galerkin approximation, to reduce the problem to a pair of nonlinear nonstationary ordinary differential equations with slowly varying parameters. These equations are given by (4-9) and (4-10). Their structure indicates a special case of the general system of two degrees of freedom discussed in Chapter III. It is observed that the coupling is only due to the nonlinear terms.

Following the procedure of Chapter III the approximate expressions for amplitudes and instantaneous frequencies can be obtained. The amplitudes given by eqns. (4-18a) and (4-18b) can be integrated in closed form and are found to be exponentially decaying functions of time.

The dynamic deflection of the beam is given by eqn. (4-21). Since $U_2(t)$ turns out to be very small in comparison to $U_1(t)$, the higher modes do not contribute significantly. Hence one term approximation may be satisfactory. Figures (18), (19) and figures (22), (23) present results for exponentially decaying transverse load and an exponential step load, respectively. The approximate response are in fair agreement with the numerical solution. In this case, the exponential step was not very rapid, unlike example 2 in Chapter III, resulting in less discrepancies between numerical and analytical results.

DISCUSSION AND CONCLUSION

The purpose of the present study was to obtain an approximate response of nonlinear vibrational systems with N degrees of freedom and slowly varying system parameters subjected to arbitrary pulse excitations. In Chapter II, an approximate solution of the first order was obtained by using the Krylov, Bogoliubov and Mitropolskii method. The averaging technique served as a tool in the development of such a solution. The procedure for obtaining amplitude and instantaneous frequency variations has been described.

In Chapter III, for the specific case of a two degrees of freedom system, approximate expressions for amplitude and instantaneous frequency variations were obtained. In general, such expressions, due to their complexity, can be integrated in closed form only under special cases. For the case where the nonlinearities of the system depend on displacement type terms only, the amplitude of vibration depends only on the characteristics of the slowly varying parameters of the system. The expressions for amplitude and instantaneous frequency can be integrated using simple quadrature schemes.

In the case where the nonlinearities of the system contain velocity type terms, the amplitude of response is affected due to the nonlinearities as well as by the characteristics of the slowly varying parameters. However, in such cases, the variations in instantaneous frequency is only due to the nature of the slowly varying parameters and in general, is quite less. For a rapidly

varying pulse load discrepancies between analytical and numerical solutions were quite high. Such observations were made based on a comparison of the results obtained by the use of a relatively smooth exponentially decaying pulse load and a rapidly rising exponential step load in the system.

In Chapter IV, it was demonstrated that the method can be applied to study the response of parametrically excited beam columns subjected to arbitrary loads. By the use of a two term Galerkin approximation; the problem reduces to a pair of nonlinear nonstationary nonhomogeneous ordinary differential equations. The system was found to be coupled only through the nonlinear terms hence forming a special case of the two degrees of freedom system presented in Chapter III. The amplitude expressions were obtained in closed forms. The instantaneous frequency expressions needed only simple quadrature scheme for integration. The contribution due to second mode is found to be small, hence one term approximation may be satisfactory. The method in general, yields fairly good results for this problem.

The structure of equations (3-22) and (3-23) reveal that the amplitude of resonance may be manipulated by a judicious choice of the nonstationary parameters to compensate for the effect of the nonlinearities of the system. Further investigation in this area is suggested since it may be of considerable value for design of such systems. It is found that a significant reduction in the value of the slowly varying parameters results in greater discrepancies between numerical and analytical results. This

needs further investigation.

In conclusion, a method has been suggested to obtain approximate analytical solutions of nonlinear nonstationary N-degrees of freedom systems subjected to arbitrary pulse excitations. Explicit expressions for the rate of change of amplitudes and instantaneous frequencies for a general two degrees of freedom system have been obtained. The method has been applied successfully to study the response of a parametrically excited beam column subjected to arbitrary transverse loads.

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RESPONSE OF NONLINEAR NONSTATIONARY
VIBRATIONAL SYSTEMS WITH N DEGREES OF FREEDOM
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by

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ABSTRACT

The present study involves the analysis of nonlinear non-stationary vibrational systems with N degrees of freedom subjected to arbitrary pulse excitations. The nonstationary parameters of the system are taken as slowly varying functions of time. The system is reduced to a set of equivalent differential equations using a transformation of the dependent variable. A general procedure for first order approximate solution has been presented, through an application of Krylov, Bogoliubov - Mitropolskii method. The special case of systems with two degrees of freedom has been studied in detail. The response of the system under several types of nonlinearities with the nonstationary parameters varying as linear functions of time, has been presented. Results from various types of pulse excitations have been included.

Further, the method has been successfully applied to obtain the response of a parametrically excited beam column subjected to arbitrary pulse loads. An exponentially decaying pulse and asymptotic step load was considered. The results have been compared with numerical solutions.