NUMERICAL METHODS OF MATRIX INVERSION

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INTRODUCTION

With the advent of high-speed electronic digital computers it became feasible to solve large systems of equations. Many techniques have been developed to solve these equations. One of these involves calculating the inverse of the coefficient matrix. This report will consider eight different methods of calculating the inverse of a matrix.

Before proceeding with the discussion of matrix inversion, it will be necessary to define several terms. A matrix can be defined as a rectangular array of elements. For this report, the matrix will be square and its elements will all be either real numbers or smaller matrices that may be rectangular. Capital letters will be used to denote matrices, and small letters will be used to denote the elements. The individual elements of a matrix A will be designated by a_{rs}, where the subscript r indicates the row in which the element is located, and the s the column. For a shorter notation the matrix A will be denoted by

$$A = (a_{rs}).$$

The size, or dimension, of a matrix with n rows and m columns will be denoted as n x m. When A is equal to (a_{rs}), then the transpose of A, denoted A', is equal to (a_{sr}). The diagonal elements of a square matrix are those whose row and column subscripts are the same. The trace of a square matrix A, denoted $T(A)$, is the sum of the diagonal elements of the matrix A. If a square matrix has zeros for all elements not on the diagonal then it is called a diagonal matrix. If all the possible non-zero elements of a diagonal matrix are equal, the matrix is called a scalar matrix. The product of a scalar c and a matrix A is given by

$$cA = (ca_{rs}).$$

This means that the constant c is multiplied by every element of A. A submatrix of a matrix A is an array formed by deleting one or more rows and/or columns of A. Partitioning a matrix is simply dividing the matrix into several submatrices. This dividing can be accomplished by drawing vertical lines between some of the columns and/or drawing horizontal lines between some of the rows. All matrices partitioned in this report will be divided into four submatrices by drawing one vertical line between two columns and one horizontal line between the corresponding rows. These submatrices then become the elements of the matrix and are denoted by capital A_{rs}.

If two matrices A and B are the same size and $a_{rs} = b_{rs}$ for all r and s, then A = B. The sum of two matrices is given by

$$A + B = (a_{rs} + b_{rs}).$$

This means that to add two matrices, it is necessary to add corresponding elements of each of the matrices. The set of all matrices the same size forms a commutative group. The identity matrix for matrix addition is the matrix Z, which has zeros for each of its elements. The product of two matrices A and B of dimension n x m and m x p, respectively, is given by

$$A \cdot B = AB = \left(\sum_{k=1}^{m} a_{rk}\, b_{ks}\right)$$

where AB is of dimension n x p. This means that the element of AB in the r-th row and the s-th column will have the above summation for its value. Matrix multiplication is not always cummutative; however, it is associative. The identity matrix, denoted by I, for multiplication is a scalar matrix with the value 1 for the diagonal elements. When the matrix A is multiplied by itself $A \cdot A$, it is denoted by A^2. Multiplying A^2 times A would be A^3. In other words, powers of a matrix are similar to powers of a number x in the real number system. The matrix A^0 is defined to be the identity matrix I. For some square matrices there exists another square matrix denoted A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

This matrix A^{-1} is called the inverse of the given matrix. A square matrix that has an inverse is called a nonsingular matrix. Even when the inverse of a given matrix exists it is not always easy to find.

For every square matrix there exists a number of the system of the elements which can be associated with it. In this paper the number will always be real. This number, denoted $|A|$, is called the determinant of the matrix. The determinant of a 2 x 2 matrix is defined to be

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}\, a_{22} - a_{12}\, a_{21}).$$

The determinant of a square submatrix of a matrix A is called a minor of A. If only the r-th row and the s-th column are deleted to form the submatrix the minor is denoted as m_{rs}. The cofactor c_{rs} of the element a_{rs} is given by $(-1)^{r+s}m_{rs}$. The cofactor differs from the minor only by sign when either r or s (but not both) is an odd integer.

For any square matrix A, and for any choice of r and s, the determinant of A can be defined as

$$|A| = \sum_{i=1}^{n} a_{ri}\, c_{ri} = \sum_{j=1}^{n} a_{js}\, c_{js}$$

where the c_{ri} and the c_{js} are the cofactors of the elements a_{ri} and a_{js} of the matrix A. The first expression is said to be the expansion by cofactors along the r-th row of A, and the second is the expansion by cofactors along the s-th column. By applying this definition to the cofactors, it is possible to evaluate the determinant of any size of matrix by calculating the determinant of a number of 2 x 2 submatrices. The cofactor matrix, denoted by C, can be determined by replacing every element of A by its corresponding cofactor.

Primarily, this report uses notation as adopted by Fuller in his book Basic Matrix Theory, (4), where a much more complete discussion of the preceding definitions can be found.

Using matrix notation, a system of n linear equations in n unknowns can be written in the matrix form

$$AX = G.$$

From this form it is easy to see how the inverse of the coefficient matrix, if it exists, can be used in solving a system of equations. By multiplying both sides of the equation by A^{-1}

$$A^{-1}AX = A^{-1}G$$
$$IX = A^{-1}G$$
$$X = A^{-1}G.$$

Thus the solution for the system can be found by one matrix multiplication. The matrix A will be used as an example to illustrate the different methods of matrix inversion in the next three sections where

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ 0 & -4 & -2 & 3 \\ 2 & 1 & 0 & 4 \end{bmatrix}.$$

INVERTING MATRICES USING ELEMENTARY OPERATIONS

Before going into methods of computing the inverse of a matrix it will be necessary to discuss what is meant by elementary operations. Elementary operations are operations which can be performed on a system of equations that will yield a new system of equations with the same solution as the original system.

When these operations are used on a matrix, they are called the elementary row operations for a matrix and are defined as follows:

1. Interchange of the i-th and j-th row denoted by R(i, j).
2. Multiplication of the elements of the i-th row by the non-zero constant c, denoted by cRi.
3. Adding to the elements of the i-th row, k times the corresponding elements of the j-th row, denoted by Ri + kRj.

If the word row is replaced by the word column and if the R is replaced by C everywhere in the above definitions, then one has the definitions for elementary column operations.

Of all the methods discussed in this paper for calculating the inverse of a matrix, those methods which involve elementary operations are the most easily adapted to machine computation. The method, (1), involved in this section is the most general, for it uses both elementary row and column operations.

It will be shown that if A, P and Q are nonsingular matrices such that $PAQ = I$, then

$$A^{-1} = QP.$$

First premultiplying both sides of the equation $PAQ = I$ by Q

$$QPAQ = Q.$$

Since Q is nonsingular, postmultiply both sides by Q^{-1}

$$QPA = I.$$

Therefore by definition of matrix inverse

$$A^{-1} = QP.$$

This method gives a technique for computing the two matrices P and Q using elementary operations. If the matrix A is reduced to I by a series of elementary row and column operations, then one obtains a matrix P by applying the same row operations to I and a matrix Q by applying the same column operations to I. These two matrices, P and Q, are the ones desired for calculating the inverse.

