Embeddings in high dimension

by

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B.Sc. (Hons), University of Delhi, 2016

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AN ABSTRACT OF A DISSERTATION

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Department of Mathematics
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Abstract

The spaces of smooth high dimensional spherical and long embeddings attracted a lot of attention starting mid-20th century. In 1966, André Haefliger proved that the group of isotopy classes of spherical (framed) embeddings of S^n in S^{n+q} for $q \geq 3$ can be described by means of homotopy groups of spheres and orthogonal groups. We show in the second chapter of this dissertation that Haefliger's result can be adapted and extended to spherical and long embeddings modulo immersions of codimension at least three. The spaces of such embeddings have been actively studied in the last 20 years. Moreover, we prove that a partial result is true for concordance classes of spherical and long embeddings modulo immersions of codimension two.

Our motivation for third and fourth chapters is to utilize the combinatorial techniques developed in 1990's and further on for classical knots and links in $\mathbb{R}^3$, to gain a geometric intuition about high dimensional knots and links. In particular, we are interested in generalizing the work of Polyak and Viro in the classical knot theory, where they provided explicit formulas for finite type knot invariants in terms of Gauss diagrams. We obtained Polyak-Viro type formulas for 2- and 3-component spherical and long links of dimension $(2\ell - 1)$ in $\mathbb{R}^{3\ell}$, $\ell \geq 2$, as discussed in chapter three. Although, the generalization to knots (one-component links) in these (co)dimensions turns out to be a more difficult problem, we give a conjectural formula for an invariant distinguishing such knots in the final fourth chapter.

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Acknowledgments

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Dedication

"To my grandparents – in loving memory."

Chapter 1

Background

1.1 History

High dimensional spherical/long embeddings, especially the ones with codimension at least three are of particular interest. Haefliger [21] proved that the smooth knots $S^n \hookrightarrow S^{n+q}$ are all trivial for $q > \frac{n+3}{2}$; while there are non-trivial smooth knots in codimension $3 \leq q \leq \frac{n+3}{2}$. Surprisingly, this contrasts with the findings in piecewise linear and topological locally flat settings, where Zeeman [47] and Stallings [43] independently showed that knots $S^n \hookrightarrow S^{n+q}$ are all trivial when $q \geq 3$. According to Haefliger [21], the set of connected components of smooth spherical embeddings in the border codimension $q = \frac{n+3}{2}$ can be characterized as follows:

$$\pi_0 Emb(S^{2\ell-1}, S^{3\ell}) = \begin{cases} \mathbb{Z} & \ell \geq 2,\ \ell \text{ even} \\ \mathbb{Z}_2 & \ell \geq 3,\ \ell \text{ odd} \end{cases}$$

In the even case, Haefliger [23] established the existence of an explicit embedding known as the "Haefliger trefoil knot", representing the generator of $\pi_0 Emb(S^{2\ell-1}, S^{3\ell})$. Recently Koytcheff [29] has extended this result to the odd case. One intriguing problem is to develop a combinatorial formula for the Haefliger invariant $\pi_0 Emb(S^{4k-1}, S^{6k}) \xrightarrow{\approx} \mathbb{Z}$ in terms of the self-intersection of the projection of the knot to one dimension less. In the classical knot theory of 1-dimensional knots in $\mathbb{R}^3$, Polyak and Viro [34, 35] expressed some finite type knot invariants (such as the Casson invariant) in terms of Gauss diagrams, which encode the information about the double points of the projection of the knot to $\mathbb{R}^2$. Our main objective was to get an analogue to Polyak-Viro type formulas for Haefliger knots $S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$.

Another interesting result by Haefliger [21] is the isomorphism between the group of isotopy classes of smooth spherical (framed) embeddings of S^n in S^{n+q} for $q \geq 3$ and certain homotopy groups involving spheres and orthogonal groups. We extend Haefliger's result to the group of isotopy classes of smooth spherical or disk embeddings modulo immersions. The spaces of embeddings modulo immersions in recent years attracted a lot of attention [4, 3, 2, 10, 5, 6, 7, 8, 16, 38, 45]. In particular, it was shown in [5, 7, 38] that there is a natural little disks operad action on the spaces of (framed) disk embeddings modulo immersions. Also, for codimension at least 3, there are several delooping results on such spaces by means of smoothing theory [38] and the Goodwillie-Weiss functor calculus on manifolds [10, 11]. Furthermore, the rational homotopy and homology of such spaces were studied in [4, 3, 2, 15, 16], and in case of spherical embeddings, the explicit computations were done in [45]. The advantage of these spaces over the usual embedding spaces is mainly the easier description of their rational homology and homotopy as well as deloopings.

1.2 Preliminaries

In this section, we provide basic definitions, results and notation that we will use throughout the manuscript. We typically use the symbol '$\approx$' for a group isomorphism; '$\simeq$' for a homotopy equivalence between two topological spaces and '$\cong$' for a diffeomorphism between manifolds. Throughout the dissertation we work in the smooth category.

Let $f : M \to N$ be a smooth map. We say f is an *immersion* if the differential $df_p : T_pM \to T_{f(p)}N$ is injective for all points $p \in M$. We denote an immersion by $f : M \looparrowright N$. An immersion $f : M \looparrowright N$ is an *embedding* denoted by $f : M \hookrightarrow N$ if it is a homeomorphism onto its image $f(M)$. Then the spaces of embeddings and immersions are denoted by $Emb(M, N)$ and $Imm(M, N)$, respectively. Note that $Emb(M, N) \subset Imm(M, N)$.

Definition 1.2.1. An *isotopy* between two embeddings $f, g : M \hookrightarrow N$ is a path $\alpha : I \to Emb(M, N)$ where $\alpha(0) = f$ and $\alpha(1) = g$. Similarly, a *regular homotopy* between two immersions $f, g : M \hookrightarrow N$ is a path $\alpha : I \to Imm(M, N)$ where $\alpha(0) = f$ and $\alpha(1) = g$.

We denote the set of isotopy (resp. regular homotopy) classes of embeddings (resp. immersions) of M in N by $\pi_0 Emb(M, N)$ (respectively, $\pi_0 Imm(M, N)$). Abusing notation, we sometimes use the same notation for an embedding and its isotopy class (as well as for an immersion and its regular homotopy class).

Definition 1.2.2. Two embeddings $f_0, f_1 : M \hookrightarrow N$ are *concordant* if there exists an embedding $F : M \times I \hookrightarrow N \times I$ such that $F(x, i) = (f_i(x), i)$ for $i = 0, 1$. Note that if F is level-preserving i.e., $F(x, t) = (f_t(x), t)$ for each $t \in I$, then f_0 is isotopic to f_1.

By [21, Theorem 1.2], two concordant embeddings of S^n in S^{n+q} are isotopic when $q \geq 3$.

Definition 1.2.3. A *high-dimensional knot* is an embedding $S^n \hookrightarrow \mathbb{R}^{n+q}$. By a *spherical knot*, we mean the embedding of S^n in S^{n+q}. A *long knot* is an embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+q}$ which is the standard inclusion outside a compact set. We denote by $Emb_\partial(\mathbb{R}^n, \mathbb{R}^{n+q})$ the space of long knots.

A *high-dimensional link* with m-components is an embedding $\underbrace{S^n \sqcup \ldots \sqcup S^n}_{m} \hookrightarrow \mathbb{R}^{n+q}$ (we only consider links with components of the same dimension). A knot (resp. link) is said to be *unknot (resp. unlink)* if it is isotopic to the trivial embedding.

Remark 1.2.1. For $q \geq 3$ and $m \geq 1$ we have

$$\pi_0 Emb(\bigsqcup_{i=1}^{m} S^n, \mathbb{R}^{n+q}) = \pi_0 Emb(\bigsqcup_{i=1}^{m} S^n, S^{n+q}) = \pi_0 Emb_\partial(\bigsqcup_{i=1}^{m} \mathbb{R}^n, \mathbb{R}^{n+q}), \tag{1.2.1}$$

each forming a finitely generated abelian group with respect to component-wise connected sum operation, see [22],[44, Lemma 3.7],[29, Lemma 4.8]. This equivalence allows us to interchangeably refer to these embeddings throughout the dissertation.

Definition 1.2.4. A *framing* of an embedding $M \hookrightarrow N$ (or an immersion $M \looparrowright N$) is the choice of the trivialization for the normal bundle along the embedded (immersed) manifold M.

Let $Emb_\partial(D^n, D^{n+q})$ denote the space of disk embeddings $D^n \hookrightarrow D^{n+q}$ with the fixed behavior near the boundary to be the standard inclusion $D^n \subset D^{n+q}$. The following definitions are in accordance to Haefliger [21].

Definition 1.2.5. The *suspension* of a map $f\colon D^n \to D^n$ is given by the map $S(f)\colon D^{n+1} \to D^{n+1}$ sending the arc of circle going from e_{n+1}, by $x \in D^n$, to $-e_{n+1}$ on the arc of circle from e_{n+1}, by $f(x)$, to $-e_{n+1}$. The suspension $S^{n+1} \to S^{n+1}$ of a map $S^n \to S^n$ is defined in the same way.

Definition 1.2.6. The *suspension of an embedding* $S^n \overset{f}{\hookrightarrow} S^{n+q}$ is the composition $S^n \overset{f}{\hookrightarrow} S^{n+q} \subset S^{n+q+1}$. For the suspension of a framed embedding $S^n \hookrightarrow S^{n+q}$, the framing is completed by adding the standard vector e_{n+q+2} as the last vector. We often say suspension for an iterated suspension defined inductively. For example, when we say $S^n \hookrightarrow S^{n+N}$ is the suspension of a framed embedding $S^n \overset{f}{\hookrightarrow} S^{n+q}$ for $N > q$, we mean that it is defined as the composition $S^n \overset{f}{\hookrightarrow} S^{n+q} \subset S^{n+N}$ and the framing is obtained by adding vectors $\{e_{n+q+2}, \dots, e_{n+N+1}\}$ to the initial framing. We define the suspension of a (framed) disk embedding similarly.

Chapter 2

Haefliger's approach for spherical knots modulo immersions

2.1 Introduction

In this chapter we study the set of isotopy classes of spherical and disk embeddings modulo immersions by extending Haefliger's work [21] on isotopy classes of (framed) spherical embeddings of codimension at least 3. Also, we establish a natural connection between our and Haefliger's results in terms of long exact sequences. The results in this chapter except those from Section 2.7 were published in [17].

We define the *space of spherical embeddings modulo immersions* as

$$\overline{Emb}(S^n, S^{n+q}) := hofiber(Emb(S^n, S^{n+q}) \hookrightarrow Imm(S^n, S^{n+q}))$$

over the trivial inclusion $id\colon S^n \subset S^{n+q}$. An element in this space is represented by a pair (f, α), where $f\colon S^n \hookrightarrow S^{n+q}$ is a smooth embedding together with a regular homotopy $\alpha\colon [0,1] \to Imm(S^n, S^{n+q})$, between f and the trivial inclusion $S^n \subset S^{n+q}$. Moreover, if $Emb_\partial(D^n, D^{n+q})$ denote the space of disk embeddings $D^n \hookrightarrow D^{n+q}$ with the fixed behavior near the boundary, then we similarly define $\overline{Emb}_\partial(D^n, D^{n+q})$ as the space of disk embeddings modulo immersions. For framed spherical or disk embeddings, we consider the spaces $Emb^{fr}(S^n, S^{n+q}), \overline{Emb}^{fr}(S^n, S^{n+q}), Emb_{\partial}^{fr}(D^n, D^{n+q})$ and $\overline{Emb}_{\partial}^{fr}(D^n, D^{n+q})$ in the same manner. Throughout the chapter for spherical embeddings we assume that the framing respects the natural orientation: if one takes the orientation of S^n and completes it with the orientation of the normal bundle induced by the framing, one obtains the standard orientation of the ambient sphere S^{n+q}. For disk embeddings the framing is standard near the boundary.

In [21, Theorems 3.4 and 5.7], Haefliger has shown that for $q \geq 3$, the group of isotopy classes of (framed) spherical embeddings of S^n in S^{n+q} can be described in terms of the homotopy group of a triad i.e., $C_n^q := \pi_0 Emb(S^n, S^{n+q}) = \pi_{n+1}(SG; SO, SG_q)$ and $FC_n^q := \pi_0 Emb^{fr}(S^n, S^{n+q}) = \tilde{\pi}_{n+1}(SG; SO, SG_q)$. We recall these homotopy groups and isomorphisms later in Section 2.2.

2.1.1 Main results

Let $\overline{FC}_n^q$ denote the group of isotopy classes of "framed disked embeddings", which we discuss in more detail in Section 2.3.

Theorem 2.1.1. *For* $q \geq 3$, $\overline{FC}_n^q = \pi_0 \overline{Emb}(S^n, S^{n+q}) = \pi_0 \overline{Emb}_{\partial}(D^n, D^{n+q}) = \pi_0 \overline{Emb}^{fr}(S^n, S^{n+q}) = \pi_0 \overline{Emb}_{\partial}^{fr}(D^n, D^{n+q}) = \pi_{n+1}(SG, SG_q)$.

The result is an immediate corollary of Theorems 2.3.1 and 2.4.1, and Lemma 2.4.1.

Alternatively, this result can be obtained using smoothing theory, as a consequence of [23, Section 6] and [38, Theorem 1.1]. There is even a stronger result:

$$\pi_i \overline{Emb}_{\partial}(D^n, D^{n+q}) = \pi_{n+i+1}(SG_{n+q}, SG_q) \text{ for } i \leq 2q - 5,$$

which follows from the work of Lashof [30], Millett [32, Theorem 2.3] and Sakai [38, Theorem 1.1 and Remark 2.3]. Moreover, for $i \leq q - 3$, $\pi_{n+i+1}(SG_{n+q}, SG_q) = \pi_{n+i+1}(SG, SG_q)$, see Lemma 2.6.1. However, our goal is to review Haefliger's construction and give a geometric meaning to $\pi_{n+1}(SG, SG_q)$ i.e., $\pi_{n+1}(SG, SG_q) = \overline{FC}_n^q$.

Another main result that does not immediately follow from smoothing theory is to geometrically interpret the long exact sequences associated with the triad $(SG; SO, SG_q)$ considered by Haefliger [21, Sections 4.4 and 5.9]. Let Im^q_n and FIm^q_n denote the group of regular homotopy classes of immersions $S^n \looparrowright S^{n+q}$ and framed immersions $S^n \looparrowright S^{n+q}$, respectively. It is natural to ask which (framed) spherical immersions can be realized as (framed) embeddings, or when two (framed) spherical embeddings are equivalent as (framed) immersions. Answers to these questions are encoded by the lower exact sequences in (2.1.1) and (2.1.2) of Theorem 2.1.2, in which $\overline{FC}^q_n$ naturally fits.

Theorem 2.1.2. *For $q \geq 3$, the two long exact sequences of the triad $(SG; SO, SG_q)$ are isomorphic to the corresponding geometric long exact sequences:*

$$\begin{tikzcd} \arrow[r] & \pi_{n+1}(SG, SG_q) \arrow[r] \arrow[d, equal] & \pi_{n+1}(SG; SO, SG_q) \arrow[r] \arrow[d, equal] & \pi_n(SO, SO_q) \arrow[r] \arrow[d, equal] & \pi_n(SG, SG_q) \arrow[r] \arrow[d, equal] & {} \\ \arrow[r] & \overline{FC}^q_n \arrow[r] & C^q_n \arrow[r] & Im^q_n \arrow[r] & \overline{FC}^q_{n-1} \arrow[r] & {} \end{tikzcd} \tag{2.1.1}$$

$$\begin{tikzcd} \arrow[r] & \pi_{n+1}(SG, SG_q) \arrow[r] \arrow[d, equal] & \tilde{\pi}_{n+1}(SG; SO, SG_q) \arrow[r] \arrow[d, equal] & \pi_n(SO) \arrow[r] \arrow[d, equal] & \pi_n(SG, SG_q) \arrow[r] \arrow[d, equal] & {} \\ \arrow[r] & \overline{FC}^q_n \arrow[r] & FC^q_n \arrow[r] & FIm^q_n \arrow[r] & \overline{FC}^q_{n-1} \arrow[r] & {} \end{tikzcd} \tag{2.1.2}$$

Note that the upper sequences in (2.1.1) and (2.1.2) are the long exact sequences of the homotopy groups of pairs $(SG/SG_q, SO/SO_q)$ and $(SG/SG_q, SO)$, respectively, see Remarks 2.2.1 and 2.2.2.

Another case of particular interest is of codimension two. In this case, concordance and isotopy are different relations. By viewing C^2_n as the group of concordance classes of embeddings $S^n \hookrightarrow S^{n+2}$, and carefully looking at the Haefliger's isomorphism argument in [21, §3.6], the natural homomorphism $\psi : C^2_n \to \pi_{n+1}(SG; SO, SG_2)$ is surjective for n odd and injective for n even. Moreover, Kervaire proved in [26, Theorems III.3, III.6] that ψ is an isomorphism in the even case. We extend these results of Kervaire and Haefliger to embeddings modulo immersions as follows.

Theorem 2.1.3. *The natural homomorphism $\xi : \overline{FC}_n^2 \to \pi_{n+1}(SG, SG_2)$ is a surjection for n odd, and is an isomorphism for n even.*

2.1.2 Terminology

Let D^n be the standard unit disk in $\mathbb{R}^n$, and $\{e_1, .., e_n\}$ denote the natural basis of $\mathbb{R}^n$. Let $S^n = \partial D^{n+1}$ be the unit sphere such that $S^n = D^n_- \cup D^n_+$ with $D^n_- = \{x \in S^n | x_1 \leq 0\}$ and $D^n_+ = \{x \in S^n | x_1 \geq 0\}$.

2.2 Embeddings of S^n in S^{n+q}

Haefliger [21] proved that the group of concordance classes of embeddings of S^n in S^{n+q} is isomorphic to $\pi_{n+1}(SG; SO, SG_q)$ for $q \geq 3$.

2.2.1 The group C_n^q

$$C_n^q := \{\text{concordance classes of smooth embeddings } S^n \hookrightarrow S^{n+q}\}.$$

Theorem 2.2.1. *[21, Theorem 1.2] Two concordant embeddings of S^n in S^{n+q} are isotopic when $q \geq 3$, i.e. $C_n^q = \pi_0 Emb(S^n, S^{n+q})$, the set of connected components of the space of embeddings of S^n in S^{n+q}.*

Furthermore, the equality $\pi_0 Emb(S^n, S^{n+q}) = \pi_0 Emb_\partial(D^n, D^{n+q})$ enables C_n^q with an additive multiplication, and the existence of inverses is guaranteed, as we consider concordance classes. Hence, C_n^q is an abelian group.

Lemma 2.2.1. *[21, Section 1] An embedding $S^n \hookrightarrow S^{n+q}$ is concordant to the trivial one if and only if it is slice, in other words, if it can be extended to an embedding $D^{n+1} \hookrightarrow D^{n+q+1}$.*

2.2.2 The group $\pi_{n+1}(SG; SO, SG_q)$

Let SG_q be the space of degree one maps $S^{q-1} \to S^{q-1}$, $SG = \cup SG_q$ under suspension, and $SO = \cup SO_q$, where SO_q is the special orthogonal group.

An element in $\pi_{n+1}(SG; SO, SG_q)$ is represented by a continuous based map $\phi\colon D^{n+1} \to SG$ i.e., for $x \in D^{n+1}$, $\phi(x)\colon S^{N-1} \to S^{N-1}$, for some large N, such that $\phi(D^n_-) \subset SO_N$ and $\phi(D^n_+) \subset SG_q$. Note that the equator $S^{n-1} = \partial D^n_- = \partial D^n_+$ goes to $SO \cap SG_q = SO_q$, and $\phi(*) = id$ for the base-point $* = e_2 \in S^{n-1}$.[1] Abusing notation, we also view ϕ as a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$, and sometimes for $\phi(x)$ we write $\phi_x = \phi(x, -)\colon S^{N-1} \to S^{N-1}$.

Two such maps $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ and $\phi'\colon D^{n+1} \times S^{N'-1} \to S^{N'-1}$ represent the same element in $\pi_{n+1}(SG; SO, SG_q)$ if there is a homotopy $\phi_t\colon D^{n+1} \times S^{M-1} \to S^{M-1}$ for some $M \geq N, N'$ and $t \in [0,1]$, satisfying the above conditions and such that for any $x \in D^{n+1}$, the maps $\phi_0(x,-), \phi_1(x,-)\colon S^{M-1} \to S^{M-1}$ are suspensions of $\phi(x,-)$ and $\phi'(x,-)$, respectively. The product operation of any two elements in $\pi_{n+1}(SG; SO, SG_q)$ is defined point-wise.

Remark 2.2.1. Recall that the upper long exact sequence in (2.1.1) is the long exact sequence of the pair $(SG/SG_q, SO/SO_q)$. Indeed, Milgram [31, Section 1] interpreted the group $\pi_{n+1}(SG; SO, SG_q)$ as $\pi_n\Big(\mathrm{hofib}(SO/SO_q \to SG/SG_q) \simeq \mathrm{hofib}(SG_q/SO_q \to SG/SO)\Big)$.[2] One way to see this interpretation is that the group $\pi_{n+1}(SG; SO, SG_q)$ is obviously isomorphic to $\pi_n(SO \times^h_{SG} SG_q, SO_q)$, where $SO \times^h_{SG} SG_q$ is the homotopy pullback of $SO \to SG \leftarrow SG_q$.

[1] Haefliger in [21] does not consider the base-point condition, but it is immediate that adding it yields the same homotopy group, since $SO_q = SO \cap SG_q$ is connected.

[2] The spaces are equivalent because they describe the total homotopy fiber of the square

$$\begin{tikzcd} BSO_q \arrow[r] \arrow[d] & BSG_q \arrow[d] \\ BSO \arrow[r] & BSG\,. \end{tikzcd}$$

2.2.3 The isomorphism $\psi\colon C_n^q \to \pi_{n+1}(SG; SO, SG_q)$

Although these two groups look completely different, there is a natural map between them. To see the relation, Haefliger considers representatives in C_n^q to be framed embeddings of D^{n+1} with different boundary conditions on $D_-^n \subset S^n = \partial D^{n+1}$ and $D_+^n \subset S^n = \partial D^{n+1}$. Framing will be crucial to relate such embeddings to $\pi_{n+1}(SG; SO, SG_q)$ by means of Pontryagin-Thom type construction [21, Section 3].

An embedding $f\colon S^n \hookrightarrow S^{n+q}$ is called a special embedding if $f|_{D_-^n} = id$ and $f(\text{int } D_+^n) \subset \text{int } D_+^{n+q}$. We can always extend $f\colon S^n \hookrightarrow S^{n+q}$ to a disk embedding $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+N+1}$, for some N large enough (in fact $N > n+2$). We refer the obtained pair $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ as a ***disked embedding***. Any element in C_n^q can be represented by a special disked embedding $(f, \bar{f})$ together with some framing on $\bar{f}$ defined as follows:

- Fix the base-point $* = e_2 \in S^{n-1} = D_-^n \cap D_+^n$ and endow it with the framing $\{e_{n+2}, \ldots, e_{n+q+1}\}$.
- Extend the framing from $* = e_2$ to D_+^n inside D_+^{n+q}. Since $* \hookrightarrow D_+^n$ is a homotopy equivalence, this extension is unique up to homotopy. Take the suspension of this framing in D^{n+N+1} by adding $\{e_{n+q+2}, \ldots, e_{n+N+1}\}$ as last vectors.
- Extend the obtained framing from D_+^n to the entire disk D^{n+1} inside D^{n+N+1}. Again this framing is defined uniquely up to homotopy.

Note that even though the knot f is trivial on D_-^n, the extended framing can be non-trivial. Moreover, the framing on $\bar{f}|_{D_-^n}$ inside D^{n+N+1} might not be a suspension, while the framing on $\bar{f}|_{D_+^n}$ inside D^{n+N+1} is the suspension of a framing inside D_+^{n+q}. We refer this boundary condition on the framing defined on $\bar{f}$ as ***Type I*** (in Sections 2.2.4 and 2.3, we will also consider framing with Type II and Type III boundary conditions). Hence, any embedding $f\colon S^n \hookrightarrow S^{n+q}$ representing an element in C_n^q can be considered as a ***special disked embedding*** $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ ***with Type I framing***, i.e., $\bar{f}|_{S^n = \partial D^{n+1}} = f$ is a special knot, and the framing on $\bar{f}$ has boundary condition defined as above.

Any two special disked embeddings $(f_0, \bar{f}_0)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N_0+1})$ and $(f_1, \bar{f}_1)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N_1+1})$ with Type I framing are *concordant* if there exists an embedding $F\colon D^{n+1} \times [0,1] \hookrightarrow D^{n+N+1} \times [0,1]$ for $N \geq max\{N_0, N_1\}$, such that $F|_{D^{n+1}\times i} = \bar{f}_i$ for $i = 0, 1$, $F|_{D^n_- \times[0,1]} = id$ and $F|_{D^n_+\times[0,1]} \subset D^{n+q}_+ \times [0,1]$. Furthermore, the framing on $F|_{D^n_+\times[0,1]}$ and $F|_{D^{n+1}\times i}$, $i = 0, 1$, is given by suspension of a framing inside $D^{n+q}_+ \times [0,1]$ and $D^{n+N_i+1} \times i$, respectively. Similarly, we define the isotopy relation to be a level-preserving concordance. All the following groups are isomorphic for $q \geq 3$:

$$C^q_n = \{\text{concordance/isotopy classes of embeddings } S^n \hookrightarrow S^{n+q}\}$$
$$\updownarrow$$
$$\{\text{concordance/isotopy classes of special embeddings } S^n \hookrightarrow S^{n+q}\}$$
$$\updownarrow$$
$$\{\text{concordance/isotopy classes of special disked embeddings } (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})\}$$
$$\updownarrow$$
$$\{\text{concordance/isotopy classes of special disked embeddings } (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1}) \text{ with Type I framing}\}$$

Furthermore, as a consequence of the tubular neighborhood theorem, one can choose a representative f in C^q_n such that $f(S^n)$ is contained in a subspace of S^{n+q} which can be identified with $S^n \times D^q$. Thus, we can consider a special knot to be $f\colon S^n \hookrightarrow S^n \times D^q$ such that $f|_{D^n_-}$ is the natural inclusion $D^n_- \hookrightarrow D^n_- \times 0$ and $f(\text{int } D^n_+) \subset \text{int}(D^n_+ \times D^q)$, together with a disk extension $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+1} \times D^N$ with a similarly defined framing of Type I. The homomorphism $\psi\colon C^q_n \to \pi_{n+1}(SG; SO, SG_q)$ is then defined as follows.

Theorem 2.2.2. *Given an element $\alpha \in C^q_n$ represented by a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with Type I framing, a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ represents $\psi(\alpha) \in \pi_{n+1}(SG; SO, SG_q)$ if there exists an extension $\bar{\phi}\colon D^{n+1} \times D^N \to D^N$ i.e., $\bar{\phi}|_{D^{n+1}\times S^{N-1}} = \phi$ such that:*

(i) $\bar{\phi}$ is regular on $0 \in D^N$ and $\bar{\phi}^{-1}(0) = \bar{f}(D^{n+1})$ as framed submanifolds,

(ii) $\bar{\phi}_x \in SO_N$ for $x \in D^n_-$,

(iii) $\bar{\phi}_x$ *is the suspension of a map* $D^q \to D^q$ *for* $x \in D^n_+$.

The homomorphism $\psi\colon C^q_n \to \pi_{n+1}(SG; SO, SG_q)$ *is well defined [21, Theorem 2.3] and is an isomorphism for* $q \geq 3$ *[21, Theorem 3.4].*

In the proof of well-definedness of ψ [21, Theorem 2.3], Haefliger shows the existence of such a map $\bar{\phi}$ as follows. Define $\bar{\phi}_-\colon D^n_- \times D^N \to D^N$ uniquely as a linear map such that $(\bar{\phi}_-)_x \in SO_N$ for $x \in D^n_-$ and $\bar{\phi}_-^{-1}(0) = f(D^n_-)$, as framed submanifolds. Using obstruction theory [21, Lemma 2.4] the restriction $\bar{\phi}_-|_{S^{n-1}\times D^q}$ can be extended to $\bar{\phi}_-|_{D^n_+\times D^q}$ with the given framing on $f(D^n_+)$. Define $\bar{\phi}_+\colon D^n_+ \times D^N \to D^N$ to be the $(N-q)$-suspension of $\bar{\phi}_-\colon D^n_+ \times D^q \to D^q$. By using [21, Lemma 2.4] again, we extend $\bar{\phi}_- \cup \bar{\phi}_+\colon S^n \times D^N \to D^N$ to a map $\bar{\phi}\colon D^{n+1} \times D^N \to D^N$ verifying $(i) - (iii)$ above. To show ψ is well defined, he uses the same argument invoking [21, Lemma 2.4] twice to construct a homotopy between two maps $\phi_0, \phi_1\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ corresponding to two concordant embeddings $(f_0, \bar{f}_0), (f_1, \bar{f}_1)\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$.

To prove the isomorphism [21, Theorem 3.4], Haefliger interprets the group $\pi_{n+1}(SG; SO, SG_q)$ in terms of cobordisms (we refer this as Pontryagin-Thom type construction). An element of $\pi_{n+1}(SG; SO, SG_q)$ represented by a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ as in Subsection 2.2.2 which is regular on e_1, corresponds to a framed $(n+1)$-submanifold $V = \phi^{-1}(e_1) \subset D^{n+1} \times S^{N-1}$ with two parts of boundary:

- $V \cap (D^n_- \times S^{N-1})$ is the graph of some map $g\colon D^n_- \to S^{N-1}$ with the framing at points $(x, g(x))$ lying inside $x \times S^{N-1}$ and orthonormal. Indeed, for $x \in D^n_-$, the map $\phi_x\colon S^{N-1} \to S^{N-1}$ is linear and therefore the preimage of e_1 is just a point.

- $V \cap (D^n_+ \times S^{N-1})$ is the suspension of a framed submanifold in $D^n_+ \times S^{q-1}$, since for any $x \in D^n_+$, the map $\phi_x\colon S^{N-1} \to S^{N-1}$ is the suspension of a map $S^{q-1} \to S^{q-1}$.

Thus, $\pi_{n+1}(SG; SO, SG_q)$ can be described as the group of cobordisms of framed $(n+1)$-manifolds with such boundary conditions.

He then considers $\bar{\phi}$ which exists by [21, Theorem 2.3]. Note that $\bar{\phi}\colon D^{n+1} \times D^N \to D^N$ can always be slightly changed so that $\bar{\phi}^{-1}(\partial D^N) \subset D^{n+1} \times \partial D^N$. The preimage $\bar{\phi}^{-1}(I) \subset$

$D^{n+1} \times D^N$ of the segment I joining 0 and e_1 in D^N is a framed $(n+2)$-manifold W with corners ($\bar{\phi}$ is chosen to be transversal to I). In particular, ∂W has the following strata:

- a free face given by the framed disk $\bar{f}(D^{n+1}) = \bar{\phi}^{-1}(0)$,
- $\partial W \cap (D^{n+1} \times S^{N-1}) = \bar{\phi}^{-1}(e_1) = V$,
- $\partial W \cap (D^n_- \times D^N)$ is the radial extension of $V \cap (D^n_- \times S^{N-1})$,
- $\partial W \cap (D^n_+ \times D^N)$ is the $(N-q)$-fold suspension of a framed submanifold in $D^n_+ \times D^q$.

As a result, he restates the homomorphism defined in Theorem 2.2.2 as follows. Given an element $\alpha \in C^q_n$ represented by a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with Type I framing, a framed submanifold $V \subset D^{n+1} \times\ S^{N-1}$ as defined above represents $\psi(\alpha) \in \pi_{n+1}(SG; SO, SG_q)$ if there exists a framed submanifold $W \subset D^{n+1} \times D^N$ with the boundary strata as given above.

According to [21, Argument 3.5], to show surjectivity he applies surgery to construct W satisfying $\partial W \cap (D^{n+1} \times S^{N-1}) = V$ for a given V. For injectivity, he shows if $[(f, \bar{f})]$ maps to the trivial element $[V]$ of $\pi_{n+1}(SG; SO, SG_q)$, then the corresponding W can be modified using surgery so that it is embedded in $D^{n+1} \times D^q$. In particular, the free face $\bar{f}(D^{n+1})$ of W is inside $D^{n+1} \times D^q \cong D^{n+q+1}$, and therefore the corresponding $f = \bar{f}|_{\partial D^{n+1}}$ is slice i.e., concordant to the trivial embedding of S^n in S^{n+q}, by Lemma 2.2.1.

2.2.4 Framed embeddings of S^n in S^{n+q}

Let us recall that we always consider framed embeddings with a framing preserving the natural orientation. For $q \geq 3$, Haefliger expressed the group FC^q_n of concordance classes of framed embeddings of S^n in S^{n+q} as $\tilde{\pi}_{n+1}(SG; SO, SG_q)$. An element in $\tilde{\pi}_{n+1}(SG; SO, SG_q)$ is represented by a continuous map $\phi\colon D^{n+1} \to SG$ i.e., for $x \in\ D^{n+1}$, $\phi(x)\colon S^{N-1} \to S^{N-1}$, for some large N, such that $\phi(D^n_-) \subset SO$, $\phi(D^n_+) \subset SG_q$ and $\phi(\partial D^n_- = \partial D^n_+) =\ id$. Again, abusing notation we also view ϕ as a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ and sometimes write ϕ_x for $\phi(x)$.

Remark 2.2.2. It is easy to see that the group $\tilde{\pi}_{n+1}(SG; SO, SG_q)$ is isomorphic to $\pi_n\Big((SO \times^h_{SG} SG_q) \simeq \mathrm{hofib}(SO \to SG/SG_q)\Big)$. Moreover, the upper long exact sequence in (2.1.2) is the long exact sequence of the pair $(SG/SG_q, SO)$.

Remark 2.2.3. [21, Section 5.1]. Two concordant framed embeddings of S^n in S^{n+q} are isotopic when $q \geq 3$ and therefore, $FC^q_n = \pi_0 Emb^{fr}(S^n, S^{n+q}) = \pi_0 Emb^{fr}_\partial(D^n, D^{n+q})$.

