

THE EFFECTS OF TENSIONING ON THE BUCKLING
AND VIBRATION OF CIRCULAR SAW BLADES

by 6791

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NOMENCLATURE

R	outside radius of disk
R_a	radius of central hole in disk
R_b	disk clamping radius
R_c	disk tensioning radius
R_1	inner tensioning radius in annulus approach
R_2	outer tensioning radius in annulus approach
R_m	mean radius of tensioning annulus
R_w	radial width of tensioning annulus
h	disk thickness
E	modulus of elasticity
ν	Poisson's ratio
D	plate rigidity = $Eh^3/12(1-\nu^2)$
T	kinetic energy
V	potential energy
H	Hamiltonian
t	time
r	radial coordinate
θ	angular coordinate
σ_r	radial in-plane normal stress
σ_θ	hoop in-plane normal stress
$\tau_{r\theta}$	in-plane shear stress
ϵ_r	radial strain
ϵ_θ	hoop strain

$\bar{\epsilon}_r$	mean radial strain in the tensioning annulus
$\bar{\epsilon}_\theta$	mean hoop strain in the tensioning annulus
W	lateral deflection of plate
P	magnitude of the in-plane load
P, P_1, P_2	theoretical induced tensioning stresses
γ	weight density
ϕ	stress function
n	number of nodal diameters
i	number of nodal circles
N, I	number of n and i terms at which series is truncated
w_{ni}	eigenfunction of free vibration
c_{ni}	coefficients in series expansion of W
R_{ni}	function of r in w_{ni}
$J_n(x)$	Bessel function of the first kind of order n
$Y_n(x)$	Bessel function of the second kind of order n
$I_n(x)$	modified Bessel function of the first kind of order n
$K_n(x)$	modified Bessel function of the second kind of order n
B_{ni}, C_{ni}, D_{ni}	constants in R_{ni}
k_{ni}	constants in R_{ni}
S_{ni}	constants defined by equation [18]
$Q_{ni,mj}$	constants defined by equation [19]
$Q'_{ni,mj}$	constants defined by equation [21]
$Q''_{ni,mj}$	constants defined by equation [21]
$[V]$	vector form of c_{ni}
$[SW]$	matrix form of S_{ni}
$[SM]$	matrix form of $(k_{ni}R)^4 S_{ni}$

$[QM]$	matrix form of $Q'_{ni,mj}$
$[QTP]$	matrix form of $PRQ'_{ni,mj}/D$
$P\sigma'$	stress due to in-plane load
σ''	stress other than that due to in-plane load
∇^2	Laplacean operator
∇^4	biharmonic operator
u	radial displacement
ω_0	angular velocity of rotation
λ	frequency of vibration
λ_1	fundamental frequency of disk vibration
λ_2	second natural frequency of disk vibration
λ_3	third natural frequency of disk vibration
λ_4	fourth natural frequency of disk vibration
λ_5	fifth natural frequency of disk vibration

INTRODUCTION

Many theoretical and experimental studies have been conducted concerning the stability and vibration of circular plates. Kirchhoff (15)* in 1850 showed that the transverse vibration of an unconstrained uniform thickness circular disk could be expressed in terms of nodal circles and evenly spaced nodal diameters. Rayleigh (24) reproduced much of Kirchhoff's work in 1877 and added comments of his own. The same type of disk rotating at constant speed about its axis of symmetry was analyzed by Lamb (16) in 1921. A most significant work on circular disks then appeared in 1922. This treatise by Southwell (25) considered the free transverse vibration of a thin rotating disk clamped at its center.

Stability investigations of circular plates were initiated in 1891 by Bryan (2). However, Bryan's studies and those that followed usually limited their scope to constant thickness plates with clamped or simply supported boundaries and uniformly distributed compressive or shearing forces.

More recent analyses have been practically applied toward improving the design of circular saw blades. These studies have accounted for stresses due to temperature gradients (17,19,20,22,23), tensioning (4,5,6,7,12), transverse loading (11), and concentrated in-plane loading (4,26). Saw blades in operation are acted upon by several of these loading conditions simultaneously. Unfortunately, the net effect of the total loading environment on the stability and vibration characteristics of the blade is not a

*Numbers in parentheses designate entries in Selected Bibliography.

linear combination of the individual effects of each of the loads. Therefore, a combination of loading conditions was chosen that approximates those encountered by a circular saw blade during operation.

Most of the previous authors have investigated the vibration of wood sawing blades. However, this study was undertaken to theoretically analyze the buckling and vibration of circular saw blades for rock cutting applications. The specific objective was to obtain some theoretical guidelines for the optimum tensioning of such a blade. Tensioning (pre-stressing by inducing local plastic deformation with hammers or rollers) is by no means a new idea, since it has been known and practiced for over a century. Nevertheless, adequate theoretical information has not been introduced and the process has predominately been carried out in trial and error fashion.

The constant thickness annular disk of interest is rigidly clamped at radius R_0 and subjected to a concentrated in-plane radial load, P , as shown in Figure 1. Since rock cutting blades are constantly cooled by a liquid stream, temperature gradients in the blade were assumed to be small and thermal stresses were neglected. Tensioning stresses were assumed and their effects on the critical buckling load and the vibrational frequencies of the blade were determined.

Preliminary calculations used theoretical tensioning stresses proposed by Mote (17,19). This initial stress distribution was assumed to be due to a constant radial compressive stress, $-p$, induced at the tensioning radius, R_c . Mote's work shows a definite optimum tensioning radius for improving the vibrational characteristics of wood saws. The preliminary calculations here indicated an almost identical optimum radius for buckling considerations, but the percentage increase attainable in the critical buckling load was disappointing.

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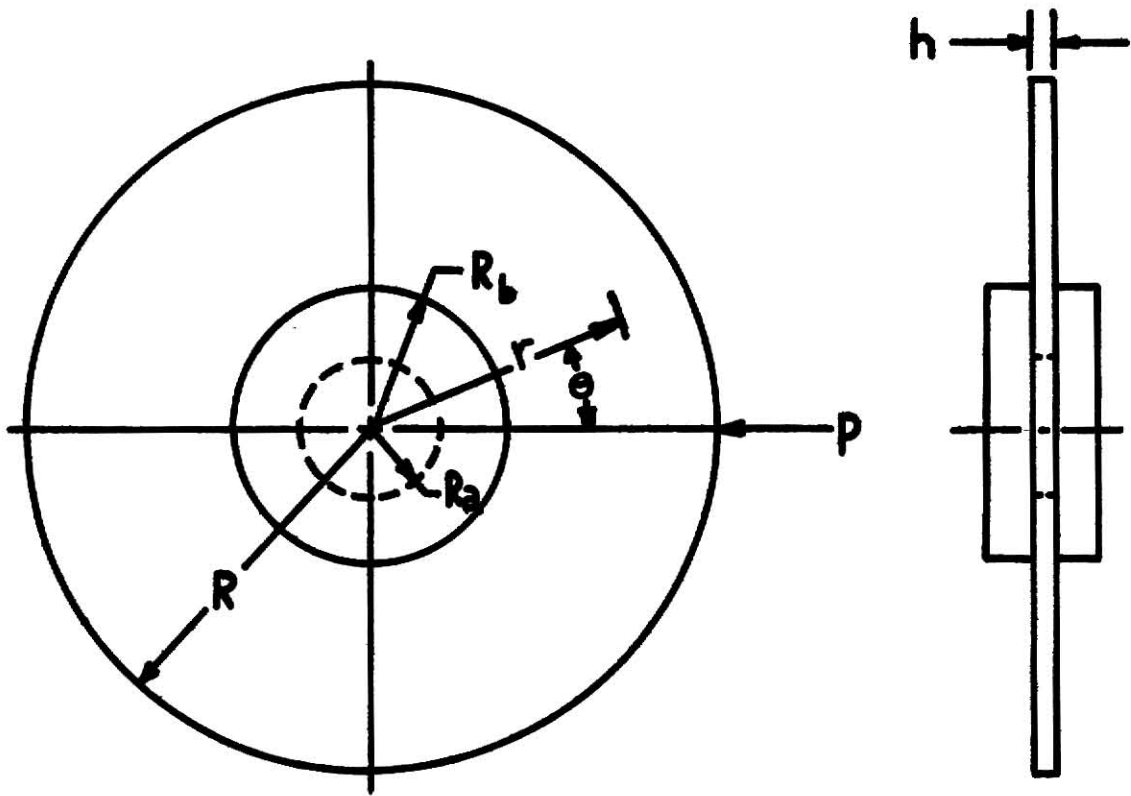


Figure 1

Sketch of blade geometry.

To generalize the tensioning theory, another parameter was added. A tensioning annulus was assumed with constant radial stresses p_1 at its inner radius R_1 and p_2 at its outer radius R_2 . Various values of p_1/p_2 were investigated and an optimum theoretical ratio for improving both the buckling and vibrational characteristics of the blade was found.

To impart more physical meaning to the tensioning parameters, the analytical relationship between the induced stresses and the corresponding average strains in the annulus was determined. Once the optimum p_1/p_2 was known, this relationship provided some indication of the type of strain desired, and thus indirectly gave some guidelines as to the proper roller geometry for tensioning saw blades.

THEORETICAL ANALYSIS

The disk was assumed to be of a homogeneous isotropic linearly elastic material and the effects of rotatory inertia and transverse shearing force were neglected. The potential energy of the deformed plate as given by Timoshenko (29,31) is then

$$\begin{aligned}
 V = & \frac{D}{2} \int_A (\nabla^2 W)^2 - 2(1-\nu) \left[W_{rr} \left(\frac{1}{r} W_r + \frac{1}{r^2} W_{\theta\theta} \right) - \left(\left(\frac{1}{r} W_\theta \right)_{,r} \right)^2 \right] dA \\
 & + \frac{h}{2} \int_A \left[\sigma_r (W_r)^2 + \sigma_\theta \left(\frac{1}{r} W_\theta \right)^2 + 2\tau_{r\theta} (W_r) \left(\frac{1}{r} W_\theta \right) \right] dA \\
 & + \frac{h}{2E} \int_A \left[(\sigma_r + \sigma_\theta)^2 - 2(1-\nu)(\sigma_r \sigma_\theta - \tau_{r\theta}^2) \right] dA.
 \end{aligned} \tag{1}$$

$W = W(r, \theta, t)$ represents the lateral deflection of the plate and σ_r , σ_θ , and $\tau_{r\theta}$ are the radial, hoop, and shear stresses acting in the plane of the plate, respectively. The three terms in V represent the strain energy of bending, the potential energy of the external loads and internal stresses as the plate undergoes a small deflection, and the energy of elastic deformation prior to any lateral deflection, in that order. The kinetic energy of the plate is

$$T = \frac{\gamma h}{2g} \int_A (W_t)^2 dA. \tag{2}$$

The governing equation of motion was formulated by Hamilton's principle, which states that the motion of a conservative system between times t_1 and t_2 proceeds in such a way that the Hamiltonian,

$$H = \int_{t_1}^{t_2} (T - V) dt,$$

is stationary with respect to arbitrary small changes in the motion which are

consistent with the constraints and vanish at t_1 and t_2 . In variational notation, Hamilton's principle is

$$\delta H = \delta \int_{t_1}^{t_2} (T-V) dt = 0. \quad [3]$$

The magnitude of the plate deflection was assumed to be small to the extent that only higher order changes in the internal stress distribution were produced. Under such an assumption, the internal stresses are independent of both time and lateral deflection and may be derived directly from the equations of elasticity. The equations of equilibrium for the in-plane stresses become

$$(\sigma_r)_{,r} + \frac{1}{r}(\tau_{r\theta})_{,\theta} + (\sigma_r - \sigma_\theta)/r = 0 \quad [4a]$$

$$\text{and} \quad (\tau_{r\theta})_{,r} + \frac{1}{r}(\sigma_\theta)_{,\theta} + 2\tau_{r\theta}/r = 0. \quad [4b]$$

The governing differential equation and natural boundary conditions of the circular plate can be obtained by directly applying the techniques of the calculus of variations to the variational equation [3]. Simplification of the governing equation by use of the equilibrium equations [4] yields

$$D\nabla^4 W - h[\sigma_r W_{rr} + \sigma_\theta(\frac{1}{r}W_r + \frac{1}{r^2}W_{\theta\theta}) + 2\tau_{r\theta}(\frac{1}{r}W_{r\theta} - \frac{1}{r^2}W_\theta)] + \frac{\gamma h}{g}W_{tt} = 0. \quad [5]$$

The exact solution of this equation of motion has been found only in the cases of uniform loading around the entire boundary. This fact, plus the availability of a complete set of orthogonal functions satisfying all of the disk boundary conditions except at the point of application of the in-plane force, makes it desirable to obtain an approximate solution to the problem by the Rayleigh-Ritz method.

Assuming a harmonically varying displacement

$$W(r, \theta, t) = w(r, \theta)\cos(\lambda t + \psi)$$

and taking the time interval $t_2 - t_1$ equal to one period, $2\pi/\lambda$, the equation

for the Hamiltonian reduces to

$$H = \frac{\gamma m \lambda}{2g} \int_A [w(r, \theta)]^2 dA - \frac{\pi}{\lambda} (V_1 + V_2 + 2V_3) \quad [6]$$

where V_1 , V_2 , and V_3 refer to the three terms in the potential energy in the order previously given. The Rayleigh-Ritz energy method was then applied in the following steps to find the natural frequencies of vibration of the plate. First, the deflection of the plate was expressed as an infinite series expansion of a set of functions having undetermined coefficients. Each term of the expansion satisfied the geometric boundary conditions of the problem, eliminating the need for constraints. Then the Hamiltonian [6] was minimized with respect to the undetermined coefficients, once the in-plane stresses were specified. The minimization led to an eigenvalue problem for the frequencies of vibration.

Since in practice the infinite series expansion for the deflection must be truncated after a finite number of terms, the Rayleigh-Ritz method will not yield the exact frequencies in most cases. It can be shown that truncation of the series stiffens the system, so that the solutions obtained are upper bounds for the true eigenvalues.

To determine the critical value of the in-plane load, the buckling load, a similar procedure was followed. For this static problem, Hamilton's principle reduced to the principle of minimum potential energy. Instead of minimizing the Hamiltonian, the potential energy was made a minimum.

The rate of convergence in the Rayleigh-Ritz solution procedure is greatly affected by the choice of generating functions for the expansion of the deflection. Ideally, these functions should satisfy both the natural and the geometric boundary conditions and form a complete orthogonal set.