In the actual computation of A^{-1} by this method, it is not necessary to keep track of the operations that are made. Instead, by setting up the matrix

$$B = \begin{bmatrix} A & I \\ I & Z \end{bmatrix},$$

where Z is the zero matrix, it is possible to operate on I at the same time A is being operated on by both row and column operations. It is important to remember that all of the operations must occur on A and that the matrix B is used merely to keep track of the result of the operation on I. Once A has been reduced to I

$$B \leftrightarrow \begin{bmatrix} I & P \\ Q & Z \end{bmatrix}$$

and P and Q have been determined.

One advantage of this method is that it can be checked at any stage. If one has obtained

$$B = \begin{bmatrix} A & I \\ I & Z \end{bmatrix} \leftrightarrow \begin{bmatrix} A_1 & N \\ M & Z \end{bmatrix}$$

then the work is correct if

$$NAM = A_1.$$

As an example of this method, consider the matrix A given in the introduction.

First form

$$
B = \begin{bmatrix} A & I \\ I & Z \end{bmatrix} = \left[\begin{array}{cccc|cccc}
1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\
-2 & 4 & 3 & 0 & 0 & 1 & 0 & 0 \\
0 & -4 & -2 & 3 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 4 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & & & & \\
0 & 1 & 0 & 0 & & & Z & \\
0 & 0 & 1 & 0 & & & & \\
0 & 0 & 0 & 1 & & & &
\end{array}\right].
$$

Now using elementary operations

$$
B \underset{\substack{C2 + 1C1 \\ C3 + 1C1 \\ C4 + 1C1 \\ R2 + 2R1 \\ R4 - 2R1}}{\longleftrightarrow} \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & -2 & 2 & 1 & 0 & 0 \\
0 & -4 & -2 & 3 & 0 & 0 & 1 & 0 \\
0 & 3 & 2 & 6 & -2 & 0 & 0 & 1 \\
\hline
1 & 1 & 1 & 1 & & & & \\
0 & 1 & 0 & 0 & & & Z & \\
0 & 0 & 1 & 0 & & & & \\
0 & 0 & 0 & 1 & & & &
\end{array}\right]
\underset{C2 - 1C3}{\longleftrightarrow} \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & -2 & 2 & 1 & 0 & 0 \\
0 & -2 & -2 & 3 & 0 & 0 & 1 & 0 \\
0 & 1 & 2 & 6 & -2 & 0 & 0 & 1 \\
\hline
1 & 0 & 1 & 1 & & & & \\
0 & 1 & 0 & 0 & & & Z & \\
0 & -1 & 1 & 0 & & & & \\
0 & 0 & 0 & 1 & & & &
\end{array}\right].
$$

$$\xleftrightarrow[\substack{C3 - 1C2 \\ C4 + 2C2 \\ R3 + 2R2 \\ R4 - 1R2}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & 8 & -4 & -1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & & & & \\ 0 & 1 & -1 & 2 & & & & \\ 0 & -1 & 2 & -2 & & \multicolumn{2}{c}{Z} & \\ 0 & 0 & 0 & 1 & & & & \end{array}\right] \xleftrightarrow[R3 + 1R4]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 8 & -4 & -1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & & & & \\ 0 & 1 & -1 & 2 & & & & \\ 0 & -1 & 2 & -2 & & \multicolumn{2}{c}{Z} & \\ 0 & 0 & 0 & 1 & & & & \end{array}\right]$$

$$\xleftrightarrow[\substack{C4 - 7C3 \\ R4 - 1R3}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 & -2 & -1 & 0 \\ \hline 1 & 0 & 1 & -6 & & & & \\ 0 & 1 & -1 & 9 & & & & \\ 0 & -1 & 2 & -16 & & \multicolumn{2}{c}{Z} & \\ 0 & 0 & 0 & 1 & & & & \end{array}\right],$$

which gives

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -4 & -2 & -1 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 1 & -6 \\ 0 & 1 & -1 & 9 \\ 0 & -1 & 2 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$A^{-1} = QP = \begin{bmatrix} 1 & 0 & 1 & -6 \\ 0 & 1 & -1 & 9 \\ 0 & -1 & 2 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -4 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix}.$$

This value of A^{-1} can be checked in the equation $AA^{-1} = I$.

If no elementary column operations are performed on A, the matrix Q will remain equal to I. Once A has been reduced to I, the inverse of A can still be calculated by

$$A^{-1} = QP.$$

However, since Q is equal to I,

$$A^{-1} = P.$$

A similar argument would show that when no row operations are performed on A the inverse of A is given by

$$A^{-1} = Q.$$

These are only two of the many variations of the method of elementary operations. One distinct advantage of both of these special cases is that it is not necessary to form the product QP.

METHODS INVOLVING THE DETERMINANT OF A MATRIX

Calculating the Inverse Using the Adjoint Matrix

The adjoint of a matrix A is given by adj $A = C'$ where C' is the transpose of the cofactor matrix. The following principal, (4), pp. 103-111, makes it possible to use the adjoint matrix for finding the inverse:

$$A \cdot \text{adj } A = \text{adj } A \cdot A = |A| I.$$

If $|A| \neq 0$, then this equation can be written as

$$A \frac{\text{adj } A}{|A|} = \frac{\text{adj } A}{|A|} A = I.$$

Therefore, by definition of the matrix inverse

$$A^{-1} = \frac{\text{adj } A}{|A|} .$$

This method can be used for all matrices whose determinants are nonzero. It can be shown that the nonvanishing of the determinant of a matrix is a necessary and sufficient condition for the matrix to be nonsingular. Therefore if a matrix has an inverse it can be found using this method.

It is obvious that once the adjoint matrix is obtained, finding the inverse will not be difficult. Multiplying the k-th column of A by the k-th row of adj A will give the $|A|$. All that remains is to divide the adj A by $|A|$. The usefulness of this method, therefore, depends upon the ease with which the adjoint matrix is obtained. In the case of a 2 x 2 matrix the adjoint matrix can be written down immediately. For given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then

$$C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}.$$

Therefore

$$\text{adj } A = C' = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

In other words, to find the adjoint of a 2 x 2 matrix, interchange the diagonal elements and change the signs of the remaining two elements. However, in the case of a 4 x 4 matrix, it is necessary to find the determinants of sixteen 3 x 3 matrices, and for a 5 x 5 matrix the determinants of twenty-five 4 x 4 matrices are needed. Because of the increasing difficulty of obtaining the adjoint matrix, this method should not be used for large matrices.

As an example of this method of finding the inverse consider the matrix A given in the introduction. Its cofactor matrix is

$$C = \begin{bmatrix} 25 & -34 & 62 & -4 \\ 13 & -18 & 33 & -2 \\ 7 & -10 & 18 & -1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \qquad \text{adj } A = C' = \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix}.$$

A good method of checking the computations for adj A is to form its product with A. For this example this gives

$$A\cdot\text{adj } A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ 0 & -4 & -2 & 3 \\ 2 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The adjoint matrix is incorrect if the product is not a scalar matrix.