Lemma 2.2.2. *[21, Section 5] A framed embedding $S^n \hookrightarrow S^{n+q}$ is concordant to the trivial one if and only if it is slice i.e., if it can be extended to an embedding $D^{n+1} \hookrightarrow D^{n+q+1}$ along with the framing.*

The isomorphism $\tilde{\psi}\colon FC^q_n \to \tilde{\pi}_{n+1}(SG; SO, SG_q)$

The natural map $\tilde{\psi}$ between the two groups is defined as in the "non-framed" case. Firstly, an element in FC^q_n can be represented by a special framed knot $f\colon S^n \hookrightarrow S^{n+q}$ which is the natural inclusion on D^n_- with trivial framing $\{e_{n+2}, \ldots, e_{n+q+1}\}$, and $f(\text{int } D^n_+) \subset \text{int}(D^{n+q}_+)$ with some non-trivial framing. Such a framed knot is assigned a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ along with a framing as follows. We extend $f\colon S^n \hookrightarrow S^{n+q}$ to a disk embedding $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+N+1}$ for N large enough. For the framing on $\bar{f}(D^{n+1})$, which is defined uniquely up to homotopy, we first suspend the framing on D^n_+ inside D^{n+q}_+ to a framing inside D^{n+N+1} by adding vectors $\{e_{n+q+2}, \ldots, e_{n+N+1}\}$. Then we extend the obtained framing to the entire disk D^{n+1} inside D^{n+N+1}. Note that the framing on $\bar{f}|_{D^n_-}$ may now be non-trivial (and does not have to be a suspension), while the framing on $\bar{f}|_{D^n_+}$ is the suspension of the framing on D^n_+ inside D^{n+q}_+. But we still obtain a trivial framing on the equator $S^{n-1} = D^n_- \cap D^n_+$. Such boundary condition on the framing defined on $\bar{f}$ is referred as ***Type II***. Therefore, a representative in FC^q_n can be considered to be a ***special disked embedding*** $(f, \bar{f})$ ***with Type II framing***. For $q \geq 3$, the following groups are isomorphic:

$$FC^q_n = \{\text{concordance/isotopy classes of framed embeddings } S^n \hookrightarrow S^{n+q}\}$$
$$\updownarrow$$
$$\{\text{concordance/isotopy classes of special framed embeddings } S^n \hookrightarrow S^{n+q}\}$$

$$\updownarrow$$

$\{$concordance/isotopy classes of special disked embeddings $(S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ with Type II framing$\}$

Using the tubular neighborhood theorem, we can transform any special framed knot $f\colon S^n \hookrightarrow S^{n+q}$ into $f\colon S^n \hookrightarrow S^n \times D^q$, with a framed disk extension $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+1} \times D^N$. Thus, an element in FC_n^q can be represented by a pair $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with a framing of Type II. We define the homomorphism $\tilde{\psi}\colon FC_n^q \to \tilde{\pi}_{n+1}(SG; SO, SG_q)$ exactly as in Theorem 2.2.2 by adding to condition (ii) that $\bar{\phi}_x = id$ when $x \in S^{n-1}$. For $q \geq 3$, $\tilde{\psi}$ is an isomorphism [21, Theorem 5.7]. This result is stated without proof because the argument follows the same lines as in the "non-framed" case. Note that Lemma 2.2.2 is used in the proof of injectivity of $\tilde{\psi}$ in the same way as Lemma 2.2.1 is necessary for injectivity of ψ.

2.3 Framed disked embeddings

We now define a new group of concordance classes of special disked embeddings with a ***Type III*** framing. Namely, this time we require the framing to be trivial along D^n_-. To be precise, we consider special disked embeddings $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ where the framing on $\bar{f}$ comes with the following boundary condition: $\bar{f}|_{D^n_-}$ has trivial framing, while the framing on $\bar{f}|_{D^n_+}$ inside D^{n+N+1} is obtained as the suspension of a framing inside D^{n+q}_+.

$$\begin{aligned}\overline{FC}_n^q := \{&\text{concordance classes of special disked embeddings}\\ &(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1}) \text{ with Type III framing}\}.\end{aligned}$$

Note that since the codimension condition $q \geq 3$ is satisfied, concordance and isotopy relations coincide for special disked embeddings with all three boundary restrictions on framing.

2.3.1 The group $\pi_{n+1}(SG, SG_q)$

An element in $\pi_{n+1}(SG, SG_q)$ is represented by a continuous map $\phi\colon D^{n+1} \to SG$ such that $\phi|_{D^n_-} = id$ and $\phi(D^n_+) \subset SG_q$.

This representation is equivalent to the usual definition of a relative homotopy group i.e., $\pi_{n+1}(SG; *, SG_q) = \pi_{n+1}(SG, SG_q)$, since D^n_- can be collapsed to get the base-point in the relative group.

2.3.2 The isomorphism $\xi\colon \overline{FC}^q_n \to \pi_{n+1}(SG, SG_q)$

Following the same argument as in Subsection 2.2.3, when an element in $\overline{FC}^q_n$ is represented by a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with Type III framing, there is a natural homomorphism $\xi\colon \overline{FC}^q_n \to \pi_{n+1}(SG, SG_q)$ defined as in Theorem 2.2.2 by replacing condition *(ii)* with $\bar{\phi}_x = id$ for $x \in D^n_-$. By Haefliger's surgery construction [21, Argument 3.5] that proves [21, Theorem 3.4], we conclude:

Theorem 2.3.1. *The homomorphism $\xi\colon \overline{FC}^q_n \to \pi_{n+1}(SG, SG_q)$ is an isomorphism for $q \geq 3$.*

The sliceness Lemma 2.3.1 is used to prove injectivity of ξ, similarly to the cases of ψ and $\tilde{\psi}$. Note that Theorem 2.3.1 can be deduced from the proof of Theorem 2.1.2 given in Section 2.5. In particular, with ψ and η as isomorphisms in (2.5.1), ξ is also an isomorphism by the five lemma.

As a review, the following tables point to the main difference among all the groups we discussed in the three cases. In terms of special disked embeddings with different boundary conditions on framing of $\bar{f}$:

C^q_n	trivial framing at the base-point $*$ (Type I)
FC^q_n	trivial framing at the equator S^{n-1} (Type II)
$\overline{FC}^q_n$	trivial framing at D^n_- (Type III)

Table 2.1: Difference in framing

The corresponding homotopy groups differ as follows:

$\pi_{n+1}(SG; SO, SG_q)$	$\phi(*) = id$
$\tilde{\pi}_{n+1}(SG; SO, SG_q)$	$\phi(S^{n-1}) = id$
$\pi_{n+1}(SG, SG_q)$	$\phi(D^n_-) = id$

Table 2.2: Difference in homotopy groups

Remark-Definition 2.3.1. Consider a disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ which is not necessarily special, i.e., without a fixed behavior at D^n_-. Assume both f and $\bar{f}$ are framed embeddings such that framing on $\bar{f}(D^{n+1})$ inside D^{n+N+1} is defined by extending the suspension of the framing of $f(S^n) \subset S^{n+q}$. We call such a pair $(f, \bar{f})$ a ***framed disked embedding***. The concordance classes of such embeddings are the same as those of special ones with Type III framing representing elements in $\overline{FC}^q_n$. It is because given any framed disked embedding, we can always isotope it, so that near the base-point $* \in \partial D^n_- = S^{n-1}$ it is the identity inclusion with the trivial framing. Then we can reparametrize the sphere so that the small neighborhood of $*$ is D^n_- and the rest is D^n_+. As a result, we get a special disked embedding $(f, \bar{f})$ with Type III framing. Therefore, we can describe $\overline{FC}^q_n$ as the group of concordance classes of framed disked embeddings $(f, \bar{f})$.

Thus, all the groups C^q_n, FC^q_n and $\overline{FC}^q_n$ can be described as groups of concordance classes of "non-special" embeddings, as given in the table below:

C^q_n	embeddings $S^n \hookrightarrow S^{n+q}$
FC^q_n	framed embeddings $S^n \hookrightarrow S^{n+q}$
$\overline{FC}^q_n$	framed disked embeddings $(S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$

Table 2.3: Description of each group in terms of "non-special" embeddings

Note that special disked embeddings with framing of Type I or Type II are not framed disked embeddings because for latter we require the framing on $\bar{f}$ to be the suspension on entire boundary $\partial D^{n+1} = S^n$, see the definition above.

2.3.3 Sliceness

In this subsection, we study an interesting property of sliceness for framed disked embeddings representing elements in the group $\overline{FC}^q_n$.

Definition 2.3.1. A framed disked embedding $(f, \alpha)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ is ***slice*** if there exists a framed embedding $H\colon D^{n+2} \hookrightarrow D^{n+N'+2}$ where $N' \geq N$, such that $H|_{(\partial_- D^{n+2} = D^{n+1}_-)} = \alpha$ and $H|_{\partial_+ D^{n+2}}$ is the suspension of a framed embedding inside D^{n+q+1} i.e., $H(\partial_+ D^{n+2}) \subset D^{n+q+1} \subset \partial_+ D^{n+N'+2} = D^{n+N+1}$.

The trivial element in $\overline{FC}^q_n$ is given by the equivalence class of the trivial framed disked embedding $(id, id)\colon (S^n, D^{n+1}) \subset (S^{n+q}, D^{n+N+1})$ i.e., the trivial pair with the trivial framing.

Lemma 2.3.1. *A framed disked embedding $(f, \alpha)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ representing an element in $\overline{FC}^q_n$ is concordant to the trivial element $(id, id)\colon (S^n, D^{n+1}) \subset (S^{n+q}, D^{n+N+1})$, if and only if (f, α) is slice.*

Proof. Let $F\colon D^{n+1} \times [0,1] \hookrightarrow D^{n+N'+1} \times [0,1]$, where $N' \geq N$ be a concordance between (f, α) and (id, id). Since at $t = 1$ we have a trivial framing, we attach a half disk $\frac{1}{2}D^{n+N'+2}$ along the trivial embedding such that it extends D^{n+1} to the disk D^{n+2}, see Figure 2.1. Since F takes the boundary inside $S^{n+q} \times [0,1]$, therefore attaching this half disk gives the sliceness of the framed knot $S^n \hookrightarrow S^{n+q}$ i.e., a framed extension $D^{n+1} \hookrightarrow D^{n+q+1}$. As a result, we get a framed embedding $H\colon D^{n+2} \hookrightarrow D^{n+N'+2}$, which on one part of ∂D^{n+2} gives α and on the other, an embedding to D^{n+q+1}. Therefore, (f, α) is slice.

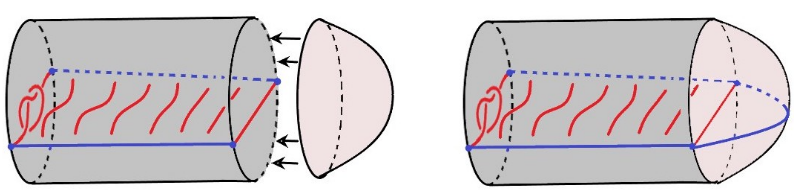


Figure 2.1: Attaching a half disk $D^{n+N'+2}$ to $D^{n+N'+1}$ at $t = 1$.

The converse is easy to prove by reversing the above argument. We remove a small half disk $\frac{1}{2}D^{n+N'+2}$ around a point in $D^{n+q+1} \subset D^{n+N'+2}$, see Figure 2.2, such that the resulting space acts as a concordance between (f, α) and (id, id).

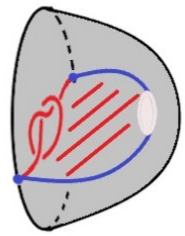


Figure 2.2: Removing a small half disk $D^{n+N'+2}$ going inside.

$\square$

2.4 Embeddings modulo immersions as special disked embeddings

Let $D^{n+\infty} := \cup_N D^{n+N}$. By a smooth embedding $D^n \hookrightarrow D^{n+\infty}$, we understand $D^n \hookrightarrow D^{n+N}$ for some N large enough. Set

$$Emb^{fr}_{\partial}(D^n, D^{n+\infty}) := \bigcup_N Emb^{fr}_{\partial}(D^n, D^{n+N}),$$

$$Imm^{fr}_{\partial}(D^n, D^{n+\infty}) := \bigcup_N Imm^{fr}_{\partial}(D^n, D^{n+N}).$$

Similarly, we define SDE^q_n to be the space of special disked embeddings $(f, \alpha)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+1+\infty})$ with Type III framing. By construction, $\pi_0 SDE^q_n = \overline{FC}^q_n$.

We claim that the space SDE^q_n has the same set of connected components as the space of embeddings modulo immersions i.e.,$\overline{FC}^q_n = \pi_0\overline{Emb}_{\partial}(D^n, D^{n+q})$. First we prove the following lemma which gives different geometric interpretations of the group $\pi_0\overline{Emb}_{\partial}(D^n, D^{n+q})$.

Lemma 2.4.1. *For $q \geq 3$,*

$$\begin{aligned}\pi_0\overline{Emb}(S^n, S^{n+q}) &= \pi_0\overline{Emb}_{\partial}(D^n, D^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}) \\ &= \pi_0\overline{Emb}^{fr}(S^n, S^{n+q}) = \pi_0\overline{Emb}^{fr}_{\partial}(D^n, D^{n+q}) = \pi_0\overline{Emb}^{fr}(S^n, \mathbb{R}^{n+q}). \quad (2.4.1)\end{aligned}$$

Proof. Let us first prove the "non-framed" case

$$\pi_0\overline{Emb}(S^n, S^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

Consider the following diagram, where the horizontal lines are fiber sequences:

$$\begin{array}{ccccc} \overline{Emb}(S^n, \mathbb{R}^{n+q}) & \longrightarrow & Emb(S^n, \mathbb{R}^{n+q}) & \longrightarrow & Imm(S^n, \mathbb{R}^{n+q}) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{Emb}(S^n, S^{n+q}) & \longrightarrow & Emb(S^n, S^{n+q}) & \longrightarrow & Imm(S^n, S^{n+q}) \end{array}$$

We view $\mathbb{R}^{n+q} = S^{n+q} - \{\infty\}$, where the "infinity" point is of codimension $n+q$. Given an embedding (resp. immersion) from S^n to S^{n+q}, we can perturb it slightly in a way that it misses the "infinity" point as $q \geq 3$, so that we get an embedding (resp. immersion) from S^n to $\mathbb{R}^{n+q}$. Therefore, the second (resp. third) vertical map is surjective on the level of π_0. For injectivity, note that an isotopy (resp. regular homotopy) of an embedding (resp. immersion) of S^n in S^{n+q} is $(n+1)$-dimensional, while the "infinity" point has codimension $n+q$, so it can still miss the point given $q \geq 3$, and therefore the second (resp. third) vertical map is bijective on π_0. The same argument holds for π_1 because $q \geq 3$. Therefore, the second and third vertical maps induce isomorphisms on π_0 and π_1 when $q \geq 3$. By five lemma, we get

$$\pi_0\overline{Emb}(S^n, S^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

It is proved in [45, Theorem 1.1] that

$$\pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}) = \pi_0\overline{Emb}_\partial(D^n, D^{n+q}).$$

Using the argument from [44, Proposition 1.2], we have that the following natural projections are weak equivalences:

$$\overline{Emb}_\partial^{fr}(D^n, D^{n+q}) \to \overline{Emb}_\partial(D^n, D^{n+q}),$$

$$\overline{Emb}^{fr}(S^n, \mathbb{R}^{n+q}) \to \overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

Similarly, one can show that $\overline{Emb}^{fr}(S^n, S^{n+q}) \to \overline{Emb}(S^n, S^{n+q})$ is a weak equivalence. Thus, we get different representations for $\pi_0\overline{Emb}_\partial(D^n, D^{n+q})$ as in (2.4.1).

□

By Smale-Hirsch theory [25, 41], we have $Imm^{fr}_\partial(D^n, D^{n+q}) \simeq \Omega^n SO(n+q)$, and since we consider the ambient dimension tend to infinity, we get $Imm^{fr}_\partial(D^n, D^{n+\infty}) \simeq \Omega^n SO$. Note that $Emb^{fr}_\partial(D^n, D^{n+N})$ is an open, dense subset of $Imm^{fr}_\partial(D^n, D^{n+N})$ of codimension $N-n$. As N gets large, the inclusion $Emb^{fr}_\partial(D^n, D^{n+N}) \hookrightarrow Imm^{fr}_\partial(D^n, D^{n+N})$ becomes highly connected, and we get $Emb^{fr}_\partial(D^n, D^{n+\infty}) \simeq Imm^{fr}_\partial(D^n, D^{n+\infty}) \simeq \Omega^n SO$.

Lemma 2.4.2. *For $q \geq 3$,*

$$\pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) = \pi_0 hofib(Emb^{fr}_\partial(D^n, D^{n+q}) \to Emb^{fr}_\partial(D^n, D^{n+\infty})).$$

Proof. By definition, $\pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q})$ is equal to $\pi_0\mathrm{hofib}(Emb^{fr}_\partial(D^n, D^{n+q}) \to Imm^{fr}_\partial(D^n, D^{n+q}) \simeq \Omega^n SO(n+q))$, which is isomorphic to $\pi_0\mathrm{hofib}(Emb^{fr}_\partial(D^n, D^{n+q}) \to \Omega^n SO)$ using the stability of the homotopy groups of SO:

$$\pi_i SO(n+q) = \pi_i SO, \quad \text{if } i \leq n+q-2.$$

Therefore, for $q \geq 3$, we have that $\pi_0\Omega^n SO(n+q) = \pi_n SO(n+q) = \pi_n SO = \pi_0\Omega^n SO$ and similarly $\pi_1\Omega^n SO(n+q) = \pi_1\Omega^n SO$. Since $\Omega^n SO \simeq Emb^{fr}_\partial(D^n, D^{n+\infty})$, we get the result as a consequence of five lemma.

□

Thus, for any element $[(f,\alpha)]$ in $\pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q})$ there corresponds an equivalence class of a pair $(\tilde{f}, \tilde{\alpha})$ where $\tilde{f}\colon D^n \hookrightarrow D^{n+q}$ and $\tilde{\alpha}\colon [0,1] \to Emb^{fr}_\partial(D^n, D^{n+\infty})$ i.e, $\tilde{\alpha}\colon D^n \times [0,1] \hookrightarrow D^{n+N} \times [0,1]$ such that $\tilde{\alpha}|_{D^n\times 0} = id$ and $\tilde{\alpha}|_{D^n\times 1} = \tilde{f}$, together with framing.

We consider $D^{n+1} \cong D^n \times [0,1]$ obtained by identifying D^n_+ to $D^n \times \{1\}$ and D^n_- to $D^n \times \{0\} \cup S^{n-1} \times [0,1]$, and then smoothening the corners. We similarly identify $D^{n+N+1} \cong D^{n+N} \times [0,1]$, for some large N. Therefore, each pair $(\tilde{f}, \tilde{\alpha})$ can be thought of as a special disked embedding with Type III framing i.e., a pair $(id \cup \tilde{f}, \tilde{\alpha})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ such that $\tilde{\alpha}|_{D^n_-}$ is the trivial inclusion $id\colon D^n_- \hookrightarrow D^{n+q}_-$ with trivial framing, and $\tilde{\alpha}|_{D^n_+}$ is the framed embedding $\tilde{f}\colon D^n_+ \hookrightarrow D^{n+q}_+$. In other words, one has a natural map

$$\mu\colon \ \mathrm{hofib}(Emb^{fr}_\partial(D^n, D^{n+q}) \to Emb^{fr}_\partial(D^n, D^{n+\infty})) \longrightarrow SDE^q_n. \tag{2.4.2}$$

On the level of π_0, we obtain

$$\mu_*\colon \pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) \to \overline{FC}^q_n,$$

$$[(f,\alpha)] \mapsto [(id \cup \tilde{f}, \tilde{\alpha})].$$

Theorem 2.4.1. *For $q \geq 3$, μ_* is an isomorphism, and therefore*

$$\pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) = \overline{FC}^q_n.$$

Proof. To show that μ_* is bijective, it suffices to show that $\mathrm{hofib}(Emb^{fr}_\partial(D^n, D^{n+q}) \to Emb^{fr}_\partial(D^n, D^{n+\infty}))$ and SDE^q_n are weakly homotopy equivalent, and then Lemma 2.4.2 concludes the result.

Consider the following diagram, where the vertical lines are fiber sequences, and the map μ is defined above (2.4.2).

$$\begin{tikzcd}
\Omega Emb^{fr}_\partial(D^n, D^{n+\infty}) \arrow[r] \arrow[d] & Emb^{fr}_\partial(D^{n+1}, D^{n+1+\infty}) \arrow[d] \\
\mathrm{hofib}(Emb^{fr}_\partial(D^n, D^{n+q}) \to Emb^{fr}_\partial(D^n, D^{n+\infty})) \arrow[r, "\mu"] \arrow[d] & SDE^q_n \arrow[d] \\
Emb^{fr}_\partial(D^n, D^{n+q}) \arrow[r, equal] & Emb^{fr}_\partial(D^n, D^{n+q})
\end{tikzcd}$$

Note that the top map is just restriction on the fibers. Moreover, it is a homotopy equivalence since $Emb^{fr}_\partial(D^{n+1}, D^{n+1+\infty}) \simeq Imm^{fr}_\partial(D^{n+1}, D^{n+1+\infty}) \simeq \Omega^{n+1}SO \simeq \Omega\Omega^n SO \simeq \Omega Emb^{fr}_\partial(D^n, D^{n+\infty})$. Thus, the map in the middle μ is also a weak homotopy equivalence. By Lemma 2.4.2, we get $\pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) = \overline{FC}^q_n$.

□

Theorem 2.1.1 is immediate by combining Lemma 2.4.1 and Theorems 2.3.1 and 2.4.1.

2.5 Geometric interpretation of long exact sequences associated with the triad $(SG; SO, SG_q)$

In this section, we prove Theorem 2.1.2 for the "non-framed" case i.e., we show the isomorphism between the sequences in (2.1.1). The proof for the framed case is similar.

Recall that Im_n^q is the group of concordance (or equivalently regular homotopy) classes of immersions of S^n in S^{n+q}. According to Haefliger [23, Section 4], any representative in Im_n^q is regular homotopic to a special immersion i.e., an immersion $f\colon S^n \looparrowright S^{n+q}$ such that $f|_{D^n_-}$ is the natural inclusion in D^{n+q}_- and $f|_{D^n_+}$ is an immersion in D^{n+q}_+. We can extend this immersion as a disk immersion $\bar{f}\colon D^{n+1} \looparrowright D^{n+N+1}$ for N large enough. Furthermore, we add framing on $\bar{f}$ by first extending the framing from the base-point $* = e_2$ to D^n_+ inside D^{n+q}_+, and then we extend this framing to D^{n+1} inside D^{n+N+1} after taking the suspension. In other words, we add disk structure and Type I framing in the same way as we did for special embeddings representing elements in C_n^q. Thus, any element in Im_n^q can be represented by a special disked immersion $(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing.

Haefliger [21, Section 4.2] has shown that Im_n^q is isomorphic to the homotopy group $\pi_n(SO, SO_q)$ where his map $\eta\colon Im_n^q \to \pi_n(SO, SO_q)$ is defined as follows: given a special disked immersion $(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing, one considers the trivialization of the normal bundle induced by the framing of $\bar{f}$. To each $x \in S^n$, one associates the $(N-q)$ frame $e_{n+q+2}, ..., e_{N+n+1}$ with respect to this trivialization. This $(N-q)$ frame defines a map $h_f\colon S^n \to V_{N,N-q}$ that represents a homotopy class $[h_f]$ in $\pi_n(V_{N,N-q}) = \pi_n(SO, SO_q)$, where $V_{N,N-q} = SO_N/SO_{N-q}$ is the Stiefel manifold.

Remark 2.5.1. Note that $h_f|_{D^n_+}$ is constantly equal to the identity inclusion $\mathbb{R}^{N-q} \subset \mathbb{R}^N$ (viewed as the base-point of $V_{N,N-q}$) because the framing on D^n_+ is given by suspension and the last $(N-q)$ vectors are $e_{n+q+2}, ..., e_{N+n+1}$. Hence, the class $[h_f]$ depends only on the framing at D^n_-.

Let us now describe the map θ appearing in the geometric long exact sequence:

$$\ldots \longrightarrow \overline{FC}^q_n \longrightarrow C^q_n \longrightarrow Im^q_n \xrightarrow{\theta} \overline{FC}^q_{n-1} \longrightarrow \ldots$$

Note that $Im^q_n = \pi_0 Imm_\partial(D^n, D^{n+q}) = \pi_n V_{n+q,n} = \pi_1 Imm_\partial(D^{n-1}, D^{n+q-1})$. The natural map $\Omega Imm_\partial(D^{n-1}, D^{n+q-1}) \to \overline{Emb}_\partial(D^{n-1}, D^{n+q-1})$ induces a map $Im^q_n = \pi_1 Imm_\partial(D^{n-1}, D^{n+q-1}) \to \pi_0 \overline{Emb}_\partial(D^{n-1}, D^{n+q-1}) = \overline{FC}^q_{n-1}$.

We can also interpret $\theta\colon Im^q_n \to \overline{FC}^q_{n-1}$ in terms of disked embeddings/immersions as follows: given a special disked immersion $(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing representing an element in Im^q_n, we consider the restriction $f|_{S^{n-1}=D^n_- \cap D^n_+} = g = id\colon S^{n-1} \hookrightarrow S^{n+q-1}$, which is the natural inclusion. Moreover, we get the disk immersion $f|_{D^n_+}\colon D^n_+ \looparrowright D^{n+q}_+$, which can be immersed inside a bigger disk D^{n+N}_+ by allowing more dimensions. As a result, we obtain a disk immersion $\bar{g} := id \circ f|_{D^n_+}\colon D^n_+ \looparrowright D^{n+N}_+$ with the restricted framing from $\bar{f}|_{D^n_+}$. Since N is large enough, we can change the framed immersion $\bar{g}$ into a framed embedding $\bar{g}'\colon D^n \hookrightarrow D^{n+N}$. The obtained pair $(g, \bar{g}')\colon (S^{n-1}, D^n) \hookrightarrow (S^{n+q-1}, D^{n+N})$ is a disked embedding where the framing on $\bar{g}'|_{S^{n-1}}$ is given by suspension of a framing inside S^{n+q-1} i.e., $(g, \bar{g}')$ is a framed disked embedding. Therefore, given a special disked immersion $(f, \bar{f})$ with Type I framing, we can assign a framed disked embedding $(g, \bar{g}')$ to it. Thus, we get a well defined map from Im^q_n to $\overline{FC}^q_{n-1}$.

The commutativity of the following diagram is given by a similar argument as in the proof of Theorem 2.4.1.

$$\begin{tikzcd}
Im^q_n \arrow[r, "\theta"] & \overline{FC}^q_{n-1} \\
\pi_1 Imm(D^{n-1}, D^{n+q-1}) \arrow[u, "\simeq"] \arrow[r] & \pi_0 \overline{Emb}_\partial(D^{n-1}, D^{n+q-1}) \arrow[u, "\simeq"']
\end{tikzcd}$$

Remark 2.5.2. Note that while defining θ, instead of $(g, \bar{g}) = (id, id \circ f|_{D^n_+})\colon (S^{n-1}, D^n) \looparrowright (S^{n+q-1}, D^{n+N})$, we can consider the pair $(id, id \circ f|_{D^n_-})$ which is same as (id, id) since $f|_{D^n_-} =$

id with framing restricted from $\bar{f}$ (such framing may not be a suspension). In $\overline{FC}^q_{n-1}$, the representative $(g, \bar{g}')$ corresponding to $(g, \bar{g}) = (id, id \circ f|_{D^n_+})$ is equivalent to the framed trivial disked embedding $(id, id) = (id, id \circ f|_{D^n_-})$ with framing as on $\bar{f}|_{D^n_-}$, since $\bar{f}$ acts as a concordance between $id \circ f|_{D^n_+}$ and $id \circ f|_{D^n_-}$. To be precise, we take a perturbation $\bar{f}'$ of $\bar{f}$ which acts as a concordance. We will use this in the following proof to show commutativity of the third square in (2.5.1).

Proof of Theorem 2.1.2. To prove the result, we need to show that each square in the following diagram commutes:

$$\begin{tikzcd}
\pi_{n+1}(SG, SG_q) \arrow[r] & \pi_{n+1}(SG; SO, SG_q) \arrow[r] & \pi_n(SO, SO_q) \arrow[r] & \pi_n(SG, SG_q) \\
\overline{FC}^q_n \arrow[u, "\xi", "\simeq"'] \arrow[r] & C^q_n \arrow[u, "\psi", "\simeq"'] \arrow[r] & Im^q_n \arrow[u, "\eta", "\simeq"'] \arrow[r, "\theta"] & \overline{FC}^q_{n-1} \arrow[u, "\xi", "\simeq"']
\end{tikzcd} \tag{2.5.1}$$

For the first square, the map $\overline{FC}^q_n \to C^q_n$ is an inclusion on the level of representatives i.e., a framed disked embedding representing an element in $\overline{FC}^q_n$ clearly represents an element in C^q_n. Therefore, the commutativity of this square is straightforward from the construction.

The commutativity of the second square is given by Haefliger [21, Section 4.4] and is easy to see. The map $C^q_n \to Im^q_n$ is obvious since an embedding is also an immersion. We have seen that the vertical map η on a given representative in Im^q_n depends only on the behavior of the representative on D^n_-, see Remark 2.5.1. Similarly, the top horizontal map $\pi_{n+1}(SG; SO, SG_q) \to \pi_n(SO, SO_q)$ is defined by restricting the representatives in $\pi_{n+1}(SG; SO, SG_q)$ to the half-disk D^n_-.

We now check the commutativity of the third square. Given an element $\alpha \in Im^q_n$ represented by a special disked immersion $(f, \bar{f}) \colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing, by Remark 2.5.2, the corresponding element $\theta(\alpha)$ in $\overline{FC}^q_{n-1}$ can be represented by the framed trivial disked embedding $(id, id) = (id, id \circ f|_{D^n_-}) \colon (S^{n-1}, D^n) \hookrightarrow (S^{n-1} \times D^q, D^n \times D^N)$, with the framing obtained as a restriction $\bar{f}|_{D^n_-}$. Recall that on $S^{n-1} = D^n_- \cap D^n_+$, the framing is given by suspension of a framing inside S^{n+q-1}. We can homotope the obtained framing on S^{n-1} so that it becomes trivial on D^{n-1}_-. Now, under the vertical map ξ, the image of $\theta(\alpha)$ is represented by a map $\phi\colon D^n \times S^{N-1} \to S^{N-1}$ with an extension $\bar{\phi}\colon D^n \times D^N \to D^N$ defined linearly by $(x, y) \mapsto r(x)(y)$, for some rotation r given by the framing on $\bar{f}|_{D^n_-}$. More precisely, $r\colon D^n \to SO(N)$ is such that $r|_{\partial D^n = S^{n-1}}$ is a suspension of rotation in $SO(q)$ with $r = id$ on D^{n-1}_-, by construction. The map $\bar{\phi}$ satisfies the definition of ξ (see Subsection 2.3.2), since $\bar{\phi}^{-1}(0) = \bar{f}(D^n_-) = D^n \times 0$, with $\bar{\phi}|_{S^{n-1}} \in SO(q)$ such that $\bar{\phi}_x = id$ for $x \in D^{n-1}_-$ and $\bar{\phi}_x$ is the suspension of a map $D^q \to D^q$ for any $x \in D^{n-1}_+$. Moreover, $\bar{\phi}$ also represents an element in $\pi_n(SO, SO_q)$ and is precisely the representative that we get for $\eta(\alpha)$, as η also depends only on the non-trivial framing on D^n_- (see Remark 2.5.1). Therefore, the square commutes. $\square$

2.6 Applications

2.6.1 Known computations

For C^q_n, the well-known computations were done by Haefliger [19, 20, 21] in the 1960s and later by Milgram [31] in the early 1970s. To the best of our knowledge, no computations were done ever since. The Manifold Atlas webpage [1] describes all the known groups C^q_n. Haefliger [19] has shown that $C^q_n = 0$ for $n < 2q - 3$. Furthermore, he proved that for $q \geq 3$ (see [21, Corollary 8.14]),

$$C^q_{2q-3} = \begin{cases} \mathbb{Z} & q \ odd, \\ \mathbb{Z}_2 & q \ even. \end{cases}$$

For odd q, the generator is given by the Haefliger trefoil knot [20]. It is an interesting question whether the Haefliger trefoil is a generator for the even case.

There are only a few computations for FC^q_n in the literature. For example, Haefliger [21, Theorem 5.17] has shown that $FC^3_3 = \mathbb{Z} \oplus \mathbb{Z}$. Moreover, it is easy to see that for $n < 2q - 3$, we get $FC^q_n = \pi_n(SO_q)$, see Proposition 2.6.2.

The groups $\overline{FC}^q_n = \pi_{n+1}(SG, SG_q)$ are related to the homotopy groups of spheres.

Some of these groups are known, in particular, $\overline{FC}_2^3 = \pi_3(SG, SG_3) = \mathbb{Z}_2$ and $\overline{FC}_3^3 = \pi_4(SG, SG_3) = \mathbb{Z}$, found in [21, Section 5.16] and [40, Proof of Lemma 3.1].

The rational computations of $\overline{FC}_n^q$ are known [2, Corollary 20], [16, Section 5.7], and can also be computed directly as follows.

Proposition 2.6.1. *For $q \geq 3$,*

$$\overline{FC}_n^q \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & n = q-1, \quad q \ \textit{even}, \\ \mathbb{Q} & n = 2q-3, \quad q \ \textit{odd}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since $\overline{FC}_n^q = \pi_{n+1}(SG, SG_q)$, we consider the long exact sequence of the pair (SG, SG_q):

$$\ldots \longrightarrow \pi_{n+1}(SG) \longrightarrow \pi_{n+1}(SG, SG_q) \longrightarrow \pi_n(SG_q) \longrightarrow \pi_n(SG) \longrightarrow \ldots \tag{2.6.1}$$

The rational homotopy groups $\pi_n^{\mathbb{Q}}(SG_q)$ can be easily computed by considering the long exact sequence associated with the fibration $\Omega_*^{q-1}S^{q-1} \to SG_q \to S^{q-1}$, where $\Omega_*^{q-1}S^{q-1}$ is the component of loops of degree one. The connecting homomorphism $\pi_{n+1}(S^{q-1}) \to \pi_n(\Omega_*^{q-1}S^{q-1}) = \pi_{n+q-1}(S^{q-1})$ is given by the Whitehead bracket $[id_{S^{q-1}}, -]$. Note that all $\pi_n(SG)$ are torsions being the stable homotopy groups of spheres, hence, $\pi_n^{\mathbb{Q}}(SG) = 0$. Using the rational homotopy groups of spheres, we get

$$\pi_n^{\mathbb{Q}}(SG_q) = \begin{cases} \mathbb{Q} & n = q-1, \quad q \ \textit{even}, \\ \mathbb{Q} & n = 2q-3, \quad q \ \textit{odd}, \\ 0 & \textit{otherwise}. \end{cases}$$

Thus, from the long exact sequence (2.6.1) we get $\pi_{n+1}^{\mathbb{Q}}(SG, SG_q) = \pi_n^{\mathbb{Q}}(SG_q)$ and that concludes the result. $\square$

2.6.2 Metastable range

From [21, Section 4.4], one can deduce the following stability result for the groups $\overline{FC}_n^q$ and FC_n^q:

Proposition 2.6.2. *For $n < 2q-3$, $\overline{FC}_n^q = \pi_{n+1}(SO, SO_q)$ and $FC_n^q = \pi_n SO_q$.*

Proof. Consider the long exact sequence associated with the triad $(SG; SO, SG_q)$ given in [21, Section 4.4]:

$$\ldots \to \pi_{n+1}(SG, SG_q) \to \pi_{n+1}(SG; SO, SG_q) \to \pi_n(SO, SO_q) \to \pi_n(SG, SG_q) \to \ldots \tag{2.6.2}$$

By [21, Corollary 6.6], the groups $\pi_{n+1}(SG; SO, SG_q) = C_n^q = 0$ for $n < 2q-3$. Therefore, from the above sequence, we get $\pi_{n+1}(SO, SO_q) = \pi_{n+1}(SG, SG_q) = \overline{FC}_n^q$ for $n < 2q-4$. Moreover, any element of C_n^q is trivial as immersion for $n < 2q-1$, see [21, Corollary 6.10]. Thus, the homomorphism $\pi_{2q-2}(SG; SO, SG_q) \to \pi_{2q-3}(SO, SO_q)$ in (2.6.2) is trivial, and we get $\pi_{2q-3}(SO, SO_q) = \pi_{2q-3}(SG, SG_q) = \overline{FC}_{2q-4}^q$.