In the manner of St. Cyr (26) and DuBois (4), the lateral deflection of

the disk was represented as

$$w(r, \theta) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} c_{ni} w_{ni}(r, \theta) \quad [7]$$

where n and i refer to the number of nodal diameters and nodal circles, respectively, and w_{ni} represents an eigenfunction of free vibration of the disk. The eigenfunctions are separable into functions of r and θ only.

$$w_{ni}(r, \theta) = R_{ni}(r) \cos n\theta \quad [8]$$

$$\text{where } R_{ni}(r) = J_n(k_{ni}r) + B_{ni}Y_n(k_{ni}r) + C_{ni}I_n(k_{ni}r) + D_{ni}K_n(k_{ni}r). \quad [9]$$

The B 's, C 's, D 's, and k 's are constants while $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$ are Bessel functions as defined in the nomenclature. As required by the Rayleigh-Ritz technique, each term in the assumed series representation of the deflection satisfies the geometric boundary conditions of zero deflection and zero slope at the clamping radius. The eigenfunctions of free vibration were chosen for the series because they not only fulfill this requirement, but also satisfy the natural boundary conditions of the free edge except for the point at which the concentrated radial load is applied. DuBois (4) describes the calculation of the constants in the series, and St. Cyr (26) gives tabulated values for several clamping ratios. Southwell (25) also treats this free vibration problem in detail.

To complete the evaluation of the Hamiltonian, the stresses were formulated in terms of the in-plane load and the tensioning parameters. The stress distribution in the disk was obtained by superimposing the stress distribution due to tensioning upon that due to the in-plane load. In a few cases, the centrifugal stresses due to rotation of the disk were also added to determine the magnitude of their effects on the buckling and vibration characteristics of the disk. Residual stresses from the manufacturing process were neglected.

The equilibrium equations [4] were satisfied by choosing a stress

function in polar coordinates, ϕ , such that

$$\sigma_r = \frac{1}{r} \phi_r + \frac{1}{r^2} \phi_{\theta\theta} , \quad [10a]$$

$$\sigma_\theta = \phi_{rr} , \quad [10b]$$

$$\text{and } \tau_{r\theta} = -\left(\frac{1}{r} \phi_\theta\right)_{,r} . \quad [10c]$$

To yield a possible stress distribution, ϕ must satisfy the compatibility equation

$$\nabla^4 \phi = 0. \quad [11]$$

The lengthy general solution to equation [11] for the two-dimensional case in polar coordinates is listed by Timoshenko and Goodier (30) on page 133 of their latest edition. To find the stress distribution due to the concentrated load, the constants in the general solution were calculated by imposing the disk boundary conditions. The stress boundary condition at the outer edge was found by first considering the Fourier series expansion of a uniformly distributed load over an arc of length 2α and intensity such that the total magnitude of the load was P . Then α was allowed to approach zero and the resulting circumferential stress distribution due to the concentrated load became

$$\sigma_r|_{r=R} = \frac{-P}{\pi R h} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta \right]. \quad [12]$$

The other boundary conditions were zero shear stress at the outer edge, zero radial displacement at the clamping radius, and zero tangential displacement at the clamping radius.

With these boundary conditions, the stress distribution due to the concentrated load, P , was calculated by St. Cyr (26) and DuBois (4) as

$$\sigma_r = \frac{-P}{\pi R h} \sum_{k=0}^{\infty} \left[a_k^1 \left(\frac{r}{k}\right)^{k-2} + a_k^2 \left(\frac{r}{k}\right)^k + a_k^3 \left(\frac{r}{k}\right)^{-k-2} + a_k^4 \left(\frac{r}{k}\right)^{-k} \right] \cos k\theta, \quad [13a]$$

$$\sigma_\theta = \frac{-P}{\pi R h} \sum_{k=0}^{\infty} \left[b_k^1 \left(\frac{r}{k}\right)^{k-2} + b_k^2 \left(\frac{r}{k}\right)^k + b_k^3 \left(\frac{r}{k}\right)^{-k-2} + b_k^4 \left(\frac{r}{k}\right)^{-k} \right] \cos k\theta, \quad [13b]$$

$$\tau_{r\theta} = \frac{-P}{\pi R h} \sum_{k=0}^{\infty} \left[c_k^1 \left(\frac{r}{k}\right)^{k-2} + c_k^2 \left(\frac{r}{k}\right)^k + c_k^3 \left(\frac{r}{k}\right)^{-k-2} + c_k^4 \left(\frac{r}{k}\right)^{-k} \right] \cos k\theta, \quad [13c]$$

where $K_1 = R_b/R$, $K_2 = (3-\nu)/(1+\nu)$, $K_3 = (1+\nu)/(1-\nu)$,

$$a_0^1 = -b_0^1 = K_1^2(1/K_3 + K_1^2)/2,$$

$$a_0^2 = b_0^2 = K_3(1/K_3 + K_1^2)/2,$$

$$a_0^3 = a_0^4 = b_0^3 = b_0^4 = 0,$$

$$a_1^1 = 1,$$

$$b_1^1 = c_1^1 = 0,$$

$$a_1^2 = b_1^2/3 = c_1^2 = (1-\nu)(K_3K_1^2 + 1)/4(1 + K_2K_1^4),$$

$$a_1^3 = -b_1^3 = c_1^3 = -K_1^2(1-\nu)(K_3 - K_2K_1^2)/4(1 + K_2K_1^4),$$

$$a_1^4 = b_1^4 = c_1^4 = -(1-\nu)/4,$$

and for k greater than 1:

$$d_k = 2K_2(K_1^k + K_1^{-k})^2 + (k^2 - 1)(K_1 - K_1^{-1})^2 + (K_2K_1 - K_1^{-1})^2,$$

$$e_k = K_2K_1^{-2k} + k + 1 - kK_1^{-2},$$

$$f_k = kK_1^{-2} + K_2K_1^{2k} - k + 1,$$

$$a_k^1 = -b_k^1 = -c_k^1 = [kK_2K_1^{-2k} + (k^2 - 1 + K_2^2)K_1^2 - k(k-1)]/d_k,$$

$$a_k^2 = -(k-2)e_k/d_k,$$

$$b_k^2 = (k+2)e_k/d_k,$$

$$c_k^2 = ke_k/d_k,$$

$$a_k^3 = -b_k^3 = c_k^3 = [(k^2 - 1 + K_2^2)K_1^2 - k(k+1) - kK_2K_1^{2k}]/d_k,$$

$$a_k^4 = (k+2)f_k/d_k,$$

$$b_k^4 = -(k-2)f_k/d_k,$$

and $c_k^4 = kf_k/d_k.$

The stress distribution resulting from tensioning is highly dependent upon the method of tensioning and the location and amount of plastic deformation. The currently practiced tensioning method consists of rolling the disk at some radius, R_c , as sketched in Figure 2. Mote (17,19) assumed, with some experimental justification, that the radially symmetric initial stress distribution due to such tensioning was the result of a constant radial

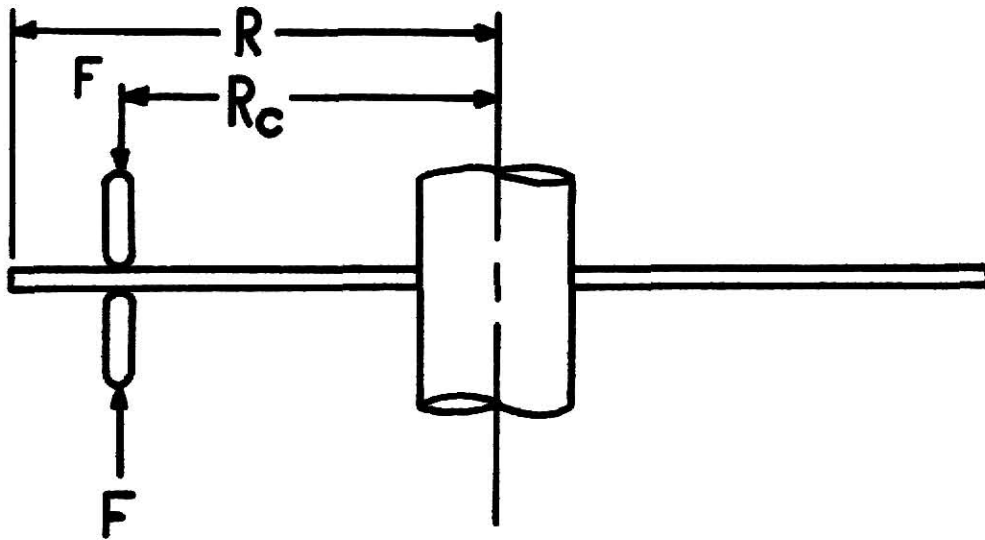


Figure 2

Assumed tensioning method.

compressive stress, $-p$, imposed at R_c . This is the same type of stress distribution that occurs in the formation of a composite disk, and is tabulated by Den Hartog (3) as shown below.

$$\tau_{r\theta} = 0 \text{ everywhere.} \quad [14a]$$

For r between R_a and R_c :

$$\sigma_r = p \left[\frac{R_a^2 R_c^2}{r^2 (R_c^2 - R_a^2)} - \frac{R_c^2}{(R_c^2 - R_a^2)} \right] \quad [14b]$$

$$\text{and } \sigma_\theta = -p \left[\frac{R_a^2 R_c^2}{r^2 (R_c^2 - R_a^2)} + \frac{R_c^2}{(R_c^2 - R_a^2)} \right]. \quad [14c]$$

For r between R_c and R :

$$\sigma_r = p \left[\frac{R_c^2}{(R^2 - R_c^2)} - \frac{R^2 R_c^2}{r^2 (R^2 - R_c^2)} \right] \quad [14d]$$

$$\text{and } \sigma_\theta = p \left[\frac{R_c^2}{(R^2 - R_c^2)} + \frac{R^2 R_c^2}{r^2 (R^2 - R_c^2)} \right]. \quad [14e]$$

The theoretical initial stress distribution derived above is difficult or impossible to obtain physically, since the rollers cause plastic deformation in an annulus of finite width instead of at just one radius alone. Thus, an annulus approach was adopted to more closely model the physical situation. Two constant radial stresses, p_1 at inner annulus radius R_1 and p_2 at outer annulus radius R_2 , were assumed. For the method of tensioning considered, p_2 appears to be limited to compression while p_1 could conceivably become tensile. Thus, the only constraint imposed on the stresses was that p_2 remain compressive. The resultant tensioning stress distribution in the disk was determined by applying equations [14] for p_1 and p_2 separately and superimposing the two results.

Centrifugal stresses due to rotation at angular speed ω_0 are always tensile. There is no shear stress, and the normal stresses are given by Den Hartog (3) as

$$\sigma_r = \frac{(3+\nu)\gamma\omega_0^2}{8g} (R^2 + R_a^2 - R^2 R_a^2 / r^2 - r^2) \quad [15a]$$

$$\text{and } \sigma_\theta = \frac{(3+\nu)\gamma\omega_0^2}{8g} [R^2 + R_a^2 + R^2 R_a^2 / r^2 - r^2(1+\nu)/(3+\nu)]. \quad [15b]$$

Substituting the stress distributions and the assumed deflection into [6] transforms the Hamiltonian into a function of the c_{ni} 's with parameters P and λ . Necessary and sufficient conditions for an extremum are that

$$\frac{\delta H}{\delta c_{ni}} = 0 \quad \text{for } n, i = 0, 1, 2, \dots \quad [16]$$

Performing the differentiation and applying the orthogonality relations for the w_{ni} 's as given by St. Cyr (26) results in a set of linear homogeneous equations for the c_{ni} 's.

$$c_{ni} [(k_{ni} R)^4 - \gamma h \lambda^2 R^4 / g D] S_{ni} + \frac{PR}{D} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{mj} Q_{ni,mj} = 0 \quad [17]$$

$$\text{where } S_{ni} = (1/R^2) \int_A w_{ni}^2 dA \quad \text{and} \quad [18]$$

$$Q_{ni,mj} = Q_{mj,ni} = \frac{Rh}{P} \int_A [\sigma_r(w_{mj}),_r(w_{ni}),_r + \sigma_\theta(w_{mj}),_\theta(w_{ni}),_\theta(1/r^2) + \tau_{r\theta}(w_{mj}),_r(w_{ni}),_\theta(1/r) + \tau_{\theta r}(w_{ni}),_r(w_{mj}),_\theta(1/r)] dA. \quad [19]$$

A numerical approximation to the deflection was obtained by terminating the series expansion at $n = N$ and $i = I$. It was advantageous for computational purposes to break up the stresses into two parts as described below. Let

$$\sigma = P\sigma' + \sigma'' \quad [20]$$

where σ' and σ'' are functions of r and θ only. Similarly, let

$$Q_{ni,mj} = Q'_{ni,mj} + PQ''_{ni,mj}. \quad [21]$$

Using [20] and [21] and defining

$$V(n*N+i+1) = c_{ni}, \quad [22a]$$

$$SM(n*N+i+1, n*N+i+1) = (k_{ni}R)^4 S_{ni}, \quad [22b]$$

$$SW(n*N+i+1, n*N+i+1) = S_{ni}, \quad [22c]$$

$$QM(n*N+i+1, m*N+j+1) = Q'_{ni,mj}, \quad [22d]$$

$$\text{and } QTP(n*N+i+1, m*N+j+1) = PRQ''_{ni,mj}/D, \quad [22e]$$

equation [17] can be written in matrix form as

$$[SM][V] + (PR/D)[QM][V] + [QTP][V] = (\gamma h \lambda^2 R^4 / Dg)[SW][V]. \quad [23]$$

To find the natural frequencies of vibration, this matrix eigenvalue problem was manipulated into the form

$$[SW][SM + (PR/D)QM + QTP]^{-1}[V] = (gD/\gamma h \lambda^2 R^4)[V] \quad [24]$$

from which the eigenvalues, $gD/\gamma h \lambda^2 R^4$, and eigenvectors, $[V]$, were calculated by a standard computer routine.

For the buckling loads, equation [17] still applies with $\lambda^2 = 0$. This corresponds to minimum potential energy. The problem was put into the form

$$(D/PR)[V] = -[QM][QTP + SM]^{-1}[V] \quad [25]$$

with eigenvalues D/PR and eigenvectors $[V]$.