The $|A|$ is equal to the value of the diagonal elements of the product of $A \cdot \text{adj } A$. In this example,

$$|A| = 1.$$

Hence

$$A^{-1} = \text{adj } A.$$

Inversion of a Matrix using the Characteristic Equation

The following method, (5), like the method using the adjoint matrix, is not advantageous except with small matrices or matrices for which the characteristic equation is known. The basis for using the characteristic equation to find the inverse is the Hamilton-Cayley theorem, which states that every matrix satisfies its characteristic equation.

The characteristic equation of the matrix A is the polynomial equation $|rI - A| = 0$. The characteristic equation of the example matrix A is

$$|rI - A| = \begin{vmatrix} r-1 & 1 & +1 & -1 \\ 2 & r-4 & -3 & 0 \\ 0 & 4 & r+2 & -3 \\ -2 & -1 & 0 & r-4 \end{vmatrix} = 0.$$

Evaluation of this determinant gives

$$|rI - A| = r^4 - 7r^3 + 18r^2 - 25r + 1 = 0$$

which is the characteristic equation of the matrix A.

According to the Hamilton-Cayley theorem, "satisfying" the characteristic equation means that this same equation with corresponding powers of A will give the zero matrix when A^0 is defined as I. Therefore, in the example

$$A^4 - 7A^3 + 18A^2 - 25A + I = Z$$

subtracting I from both sides of the equation leaves

$$A^4 - 7A^3 + 18A^2 - 25A = -I.$$

Factoring out an A on the right side of the equality gives

$$(A^3 - 7A^2 + 18A - 25I)A = -I$$

then

$$(-A^3 + 7A^2 - 18A + 25I)A = I.$$

Since the product on the right is equal to the identity matrix, the inverse of A is by definition:

$$A^{-1} = (-A^3 + 7A^2 - 18A + 25I).$$

The inverse of any non-singular matrix could be calculated in a similar manner once the characteristic equation is known. All that remains to compute A^{-1} is to substitute the values of A^3, A^2, A and I into the equation and add.

$$A^{-1} = \begin{bmatrix} 11 & 9 & 3 & 39 \\ 0 & -13 & -12 & -78 \\ -36 & -4 & 13 & 14 \\ -24 & -26 & -8 & -51 \end{bmatrix} + \begin{bmatrix} 7 & -14 & -14 & -56 \\ -70 & 42 & 56 & 77 \\ 98 & -35 & -56 & 42 \\ 56 & 42 & 7 & 98 \end{bmatrix}$$

$$+ \begin{bmatrix} -18 & 18 & 18 & 18 \\ 36 & -72 & -54 & 0 \\ 0 & 72 & 36 & -54 \\ -36 & -18 & 0 & -72 \end{bmatrix} + \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix}.$$

This value checks with results calculated previously. One disadvantage of this method is that there are no checks on the calculations until they are completed.

For some matrices there exists an equation, which the matrix satisfies, whose degree is lower than the characteristic equation. If this is the case then this minimum equation may be used to find the inverse instead of the characteristic equation.

Matrix Inversion Using the Trace of a Matrix

The last method discussed in this section is a method for computing the inverse which involves the trace of the matrix. The fact that it requires a greater number of operations than most of the methods discussed in this paper is a disadvantage, but it has two definite advantages. First, it is not affected by individual peculiarities of different matrices such as zero elements on the diagonal or determinants of submatrices being equal to zero. Second, as a by-product this method will give the characteristic equation and the characteristic vectors of the matrix; therefore, if any or all of these are desired this method could be used to advantage.

Let the n x n matrix A have the following characteristic equation:

$$|rI - A| = r^n - p_1 r^{n-1} - p_2 r^{n-2} - \cdots - p_{n-1} r^1 - p_n$$

with characteristic roots being denoted as $r_1, r_2, \ldots, r_n$. Let the following series be equal to s_k:

$$s_k = r_1^k + r_2^k + \cdots + r_n^k .$$

These are the sum of the characteristic roots of A^k so

$$s_k = T(A^k).$$

Once the s_k are known, the coefficients of the characteristic equation can be calculated using Newton's recursion formula.

$$p_1 = s_1$$
$$2p_2 = s_2 - p_1 s_1$$
$$3p_3 = s_3 - p_1 s_2 - p_2 s_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$kp_k = s_k - p_1 s_{k-1} - \cdots - p_{k-1} s_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$np_n = s_n - p_1 s_{n-1} - \cdots - p_{n-1} s_1$$

where the p_k are the coefficients of the characteristic equation. This is a way of deriving the characteristic equation in which it is first necessary to compute the powers of A.

A method, (3), has been developed that eliminates the necessity of computing the powers of A and that will also give the inverse of A. Instead of the powers, a set of n matrices $A_1, A_2, \ldots, A_n$ will be computed using the following formulae:

$$c_1 = T(AA_0) \qquad\qquad A_1 = AA_0 - c_1 I$$

$$c_2 = 1/2\ T(AA_1) \qquad\qquad A_2 = AA_1 - c_2 I$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$c_k = 1/k\ T(AA_{k-1})$ $\qquad\qquad$ $A_k = AA_{k-1} - c_kI$

. . . .

$c_n = 1/n\ T(AA_{n-1})$ $\qquad\qquad$ $A_n = AA_{n-1} - c_nI$

where A_0 is defined to be I. It will be shown that

$$c_k = p_k \qquad k = 1, 2, \ldots, n$$

and that

$$A_n = Z.$$

If A is nonsingular and $c_n \neq 0$, then

$$AA_{n-1} = c_nI$$
$$A(A_{n-1})/c_n = I.$$

Therefore

$$A^{-1} = A_{n-1}/c_n.$$

Note: The first step of the above proof shows that the matrix AA_{n-1} is a scalar matrix. This fact will be used later as a check for this method.

Mathematical induction will be used to prove that c_k is equal to p_k. First of all by choice:

$$c_1 = T(AA_0) = T(AI) = T(A) = s_1 = p_1$$

also

$$AA_1 = A(AA_0 - c_1I) = A^2 - c_1A = A^2 - p_1A$$

so

$$2c_2 = T(AA_1) = T(A^2) - p_1T(A) = s_2 - p_1s_1 = 2p_2.$$

Therefore

$$c_2 = p_2.$$

Next

$$AA_2 = A\left[A(AA_0 - c_1I) - c_2I\right] = A^3 - c_1A^2 - c_2A = A^3 - p_1A^2 - p_2A$$

so

$$3c_3 = T(AA_2) = T(A^3) - p_1T(A^2) - p_2T(A) = s_3 - p_1s_2 - p_2s_1 = 3p.$$

Therefore

$$c_3 = p_3$$

. . . .

$$AA_k = A(AA_{k-1} - c_kI) = A\left[A(AA_{k-2} - c_{k-1}I) - c_kI\right] = A^{k+1} - p_1A^k - \cdots - p_kA$$

$$(k+1)c_{k+1} = T(AA_k) = T(A^{k+1}) - p_1T(A^k) - \cdots - p_kT(A)$$

$$= s_{k+1} - p_1s_k - \cdots - p_ks_1 = (k+1)p_{k+1} .$$

Therefore

$$c_{k+1} = p_{k+1}.$$

Also by the induction

$$A_1 = A - p_1 I$$

$$A_2 = AA_1 - p_2 I = A^2 - p_1 A - p_2 I$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_n = AA_{n-1} - p_n I = A^n - p_1 A^{n-1} - \cdot \cdot \cdot \cdot - p_n I.$$