For the second equality, we consider the geometric long exact sequence given by Haefliger [21, Section 5.9]:

$$\ldots \longrightarrow \pi_n SO_q \longrightarrow FC_n^q \longrightarrow C_n^q \longrightarrow \pi_{n-1} SO_q \longrightarrow \ldots \tag{2.6.3}$$

The result easily follows for $n < 2q-4$ since $C_n^q = 0$ for $n < 2q-3$. When $n = 2q-4$, the homomorphism $C_{2q-3}^q \to \pi_{2q-4} SO_q$ in (2.6.3) is the composition $C_{2q-3}^q = \pi_{2q-2}(SG; SO, SG_q) \xrightarrow{0} \pi_{2q-3}(SO, SO_q) \to \pi_{2q-4}(SO_q)$, and therefore is also trivial. $\square$

Lemma 2.6.1. *For $i \leq q-2$, $\pi_i(SG_q) = \pi_i(SG)$.*

Proof. When $i \leq q-1$, we have $\pi_i(SG, SG_q) = \pi_i(SO, SO_q) = 0$, where we get the first equality from Proposition 2.6.2, and the second one using the fact that $\pi_i(SO, SO_q) = 0$ for $i < q$. Therefore, from the long exact sequence (2.6.1), we conclude that for $i \leq q-2$, $\pi_i(SG_q) = \pi_i(SG)$. $\square$

2.7 Codimension two case

In this section, we consider the concordance groups C_n^2 and $\overline{FC}_n^2$. Haefliger briefly mentions that the homomorphism $\psi\colon C_n^2 \to \pi_{n+1}(SG; SO, SG_2) = \pi_{n+1}(SG/SG_2, SO/SO_2) = \pi_{n+1}(SG, SO)$ is an isomorphism for n even (see [21, Remark 3.8]). This follows from Kervaire results in [26, Theorems III.3, III.6] and the Kervaire-Milnor exact sequence in [27] (see also [21, §4.10])

$$\ldots \to P_{n+1} \xrightarrow{b} \theta_n \to \pi_n(SG, SO) \to P_n \xrightarrow{b} \ldots,$$

where θ_n is the group of h-cobordism classes of homotopy n-spheres, and P_n is the group of cobordism classes of compact framed manifolds bounded by homotopy spheres. In particular, $P_n = 0$ for n odd, $\mathbb{Z}_2$ for $n = 2(\text{mod } 4)$, $\mathbb{Z}$ for $n = 0(\text{mod } 4)$. Moreover, according to Kervaire [26], every codimension two knot $S^n \hookrightarrow S^{n+2}$ is concordant to a reparametrization of the trivial knot, when n is even, while the group of concordance classes in this case is $C_n^2 = \theta_{n+1}/bP_{n+2}$. Note that the group $\theta_{n+1} = \pi_0 \mathrm{Diff}_\partial(D^n)$, $n \neq 4$, is responsible for reparametrization of knots.

Haefliger's argument in [21, §3.6] for the surjection of $\psi\colon C_n^q \to \pi_{n+1}(SG; SO, SG_q)$, $q > 2$, works when n is odd and $q = 2$. The surjectivity in this case is used in the proof below.

Proof of Theorem 2.1.3. Consider the commutative diagram in (2.5.1) with $q = 2$ and n odd (without loss of generality):

$$\begin{tikzcd}
Im^2_{n+1} \arrow[r] \arrow[d, "\eta"', "\simeq"] & \overline{FC}^2_n \arrow[r] \arrow[d, "\xi"] & C^2_n \arrow[r] \arrow[d, "\psi", two heads] & Im^2_n \arrow[r] \arrow[d, "\eta"', "\simeq"] & \overline{FC}^2_{n-1} \arrow[r] \arrow[d, "\xi"] & C^2_{n-1} \arrow[d, "\psi"', "\simeq"] \\
\pi_{n+1}(SO) \arrow[r] & \pi_{n+1}(SG) \arrow[r] & \pi_{n+1}(SG, SO) \arrow[r] & \pi_n(SO) \arrow[r] & \pi_n(SG) \arrow[r] & \pi_n(SG, SO).
\end{tikzcd} \tag{2.7.1}$$

To be precise, in (2.7.1) instead of $\pi_*(SO)$ and $\pi_*(SG)$ one should use $\pi_*(SO, SO_2)$ and $\pi_*(SG, SG_2)$ respectively, in the special cases $* = 1$ and 2. Using the four lemma on the left side implies ξ is a surjection. Moreover, the four lemma on the right side implies ξ is an isomorphism. This concludes the result. $\square$

2.7.1 Some computations

For $n \leq 6$, using the known stable homotopy groups of SG and SO, and the J homomorphism we get the following groups:

n	1	2	3	4	5	6
$\pi_n(SG, SO)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
$\pi_n(SG, SG_2)$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$

Table 2.4: Computations in codimension two

Note that for $n \geq 3$, the group $\pi_n(SG, SG_2) = \pi_n(SG) = \pi_n^S$, the stable homotopy groups of the sphere.

Chapter 3

Polyak-Viro type formulas for 2- and 3-component links

3.1 Introduction

In the classical case of knots i.e., embeddings $S^1 \hookrightarrow \mathbb{R}^3$, Goussarov, Polyak and Viro [18, 34] gave explicit formulas for some finite type knot invariants in terms of Gauss diagrams. A ***Gauss diagram*** is an oriented circle with the information on overcrossings and undercrossings at double points of the projection. The preimages of each double point is connected with an oriented chord, oriented from overcrossing to undercrossing. Moreover, each chord in the diagram is equipped with a sign of the corresponding double point given by the local writhe number. The following figure shows a Gauss diagram associated the 4_1 knot (picture from [34, 35]):

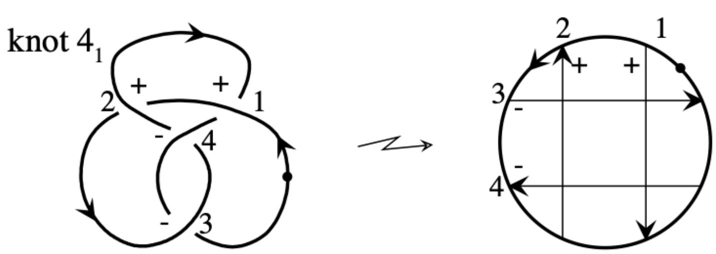


Figure 3.1: Gauss diagram of 4_1 knot

An example of Polyak-Viro type formula is given for the Casson invariant in [35]. It is defined as follows: If G is any based Gauss diagram of a knot K, then $\upsilon_{\mathbf{2}}(K)$ is the sum $\Sigma\epsilon(c_1)\epsilon(c_2)$ over all subdiagrams of G isomorphic to

and $\epsilon(c_1)$,$\epsilon(c_2)$ are the signs of the chords c_1, c_2 of the subdiagrams.

In particular, $\upsilon_{\mathbf{2}}(\mathbf{4_1}) = \mathbf{-1}$, given by the contribution of the following subdiagram:

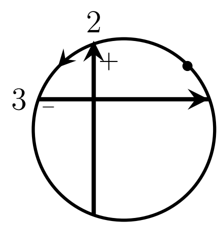


Figure 3.2: Subdiagram for computing $\upsilon_{\mathbf{2}}(4_1)$

Another way to describe a Gauss diagram is that it is the set of double points under the projection forming a 0-submanifold with involution with some additional information of overcrossing and undercrossing. For example, the Gauss diagram corresponding to 4_1-knot can be described as

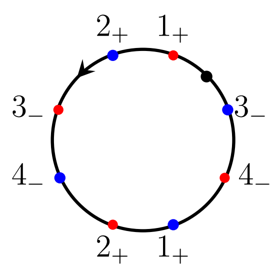


Figure 3.3: Another description of the Gauss diagram of 4_1-knot

where red denotes the overcrossing and blue denotes the undercrossing.

The goal for us is to find an analogue to the Polyak-Viro type formulas for high dimensional embeddings $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$, where M is closed, oriented, but possibly disconnected manifold. As done in the classical case when $\ell = 1$, we first project the embedding $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$ to $\mathbb{R}^{3\ell-1}$. The set of double points under this projection forms an $(\ell-1)$-submanifold L, which we can view as a link in $M^{2\ell-1}$. This link is a manifold with involution such that one half of the link is overcrossing, and the other half is undercrossing. Moreover, the fixed points of involution form a codimension one submanifold in L, containing the points of Whitney umbrella. This embedded manifold with involution is an analogue to the Gauss diagram in the classical case, because the Gauss diagram is also a 0-dimensional manifold with involution encoding the information of overcrossing and undercrossing. Thus, the task is to get an invariant expressed in terms of this link of double points under the projection $M^{2\ell-1} \to \mathbb{R}^{3\ell-1}$ for $\ell \geq 2$.

The invariants of 2- and 3-component links of dimension $(2\ell-1)$ in $\mathbb{R}^{3\ell}$ that we get are described in terms of the linking numbers between different components of L.

3.2 The set of double points

For an embedding $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$, we take its generic projection to $\mathbb{R}^{3\ell-1}$. Here is a general theorem for generic maps:

Theorem 3.2.1. *[46, Theorem 4.7.6] Any generic smooth map $f : M^{2\ell-1} \to \mathbb{R}^{3\ell-1}$ is an embedding except as follows: there are double points, forming an $(\ell-1)$-submanifold $L(f)$, and singular points forming an $(\ell-2)$-submanifold $\Sigma^1(f)$.*

Near $\Sigma^1(f)$, f is locally given by

$$y_1 = \frac{1}{2}x_1^2, \quad y_j = x_j \ (2 \leq j \leq 2\ell-1), \quad y_{i+2\ell-2} = x_1 x_i \ (2 \leq i \leq \ell+1),$$

Here, the rank of the differential of f drops by one on $\Sigma^1(f)$. Such singular points are known as Whitney umbrella points. Hence, $\overline{L(f)} = L(f) \cup \Sigma^1(f)$ is a smooth closed submanifold with involution in M (with $\Sigma^1(f)$ as the set of fixed points), and $f(\overline{L(f)})$ is an $(\ell-1)$-submanifold of $\mathbb{R}^{3\ell-1}$ with boundary $f(\Sigma^1(f))$.

We can say more about $\overline{L(f)}$ when $f : M^{2\ell-1} \to \mathbb{R}^{3\ell-1}$ is obtained by projecting an embedding. In particular, we have

$$L(f) = L_+(f) \cup L_-(f)$$

where $L_+(f)$ is the overcrossing and $L_-(f)$ is the undercrossing. Moreover, $\overline{L_+(f)} = L_+(f) \cup \Sigma^1(f)$ is a manifold with boundary $\partial\overline{L_+(f)} = \Sigma^1(f) = \partial\overline{L_-(f)}$. Therefore, $\overline{L(f)}$ is the double of the manifold $\overline{L_+(f)}$ with boundary, embedded in M, thus forming an $(\ell-1)$-link in M.

3.2.1 Orientation conventions

1. Note that the codimension of $L(f)$ in $M^{2\ell-1}$ is ℓ. When the codimension is even i.e., $\ell = 2k$, the submanifold $L(f) \subset M^{4k-1}$ and its image $f(L(f)) \subset \mathbb{R}^{6k-1}$ admit natural orientations. Given any generic map $f : M^{4k-1} \to \mathbb{R}^{6k-1}$, we define *Ekholm orientation* (see [13]) on the set of double points and its image as follows:

   For a self intersection $x \in f(L(f))$, let $\vec{u} = (u_1, \ldots, u_{2k-1})$ be a basis of $T_x f(L(f))$. We can choose partial tangent frames $\vec{v} = (v_{2k}, \ldots, v_{4k-1})$ and $\vec{w} = (w_{2k}, \ldots, w_{4k-1})$ of the corresponding two sheets coming together along $x \in f(L(f))$ so that $(\vec{u}, \vec{v})$ and $(\vec{u}, \vec{w})$ are the positive bases of these two sheets. We say $\vec{u}$ represents the positive orientation of $f(L(f))$ if the frame $(\vec{u}, \vec{v}, \vec{w})$ forms a positive basis of $\mathbb{R}^{6k-1}$. Since the codimension of f is even, this induced orientation is independent of the ordering (since $\vec{v}$ and $\vec{w}$ are $2k$-dimensional). Thus, the orientation on $L(f)$ is defined such that $f : L(f) \to f(L(f))$ preserves the orientation. This orientation is preserved by the involution and therefore it is not globally defined on the components of $\overline{L(f)}$ with Whitney Umbrellas points. This can be fixed in case the map $f : M^{4k-1} \to \mathbb{R}^{6k-1}$ is obtained by projecting an embedding, see below.

2. When f corresponds to the projection of an embedding i.e., $f : M^{4k-1} \hookrightarrow \mathbb{R}^{6k} \to \mathbb{R}^{6k-1}$, we can define an orientation on the link $\overline{L(f)}$ by the following convention: we take Ekholm orientation on $\overline{L_+(f)}$ and the opposite one on $\overline{L_-(f)}$,

   $$\overline{L(f)} = \overline{L_+(f)} \cup -\overline{L_-(f)}$$

   This gives a way to define orientation on components with Whitney Umbrella points. We call such an orientation as *adjusted Ekholm orientation.*

3. When ℓ is odd i.e., $\ell = 2k+1$, and $f : M^{4k+1} \to \mathbb{R}^{6k+2}$ corresponds to the projection of an embedding, we can still define orientation on $L(f)$ by fixing the order on the two sheets coming along the self intersection point.

   For a self intersection $x \in f(L(f))$, let $\vec{u} = (u_1, \ldots, u_{2k})$ be a basis of $T_x f(L(f))$. We fix the order and orientation on the two sheets coming together along $x \in f(L(f))$ as follows. At each double point x, one of the sheets is above the other sheet with respect to the direction of the projection. Choose a tangent frame $\vec{v} = (v_{2k+1}, \ldots, v_{4k+1})$ for the sheet which is above, and $\vec{w} = (w_{2k+1}, \ldots, w_{4k+1})$ for the sheet which lie under, so that $(\vec{u}, \vec{v})$ and $(\vec{u}, \vec{w})$ are the positive bases of these two sheets. We say $\vec{u}$ represents the positive orientation of $f(L(f))$ if the frame $(\vec{u}, \vec{v}, \vec{w})$ forms a positive basis of $\mathbb{R}^{6k+2}$. Thus, the orientation on $L(f)$ is defined such that $f : L(f) \to f(L(f))$ preserves the orientation and we have $\overline{L(f)} = \overline{L_+(f)} \cup \overline{L_-(f)}$.

Applying the Theorem 3.2.1 to $f : M^3 \to \mathbb{R}^5$, we get that $\overline{L(f)}$ is 1-dimensional with a 0-dimensional submanifold $\Sigma^1(f)$. The possible singularities are double points and the points of Whitney umbrella. Since the codimension is even, there is a natural orientation on $\overline{L(f)}$ and $f(\overline{L(f)})$. Thus, $\overline{L(f)}$ is a 1-manifold with involution in M which is an oriented link with the components of following three types:

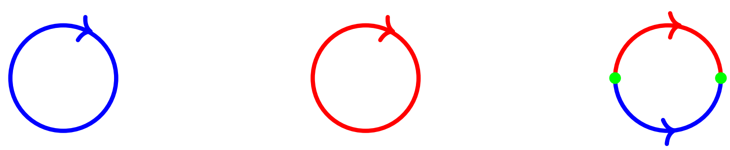


Figure 3.4: Possible components of $\overline{L(f)}$ when $\ell = 1$

We denote the undercrossing in blue, overcrossing in red and the points of Whitney umbrellas in green. The images of these link components under f are given in $\mathbb{R}^5$ by

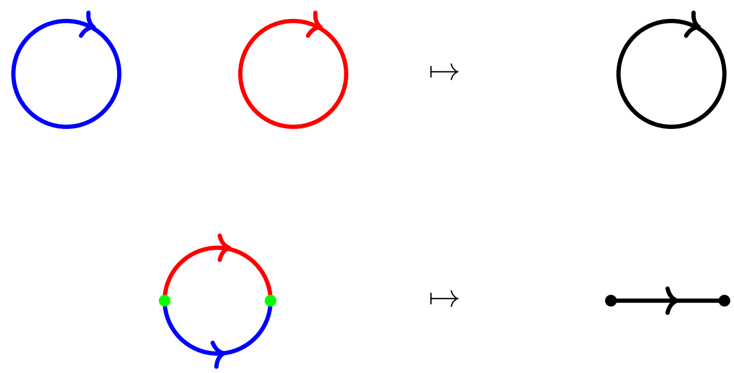


Figure 3.5: Corresponding $f(\overline{L(f)})$ for $\ell = 1$

For pictures, we use Ekholm orientation.

3.3 Cobordisms induced by isotopy

We are looking for an isotopy invariant, so now we see what happens when we isotope $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$. An analogue of Theorem 3.2.1 is as follows:

Theorem 3.3.1. *A generic smooth map $H : M^{2\ell-1} \times [0,1] \to \mathbb{R}^{3\ell-1} \times [0,1]$ is an embedding except as follows: the set of double points $L(H)$ is an immersed ℓ-manifold with transverse self-intersection, and $H(L(H))$ can have a discrete set of triple points with transverse self-intersection. Moreover, $\Sigma^1(H)$ is an $(\ell-1)$-submanifold containing the points of Whitney umbrella. Hence, $\overline{L(H)} = L(H) \cup \Sigma^1(H)$ is a smooth immersed compact ℓ-manifold with involution in $M \times [0,1]$ (with the set of fixed points $\Sigma^1(H)$), and $H(\overline{L(H)})$ is an immersed ℓ-manifold in $\mathbb{R}^{3\ell-1} \times [0,1]$.*

In particular, when we project an isotopy, we get a cobordism between $\overline{L(f_0)}$ and $\overline{L(f_1)}$, which can be described as

$$\overline{L(H)} = \overline{L_+(H)} \cup \overline{L_-(H)}.$$

Thus, $\overline{L(H)}$ is a double of an ℓ-manifold with corners $\overline{L_+(H)}$, with internal boundary $\partial\overline{L_+(H)} = \Sigma^1(H)$ and other parts of boundary $\overline{L_+(f_0)}$ and $\overline{L_+(f_1)}$. Moreover, $\overline{L(H)}$ itself is a cobordism with involution, with a codimension one submanifold of fixed points of Whitney Umbrellas.

3.3.1 Elementary cobordisms when $\ell = 2$

We first give an analogy of the Reidemeister moves from the classical case $\ell = 1$ to $\ell = 2$.

By Theorem 3.3.1, we get a 2-dimensional cobordism $\overline{L(H)} \subset M^3 \times [0,1]$ as a double of a manifold with boundary $\Sigma^1(H)$, whose image under H may have triple points. Under isotopy, we get the following possible cobordisms as the double $\overline{L(H)}$.

1) The analogue of the first Reidemeister move:

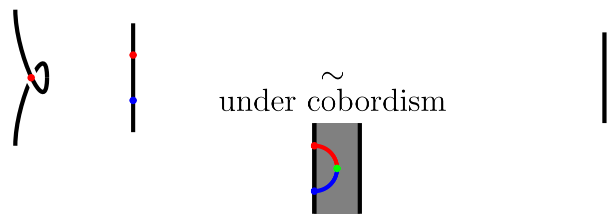


Figure 3.6: First Reidemeister move for $\mathbb{R} \hookrightarrow \mathbb{R}^3$

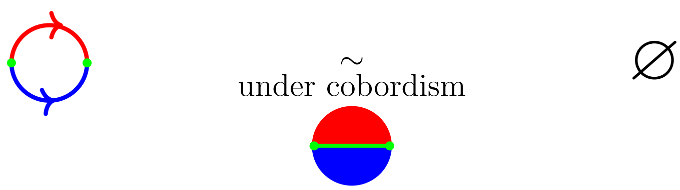


Figure 3.7: First Reidemeister move for $M^3 \hookrightarrow \mathbb{R}^6$

2) The analogue of the second Reidemeister move:

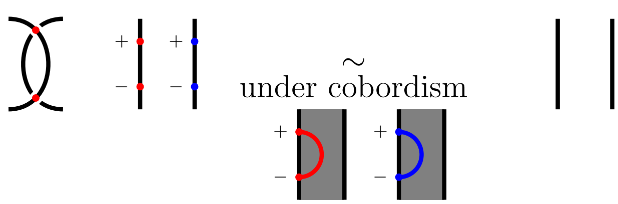


Figure 3.8: Second Reidemeister move for $\mathbb{R} \hookrightarrow \mathbb{R}^3$

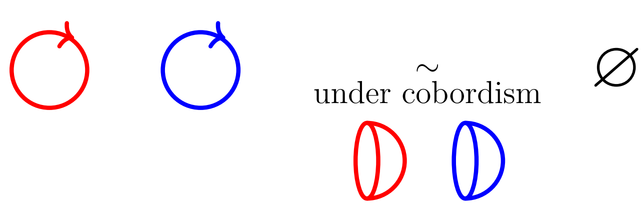


Figure 3.9: Second Reidemeister move for $M^3 \hookrightarrow \mathbb{R}^6$

There are other possible embedded cobordisms which do not appear in the classical case.

3) Saddle cobordism containing points of Whitney umbrella:

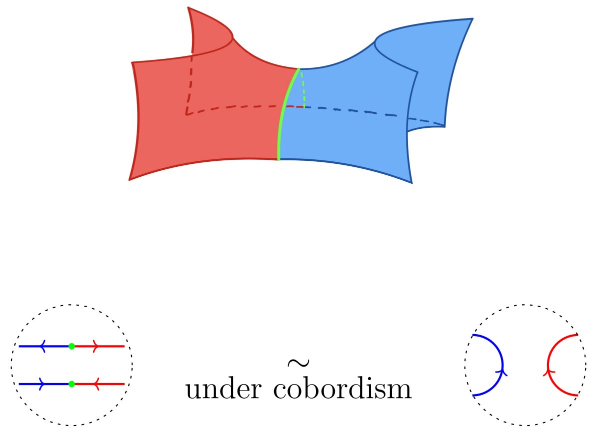


Figure 3.10: Saddle (and its local picture) with Whitney umbrella points

We can get rid of the points of Whitney umbrella using this cobordism.

4) Two saddle cobordisms occurring simultaneously:

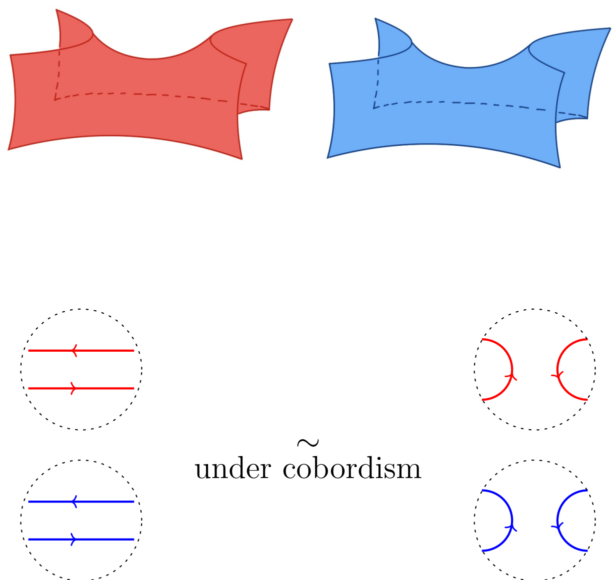
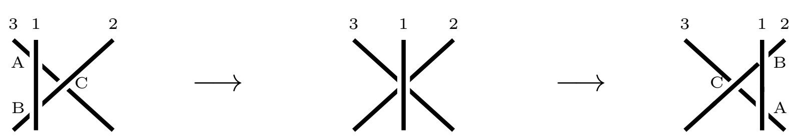


Figure 3.11: Saddles (and their local pictures) that occur simultaneously

In addition to embedded cobordisms, there is another cobordism coming from the triple points, which is not embedded.

5) the analogue of the third Reidemeister move:



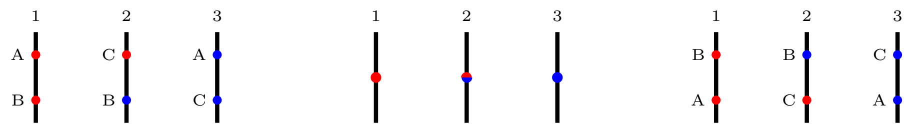


Figure 3.12: Third Reidemeister move (and its local picture) for $\mathbb{R} \hookrightarrow \mathbb{R}^3$

For $M^3 \hookrightarrow \mathbb{R}^6$, each strand $1, 2$ and 3 is 3-dimensional and their double-intersections A, B, and C are 1-dimensional.

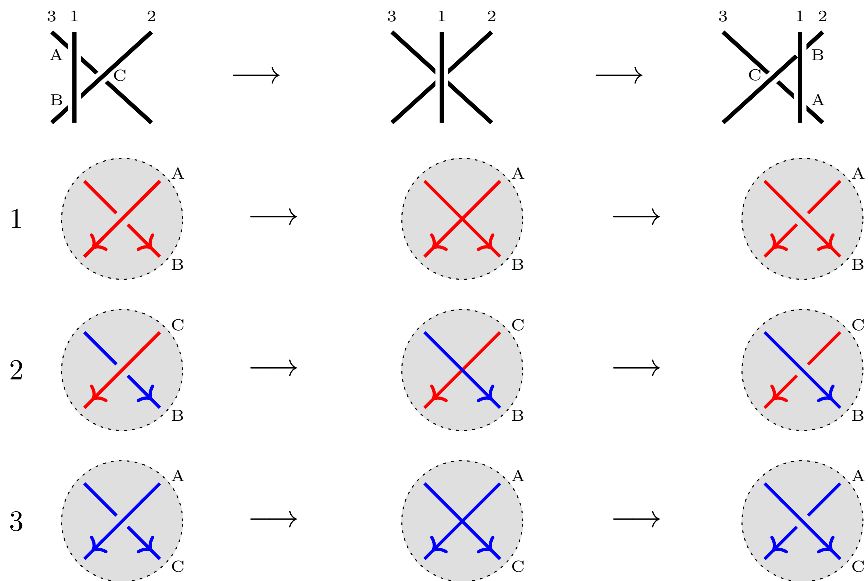


Figure 3.13: Third Reidemeister move (and its local picture) for $M^3 \hookrightarrow \mathbb{R}^6$

3.3.2 General cobodisms

In general, there are two different ways to describe the cobordisms we get under isotopy. In $M^{2\ell-1} \times [0,1]$, we can consider the cobordism $\overline{L_+(H)}$, which is an ℓ-manifold with boundary $\partial\overline{L_+(H)} = \Sigma^1(H) \cup \overline{L_+(f_0)} \cup \overline{L_+(f_1)}$. On the other hand, one can consider the cobordism given by the double of $\overline{L_+(H)}$ i.e., $\overline{L(H)}$, which is an ℓ-manifold with involution, with a codimension one submanifold $\Sigma^1(H)$ of fixed points of Whitney Umbrellas. In the next section, we discuss the first point of view i.e., cobordisms of manifolds with boundary. Additionally to cobordisms, there is a high dimensional analogue to the third Reidemeister move which corresponds to triple points in the isotopy of $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$. Let A, B, and C denote the $(\ell-1)$-double intersection of $1 \cap 3$, $1 \cap 2$ and $2 \cap 3$, respectively, when projected to $\mathbb{R}^{3\ell-1}$. Under isotopy, we also get points of triple intersection in $\mathbb{R}^{3\ell-1} \times [0,1]$.

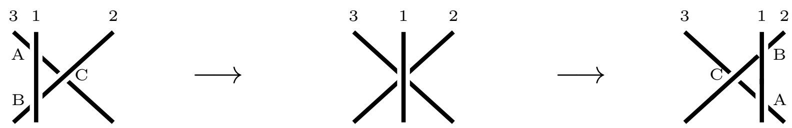


Figure 3.14: Third Reidemeister move for $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$

At the moment of the triple intersection, the three preimages in each component occur as double intersections inside $(2\ell-1)$-balls given as follows:

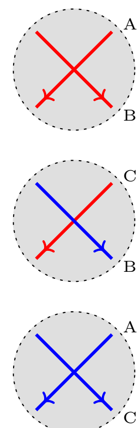


Figure 3.15: The preimages of the triple intersection

Under the isotopy, the preimage of passing through the triple point corresponds to the local change in the crossing of the double intersection in each component:

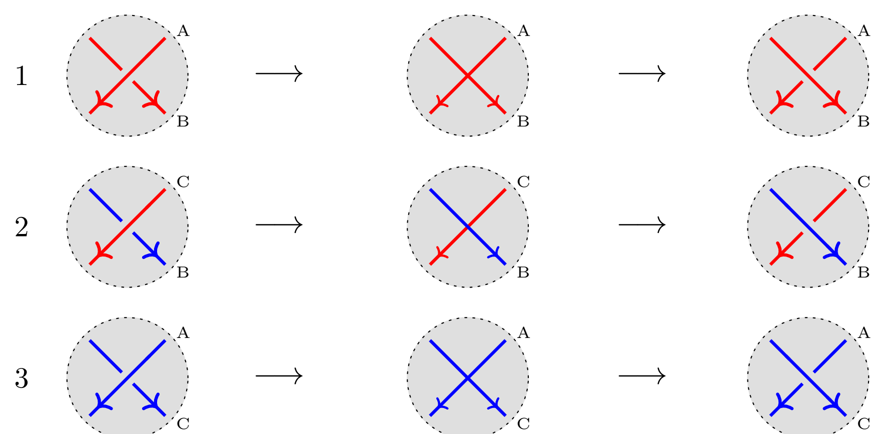


Figure 3.16: Crossing change when we pass the triple point

The sign of the crossings change under this move.

3.3.3 Cobordisms of manifolds with boundary

Let us first recall the Morse theory for manifolds with boundary.

Let $(M_0, \partial M_0)$ and $(M_1, \partial M_1)$ be two compact, oriented $(\ell-1)$-manifolds with boundary. We say $(\Omega, \partial_I \Omega)$ is a cobordism between $(M_0, \partial M_0)$ and $(M_1, \partial M_1)$, if Ω is a compact, oriented ℓ-manifold with boundary $\partial\Omega = \partial_I\Omega \cup M_0 \cup M_1$ such that $M_0 \cap M_1 = \emptyset$, $\partial_I\Omega \cap M_0 = \partial M_0$ and $\partial_I\Omega \cap M_1 = \partial M_1$. Alternatively, Ω can be thought of as a manifold with corners $M_0 \sqcup M_1$, and $\partial_I\Omega$ is what we call the internal part of the boundary of Ω. A function $f : \Omega \to [0, 1]$ is called a *Morse function* if $f|_{M_0} = 0$, $f|_{M_1} = 1$, and all of its critical points are non-degenerate and are contained in the interior of Ω. Moreover, we require that $f|_{\partial_I\Omega}$ is a Morse function on $\partial_I\Omega$. All critical points (both on Ω and on $\partial_I\Omega$) are assumed to occur at distinct levels.

Given a Morse function $f : \Omega \to [0, 1]$ and a critical point $p \in \Omega - \partial\Omega$ of index i, $0 \le i \le \ell$, there exist local coordinates $x_1, x_2, \ldots, x_\ell$ centered at p, such that

$$f(x_1, \ldots, x_\ell) = f(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_\ell^2.$$

We define a *boundary critical point* as a critical point of $f|_{\partial_I\Omega}$. We can always make a choice of coordinates centered at a boundary critical point p^∂ to be $x_1, x_2, \ldots, x_\ell$ such that $x_\ell \ge 0$. Then, near the boundary critical point p^∂ of index i, $0 \le i \le \ell - 1$, the function will be given by

$$f(x_1, \ldots, x_\ell) = f(p^\partial) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_{\ell-1}^2 \pm x_\ell.$$

Surgeries on the level sets of Ω from interior critical points

Let $f : \Omega \to [0, 1]$ be a Morse function. Consider the sublevel sets

$$\Omega_{(-\infty,a]} := \{x \ \in \Omega | f(x) \le \ a\}$$

and

$$\Omega_{[a,b]} := \{x \in \Omega | a \le f(x) \le b\}.$$

Note that the boundary part $f^{-1}(a) := \Omega_a \subset \partial\Omega_{(-\infty,a]}$ is the level set. If there is one critical point p of index i, $0 \le i \le \ell$, in the interior of $\Omega_{[a,b]}$, then $\Omega_{(-\infty,b]} \cong \Omega_{(-\infty,a]} \cup H_i$, where $H_i = D^i \times D^{\ell-i}$ is the ℓ-dimensional handle of index i, attached along some framed embedding $\phi : S^{i-1} \times D^{\ell-i} \hookrightarrow \Omega_a$. In particular, the level set Ω_b is obtained from Ω_a by a spherical surgery: we remove $\phi(S^{i-1} \times D^{\ell-i})$ from Ω_a and replace it by $D^i \times S^{\ell-i-1}$,

$$\Omega_b \cong \Omega_a \backslash \phi(S^{i-1} \times D^{\ell-i}) \bigcup_{S^{i-1} \times S^{\ell-i-1}} D^i \times S^{\ell-i-1}.$$

Note that when we pass the critical point of index 0, we attach a handle $H_0 = D^0 \times D^\ell$ of index 0, along the framed embedding $\emptyset = S^{-1} \times D^\ell \hookrightarrow \Omega_a$. This surgery results in taking the disjoint union with an ℓ-disk i.e., $\Omega_b \cong \Omega_a \sqcup D^0 \times D^\ell$. Moreover, when we pass the critical point of the top index ℓ, we perform a surgery which fills in $S^{\ell-1} \times D^0 \hookrightarrow \Omega_a$ with a disk $D^\ell \times D^0$. Thus, $\Omega_b \cong \Omega_a \cup_{S^{\ell-1} \times D^0} D^\ell \times D^0$.

Remark 3.3.1. The surgeries corresponding to the attachment of handles of indices 0 and ℓ, when applied to cobordisms induced by isotopy, see section 3.3.1, give an analogue to the second Reidemeister move in higher dimension.

Surgeries on the level sets of Ω from boundary critical points

Recall that, the Morse function around a boundary critical point p^∂ of index i, $0 \le i \le \ell - 1$ is locally given by

$$f(x_1, ..., x_\ell) = f(p^\partial) - x_1^2 - ... - x_i^2 + x_{i+1}^2 + .. + x_{\ell-1}^2 + m x_\ell,$$

where $m = \pm 1$. In Ω, passing the boundary critical points corresponds to attaching half-handles on the sublevel sets of Ω. For the level sets of Ω, the result is obtained by performing a *discal* surgery, which is described as follows.

Consider the half-disk $D^i_+ := D^i \cap \{x_i \ge 0\}$ with boundary $\partial D^i_+ = D^{i-1} \cup S^{i-1}_{\ge 0}$, where $S^{i-1}_{\ge 0}$ is the upper hemisphere of S^{i-1}. We define a *right half-handle* of index i by $R_i := D^i \times D^{\ell-i}_+$ and a *left half-handle* of index i by $L_i := D^i_+ \times D^{\ell-i}$.