NUMERICAL CALCULATIONS

Matrices SM, SW, QM, and QTP were calculated by a high speed digital computer. The series expansion for the deflection was truncated at $N=I=3$, resulting in a 16 element eigenvector and 16 by 16 matrices. Even so, the computer calculations were rather lengthy, since they involved evaluation of Bessel functions, numerical integration, and many matrix operations. Only a few comments on the details of the computations will be given here, but DuBois (4) has a more complete account of some of the required matrix calculations.

Diagonal matrices SM and SW depend solely upon the disk geometry, since it can be shown that S_{ni} varies only with R_b and R and is independent of the stresses. The symmetric matrix QM is independent of the tensioning stresses and was calculated from equation [19] using the stresses $P\sigma'$ due to the in-plane load alone. Symmetric matrix QTP is independent of the in-plane load. The PQ" portion of QTP was calculated from equation [19] using stresses σ'' due to loads other than P. Required integrations were done numerically by Romberg integration.

The computer program of DuBois (4) was employed to calculate the SW, SM, QM, and QTP matrices and solution of the resulting eigenvalue problems was then facilitated by a subprogram in the IBM Scientific Subroutine Package. The use of 16 terms in the series expansion was felt to be a reasonable choice. That number should give an adequate representation of the deflection while holding the number of arithmetic operations within a practical bound.

RESULTS

The significant buckling load results using Mote's (17,19) tensioning theory are plotted in Figure 3, with the complete data listing in Table 1 in the Appendix. The peak buckling loads for each tensioning radius, as shown in Figure 4, clearly indicate the optimum location of $R_c = 9.66$ inches. The desired non-dimensional tensioning radius for buckling is then $R_c/R = .878$, which closely agrees with the optimum of .87 found by Mote (17,19) for vibration. The location of the tensioning is obviously very critical, as seen from Figure 4. Some vibrational frequency data using this preliminary theory are shown in Table 2, and Table 3 gives the frequencies and buckling loads of the untensioned blade for comparison.

As previously discussed, the remainder of the calculations were carried out using the tensioning annulus approach with induced stresses p_1 and p_2 . The average strains in the annulus were calculated in terms of p_1 and p_2 as shown below.

$$\bar{\epsilon}_\theta = \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \epsilon_\theta \, dr = \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{u}{r} \, dr \quad [26]$$

$$\text{and } \bar{\epsilon}_r = \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \epsilon_r \, dr = \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{du}{dr} \, dr = \frac{1}{R_2 - R_1} u \Big|_{R_1}^{R_2}. \quad [27]$$

These average strains were evaluated with the help of Den Hartog's (3) expressions for the radial displacement, u , in a composite disk. Using .3 for Poisson's ratio, the ratio of the average strains in the annulus becomes

$$\frac{\bar{\epsilon}_\theta}{\bar{\epsilon}_r} = \frac{p_1(1.3R_1R_2^2 - .7R_1^3 - .6R_1^2R_2) + p_2(1.3R_1^2R_2 - .7R_2^3 - .6R_2^2R_1)}{p_1(2R_1^2R_2 - .7R_1^3 - 1.3R_1R_2^2) + p_2(2R_2^2R_1 - .7R_2^3 - 1.3R_2R_1^2)}. \quad [28]$$

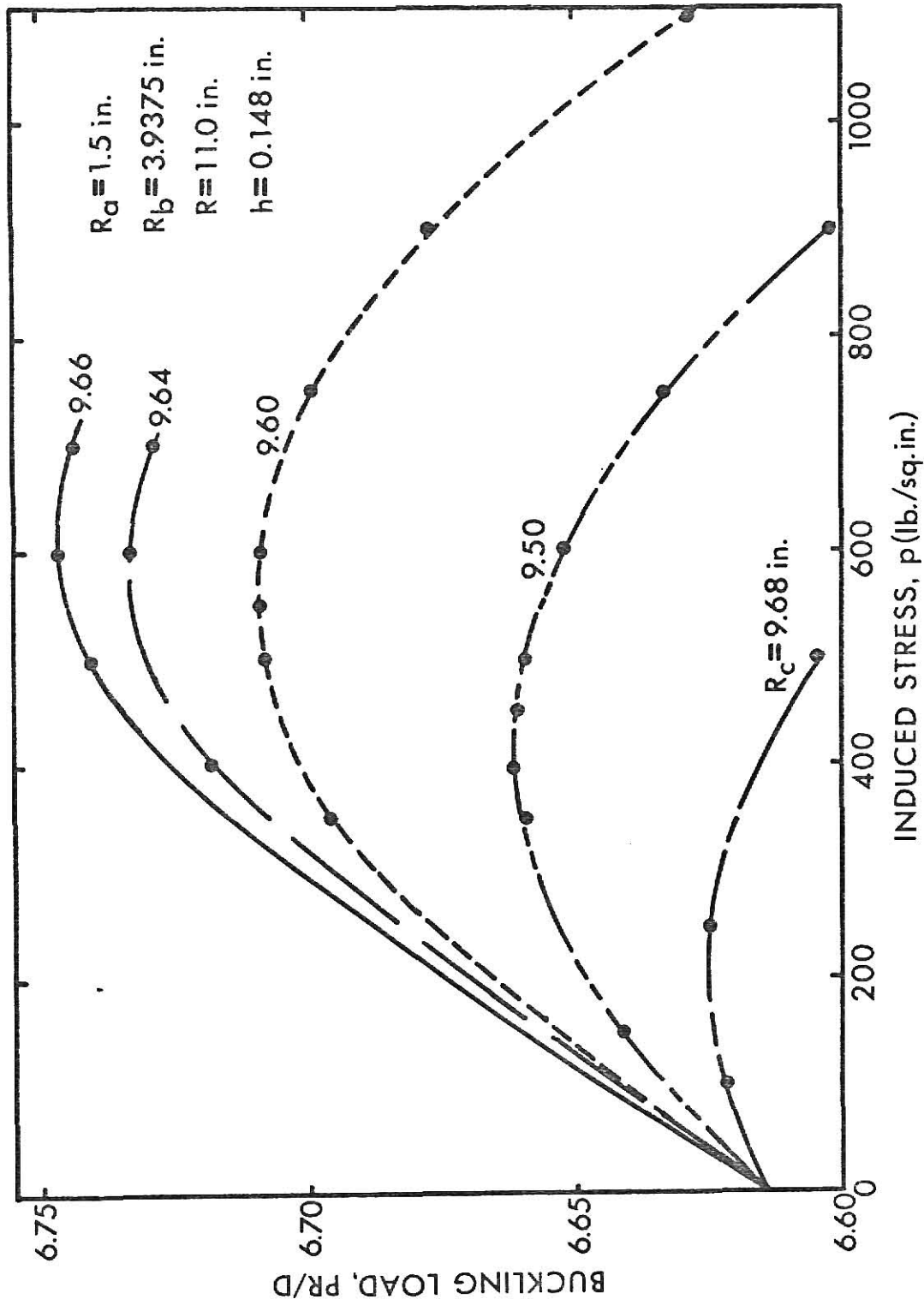


Figure 3

Buckling loads for one tensoning radius.

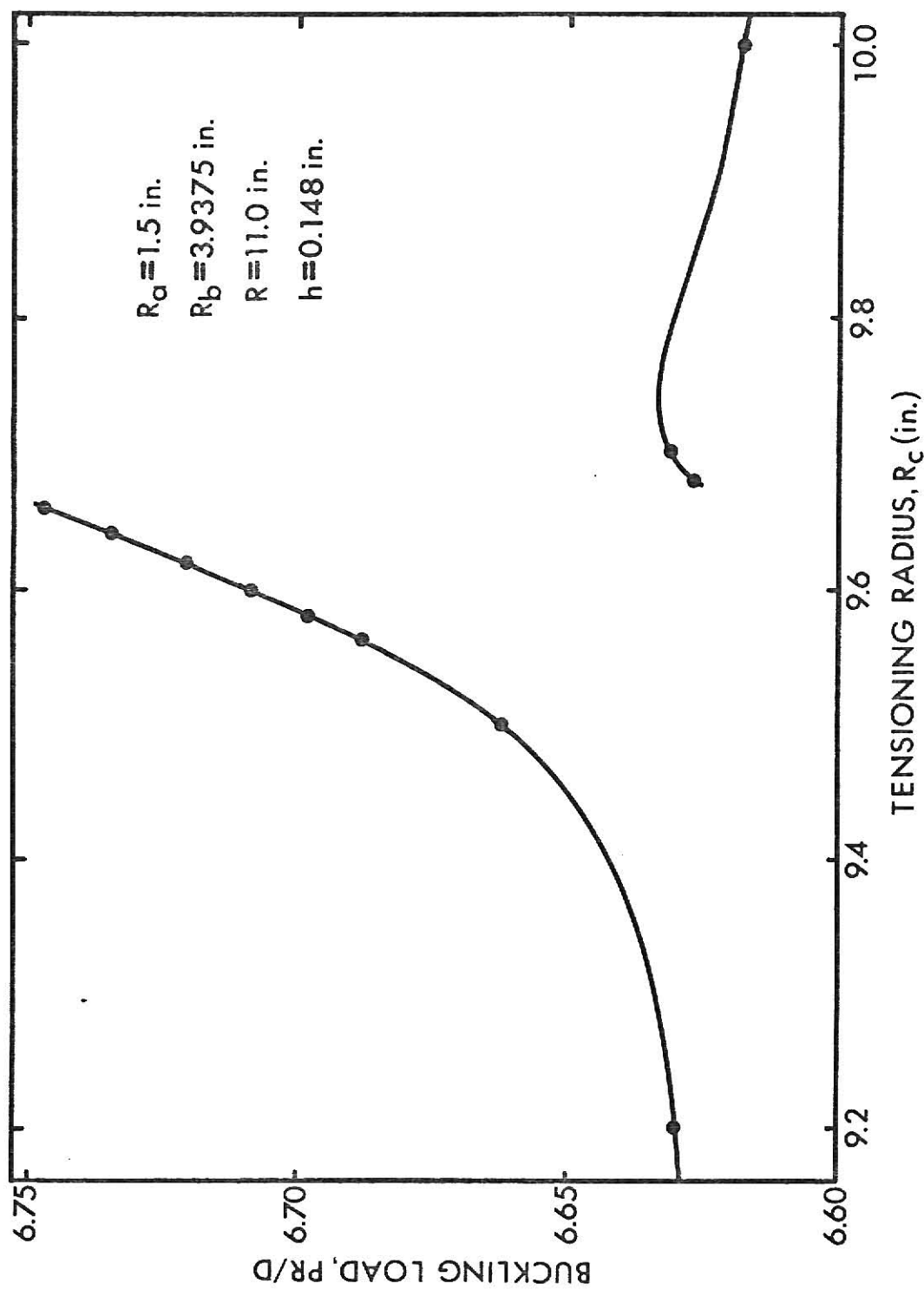


Figure 4

Peak buckling loads for the tensioning radii of interest.

A representative annulus width of one-half inch was selected as a starting point for the calculations and the mean radius was taken as 9.66 inches due to the findings in the preliminary calculations. Figure 5 shows the resulting relationship between the average strains and the induced stresses. Numerical results of the computer calculations are depicted graphically in Figures 6 through 13, with complete data tables in the Appendix. Time did not permit the investigation of other annulus dimensions.

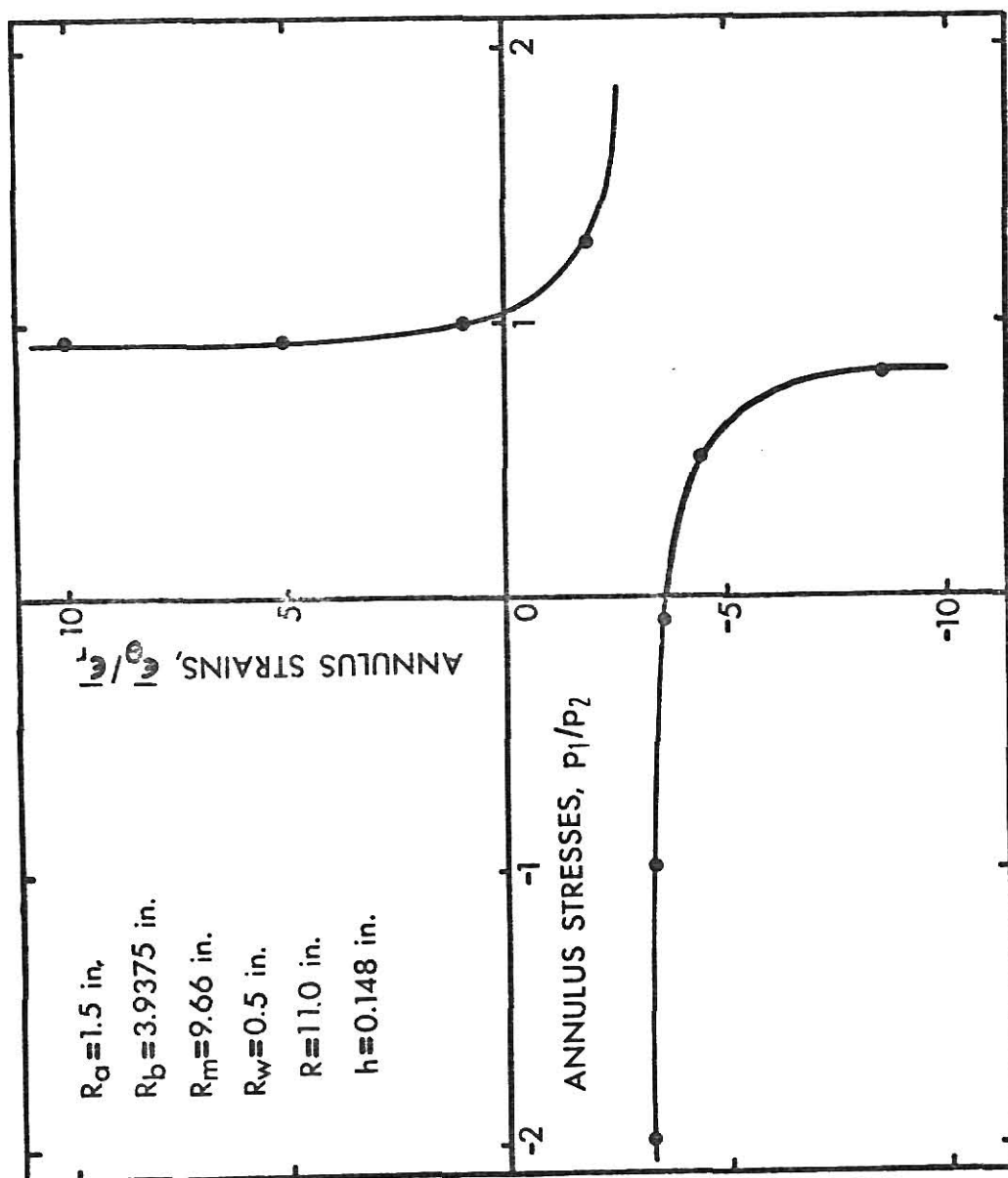


Figure 5
Average strains in the annulus.