By the Hamilton-Cayley theorem then,

$$A_n = Z.$$

As an example of this method, consider again the matrix A. First computing

$c_1 = 1/1 \; T(AA_0) = 7$

$$A_1 = AA_0 - 7I = \begin{bmatrix} -6 & -1 & -1 & -1 \\ -2 & -3 & 3 & 0 \\ 0 & -4 & -9 & 3 \\ 2 & 1 & 0 & -3 \end{bmatrix}.$$

$c_2 = 1/2 \; T(AA_1) = -18$

$$A_2 = AA_1 + 18I = \begin{bmatrix} 12 & 5 & 5 & -1 \\ 4 & -4 & -13 & 11 \\ 14 & 23 & 24 & -15 \\ -6 & -1 & 1 & 4 \end{bmatrix}$$

$c_3 = 1/3\ T(AA_2) = 25$

$$A_3 = AA_2 - 25I = \begin{bmatrix} -25 & -13 & -7 & -1 \\ 34 & 18 & 10 & 1 \\ -62 & -33 & -18 & -2 \\ 4 & 2 & 1 & 0 \end{bmatrix}$$

$$AA_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad c_4 = 1/4\ T(AA_3) = -1$$

If AA_3 had not been a scalar matrix then this would have indicated an error in the calculation.

Finally,

$$A^{-1} = A_{n-1}/c_n = A_3/c_4 = \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix}.$$

Substituting the above values of c_1, c_2, c_3 and c_4 into the characteristic equation gives

$$|rI - A| = r^4 - 7r^3 + 18r^2 - 25r + 1 = 0.$$

This value of the inverse of A checks with results calculated previously.

INVERSION OF A MATRIX BY PARTITIONING

Partitioning of a Matrix to Compute the Inverse

The motivation for the method of inversion by partitioning is the fact that smaller matrices are easier to invert than large ones. With this method, (2), it is possible to invert a matrix by computing the inverses of two smaller matrices and performing some matrix multiplication and addition. Two different sets of equations will be developed which will each give the inverse.

Let D be the inverse of the matrix A. Partition A between a pair of rows and corresponding columns. Also partition D and I similarly. The product AD can then be written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} I & Z \\ Z & I \end{bmatrix} .$$

Performing the matrix multiplication results in the following equality:

$$\begin{bmatrix} A_{11}D_{11} + A_{12}D_{21} & A_{11}D_{12} + A_{12}D_{22} \\ A_{21}D_{11} + A_{22}D_{21} & A_{21}D_{12} + A_{22}D_{22} \end{bmatrix} = \begin{bmatrix} I & Z \\ Z & I \end{bmatrix} .$$

In order for these two matrices to be identical, the corresponding submatrices must be equal; that is, the following matrix equations must hold:

1) $A_{11}D_{11} + A_{12}D_{21} = I$ 2) $A_{11}D_{12} + A_{12}D_{22} = Z$

3) $A_{21}D_{11} + A_{22}D_{21} = Z$ 4) $A_{21}D_{12} + A_{22}D_{22} = I$.

When the product AD is reversed a different set of four matrix equations will result. The equations for the reversed product, DA, are given below.

1') $D_{11}A_{11} + D_{12}A_{21} = I$ 2') $D_{11}A_{12} + D_{12}A_{22} = Z$

3') $D_{21}A_{11} + D_{22}A_{21} = Z$ 4') $D_{21}A_{12} + D_{22}A_{22} = I$

To find D it is necessary to solve the two sets of matrix equations for D_{11}, D_{12}, D_{21} and D_{22}. Solving for D_{12} in equation 2, and for D_{21} in equation 3' gives

5) $D_{12} = -A_{11}^{-1}A_{12}D_{22}$ 6) $D_{21} = -D_{22}A_{21}A_{11}^{-1}$.

Substituting the value of D_{12} in equation 5 into equation 4 one obtains

$$A_{21}(-A_{11}^{-1}A_{12}D_{22}) + A_{22}D_{22} = I.$$

Collecting coefficients of D_{22} gives

$$(-A_{21}A_{11}^{-1}A_{12} + A_{22})D_{22} = I.$$

By definition of the inverse of matrix

$$D_{22}^{-1} = (A_{22} - A_{21}A_{11}^{-1}A_{12})$$

or

$$7)\quad D_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}.$$

Solving for D_{11} in equation 1 gives

$$8)\quad D_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}D_{21}.$$

Using equations 5, 6, 7 and 8 one can calculate D. In this case it is necessary to find the inverses of two matrices, A_{11} and the inverse of the matrix in equation 7. If either of these two matrices are singular then this set of equations can not be used. However, by using a different combination of the original two sets of equations, another set of four equations may be obtained in which two different inverses must be found. These equations are the following:

$$5') \quad D_{12} = -D_{11}A_{12}A_{22}^{-1} \qquad\qquad 6') \quad D_{21} = -A_{22}^{-1}A_{21}D_{11}$$

$$7') \quad D_{22} = A_{22}^{-1} - A_{22}^{-1}A_{21}D_{12} \qquad\qquad 8') \quad D_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}.$$

In this set the inverses of A_{22} and the matrix in equation 8' must be calculated. This shows that in order to use partitioning to compute the inverse of a matrix at least one of the diagonal matrices must be non-singular.

It is interesting to note that by using equations 5, 6', 7 and 8' it is also possible to calculate D. However, in this case the inverses of all four of the matrices discussed above would have to be calculated. Besides the difficulty of calculating two additional matrices, this set of equations is more restrictive since it requires two more matrices to be non-singular.

An example of a matrix for which partitioning would be advantageous is a 4 x 4 matrix. This matrix could be divided into four submatrices with 2 x 2 matrices on the diagonal. As was shown in the second method of this report, the inverse of a 2 x 2 matrix is easy to compute using the adjoint matrix.

Below is an example of this procedure using the first set of equations. Suppose that the matrix A is partitioned so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ \hline 0 & -4 & -2 & 3 \\ 2 & 1 & 0 & 4 \end{array}\right].$$

Next calculate A_{11}^{-1} using the adjoint matrix

$$A_{11}^{-1} = 1/|A_{11}| \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = 1/2 \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

and check by substituting into $A_{11}A_{11}^{-1} = I$:

$$1/2 \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(It is important to check every inverse matrix that is computed. However, to save space in this paper the check will not always be shown.)