When $m = 1$ and we pass a boundary critical point of index i, $0 \le i \le \ell - 1$, then $\Omega_{(-\infty,b]}$

is obtained from $\Omega_{(-\infty,a]}$ by attaching $R_i = D^i \times D_+^{\ell-i}$ along some framed embedding of a pair $(\phi, \phi^\partial) : (S^{i-1} \times D_+^{\ell-i}, S^{i-1} \times D^{\ell-i-1}) \hookrightarrow (\Omega_a, \partial_I \Omega_a = \partial\Omega_a)$. The level set $(\Omega_b, \partial\Omega_b)$ is obtained by removing the image of (ϕ, ϕ^∂) from $(\Omega_a, \partial\Omega_a)$ and replacing it by $(D^i \times D^{\ell-i-1}, S^{i-1} \times D^{\ell-i-1})$. Note that the effect of this surgery is equivalent to attaching a handle $D^i \times D^{\ell-i-1}$ along $\phi^\partial : S^{i-1} \times D^{\ell-i-1} \hookrightarrow \partial\Omega_a$ i.e., $\Omega_b \cong \Omega_a \cup_{S^{i-1}\times D^{\ell-i-1}} D^i \times D^{\ell-i-1}$.

When $m = -1$ and we pass a boundary critical point of index i, $0 \leq i \leq \ell - 1$, then $\Omega_{(-\infty,b]}$ is obtained from $\Omega_{(-\infty,a]}$ by attaching $L_{i+1} = D_+^{i+1} \times D^{\ell-i-1}$ along a framed embedding $(\phi, \phi^\partial) : (S^i_{\geq 0} \times D^{\ell-i-1}, S^{i-1} \times D^{\ell-i-1}) \hookrightarrow (\Omega_a, \partial\Omega_a)$. We obtain the level set $(\Omega_b, \partial\Omega_b)$ from $(\Omega_a, \partial\Omega_a)$ by removing $(\phi, \phi^\partial)(S^i_{\geq 0} \times D^{\ell-i-1}, S^{i-1} \times D^{\ell-i-1})$ and replacing it by $(D_+^{i+1} \times S^{\ell-i-2}, S^i_{\geq 0} \times S^{\ell-i-2})$. Note that the effect of this surgery is equivalent to removing from Ω_a the image of ϕ, which is the same as cutting off a handle of index $\ell - i - 1$ from Ω_a.

Thus, passing a boundary critical point of index i, $0 \leq i \leq \ell-1$ results in either attaching a handle of index i or removing a handle of index $\ell - i - 1$ from Ω_a, depending on whether $m = 1$ or $m = -1$, respectively.

Remark 3.3.2. Note that passing the boundary critical point of index 0 with $m = 1$ results in attaching a right half-handle $D^0 \times D_+^\ell$ along the framed embedding $\emptyset = S^{-1} \times D_+^\ell \hookrightarrow \Omega_a$. Moreover, passing the boundary critical point of the top index $\ell - 1$ with $m = -1$ results in the removal of a left half-handle $D_+^\ell \times D^0$ of index ℓ. Thus, the surgeries corresponding to boundary critical points of index 0 (with $m = 1$) and index $\ell - 1$ (with $m = -1$), when applied to cobordisms induced by isotopy, see Section 3.3.1, gives an analogue to the first Reidemeister move in higher dimension.

Remark 3.3.3. For $\ell = 2$, we get both saddle cobordisms given in Section 3.3.1. The surgery corresponding to an interior critical point of index $i = 1$, gives the two saddles that occur simultaneously. While the surgeries corresponding to the boundary critical points of index $i = 0$ ($m = -1$), and index $i = 1$ ($m = 1$) gives the saddle cobordism containing Whitey Umbrella points.

3.4 Brunnian splitting

In the previous section, we considered a general situation of an embedding $M^{2\ell-1} \hookrightarrow \mathbb{R}^{3\ell}$, where M is a closed oriented manifold with possibly several path components i.e., $M = \bigsqcup_{i=1}^{k} M_i$. Starting from now, we focus our attention to either spherical or long links spaces, denoted by $Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$ or $Emb(\bigsqcup_k \mathbb{R}^{2\ell-1}, \mathbb{R}^{3\ell})$, respectively.

It is known [44, Lemma 3.7][29, Lemma 4.8] that for $\ell \geq 2$, the set of connected components of spherical links is same as the set of connected components of the long links i.e., for any $k \in \mathbb{N}$

$$\pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell}) = \pi_0 Emb_\partial(\bigsqcup_k \mathbb{R}^{2\ell-1}, \mathbb{R}^{3\ell}). \tag{3.4.1}$$

The latter is a finitely generated abelian group (see [21, 22]) for $\ell \geq 2$. The product on $\pi_0 Emb_\partial(\bigsqcup_k \mathbb{R}^{2\ell-1}, \mathbb{R}^{3\ell})$ is induced by concatenation of links. The product in $\pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$ is defined by Haefliger [22, §2.5] by component-wise connected sum given as follows: for $[f], [g] \in \pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$ with images of f and g contained in two disjoint balls, we connect these images component-wise by thin tubes. Since the complement of the images of f and g are simply connected (in fact $(\ell-1)$-connected), this operation is well-defined. Moreover, the inverses exist because isotopy classes of such embeddings coincide with concordance classes when $\ell \geq 2$ (see [42]). It is easy to see that the isomorphism in (3.4.1) respects the product.

Consider the homomorphism $\pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell}) \to \bigoplus_{i=1}^{k} \pi_0 Emb(S_i^{2\ell-1}, \mathbb{R}^{3\ell})$ induced by the maps where we forget all but one component. The kernel of this map is $\pi_0^U Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$, the subgroup consisting of k- component links where each component is unknotted. Moreover, there is a map $\bigoplus_{i=1}^{k} \pi_0 Emb(S_i^{2\ell-1}, \mathbb{R}^{3\ell}) \to \pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$ that creates a k-component link where each $S_i^{2\ell-1}$ is contained in a small (3ℓ)-ball disjoint from the other balls. Thus, $\bigoplus_{i=1}^{k} \pi_0 Emb(S_i^{2\ell-1}, \mathbb{R}^{3\ell})$ is a retract of the group $\pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell})$, and we get the following splitting

$$\pi_0 Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell}) = \Big(\bigoplus_{i=1}^{k} \pi_0 Emb(S_i^{2\ell-1}, \mathbb{R}^{3\ell})\Big) \oplus \pi_0^U Emb(\bigsqcup_k S^{2\ell-1}, \mathbb{R}^{3\ell}). \tag{3.4.2}$$

A *Brunnian link* is a non-trivial link that becomes trivial if any one of the components is removed. Denote by $\pi_0^{br} Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell})$ the subgroup of $\pi_0^U Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell})$ consisting of Brunnian links if $k \geq 2$, and for $k = 1$ we set $\pi_0^{br} Emb(S^{2\ell-1}, \mathbb{R}^{3\ell}) = \pi_0 Emb(S^{2\ell-1}, \mathbb{R}^{3\ell})$.

Proposition 3.4.1. *For $\ell \geq 2$,*

$$\pi_0 Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) = \bigoplus_{i=1}^{k} \Big(\underset{\binom{k}{i}}{\oplus} \pi_0^{br} Emb(\underset{i}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \Big) \tag{3.4.3}$$

Proof. Firstly, there is a natural map $\bigoplus_{i=1}^{k} \Big(\underset{\binom{k}{i}}{\oplus} \pi_0^{br} Emb(\underset{i}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \Big) \to \pi_0 Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell})$ defined using the component-wise connected sum defined earlier. Moreover, we can further continue the splitting in (3.4.2) as follows. For any pair $\{i, j\} \subset \{1, \dots, k\}$, we consider a map $\pi_0^U Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \to \pi_0^U Emb(\underset{2}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) = \pi_0^{br} Emb(\underset{2}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell})$. There are $\binom{k}{2}$ choices of such maps. Thus, we get a map $\pi_0^U Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \to \underset{\binom{k}{2}}{\oplus} \pi_0^{br} Emb(\underset{2}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell})$, where the kernel is given by the subgroup of links such that if one removes all but (any) two components the resulting link is trivial. We can continue to split the obtained group and iteratively doing so we get $\pi_0^U Emb(\underset{k}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) = \bigoplus_{i=2}^{k} \Big(\underset{\binom{k}{i}}{\oplus} \pi_0^{br} Emb(\underset{i}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \Big)$, which concludes the result. ☐

3.5 3-component links

Proposition 3.4.1 for 3-component links implies the following Brunnian splitting provided $\ell \geq 2$:

$$\pi_0 Emb(\underset{3}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) = \Big(\underset{3}{\oplus} \pi_0 Emb(S^{2\ell-1}, \mathbb{R}^{3\ell}) \Big) \oplus \Big(\underset{3}{\oplus} \pi_0^{br} Emb(\underset{2}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) \Big) \oplus \pi_0^{br} Emb(\underset{3}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}). \tag{3.5.1}$$

Using Haefliger's work, the following is known about each of the summands. By [21, Corollary 8.14],

$$\pi_0 Emb(S^{2\ell-1}, \mathbb{R}^{3\ell}) = \begin{cases} \mathbb{Z} & \ell \geq 2,\ \ell \text{ even}, \\ \mathbb{Z}_2 & \ell \geq 3,\ \ell \text{ odd}. \end{cases}$$

The generator for the Haefliger invariant $\pi_0 Emb(S^{2\ell-1}, \mathbb{R}^{3\ell}) \to \mathbb{Z}$ is given by the Haefliger trefoil knot (see [20] for even ℓ and [29] for odd ℓ.) By [22, Corollary 10.3] and using the homotopy groups of spheres, we get

$$\pi_0^{br} Emb(\bigsqcup_2 S^{2\ell-1}, \mathbb{R}^{3\ell}) = \begin{cases} \mathbb{Z}^2 & \ell = 2, \\ \mathbb{Z}^2 \oplus torsion & \ell \geq 4 \ \ even, \\ torsion & \ell \geq 3 \ \ odd. \end{cases}$$

In Section 3.6, we find an explicit invariant which detects the non-torsion part of $\pi_0^{br} Emb(\bigsqcup_2 S^{2\ell-1}, \mathbb{R}^{3\ell})$ along with (non-torsion) generators. From [22, Theorem 9.4], one can easily deduce

$$\pi_0^{br} Emb(\bigsqcup_3 S^{2\ell-1}, \mathbb{R}^{3\ell}) = \mathbb{Z}.$$

In Subsection 3.5.2, we find an invariant which produces the above isomorphism. Moreover, in Subsection 3.5.3 we show that the generator in this case is given by the high dimensional Borromean link.

3.5.1 Linking number

We now give different but equivalent definitions for a high dimensional linking number.

- For two disjoint oriented manifolds M^m and N^n in $\mathbb{R}^{m+n+1}$, consider the Gauss map

$$\phi : M \times N \to S^{m+n}$$

$$(x, y) \mapsto \frac{y - x}{|y - x|}.$$

The linking number $lk(M, N)$ is defined as the degree of ϕ i.e., $lk(M, N) = deg(\phi)$. We orient S^{m+n} as the boundary of the unit ball in $\mathbb{R}^{m+n+1}$ using the outward normal first convention: the outward normal vector followed by $Or(S^{m+n})$, the orientation of S^{m+n} gives the standard orientation of $\mathbb{R}^{m+n+1}$.

- Consider $M^m \sqcup N^n$ in $\mathbb{R}^{m+n+1}$, and project it to the orthogonal hyperplane $v^{\perp}$ along some non-zero vector v. Given two points $m \in M$ and $n \in N$ with the same projection on $v^{\perp}$, we say point n is above m if $n - m = \lambda v$ for $\lambda > 0$. We then define the linking number $lk(M, N)$ as the algebraic number of crossings where M passes under N. At each point where M crosses under N, we count the sign as follows: a crossing is $+1$ (respectively -1) if the orientation of M followed by v and then the orientation of N i.e., $(Or(M), v, Or(N))$, gives the positive (respectively negative) orientation of $\mathbb{R}^{m+n+1}$.

- Let M^m and N^n be triangulated and thus can be considered as singular cycles. Let $\mathcal{N}^{n+1}$ be a singular chain such that $\partial \mathcal{N} = N$, often but not always one can choose $\mathcal{N}$ to be a manifold with boundary N, where $\mathcal{N}$ is oriented using the outward normal first convention. Note that M and $\mathcal{N}$ intersect generically at finitely many points, and thus we define $lk(M, N)$ as the algebraic number of intersections of M and $\mathcal{N}$. At an intersection point, we assign $+1$ (respectively -1) if at that point $(Or(M), Or(\mathcal{N}))$ gives the positive (respectively negative) orientation of $\mathbb{R}^{m+n+1}$.

With the chosen conventions, one can check that all three definitions above are equivalent. The second definition is equivalent to the first one: Suppose we project $M^m \sqcup N^n$ to $v^{\perp}$ along some non-zero vector v, then the degree of ϕ can be computed as the algebraic number of points in the preimage of $\frac{v}{|v|} \in S^{m+n}$, where each preimage corresponds to a crossing where M passes under N. Moreover, to count the sign, we first take the outward normal to S^{m+n}, which is the vector v, then $(-1)^m$ times the orientation of M followed by the orientation of N i.e., $(v, (-1)^m Or(M), Or(N))$ and compare it with the standard orientation of $\mathbb{R}^{m+n+1}$ (the sign $(-1)^m$ is due to the fact that $x \in M$ appear with negative sign in $\frac{y-x}{|y-x|}$). Note that this sign convention matches with the one given in the second definition.

The second definition is equivalent to the third one: we construct the singular chain $\mathcal{N}$ as follows. Since the projection is on $v^{\perp}$, we can take a cylinder for N along the $-v$ direction, making sure that the cylinder is long enough so that its bottom side is below M, and it intersects M at all points where M is under N. Then we can cap off the bottom end of the cylinder by the cone of N. Thus, the result is an $(n+1)$-singular chain $\mathcal{N}$ with boundary N. Because of the outward normal first convention, the cylindrical part of $\mathcal{N}$ at the points of intersection with M, is oriented by $(v, Or(N))$. Therefore, the sign at any intersection point is given by the sign of orientation $(Or(M), v, Or(N))$, compared with the standard orientation of $\mathbb{R}^{m+n+1}$. This matches the sign convention in the second definition.

Note that the linking number is anti-symmetric:

$$lk(N, M) = (-1)^{(m+1)(n+1)} lk(M, N) \tag{3.5.2}$$

To see this, let us define $lk(N, M)$ using the first definition as the degree of the Gauss map $\tilde{\phi} : N \times M \to S^{m+n}$ such that $(y, x) \mapsto \frac{x-y}{|x-y|}$. The degree of $\tilde{\phi}$ differs from the degree of ϕ exactly by a factor of $(-1)^{(m+1)(n+1)}$ since the orientation of $N \times M$ differs from $M \times N$ by $(-1)^{mn}$, and the central symmetry of S^{m+n} contributes by $(-1)^{m+n+1}$. Thus, we have $lk(N, M) = deg(\tilde{\phi}) = (-1)^{(m+1)(n+1)} deg(\phi) = lk(M, N)$.

3.5.2 The invariant

Consider a 3-component link $X \sqcup Y \sqcup Z \hookrightarrow \mathbb{R}^{3\ell}$ with each component a $(2\ell-1)$-sphere. We project the embedding to $\mathbb{R}^{3\ell-1}$ and look at the preimage of the intersection points. We recall the orientation conventions for the set of double points given in Remark 3.2.1. For any component $i = X, Y, Z$, the preimage of the intersection with $j = X, Y, Z$ is given by the double point sets L^+_{ij} and L^-_{ij} lying in component i, where L^+_{ij} denotes the overcrossing (i above j), and L^-_{ij} denotes the undercrossing (i below j).

Theorem 3.5.1. *The following formula gives a homomorphism $\mathcal{V} : \pi_0 Emb(\bigsqcup_3 S^{2\ell-1}, \mathbb{R}^{3\ell}) \to \mathbb{Z}$ which is 0 on the first two summands of the Brunnian splitting (3.5.1), and is an isomorphism on the last summand $\pi_0^{br} Emb(\bigsqcup_3 S^{2\ell-1}, \mathbb{R}^{3\ell}) \approx \mathbb{Z}$. Given a representative F of a* *spherical* 3*-component link* $X \sqcup Y \sqcup Z \hookrightarrow \mathbb{R}^{3\ell}$*, one assigns to it*

$$\begin{aligned}\mathcal{V}(F) = \frac{1}{6}\Big[&lk(L^+_{XY}, L^+_{XZ}) + lk(L^-_{XY}, L^-_{XZ}) + (-1)^{\ell-1}2lk(L^+_{XY}, L^-_{XZ}) + (-1)^{\ell-1}2lk(L^-_{XY}, L^+_{XZ}) \\ &+ lk(L^+_{YZ}, L^+_{YX}) + lk(L^-_{YZ}, L^-_{YX}) + (-1)^{\ell-1}2lk(L^+_{YZ}, L^-_{YX}) + (-1)^{\ell-1}2lk(L^-_{YZ}, L^+_{YX}) \\ &+ lk(L^+_{ZX}, L^+_{ZY}) + lk(L^-_{ZX}, L^-_{ZY}) + (-1)^{\ell-1}2lk(L^+_{ZX}, L^-_{ZY}) + (-1)^{\ell-1}2lk(L^-_{ZX}, L^+_{ZY})\Big]\end{aligned} \tag{3.5.3}$$

Remark 3.5.1. The reason we consider such a long formula is that it is symmetric under the following changes: changing orientation of any of the components, permuting the components, and changing the ambient orientation. The symmetries are encoded by the group $(\mathbb{Z}_2 \wr S_3) \times \mathbb{Z}_2$ containing $8 \times 6 \times 2 = 96$ elements. Changing the orientation of any one of the components changes the sign of all linking numbers in the formula (3.5.3), thus the overall formula changes by a negative sign. For example, if we change the orientation of X, all the double point set L_{ij} in the formula with one of the entries as X change orientation, thus becoming $-L_{ij}$. Moreover, the linking numbers defined in X also change sign, thus the overall formula changes by a negative sign. Changing the ambient orientation i.e., performing the mirror symmetry where overcrossings become undercrossings and vice-versa, does not change the formula. More precisely, with our orientation conventions as in Remark 3.2.1, L^+_{ij} swaps with L^-_{ij} when ℓ is even, and L^+_{ij} swaps with $-L^-_{ij}$ when ℓ is odd. But in each case this transformation does not affect the formula. The S_3 action for permuting the components is trivial when ℓ is even, thus the formula remains unchanged since linking number is symmetric when ℓ is even. When ℓ is odd, the action is given by the sign representation of S_3. For example, if X and Y are permuted, then we get back all the linking numbers present in the formula (3.5.3), each with the reverse order, like the first linking number becomes $lk(L_{YX}, L_{YZ}) = -lk(L_{YZ}, L_{YX})$. Since the linking number is anti-symmetric in this case, we get a negative sign from each term, thus the overall formula changes by a negative sign. It is easy to check that when we permute X to Y and Y to Z, then we get back all the linking numbers in the formula, each with the same order, thus the formula is preserved.

Proof. Note that the Whitney Umbrella points appear as a limit of self-intersection. On the other hand, the formula does not involve self-intersections, and therefore is unchanged under any cobordism moves which correspond to the surgeries induced by the boundary critical points, see Section 3.3.3. Moreover, the cobordism moves corresponding to the interior critical points (see Section 3.3.3) preserve the linking numbers of the formula (3.5.3), leaving $\mathcal{V}(F)$ unchanged. However, the analogue to the third Reidemeister move described in Section 3.3.2 changes the linking numbers. It is easy to see that the third Reidemeister move in which one of component is repeated (for example, at the triple point if two sheets of X and one sheet of Y come together) do not change any of the linking numbers in $\mathcal{V}(F)$. We first consider the case when at the triple point, the component X is above Y, and Y is above Z. The following is a local model for this situation considered as a projection to $\mathbb{R}^{3\ell-1}$.

Consider a 1-parameter family $\{F_t\}$ of a link $X \sqcup Y \sqcup Z \hookrightarrow \mathbb{R}^{3\ell}$ for $t \in (-\epsilon, \epsilon)$ such that its projection to $\mathbb{R}^{3\ell-1}$ along the last coordinate is a 1-parameter family $\{f_t\}$ of immersions $X \sqcup Y \sqcup Z \looparrowright \mathbb{R}^{3\ell-1}$. Since X is above Y, and Y is above Z, we can assume that the last coordinates of X, Y, Z are constants a, b, c respectively, where $a > b > c$. Suppose that a triple point occurs at $t = 0$ i.e., f_0 has a triple point say $0 \in \mathbb{R}^{3\ell-1}$. One can find the following local coordinates

$$(x, y_1, \dots, y_{\ell-1}, z_1, \dots z_{\ell-1}),$$
$$(y, z_1, \dots, z_{\ell-1}, x_1, \dots x_{\ell-1}),$$
$$(z, x_1, \dots, x_{\ell-1}, y_1, \dots, y_{\ell-1}),$$

on X, Y, and Z respectively, and local coordinates

$$(x, y, x_1, \dots, x_{\ell-1}, y_1, \dots, y_{\ell-1}, z_1, \dots, z_{\ell-1})$$

on $\mathbb{R}^{3\ell-1}$, where all coordinate systems are assumed to be positively oriented. Near the triple point $0 \in \mathbb{R}^{3\ell-1}$ we have

$$f_t|_X = (x, 0, \underbrace{0, \ldots, 0}_{\ell-1}, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1}),$$

$$f_t|_Y = (t, y, x_1, \ldots, x_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}, z_1, \ldots, z_{\ell-1}),$$

$$f_t|_Z = (z, z, x_1, \ldots, x_{\ell-1}, y_1, \ldots, y_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}).$$

Note that only the component Y is moved by this isotopy. The double intersections of f_t are given by $(\ell - 1)$ manifolds as follows:

$$f_t|_X \cap f_t|_Y = (t, 0, \underbrace{0, \ldots, 0}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}, s_1, \ldots, s_{\ell-1}),$$

$$f_t|_X \cap f_t|_Z = (0, 0, \underbrace{0, \ldots, 0}_{\ell-1}, \tilde{s}_1, \ldots, \tilde{s}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}),$$

$$f_t|_Y \cap f_t|_Z = (t, t, \tilde{\tilde{s}}_1, \ldots, \tilde{\tilde{s}}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}).$$

According to our convention in Remark 3.2.1, we orient $f_t|_X \cap f_t|_Y$ in $\mathbb{R}^{3\ell-1}$ by first taking the orientation of the intersection, then completing the orientation of X followed by the remaining orientation of Y, such that the overall is the positive orientation of $\mathbb{R}^{3\ell-1}$. More precisely, if $(e_1, e_2, \ldots, e_{3\ell-1})$ denotes the basis for $\mathbb{R}^{3\ell-1}$ in the new coordinates, then one can check that

$$(\underbrace{(-1)^{\ell-1}e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1}}_{Or(f_t|_X \cap f_t|_Y)}, \underbrace{(-1)^{\ell-1}e_1, e_{\ell+2}, \ldots, e_{2\ell}}_{\text{rest of } Or(X)}, \underbrace{e_2, e_3, \ldots, e_{\ell+1}}_{\text{rest of } Or(Y)})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$, thus $((-1)^{\ell-1}e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1})$ is the orientation of $f_t|_X \cap f_t|_Y$. Similarly, $(e_{\ell+2}, \ldots, e_{2\ell})$ is the orientation of $f_t|_X \cap f_t|_Z$, since

$$(\underbrace{e_{\ell+2}, \ldots, e_{2\ell}}_{Or(f_t|_X \cap f_t|_Z)}, \underbrace{(-1)^{\ell-1}e_1, e_{2\ell+1}, \ldots, e_{3\ell-1}}_{\text{rest of } Or(X)}, \underbrace{e_1 + e_2, e_3, \ldots, e_{\ell+1}}_{\text{rest of } Or(Z)})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$. The orientation of $f_t|_Y \cap f_t|_Z$ is given by $((-1)^\ell e_3, e_4, \ldots, e_{\ell+1})$, where

$$(\underbrace{(-1)^\ell e_3, e_4, \ldots, e_{\ell+1}}_{Or(f_t|_Y \cap f_t|_Z))}, \underbrace{(-1)^\ell e_2, e_{2\ell+1}, \ldots, e_{3\ell-1}}_{\text{rest of } Or(Y)}, \underbrace{-(e_1+e_2), e_{\ell+2}, \ldots, e_{2\ell}}_{\text{rest of } Or(Z)})$$

is the positive orientation on $\mathbb{R}^{3\ell-1}$.

The oriented preimages of the double intersection of f_t in each component are given as follows:

$$\begin{aligned}
\text{In } X: \quad L^+_{XY}(t) &= (t, \underbrace{0, \ldots, 0}_{\ell-1}, (-1)^{\ell-1} s_1, s_2, \ldots, s_{\ell-1}), \\
L^+_{XZ} &= (0, \tilde{s}_1, \ldots, \tilde{s}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}), \\
\text{In } Y: \quad L^-_{YX} &= (0, (-1)^{\ell-1} s_1, s_2, \ldots, s_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}), \\
L^+_{YZ}(t) &= (t, \underbrace{0, \ldots, 0}_{\ell-1}, (-1)^\ell \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{\ell-1}), \\
\text{In } Z: \quad L^-_{ZX} &= (0, \underbrace{0, \ldots, 0}_{\ell-1}, \tilde{s}_1, \ldots, \tilde{s}_{\ell-1}), \\
L^-_{ZY}(t) &= (t, (-1)^\ell \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}).
\end{aligned}$$

Note that the only linking numbers that contribute are $lk(L^+_{XY}, L^+_{XZ}), lk(L^+_{YZ}, L^-_{YX})$, and $lk(L^-_{ZX}, L^-_{ZY})$.

For $t = -\delta$, where δ is a small positive number, we get three local pictures

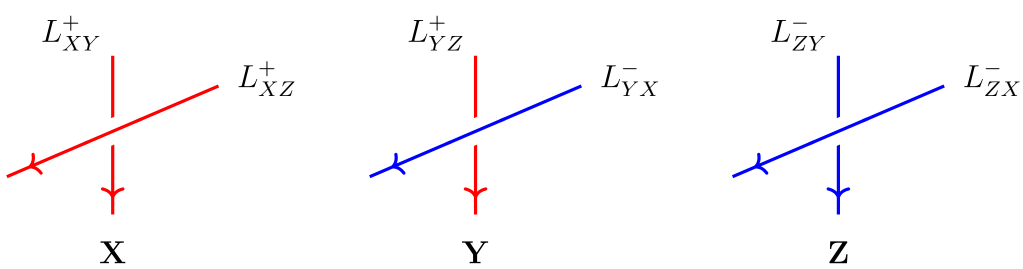


Figure 3.17: The inverse image of double-intersection of f_t for $t < 0$

We calculate the change in linking numbers in X going from $f_{-\delta}$ and f_δ. Assume L^+_{XZ} is a boundary of some singular ℓ-chain $C(L^+_{XZ})$ which is given by $\{(r, \tilde{s}_1, \ldots, \tilde{s}_{\ell-1}, 0, \ldots, 0) | r \leq 0\}$ near L^+_{XZ}. For $t = -\delta$, $C(L^+_{XZ})$ intersects L^+_{XY} exactly at one point, as shown in the figure below. Recall that the linking number $lk(L^+_{XY}, L^+_{XZ})$ is given by the algebraic intersection number of $L^+_{XY} \cap C(L^+_{XZ})$, where at the intersection point we take $Or(L^+_{XY})$, then $\frac{\partial}{\partial x}$ (the outward normal to L^+_{XZ}), followed by $Or(L^+_{XZ})$ i.e.,

$$\Big(\underbrace{(-1)^{\ell-1}\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{\ell-1}}}_{Or(L^+_{XY})}, \frac{\partial}{\partial x}, \underbrace{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{\ell-1}}}_{Or(L^+_{XZ})}\Big).$$

It is easy to check that this contributes $(-1)^{\ell-1}$ to $lk(L^+_{XY}, L^+_{XZ})$. Note that for $t = \delta$, this intersection point disappears. Thus, the change $\Delta lk(L^+_{XY}, L^+_{XZ}) = (-1)^\ell$.

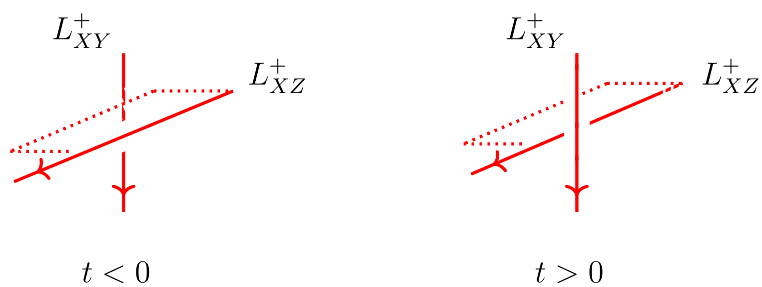


Figure 3.18: $C(L^+_{XZ})$ intersected with L^+_{XY}

In the component Y, both L^+_{YZ} and L^-_{YX} move along with Y going from $f_{-\delta}$ and f_δ. Assume L^-_{YX} is a boundary of a singular ℓ-chain $C(L^-_{YX}) = \{(r, (-1)^{\ell-1}s_1, s_2, \ldots, s_{\ell-1}, 0, \ldots, 0) | r \leq 0\}$ that intersect L^+_{YZ} exactly at one point for $t = -\delta$, see the figure below. At the intersection point, we take

$$\Big(Or(L^+_{YZ}), \frac{\partial}{\partial y}, Or(L^-_{YX})\Big) = \Big((-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\ell-1}}, \frac{\partial}{\partial y}, (-1)^{\ell-1}\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{\ell-1}}\Big),$$

which contributes -1 to $lk(L^+_{YZ}, L^-_{YX})$. For $t = \delta$, this intersection point disappears, thus $\Delta lk(L^+_{YZ}, L^-_{YX}) = +1$.

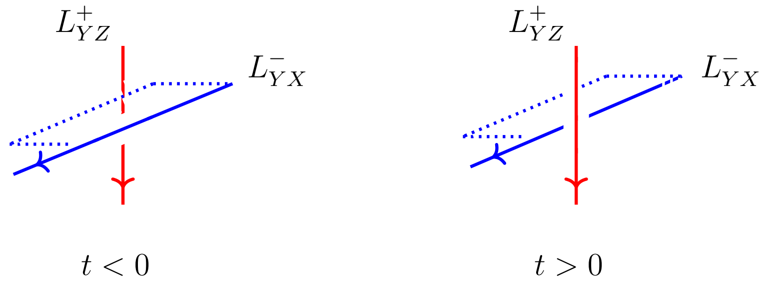


Figure 3.19: $C(L^-_{YX})$ intersected with L^+_{YZ}

To see the change in Z, we consider a singular ℓ-chain

$$C(L^-_{ZY}) = \{(t + r, (-1)^\ell \tilde{\tilde{\mathrm{s}}}_1, \tilde{\tilde{\mathrm{s}}}_2, \ldots, \tilde{\tilde{\mathrm{s}}}_{\ell-1}, 0, \ldots, 0) | r \geq 0\}$$

with boundary L^-_{ZY}. For $t = -\delta$, $C(L^-_{ZY})$ intersects L^-_{ZX} exactly at one point, see figure below, where at the intersection point we get

$$\Big(Or(L^-_{ZX}), -\frac{\partial}{\partial z}, Or(L^-_{ZY})\Big) = \Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_{\ell-1}}, -\frac{\partial}{\partial z}, (-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\ell-1}}\Big).$$

This contributes $(-1)^{\ell-1}$ to $lk(L^-_{ZX}, L^-_{ZY})$, and since this intersection point vanishes for $t = \delta$, we get $\Delta lk(L^-_{ZX}, L^-_{ZY}) = (-1)^\ell$.

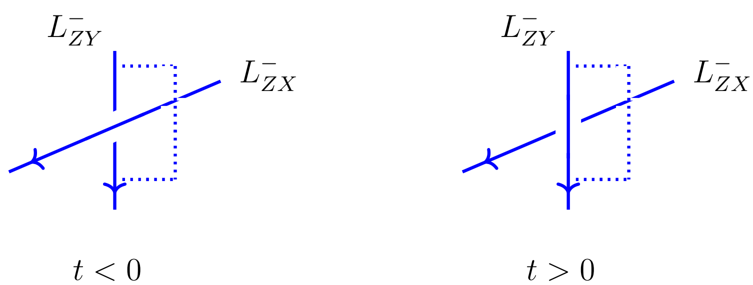


Figure 3.20: $C(L^-_{ZY})$ intersected with L^-_{ZX}

While passing the triple point, the overall change in the formula is 0:

$$
\begin{aligned}
\Delta\mathcal{V}(F) &= \mathcal{V}(F_\delta) - \mathcal{V}(F_{-\delta}) \\
&= \Delta lk(L^+_{XY}, L^+_{XZ}) + (-1)^{\ell-1} \cdot 2\Delta lk(L^+_{YZ}, L^-_{YX}) + \Delta lk(L^-_{ZX}, L^-_{ZY}) \\
&= (-1)^\ell + (-1)^{\ell-1} \cdot 2 + (-1)^\ell \\
&= 0.
\end{aligned}
$$

Thus, the formula $\mathcal{V}(F)$ is invariant under the third Reidemeister move with the convention that at the triple point, X is above Y, and Y is above Z.

Note that there are seven more cases to be checked when X is above Y, and Y is above Z corresponding to different orientations of X, Y and Z at the triple point. Additionally, one should also check that the formula is invariant when the heights of X, Y and Z have a different order at the triple point. So, formally speaking these different versions of the third Reidemeister move are encoded by the action of the group $(\mathbb{Z}_2 \wr S_3)$ with $8 \times 3! = 48$ elements. However, note that our formula has these symmetries according to Remark 3.5.1, and the action of this group must be a sign action. To compute the exact signs, we first note that the cycle (123) of S_3 acts trivially in both one-dimensional representation of S_3. Note that our formula (3.5.3) is preserved by the cyclic (123) action. Thus, it suffices to check the changes under the action of a transposition in S_3, and when the orientation of one of the components is reversed (together with the cyclic permutations, they generate $(\mathbb{Z}_2 \wr S_3)$).

Let us take the component X with opposite orientation, so now $-X$ is above Y, and Y is above Z at the triple point. One can check that the following changes occur:
With the same local coordinates of X, Y, Z and $\mathbb{R}^{3\ell-1}$ as before, near the triple point the map $f_t|_X$ changes as follows:

$$
f_t|_X = (-x, 0, \underbrace{0, \ldots, 0}_{\ell-1}, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1}).
$$

Note that all double intersections of f_t in $\mathbb{R}^{3\ell-1}$ are the same. However, the orientations of both $f_t|_X \cap f_t|_Y$ and $f_t|_X \cap f_t|_Z$ changes using the facts that

$$(\underbrace{(-1)^\ell e_{2\ell+1}, e_{2\ell+2}, \dots, e_{3\ell-1}}_{Or(f_t|X \cap f_t|Y)}, \underbrace{(-1)^{\ell-1} e_1, e_{\ell+2}, \dots, e_{2\ell}}_{\text{rest of } Or(X)}, \underbrace{-e_2, e_3, \dots, e_{\ell+1}}_{\text{rest of } Or(Y)})$$

and

$$(\underbrace{-e_{\ell+2}, e_{\ell+3}, \dots, e_{2\ell}}_{Or(f_t|X \cap f_t|Z)}, \underbrace{(-1)^{\ell-1} e_1, e_{2\ell+1}, \dots, e_{3\ell-1}}_{\text{rest of } Or(X)}, \underbrace{-(e_1+e_2), e_3, \dots, e_{\ell+1}}_{\text{rest of } Or(Z)})$$

give the positive orientation on $\mathbb{R}^{3\ell-1}$.