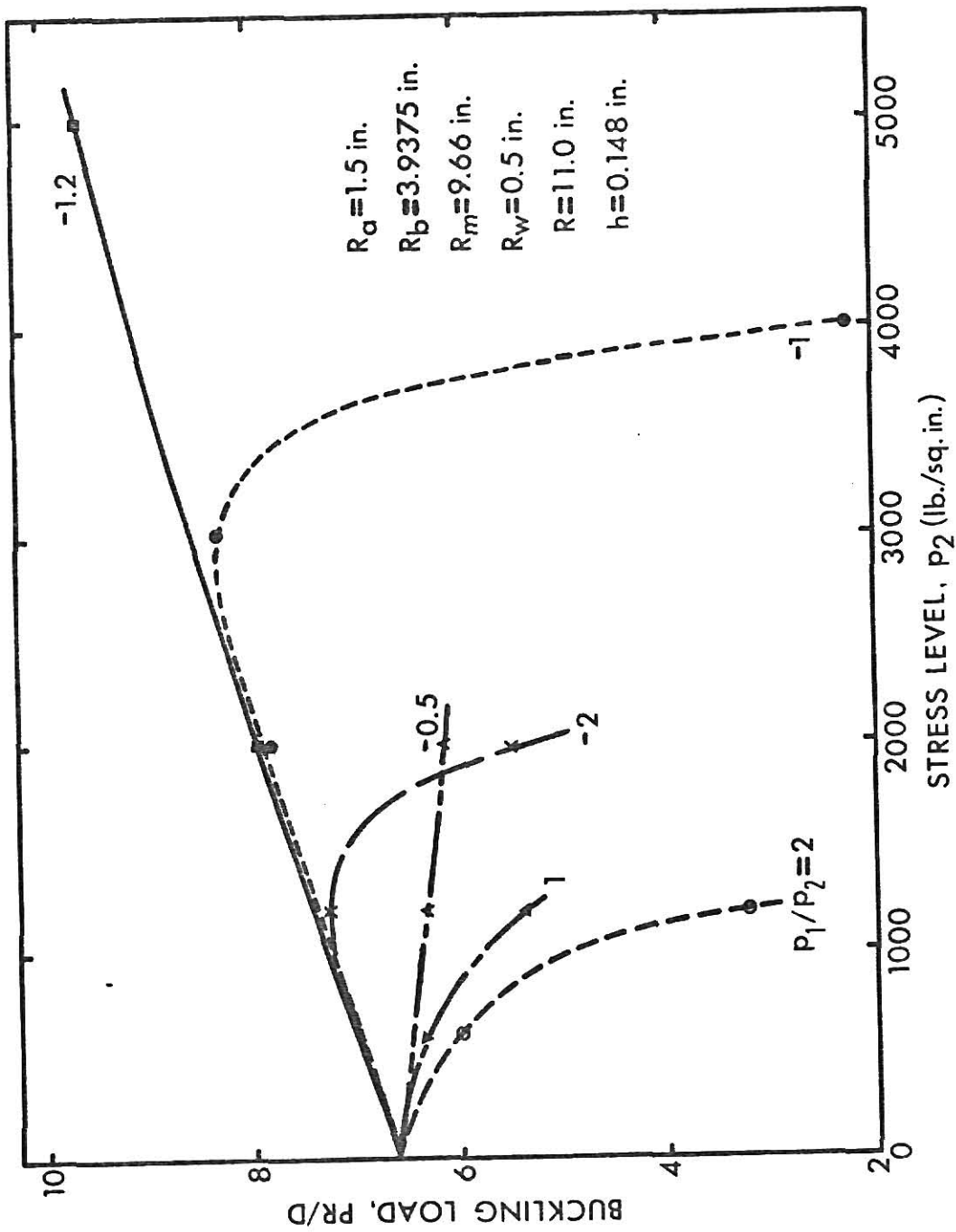


Figure 6

Buckling loads of tensioned disk.

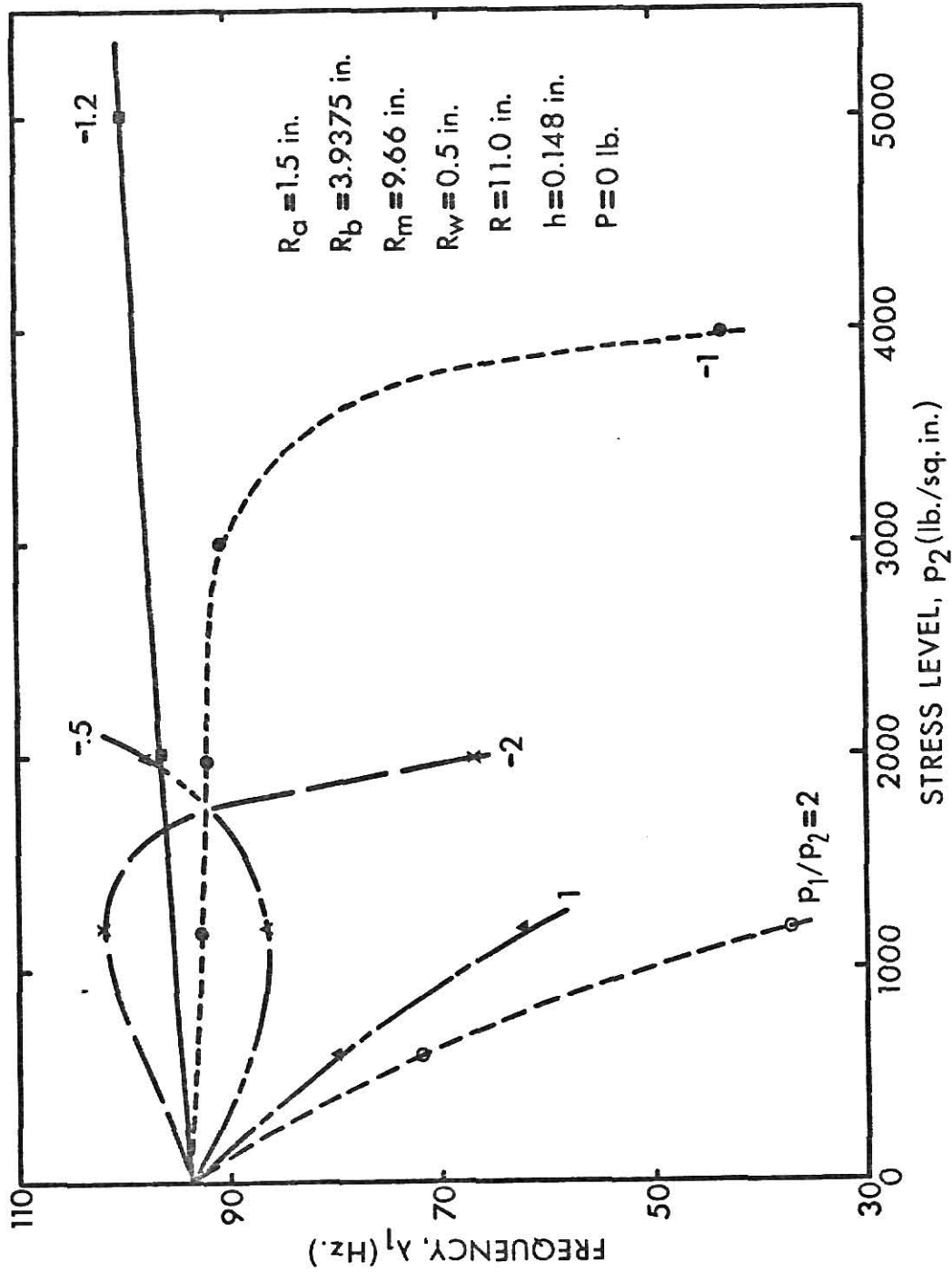


Figure 7

Fundamental frequencies of tensioned disk.

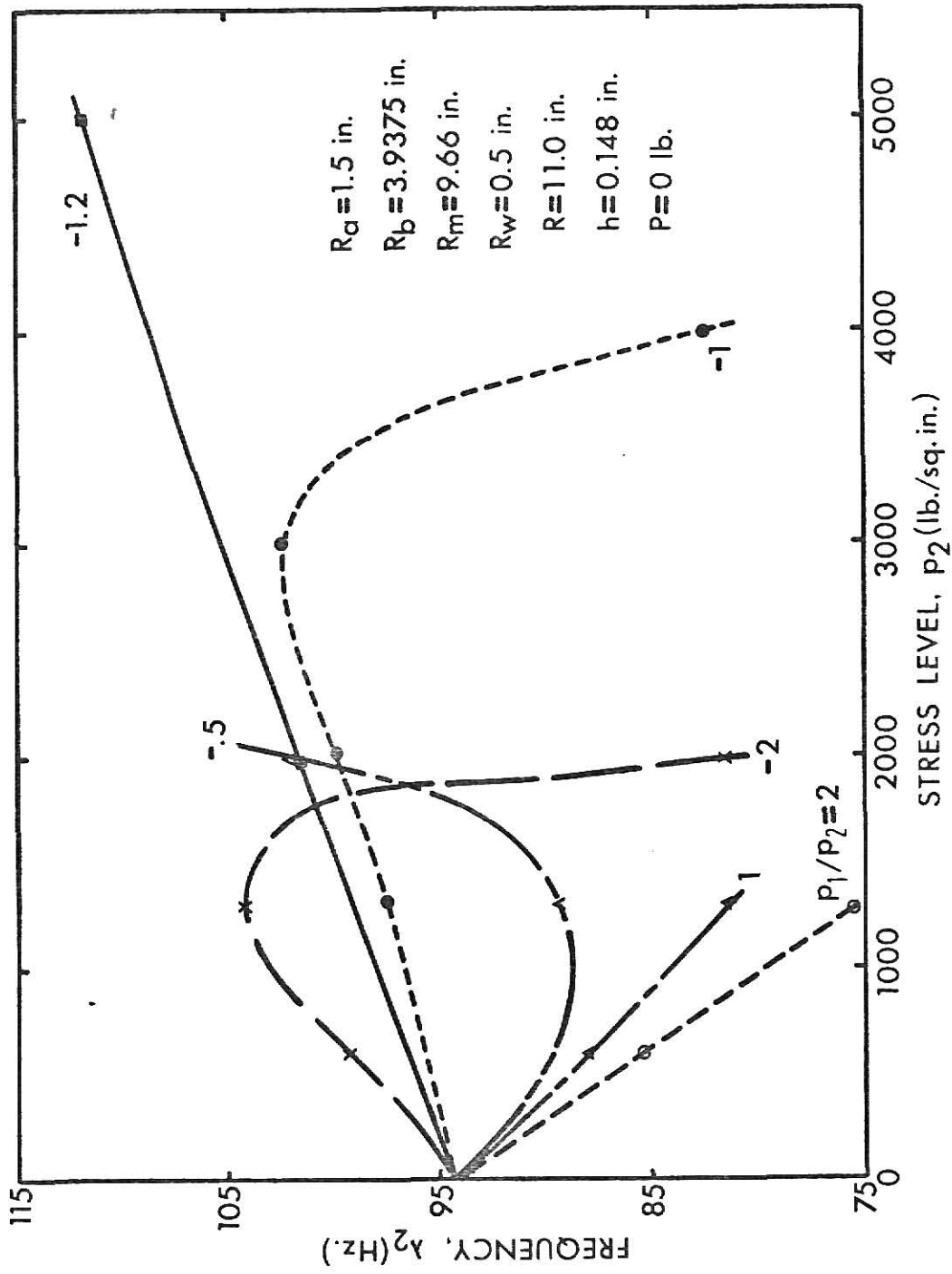


Figure 8
Second natural frequencies of tensioned disk.

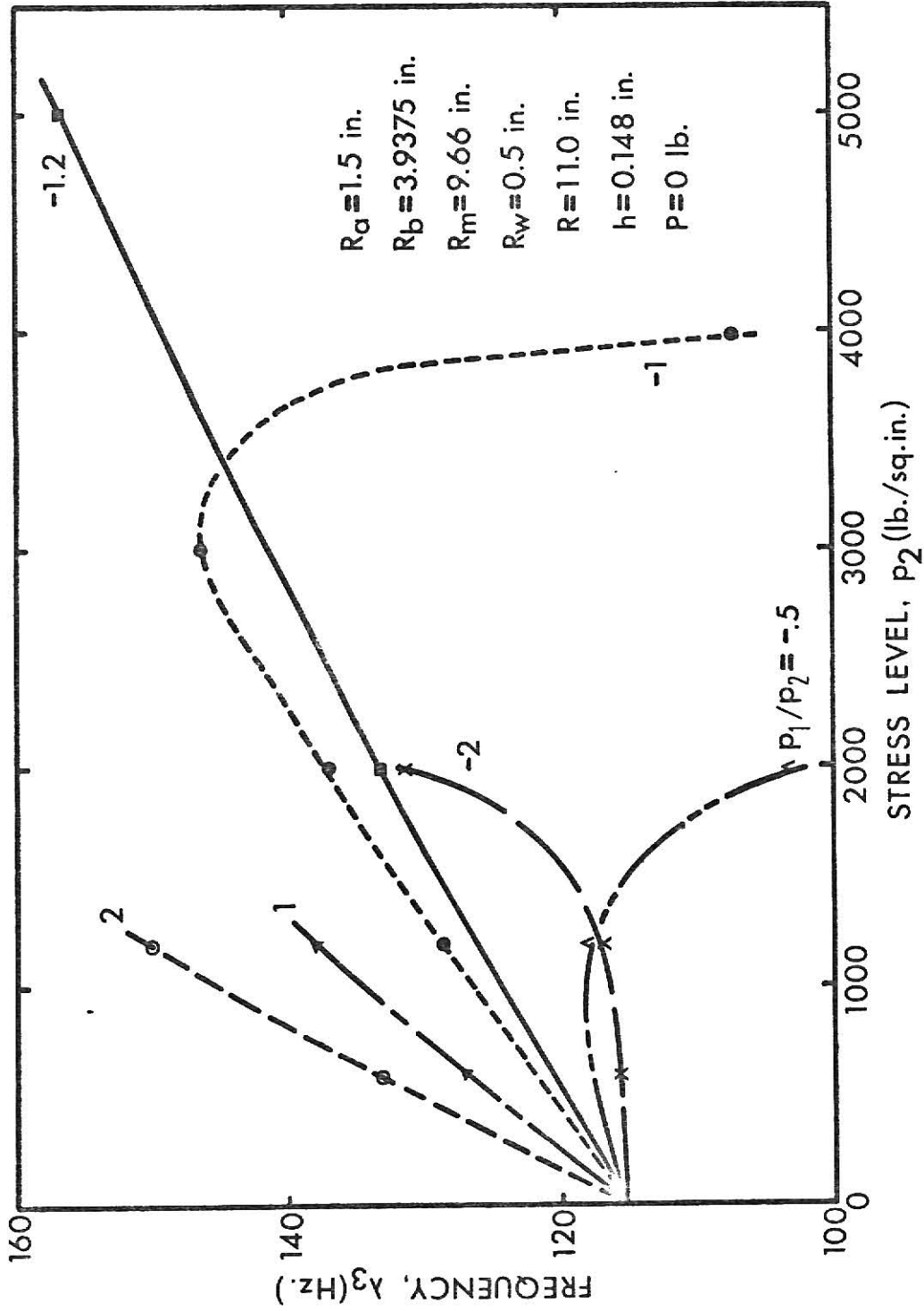


Figure 9

Third natural frequencies of tensioned disk.