Now find

$$A_{11}^{-1}A_{12} = 1/2 \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} = 1/2 \begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix}$$

$$A_{21}A_{11}^{-1} = 1/2 \begin{bmatrix} 0 & -4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = 1/2 \begin{bmatrix} -8 & -4 \\ 10 & 3 \end{bmatrix}$$

$$A_{21}A_{11}^{-1}A_{12} = 1/2 \begin{bmatrix} 0 & -4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1/2 & -5 \end{bmatrix}$$

$$A_{22} - A_{21}A_{11}^{-1}A_{12} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 1/2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1/2 & 9 \end{bmatrix}$$

Finally

$$D_{22} = (A_{22} - A_{21}A_{11}A_{12})^{-1} = 2 \begin{bmatrix} 9 & 1 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 2 \\ -1 & 0 \end{bmatrix}.$$

Next find

$$D_{12} = -A_{11}^{-1}A_{12}D_{22} = -1/2 \begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 18 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -10 & -1 \end{bmatrix}$$

$$D_{21} = -D_{22}A_{21}A_{11}^{-1} = -1/2 \begin{bmatrix} 18 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -8 & -4 \\ 10 & 3 \end{bmatrix} = \begin{bmatrix} 62 & 33 \\ -4 & -2 \end{bmatrix}$$

$$A_{11}^{-1}A_{12}D_{21} = 1/2 \begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 62 & 33 \\ -4 & -2 \end{bmatrix} = 1/2 \begin{bmatrix} -46 & -25 \\ 70 & 37 \end{bmatrix}$$

$$D_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}D_{21} = 1/2 \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} + 1/2 \begin{bmatrix} 46 & 25 \\ -70 & -37 \end{bmatrix} = \begin{bmatrix} 25 & 13 \\ -34 & -18 \end{bmatrix}.$$

Therefore

$$A^{-1} = D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ \hline 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{array}\right].$$

This value for A^{-1} checks with results computed previously.

The example shows how this method can be used to aid in hand calculation of the inverses of small matrices. It also has advantages for calculating inverses with digital computers. Assume that it is necessary to find the inverse of a 100 x 100 matrix. The storage capacity of the available computer limits the size of the matrix to be inverted to 60 x 60 and the size of matrices to be multiplied to 50 x 50. By partitioning the 100 x 100 matrix into four 50 x 50 submatrices it would be possible to calculate the inverse part by part and assemble the final result outside the computer. This shows that by the use of partitioning a computer's capacity for finding the inverse is almost doubled.

This method can be extended by repeated partitioning to compute the inverses of any size of matrix. As an illustration consider the following 6 x 6 matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array}\right].$$

After A has been subdivided in the above manner, A_{11} can be subdivided into four 2 x 2 matrices. The inverse of A_{11} can then be calculated using the method of partitioning. Once A_{11}^{-1} is known the inverse of A can be computed using equations 5, 6, 7 and 8. By using this process the inverses of all the matrices to be inverted are easy to compute. However, the increase of the number of matrices to be inverted is 2n where n is the number of times the original matrix is partitioned.

The Inversion of a Matrix by Bordering

As will be seen shortly, bordering is very closely related to the method of partitioning described in the first part of this section. In this method the given matrix A_n will be regarded as the result of bordering a matrix of order n-1. This submatrix will be designated by A_{n-1}, and its inverse A_{n-1}^{-1} will be assumed to be known. Thus

$$A_n = \begin{bmatrix} A_{n-1} & A_{12n} \\ A_{21n} & a_{nn} \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} \cdots & & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} \cdots & & a_{2,n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} \cdots & & a_{n-1,n-1} & a_{n-1,n} \\ \hline a_{n1} & a_{n2} \cdots & & a_{n,n-1} & a_{nn} \end{array}\right]$$

where A_{12n} is a column matrix and A_{21n} is a row matrix.

The inverse of A_n will be designated by D_n and will be partitioned in the form

$$D_n = \begin{bmatrix} D_{n-1} & D_{12n} \\ D_{21n} & d_n \end{bmatrix}$$

where D_{n-1} is a matrix of order n-1, D_{12n} is a column matrix, D_{21n} is a row matrix, and d_n is a number. The product A_nD_n can then be written as

$$\begin{bmatrix} A_{n-1} & A_{12n} \\ A_{21n} & a_{nn} \end{bmatrix} \begin{bmatrix} D_{n-1} & D_{12n} \\ D_{21n} & d_n \end{bmatrix} = \begin{bmatrix} I & Z \\ Z & I \end{bmatrix}.$$

Substituting the elements into equations 5, 6, 7 and 8 from the method of partitioning gives

$$D_n = \begin{bmatrix} A_{n-1}^{-1} + (A_{n-1}^{-1}A_{12n}A_{21n}A_{n-1}^{-1})d_n & -A_{n-1}^{-1}A_{12n}d_n \\ -A_{21n}A_{n-1}^{-1}d_n & d_n \end{bmatrix}$$

where $d_n = (a_{nn} - A_{21}A_{n-1}^{-1}A_{12})^{-1}$.

The inverse of a matrix A can be obtained by using the above equation to compute successively D_1, D_2, D_3, . . . , D_n where

$$D_1 = \begin{bmatrix} a_{11} \end{bmatrix}^{-1}$$

$$D_2 = \left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right]^{-1}$$

$$D_3 = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]^{-1}$$

. . . .

$$
D_n = \left[\begin{array}{cccc|c}
a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2,n-1} & a_{2n} \\
\cdot & \cdot & \cdots & \cdot & \cdots \\
a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\
\hline
a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{nn}
\end{array}\right]^{-1}.
$$

As an example of this method consider the following matrix:

$$
A_4 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ 0 & -4 & -2 & 3 \\ 2 & 1 & 0 & 4 \end{bmatrix}.
$$

To compute D_4 using this method it would first be necessary to find the inverses of the following three matrices:

$$
A_1 = \begin{bmatrix} 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 4 & 3 \\ 0 & -4 & -2 \end{bmatrix}.
$$

However, since it is easy to find the inverse of A_2 using the adjoint matrix this will be done, and will leave only two borderings necessary to find D_4. Bordering A_3 gives

$$A_3 = \begin{bmatrix} A_2 & A_{123} \\ A_{213} & a_{33} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -1 & -1 \\ -2 & 4 & 3 \\ \hline 0 & -4 & -2 \end{array}\right]$$

where

$$A_2^{-1} = 1/2 \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}.$$

First find $-A_3^{-1} A_{123}$ and $-A_{213}A_2^{-1}$.

$$-A_2^{-1}A_{123} = -1/2\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -1/2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad -A_{213}A_2^{-1} = -1/2\begin{bmatrix} 0 & -4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = -1/2\begin{bmatrix} -8 & -4 \end{bmatrix}$$

Next, as a check on these values, calculate $A_{213}(-A_2^{-1}A_{123})$ and $(-A_{213}A_2^{-1})A_{123}$.

$$A_{213}(-A_2^{-1}A_{123}) = -1/2\begin{bmatrix} 0 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \qquad (-A_{213}A_2^{-1})A_{123} = -1/2\begin{bmatrix} -8 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = 2$$

Since these values are equal, calculate d_3^{-1} where

$$d_3^{-1} = a_{33} + (-A_{21}A_2^{-1}A_{12}) = -2 + 2 = 0.$$

Notice from the equation for computing D_3 that every term of D_3 must be multiplied by d_3, which in this case is undefined. Since this is impossible, the inverse of A_4 cannot be computed in its present form by this method. Remembering that two rows of A may be interchanged when finding the inverse if after the inverse is calculated the corresponding columns of the inverse are changed back; A_4 will be rewritten by interchanging row 3 and row 4.