Thus, the new oriented preimages are

$$\begin{aligned} L^+_{XY}(t) &= (-t, \underbrace{0, \dots, 0}_{\ell-1}, (-1)^\ell s_1, s_2, \dots, s_{\ell-1}), \\ L^+_{XZ} &= (0, -\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{\ell-1}, \underbrace{0, \dots, 0}_{\ell-1}), \\ L^-_{YX} &= (0, (-1)^\ell s_1, s_2, \dots, s_{\ell-1}, \underbrace{0, \dots, 0}_{\ell-1}), \\ L^-_{ZX} &= (0, \underbrace{0, \dots, 0}_{\ell-1}, -\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{\ell-1}), \end{aligned}$$

and L^+_{YZ} and L^-_{ZY} are the same as before.

This only affects the orientations defined earlier at intersection points (when $t = -\delta$) in each case. In X, at the intersection point we get

$$\Big(Or(L^+_{XY}), \frac{\partial}{\partial y}, Or(L^+_{XZ})\Big) = \Big((-1)^\ell \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{\ell-1}}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{\ell-1}}\Big),$$

which contributes $(-1)^\ell$ to $lk(L^+_{XY}, L^+_{XZ})$, thus $\Delta lk(L^+_{XY}, L^+_{XZ}) = (-1)^{\ell-1}$.

In Y, at the intersection point we get

$$\Big(Or(L^+_{YZ}), \frac{\partial}{\partial y}, Or(L^-_{YX})\Big) = \Big((-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_{\ell-1}}, \frac{\partial}{\partial y}, (-1)^\ell \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{\ell-1}}\Big),$$

which gives the positive orientation of Y, thus $\Delta lk(L^+_{YZ}, L^-_{YX}) = -1$.

In Z, at the intersection point we get

$$\Big(Or(L^-_{ZX}), -\frac{\partial}{\partial z}, Or(L^-_{ZY})\Big) = \Big(-\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{\ell-1}}, -\frac{\partial}{\partial z}, (-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_{\ell-1}}\Big).$$

that contributes $(-1)^\ell$ to $lk(L^-_{ZX}, L^-_{ZY})$, therefore $\Delta lk(L^-_{ZX}, L^-_{ZY}) = (-1)^{\ell-1}$.

Hence, the overall change in the formula is given by

$$\begin{aligned}
\Delta\mathcal{V}(F) &= \mathcal{V}(F_\delta) - \mathcal{V}(F_{-\delta}) \\
&= \Delta lk(L^+_{XY}, L^+_{XZ}) + (-1)^{\ell-1} \cdot 2\Delta lk(L^+_{YZ}, L^-_{YX}) + \Delta lk(L^-_{ZX}, L^-_{ZY}) \\
&= (-1)^{\ell-1} + (-1)^{\ell-1} \cdot (-2) + (-1)^{\ell-1} \\
&= 0.
\end{aligned}$$

Thus, the formula remains unchanged under the third Reidemeister move when we reverse the orientation of one of the components.

Now consider the case when Y is above X, and X is above Z. The immersions $f_t|_X, f_t|_Y$ and $f_t|_Z$ are the same as before. All intersections are the same except that the orientation on $f_t|_X \cap f_t|_Y$ changes by $(-1)^\ell$. More precisely, the orientation on $f_t|_X \cap f_t|_Y$ is computed as follows

$$(\underbrace{-e_{2\ell+1}, e_{\ell+2}, \dots, e_{3\ell-1}}_{Or(f_t|_X \cap f_t|_Y)}, \underbrace{(-1)^\ell e_2, e_3, \dots, e_{\ell+1}}_{\text{rest of } Or(Y)}, \underbrace{-e_1, e_{\ell+2}, \dots, e_{2\ell}}_{\text{rest of } Or(X)}),$$

which gives the positive orientation on $\mathbb{R}^{3\ell-1}$. Note that compared to the earlier case, we get an additional $(-1)^\ell$ sign because we swapped the rest of the orientation of X with the orientation of Y. Therefore, $(-e_{2\ell+1}, e_{2\ell+2}, \dots, e_{3\ell-1})$ is now the orientation on $f_t|_X \cap f_t|_Y$, with the following oriented preimages where instead of L^+_{XY}, we now get L^-_{XY}, and similarly instead of L^-_{YX}, we get L^+_{YX}:

$$L^-_{XY}(t) = (t, \underbrace{0, \ldots, 0}_{\ell-1}, -s_1, s_2, \ldots, s_{\ell-1}),$$

$$L^+_{YX} = (0, -s_1, s_2 \ldots, s_{\ell-1}, \underbrace{0, \ldots, 0}_{\ell-1}).$$

The change in linking number in Z is the same i.e., $\Delta lk(L^-_{ZX}, L^-_{ZY}) = (-1)^\ell$. To see the change in linking numbers in X, note that for $t = -\delta$, the singular chain $C(L^+_{XZ})$ as defined earlier, intersects L^-_{XY} at one point. At the intersection point we get

$$\Big(Or(L^-_{XY}), \frac{\partial}{\partial x}, Or(L^+_{XZ})\Big) = \Big(- \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \ldots, \frac{\partial}{\partial z_{\ell-1}}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1} \ldots, \frac{\partial}{\partial y_{\ell-1}}\Big).$$

It contributes -1 to the $lk(L^-_{XY}, L^+_{XZ})$, and since for $t = \delta$ this intersection disappears, we get $\Delta lk(L^-_{XY}, L^+_{XZ}) = +1$. In Y, we assume L^+_{YX} bounds a singular ℓ-chain $C(L^+_{YX}) = \{(r, -s_1, s_2 \ldots, s_{\ell-1}, 0, \ldots, 0)|r \leq 0\}$ that intersects L^+_{YZ} at a point when $t = -\delta$, and the intersection disappears for $t = \delta$. At the intersection point, we get

$$\Big(Or(L^+_{YZ}), \frac{\partial}{\partial y}, Or(L^+_{YX})\Big) = \Big((-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \ldots, \frac{\partial}{\partial x_{\ell-1}}, \frac{\partial}{\partial y}, -\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{\ell-1}}\Big),$$

which contributes $(-1)^{\ell-1}$ to $lk(L^+_{YZ}, L^+_{YX})$. Thus, $\Delta lk(L^+_{YZ}, L^+_{YX}) = (-1)^\ell$. Hence, the overall change in the formula is given by

$$\begin{aligned}
\Delta\mathcal{V}(f) &= \mathcal{V}(F_\delta) - \mathcal{V}(F_{-\delta}) \\
&= (-1)^{\ell-1} \cdot 2\Delta lk(L^-_{XY}, L^+_{XZ}) + \Delta lk(L^+_{YZ}, L^+_{YX}) + \Delta lk(L^-_{ZX}, L^-_{ZY}) \\
&= 2(-1)^{\ell-1} + (-1)^\ell + (-1)^\ell \\
&= 0.
\end{aligned}$$

Thus, the formula remains unchanged under the third Reidemeister move when we permute the components.

Moreover, this homomorphism is an isomorphism to $\mathbb{Z}$ because the formula evaluated on the generator of $\pi_0^{br} Emb(\bigsqcup_3 S^{2\ell-1}, \mathbb{R}^{3\ell})$, the high dimensional Borromean link gives one, see Theorem 3.5.2. Note, however, that Theorem 3.5.2 is fully proved only in Section 3.7. $\square$

3.5.3 The generator

In this subsection, we describe the high dimensional Borromean link and state that it is the generator of $\pi_0^{br} Emb(\bigsqcup_3 S^{2\ell-1}, \mathbb{R}^{3\ell}) = \mathbb{Z}$. Moreover, we show that the invariant of Theorem 3.5.1 evaluated on the Borromean link is negative one.

Denote coordinates in $\mathbb{R}^{3\ell}$ by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x_1, \ldots, x_\ell, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell)$, the Borromean link $\mathcal{B}$ are three spheres given by following systems of equations:

$$\begin{aligned} X : \mathbf{x} = 0, &\quad \frac{|\mathbf{y}|^2}{\alpha^2} + \frac{|\mathbf{z}|^2}{\beta^2} = 1; \\ Y : \mathbf{y} = 0, &\quad \frac{|\mathbf{z}|^2}{\alpha^2} + \frac{|\mathbf{x}|^2}{\beta^2} = 1; \\ Z : \mathbf{z} = 0, &\quad \frac{|\mathbf{x}|^2}{\alpha^2} + \frac{|\mathbf{y}|^2}{\beta^2} = 1; \end{aligned}$$

where α and β are real numbers with $\alpha > \beta > 0$, as shown in the figure below.

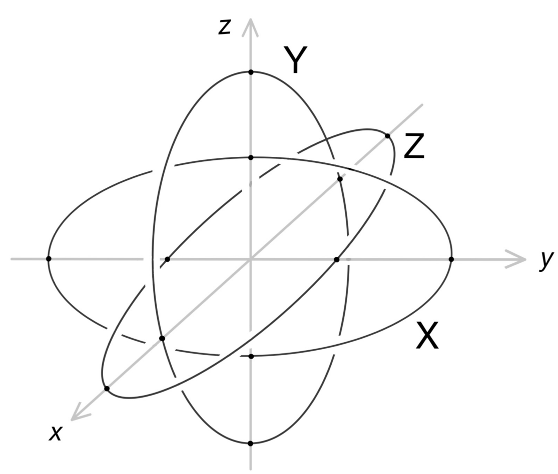


Figure 3.21: High dimensional Borromean link $\mathcal{B}$

Note that if any of the components is removed, then $\mathcal{B}$ is a trivial link, and thus $\mathcal{B}$ is an example of a Brunnian link. There is another equivalent description of Borromean link $\mathcal{B}$ given by Sakai in [39, §3], in which it is easier to analyze the double points of the projection to $\mathbb{R}^{3\ell-1}$. Let $\mathcal{B} = X \sqcup Y \sqcup Z \subset \mathbb{R}^{3\ell}$ where (fix $\alpha, \beta > 0$ so that $2\beta < \alpha$)

$$X := \partial\{(\mathbf{0}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^\ell)^{\times 3} \mid |\mathbf{y}| \leq \alpha, |\mathbf{z}| \leq \beta\} \simeq S^{2\ell-1},$$

$$Y := \partial\{(\mathbf{x}, \mathbf{0}, \mathbf{z}) \in (\mathbb{R}^\ell)^{\times 3} \mid |\mathbf{z}| \leq \alpha, |\mathbf{x}| \leq \beta\} \simeq S^{2\ell-1},$$

$$Z := \partial\{(\mathbf{x}, \mathbf{y}, \mathbf{0}) \in (\mathbb{R}^\ell)^{\times 3} \mid |\mathbf{x}| \leq \alpha, |\mathbf{y}| \leq \beta\} \simeq S^{2\ell-1}.$$

Smoothing the corners gives smooth $(2\ell - 1)$-spheres that we denote by X, Y, Z again. To orient these spheres, we fix that $(\partial_{\mathbf{y}}, \partial_{\mathbf{z}})$, $(\partial_{\mathbf{z}}, \partial_{\mathbf{x}})$, and $(\partial_{\mathbf{x}}, \partial_{\mathbf{y}})$ give the orientation of the coordinate planes $(0 \times \mathbb{R}^\ell_{\mathbf{y}} \times \mathbb{R}^\ell_{\mathbf{z}}) \supset X$, $(\mathbb{R}^\ell_{\mathbf{x}} \times 0 \times \mathbb{R}^\ell_{\mathbf{z}}) \supset Y$, and $(\mathbb{R}^\ell_{\mathbf{x}} \times \mathbb{R}^\ell_{\mathbf{y}} \times 0) \supset Z$, respectively. Each sphere $S^{2\ell-1}$ is oriented as the boundary of the disk using the outward normal first convention. Moreover, each sphere has two parts, for example:

$$X = (D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta)) \sqcup (S^{\ell-1}_{\mathbf{y}}(\alpha) \times D^\ell_{\mathbf{z}}(\beta)),$$

where $D^\ell_{\mathbf{y}}(\alpha)$ and $D^\ell_{\mathbf{z}}(\beta)$ are disks of radius α and β in $(0 \times \mathbb{R}^\ell_{\mathbf{y}} \times 0)$ and $(0 \times 0 \times \mathbb{R}^\ell_{\mathbf{z}})$, respectively, and $S^{\ell-1}_{\mathbf{y}}(\alpha) = \partial D^\ell_{\mathbf{y}}(\alpha)$ and $S^{\ell-1}_{\mathbf{z}}(\beta) = \partial D^\ell_{\mathbf{z}}(\beta)$.

Theorem 3.5.2. *The Borromean link $\mathcal{B}$ is a generator for the group $\pi_0^{br} Emb(\underset{3}{\sqcup} S^{2\ell-1}, \mathbb{R}^{3\ell}) = \mathbb{Z}$. Moreover, $\mathcal{V}(\mathcal{B}) = -1$.*

Proof. We will prove that $\mathcal{B}$ is a generator in the next section. To evaluate $\mathcal{V}(\mathcal{B})$, we follow Sakai's interpretation for $\mathcal{B}$ and computations done in [39, §3]. Consider the projection $p : \mathbb{R}^{3\ell} \to \mathbb{R}^{3\ell-1} \times 0$ along the vector $\mathbf{n} = (1, \dots, 1) \in \mathbb{R}^{3\ell}$. Under the projection, the points in X, Y, Z with coordinates

$$(\underbrace{0, \dots, 0}_{\ell}, y_1, \dots, y_\ell, z_1, \dots, z_\ell),$$

$$(\tilde{x}_1, \dots, \tilde{x}_\ell, \underbrace{0, \dots, 0}_{\ell}, \tilde{z}_1, \dots, \tilde{z}_\ell),$$

$$(\tilde{\tilde{\mathrm{x}}}_1, \dots, \tilde{\tilde{\mathrm{x}}}_\ell, \tilde{\tilde{\mathrm{y}}}_1, \dots, \tilde{\tilde{\mathrm{y}}}_\ell, \underbrace{0, \dots, 0}_{\ell})$$

are mapped to

$$(-z_\ell, \dots, -z_\ell, y_1 - z_\ell, \dots, y_\ell - z_\ell, z_1 - z_\ell, \dots, z_{\ell-1} - z_\ell),$$

$$(\tilde{x}_1 - \tilde{z}_\ell, \dots, \tilde{x}_\ell - \tilde{z}_\ell, -\tilde{z}_\ell, \dots, -\tilde{z}_\ell, \tilde{z}_1 - \tilde{z}_\ell, \dots, \tilde{z}_{\ell-1} - \tilde{z}_\ell),$$

$$(\tilde{\tilde{\mathrm{x}}}_1, \dots, \tilde{\tilde{\mathrm{x}}}_\ell, \tilde{\tilde{\mathrm{y}}}_1, \dots, \tilde{\tilde{\mathrm{y}}}_\ell, \underbrace{0, \dots, 0}_{\ell-1}),$$

respectively. Then $p \circ \mathcal{B}$ is generic and has double intersections of dimension $\ell - 1$. We will show in our computations that the double intersections are given by six disjoint spheres $A_1 \sqcup \ldots \sqcup A_6$, each of dimension $\ell - 1$. Moreover, the set of double points in each component X, Y, Z is given by two disjoint Hopf links. For example, inside X

$$L^+_{XY} = \{(\mathbf{0}, \beta'\mathbf{n}', \mathbf{z}) \in X \mid |\mathbf{z}| = \beta\}, \qquad L^-_{XY} = \{(\mathbf{0}, -\beta'\mathbf{n}', \mathbf{z}) \in X \mid |\mathbf{z}| = \beta\},$$

$$L^+_{XZ} = \{(\mathbf{0}, \mathbf{y}, \beta'\mathbf{n}') \in X \mid |\mathbf{y} - \beta'\mathbf{n}'| = \beta\}, \quad L^-_{XZ} = \{(\mathbf{0}, \mathbf{y}, -\beta'\mathbf{n}') \in X \mid |\mathbf{y} + \beta'\mathbf{n}'| = \beta\}$$

form two disjoint Hopf links as shown in the picture below, see also figure 3.23.

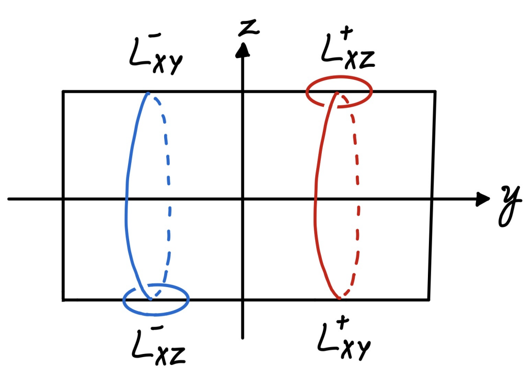


Figure 3.22: The set of double points in X

In particular, $p(X) \cap p(Z)$ is given by the following equations

$$\tilde{\tilde{\mathrm{x}}}_1 = \ldots = \tilde{\tilde{\mathrm{x}}}_\ell = -z_\ell;$$

$$\tilde{\tilde{\mathrm{y}}}_i = y_i - z_\ell, 1 \leq i \leq \ell;$$

$$z_1 = \ldots = z_{\ell-1} = z_\ell.$$

Note that since $X = (D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta)) \sqcup (S^{\ell-1}_{\mathbf{y}}(\alpha) \times D^\ell_{\mathbf{z}}(\beta))$, either

$$y_1^2 + \ldots + y_\ell^2 \leq \alpha^2, \quad z_1^2 + \ldots + z_\ell^2 = \beta^2; \tag{3.5.4}$$

or

$$y_1^2 + \ldots + y_\ell^2 = \alpha^2, \quad z_1^2 + \ldots + z_\ell^2 \leq \beta^2. \tag{3.5.5}$$

Similarly, since $Z = (D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta)) \sqcup (S^{\ell-1}_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times D^\ell_{\tilde{\tilde{\mathrm{y}}}}(\beta))$, we get either

$$\tilde{\tilde{\mathrm{x}}}_1^2 + \ldots + \tilde{\tilde{\mathrm{x}}}_\ell^2 \leq \alpha^2, \quad \tilde{\tilde{\mathrm{y}}}_1^2 + \ldots + \tilde{\tilde{\mathrm{y}}}_\ell^2 = \beta^2; \tag{3.5.6}$$

or

$$\tilde{\tilde{\mathrm{x}}}_1^2 + \ldots + \tilde{\tilde{\mathrm{x}}}_\ell^2 = \alpha^2, \quad \tilde{\tilde{\mathrm{y}}}_1^2 + \ldots + \tilde{\tilde{\mathrm{y}}}_\ell^2 \leq \beta^2. \tag{3.5.7}$$

In $p(X) \cap p(Z)$, $\tilde{\tilde{\mathrm{x}}}_i = -z_i$, thus (3.5.7) cannot intersect (3.5.4) and (3.5.5) (recall that $2\beta < \alpha$). Therefore, in Z this intersection happens only in the first part i.e., $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta)$ given by (3.5.6). However, (3.5.6) cannot intersect (3.5.5) because $\tilde{\tilde{\mathrm{y}}}_i = y_i - z_i$ thus we get

$$y_1^2 + \ldots + y_\ell^2 = \alpha^2;$$

$$z_1^2 + \ldots + z_\ell^2 \leq \beta^2;$$

$$(y_1 - z_1)^2 + \ldots + (y_\ell - z_\ell)^2 = \beta^2;$$

which by triangle inequality (for the points $\mathbf{0}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^\ell$) implies that $\alpha \leq 2\beta$, contradicting $2\beta < \alpha$.

Thus, $p(X) \cap p(Z)$ happens only when (3.5.4) and (3.5.6) hold i.e., only projections of $D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta) \subset X$ and $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta) \subset Z$ may have a non-trivial intersection.

If $(\underbrace{w,\ldots,w}_{\ell}, w_1, w_2, \ldots, w_\ell, \underbrace{0,\ldots,0}_{\ell-1})$ are the coordinates on $p(X)\cap p(Z) \subset \mathbb{R}^{3\ell-1}$, then with $w = \tilde{\tilde{\mathrm{x}}}_i = -z_i = \pm\frac{\beta}{\sqrt{\ell}}$ and $w_i = \tilde{\tilde{\mathrm{y}}}_i = y_i \pm \frac{\beta}{\sqrt{\ell}}$ for $1 \le i \le \ell$, the intersection $p(X)\cap p(Z)$ is given by

$$A_1 = \{(\underbrace{\frac{\beta}{\sqrt{\ell}},\ldots,\frac{\beta}{\sqrt{\ell}}}_{\ell}, w_1,\ldots,w_\ell, \underbrace{0,\ldots,0}_{\ell-1}) \in \mathbb{R}^{3\ell-1} | w_1^2 + \ldots + w_\ell^2 = \beta^2\},$$

and

$$A_2 = \{(\underbrace{-\frac{\beta}{\sqrt{\ell}},\ldots,-\frac{\beta}{\sqrt{\ell}}}_{\ell}, w_1,\ldots,w_\ell, \underbrace{0,\ldots,0}_{\ell-1}) \in \mathbb{R}^{3\ell-1} | w_1^2 + \ldots + w_\ell^2 = \beta^2\}.$$

Thus, $p(X)\cap p(Z) = A_1 \sqcup A_2 \subset \mathbb{R}^{3\ell-1}$ where each A_i is a sphere $S^{\ell-1}$, with the following preimages: (set $\mathbf{n}' = (1,\ldots,1) \in \mathbb{R}^\ell$ and $\beta' = \frac{\beta}{\sqrt{\ell}}$)

$$\begin{aligned}
\text{in } Z: \quad & L^+_{ZX} = \{(\beta'\mathbf{n}', \tilde{\tilde{\mathrm{y}}}, \mathbf{0}) \in Z \mid |\tilde{\tilde{\mathrm{y}}}| = \beta\}, \\
& L^-_{ZX} = \{(-\beta'\mathbf{n}', \tilde{\tilde{\mathrm{y}}}, \mathbf{0}) \in Z \mid |\tilde{\tilde{\mathrm{y}}}| = \beta\}, \\
\text{in } X: \quad & L^-_{XZ} = \{(\mathbf{0}, \mathbf{y}, -\beta'\mathbf{n}') \in X \mid |\mathbf{y} + \beta'\mathbf{n}'| = \beta\}, \\
& L^+_{XZ} = \{(\mathbf{0}, \mathbf{y}, \beta'\mathbf{n}') \in X \mid |\mathbf{y} - \beta'\mathbf{n}'| = \beta\},
\end{aligned}$$

satisfying $A_1 = p(\mathcal{B}(L^+_{ZX})) = p(\mathcal{B}(L^-_{XZ}))$ and $A_2 = p(\mathcal{B}(L^-_{ZX})) = p(\mathcal{B}(L^+_{XZ}))$.

Let $(e_1,\ldots,e_{3\ell-1})$ be the standard basis on $\mathbb{R}^{3\ell-1}$. We orient the sphere A_1 at the point $a_1 = (\underbrace{\frac{\beta}{\sqrt{\ell}},\ldots,\frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{\beta, 0\ldots,0}_{\in S^{\ell-1}}, \underbrace{0,\ldots,0}_{\ell-1})$, where we choose $e_{\ell+1}$ as the outward normal vector. Then $(e_{\ell+2},\ldots,e_{2\ell})$ lie in the tangent space of A_1 at a_1. We need to check that at a_1, the chosen orientation of A_1 followed by the rest of the orientation on $p(Z)$, followed by the rest of the orientation $p(X)$ (Z is above X under the projection) agrees with the natural orientation of $\mathbb{R}^{3\ell-1}$. Under the projection, $p(D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta))$ has the same orientation as $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times$

$S^{\ell-1}_{\tilde{\tilde{y}}}(\beta)$ oriented at the preimage of a_1 i.e., $(\underbrace{\frac{\beta}{\sqrt{\ell}},\ldots,\frac{\beta}{\sqrt{\ell}}}_{\ell},\underbrace{\beta,0\ldots,0}_{\in S^{\ell-1}},\underbrace{0,\ldots,0}_{\ell})\in L^{+}_{ZX}$. Thus, the orientation of $p(D^{\ell}_{\tilde{\tilde{x}}}(\alpha)\times S^{\ell-1}_{\tilde{\tilde{y}}}(\beta))$ is given by $(e_1,\ldots,e_\ell,(-1)^\ell e_{\ell+2},e_{\ell+3},\ldots,e_{2\ell})$, using the outward normal first convention. Moreover, the preimage of a_1 in $L^{-}_{XZ}\subset D^{\ell}_{\mathbf{y}}(\alpha)\times S^{\ell-1}_{\mathbf{z}}(\beta)$ is $(\underbrace{0,\ldots,0}_{\ell},\underbrace{\beta-\frac{\beta}{\sqrt{\ell}},-\frac{\beta}{\sqrt{\ell}},\ldots,-\frac{\beta}{\sqrt{\ell}}}_{\ell},\underbrace{-\frac{\beta}{\sqrt{\ell}},\ldots,-\frac{\beta}{\sqrt{\ell}}}_{\ell})$. To get the orientation of $p(D^{\ell}_{\mathbf{y}}(\alpha)\times S^{\ell-1}_{\mathbf{z}}(\beta))$ at a_1, we need to know how the orientation of $S^{\ell-1}_{\mathbf{z}}(\beta)$ at $(\underbrace{-\frac{\beta}{\sqrt{\ell}},\ldots,-\frac{\beta}{\sqrt{\ell}}}_{\ell})$ is mapped to $\mathbb{R}^{\ell-1}\subset\mathbb{R}^{3\ell-1}$. Given a point $(s_1,\ldots,s_\ell)$ in the tangent space $T_{-\mathbf{n}}S^{\ell-1}_{z}(\beta)$ for $-\mathbf{n}=(-1,\ldots,-1)\in\mathbb{R}^\ell$, the projection maps $(s_1,\ldots,s_\ell)$ to $(s_1-s_\ell,\ldots,s_{\ell-1}-s_\ell)\in\mathbb{R}^{\ell-1}$. Thus, the inverse map takes $(t_1,\ldots,t_{\ell-1})\in\mathbb{R}^{\ell-1}$ to $(t_1+\lambda,\ldots,t_{\ell-1}+\lambda,\lambda)$ in $T_{-\mathbf{n}}S^{\ell-1}_{z}(\beta)$. Since $(t_1+\lambda,\ldots,t_{\ell-1}+\lambda,\lambda)$ is orthogonal to $-\mathbf{n}$, we get $t_1+\ldots+t_{\ell-1}+\ell\lambda=0$, which implies $\lambda=-\frac{t_1+\ldots+t_{\ell-1}}{\ell}$. Thus, $(e_{2\ell+1}-\frac{1}{\ell}\mathbf{n},\ldots,e_{3\ell-1}-\frac{1}{\ell}\mathbf{n})\in T_{-\mathbf{n}}S^{\ell-1}_{z}(\beta)$ are the basis vectors corresponding to $(e_{2\ell+1},\ldots,e_{3\ell-1})\in\mathbb{R}^{3\ell-1}$. With the outward normal first, we get $(-\mathbf{n},e_{2\ell+1}-\frac{1}{\ell}\mathbf{n},\ldots,e_{3\ell-1}-\frac{1}{\ell}\mathbf{n})\sim(-e_{3\ell},e_{2\ell+1},\ldots,e_{3\ell-1})\sim((-1)^\ell e_{2\ell+1},e_{2\ell+2}\ldots,e_{3\ell})$. Thus, the orientation of $D^{\ell}_{\mathbf{y}}(\alpha)\times S^{\ell-1}_{\mathbf{z}}(\beta)$ is given by $(e_{\ell+1},\ldots,e_{2\ell},(-1)^\ell(-1)^\ell e_{2\ell+1},e_{2\ell+2}\ldots,e_{3\ell-1})$, so $(e_{\ell+1},\ldots,e_{2\ell},e_{2\ell+1},\ldots,e_{3\ell-1})$ is the orientation of $p(D^{\ell}_{\mathbf{y}}(\alpha)\times S^{\ell-1}_{\mathbf{z}}(\beta))$.

Thus, the following

$$(\underbrace{(-1)^\ell e_{\ell+2},e_{\ell+3},\ldots,e_{2\ell},}_{Or(A_1)}\ \underbrace{e_1,\ldots,e_\ell}_{\text{rest of }Or(p(Z))}\ ,\underbrace{-e_{\ell+1},e_{2\ell+1},\ldots,e_{3\ell-1}}_{\text{rest of }Or(p(X))})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$.

Similarly, we orient A_2 at the point $a_2=(\underbrace{-\frac{\beta}{\sqrt{\ell}},\ldots,-\frac{\beta}{\sqrt{\ell}}}_{\ell},\underbrace{\beta,0\ldots,0}_{\in S^{\ell-1}},\underbrace{0,\ldots,0}_{\ell-1})$ with $e_{\ell+1}$ as the outward normal vector. At a_2, the orientation of A_2 followed by the rest of the orientation of $p(X)$ followed by the rest of the orientation of $p(Z)$(now X is above Z under the projection) must agree with the natural orientation of $\mathbb{R}^{3\ell-1}$. The orientation of $p(D^{\ell}_{\tilde{\tilde{x}}}(\alpha)\times S^{\ell-1}_{\tilde{\tilde{y}}}(\beta))$ is the same as before, given by $(e_1,\ldots,e_\ell,(-1)^\ell e_{\ell+2},e_{\ell+3},\ldots,e_{2\ell})$. The preimage of a_2 in

$L^+_{XZ} \subset D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta)$ is $(\underbrace{0,\ldots,0}_{\ell}, \underbrace{\beta + \frac{\beta}{\sqrt{\ell}}, \frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell})$. By a similar argument as before, the tangent space $T_{\mathbf{n}} S^{\ell-1}_z(\beta)$ is oriented by $(e_{2\ell+1} - \frac{1}{\ell}\mathbf{n}, \ldots, e_{3\ell-1} - \frac{1}{\ell}\mathbf{n})$, with the outward normal vector as $\mathbf{n}$, and $(\mathbf{n}, e_{2\ell+1} - \frac{1}{\ell}\mathbf{n}, \ldots, e_{3\ell-1} - \frac{1}{\ell}\mathbf{n}) \sim (e_{3\ell}, e_{2\ell+1}, \ldots, e_{3\ell-1}) \sim ((-1)^{\ell-1} e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell})$. Thus, the orientation of $D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta)$ is given by $(e_{\ell+1}, \ldots, e_{2\ell}, (-1)^\ell (-1)^{\ell-1} e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1})$ i.e., $(e_{\ell+1}, \ldots, e_{2\ell}, -e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1})$, which is also the orientation of $p(D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta))$.

Thus, the following

$$(\underbrace{-e_{\ell+2}, e_{\ell+3}, \ldots, e_{2\ell}}_{Or(A_2)}, \underbrace{(-1)^\ell e_{\ell+1}, -e_{2\ell+1}, e_{2\ell+2} \ldots, e_{3\ell-1}}_{\text{rest of } Or(p(X))}, \underbrace{(-1)^{\ell-1} e_1, e_2, \ldots, e_\ell}_{\text{rest of } Or(p(Z))})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$.

Now consider the intersection $p(Y) \cap p(Z)$ which is given by the following equations

$$\tilde{\tilde{\mathrm{x}}}_i = \tilde{x}_i - \tilde{z}_\ell, 1 \leq i \leq \ell;$$

$$\tilde{\tilde{\mathrm{y}}}_1 = \ldots = \tilde{\tilde{\mathrm{y}}}_\ell = -\tilde{z}_\ell;$$

$$\tilde{z}_1 = \ldots = \tilde{z}_{\ell-1} = \tilde{z}_\ell.$$

Again, since $2\beta < \alpha$, it is easy to check that $p(Y) \cap p(Z)$ occurs only between $D^\ell_{\tilde{\tilde{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{x}}}(\beta) \subset Y$ and $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta) \subset Z$. Thus, the above equations must satisfy

$$\tilde{\tilde{\mathrm{y}}}_1^2 + \ldots + \tilde{\tilde{\mathrm{y}}}_\ell^2 = \beta^2 \implies \ell \tilde{\tilde{\mathrm{y}}}_\ell^2 = \beta^2 \ quad\implies \tilde{\tilde{\mathrm{y}}}_\ell = \pm \frac{\beta}{\sqrt{\ell}};$$

$$\tilde{x}_1^2 + \ldots + \tilde{x}_\ell^2 = \beta^2.$$

Let $(w_1, w_2, \ldots, w_\ell, \underbrace{w, \ldots, w}_{\ell}, \underbrace{0, \ldots, 0}_{\ell-1})$ be the coordinates on $p(Y) \cap p(Z)$ in $\mathbb{R}^{3\ell-1}$, where

$w_i = \tilde{\tilde{x}}_i = \tilde{x}_i \pm \frac{\beta}{\sqrt{\ell}}$ and $w = -\tilde{z}_i = \tilde{\tilde{y}}_i = \pm\frac{\beta}{\sqrt{\ell}}$ for $1 \leq i \leq \ell$. Thus, $p(Y) \cap p(Z)$ is given by

$$A_3 = \Big\{(w_1, \ldots, w_\ell, \underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell-1}) | (w_1 - \frac{\beta}{\sqrt{\ell}})^2 + \ldots + (w_\ell - \frac{\beta}{\sqrt{\ell}})^2 = \beta^2\Big\},$$

$$A_4 = \Big\{(w_1, \ldots, w_\ell, \underbrace{-\frac{\beta}{\sqrt{\ell}}, \ldots, -\frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell-1}) | (w_1 + \frac{\beta}{\sqrt{\ell}})^2 + \ldots + (w_\ell + \frac{\beta}{\sqrt{\ell}})^2 = \beta^2\Big\},$$

with the following preimages

$$\begin{aligned} \text{in } Y: \quad & L^+_{YZ} = \{(\tilde{\mathbf{x}}, \mathbf{0}, \beta'\mathbf{n}') \in Y \mid |\tilde{\mathbf{x}}| = \beta\}, \\ & L^-_{YZ} = \{(\tilde{\mathbf{x}}, \mathbf{0}, -\beta'\mathbf{n}') \in Y \mid |\tilde{\mathbf{x}}| = \beta\}, \\ \text{in } Z: \quad & L^-_{ZY} = \{(\tilde{\tilde{\mathrm{x}}}, -\beta'\mathbf{n}', \mathbf{0}) \in Z \mid |\tilde{\tilde{\mathrm{x}}} + \beta'\mathbf{n}'| = \beta\}, \\ & L^+_{ZY} = \{(\tilde{\tilde{\mathrm{x}}}, \beta'\mathbf{n}', \mathbf{0}) \in Z \mid |\tilde{\tilde{\mathrm{x}}} - \beta'\mathbf{n}'| = \beta\}, \end{aligned}$$

satisfying $A_3 = p(\mathcal{B}(L^-_{YZ})) = p(\mathcal{B}(L^+_{ZY}))$ and $A_4 = p(\mathcal{B}(L^+_{YZ})) = p(\mathcal{B}(L^-_{ZY}))$.

To orient A_3, we choose a point $a_3 = \Big(\underbrace{\frac{2\beta}{\sqrt{\ell}}, 0 \ldots, 0}_{\in S^{\ell-1}}, \underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell-1}\Big)$, with e_1 as the outward normal. At a_3, the chosen orientation of A_3 followed by the rest of the orientation of $p(Z)$ followed by the rest of the orientation of $p(Y)$(now Z is above Y under the projection) must agree with the natural orientation of $\mathbb{R}^{3\ell-1}$.