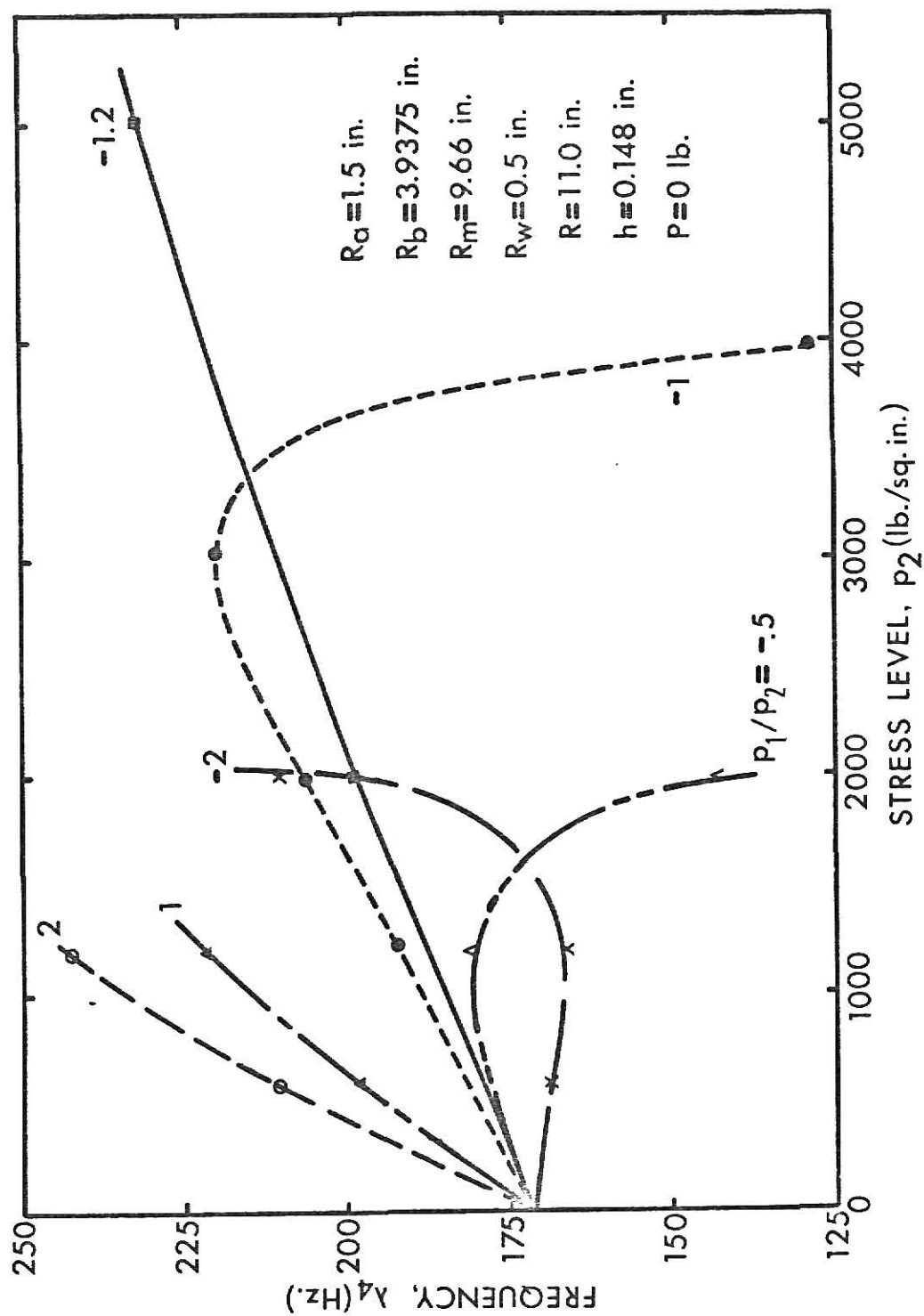


Figure 10

Fourth natural frequencies of tensioned disk.

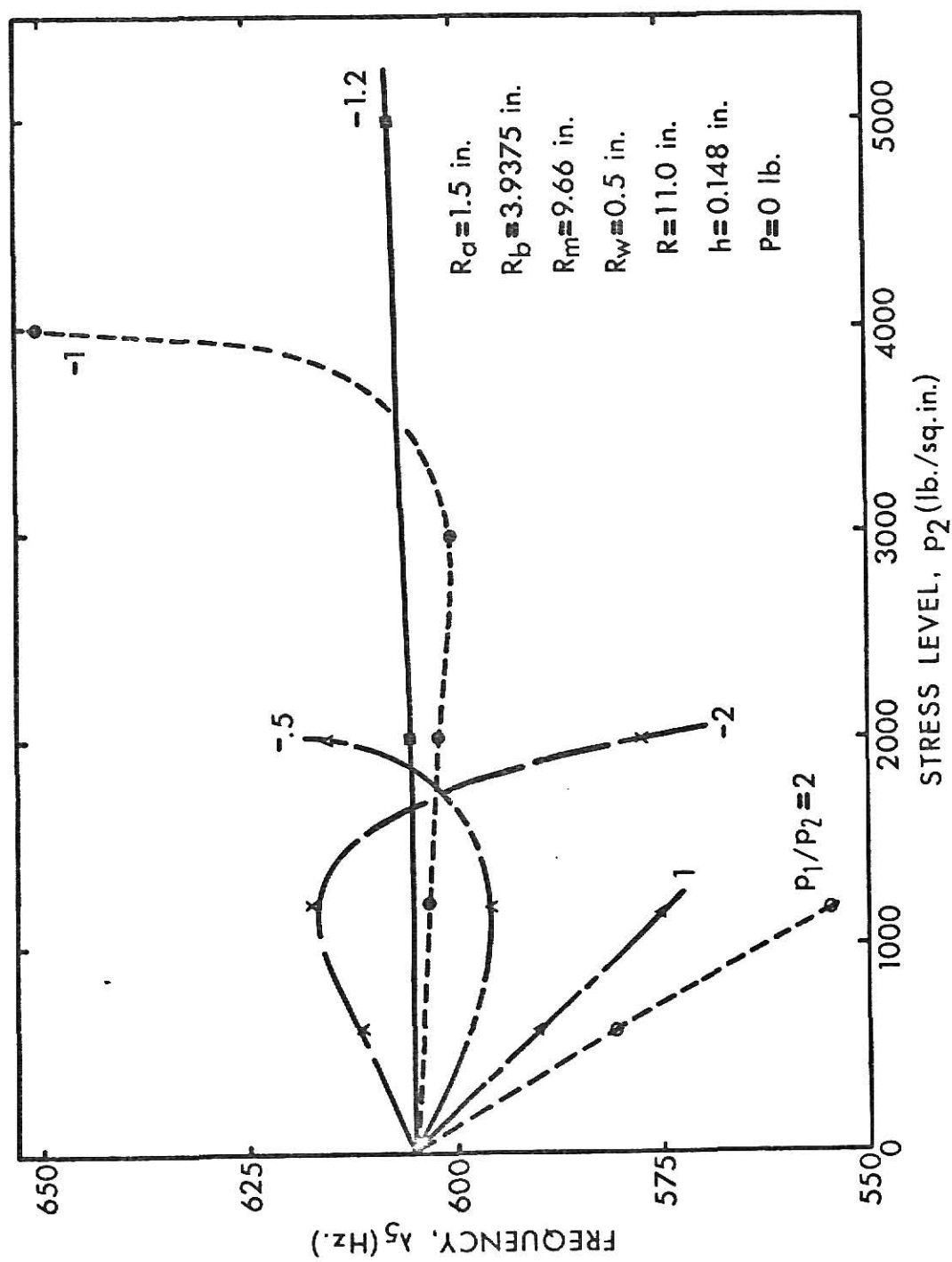


Figure 11

Fifth natural frequencies of tensioned disk.

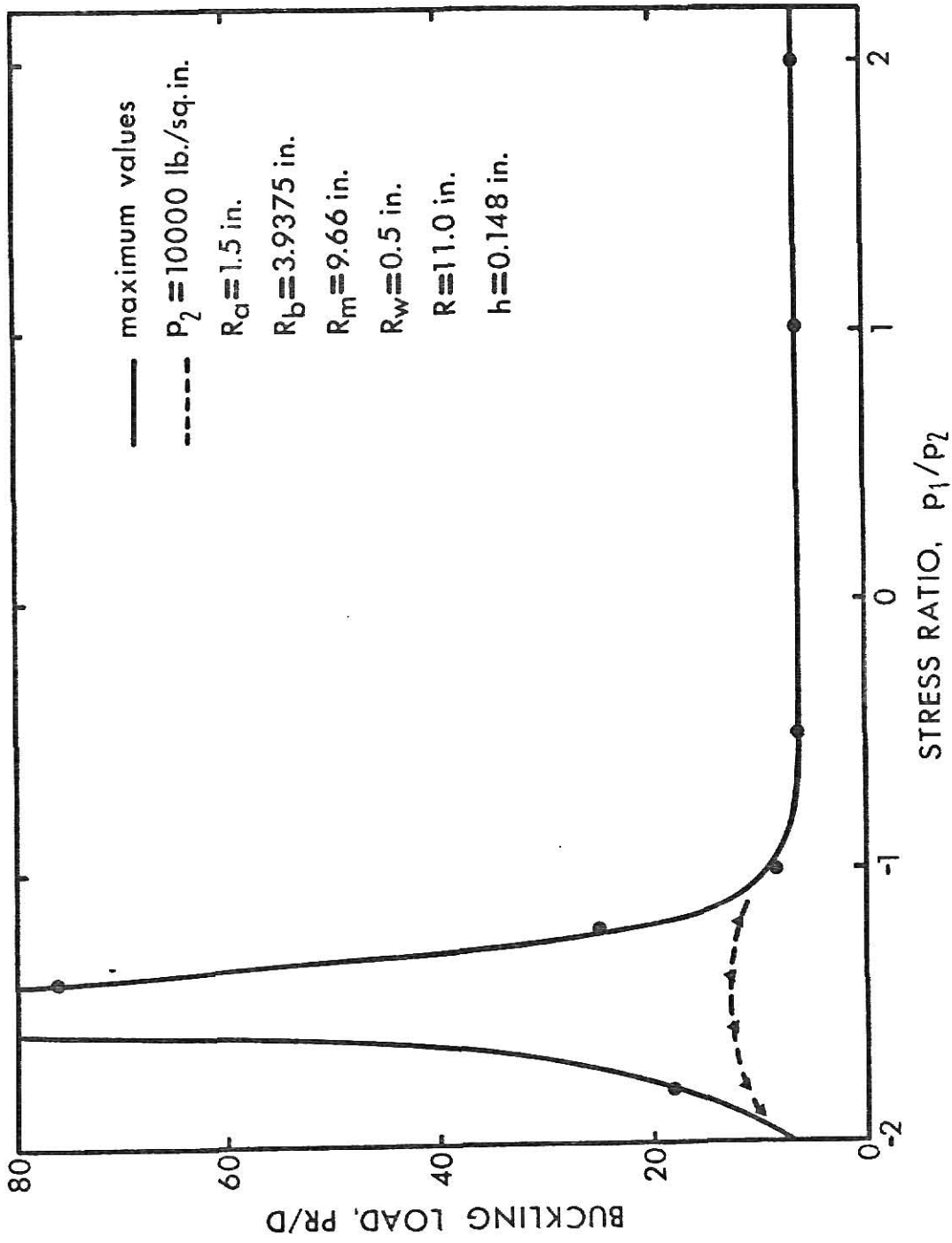


Figure 12
 Peak buckling loads for various ratios of induced stress.