$$A_4' = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ 2 & 1 & 0 & 4 \\ 0 & -4 & -2 & 3 \end{bmatrix}$$

Since A_2 remains the same, the first step is to border A_3. This gives

$$A_3 = \begin{bmatrix} A_2 & A_{123} \\ A_{213} & A_{33} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -1 & -1 \\ -2 & 4 & 3 \\ \hline 2 & 1 & 0 \end{array}\right].$$

As A_2^{-1} has already been calculated, solve again for $-A_2^{-1}A_{123}$ and $-A_{213}A_2^{-1}$.

$$-A_2^{-1}A_{123} = -1/2\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -1/2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-A_{213}A_2^{-1} = -1/2\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = -1/2\begin{bmatrix} 10 & 3 \end{bmatrix}$$

Next calculate $A_{213}(-A_2^{-1}A_{123})$ and $(-A_{213}A_2^{-1})A_{123}$.

$$A_{213}(-A_2^{-1}A_{123}) = -1/2\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1/2$$

$$(-A_{213}A_2^{-1})A_{123} = -1/2\begin{bmatrix} 10 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = 1/2$$

Since these are equal, solve for d_3^{-1} where

$$d_3^{-1} = a_{33} + (-A_{213}A_2^{-1}A_{123}) = 0 + 1/2 = 1/2.$$

Hence

$$d_3 = 2$$

Now

$$D_2 = A_2^{-1} + (A_2^{-1}A_{123}A_{213}A_2^{-1})d_3 = 1/2\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} + 2(-1/2)(-1/2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 10 & 3 \end{bmatrix}$$

$$= 1/2\begin{bmatrix} -6 & -2 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix}$$

$$D_{123} = -A_2^{-1}A_{123}d_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad D_{213} = -A_{213}A_2^{-1}d_3 = \begin{bmatrix} -10 & -3 \end{bmatrix}$$

Therefore

$$A_3^{-1} = D_3 = \begin{bmatrix} D_2 & D_{123} \\ D_{213} & d_3 \end{bmatrix} = \left[\begin{array}{cc|c} -3 & -1 & 1 \\ 6 & 2 & -1 \\ \hline -10 & -3 & 2 \end{array}\right].$$

Next, repeat this process after bordering A_4'.

$$A_4' = \begin{bmatrix} A_3 & A_{124} \\ A_{214} & a_{44} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ -2 & 4 & 3 & 0 \\ 2 & 1 & 0 & 4 \\ \hline 0 & -4 & -2 & 3 \end{array}\right]$$

Solve for $-A_3^{-1}A_{124}$ and $-A_{214}A_3^{-1}$.

$$-A_3^{-1}A_{124} = -1\begin{bmatrix} -3 & -1 & 1 \\ 6 & 2 & -1 \\ -10 & -3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \\ -18 \end{bmatrix}$$

$$-A_{214}A_3^{-1} = \begin{bmatrix} 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} -3 & -1 & 1 \\ 6 & 2 & -1 \\ -10 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix}$$

Check by showing that $A_{214}(-A_3^{-1}A_{124})$ and $(-A_{214}A_3^{-1})A_{124}$ are equal.

$$A_{214}(-A_3^{-1}A_{124}) = \begin{bmatrix} 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} -7 \\ 10 \\ -18 \end{bmatrix} = -4 \qquad (-A_{214}A_3^{-1})A_{124} = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = -4$$

Now solve for d_4^{-1}.

$$d_4^{-1} = a_{44} + (-A_{214}A_3^{-1}A_{124}) = 3 - 4 = -1$$

$$d_4 = -1$$

Next calculate D_3, D_{124} and D_{214}.

$$D_3 = \begin{bmatrix} -3 & -1 & 1 \\ 6 & 2 & -1 \\ -10 & -3 & 2 \end{bmatrix} - \begin{bmatrix} -7 \\ 10 \\ -18 \end{bmatrix}\begin{bmatrix} 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 13 & 1 \\ -34 & -18 & -1 \\ 62 & 33 & 2 \end{bmatrix}$$

$$D_{124} = -1\begin{bmatrix} -7 \\ 10 \\ -18 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 18 \end{bmatrix} \qquad D_{214} = -1\begin{bmatrix} 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 0 \end{bmatrix}$$

Therefore

$$D_4' = \begin{bmatrix} D_3 & D_{124} \\ D_{214} & d_4 \end{bmatrix} = \left[\begin{array}{ccc|c} 25 & 13 & 1 & 7 \\ -34 & -18 & -1 & -10 \\ 62 & 33 & 2 & 18 \\ \hline -4 & -2 & 0 & -1 \end{array}\right].$$

All that remains in calculating D_4 is to interchange column three and column four of D_4'. Thus

$$D_4 = \begin{bmatrix} 25 & 13 & 7 & 1 \\ -34 & -18 & -10 & -1 \\ 62 & 33 & 18 & 2 \\ -4 & -2 & -1 & 0 \end{bmatrix}.$$

This value checks with results computed previously.

One advantage of this method is the fact that the computations may be checked when calculating each d_n and after every D_n is computed. One disadvantage, as illustrated in the example just completed, is that it is undesirable to have a submatrix on the diagonal which does not have an inverse.

ITERATIVE METHODS OF MATRIX INVERSION

Direct methods of matrix inversion were considered in the three previous sections of this report. Since it is possible for roundoff errors and errors in computation to occur, a method of iteration for improving the accuracy of the inverse of a given matrix will be presented. Iterative methods are characterized by a cyclicing process in which the same equations are used repeatedly to obtain the inverse. An initial matrix for the inverse must be given. When the desired accuracy is obtained the process is stopped. Before this method of iteration is discussed, the basic principle, (4), pp. 177-182, of the method will be developed.

This principle involves the raising of an error matrix to higher and higher powers. In order for the approximations for the inverse to approach the desired accuracy, the elements of these higher powers of the error matrix must become numerically smaller and approach zero. It will be shown that if the sums of the absolute values of the elements in each row or each column of a given matrix are less than 1, then the elements of the higher powers of the matrix approach zero. The r-th row sum of an n x n matrix, rRSA, is defined as this sum of the absolute values of the elements in the r-th row. A simpler statement of the above theorem is, "powers of a matrix whose row sums are less than 1 will approach Z." The proof to be given will consider the case of the "row sums" only. The proof of the column sums would follow in a similar manner.

Consider a matrix B whose row sums are less than 1. Then there will exist at least one row sum of B of largest value p where $p<1$. It will be shown that the row sums of B^m are no larger than p^m. This is done by noting that all row sums of a product of two matrices AB are no larger than p times the corresponding row sums of A. It will follow that all the row sums of B^m approach 0 as m is increased because p^m will approach 0.