The orientation on $p(D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta))$ at a_3 is given by $(e_1, \ldots, e_\ell, (-1)^\ell e_{\ell+2}, e_{\ell+3}, \ldots, e_{2\ell})$. To get the orientation of $p(D^\ell_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta))$, we first take the preimage of a_3 in $L^-_{YZ} \subset D^\ell_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta)$, given by

$$(\underbrace{\frac{\beta}{\sqrt{\ell}}, -\frac{\beta}{\sqrt{\ell}}, \ldots, -\frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell}, \underbrace{-\frac{\beta}{\sqrt{\ell}}, \ldots, -\frac{\beta}{\sqrt{\ell}}}_{\ell}).$$

Note that the orientation $(e_{2\ell+1}, \ldots, e_{3\ell}, (-1)^\ell e_2, e_3, \ldots, e_\ell)$ on $D^\ell_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta)$ is mapped to $(e_{2\ell+1}, \ldots, e_{3\ell-1}, -e_1 - e_2 \ldots - e_{3\ell-1}, (-1)^\ell e_2, e_3, \ldots, e_\ell)$ under the projection. Thus, the following

$$(\underbrace{-e_2, e_3, \ldots, e_\ell}_{Or(A_3)}, \underbrace{(-1)^\ell e_1, (-1)^\ell e_{\ell+2}, e_{\ell+3}, \ldots, e_{2\ell}}_{\text{rest of } Or(p(Z))}, \underbrace{(-1)^{\ell-1} e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1}, -e_1 - e_2 - \ldots - e_{3\ell-1}}_{\text{rest of } Or(p(Y))})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$.

We orient A_4 at $a_4 = \Big(\underbrace{\frac{-2\beta}{\sqrt{\ell}}, 0 \ldots, 0}_{\in S^{\ell-1}}, \underbrace{-\frac{\beta}{\sqrt{\ell}}, \ldots, -\frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell-1}\Big)$, with $-e_1$ as the outward normal. The preimage of a_4 in $L^+_{YZ} \subset D^\ell_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta)$ is given by

$$(\underbrace{-\frac{\beta}{\sqrt{\ell}}, \frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell}, \underbrace{0, \ldots, 0}_{\ell}, \underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{\ell}).$$

Thus, the orientation on $D^\ell_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta)$ is given by $(e_{2\ell+1}, \ldots, e_{3\ell}, (-1)^\ell e_2, e_3, \ldots, e_\ell)$, which under the projection is $(e_{2\ell+1}, \ldots, e_{3\ell-1}, -e_1 - e_2 \ldots - e_{3\ell-1}, (-1)^\ell e_2, e_3, \ldots, e_\ell)$. Thus, for the A_4 we get that

$$(\underbrace{(-1)^{\ell-1} e_2, e_3, \ldots, e_\ell}_{Or(A_4)}, \underbrace{-e_{2\ell+1}, e_{2\ell+2}, \ldots, e_{3\ell-1}, -e_1 - e_2 - \ldots - e_{3\ell-1}}_{\text{rest of } Or(p(Y))}, \underbrace{e_1, (-1)^\ell e_{\ell+2}, e_{\ell+3}, \ldots, e_{2\ell}}_{\text{rest of } Or(p(Z))})$$

gives the positive orientation on $\mathbb{R}^{3\ell-1}$.

Note that in $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta) \subset Z$, spheres $L^\pm_{ZX}$ are obtained from each other by translation by the vector $(2\beta'\mathbf{n}', \mathbf{0}, \mathbf{0}) \in \mathbb{R}^{3\ell}$. Similarly, $L^\pm_{ZY}$ are also obtained from each other by translation by $(2\beta'\mathbf{n}', 2\beta'\mathbf{n}', \mathbf{0}) \in \mathbb{R}^{3\ell}$. Since $2\beta < \alpha$, L^+_{ZY} and L^-_{ZY} do not touch the boundary of $D^\ell_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta)$. Moreover, L^+_{ZX} links with L^+_{ZY}, and not with L^-_{ZY}, and similarly L^-_{ZX} links with only L^-_{ZY}. Therefore, since all of these $L^\pm_i$'s are $S^{\ell-1}$, we get two disjoint Hopf links in Z.

Similarly, we can find that $p(X) \cap p(Y)$ are two spheres, and by symmetry we get that the double points set of $p \circ \mathcal{B}$ is given by six disjoint Hopf links, two in each X, Y, and Z:

$$L^+_{XY} \sqcup L^+_{XZ}, L^-_{XY} \sqcup L^-_{XZ} \subset D^\ell_{\mathbf{y}}(\alpha) \times S^{\ell-1}_{\mathbf{z}}(\beta) \subset X,$$

$$L^+_{YX} \sqcup L^+_{YZ}, L^-_{YX} \sqcup L^-_{YZ} \subset D^{\ell}_{\tilde{\mathbf{z}}}(\alpha) \times S^{\ell-1}_{\tilde{\mathbf{x}}}(\beta) \subset Y,$$

$$L^+_{ZX} \sqcup L^+_{ZY}, L^-_{ZX} \sqcup L^-_{ZY} \subset D^{\ell}_{\tilde{\tilde{\mathrm{x}}}}(\alpha) \times S^{\ell-1}_{\tilde{\tilde{\mathrm{y}}}}(\beta) \subset Z,$$

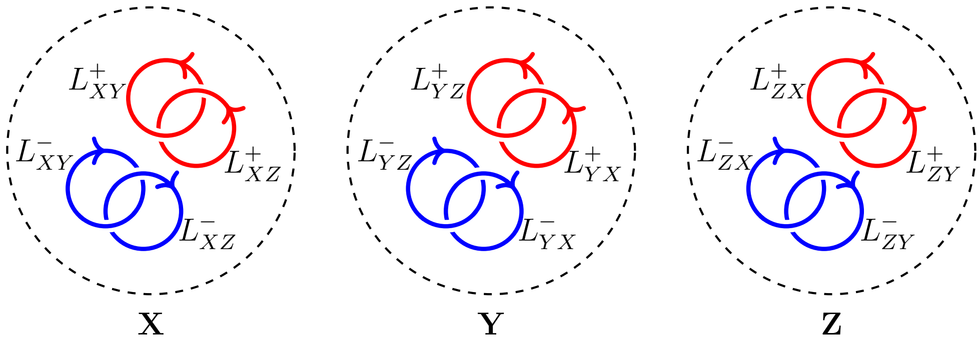


Figure 3.23: The set of double points of $p \circ \mathcal{B}$

Therefore, the only linking numbers contributing to the formula 3.5.3 are:

$$lk(L^+_{XY}, L^+_{XZ}), lk(L^-_{XY}, L^-_{XZ}), lk(L^+_{YZ}, L^+_{YX}), lk(L^-_{YZ}, L^-_{YX}), lk(L^+_{ZX}, L^+_{ZY}), lk(L^-_{ZX}, L^-_{ZY}).$$

Note that by the cyclic symmetry, see Remark 3.5.1, $\mathcal{T}(\mathcal{B})$ is three times the contribution of the linking numbers in Z (the third line in (3.5.3)).

By definition, $lk(L^+_{ZX}, L^+_{ZY})$ can be computed as the algebraic intersection number of L^+_{ZX} with the disk $D(L^+_{ZY}) = \{(\tilde{\tilde{\mathrm{x}}}, \beta'\mathbf{n}', \mathbf{0}) \in Z \mid |\tilde{\tilde{\mathrm{x}}} - \beta'\mathbf{n}'| \leq \beta\}$ bounding L^+_{ZY}. There is only one intersection point at $(\underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{2\ell}, \underbrace{0 \ldots, 0}_{\ell})$. Note that this intersection point is the center of $D(L^+_{ZY})$, and L^+_{ZX} passes through this point.

The orientation at the intersection point $(\underbrace{\frac{\beta}{\sqrt{\ell}}, \ldots, \frac{\beta}{\sqrt{\ell}}}_{2\ell}, \underbrace{0 \ldots, 0}_{\ell})$ is given by

$$(\underbrace{(-1)^{\ell} e_{\ell+2}, e_{\ell+3}, \ldots, e_{2\ell}}_{Or(L^+_{ZX})}, e_1, \underbrace{-e_2, e_3, \ldots, e_{\ell}}_{Or(L^+_{ZY})}),$$

which gives the negative orientation of Z. Thus, we get $lk(L^+_{ZX}, L^+_{ZY}) = -1$

Similarly, $lk(L^-_{ZX}, L^-_{ZY}) = -1$, given by the contribution of the following orientation at the intersection point $(\underbrace{-\frac{\beta}{\sqrt{\ell}},\dots,-\frac{\beta}{\sqrt{\ell}}}_{2\ell},\underbrace{0\dots,0}_{\ell})$:

$$(\underbrace{-e_{\ell+2},e_{\ell+3},\dots,e_{2\ell}}_{Or(L^-_{ZX})},-e_1,\underbrace{(-1)^{\ell-1}e_2,e_3,\dots,e_\ell}_{Or(L^-_{ZY})}).$$

Thus, $\mathcal{V}(\mathcal{B})=\frac{1}{6}[3\cdot(lk(L^+_{ZX},L^+_{ZY})+lk(L^-_{ZX},L^-_{ZY})]=-1$.

□

3.6 2-component links

Consider the Brunnian splitting in Proposition 3.4.1 for 2-component links:

$$\pi_0 Emb(\sqcup_2 S^{2\ell-1},\mathbb{R}^{3\ell})=\Big(\bigoplus_2\pi_0 Emb(S^{2\ell-1},\mathbb{R}^{3\ell})\Big)\oplus\Big(\pi_0^{br}Emb(\sqcup_2 S^{2\ell-1},\mathbb{R}^{3\ell})\Big). \tag{3.6.1}$$

Recall that (by [22, Corollary 10.3])

$$\pi_0^{br}Emb(\sqcup_2 S^{2\ell-1},\mathbb{R}^{3\ell})=\begin{cases}\mathbb{Z}^2 & \ell=2,\\ \mathbb{Z}^2\oplus torsion & \ell\geq 4\ \ even,\\ torsion & \ell\geq 3\ \ odd.\end{cases}$$

In this case, we find an invariant which detects the non-torsion part of $\pi_0^{br}Emb(\sqcup_2 S^{2\ell-1},\mathbb{R}^{3\ell})$, thus we let $\ell=2k$.

3.6.1 The invariant

Consider a 2-component link $X\sqcup Y\hookrightarrow\mathbb{R}^{6k}$ with each component a $(4k-1)$-sphere. Under the projection to $\mathbb{R}^{6k-1}$, the set of double points is $(2k-1)$-dimensional and is oriented according to Remark 3.2.1, since the codimension is even thus the order of the two sheets at the intersection does not matter. For any component $i=X,Y$, the preimage of the intersection with $j = X, Y$ is given by the double point set

$$L_{ij} := L_{ij}^{+} \sqcup -L_{ij}^{-},$$

where L_{ij} lies inside the component i, L_{ij}^{+} denotes the overcrossing (i above j), and L_{ij}^{-} denotes the undercrossing (i below j).

Furthermore,

$$\begin{aligned} lk(L_{ij}, L_{ik}) &= lk(L_{ij}^{+} \sqcup -L_{ij}^{-}, L_{ik}^{+} \sqcup -L_{ik}^{-}) \\ &= lk(L_{ij}^{+}, L_{ik}^{+}) + lk(L_{ij}^{-}, L_{ik}^{-}) - lk(L_{ij}^{+}, L_{ik}^{-}) - lk(L_{ij}^{-}, L_{ik}^{+}). \end{aligned}$$

Note that since each L_{ij} is $(2k-1)$-dimensional, the linking number is symmetric (see (3.5.2)).

Theorem 3.6.1. *The following formulas give a homomorphism $\mathcal{W} : \pi_0 Emb(\bigsqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \to \mathbb{Q}^2$ which is 0 on the first summand of the Brunnian splitting (3.6.1), and is an isomorphism rationally on the second summand: Given a representative F of a spherical 2-component link $X \sqcup Y \hookrightarrow \mathbb{R}^{6k}$, we assign two invariants:*

$$\mathcal{W}_1(F) := \frac{1}{2} lk(L_{XX}, L_{XY}) - lk(L_{YX}^{+}, L_{YX}^{-}); \tag{3.6.2}$$

$$\mathcal{W}_2(F) := \frac{1}{2} lk(L_{YY}, L_{YX}) - lk(L_{XY}^{+}, L_{XY}^{-}). \tag{3.6.3}$$

The image of homomorphism $\mathcal{W}$ is

$$\begin{cases} \{(a,b) \in \mathbb{Z}^2 | a+b \in 2\mathbb{Z}\} & k = 1,2,4 \\ 2\mathbb{Z} \times 2\mathbb{Z} & \textit{otherwise.} \end{cases}$$

Proof. Note that the formulas involve self-intersections of components X and Y, however the linking numbers in the formulas are preserved by cobordism moves corresponding to both interior and boundary critical points. Thus, $\mathcal{W}_1$ and $\mathcal{W}_2$ are unchanged under any cobordism moves. However, the analogue to the third Reidemeister move (see Section 3.3.3) changes these linking numbers. Since the formulas for $\mathcal{W}_1$ and $\mathcal{W}_2$ are similar, without loss of generality, we will only prove that $\mathcal{W}_1$ is invariant under the third Reidemeister move. Note that for $\mathcal{W}_1$, only the Reidemeister move in which X is repeated exactly twice changes the linking numbers.

Let us consider the case when at the triple point, two sheets of component X and one sheet of Y come together, such that X_1 is above X_2 and X_2 is above Y. The following is a local model for this situation considered as a projection to $\mathbb{R}^{6k-1}$.

We consider a 1-parameter family $\{f_t\}$ of immersions for $t \in (-\epsilon, \epsilon)$ of some link $X \sqcup Y \hookrightarrow \mathbb{R}^{6k}$ projected to $\mathbb{R}^{6k-1}$ along the last coordinate. Suppose that a triple point occurs at $t = 0$ i.e., f_0 has a triple point say $0 \in \mathbb{R}^{6k-1}$, and since at that point X_1 is above X_2, and X_2 is above Y, we can assume that the last coordinates of X_1, X_2, Y are constants a, b, c respectively, where $a > b > c$. One can find the following local coordinates

$$(x_1, x_1^2, \ldots, x_{2k-1}^2, y_1, \ldots y_{2k-1})$$

$$(x_2, y_1, \ldots, y_{2k-1}, x_1^1, \ldots x_{2k-1}^1)$$

$$(y, x_1^1, \ldots, x_{2k-1}^1, x_1^2, \ldots, x_{2k-1}^2)$$

$$(x_1, x_2, x_1^1, \ldots, x_{2k-1}^1, x_1^2, \ldots, x_{2k-1}^2, y_1, \ldots, y_{2k-1})$$

on X_1, X_2, Y, and $\mathbb{R}^{6k-1}$, respectively, where all coordinate systems are assumed to be positively oriented. Near the triple point $0 \in \mathbb{R}^{6k-1}$ we have

$$f_t|_{X_1} = (x_1, 0, \underbrace{0, \ldots, 0}_{2k-1}, x_1^2, \ldots, x_{2k-1}^2, y_1, \ldots, y_{2k-1}),$$

$$f_t|_{X_2} = (t, x_2, x_1^1, \ldots, x_{2k-1}^1, \underbrace{0, \ldots, 0}_{2k-1}, y_1, \ldots, y_{2k-1}),$$

$$f_t|_{Y} = (y, y, x_1^1, \ldots, x_{2k-1}^1, x_1^2, \ldots, x_{2k-1}^2, \underbrace{0, \ldots, 0}_{2k-1}).$$

Note that only the component X_2 is moved by this isotopy. The double intersections of f_t are given by $(2k-1)$ manifolds as follows:

$$f_t|_{X_1} \cap f_t|_{X_2} = (t, 0, \underbrace{0, \ldots, 0}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}, s_1, \ldots, s_{2k-1}),$$

$$f_t|_{X_1} \cap f_t|_{Y} = (0, 0, \underbrace{0, \ldots, 0}_{2k-1}, \tilde{s}_1, \ldots, \tilde{s}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}),$$

$$f_t|_{X_2} \cap f_t|_{Y} = (t, t, \tilde{\tilde{s}}_1, \ldots, \tilde{\tilde{s}}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}).$$

Since the codimension is even $(2k)$, according to our convention in Remark 3.2.1, the ordering of the two sheets at the intersection does not matter while defining the orientation on the intersection. Let $(e_1, e_2, \ldots, e_{6k-1})$ denotes the basis for $\mathbb{R}^{6k-1}$ in the new coordinates, we define orientations on each intersection above as follows:

$$(\underbrace{-e_{4k+1}, e_{4k+2}, \ldots, e_{6k-1}}_{Or(f_t|_{X_1} \cap f_t|_{X_2})}, \underbrace{-e_1, e_{2k+2}, \ldots, e_{4k}}_{\text{rest of } Or(X_1)}, \underbrace{e_2, e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(X_2)});$$

$$(\underbrace{e_{2k+2}, \ldots, e_{4k}}_{Or(f_t|_{X_1} \cap f_t|_{Y})}, \underbrace{-e_1, e_{4k+1}, \ldots, e_{6k-1}}_{\text{rest of } Or(X_1)}, \underbrace{e_1 + e_2, e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(Y)});$$

$$(\underbrace{e_3, e_4, \ldots, e_{2k+1}}_{Or(f_t|_{X_2} \cap f_t|_{Y}))}, \underbrace{e_2, e_{4k+1}, \ldots, e_{6k-1}}_{\text{rest of } Or(X_2)}, \underbrace{-(e_1 + e_2), e_{2k+2}, \ldots, e_{4k}}_{\text{rest of } Or(Y)});$$

where each one of these gives the positive orientation on $\mathbb{R}^{6k-1}$.

Thus, the oriented preimages of the double intersection of f_t in each component are given as follows:

$$\text{In } X_1: \quad L^+_{X_1X_2}(t) = (t, \underbrace{0,\dots,0}_{2k-1}, -s_1, s_2, \dots, s_{2k-1}),$$

$$L^+_{X_1Y} = (0, \tilde{s}_1, \dots, \tilde{s}_{2k-1}, \underbrace{0,\dots,0}_{2k-1}),$$

$$\text{In } X_2: \quad L^-_{X_2X_1} = (0, -s_1, s_2, \dots, s_{2k-1}, \underbrace{0,\dots,0}_{2k-1}),$$

$$L^+_{X_2Y}(t) = (t, \underbrace{0,\dots,0}_{2k-1}, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \dots, \tilde{\tilde{s}}_{2k-1}),$$

$$\text{In } Y: \quad L^-_{YX_1} = (0, \underbrace{0,\dots,0}_{2k-1}, \tilde{s}_1, \dots, \tilde{s}_{2k-1}),$$

$$L^-_{YX_2}(t) = (t, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \dots, \tilde{\tilde{s}}_{2k-1}, \underbrace{0,\dots,0}_{2k-1}).$$

Note that the only linking numbers that contribute to the change of $\mathcal{W}_1$ are $lk(L^+_{X_1X_2}, L^+_{X_1Y})$ and $lk(L^-_{X_2X_1}, L^+_{X_2Y})$ in X_1 and X_2, respectively. In particular, what is happening inside Y has no effect on $\mathcal{W}_1$.

For $t = -\delta$, where δ is a small positive number, we get two local pictures

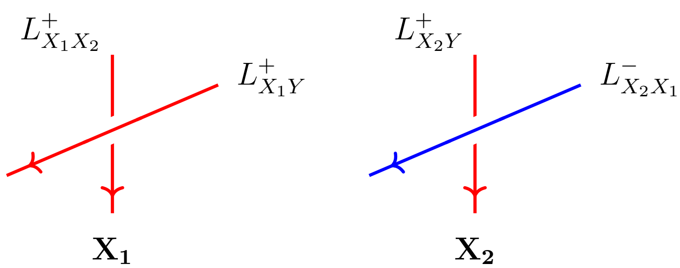


Figure 3.24: The inverse image of the self-intersection of f_t in X, $t < 0$

We calculate the change in linking numbers in X_1 and X_2 going from $f_{-\delta}$ and f_δ.

Assume $L^+_{X_1Y}$ is a boundary of some singular $2k$-chain $C(L^+_{X_1Y}) = \{(r, \tilde{s}_1, \dots, \tilde{s}_{2k-1}, 0, \dots, 0) | r \leq 0\}$. For $t = -\delta$, $C(L^+_{X_1Y})$ intersects $L^+_{X_1X_2}$ exactly at one point. At the intersection point, we take

$$\Big(\underbrace{-\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\dots,\frac{\partial}{\partial y_{2k-1}}}_{Or(L^+_{X_1X_2})},\frac{\partial}{\partial x_1},\underbrace{\frac{\partial}{\partial x_1^2},\dots,\frac{\partial}{\partial x_{2k-1}^2}}_{Or(L^+_{X_1Y})}\Big)$$

which contributes -1 to $lk(L^+_{X_1X_2},L^+_{X_1Y})$. Since this intersection point disappears for $t=\delta$, the change $\Delta lk(L^+_{X_1X_2},L^+_{X_1Y})=+1$.

Similarly, assume $L^+_{X_2Y}$ is a boundary of a singular $2k$-chain $C(L^+_{X_2Y}) = \{(r+t,0,\dots,0,\tilde{\tilde{s}}_1,\tilde{\tilde{s}}_2,\dots,\tilde{\tilde{s}}_{2k-1})|r\geq 0\}$. At the intersection point, we take

$$\Big(Or(L^-_{X_2X_1}),-\frac{\partial}{\partial x_2},Or(L^+_{X_2Y})\Big)=\Big(-\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\dots,\frac{\partial}{\partial y_{2k-1}},-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1^1},\dots,\frac{\partial}{\partial x_{2k-1}^1}\Big),$$

which contributes -1 to $lk(L^-_{X_2X_1},L^+_{X_2Y})$. For $t=\delta$, this intersection point disappears, thus $\Delta lk(L^-_{X_2X_1},L^+_{X_2Y})=+1$.

While passing the triple point, the overall change in the first formula $\mathcal{W}_1(F)$ is given by:

$$\begin{aligned}\Delta\mathcal{W}_1(F)&=\mathcal{W}_1(F_\delta)-\mathcal{W}_1(F_{-\delta})\\&=\frac{1}{2}\Big(\Delta lk(L^+_{X_1X_2},L^+_{X_1Y})-\Delta lk(L^-_{X_2X_1},L^+_{X_2Y})\Big)\\&=\frac{1}{2}(1-1)\\&=0\end{aligned}$$

Thus, $\mathcal{W}_1(F)$ is invariant under the third Reidemeister move with the convention that at the triple point, X_1 is above X_2, and X_2 is above Y. Let us denote this situation by $X_1>X_2>Y$ (we discuss later the effect of reversing the orientations of X_1,X_2,Y).

Note that $\mathcal{W}_1$ is unchanged under the mirror symmetry. Therefore, $\mathcal{W}_1(F)$ is invariant under the third Reidemeister move when at the triple point, $Y>X_1>X_2$. However, we still need to check the case when $X_1>Y>X_2$.

Assume $X_1>Y>X_2$, the immersions $f_t|_{X_1},f_t|_{X_2},f_t|_Y$ and all intersections (with orientations) are the same as before. But the oriented preimages in X_2 and Y may now be different, instead of $L^+_{X_2Y}$, we now get $L^-_{X_2Y}$, and similarly instead of $L^-_{YX_2}$, we get $L^+_{YX_2}$:

$$L^-_{X_2Y}(t) = (t, \underbrace{0, \ldots, 0}_{2k-1}, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{2k-1}),$$

$$L^+_{YX_2}(t) = (t, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}).$$

Now the linking numbers that contribute to $\mathcal{W}_1$ are $lk(L^+_{X_1X_2}, L^+_{X_1Y})$, $lk(L^-_{X_2X_1}, L^-_{X_2Y})$, and $lk(L^+_{YX_2}, L^-_{YX_1})$. Following the previous argument, $\Delta lk(L^+_{X_1X_2}, L^+_{X_1Y}) = +1$ and $\Delta lk(L^-_{X_2X_1}, L^-_{X_2Y}) = +1$ (by replacing $L^+_{X_2Y}$ by $L^-_{X_2Y}$).

To see the change in $lk(L^+_{YX_2}, L^-_{YX_1})$, let us assume $L^-_{YX_1}$ is a boundary of a singular $2k$-chain $C(L^-_{YX_1}) = \{(r, 0, \ldots, 0, \tilde{s}_1, \ldots, \tilde{s}_{2k-1}) | r \leq 0\}$, which intersects $L^+_{YX_2}$ exactly at one point when $t = -\delta$. At the intersection point, we have

$$\Big(\underbrace{\frac{\partial}{\partial x^1_1}, \ldots, \frac{\partial}{\partial x^1_{2k-1}}}_{Or(L^+_{YX_2})}, \frac{\partial}{\partial y}, \underbrace{\frac{\partial}{\partial x^2_1}, \ldots, \frac{\partial}{\partial x^2_{2k-1}}}_{Or(L^-_{YX_1})}\Big)$$

which contributes -1 to $lk(L^+_{YX_2}, L^-_{YX_1})$. Since this intersection point disappears for $t = \delta$, we get $\Delta lk(L^+_{YX_2}, L^-_{YX_1}) = +1$.

Hence, the overall change in $\mathcal{W}_1$ is:

$$\begin{aligned}
\Delta \mathcal{W}_1(F) &= \mathcal{W}_1(F_\delta) - \mathcal{W}_1(F_{-\delta}) \\
&= \frac{1}{2}\Big(\Delta lk(L^+_{X_1X_2}, L^+_{X_1Y}) + \Delta lk(L^-_{X_2X_1}, L^-_{X_2Y})\Big) - \Delta lk(L^+_{YX_2}, L^-_{YX_1}) \\
&= \frac{1}{2}(1+1) - 1 \\
&= 0.
\end{aligned}$$

Thus, the first formula is invariant under the third Reidemeister move when $X_1 > X_2 > Y$, $Y > X_1 > X_2$ and $X_1 > Y > X_2$, with the choice of orientations as above.

Additionally, in each of these three cases, it remains to check what happens when orientations of some components are reversed near the triple point. There are seven more such possibilities in each case. However, note that our formula $\mathcal{W}_1$ does not change when we reverse the orientation of the component X (i.e. when we reverse the local orientations on both X_1 and X_2). Indeed, in $lk(L_{XX}, L_{XY})$ only L_{XY} changes sign, but since this linking number is defined in X, the linking number changes sign as well. For $lk(L^+_{YX}, L^-_{YX})$, both entries change sign, thus no overall change. By a similar argument, it can be easily seen that $\mathcal{W}_1$ changes by a negative sign when we reverse the orientation on Y. Thus, it suffices to check that $\mathcal{W}_1$ is invariant when we change the orientation of only X_1, and other cases then follow using the symmetry.

Consider X_1 with the opposite orientation i.e, now $-X_1 > X_2 > Y$ at the triple point. With the same local coordinates as before, near the triple point only the map $f_t|_{X_1}$ changes as follows:

$$f_t|_{X_1} = (-x_1, 0, \underbrace{0, \ldots, 0}_{2k-1}, x_1^2, \ldots, x_{2k-1}^2, y_1, \ldots, y_{2k-1}).$$

Note that all double intersections of f_t in $\mathbb{R}^{6k-1}$ are the same. However, the orientations on both $f_t|_{X_1} \cap f_t|_{X_2}$ and $f_t|_{X_1} \cap f_t|_Y$ are given by

$$(\underbrace{e_{4k+1}, e_{4k+2}, \ldots, e_{6k-1}}_{Or(f_t|_{X_1} \cap f_t|_{X_2})}, \underbrace{-e_1, e_{2k+2}, \ldots, e_{4k}}_{\text{rest of } Or(X_1)}, \underbrace{-e_2, e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(X_2)});$$

$$(\underbrace{-e_{2k+2}, e_{2k+3}, \ldots, e_{4k}}_{Or(f_t|_{X_1} \cap f_t|_Y)}, \underbrace{-e_1, e_{4k+1}, \ldots, e_{6k-1}}_{\text{rest of } Or(X_1)}, \underbrace{-(e_1 + e_2), e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(Y)});$$

where each gives the positive orientation on $\mathbb{R}^{6k-1}$.

Thus, the new oriented preimages are

$$\begin{aligned}
L^{+}_{X_1X_2}(t) &= (-t, \underbrace{0, \ldots, 0}_{2k-1}, s_1, s_2, \ldots, s_{2k-1}),\\
L^{+}_{X_1Y} &= (0, -\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}),\\
L^{-}_{X_2X_1} &= (0, s_1, s_2, \ldots, s_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}),\\
L^{-}_{YX_1} &= (0, \underbrace{0, \ldots, 0}_{2k-1}, -\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{2k-1}),
\end{aligned}$$

and $L^{+}_{X_2Y}$ and $L^{-}_{YX_2}$ are the same as before.

This only affects the orientations defined earlier at intersection points (when $t = -\delta$). In X_1, at the intersection point we get

$$\Big(\underbrace{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_{2k-1}}}_{Or(L^{+}_{X_1X_2})}, -\frac{\partial}{\partial x_1}, \underbrace{-\frac{\partial}{\partial x_1^2}, \frac{\partial}{\partial x_2^2}, \ldots, \frac{\partial}{\partial x_{2k-1}^2}}_{Or(L^{+}_{X_1Y})}\Big)$$

which contributes $+1$ to $lk(L^{+}_{X_1X_2}, L^{+}_{X_1Y})$, thus $\Delta lk(L^{+}_{X_1X_2}, L^{+}_{X_1Y}) = -1$.

In X_2, at the intersection point we get

$$\Big(\underbrace{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_{2k-1}}}_{Or(L^{-}_{X_2X_1})}, -\frac{\partial}{\partial x_2}, \underbrace{\frac{\partial}{\partial x_1^1}, \ldots, \frac{\partial}{\partial x_{2k-1}^1}}_{Or(L^{+}_{X_2Y})}\Big),$$

which contributes $+1$ to $lk(L^{-}_{X_2X_1}, L^{+}_{X_2Y})$, thus $\Delta lk(L^{-}_{X_2X_1}, L^{+}_{X_2Y}) = -1$.

Hence, the overall change in $\mathcal{W}_1$ is given by

$$\begin{aligned}
\Delta\mathcal{W}_1(F) &= \mathcal{W}_1(F_\delta) - \mathcal{W}_1(F_{-\delta})\\
&= \frac{1}{2}\Big(\Delta lk(L^{+}_{X_1X_2}, L^{+}_{X_1Y}) - \Delta lk(L^{-}_{X_2X_1}, L^{+}_{X_2Y})\Big)\\
&= \frac{1}{2}(-1 - (-1))\\
&= 0
\end{aligned}$$

By symmetry, we can say that $\mathcal{W}_1$ is invariant under the third Reidemeister move when $Y > X_1 > -X_2$, and when $Y > -X_1 > X_2$ (by changing orientation of X).

For the case $X_1 > Y > X_2$, again it suffices to check the change when the orientation of X_1 is reversed i.e., $-X_1 > Y > X_2$. Near the triple point, we have

$$f_t|_{X_1} = (-x_1, 0, \underbrace{0, \ldots, 0}_{2k-1}, x_1^2, \ldots, x_{2k-1}^2, y_1, \ldots, y_{2k-1}).$$

The double intersections of f_t and their orientations are same as above i.e., the orientations on both $f_t|_{X_1} \cap f_t|_{X_2}$ and $f_t|_{X_1} \cap f_t|_Y$ are given by

$$(\underbrace{e_{4k+1}, e_{4k+2}, \ldots, e_{6k-1}}_{Or(f_t|_{X_1} \cap f_t|_{X_2})}, \underbrace{-e_1, e_{2k+2}, \ldots, e_{4k}}_{\text{rest of } Or(X_1)}, \underbrace{-e_2, e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(X_2)});$$

$$(\underbrace{-e_{2k+2}, e_{2k+3}, \ldots, e_{4k}}_{Or(f_t|_{X_1} \cap f_t|_Y)}, \underbrace{-e_1, e_{4k+1}, \ldots, e_{6k-1}}_{\text{rest of } Or(X_1)}, \underbrace{-(e_1 + e_2), e_3, \ldots, e_{2k+1}}_{\text{rest of } Or(Y)});$$

where each gives the positive orientation on $\mathbb{R}^{6k-1}$. Moreover, the new oriented preimages now are

$$\begin{aligned}
L^+_{X_1X_2}(t) &= (-t, \underbrace{0, \ldots, 0}_{2k-1}, s_1, s_2, \ldots, s_{2k-1}), \\
L^+_{X_1Y} &= (0, -\tilde{s}_1, \tilde{s}_2 \ldots, \tilde{s}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}), \\
L^-_{X_2X_1} &= (0, s_1, s_2, \ldots, s_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}), \\
L^-_{X_2Y}(t) &= (t, \underbrace{0, \ldots, 0}_{2k-1}, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{2k-1}), \\
L^-_{YX_1} &= (0, \underbrace{0, \ldots, 0}_{2k-1}, -\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{2k-1}), \\
L^+_{YX_2}(t) &= (t, \tilde{\tilde{s}}_1, \tilde{\tilde{s}}_2, \ldots, \tilde{\tilde{s}}_{2k-1}, \underbrace{0, \ldots, 0}_{2k-1}).
\end{aligned}$$

Recall that the linking numbers that contribute to $\mathcal{W}_1$ in this case are $lk(L^+_{X_1X_2}, L^+_{X_1Y})$, $lk(L^-_{X_2X_1}, L^-_{X_2Y})$, and $lk(L^+_{YX_2}, L^-_{YX_1})$.

In X_1, at the intersection point we have

$$\Big(\underbrace{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{2k-1}}}_{Or(L^+_{X_1X_2})}, -\frac{\partial}{\partial x_1}, \underbrace{-\frac{\partial}{\partial x^2_1}, \frac{\partial}{\partial x^2_1}, \dots, \frac{\partial}{\partial x^2_{2k-1}}}_{Or(L^+_{X_1Y})}\Big)$$

which contributes $+1$ to $lk(L^+_{X_1X_2}, L^+_{X_1Y})$, thus $\Delta lk(L^+_{X_1X_2}, L^+_{X_1Y}) = -1$.

In X_2, at the intersection point we have

$$\Big(\underbrace{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{2k-1}}}_{Or(L^-_{X_2X_1})}, -\frac{\partial}{\partial x_2}, \underbrace{\frac{\partial}{\partial x^1_1}, \dots, \frac{\partial}{\partial x^1_{2k-1}}}_{Or(L^-_{X_2Y})}\Big),$$

which contributes $+1$ to $lk(L^-_{X_2X_1}, L^+_{X_2Y})$, thus $\Delta lk(L^-_{X_2X_1}, L^+_{X_2Y}) = -1$.

In Y, at the intersection point, we have

$$\Big(\underbrace{\frac{\partial}{\partial x^1_1}, \dots, \frac{\partial}{\partial x^1_{2k-1}}}_{Or(L^+_{YX_2})}, \frac{\partial}{\partial y}, \underbrace{-\frac{\partial}{\partial x^2_1}, \frac{\partial}{\partial x^2_2}, \dots, \frac{\partial}{\partial x^2_{2k-1}}}_{Or(L^-_{YX_1})}\Big)$$

which contributes $+1$ to $lk(L^+_{YX_2}, L^-_{YX_1})$. Since this intersection point disappears for $t = \delta$, we get $\Delta lk(L^+_{YX_2}, L^-_{YX_1}) = -1$.