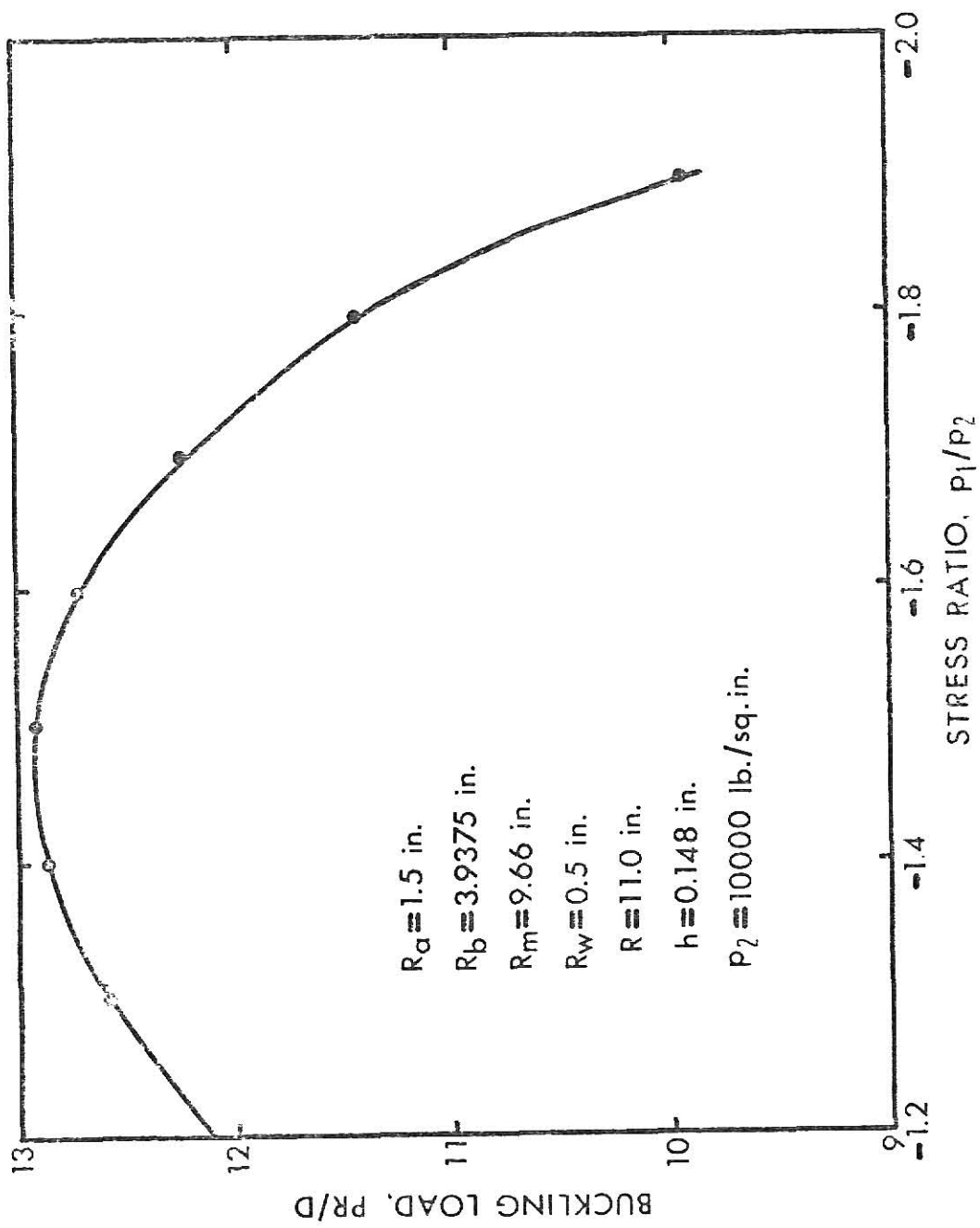


Figure 13
Optimum tensioning range for buckling.

CONCLUSIONS

The computer routine was checked for accuracy by comparing test values with those given by Vogel and Skinner (32). Agreement was very close for the test case. The eigenvectors of the first five modes of vibration were also checked for orthogonality in each of the runs.

Results presented in the previous section show that the most effective mean tensioning radius is near the outer edge of the disk at about $R_m/R = 0.87$ to 0.88. The data show that this radial location is very critical for maximum improvement in disk stability.

The output from the annulus approach indicates that it is highly desirable to induce tensile stress at the inside of the roller, especially from the buckling standpoint. The magnitude of this stress should be about one and one half times that of the compressive stress induced at the outside of the roller. Theoretically, the critical buckling load can be increased almost without limit if initial stresses of sufficient amount are induced in this proportion. An upper bound to the magnitude of the initial stresses could be found by considering an appropriate yield criterion for the disk. Obviously, most of the tensioning stresses listed in Table 17 could not possibly be reached before yielding and failure of the disk. They were considered only to find the theoretical maximum to which the buckling load could be raised.

The feasibility of physically attaining the optimum initial stress distribution is a matter that needs to be pursued. Figure 5 shows that the corresponding optimum ratio of average strains is about $\bar{\epsilon}_\theta/\bar{\epsilon}_r = -3.3$, and analysis of equation [28] reveals that $\bar{\epsilon}_r$ must be negative while $\bar{\epsilon}_\theta$ is

positive to cause the desired stress ratio. Induced hoop strain is obviously very important. The precise tensioning equipment and procedure needed to produce this strain ratio could perhaps be the topic of future theoretical and/or experimental analyses.

Other annulus locations and widths could not be considered because of time limitations. This omission was not felt to be extremely serious, since sufficient calculations had previously been done to locate the approximate optimum rolling radius. The width selected should be small enough to be practical, yet large enough to show some significant differences from the old theory.

A more complete representation of the operational environment of the saw blade could include axial load, tangential in-plane load, and rotation of the blade with the loads fixed in space. Thermal effects must also be included when studying wood saws. Axial forces are usually small in comparison with the in-plane forces, so that neglecting them is probably not serious. Tangential loads have apparently never been considered, although they could be of significant importance. Dugdale (11) considers disk rotation and investigates the critical speed at which a standing wave is formed in the disk. He shows that the lowest critical speed under single point loading is governed by one of the modes of flexural vibration, but not necessarily the fundamental. The particular mode of importance is somewhat dependent upon the clamping ratio, R_b/R . It appears that the two nodal diameter mode may control the critical speed for the untensioned blade considered here. Wave solutions of this type seem very practical and may well be the next logical step in the analysis of circular saw blade vibration and failure.

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APPENDIX

The following tables contain the pertinent computer output. Eigenvectors are not shown for obvious space reasons, but the following comments can be made concerning the mode shapes. The four lowest modes always correspond to zero nodal circles and from zero to three nodal diameters and the fifth mode is always one nodal circle with either zero or one nodal diameter. For the untensioned case, the first four modes are from zero to three nodal diameters in that order, and the fifth mode is zero nodal diameters with one nodal circle.

The computer program is not listed here, partially because of its length. The portion of the program taken from DuBois (4) occupies about 20 pages in his report.

TABLE 1

BUCKLING LOADS FOR ONE TENSIONING RADIUS

$R = 11.0$ in.
 $R_a = 1.5$ in. $D = 8906.022$ in-lb $R_b = 3.9375$ in.
 $h_b = 0.148$ in.

R_c (in.)	p (lb/in ²)	PR/D	P_{cr} (lb.)
-----	0	6.614	5355.10
9.20	150	6.620	5359.76
9.20	300	6.617	5356.97
9.50	350	6.659	5391.38
9.50	400	6.661	5392.64
9.50	450	6.660	5392.54
9.56	400	6.684	5411.95
9.56	500	6.688	5414.63
9.56	600	6.685	5412.83
9.58	400	6.693	5418.68
9.58	500	6.698	5422.71
9.58	600	6.697	5422.15
9.60	500	6.708	5431.05
9.60	600	6.709	5431.86
9.60	750	6.699	5423.77
9.64	400	6.719	5440.16
9.64	600	6.734	5451.89
9.64	750	6.729	5447.78
9.66	500	6.741	5457.77
9.66	600	6.747	5462.56
9.66	750	6.744	5460.33
9.68	100	6.622	5361.81
9.68	250	6.625	5364.07
9.68	500	6.605	5347.52
9.70	150	6.628	5366.24
9.70	250	6.631	5368.57
9.70	350	6.628	5366.55
10.00	150	6.618	5358.19
10.00	300	6.611	5352.34

TABLE 2
FREQUENCIES FOR ONE TENSIONING RADIUS

$R_c = 9.66 \text{ in.}$			$p = 600 \text{ lb/in}^2$			
$\omega_o = 0 \text{ rpm}$ $PR/D = 6.747$ $P_{cr} = 5462.555 \text{ lb.}$	$\omega_o = 1000 \text{ rpm}$ $PR/D = 7.045$ $P_{cr} = 5703.895 \text{ lb.}$	$\omega_o = 2000 \text{ rpm}$ $PR/D = 7.927$ $P_{cr} = 6418.204 \text{ lb.}$				
P/P_{cr}	$\omega_o(\text{rpm})$	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
0	0	87.17	92.17	124.38	190.26	598.08
0	1000	89.11	94.51	127.25	193.39	600.00
0	2000	94.70	101.21	135.52	202.48	605.87
1/8	0	85.37	90.46	122.65	188.76	596.55
1/8	1000	87.32	92.74	125.49	191.84	598.40
1/8	2000	92.91	99.27	133.65	200.81	604.05
1/4	0	81.74	89.76	121.10	187.32	594.46
1/4	1000	83.70	91.95	123.90	190.36	596.19
1/4	2000	89.29	98.25	131.96	199.22	601.46
3/8	0	76.64	89.54	119.75	185.95	591.69
3/8	1000	78.53	91.68	122.52	188.96	593.23
3/8	2000	83.95	97.85	130.48	197.71	597.92
1/2	0	70.16	89.48	118.58	184.65	588.19
1/2	1000	71.93	91.60	121.32	187.62	589.49
1/2	2000	77.02	97.70	129.20	196.27	593.42
5/8	0	62.05	89.49	117.59	183.11	583.99
5/8	1000	63.65	91.60	120.31	186.35	584.99
5/8	2000	68.25	97.66	128.11	194.91	588.04
3/4	0	51.65	89.53	116.75	182.24	579.15
3/4	1000	53.01	91.64	119.45	185.15	579.83
3/4	2000	56.91	97.68	127.18	193.62	581.90
7/8	0	37.19	89.59	116.04	181.12	573.78
7/8	1000	38.18	91.69	118.72	184.00	574.10
7/8	2000	41.05	97.72	126.40	192.39	575.12

TABLE 3
DATA FOR UNTENSIONED BLADE

$\omega_o = 0$ rpm
PR/D = 6.614
 $P_{cr} = 5355.118$ lb.

$\omega_o = 1000$ rpm
PR/D = 6.913
 $P_{cr} = 5596.870$ lb.

$\omega_o = 2000$ rpm
PR/D = 7.797
 $P_{cr} = 6312.639$ lb.

P/P_{cr}	ω_o (rpm)	λ_1 (Hz)	λ_2 (Hz)	λ_3 (Hz)	λ_4 (Hz)	λ_5 (Hz)
0	0	93.79	93.96	114.95	171.73	605.81
0	1000	95.59	96.25	118.05	175.18	607.71
0	2000	100.77	102.82	126.92	185.15	613.38
1/8	0	90.41	93.94	113.19	170.08	604.34
1/8	1000	92.45	95.92	116.25	173.50	606.17
1/8	2000	98.16	101.80	125.01	183.35	611.62
1/4	0	85.86	94.04	111.78	168.55	602.34
1/4	1000	87.85	96.00	114.80	171.92	604.04
1/4	2000	93.49	101.74	123.43	181.67	609.12
3/8	0	80.15	94.13	110.70	167.12	599.69
3/8	1000	82.04	96.10	113.67	170.46	601.21
3/8	2000	87.45	101.79	122.17	180.10	605.72
1/2	0	73.11	94.23	109.88	165.80	596.34
1/2	1000	74.87	96.19	112.81	169.10	597.63
1/2	2000	79.93	101.87	121.19	178.65	601.41
5/8	0	64.44	94.32	109.25	164.56	592.33
5/8	1000	66.03	96.29	112.15	167.84	593.33
5/8	2000	70.59	101.96	120.42	177.30	596.24
3/4	0	53.47	94.42	108.76	163.42	587.72
3/4	1000	54.82	96.38	111.63	166.66	588.40
3/4	2000	58.68	102.05	119.82	176.04	590.34
7/8	0	38.37	94.51	108.38	162.35	582.58
7/8	1000	39.35	96.48	111.22	165.57	582.91
7/8	2000	42.20	102.15	119.33	174.87	583.82

TABLE 4
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$	$R_b = 3.9375 \text{ in.}$	$R = 11.0 \text{ in.}$	$h = 0.148 \text{ in.}$			
$R_m = 9.66 \text{ in.}$	$R_w = 0.5 \text{ in.}$			$p_1/p_2 = 2$		
	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 200 \text{ lb/in}^2$ $PR/D = 6.584$ $P_{cr} = 5330.43 \text{ lb.}$	0	87.26	91.22	121.35	185.79	598.26
	1/8	85.34	89.68	119.63	184.28	596.79
	1/4	81.51	89.21	118.11	182.85	594.77
	3/8	76.31	89.11	116.80	181.49	592.11
	1/2	69.79	89.11	115.69	180.20	588.75
	5/8	61.67	89.15	114.76	178.98	584.72
	3/4	51.30	89.21	113.98	177.82	580.08
	7/8	36.91	89.29	113.33	176.72	574.92
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.011$ $P_{cr} = 4866.95 \text{ lb.}$	0	72.03	85.31	133.22	210.85	582.91
	1/8	70.57	83.27	131.75	209.65	581.55
	1/4	68.18	81.55	130.36	208.49	579.74
	3/8	64.59	80.27	129.05	207.36	577.38
	1/2	59.61	79.43	127.82	206.26	574.44
	5/8	53.03	78.90	126.68	205.20	570.90
	3/4	44.34	78.57	125.63	204.16	566.82
	7/8	32.04	78.36	124.67	203.15	562.24
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 3.218$ $P_{cr} = 2605.45 \text{ lb.}$	0	37.02	75.06	149.21	242.99	559.19
	1/8	35.65	73.72	148.50	242.45	558.49
	1/4	33.99	72.38	147.81	241.92	557.66
	3/8	31.96	71.04	147.12	241.39	556.70
	1/2	29.45	69.72	146.44	240.87	555.58
	5/8	26.27	68.45	145.77	240.35	554.30
	3/4	22.08	67.22	145.11	239.83	552.84
	7/8	16.07	66.06	144.46	239.32	551.21

TABLE 5
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$	$R_b = 3.9375 \text{ in.}$	$R = 11.0 \text{ in.}$	$h = 0.148 \text{ in.}$
$R_m = 9.66 \text{ in.}$	$R_w = 0.5 \text{ in.}$	$p_1/p_2 = 1.275$	