This result will be proved for the product of two n x n matrices A and B. The product is given by

$$AB = (a_{rs})(b_{rs}) = (\sum_{k=1}^{n} a_{rk}b_{ks}).$$

Considering only the elements in the r-th row; that is, the elements obtained for a fixed r and for s varying from 1 to n,

$$rRSAB = \left|\sum_{k=1}^{n} a_{rk}b_{k1}\right| + \left|\sum_{k=1}^{n} a_{rk}b_{k2}\right| + \dots + \left|\sum_{k=1}^{n} a_{rk}b_{kn}\right|$$

$$= \sum_{s=1}^{n}\left|\sum_{k=1}^{n} a_{rk}b_{ks}\right| \leq \sum_{s=1}^{n}\sum_{k=1}^{n}\left|a_{rk}\right|\left|b_{ks}\right|$$

$$\leq \sum_{k=1}^{n}\left|a_{rk}\right|\sum_{s=1}^{n}\left|b_{ks}\right| = \sum_{k=1}^{n}\left|a_{rk}\right|(kRSB)$$

$$\leq \sum_{k=1}^{n}p\left|a_{rk}\right| = p\sum_{k=1}^{n}\left|a_{rk}\right| = p(rRSA).$$

This result shows that the row sums of AB are no greater than the product of the maximum row sum of B and the corresponding row sum of A.

If the matrix A = B, the discussion above indicates that the row sums of B^2 are less than or equal to the product of p and the corresponding row sum of B. This means for this case that the row sums of B^2 are no larger than p^2. Next letting $A=B^2$, one can conclude that the row sums of B^3 are no larger than the product of p and the corresponding row sum of B^2. Since the largest row sum of B^2 is not greater than p^2, the row sums of B^3 are all less than or equal to p^3. Similarly, one can conclude by letting $A = B^k$ that the row sums of B^{k+1} are no larger than the product of p and the corresponding row sums of B^k. Since the row sums of B^k are no larger than p^k, the row sums of B^{k+1} are no larger than p^{k+1}. Since p was assumed to be less than 1, the powers of p must approach 0. Since this is the case, there must exist an m such that p^m is approximately equal to 0. This means that for this m the row sums of B^m must be approximately equal to 0. Since by definition the row sums are the sum of the absolute values of the elements, the row sum approaching 0 as the power is increased implies that each element must approach 0. Therefore, if the row sums of B^m are approximately equal to 0, then B^m is approximately equal to Z. Thus, the theorem is proven for the row sum. As was stated earlier, the part for the column sums can be proven in a similar manner using column sums instead of row sums.

Improving the Accuracy of the Inverse of a Matrix

With the background knowledge of matrices whose powers approach the zero matrix, the basic iterative procedure, (4), pp. 182-187, for improving the accuracy of the inverse of a matrix will be discussed. The first approximation of the inverse of A will be designated by D_0. This approximation will be improved upon using the following sequence:

$$D_1 = (2I - D_0A)D_0$$

$$D_2 = (2I - D_1A)D_1$$

$$\cdots\cdots\cdots$$

$$D_m = (2I - D_{m-1}A)D_{m-1}.$$

It is obvious that the error matrix for D_0 can be written as

$$E_0 = C = I - D_0A.$$

It can be shown that the error matrices of subsequent approximations of the inverse are given by

$$E_m = C^{2^m} = I - D_mA.$$

Therefore if the first approximation is such that the row sums or column sums of the error matrix C are all less than 1, then the higher powers of C will approach Z. Since the error matrix decreases geometrically with successive D_k, the accuracy should improve rapidly. Once the elements of E_m are equal to zero when rounded to the desired number of decimal places then D_m will be equal to A^{-1} to the desired accuracy.

As an example of this method consider the following matrix:

$$B = \begin{bmatrix} 1 & -2 & -1 \\ 4 & -3 & 1 \\ 3 & -1 & -2 \end{bmatrix}.$$

As a first approximation of the inverse, let

$$D_0 = \begin{bmatrix} -0.3 & 0.2 & 0.2 \\ -0.5 & 0.0 & 0.1 \\ -0.2 & 0.3 & -0.3 \end{bmatrix}.$$

Form the product

$$D_0B = \begin{bmatrix} -0.3 & 0.2 & 0.2 \\ -0.5 & 0.0 & 0.1 \\ -0.2 & 0.3 & -0.3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 4 & -3 & 1 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1.1 & -0.2 & 0.1 \\ -0.2 & 0.9 & 0.3 \\ 0.1 & -0.2 & 1.1 \end{bmatrix}.$$

Subtracting this product from the identity matrix gives the error matrix

$$C = I - D_0B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.1 & -0.2 & 0.1 \\ -0.2 & 0.9 & 0.3 \\ 0.1 & -0.2 & 1.1 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.2 & -0.1 \\ 0.2 & 0.1 & -0.3 \\ -0.1 & 0.2 & -0.1 \end{bmatrix}.$$

Since the row sums of C are all less than 1, this value of D_0 will work. If six place accuracy is desired compute certain powers of C to determine how many iterations will be necessary. These powers will also serve as a check on the computations. The powers needed are

$$C^2 = \begin{bmatrix} 0.06 & -0.02 & -0.04 \\ 0.03 & -0.01 & -0.02 \\ 0.06 & -0.02 & -0.04 \end{bmatrix} \qquad C^4 = \begin{bmatrix} 0.0006 & -0.0002 & -0.0004 \\ 0.0003 & -0.0001 & -0.0002 \\ 0.0006 & -0.0002 & -0.0004 \end{bmatrix}$$

$$C^8 = \begin{bmatrix} 0.00000006 & -0.00000002 & -0.00000004 \\ 0.00000003 & -0.00000001 & -0.00000002 \\ 0.00000006 & -0.00000002 & -0.00000004 \end{bmatrix}.$$

Since C^8 equals Z to six places D_3 will equal B^{-1} to the same accuracy. To compute D_3 it is first necessary to compute D_1.

$$D_1 = (2I - D_0B)D_0 = \begin{bmatrix} 0.9 & 0.2 & -0.1 \\ 0.2 & 1.1 & -0.3 \\ -0.1 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} -0.3 & 0.2 & 0.2 \\ -0.5 & 0.0 & 0.1 \\ -0.2 & 0.3 & -0.3 \end{bmatrix}$$

and

$$D_1B = \begin{bmatrix} -0.35 & 0.15 & 0.23 \\ -0.55 & -0.05 & 0.24 \\ -0.25 & 0.25 & -0.27 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 4 & -3 & 1 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0.94 & 0.02 & 0.04 \\ -0.03 & 1.01 & 0.02 \\ -0.06 & 0.02 & 1.04 \end{bmatrix}.$$

As a check D_1B subtracted from the identity matrix should equal C^2. It is easy to see that this is true.