Hence, the overall change in $\mathcal{W}_1$ is:

$$\begin{aligned}
\Delta\mathcal{W}_1(F) &= \mathcal{W}_1(F_\delta) - \mathcal{W}_1(F_{-\delta}) \\
&= \frac{1}{2}\Big(\Delta lk(L^+_{X_1X_2}, L^+_{X_1Y}) + \Delta lk(L^-_{X_2X_1}, L^-_{X_2Y})\Big) - \Delta lk(L^+_{YX_2}, L^-_{YX_1}) \\
&= \frac{1}{2}(-1-1) - (-1) \\
&= 0.
\end{aligned}$$

Thus, we can conclude that $\mathcal{W}_1$ is invariant under all possible third Reidemeister moves (in which X is repeated exactly twice). Similarly, $\mathcal{W}_2$ is invariant under all possible third Reidemeister moves (in which Y is repeated exactly twice).

To see the image of $\mathcal{W} : \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \xrightarrow{(\mathcal{W}_1, \mathcal{W}_2)} \mathbb{Q} \oplus \mathbb{Q}$, we first note that two linearly independent elements of $\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})$ are obtained from the Borromean link $\mathcal{B}$ by means of tubing along a chosen path. In particular, the first element $\mathcal{B}_1$ is obtained by connecting the component X with Y in $\mathcal{B}$ via a thin tube i.e., $\mathcal{B}_1 = (X \# Y) \sqcup Z$. To get $\mathcal{B}_2$, we tube the component Y with Z in $\mathcal{B}$ i.e., $\mathcal{B}_2 = X \sqcup (Y \# Z)$. The result of tubing does not depend on the choice of paths since the complement is simply connected. We will show in the next subsection that $\mathcal{W}(\mathcal{B}_1) = (-2, 0)$ and $\mathcal{W}(\mathcal{B}_2) = (0, -2)$. This implies that $\mathcal{B}_1$ and $\mathcal{B}_2$ generate $\mathcal{W} : \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})$ rationally.

There is another map $T : \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \xrightarrow{(T_1, T_2)} \mathbb{Z} \oplus \mathbb{Z}$, where $T_1 : \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \to \pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) = \mathbb{Z}$ is defined as the tubing of the two components while preserving the orientation of the components i.e., $X \sqcup Y \mapsto X \# Y$. Similarly, the map $T_2 : \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \to \pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) = \mathbb{Z}$ is defined as the tubing of the two components where we reverse the orientation on one of the components i.e., $X \sqcup Y \mapsto X \# (-Y)$. Both T_1 and T_2 map the rational generators $\mathcal{B}_1$ and $\mathcal{B}_2$ up to a sign, to the Haefliger trefoil knot (the generator of $\pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) = \mathbb{Z}$). In particular, $T(\mathcal{B}_1) = (1, -1)$ and $T(\mathcal{B}_2) = (1, 1)$. This is because of the symmetry of $\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})$ which gives that $X \sqcup Y \sqcup Z = X \sqcup (-Y) \sqcup (-Z) = -(X \sqcup Y \sqcup (-Z))$, see Remark 3.5.3. Note that the elements $(1, -1)$ and $(1, 1)$ generate a sub-lattice L in $\mathbb{Z}^2$, where $L = \{(a, b) \in \mathbb{Z}^2 | a + b \in 2\mathbb{Z}\}$. Moreover, L is a sub-lattice of index 2 in $\mathbb{Z}^2$ as it is the kernel of the map $\mathbb{Z}^2 \to \mathbb{Z}_2$ which takes $(a, b) \mapsto (a + b) \mod 2$. Thus, we get two possibilities: either $T(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = \mathbb{Z}^2$ or $T(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = L$. Note that we have the following commutative diagram:

$$\begin{tikzcd}
 & \pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \arrow[dl, "T"'] \arrow[dr, "\mathcal{W}"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathbb{Z}^2 \arrow[rr, "A"'] & & \mathbb{Q}^2
\end{tikzcd}$$

which in terms of the rational generators is

$$
\begin{tikzcd}
& \mathcal{B}_1, \mathcal{B}_2 \arrow[dl, "T"'] \arrow[dr, "\mathcal{W}"] \arrow[d, phantom, "\circlearrowleft"] & \\
(1,-1),(1,1) \arrow[rr, "A"'] & & (-2,0),(0,-2)
\end{tikzcd}
$$

where the linear map A is multiplication by the matrix

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

(since $(1,0) = \frac{(1,1)+(1,-1)}{2}$, $(0,1) = \frac{(1,1)-(1,-1)}{2}$ in $\mathbb{Z}^2$ must be mapped to $(-1,-1) = \frac{(0,-2)+(-2,0)}{2}$, $(1,-1) = \frac{(0,-2)-(-2,0)}{2}$ in $\mathbb{Q}^2$, respectively).

We will prove in Section 3.7 that for $k = 1,2,4$ we get $T(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = \mathbb{Z}^2$, which then implies that $\mathcal{W}(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = A(\mathbb{Z}^2) = L$; and for $k \neq 1,2,4$ we get $T(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = L$, which implies $\mathcal{W}(\pi_0^{br} Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = A(L) = 2\mathbb{Z} \times 2\mathbb{Z}$. This will conclude the proof.

$\square$

3.6.2 The generators $\mathcal{B}_1$ and $\mathcal{B}_2$

Recall that by tubing the component Y in the Borromean link with either X or Z results in $\mathcal{B}_1 = (X\#Y) \sqcup Z$ and $\mathcal{B}_2 = X \sqcup (Y\#Z)$, respectively. Note that the description of the double points set in $\mathcal{B}_1$ and $\mathcal{B}_2$ are the same as for $\mathcal{B}$ given by six disjoint Hopf links. For $\mathcal{B}_1$, we replace X by X_1, Y by X_2 and Z by Y, so that $\mathcal{B}_1 = \underbrace{X_1\#X_2}_{X} \sqcup Y$ and now the set of double points have four Hopf links in X and two in Y, given as follows

$$L^+_{X_1X_2} \sqcup L^+_{X_1Y}, L^-_{X_1X_2} \sqcup L^-_{X_1Y}, L^+_{X_2X_1} \sqcup L^+_{X_2Y}, L^-_{X_2X_1} \sqcup L^-_{X_2Y} \subset X,$$

$$L^+_{YX_1} \sqcup L^+_{YX_2}, L^-_{YX_1} \sqcup L^-_{YX_2} \subset Y.$$

With the same argument as given in Subsection 3.5.3, the only non-zero linking numbers in X are

$$lk(L^+_{X_1X_2}, L^+_{X_1Y}), lk(L^+_{X_1X_2}, L^-_{X_1Y}), lk(L^+_{X_2X_1}, L^+_{X_2Y}) \text{ and } lk(L^-_{X_2X_1}, L^-_{X_2Y});$$

where all of them equal to -1. Note that $lk(L^+_{YX}, L^-_{YX}) = 0$, therefore we get

$$\begin{aligned}
\mathcal{W}_1(\mathcal{B}_1) &= \frac{1}{2} lk(L_{XX}, L_{XY}) - \underbrace{lk(L^+_{YX}, L^-_{YX})}_{=0} \\
&= \frac{1}{2} lk(L_{X_1X_2} \sqcup L_{X_2X_1}, L_{X_1Y} \sqcup L_{X_2Y}) - 0 \\
&= \frac{1}{2} \cdot -4 \\
&= -2.
\end{aligned}$$

It is easy to see that $\mathcal{W}_2(\mathcal{B}_1) = 0$ and therefore, $\mathcal{W}(\mathcal{B}_1) = (-2, 0)$. By symmetry, we can conclude that $\mathcal{W}(\mathcal{B}_2) = (0, -2)$.

3.7 Connection to configuration spaces

Let $C(k, X)$ denote the configuration space of ordered distinct k points in X i.e.,

$$C(k, X) := \{(x_1, \dots, x_k) \in X^k, x_i \neq x_j \text{ if } i \neq j\}.$$

For $\ell \geq 3$, let $\Omega^m C(k, \mathbb{R}^\ell)$ be the space of smooth maps $D^m \to C(k, \mathbb{R}^\ell)$ which send a neighborhood of ∂D^m to a chosen base point. One can view $\Omega^m C(k, \mathbb{R}^\ell)$ as the space m-dimensional braids in $\mathbb{R}^{m+\ell}$ using the graphing map

$$G : \Omega^m C(k, \mathbb{R}^\ell) \to Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell}),$$

as defined in [6, 28]. This map induces a morphism

$$G_*:\pi_0\Omega^m C(k,\mathbb{R}^\ell)=\pi_m C(k,\mathbb{R}^\ell)\to\pi_0 Emb_\partial(\bigsqcup_k\mathbb{R}^m,\mathbb{R}^{m+\ell}),$$

which was conjectured in [28, Conjecture 5.7] to be an injection rationally. In fact, it is also true integrally and moreover, G_* is split-injective, which immediately follows from the following theorem.

Theorem 3.7.1. *For $\ell\geq 3$, the space $\Omega^m C(k,\mathbb{R}^\ell)$ is a homotopy retract of $Emb_\partial(\bigsqcup_k\mathbb{R}^m,\mathbb{R}^{m+\ell})$.*

Proof. From [9, Theorem 2.1], there is a homotopy equivalence

$$\Omega C(k,\mathbb{R}^\ell)\to\prod_{i=1}^{k-1}\Omega(\bigvee_i S^{\ell-1}). \tag{3.7.1}$$

Consider the map $C(k,\mathbb{R}^\ell)\to C(k-1,\mathbb{R}^\ell)$ which forgets the last configuration point. This gives a fibration

$$\bigvee_{k-1}S^{\ell-1}\to C(k,\mathbb{R}^\ell)\to C(k-1,\mathbb{R}^\ell),$$

with a section $C(k-1,\mathbb{R}^\ell)\to C(k,\mathbb{R}^\ell)$ that includes a point in $\mathbb{R}^\ell$ which is disjoint from the others. We get an induced fibration in loop spaces

$$\Omega(\bigvee_{k-1}S^{\ell-1})\to\Omega C(k,\mathbb{R}^\ell)\to\Omega C(k-1,\mathbb{R}^\ell),$$

along with an induced section. This implies that

$$\pi_0\Omega C(k,\mathbb{R}^\ell)\approx\pi_0\Omega C(k-1,\mathbb{R}^\ell)\oplus\pi_0\Omega(\bigvee_{k-1}S^{\ell-1}).$$

Since all these loop spaces are H-spaces and π_0 of all of them are abelian groups, we get

$$\Omega C(k,\mathbb{R}^\ell)\simeq\Omega C(k-1,\mathbb{R}^\ell)\times\Omega(\bigvee_{k-1}S^{\ell-1}).$$

Iteratively, this argument holds for higher loop spaces, and using (3.7.1) we get

$$\Omega^m C(k,\mathbb{R}^\ell) \simeq \prod_{i=1}^{k-1} \Omega^m(\bigvee_i S^{\ell-1}), \tag{3.7.2}$$

Moreover, a similar result holds for embedding spaces as well. Consider the map $Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell}) \to Emb_\partial(\sqcup_{k-1} \mathbb{R}^m, \mathbb{R}^{m+\ell})$ which forgets the last string. This gives a fibration

$$Emb_\partial(\mathbb{R}^m, \mathbb{R}^{m+\ell} \backslash (\sqcup_{k-1} \mathbb{R}^m)) \to Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell}) \to Emb_\partial(\sqcup_{k-1} \mathbb{R}^m, \mathbb{R}^{m+\ell}),$$

with a section $Emb_\partial(\sqcup_{k-1} \mathbb{R}^m, \mathbb{R}^{m+\ell}) \to Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell})$ that includes a trivial string in $\mathbb{R}^{m+\ell}$ which is disjoint from the others. By the same argument as above,

$$Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell}) \simeq \prod_{i=0}^{k-1} Emb_\partial(\mathbb{R}^m, \mathbb{R}^{m+\ell} \backslash (\sqcup_i \mathbb{R}^m)) \tag{3.7.3}$$

(we use here that $\pi_0 Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell})$ is an abelian group).

We now show that each factor from (3.7.2) is a retract of the corresponding factor from (3.7.3), where for $i=0$, we take a point as a factor in (3.7.2). Note that there is a natural graphing map

$$\Omega^m(\bigvee_i S^{\ell-1}) \simeq \Omega^m(\mathbb{R}^\ell \backslash \{i \text{ pts}\}) \xrightarrow{G} Emb_\partial(\mathbb{R}^m, \mathbb{R}^{m+\ell} \backslash (\sqcup_i \mathbb{R}^m)).$$

Moreover, given an embedding $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+\ell} \backslash (\sqcup_i \mathbb{R}^m)$, we project $\mathbb{R}^{m+\ell} \backslash (\sqcup_i \mathbb{R}^m)$ to $\mathbb{R}^\ell \backslash \{i \text{ pts}\}$, thus giving us a map

$$Emb_\partial(\mathbb{R}^m, \mathbb{R}^{m+\ell} \backslash (\sqcup_i \mathbb{R}^m)) \xrightarrow{F} Map_\partial(\mathbb{R}^m, \mathbb{R}^\ell \backslash \{i \text{ pts}\}) \simeq \Omega^m(\bigvee_i S^{\ell-1}).$$

It is easy to see that G is a retract i.e., $F \circ G = id$ on $\Omega^m(\mathbb{R}^\ell \backslash \{i \text{ pts}\})$. Thus, we get $\Omega^m C(k, \mathbb{R}^\ell)$ is a homotopy retract of $Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell})$.

□

The homotopy group $\pi_*C(k,\mathbb{R}^\ell)$ has a Brunnian splitting, similar to the one for embedding spaces, see Proposition 3.4.1. In particular, for $\ell \geq 3$,

$$\pi_*C(k,\mathbb{R}^\ell) = \bigoplus_{i=1}^{k}\Big(\bigoplus_{\binom{k}{i}} \pi_*^{br}C(i,\mathbb{R}^\ell)\Big), \tag{3.7.4}$$

where $\pi_*^{br}C(i,\mathbb{R}^\ell) = \bigcap\limits_{j=1}^{i} ker\big(\pi_*C(i,\mathbb{R}^\ell) \xrightarrow{s_j} \pi_*C(i-1,\mathbb{R}^\ell)\big)$ with map s_j forgetting the j^{th} point. The graphing $G_* : \pi_m C(k,\mathbb{R}^\ell) \to \pi_0 Emb_\partial(\sqcup_k \mathbb{R}^m, \mathbb{R}^{m+\ell})$ respects the splitting.

3.7.1 3-component links

It is known that $\pi_{2\ell-1}^{br}C(3,\mathbb{R}^{\ell+1})$ is isomorphic to $\mathbb{Z}$ with a generator given by $[\alpha_{12},\alpha_{23}]$ (see [24]), where $[-,-]$ is the Whitehead bracket on $\pi_*C(3,\mathbb{R}^{\ell+1})$ and $\alpha_{12},\alpha_{23} \in \pi_\ell C(3,\mathbb{R}^{\ell+1})$ correspond to the braids where strand 2 goes around strand 1 or 3, respectively. Abusing notation, α_{12},α_{23} also denote the corresponding maps $\alpha_{12},\alpha_{23} : S^\ell \to C(3,\mathbb{R}^{\ell+1})$.

Theorem 3.7.2. *The map $G_* : \pi_{2\ell-1}^{br}C(3,\mathbb{R}^{\ell+1}) \to \pi_0^{br}Emb_\partial(\sqcup_3 \mathbb{R}^{2\ell-1},\mathbb{R}^{3\ell})$ takes the generator $[\alpha_{12},\alpha_{23}]$ to the high-dimensional Borromean link $\mathcal{B}$, taken with the negative sign. Thus, $\mathcal{B}$ is the generator of $\pi_0 Emb_\partial(\sqcup_3 \mathbb{R}^{2\ell-1},\mathbb{R}^{3\ell}) = \mathbb{Z}$.*

Proof. The graphing map is given as follows:

$$G : \Omega^{2\ell-1}C(3,\mathbb{R}^{\ell+1}) \to Emb_\partial(\sqcup_3 \mathbb{R}^{2\ell-1},\mathbb{R}^{3\ell}),$$

$$f \mapsto F$$

For $f = (f_1,f_2,f_3) : D^{2\ell-1} \to C(3,\mathbb{R}^{\ell+1})$, we have $f_i = \pi_i \circ f : D^{2\ell-1} \to \mathbb{R}^{\ell+1}$ for $i = 1,2,3$, where $\pi_i : C(3,\mathbb{R}^{\ell+1}) \to \mathbb{R}^{\ell+1}$ remembers only the i^{th}-configuration point. We extend each f_i on the complement of $D^{2\ell-1}$ as a constant map to a chosen base point, thus giving us a map $\mathbb{R}^{2\ell-1} \to \mathbb{R}^{\ell+1}$. Then the corresponding embedding $F \in Emb_\partial(\sqcup_3 \mathbb{R}^{2\ell-1},\mathbb{R}^{3\ell})$ is given by

$F = (F_1, F_2, F_3)$, where $F_i \in Emb_\partial(\mathbb{R}^{2\ell-1}, \mathbb{R}^{3\ell})$ is defined as

$$F_i(x_1, \ldots, x_{2\ell-1}) := (x_1, \ldots, x_{2\ell-1}, f_i(x_1, \ldots, x_{2\ell-1})) \in D^{2\ell-1} \times \mathbb{R}^{\ell+1} \subset \mathbb{R}^{3\ell},$$

for $(x_1, \ldots, x_{2\ell-1}) \in D^{2\ell-1}$. Note that since F_i is a long embedding, we assume that everything happens inside the disk $D^{2\ell-1}$, and thus we can disregard the complement of $D^{2\ell-1}$ in $\mathbb{R}^{2\ell-1}$.

Given $\alpha_{12}, \alpha_{23} \in \pi_\ell C(3, \mathbb{R}^{\ell+1})$, we have the following fiber sequence

$$S^\ell \vee S^\ell \xrightarrow{\alpha_{12} \vee \alpha_{23}} C(3, \mathbb{R}^{\ell+1}) \xrightarrow{\pi_{13}} C(2, \mathbb{R}^{\ell+1}) \simeq S^\ell,$$

where π_{13} is the projection that forgets the second point, and $\alpha_{12} \vee \alpha_{23}$ is the fiber inclusion where the second point rotates around the first and the third point, respectively. The element $[\alpha_{12}, \alpha_{23}]$ is then represented by $f = (f_1, f_2, f_3)$ given by the composition

$$D^{2\ell-1} \to S^{2\ell-1} \simeq \partial(D^\ell \times D^\ell) \xrightarrow{\phi} S^\ell \vee S^\ell \xrightarrow{\alpha_{12} \vee \alpha_{23}} C(3, \mathbb{R}^{\ell+1}),$$

where the first map collapses the boundary $\partial D^{2\ell-1}$ to the base point in $S^{2\ell-1}$: $D^{2\ell-1} \to D^{2\ell-1}/\partial D^{2\ell-1} \cong S^{2\ell-1}$, and ϕ is the attaching map of the top 2ℓ-cell in the product $S^\ell \times S^\ell$. Since the first and third points are fixed, maps f_1 and f_3 are constant in $\mathbb{R}^{\ell+1}$ given by $f_1(x) = (-1, \underbrace{0 \ldots, 0}_{\ell})$ and $f_3(x) = (1, \underbrace{0 \ldots, 0}_{\ell})$ for all $x \in D^{2\ell-1}$.

For the corresponding embedding we first note that to evaluate our invariant 3.5.3, we only care about the set of double points, which is easier to understand if we replace $D^{2\ell-1}$ by $\partial(D^\ell \times D^\ell) \simeq S^{2\ell-1}$ and view the link as a map given by

$$F : \sqcup_3 \partial(D^\ell \times D^\ell) \to \partial(D^\ell \times D^\ell) \times \mathbb{R}^{\ell+1}$$

(note that we do not study links in $S^{2\ell-1} \times \mathbb{R}^{\ell+1}$, this replacement is done only for computing the invariant). The representative $F = (F_1, F_2, F_3)$ is such that $F_1(x) = (x, f_1(x)) =$

$(x,-1,0,\ldots,0)$ and $F_3(x)=(x,f_3(x))=(x,1,0,\ldots,0)$ for all $x\in\partial(D^\ell\times D^\ell)$. Moreover, F_2 is given by two maps defined on the two parts of $\partial(D^\ell\times D^\ell)\simeq\big(S^{\ell-1}\times D^\ell\big)\cup\big(D^\ell\times S^{\ell-1}\big)$ as follows:

$$id_{S^{\ell-1}\times D^\ell}\times(h_{-1}\circ p_2):S^{\ell-1}\times D^\ell\longrightarrow S^{\ell-1}\times D^\ell\times\mathbb{R}^{\ell+1},$$

$$id_{D^\ell\times S^{\ell-1}}\times(h_1\circ p_1):D^\ell\times S^{\ell-1}\longrightarrow D^\ell\times S^{\ell-1}\times\mathbb{R}^{\ell+1};$$

where p_1 and p_2 are projections of the domains above to the first and second factor, respectively. Moreover, h_{-1} and h_1 map the respective disk D^ℓ to $\mathbb{R}^{\ell+1}$ by collapsing its boundary to the origin such that the image is a unit sphere around $(-1,\underbrace{0\ldots,0}_{\ell})\in\mathbb{R}^{\ell+1}$ and $(1,\underbrace{0\ldots,0}_{\ell})\in\mathbb{R}^{\ell+1}$, respectively. Thus, when we project the image of F_2 on $\mathbb{R}^{\ell+1}$ we get $S^\ell_{-1}\vee S^\ell_1$ with the base point at the origin in $\mathbb{R}^{\ell+1}$. Note that the maps $h_\pm$ are given by the composition $D^\ell\to D^\ell/\partial D^\ell\xrightarrow{\cong}S^\ell$ of degree one, thus they respect the orientation.

Let us call the three components of the embedding F as X, Y, and Z. We now project F along the last $(\ell+1)$-coordinate, given by the map

$$p:\partial(D^\ell\times D^\ell)\times\mathbb{R}^{\ell+1}\to\partial(D^\ell\times D^\ell)\times\mathbb{R}^\ell.$$

Under the projection to $\mathbb{R}^\ell$, the first and third components are sent to points $(-1,\underbrace{0\ldots,0}_{\ell-1})$ and $(1,\underbrace{0\ldots,0}_{\ell-1})$, respectively. The projection of the second component has image $D^\ell_{-1}\vee D^\ell_1$ in $\mathbb{R}^\ell$ with the base point at the origin, where these two unit disks D^ℓ_{-1} and D^ℓ_1 are centered at $(-1,\underbrace{0,\ldots,0}_{\ell-1})$ and $(1,\underbrace{0,\ldots,0}_{\ell-1})$ in $\mathbb{R}^\ell$, respectively. Note that $p(F_1(X))\cap p(F_2(Y))$ and $p(F_2(Y))\cap p(F_3(Z))$ are the only non-trivial double intersections that occur i.e, the projections of X and Z do not intersect. The intersection $p(F_1(X))\cap p(F_2(Y))$ is given by

$$p(F_1(X))\cap p(F_2(Y))=\big((S^{\ell-1}\times A_+)\sqcup(S^{\ell-1}\times A_-)\big)\times(-1,\underbrace{0,\ldots,0}_{\ell-1}),$$

where points A_+ and A_- lie in the upper and lower halves of the disk D^ℓ see figure, respectively. Indeed, on $S^{\ell-1}\times D^{\ell}$, the projection $p(F_2(Y))$ satisfy the following:

$$S^{\ell-1}\times D^{\ell} \xrightarrow{F_2\,=\,id_{S^{\ell-1}\times D^{\ell}}\times(h_{-1}\circ p_2)} S^{\ell-1}\times D^{\ell}\times\mathbb{R}^{\ell+1} \xrightarrow{\ p\ } S^{\ell-1}\times D^{\ell}\times\mathbb{R}^{\ell}$$

$$S^{\ell-1}\times A_{\pm} \longmapsto S^{\ell-1}\times A_{\pm}\times(-1,\underbrace{0,\ldots,0}_{\ell-1},\pm1) \longmapsto S^{\ell-1}\times A_{\pm}\times(-1,\underbrace{0,\ldots,0}_{\ell-1}),$$

where the underlying map to $\mathbb{R}^{\ell}$ is given by first taking the projection $p_2 : S^{\ell-1}\times D^{\ell} \to D^{\ell}$ followed by the map

$$p_{|_{\mathbb{R}^{\ell+1}}} \circ h_{-1} : D^{\ell} \longrightarrow \mathbb{R}^{\ell}.$$

The latter takes ∂D^{ℓ} to the origin $(0,\ldots,0) \in D^{\ell}_{-1} \subset \mathbb{R}^{\ell}$, the disk $H^{\ell-1}$ which cuts D^{ℓ} into two halves is sent to ∂D^{ℓ}_{-1}, and the points A_+ and A_- are sent to the center $(-1,\underbrace{0,\ldots,0}_{\ell-1})$ of D^{ℓ}_{-1} as shown in the picture below.

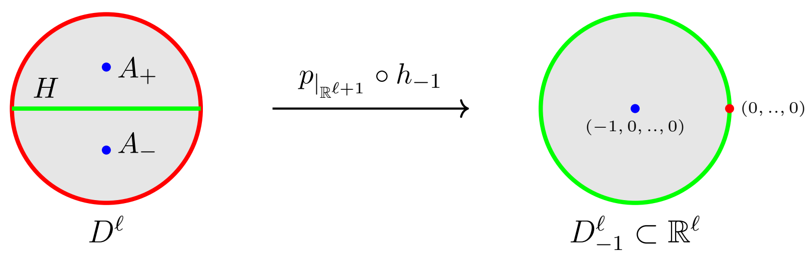


Figure 3.25: The underlying map from D^{ℓ} to D^{ℓ}_{-1}

Moreover, the upper half of D^{ℓ} is sent to D^{ℓ}_{-1} respecting the orientation and the lower half is sent reversing the orientation. Since $p(F_1(X)) = S^{\ell-1}\times D^{\ell}\times(-1,\underbrace{0,\ldots,0}_{\ell-1})$, it is clear that there are two components of intersection exactly as mentioned above. By a similar argument, one can see that the intersection $p(F_2(Y)) \cap p(F_3(Z))$ is given by

$$p(F_2(Y)) \cap p(F_3(Z)) = \big((B_+ \times S^{\ell-1}) \sqcup (B_- \times S^{\ell-1})\big) \times (1,\underbrace{0,\ldots,0}_{\ell-1}),$$

where points B_+ and B_- lie in the upper and lower halves of the disk in $D^{\ell}\times S^{\ell-1}$, respectively. Thus, the double intersection comprises of four spheres, with their preimages in X, Y, Z as follows. In $X = \partial(D^\ell \times D^\ell)$, we get

$$L^-_{XY} = S^{\ell-1} \times A_+,$$

$$L^+_{XY} = S^{\ell-1} \times A_-;$$

in $Y = \partial(D^\ell \times D^\ell)$,

$$L^+_{YX} = S^{\ell-1} \times A_+,$$

$$L^-_{YX} = S^{\ell-1} \times A_-,$$

$$L^+_{YZ} = B_+ \times S^{\ell-1},$$

$$L^-_{YZ} = B_- \times S^{\ell-1};$$

in $Z = \partial(D^\ell \times D^\ell)$,

$$L^-_{ZY} = B_+ \times S^{\ell-1},$$

$$L^+_{ZY} = B_- \times S^{\ell-1}.$$

The two preimages in both X and Z are parallel spheres, while the four preimages in Y form a double of a Hopf link. The following depicts the set of double points of $p \circ F$ with each link component (with some orientation) inside X, Y, and Z as a sphere $S^{\ell-1}$:

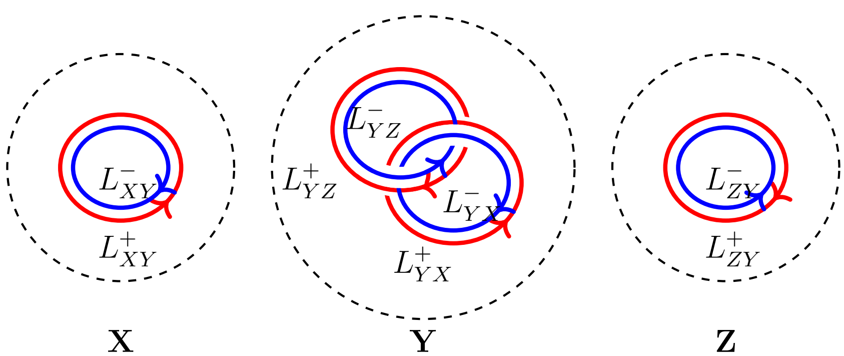


Figure 3.26: The set of double points of $p \circ G_*([\alpha_{12}, \alpha_{23}])$

Note that only the linking numbers in Y contribute to our invariant 3.5.3, i.e., only

$$lk(L^+_{YZ}, L^+_{YX}), lk(L^-_{YZ}, L^-_{YX}), lk(L^+_{YZ}, L^-_{YX}), \text{ and } lk(L^-_{YZ}, L^+_{YX}).$$

Each of these four linking numbers corresponds to the linking number of the standard Hopf link in Y, and thus we only need to find the orientation on each link component. Note that L^+_{YX} and L^-_{YX} are two parallel spheres $S^{\ell-1}$ (and so are L^+_{YZ} and L^-_{YZ}). We claim that these parallel spheres have the same orientation when ℓ is odd and opposite orientation when ℓ is even. The ambient space $\partial(D^\ell \times D^\ell) \times \mathbb{R}^\ell$ is oriented by taking first the orientation of $\partial(D^\ell \times D^\ell)$ (which uses the outward normal first convention) and then the standard orientation of $\mathbb{R}^\ell$. More precisely, the orientation on $S^{\ell-1} \times D^\ell \times \mathbb{R}^\ell$ is given by $(Or(S^{\ell-1}), Or(D^\ell), Or(\mathbb{R}^\ell))$; whereas the orientation on $D^\ell \times S^{\ell-1} \times \mathbb{R}^\ell$ is given by $(-1)^\ell$ times the standard orientation on each factor i.e., $(-1)^\ell(Or(D^\ell), Or(S^{\ell-1}), Or(\mathbb{R}^\ell))$. The sign $(-1)^\ell$ is due to pulling the outward normal to $S^{\ell-1}$ as the first vector.

In $S^{\ell-1} \times D^\ell \times \mathbb{R}^\ell$, we orient the double intersection $S^{\ell-1} \times A_+$ (we ignore the point in $\mathbb{R}^\ell$) according to our orientation convention in Remark 3.2.1 (note that Y is above X), so we take $(Or(S^{\ell-1} \times A_+),$ rest of $Or(Y),$ rest of $Or(X))$ and compare it with the orientation on $S^{\ell-1} \times D^\ell \times \mathbb{R}^\ell$. Note that the image of X is constant in the direction of $\mathbb{R}^\ell$, therefore only $Or(D^\ell)$ contributes to $Or(X)$. On the other hand, the tangent vectors on Y are a linear combination of tangent vectors on D^ℓ and $\mathbb{R}^\ell$, and since the direction of D^ℓ is taken by $Or(X)$, the remaining $Or(\mathbb{R}^\ell)$ contributes to $Or(Y)$. Therefore, for the intersection $S^{\ell-1} \times A_+$, we take

$$\Big(\underbrace{(-1)^\ell Or(S^{\ell-1})}_{Or(S^{\ell-1}\times A_+)},\ \underbrace{Or(\mathbb{R}^\ell)}_{\text{rest of } Or(Y)},\ \underbrace{Or(D^\ell)}_{\text{rest of } Or(X)}\Big),$$

which gives the standard orientation on $S^{\ell-1} \times D^\ell \times \mathbb{R}^\ell$. Thus, the orientation on the preimages L^+_{YX} and L^-_{XY} is given by $(-1)^\ell Or(S^{\ell-1})$. For $S^{\ell-1} \times A_-$, recall that A_- lies in the lower half of D^ℓ, which contributes with negative orientation to $Or(Y)$, thus we take

$$\big(\underbrace{-Or(S^{\ell-1})}_{Or(S^{\ell-1}\times A_-)},\ \underbrace{Or(D^\ell)}_{\text{rest of } Or(X)},\ \underbrace{-Or(\mathbb{R}^\ell)}_{\text{rest of } Or(Y)}\big),$$

which is equivalent to the standard orientation on $S^{\ell-1}\times D^\ell\times\mathbb{R}^\ell$. Thus, the preimages L^-_{YX} and L^+_{XY} are oriented as $-Or(S^{\ell-1})$.

In $D^\ell\times S^{\ell-1}\times\mathbb{R}^\ell$, by the same argument, we orient $B_+\times S^{\ell-1}$ by taking

$$\big(\underbrace{Or(S^{\ell-1})}_{Or(B_+\times S^{\ell-1})},\ \underbrace{Or(\mathbb{R}^\ell)}_{\text{rest of } Or(Y)},\ \underbrace{Or(D^\ell)}_{\text{rest of } Or(Z)}\big),$$

which gives the standard orientation on $D^\ell\times S^{\ell-1}\times\mathbb{R}^\ell$, that is $(-1)^\ell(Or(D^\ell),Or(S^{\ell-1}),Or(\mathbb{R}^\ell))$. Thus, the preimages L^+_{YZ} and L^-_{ZY} both get the standard orientation $Or(S^{\ell-1})$. Similarly, to orient $B_-\times S^{\ell-1}$, we take

$$\big(\underbrace{(-1)^{\ell-1}Or(S^{\ell-1})}_{B_-\times Or(S^{\ell-1})},\ \underbrace{Or(D^\ell)}_{\text{rest of } Or(Z)},\ \underbrace{-Or(\mathbb{R}^\ell)}_{\text{rest of } Or(Y)}\big),$$

which again gives the standard orientation on $D^\ell\times S^{\ell-1}\times\mathbb{R}^\ell$. Thus, the orientation on L^-_{YZ} and L^+_{ZY} is given by $(-1)^{\ell-1}Or(S^{\ell-1})$.

Since the four linking numbers that we want to compute are of a standard Hopf link inside Y, we can view $Y = S^{2\ell-1}$ as $\partial D^{2\ell}$ instead of $\partial(D^\ell\times D^\ell)$. For each Hopf link, we assume that the first component (either L^+_{YZ} or L^-_{YZ}) is given by the sphere $\{(\underbrace{0,\dots,0}_{\ell},x_{\ell+1},\dots,x_{2\ell})|x^2_{\ell+1}+\ldots+x^2_{2\ell} = 1\}\subset S^{2\ell-1}$, and the second component (either L^+_{YX} or L^-_{YX}) can be described as $\{(x_1,\dots,x_\ell,\underbrace{0,\dots,0}_{\ell})|x_1^2+\ldots+x_\ell^2=1\}\subset S^{2\ell-1}$.

By the third definition of the linking number given in Section 3.5.1, $lk(L^+_{YZ},L^+_{YX})$ is the algebraic number of intersections of L^+_{YZ} with $D(L^+_{YX})$, where $D(L^+_{YX}) = \{(x_1,\dots,x_\ell,x_{\ell+1},\underbrace{0,\dots,0}_{\ell-1})|x_{\ell+1}\geq 0, x_1^2+\ldots+x_\ell^2\leq 1\}$ is the disk bounded by L^+_{YX}. The intersection $L^+_{YZ} \cap D(L^+_{YX})$ happens exactly at one point $(\underbrace{0,\ldots,0}_{\ell},1,\underbrace{0,\ldots,0}_{\ell-1})$ as shown below.