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 100 \text{ lb/in}^2$ $PR/D = 6.620$ $P_{cr} = 5359.76 \text{ lb.}$	0	91.37	92.91	117.38	177.15	602.93
	1/8	88.74	92.12	115.62	175.56	601.45
	1/4	84.36	92.12	114.15	174.06	599.43
	3/8	78.81	92.19	112.95	172.65	596.77
	1/2	71.97	92.27	112.00	171.34	593.40
	5/8	63.50	92.36	111.24	170.10	589.37
	3/4	52.74	92.45	110.64	168.94	584.72
	7/8	37.89	92.54	110.15	167.86	579.55
$p_2 = 300 \text{ lb/in}^2$ $PR/D = 6.557$ $P_{cr} = 5309.09 \text{ lb.}$	0	86.27	90.76	122.10	187.48	597.12
	1/8	84.43	89.14	120.39	186.00	595.65
	1/4	80.74	88.54	118.87	184.58	593.64
	3/8	75.63	88.38	117.54	183.24	590.99
	1/2	69.20	88.35	116.41	181.96	587.65
	5/8	61.17	88.38	115.45	180.75	583.64
	3/4	50.89	88.43	114.64	179.59	579.02
	7/8	36.62	88.50	113.96	178.50	573.88
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.281$ $P_{cr} = 5085.46 \text{ lb.}$	0	77.85	87.37	128.84	201.88	588.31
	1/8	76.33	85.40	127.27	200.57	586.89
	1/4	73.60	83.95	125.80	199.31	584.98
	3/8	69.43	83.10	124.44	198.09	582.47
	1/2	63.81	82.64	123.21	196.91	579.33
	5/8	56.59	82.40	122.09	195.77	575.55
	3/4	47.21	82.28	121.10	194.68	571.19
	7/8	34.05	82.22	120.21	193.62	566.32
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 4.970$ $P_{cr} = 4023.64 \text{ lb.}$	0	56.63	79.93	141.33	227.60	570.34
	1/8	55.20	78.04	140.19	226.70	569.22
	1/4	53.23	76.22	139.07	225.82	567.80
	3/8	50.55	74.53	137.99	224.95	566.02
	1/2	46.91	73.06	136.94	224.09	563.84
	5/8	42.03	71.82	135.93	223.25	561.24
	3/4	35.38	70.83	134.96	222.42	558.23
	7/8	25.71	70.06	134.03	221.61	554.84

TABLE 6
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$	$R_b = 3.9375 \text{ in.}$	$R = 11.0 \text{ in.}$	$h = 0.148 \text{ in.}$			
$R_m = 9.66 \text{ in.}$	$R_w = 0.5 \text{ in.}$	$p_1/p_2 = 1.037$				
	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 300 \text{ lb/in}^2$ $PR/D = 6.566$ $P_{cr} = 5316.18 \text{ lb.}$	0	87.06	91.05	121.27	185.78	597.97
	1/8	85.15	89.51	119.55	184.28	596.50
	1/4	81.33	89.04	118.03	182.85	594.49
	3/8	76.14	88.93	116.72	181.50	591.84
	1/2	69.64	88.93	115.62	180.21	588.50
	5/8	61.54	88.97	114.68	179.00	584.49
	3/4	51.19	89.03	113.90	177.84	579.87
	7/8	36.82	89.10	113.25	176.75	574.72
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.343$ $P_{cr} = 5135.19 \text{ lb.}$	0	79.64	87.99	127.27	198.74	590.05
	1/8	78.08	86.05	125.67	197.39	588.61
	1/4	75.19	84.74	124.18	196.10	586.68
	3/8	70.81	84.05	122.82	194.85	584.14
	1/2	65.00	83.72	121.60	193.65	580.95
	5/8	57.60	83.55	120.50	192.50	577.12
	3/4	48.02	83.48	119.54	191.39	572.70
	7/8	34.62	83.47	118.68	190.32	567.77
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 5.319$ $P_{cr} = 4306.27 \text{ lb.}$	0	61.58	81.33	138.47	222.09	573.89
	1/8	60.15	79.36	137.22	221.10	572.69
	1/4	58.09	77.52	136.01	220.13	571.14
	3/8	55.17	75.89	134.84	219.18	569.17
	1/2	51.16	74.55	133.72	218.24	566.75
	5/8	45.74	73.52	132.65	217.33	563.85
	3/4	38.42	72.74	131.64	216.43	560.48
	7/8	27.87	72.18	130.68	215.56	556.70
$p_2 = 2000 \text{ lb/in}^2$ $PR/D = 0.809$ $P_{cr} = 655.28 \text{ lb.}$	0	14.80	70.96	152.06	249.18	551.71
	1/8	13.95	70.60	151.89	249.05	551.54
	1/4	13.02	70.24	151.71	248.92	551.37
	3/8	11.99	69.88	151.54	248.80	551.18
	1/2	10.81	69.52	151.37	248.67	550.99
	5/8	9.44	69.15	151.20	248.54	550.80
	3/4	7.77	68.79	151.03	248.41	550.59
	7/8	5.54	68.42	150.86	248.28	550.37

TABLE 7
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $p_1/p_2 = 1.00$

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 200 \text{ lb/in}^2$ $PR/D = 6.602$ $P_{cr} = 5345.12 \text{ lb.}$	0	89.45	92.07	119.11	181.05	600.69
	1/8	87.23	90.86	117.37	179.50	599.21
	1/4	83.06	90.69	115.87	178.03	597.19
	3/8	77.66	90.70	114.61	176.65	594.53
	1/2	70.96	90.75	113.58	175.34	591.17
	5/8	62.65	90.83	112.74	174.11	587.14
	3/4	52.07	90.91	112.05	172.95	582.50
	7/8	37.43	91.00	111.48	171.86	577.33
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.351$ $P_{cr} = 5142.19 \text{ lb.}$	0	79.91	88.08	127.03	198.25	590.31
	1/8	78.35	86.15	125.42	196.90	588.88
	1/4	75.43	84.87	123.93	195.59	586.94
	3/8	71.02	84.21	122.57	194.34	584.40
	1/2	65.18	83.89	121.35	193.14	581.20
	5/8	57.75	83.73	120.26	191.98	577.37
	3/4	48.14	83.67	119.30	190.87	572.94
	7/8	34.70	83.66	118.45	189.80	568.00
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 5.367$ $P_{cr} = 4345.36 \text{ lb.}$	0	62.31	81.54	138.02	221.21	574.44
	1/8	60.88	79.57	136.75	220.21	573.23
	1/4	58.80	77.72	135.53	219.23	571.66
	3/8	55.85	76.11	134.35	218.27	569.67
	1/2	51.77	74.80	133.22	217.32	567.21
	5/8	46.28	73.80	132.14	216.40	564.26
	3/4	38.86	73.06	131.12	215.49	560.85
	7/8	28.17	72.52	130.16	214.60	557.01

TABLE 8
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$	$R_b = 3.9375 \text{ in.}$	$R = 11.0 \text{ in.}$	$h = 0.148 \text{ in.}$			
$R_m = 9.66 \text{ in.}$	$R_w = 0.5 \text{ in.}$	$p_1/p_2 = 0.886$				
	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 100 \text{ lb/in}^2$ $PR/D = 6.617$ $P_{cr} = 5357.01 \text{ lb.}$	0	91.77	93.07	116.91	176.17	603.39
	1/8	89.03	92.39	115.15	174.57	601.91
	1/4	84.61	92.42	113.69	173.06	599.90
	3/8	79.03	92.50	112.51	171.65	597.24
	1/2	72.15	92.59	111.58	170.33	593.88
	5/8	63.65	92.68	110.84	169.10	589.85
	3/4	-----	-----	-----	-----	-----
	7/8	-----	-----	-----	-----	-----
$p_2 = 300 \text{ lb/in}^2$ $PR/D = 6.570$ $P_{cr} = 5319.65 \text{ lb.}$	0	87.56	91.23	120.74	184.70	598.52
	1/8	85.59	89.76	119.02	183.18	597.04
	1/4	81.69	89.35	117.50	181.75	595.03
	3/8	76.46	89.28	116.21	180.39	592.38
	1/2	69.91	89.29	115.11	179.10	589.04
	5/8	61.76	89.34	114.20	177.88	585.03
	3/4	51.37	89.41	113.44	176.72	580.41
	7/8	36.95	89.49	112.81	175.63	575.26
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.376$ $P_{cr} = 5162.57 \text{ lb.}$	0	80.74	88.37	126.27	196.72	591.14
	1/8	79.16	86.47	124.64	195.35	589.70
	1/4	76.15	85.27	123.14	194.03	587.76
	3/8	71.64	84.68	121.79	192.77	585.20
	1/2	65.72	84.41	120.57	191.56	581.99
	5/8	58.21	84.29	119.50	190.39	578.13
	3/4	48.51	84.25	118.56	189.27	573.69
	7/8	34.96	84.25	117.73	188.20	568.72
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 5.489$ $P_{cr} = 4444.46 \text{ lb.}$	0	64.23	82.11	136.80	218.86	575.92
	1/8	62.79	80.12	135.49	217.82	574.68
	1/4	60.67	78.28	134.24	216.81	573.06
	3/8	57.61	76.72	133.03	215.81	571.01
	1/2	53.36	75.49	131.88	214.84	568.46
	5/8	47.65	74.59	130.79	213.89	565.40
	3/4	39.97	73.94	129.76	212.95	561.87
	7/8	28.95	73.48	128.79	212.04	557.89

TABLE 9
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $p_1/p_2 = 0.800$

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.394$ $P_{cr} = 5167.74 \text{ lb.}$	0	81.37	88.59	125.69	195.56	591.77
	1/8	79.77	86.71	124.05	194.17	590.32
	1/4	76.69	85.57	122.55	192.85	588.37
	3/8	72.10	85.04	121.20	191.57	585.81
	1/2	66.12	84.81	119.99	190.35	582.59
	5/8	58.54	84.71	118.93	189.18	578.71
	3/4	48.77	84.69	118.00	188.06	574.25
	7/8	35.14	84.70	117.19	186.99	569.27
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 5.604$ $P_{cr} = 4536.99 \text{ lb.}$	0	66.11	82.68	135.55	216.44	577.41
	1/8	64.67	80.68	134.21	215.37	576.15
	1/4	62.49	78.86	132.92	214.32	574.49
	3/8	59.31	77.36	131.68	213.29	572.37
	1/2	54.89	76.23	130.51	212.29	569.74
	5/8	48.97	75.42	129.41	211.31	566.58
	3/4	41.03	74.86	128.37	210.36	562.93
	7/8	29.70	74.48	127.40	209.42	558.83

TABLE 10
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $p_1/p_2 = 0.500$

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 600 \text{ lb/in}^2$ $PR/D = 6.444$ $P_{cr} = 5217.12 \text{ lb.}$	0	83.50	89.34	123.65	191.42	593.94
	1/8	81.81	87.56	121.99	190.00	592.49
	1/4	78.46	86.68	120.48	188.64	590.52
	3/8	73.62	86.34	119.14	187.33	587.93
	1/2	67.42	86.22	117.96	186.09	584.68
	5/8	59.64	86.20	116.95	184.90	580.77
	3/4	49.66	86.21	116.07	183.77	576.26
	7/8	35.75	86.26	115.32	182.69	571.24
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 5.887$ $P_{cr} = 4766.68 \text{ lb.}$	0	71.37	84.34	131.76	209.01	581.84
	1/8	69.91	82.33	130.32	207.84	580.51
	1/4	67.54	80.63	128.94	206.70	578.75
	3/8	63.95	79.39	127.65	205.59	576.47
	1/2	58.99	78.58	126.44	204.51	573.63
	5/8	52.45	78.07	125.32	203.46	570.22
	3/4	43.84	77.75	124.29	202.45	566.27
	7/8	31.67	77.55	123.34	201.46	561.85

TABLE 11
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $p_1/p_2 = -0.50$

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 1200 \text{ lb/in}^2$ $PR/D = 6.341$ $P_{cr} = 5134.17 \text{ lb.}$	0	86.33	89.50	118.22	182.00	595.79
	1/8	84.28	88.15	116.53	180.53	594.36
	1/4	80.32	87.85	115.06	179.13	592.45
	3/8	75.11	87.82	113.80	177.81	589.95
	1/2	68.64	87.85	112.74	176.55	586.80
	5/8	60.61	87.92	111.86	175.37	583.02
	3/4	50.38	87.99	111.14	174.24	578.66
	7/8	36.22	88.07	110.53	173.18	573.79
$p_2 = 2000 \text{ lb/in}^2$ $PR/D = 6.148$ $P_{cr} = 4977.72 \text{ lb.}$	0	97.70	101.94	103.33	143.23	617.57
	1/8	94.49	100.43	103.13	141.42	616.25
	1/4	89.19	99.99	103.15	139.84	614.47
	3/8	82.65	99.84	103.19	138.46	612.17
	1/2	74.89	99.79	103.24	137.25	609.28
	5/8	65.62	99.77	103.29	136.19	605.81
	3/4	54.16	99.77	103.34	135.26	601.81
	7/8	38.69	99.78	103.40	134.42	597.34

TABLE 12
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5$ in. $R_b = 3.9375$ in. $R = 11.0$ in. $h = 0.148$ in.
 $R_m = 9.66$ in. $R_w = 0.5$ in. $p_1/p_2 = -1.00$

	P/P_{cr}	λ_1 (Hz)	λ_2 (Hz)	λ_3 (Hz)	λ_4 (Hz)	λ_5 (Hz)
$p_2 = 1200$ lb/in ² PR/D = 7.392 $P_{cr} = 5984.87$ lb.	0	92.79	97.53	128.50	193.11	603.49
	1/8	90.89	95.83	126.68	191.49	601.81
	1/4	87.00	95.19	125.07	189.94	599.45
	3/8	81.58	95.01	123.68	188.48	596.26
	1/2	74.70	94.98	122.51	187.10	592.21
	5/8	66.09	95.00	121.52	185.79	587.36
	3/4	55.03	95.05	120.70	184.55	581.81
	7/8	39.63	95.12	120.02	183.38	575.66
$p_2 = 2000$ lb/in ² PR/D = 7.836 $P_{cr} = 6344.09$ lb.	0	92.11	99.84	136.78	206.09	601.94
	1/8	90.40	97.81	134.95	204.48	600.13
	1/4	87.03	96.57	133.28	202.94	597.55
	3/8	81.96	95.99	131.79	201.47	594.04
	1/2	75.27	95.75	130.47	200.06	589.56
	5/8	66.75	95.64	129.33	198.73	584.20
	3/4	55.70	95.62	128.35	197.45	578.09
	7/8	40.20	95.63	127.50	196.23	571.34
$p_2 = 3000$ lb/in ² PR/D = 8.323 $P_{cr} = 6738.51$ lb.	0	91.23	102.65	146.46	221.18	599.98
	1/8	89.58	100.41	144.63	219.59	598.04
	1/4	86.60	98.68	142.93	218.07	595.20
	3/8	81.95	97.60	141.36	216.60	591.31
	1/2	75.57	97.00	139.93	215.18	586.34
	5/8	67.23	96.67	138.65	213.82	580.41
	3/4	56.25	96.49	137.51	212.50	573.66
	7/8	40.70	96.40	136.50	211.23	566.23
$p_2 = 4000$ lb/in ² PR/D = 2.243 $P_{cr} = 1815.91$ lb.	0	43.71	82.38	107.65	126.82	651.68
	1/8	41.93	83.67	107.10	126.63	651.26
	1/4	40.08	84.98	106.62	126.46	650.80
	3/8	38.16	86.27	106.22	126.31	650.28
	1/2	36.16	87.53	105.93	126.18	649.70
	5/8	34.09	88.69	105.82	126.06	649.07
	3/4	31.92	89.70	105.93	125.96	648.37
	7/8	29.66	90.50	106.33	125.87	647.60