The computation for D_2 and D_2B gives

$$D_2 = (2I - D_1B)D_1 = \begin{bmatrix} 1.06 & -0.02 & -0.04 \\ 0.03 & 0.99 & -0.02 \\ 0.06 & -0.02 & 0.96 \end{bmatrix} \begin{bmatrix} -0.35 & 0.15 & 0.23 \\ -0.55 & -0.05 & 0.24 \\ -0.25 & 0.25 & -0.27 \end{bmatrix}$$

and

$$D_2B = \begin{bmatrix} -0.3500 & 0.1500 & 0.2498 \\ -0.5500 & -0.0500 & 0.2499 \\ -0.2500 & 0.2500 & -0.2502 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 4 & -3 & 1 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0.9994 & 0.0002 & 0.0004 \\ -0.0003 & 1.0001 & 0.0002 \\ -0.0006 & 0.0002 & 1.0004 \end{bmatrix}.$$

In comparing this with the identity matrix one sees that the difference is equal to C^4. Finally

$$D_3 = (2I - D_2B)D_2$$

$$= \begin{bmatrix} 1.0006 & -0.0002 & -0.0004 \\ 0.0003 & 0.9999 & -0.0002 \\ 0.0006 & -0.0002 & 0.9996 \end{bmatrix} \begin{bmatrix} -0.3500 & 0.1500 & 0.2498 \\ -0.5500 & -0.0500 & 0.2499 \\ -0.2500 & 0.2500 & -0.2502 \end{bmatrix}$$

$$= \begin{bmatrix} -0.35000000 & 0.15000000 & 0.24999998 \\ -0.55000000 & -0.05000000 & 0.24999999 \\ -0.25000000 & 0.25000000 & -0.25000002 \end{bmatrix}.$$

If the product D_3B is formed and subtracted from the identity matrix, the result will equal C^8. This is a final check of computational errors.

Since six place accuracy was desired D_3 can be rounded off as

$$D_3 = \begin{bmatrix} -0.350000 & 0.150000 & 0.250000 \\ -0.550000 & -0.050000 & 0.250000 \\ -0.250000 & 0.250000 & -0.250000 \end{bmatrix}.$$

In this case D_3 is the exact inverse of B.

It is interesting to note what would happen if some D_k is the exact solution so that $D_kA = I$. If this is the case then,

$$D_{k+1} = (2I - D_kA)D_k = (2I - I)D_k = ID_k = D_k.$$

Thus one criterion for halting the iteration process is that two subsequent approximations are equal. Another process for determining the number of iterations needed is the one used in the example. This was computing the powers of C until an error matrix with small enough elements is obtained.

Inversion Method for Strongly Diagonal Matrices

For certain matrices it is easy to guess an initial approximation of the inverse which leaves the row sums of the error matrix C, less than 1. These matrices, (4), pp. 187-189, are called strongly diagonal matrices. They are characterized by diagonal elements whose absolute values are larger than the sums of the absolute values of the remaining elements of each of their rows, that is, $iRSA < 2|a_{ii}|$ for each row. If each of the rows of a strongly diagonal matrix is divided by the diagonal element then the row sums of the new matrix are less than 2. If this matrix is subtracted from the identity matrix then the row sums of the resulting matrix are less than 1. Remembering the equation for the error matrix is $I - D_0A$ it follows that the matrix to pick for D_0 is

$$\begin{bmatrix} a_{11}^{-1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_{22}^{-1} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & a_{nn}^{-1} \end{bmatrix}.$$

This value for D_0 makes C the matrix described in the above paragraph whose row sums were less than one.

As an example of a strongly diagonal matrix consider the following matrix:

$$B = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -5 & 1 \\ -2 & 4 & 8 \end{bmatrix}.$$

First checking the rows gives

$$\text{row 1:} \quad 1 + 2 = 3 < 4$$

$$\text{row 2:} \quad 3 + 1 = 4 < 5$$

$$\text{row 3:} \quad 2 + 4 = 6 < 8$$

which verifies that matrix B is indeed a strongly diagonal matrix. Let

$$D_0 = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & 0.125 \end{bmatrix}.$$

Then

$$C = I - D_0A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.0 & -0.5 & -0.25 \\ -0.6 & 1.0 & -0.2 \\ -0.25 & 0.5 & 1.0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.25 \\ 0.6 & 0 & 0.2 \\ 0.25 & -0.5 & 0 \end{bmatrix}.$$

Checking the row sums of C

$$1RSC = 0.50 + 0.25 = 0.75$$
$$2RSC = 0.60 + 0.20 = 0.80$$
$$3RSC = 0.25 + 0.50 = 0.75 .$$

Since for this value of D_0 the error matrix C satisfies the property described in the introduction of this section, B^{-1} can be found using the iterative method discussed earlier in this section.

This method can be used for calculating the inverse of a matrix that is not strongly diagonal if the matrix can be transformed into a strongly diagonal matrix by interchanging rows. The first step is to place the element of largest absolute value on the diagonal. This can be done by interchanging two rows. Next, eliminate the row and column where this element is contained, and place the element of largest absolute value in the remaining submatrix on the diagonal. Continue this process until the size of the submatrix is one.

Next, check the row sums of the newly formed matrix to determine whether the matrix is strongly diagonal. If so, then its inverse can be computed using the procedure discussed in this part. To find the inverse of the original matrix, it is necessary to interchange the corresponding columns of the calculated inverse in reverse order.

Instead of interchanging rows, it would work equally well if the columns of the original matrix were interchanged. Then the corresponding rows of the calculated inverse would be interchanged in reverse order. An obvious disadvantage of this procedure is the necessity of remembering the interchange and their order.

CONCLUSION

Only a few of the many different methods of computing the inverse of a matrix are given in this report. Some of the methods not included are variations of those contained here and others are methods which can only be used on special kinds of matrices.

Solving the system of linear equations

$$AX = G$$

using the inverse of the coefficient matrix A is only one technique of solution. It becomes advantageous to use this technique only when the coefficient matrix is large or when the same coefficient matrix A is used for several different constant matrices G.

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NUMERICAL METHODS OF MATRIX INVERSION

by

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With the advent of high-speed electronic digital computers it became feasible to solve large systems of equations. Many techniques have been developed to solve these equations. One of these involves calculating the inverse of the coefficient matrix.

Using matrix notation, a system of n linear equations in n unknowns can be written in the matrix form:

$$AX = G.$$

By multiplying both sides of the equation by A^{-1}

$$A^{-1}AX = A^{-1}G,$$

$$X = A^{-1}G.$$

Thus, if the value of the inverse of the coefficient matrix is known, the solution for the system can be found by one matrix multiplication.

The purpose of this report is to consider eight different methods by which the inverse of the coefficient matrix may be calculated. For each method an effort is made to explain the method, prove its validity, and give an example using the method. Also, for most of the methods advantages and disadvantages are given. The eight different methods of matrix inversion are discussed in four sections. The first section considers matrix inversion using elementary operations. Of all the methods discussed in this paper, those methods which involve elementary operations are the most easily adapted to machine computation. Only the most general case of these methods is discussed here.

The second section considers methods of matrix inversion involving the determinant of a matrix. The three methods included use the adjoint matrix, the characteristic equation, and the trace to calculate the inverse.

The inversion of a matrix by partitioning and a special case in which the matrix is partitioned by bordering are developed in the third section.

Iterative methods of matrix inversion are considered in the fourth section. These methods are important for they can be used to improve the accuracy of inverses calculated by other methods. A basic method of iteration and a particular type of matrix for which a good starting value is easily determined are discussed.

Only a few of the many different methods of computing the inverse of a matrix are contained in this report. Some of the methods not included are variations of those contained here and others are methods which can only be used on special kinds of matrices.

Solving a system of linear equations using the inverse of the coefficient matrix is only one technique of solution. It becomes advantageous only when the coefficient matrix is large or when the same coefficient matrix A is used for several different constant matrices G.