TABLE 13
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$	$R_b = 3.9375 \text{ in.}$	$R = 11.0 \text{ in.}$	$h = 0.148 \text{ in.}$			
$R_m = 9.66 \text{ in.}$	$R_w = 0.5 \text{ in.}$		$p_1/p_2 = -1.20$			
	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_2 = 2000 \text{ lb/in}^2$ $PR/D = 7.923$ $P_{cr} = 6414.63 \text{ lb.}$	0	96.17	101.49	133.22	198.21	606.70
	1/8	94.28	99.67	131.34	196.51	604.88
	1/4	90.38	98.91	129.67	194.90	602.28
	3/8	84.84	98.67	128.23	193.38	598.72
	1/2	77.76	98.61	127.01	191.94	594.20
	5/8	68.84	98.62	125.99	190.57	588.79
	3/4	57.37	98.66	125.14	189.28	582.62
	7/8	41.35	98.72	124.42	188.06	575.82
$p_2 = 5000 \text{ lb/in}^2$ $PR/D = 9.644$ $P_{cr} = 7808.40 \text{ lb.}$	0	99.61	111.80	156.62	232.15	607.97
	1/8	97.88	109.43	154.65	230.41	605.67
	1/4	94.73	107.57	152.81	228.73	602.17
	3/8	89.79	106.39	151.11	227.11	597.27
	1/2	82.93	105.73	149.58	225.55	591.01
	5/8	73.90	105.37	148.20	224.04	583.59
	3/4	61.95	105.17	146.98	222.59	575.23
	7/8	44.90	105.07	145.90	221.18	566.08
$p_2 = 10000 \text{ lb/in}^2$ $PR/D = 12.095$ $P_{cr} = 9792.87 \text{ lb.}$	0	105.01	127.09	189.11	279.30	609.95
	1/8	103.15	124.36	187.09	277.53	606.91
	1/4	100.23	121.80	185.12	275.78	601.96
	3/8	95.77	119.65	183.23	274.06	594.85
	1/2	89.31	118.02	181.43	272.37	585.85
	5/8	80.35	116.87	179.74	270.69	575.33
	3/4	67.97	116.08	178.15	269.03	563.59
	7/8	49.71	115.54	176.66	267.37	550.82
$p_2 = 20000 \text{ lb/in}^2$ $PR/D = 15.970$ $P_{cr} = 12929.85 \text{ lb.}$	0	114.80	152.95	240.83	354.01	613.36
	1/8	112.66	149.92	238.79	352.26	609.03
	1/4	109.63	146.86	236.76	350.50	601.42
	3/8	105.32	143.90	234.74	348.71	590.37
	1/2	99.16	141.19	232.73	346.87	576.56
	5/8	90.37	138.85	230.73	344.97	560.65
	3/4	77.59	136.92	228.75	342.96	543.02
	7/8	57.69	135.39	226.77	340.80	523.93

TABLE 14
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5$ in. $R_b = 3.9375$ in. $R = 11.0$ in. $h = 0.148$ in.
 $R_m = 9.66$ in. $R_w = 0.5$ in. $p_1/p_2 = -2.00$

	P/P_{cr}	λ_1 (Hz)	λ_2 (Hz)	λ_3 (Hz)	λ_4 (Hz)	λ_5 (Hz)
$p_2 = 600$ lb/in ² PR/D = 6.965 $P_{cr} = 5639.35$ lb.	0	98.32	99.20	115.84	169.03	611.75
	1/8	94.98	99.06	114.06	167.26	610.20
	1/4	90.23	99.13	112.73	165.64	608.06
	3/8	84.18	99.22	111.79	164.15	605.21
	1/2	76.72	99.31	111.12	162.78	601.60
	5/8	67.56	99.40	110.64	161.52	597.27
	3/4	56.01	99.49	110.28	160.37	592.31
	7/8	40.17	99.59	110.01	159.30	586.79
$p_2 = 1200$ lb/in ² PR/D = 7.275 $P_{cr} = 5890.36$ lb.	0	102.46	104.28	116.73	166.24	617.63
	1/8	99.22	103.96	114.96	164.37	616.02
	1/4	94.24	103.97	113.78	162.67	613.76
	3/8	87.84	104.03	113.03	161.13	610.73
	1/2	79.97	104.10	112.54	159.74	606.89
	5/8	70.36	104.18	112.20	158.48	602.29
	3/4	58.28	104.27	111.96	157.33	597.02
	7/8	41.77	104.36	111.78	156.29	591.18
$p_2 = 2000$ lb/in ² PR/D = 5.498 $P_{cr} = 4451.00$ lb.	0	67.05	81.54	131.35	210.21	577.63
	1/8	65.62	79.56	130.00	209.13	576.39
	1/4	63.39	77.82	128.70	208.08	574.78
	3/8	60.08	76.47	127.47	207.05	572.73
	1/2	55.49	75.51	126.31	206.05	570.20
	5/8	49.40	74.87	125.23	205.08	567.16
	3/4	41.32	74.45	124.22	204.13	563.63
	7/8	29.86	74.17	123.29	203.20	559.67

TABLE 15
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5$ in. $R_b = 3.9375$ in. $R = 11.0$ in. $h = 0.148$ in.
 $R_m = 9.66$ in. $R_w = 0.5$ in. $p_2 = 10000$ lb/in²

	P/P_{cr}	λ_1 (Hz)	λ_2 (Hz)	λ_3 (Hz)	λ_4 (Hz)	λ_5 (Hz)
$p_1/p_2 = -1.3$ $PR/D = 12.596$ $P_{cr} = 10197.96$ lb.	0	113.63	130.19	182.72	265.04	621.58
	1/8	111.78	127.51	180.53	263.06	618.42
	1/4	108.58	125.17	178.45	261.13	613.22
	3/8	103.49	123.46	176.49	259.26	605.73
	1/2	96.17	122.37	174.68	257.43	596.27
	5/8	86.22	121.70	173.02	255.64	585.27
	3/4	72.73	121.29	171.51	253.88	573.03
	7/8	53.06	121.04	170.15	252.16	559.73
$p_1/p_2 = -1.4$ $PR/D = 12.873$ $P_{cr} = 10422.81$ lb.	0	121.47	133.17	176.08	249.75	633.10
	1/8	119.57	130.60	173.76	247.57	629.91
	1/4	115.89	128.69	171.62	245.48	624.62
	3/8	109.99	127.60	169.67	243.48	616.99
	1/2	101.79	127.03	167.95	241.55	607.39
	5/8	90.95	126.75	166.44	239.71	596.25
	3/4	76.49	126.61	165.13	237.93	583.86
	7/8	55.66	126.55	163.99	236.21	570.42
$p_1/p_2 = -1.5$ $PR/D = 12.928$ $P_{cr} = 10466.93$ lb.	0	128.68	136.02	169.15	233.25	644.49
	1/8	126.60	133.73	166.76	230.88	641.35
	1/4	122.11	132.54	164.64	228.66	636.12
	3/8	115.29	132.10	162.85	226.57	628.59
	1/2	106.25	131.94	161.35	224.62	619.13
	5/8	94.60	131.90	160.13	222.78	608.15
	3/4	79.30	131.92	159.12	221.06	595.96
	7/8	57.52	131.98	158.29	219.42	582.74
$p_1/p_2 = -1.6$ $PR/D = 12.734$ $P_{cr} = 10313.23$ lb.	0	135.38	138.77	161.91	215.27	655.63
	1/8	132.66	137.19	159.52	212.73	652.60
	1/4	127.03	136.93	157.61	210.42	647.59
	3/8	119.33	136.92	156.18	208.33	640.39
	1/2	109.49	136.99	155.11	206.43	631.33
	5/8	97.08	137.07	154.30	204.71	620.81
	3/4	81.06	137.17	153.69	203.13	609.13
	7/8	58.58	137.27	153.20	201.68	596.45

TABLE 16
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $p_2 = 10000 \text{ lb/in}^2$

	P/P_{cr}	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
$p_1/p_2 = -1.7$ $PR/D = 12.263$ $P_{cr} = 9928.28 \text{ lb.}$	0	141.41	141.65	154.29	195.44	666.60
	1/8	137.00	141.70	152.13	192.78	663.76
	1/4	130.38	141.81	150.84	190.49	659.13
	3/8	121.77	141.92	150.10	188.53	652.50
	1/2	111.13	142.04	149.64	186.85	644.14
	5/8	98.04	142.15	149.34	185.39	634.41
	3/4	81.48	142.27	149.11	184.11	623.58
	7/8	58.62	142.38	148.94	182.95	611.83
$p_1/p_2 = -1.8$ $PR/D = 11.450$ $P_{cr} = 9270.56 \text{ lb.}$	0	143.96	146.27	147.55	173.15	677.43
	1/8	138.85	145.17	147.39	170.49	674.86
	1/4	131.00	145.07	147.44	168.51	670.79
	3/8	121.48	145.05	147.50	167.01	665.01
	1/2	110.19	145.06	147.58	165.84	657.67
	5/8	96.70	145.08	147.65	164.88	649.09
	3/4	79.97	145.09	147.73	164.06	639.51
	7/8	57.28	145.11	147.80	163.34	629.09
$p_1/p_2 = -1.9$ $PR/D = 9.945$ $P_{cr} = 8051.92 \text{ lb.}$	0	137.74	145.81	146.65	153.15	688.06
	1/8	133.23	142.44	146.27	152.79	685.95
	1/4	125.30	141.20	146.13	152.68	682.75
	3/8	115.58	140.69	146.00	152.66	678.29
	1/2	-----	-----	-----	-----	-----
	5/8	-----	-----	-----	-----	-----
	3/4	-----	-----	-----	-----	-----
	7/8	-----	-----	-----	-----	-----

TABLE 17
TENSIONED FREQUENCIES AND BUCKLING LOADS

$R_a = 1.5 \text{ in.}$ $R_b = 3.9375 \text{ in.}$ $R = 11.0 \text{ in.}$ $h = 0.148 \text{ in.}$
 $R_m = 9.66 \text{ in.}$ $R_w = 0.5 \text{ in.}$ $P = 0.0 \text{ lb.}$

P_1/P_2	$P_2(\text{lb/in}^2)$	PR/D	$\lambda_1(\text{Hz})$	$\lambda_2(\text{Hz})$	$\lambda_3(\text{Hz})$	$\lambda_4(\text{Hz})$	$\lambda_5(\text{Hz})$
-1.2	100000	25.13	164.10	280.96	473.89	615.40	661.14
-1.2	500000	14.07	390.22	431.20	640.58	690.03	704.68
-1.2	1000000	25.32	545.90	612.83	649.92	864.25	1334.05
-1.4	300000	58.51	392.84	493.06	717.68	988.00	1033.69
-1.4	1000000	74.89	659.60	857.25	1251.19	1280.15	1371.97
-1.4	5000000	75.75	789.93	1450.50	1661.04	2035.71	2509.74
-1.4	10000000	65.30	668.51	1546.92	2258.96	2810.26	3220.20
-1.6	80000	41.48	275.39	292.14	338.17	401.29	925.85
-1.6	300000	89.48	484.76	524.96	616.29	706.77	1439.84
-1.6	1000000	197.48	848.37	928.28	1095.53	1225.46	2345.71
-1.6	10000000	1468.15	2624.76	2887.76	3406.97	3714.85	6840.50
-1.8	20000	14.42	171.54	172.98	178.84	183.69	742.19
-1.8	50000	18.09	166.00	229.85	252.59	258.27	908.11
-1.8	60000	18.25	161.99	245.89	272.09	277.76	956.59
-1.8	80000	17.48	151.76	274.82	306.87	312.40	1046.20

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THE EFFECTS OF TENSIONING ON THE BUCKLING
AND VIBRATION OF CIRCULAR SAW BLADES

by

JERRY F. CARLIN

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AN ABSTRACT OF A THESIS

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MASTER OF SCIENCE

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Scope and Method of Study: A thin annular disk was subjected to boundary conditions and loads approximating those encountered by a circular saw blade. A more general approach to theoretical tensioning stresses than that found in the literature was adopted. The governing differential equation for the disk was derived by Hamilton's Principle and solved by the Rayleigh-Ritz technique. The lateral deflection of the disk was represented as a series expansion of its eigenfunctions of free vibration and the in-plane stresses were formulated in terms of the tensioning parameters and the in-plane load. Minimization of the Hamiltonian then led to an eigenvalue problem for the natural frequencies and mode shapes of vibration. An eigenvalue problem for the buckling loads was formulated in like manner by minimization of the potential energy of the disk. The effect of different amounts and locations of tensioning on the buckling and vibration characteristics of the blade was determined.

Findings and Conclusions: An optimum radial location for tensioning was calculated which agreed with previously published theoretical results. A particular form of initial stress distribution was found which significantly raised both the critical buckling load and the vibrational frequencies of the disk. The buckling load was theoretically raised to 200% of its untensioned value when subjected to this form of

initial stress distribution with realistic magnitude. More studies are needed to determine the feasibility of physically inducing this form of stress distribution. The critical speed of rotation which produces a standing wave in the disk due to rotation of the blade past stationary loads could also be investigated.

MAJOR PROFESSOR'S APPROVAL

J. C. Appl