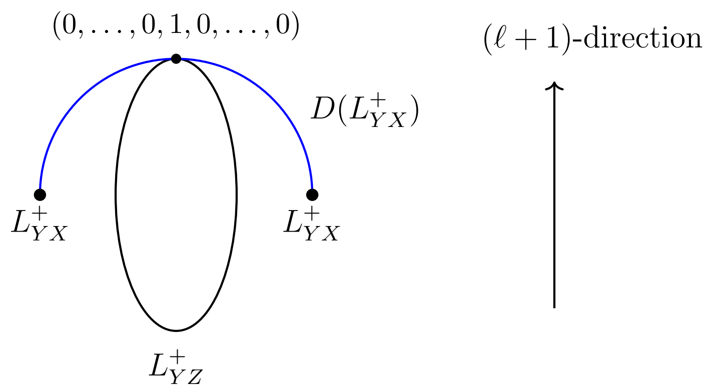


Figure 3.27: $D(L^+_{YX})$ intersected with L^+_{YZ}

Since L^+_{YX} is oriented as the boundary of $D(L^+_{YX})$ with the outward normal first convention, and we have shown that the orientation on L^+_{YX} is given by $(-1)^\ell Or(S^{\ell-1})$, therefore the orientation on $D(L^+_{YX})$ must be $((-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\ell})$. At the intersection point, we get orientation

$$\Big(\underbrace{\frac{\partial}{\partial x_{\ell+2}},\ldots,\frac{\partial}{\partial x_{2\ell}}}_{Or(L^+_{YZ})},\underbrace{(-1)^\ell\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_\ell}}_{Or(D(L^+_{YX}))}\Big),$$

which is equivalent to $\Big((-1)^\ell \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_{\ell+2}}, \ldots, \frac{\partial}{\partial x_{2\ell}}\Big)$, the orientation on $Y \simeq S^{2\ell-1}$ at the intersection point. Thus, $lk(L^+_{YZ}, L^+_{YX}) = +1$.

Similarly, we compute $lk(L^-_{YZ}, L^+_{YX})$ by taking the intersection $L^-_{YZ} \cap D(L^+_{YX})$, which again happens at the point $(\underbrace{0,\ldots,0}_{\ell},1,\underbrace{0,\ldots,0}_{\ell-1})$. At the intersection, we get

$$\Big(\underbrace{(-1)^{\ell-1}\frac{\partial}{\partial x_{\ell+2}},\ldots,\frac{\partial}{\partial x_{2\ell}}}_{Or(L^-_{YZ})},\underbrace{(-1)^\ell\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_\ell}}_{Or(D(L^+_{YX}))}\Big),$$

which is $(-1)^{\ell-1}$ times the orientation of Y at the intersection point. Thus, $lk(L^-_{YZ}, L^+_{YX}) = (-1)^{\ell-1}$.

Note that the disk $D(L^-_{YX})$ can also be described as $\{(x_1,\ldots,x_\ell,x_{\ell+1},\underbrace{0,\ldots,0}_{\ell-1})|x_{\ell+1} \geq 0, x_1^2+\ldots+x_\ell^2 \leq 1\}$ bounded by L^-_{YX} ($D(L^-_{YX})$ is parallel to $D(L^+_{YX})$). The orientation on

$D(L^-_{YX})$ is given by $\left(-\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\ell}\right)$ in accordance with the orientation of L^-_{YX}, which is $-Or(S^{\ell-1})$. At the intersection point $L^+_{YZ} \cap D(L^-_{YX}) = (\underbrace{0,\ldots,0}_{\ell}, 1, \underbrace{0,\ldots,0}_{\ell-1})$, we have

$$\Big(\underbrace{\frac{\partial}{\partial x_{\ell+2}}, \ldots, \frac{\partial}{\partial x_{2\ell}}}_{Or(L^+_{YZ})}, \underbrace{-\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\ell}}_{Or(D(L^-_{YX}))}\Big),$$

which gives $(-1)^{\ell-1}$ times the orientation of Y. Thus, $lk(L^+_{YZ}, L^-_{YX}) = (-1)^{\ell-1}$. Similarly, to compute $lk(L^-_{YZ}, L^-_{YX})$, the orientation at the intersection point is

$$\Big(\underbrace{(-1)^{\ell-1}\frac{\partial}{\partial x_{\ell+2}}, \ldots, \frac{\partial}{\partial x_{2\ell}}}_{Or(L^-_{YZ})}, \underbrace{-\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\ell}}_{Or(D(L^-_{YX}))}\Big),$$

which matches with the orientation of Y. Thus, $lk(L^-_{YZ}, L^-_{YX}) = +1$.

Hence the invariant $\mathcal{V}$ (given in 3.5.3) evaluated on $G_*([\alpha_{12}, \alpha_{23}])$ is

$$\begin{aligned}
\mathcal{V}(G_*([\alpha_{12}, \alpha_{23}])) &= \frac{1}{6}[lk(L^+_{YZ}, L^+_{YX}) + lk(L^-_{YZ}, L^-_{YX}) + (-1)^{\ell-1}2lk(L^+_{YZ}, L^-_{YX}) + (-1)^{\ell-1}2lk(L^-_{YZ}, L^+_{YX})] \\
&= \frac{1}{6}[1 + 1 + (-1)^{\ell-1}\cdot 2(-1)^{\ell-1} + (-1)^{\ell-1}\cdot 2(-1)^{\ell-1}] \\
&= \frac{1}{6}\cdot 6 \\
&= 1
\end{aligned}$$

Therefore, using the invariant $\mathcal{T}$, one can conclude that under the graphing map, $[\alpha_{12}, \alpha_{23}]$ is mapped to $-\mathcal{B}$, the negative of the Borromean link.

□

3.7.2 2-component links

Theorem 3.7.3. *The graphing* $G_* : \pi_{4k-1}S^{2k} = \pi_{4k-1}C(2, \mathbb{R}^{2k+1}) \to \pi_0 Emb_\partial(\bigsqcup_2 \mathbb{R}^{4k-1}, \mathbb{R}^{6k})$ *takes the (rational) generator* $[\alpha_{12}, \alpha_{12}]$ *of* $\pi_{4k-1}S^{2k}$ *to* $-\mathcal{B}_1 - \mathcal{B}_2$*, up to a torsion element.*

Proof. Let $\alpha_{12} \in \pi_{2k}C(2, \mathbb{R}^{2k+1})$ correspond to the braid where strand 1 goes around strand 2. Abusing notation, we also let α_{12} denote the corresponding map $\alpha_{12} : S^{2k} \to C(2, \mathbb{R}^{2k+1})$.

Then the element $[\alpha_{12}, \alpha_{12}]$ represented by $f = (f_1, f_2) : D^{4k-1} \to C(2, \mathbb{R}^{2k+1})$ is given by the composition

$$D^{4k-1} \to S^{4k-1} \simeq \partial(D^{2k} \times D^{2k}) \xrightarrow{\phi} S^{2k} \vee S^{2k} \xrightarrow{\alpha_{12} \vee \alpha_{12}} C(2, \mathbb{R}^{2k+1}) \simeq S^{2k},$$

where the first map collapses the boundary ∂D^{4k-1} to the base point in S^{4k-1}, and ϕ is the attaching map of the top $4k$-cell in the product $S^{2k} \times S^{2k}$. We may assume that the map f_2 is constant in $\mathbb{R}^{2k+1}$ given by $f_2(x) = (\underbrace{0 \ldots, 0}_{2k+1})$ for all $x \in D^{4k-1}$.

As done in the previous theorem, we assume that the corresponding embedding is represented by $F = (F_1, F_2) : \sqcup_2 \partial(D^{2k} \times D^{2k}) \hookrightarrow \partial(D^{2k} \times D^{2k}) \times \mathbb{R}^{2k+1}$, where $F_2(x) = (x, f_2(x)) = (x, 0, \ldots, 0)$ for all $x \in \partial(D^{2k} \times D^{2k})$. The map F_1 comprises of two maps on $\partial(D^{2k} \times D^{2k}) \simeq (S^{2k-1} \times D^{2k}) \cup (D^{2k} \times S^{2k-1})$ given by

$$id_{S^{2k-1} \times D^{2k}} \times (h \circ p_2) : S^{2k-1} \times D^{2k} \longrightarrow S^{2k-1} \times D^{2k} \times \mathbb{R}^{2k+1},$$

$$id_{D^{2k} \times S^{2k-1}} \times (h \circ p_1) : D^{2k} \times S^{2k-1} \longrightarrow D^{2k} \times S^{2k-1} \times \mathbb{R}^{2k+1},$$

where maps p_1 and p_2 are projections to the first and second factor of $S^{2k-1} \times D^{2k}$ and $D^{2k} \times S^{2k-1}$, respectively. The map $h : D^{2k} \to \mathbb{R}^{2k+1}$ sends the respective disk D^{2k} to $\mathbb{R}^{2k+1}$ by collapsing its boundary to the point $(1, \underbrace{0, \ldots, 0}_{2k})$ such that the image is a unit sphere around the origin in $\mathbb{R}^{2k+1}$. Note that the map h is given by the composition $D^{2k} \to D^{2k}/\partial D^{2k} \xrightarrow{\cong} S^{2k}$ of degree one, thus it respects the orientation.

Let the two components of F be X and Y. Consider the projection

$$p : \partial(D^{2k} \times D^{2k}) \times \mathbb{R}^{2k+1} \to \partial(D^{2k} \times D^{2k}) \times \mathbb{R}^{2k}$$

of F along the last $(2k+1)$-coordinate. Note that the image of the projection $p(F_2(Y)) = \partial(D^{2k} \times D^{2k}) \times (\underbrace{0, \ldots, 0}_{2k})$. Moreover, the projection $p(F_1(X))$ restricted to $S^{2k-1} \times D^{2k}$ must satisfy the following:

$$S^{2k-1}\times D^{2k} \xrightarrow{F_1 = id_{S^{2k-1}\times D^{2k}}\times(h\circ p_2)} S^{2k-1}\times D^{2k}\times\mathbb{R}^{2k+1} \xrightarrow{\quad p \quad} S^{2k-1}\times D^{2k}\times\mathbb{R}^{2k}$$

$$S^{2k-1}\times A_\pm \longmapsto S^{2k-1}\times A_\pm\times(\underbrace{0,\dots,0}_{2k},\pm1) \longmapsto S^{2k-1}\times A_\pm\times(\underbrace{0,\dots,0}_{2k}),$$

for points A_+ and A_- lying in the upper and lower halves of the disk in $S^{2k-1}\times D^{2k}$, respectively. The underlying map to $\mathbb{R}^{2k}$ is given by the projection $S^{2k-1}\times D^{2k} \xrightarrow{p_2} D^{2k}$ followed by the map $p_{|_{\mathbb{R}^{2k+1}}} \circ h : D^{2k} \to \mathbb{R}^{2k}$. The latter takes ∂D^{2k} to the point $(1,\underbrace{0,\dots,0}_{2k-1})$, the plane H which cuts D^{2k} into two halves is sent to $\partial D^{2k} \subset \mathbb{R}^{2k}$, and the points A_+ and A_- are sent to the origin in $\mathbb{R}^{2k}$. Moreover, the upper half of D^{2k} is sent respecting the orientation and the lower half is sent reversing the orientation. Similarly, $p(F_1(X))$ restricted to $D^{2k}\times S^{2k-1}$ must satisfy

$$D^{2k}\times S^{2k-1} \xrightarrow{F_1 = id_{D^{2k}\times S^{2k-1}}\times(h\circ p_1)} D^{2k}\times S^{2k-1}\times\mathbb{R}^{2k+1} \xrightarrow{\quad p \quad} D^{2k}\times S^{2k-1}\times\mathbb{R}^{2k}$$

$$B_\pm\times S^{2k-1} \longmapsto B_\pm\times S^{2k-1}\times(\underbrace{0,\dots,0}_{2k},\pm1) \longmapsto B_\pm\times S^{2k-1}\times(\underbrace{0,\dots,0}_{2k}),$$

for B_+ and B_- lying in the upper and lower halves of the disk in $D^{2k}\times S^{2k-1}$, respectively. Thus, the intersection in $\partial(D^{2k}\times D^{2k})\times\mathbb{R}^{2k}$ is given by four spheres:

$$p(F_1(X)) \cap p(F_2(Y)) = \big((S^{2k-1}\times A_\pm) \sqcup (B_\pm\times S^{2k-1})\big)\times(\underbrace{0,\dots,0}_{2k}), \tag{3.7.5}$$

with the following preimages in X and Y, each forming a double of a Hopf link:

In $X \simeq \partial(D^{2k}\times D^{2k})$,

$$L^+_{XY} = (L^+_{XY})_A \sqcup (L^+_{XY})_B := (S^{2k-1}\times A_+) \sqcup (B_+\times S^{2k-1}),$$

$$L^-_{XY} = (L^-_{XY})_A \sqcup (L^-_{XY})_B := (S^{2k-1}\times A_-) \sqcup (B_-\times S^{2k-1});$$

In $Y \simeq \partial(D^{2k} \times D^{2k})$,

$$L_{YX}^{-} = (L_{YX}^{-})_A \sqcup (L_{YX}^{-})_B := (S^{2k-1} \times A_+) \sqcup (B_+ \times S^{2k-1}),$$

$$L_{YX}^{+} = (L_{YX}^{+})_A \sqcup (L_{YX}^{+})_B := (S^{2k-1} \times A_-) \sqcup (B_- \times S^{2k-1}).$$

Note that $(L_{XY}^{+})_A$ and $(L_{XY}^{-})_A$ (respectively $(L_{XY}^{+})_B$ and $(L_{XY}^{-})_B$) are parallel spheres in X. By the same argument given in the previous theorem, we conclude that these parallel spheres are oppositely oriented (since ℓ is even) i.e., the orientation on $(L_{XY}^{+})_A$ and $(L_{XY}^{-})_A$ (also on $(L_{XY}^{+})_B$ and $(L_{XY}^{-})_B$) is given by $Or(S^{2k-1})$ and $-Or(S^{2k-1})$, respectively. Similarly, the corresponding preimages in Y are parallel spheres with opposite orientations. Thus, with each link component as a sphere S^{2k-1}, the set of double points of $p \circ F$ is depicted by the picture below:

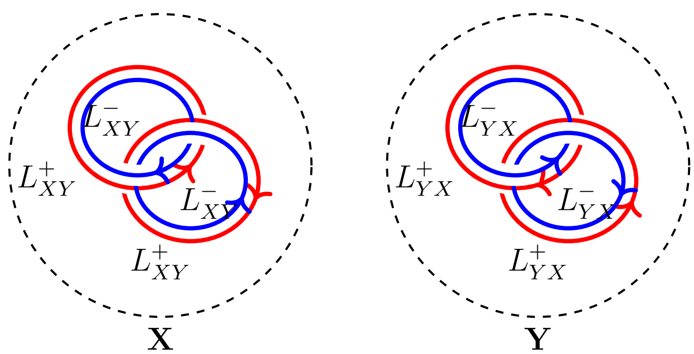


Figure 3.28: The set of double points of $p \circ G_*([\alpha_{12}, \alpha_{12}])$

The only linking number contributing to the two invariants $\mathcal{W}_1$ and $\mathcal{W}_2$ in Theorem 3.6.1 are

$$lk(L_{YX}^{+}, L_{YX}^{-}) = lk\big((L_{YX}^{+})_A, (L_{YX}^{-})_B\big) + lk\big((L_{YX}^{+})_B, (L_{YX}^{-})_A\big) \text{ and}$$

$$lk(L_{XY}^{+}, L_{XY}^{-}) = lk\big((L_{XY}^{+})_A, (L_{XY}^{-})_B\big) + lk\big((L_{XY}^{+})_B, (L_{XY}^{-})_A\big),$$

respectively, where each of the linking number correspond to the linking number of a standard Hopf link. By repeating the computations done for such linking numbers in the previous theorem, where we find the linking number as the algebraic intersection number of a sphere and a disk, we conclude that each of the four linking numbers above is -1.

Hence the invariant $\mathcal{W}_1$ on $G_*([\alpha_{12}, \alpha_{12}])$ is

$$\begin{aligned}
\mathcal{W}_1(G_*(\big[\alpha_{12}, \alpha_{12}])) &= \frac{1}{2} \underbrace{lk(L_{XX}, L_{XY})}_{0} - lk(L^+_{YX}, L^-_{YX}) \\
&= -\Big(lk\big((L^+_{YX})_A, (L^-_{YX})_B\big) + lk\big((L^+_{YX})_B, (L^-_{YX})_A\big)\Big) \\
&= -(-1-1) \\
&= 2
\end{aligned}$$

Similarly, we get $\mathcal{W}_2(G_*(\big[\alpha_{12}, \alpha_{12}])) = 2$. Thus, $\mathcal{W}([\alpha_{12}, \alpha_{12}]) = (2,2)$, so under the graphing map, $[\alpha_{12}, \alpha_{12}]$ is mapped to $-\mathcal{B}_1 - \mathcal{B}_2$.

$\square$

Remark 3.7.1. For the cases when $k = 1, 2, 4$, the generator of the $\mathbb{Z}$ summand of $\pi_{4k-1}S^{2k}$ is given by the Hopf fibration $Hopf : S^{4k-1} \to S^{2k}$, which under $G_* : \pi_{4k-1}S^{2k} \to \pi_0 Emb_\partial(\sqcup_2 \mathbb{R}^{4k-1}, \mathbb{R}^{6k})$ gets mapped to $\frac{1}{2}(-\mathcal{B}_1 - \mathcal{B}_2)$, up to a torsion element. This follows from the fact that the class $[Hopf] \in \pi_{4k-1}S^{2k}$ is one half of $[\alpha_{12}, \alpha_{12}]$. We can also see it through our invariant $\mathcal{W}$. Indeed, the only difference is in the set of double points, which now is given by one Hopf link in each X and Y:

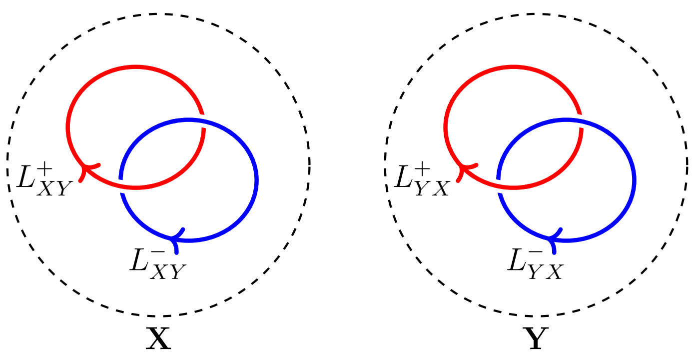


Figure 3.29: The set of double points of $p \circ G_*([Hopf])$

And it easy to compute that $lk(L^+_{XY}, L^-_{XY}) = -1 = lk(L^+_{YX}, L^-_{YX})$, thus $\mathcal{W}(G_*([Hopf])) = (1,1) = \frac{1}{2}\mathcal{W}(G_*([\alpha_{12}, \alpha_{12}]))$.

3.7.3 End of the proof of Theorem 3.6.1

We have the following diagram:

$$\begin{tikzcd}
\pi_{4k-1}S^{2k} \arrow[r, "G_*"] & \pi_0^{br}Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \arrow[r, "\mathcal{W}"] \arrow[d, "T"'] & \mathbb{Q}^2 \\
& \mathbb{Z}^2 \arrow[ur, "A"'] &
\end{tikzcd}$$

In Theorem 3.7.3 we have shown that under G_*, the generator $[\alpha_{12}, \alpha_{12}]$ gets mapped to $-\mathcal{B}_1 - \mathcal{B}_2$, up to a torsion, where the latter gets mapped to $(2, 2)$ under $\mathcal{W}$. Moreover, note that T maps $-\mathcal{B}_1-\mathcal{B}_2$ to $-(1,-1)-(1,1) = (-2,0)$. Recall that the image of T is either $\mathbb{Z}^2$ or L depending on whether the element $(-1,0)$ belongs to the image of T or not. Let $k \neq 1,2,4$, suppose that $(-1,0)$ is in image of T, then there exists an element $J \in \pi_0^{br}Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})$ such that $T(J) = (-1,0)$ (since T is a homomorphism we get $J = \frac{1}{2}(-\mathcal{B}_1 - \mathcal{B}_2)$). By Theorem 3.7.1, there exists a retraction $r : \pi_0^{br}Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k}) \to \pi_{4k-1}S^{2k}$, which then maps J to $\frac{1}{2}[\alpha_{12}, \alpha_{12}]$. This contradicts the fact that $[\alpha_{12}, \alpha_{12}]$ is the generator of $\pi_{4k-1}S^{2k}$. Thus, for $k \neq 1,2,4$, we get $T(\pi_0^{br}Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = L$. Following Remark 3.7.1, it is clear that for $k = 1,2,4$, we have $T(\pi_0^{br}Emb(\sqcup_2 S^{4k-1}, \mathbb{R}^{6k})) = \mathbb{Z}^2$. Hence, we get the desired image of W and this concludes the proof of Theorem 3.6.1.

Chapter 4

Conjectural formula for the Haefliger invariant

Haefliger [20, 21] has shown that $(4k-1)$-sphere can be knotted in $6k$-space for $k \geq 1$. Moreover, the isotopy classes of smooth embeddings of S^{4k-1} in $\mathbb{R}^{6k}$ is in one-to-one correspondence with the integers, given by the Haefliger invariant (see [20] and [21, §5.16 and Corollary 8.14])

$$\mathcal{H} : \pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) \xrightarrow{\approx} \mathbb{Z}.$$

He also gave an explicit embedding representing a generator of $\pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k})$ known as the Haefliger trefoil knot [20, §4], which is constructed by joining the three components of the Borromean link using two tubes. From [21, Theorem 5.17 and Corollary 6.7], it is known that some non-zero multiple (twice in the case when $k = 1$) of the Haefliger trefoil can be compressed to one dimension less, where the compressed embedding is non-trivial as an immersion. This tells us that the desired combinatorial formula for the Haefliger invariant cannot be solely expressed through the set of double points, and we have to take the tangential data into consideration.

In this chapter, we give two (slightly different but equivalent) conjectural formulas for the Haefliger invariant $\mathcal{H}$. In Sections 4.1 and 4.3, we explain in detail these conjectural formulas for the case $k = 1$ i.e., embeddings $S^3 \hookrightarrow \mathbb{R}^6$, when their projection to one dimension less is an immersion $S^3 \looparrowright \mathbb{R}^5$. This case of immersion is well understood by Ekholm [13] using some interesting invariants that we discuss in the next section. In the last section, we give a generalization of our conjectural formulas to higher dimensional embeddings $S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ ($k > 1$), when their projection to $\mathbb{R}^{6k-1}$ is an immersion.

4.1 First Formula $\mathcal{H}_1$

According to Smale [41], any immersion $f : S^3 \looparrowright \mathbb{R}^5$ is determined up to regular homotopy by its Smale invariant $\Omega(f) \in \pi_3(V_{5,3}) \approx \mathbb{Z}$, where $V_{5,3}$ is the Stiefel manifold of orthonormal 3-frames in $\mathbb{R}^5$. Note that $V_{5,3}$ is a fiber bundle over S^4 with fiber $V_{4,2} \simeq S^3 \times S^2$. The generator of $\pi_3(V_{5,3}) = \mathbb{Z}$ is the element $[\sigma]$ which is represented by the identity map $S^3 \to S^3$ mapped to the first factor of the fiber. The Hopf fibration $\rho : S^3 \to S^2$ over the second factor of the fiber is related to $[\sigma]$ as $[\rho] = -2[\sigma]$.[1] Here, we consider the Hopf map $\rho : S^3 \to \mathbb{C}P^1 = S^2$ given by $(z_0, z_1) \mapsto [z_0 : z_1]$, where the orientation on S^2 is induced by the complex structure (note that the Hopf invariant of ρ is 1).

Ekholm [13, §6.2] has introduced a "linking invariant" $E_1(f)$ (our notation is different than Ekholm) associated to generic immersions $f : S^3 \looparrowright \mathbb{R}^5$. It changes by ± 3 under triple point moves and remains unchanged under all the other generic moves, and thus when taken modulo 3 it is an invariant of regular homotopy. Let $L \subset S^3$ be the set of double points of f, a closed 1-manifold i.e., a link in S^3 with possibly several components and let $f(L)$ be its image in $\mathbb{R}^5$. Then $f|_L : L \to f(L)$ is a 2-fold covering. Note that there is an induced orientation on $f(L)$ and hence on L (see Remark 3.2.1 (1)). The invariant $E_1(f)$ is defined as the linking number of the image $f(S^3)$ with $\widetilde{f(L)}$, a copy of $f(L)$ shifted slightly along a suitably chosen normal (to $f(L)$) vector field. In particular, we first start by choosing a normal vector field v along $L \subset S^3$ such that $[\widetilde{L}] = 0 \in H_1(S^3 - L)$, where $\widetilde{L}$ denotes the

[1] Ekholm in [13, §3] got this relation as $[\rho'] = 2[\sigma]$, where he considered a slightly different Hopf fibration $\rho' : S^3_{\mathbb{H}} \to S^2_{Im(\mathbb{H})}$ defined by $\rho'(x) = x \cdot i \cdot x^{-1}$. Here, $S^3_{\mathbb{H}}$ is the set of unitary quaternions in $\mathbb{H} \cong \mathbb{R}^4$, which acts by conjugation on $Im(\mathbb{H}) \cong \mathbb{R}^3$, the set of imaginary or pure quaternions. Moreover, he has shown that Hopf invariant of ρ' is -1, see [13, § 9.3].

result of pushing L slightly along v. That is, for every path-component L_i of L we get

$$lk_{S^3}(L_i, \widetilde{L}) = lk_{S^3}(L_i, \widetilde{L}_i) + \sum_{j \neq i} lk_{S^3}(L_i, L_j) = 0. \tag{4.1.1}$$

Since $lk(\widetilde{L}_i, L_j) = lk(L_i, \widetilde{L_j})$, the above equation can be rewritten as

$$lk_{S^3}(\widetilde{L}_i, L) = lk_{S^3}(\widetilde{L}_i, L_i) + \sum_{j \neq i} lk_{S^3}(L_i, L_j) = 0. \tag{4.1.2}$$

Define a vector field w along $f(L)$ as follows: for $p \in f(L)$ such that $p = f(p_1) = f(p_2)$, let

$$w(p) = df(v(p_1)) + df(v(p_2)).$$

Let $\widetilde{f(L)} \subset \mathbb{R}^5$ be the result of pushing $f(L)$ slightly along w, then $\widetilde{f(L)} \cap f(S^3) = \emptyset$. Define

$$E_1(f) := lk_{\mathbb{R}^5}(\widetilde{f(L)}, f(S^3))$$

(or equivalently, $E_1(f) = [\widetilde{f(L)}] \in H_1(\mathbb{R}^5 - f(S^3)) = \mathbb{Z}$). Let us call these choices of normal vector fields for L and $f(L)$ as framing fr_1.

Remark 4.1.1. If f in the definition of $E_1(f)$ is obtained as the projection of an embedding $S^3 \hookrightarrow \mathbb{R}^6$, then we can assign an overcrossing and undercrossing part to L i.e., $L = L^+ \sqcup L^-$ and similarly $\widetilde{L} = \widetilde{L}^+ \sqcup \widetilde{L}^-$, where both overcrossing and undercrossing parts are endowed with the Ekholm orientation (see Remark 3.2.1 (1)). Then for each component L_i^+ and L_i^-, the condition in (4.1.1) gives the following:

$$\Longrightarrow lk_{S^3}(L^+, \widetilde{L}^+) = -lk_{S^3}(L^+, L^-), \tag{4.1.3}$$

and similarly

$$lk_{S^3}(L^-, \widetilde{L}^-) = -lk_{S^3}(L^+, L^-). \tag{4.1.4}$$

Now we give the first conjectural formula. Let $p : \mathbb{R}^6 \to \mathbb{R}^5$ be the projection map.

Conjecture 4.1.1. *Given a knot $F : S^3 \hookrightarrow \mathbb{R}^6$ such that its projection $f = p \circ F : S^3 \looparrowright \mathbb{R}^5$ is a generic immersion, the first conjectural formula is given by*

$$\mathcal{H}_1(F) := \frac{1}{2} lk_{S^3}(L^+, L^-) + \frac{1}{6} E_1(f) - \frac{1}{12} \Omega(f), \tag{4.1.5}$$

which produces the Haefliger isomorphism $\mathcal{H}_1 : \ \pi_0 Emb(S^3, \mathbb{R}^6) \xrightarrow{\approx} \mathbb{Z}$.

Note that the first term of the formula depends on the embedding, while the other two are defined solely in terms of the immersion f. This formula as compared to the second conjectural formula $\mathcal{H}_2(F)$ defined in Section 4.3 is easier to do computations. One limitation of this formula is that it does not easily generalize to the case where the projection f is a non-immersion and includes Whitney Umbrellas. To address this, we can slightly modify the formula to align its convention with $\mathcal{H}_2(F)$, which has more chances in the potential generalization to non-immersions with Whitney Umbrella points. We prove later in Theorem 4.3.1 that the adjusted version of the first conjectural formula $\mathcal{H}_1(F)$ is equivalent to the second conjectural formula $\mathcal{H}_2(F)$. The following is the adjusted version:

$$\mathcal{H}_1(F) = -\frac{1}{8} lk_{S^3}(L, \widetilde{L}) + \frac{1}{6} E_1(f) - \frac{1}{12} \Omega(f). \tag{4.1.6}$$

For (4.1.6), abusing notation, we redefine $L := L^+ \sqcup -L^-$, and similarly $\widetilde{L} := \widetilde{L}^+ \sqcup -\widetilde{L}^-$, using the adjusted Ekholm orientation, according to Remark 3.2.1(2). Recall that $\widetilde{L}$ is the shifted copy of L using fr_1 (see the definition of $E_1(f)$ for notation).

The two versions of $\mathcal{H}_1(F)$ produce the same value because

$$\begin{aligned} lk_{S^3}(L, \widetilde{L}) &= lk(L^+ \sqcup -L^-, \widetilde{L}^+ \sqcup -\widetilde{L}^-) \\ &= lk(L^+, \widetilde{L}^+) - \underbrace{lk(L^+, \widetilde{L}^-)}_{lk(L^+, L^-)} - \underbrace{lk(L^-, \widetilde{L}^+)}_{lk(L^+, L^-)} + lk(L^-, \widetilde{L}^-) \\ &= -4 lk(L^+, L^-), \end{aligned}$$

by using equations (4.1.3) and (4.1.4), and the fact that the linking number is symmetric.

4.2 Justification for $\mathcal{H}_1$

In this section, we justify our formula for $\mathcal{H}_1$ given in (4.1.5) through various arguments. Firstly, in Subsection 4.2.1, we provide several computational examples that validate our choice of coefficients in each term of $\mathcal{H}_1(f)$. Then, in Subsection 4.2.2, we show that our formula is invariant under all elementary cobordisms (as outlined in Section 3.3.1), without Whitney umbrella points. Furthermore, in Subsection 4.2.3, we establish that $\mathcal{H}_1(F)$ aligns with Sakai's Lin-Wang type formula for the Haefliger invariant, which measures the jump in the Haefliger invariant when overcrossing changes to undercrossing and vice-versa. Lastly, in Subsection 4.2.4, we discuss the compatibility of our formula with the concept of cabling, which lends support to the coefficient in the last term of our formula.

4.2.1 Some computations

All the examples that we consider (except for the last one) include embeddings with the projection trivial as immersion, resulting in a Smale invariant $\Omega(f) = 0$. In this subsection, Examples 1, 2, and 3 use our computations we did in Theorem 3.5.2, Theorem 3.7.2, and Theorem 3.7.3, respectively. Example 4 is due to Ekholm which corresponds to trivial knots $S^3 \hookrightarrow \mathbb{R}^6$ that project to non-trivial immersions of arbitrary Smale invariant.

Before we explore our examples, we discuss following lemmas that will be useful for the computations of $E_1(f)$.

Lemma 4.2.1. *Suppose $f : X \sqcup Y \looparrowright \mathbb{R}^5$ for each $X, Y = S^3$ is a generic immersion such that the restrictions $f_1 := f|_X$ and $f_2 := f|_Y$ are embeddings of S^3 in $\mathbb{R}^5$. Let $f_\# : S^3 \looparrowright \mathbb{R}^5$ denote the immersion obtained by tubing X and Y. Then $E_1(f_\#) = 0$.*

Proof. Let $A = A_1 \sqcup \ldots \sqcup A_m$ be the self-intersection of $f_\#(S^3)$ in $\mathbb{R}^5$ (which is the same as $f_1(X) \cap f_2(Y)$ since we choose the tube to be in general position) with preimages $L_{XY} = \bigsqcup_{i=1}^{m} L_{XY,i}$ and $L_{YX} = \bigsqcup_{i=1}^{m} L_{YX,i}$ in X and Y, respectively. Here, each A_i and its preimages $L_{XY,i}$ and $L_{YX,i}$ are circles. Let $\widetilde{A_i}$ be the shift of A_i along the sum of the shifts corresponding to $\widetilde{L_{XY,i}}$ and $\widetilde{L_{YX,i}}$, using the framing fr_1. We define $\widetilde{A_{X,i}}$ to be the shift of A_i only along the framing inside X (that is, it is the image of $\widetilde{L_{XY,i}}$). Similarly, we define $\widetilde{A_{Y,i}}$ to be the shift of A_i only along the framing inside Y (it is the image of $\widetilde{L_{YX,i}}$). Note that if $D(\widetilde{L_{XY,i}})$ is a 2-chain in X with boundary $\widetilde{L_{XY,i}}$, then we can choose $D(\widetilde{L_{XY,i}})$ to be in general position with respect to $f_1^{-1}(f_2(Y)) = L_{XY}$. This implies that under f_1, the images $D(A_{X,i}) := f_1(D(\widetilde{L_{XY,i}}))$ and $f_2(Y)$ are in general position in $\mathbb{R}^5$, where $D(A_{X,i})$ is a 2-chain bounded by $A_{X,i}$. Thus, we get

$$lk_X(\widetilde{L_{XY,i}}, L_{XY}) = lk_{\mathbb{R}^5}(\widetilde{A_{X,i}}, f_2(Y)).$$

Note that each intersection point of $D(\widetilde{L_{XY,i}})$ with L_{XY} in X is sent by f_1 to exactly one point of intersection of $D(\widetilde{A_{X,i}})$ and $f_2(Y)$ in $\mathbb{R}^5$. In both cases, the sign of the intersection point is positive or negative depending on whether the orientation of $D(\widetilde{L_{XY,i}})$ coincides or not with the coorientation of L_{XY} in X.

Similarly, one can argue that $lk_Y(\widetilde{L_{YX,i}}, f_2^{-1}(f_1(X)) = L_{YX}) = lk_{\mathbb{R}^5}(\widetilde{A_{Y,i}}, f_1(X))$. Then

$$\begin{aligned} E_1(f_\#) = \quad & lk_{\mathbb{R}^5}(\widetilde{A}, f_\#(S^3)) \\ = \quad & \sum_{i=1}^{m} lk_{\mathbb{R}^5}(\widetilde{A}_{X,i}, f_2(Y)) + \sum_{i=1}^{m} lk_{\mathbb{R}^5}(\widetilde{A}_{Y,i}, f_1(X)) \\ = \quad & \sum_{i=1}^{m} lk_X\left(\widetilde{L_{XY,i}}, L_{XY}\right) + \sum_{i=1}^{m} lk_Y\left(\widetilde{L_{YX,i}}, L_{YX}\right) \\ = \quad & 0 + 0 = 0. \qquad \text{(using (4.1.2))} \end{aligned}$$

$\square$

Lemma 4.2.2. *Suppose $f : X \sqcup Y \sqcup Z \looparrowright \mathbb{R}^5$ for each $X, Y, Z = S^3$ is a generic immersion such that the restrictions $f_1 := f|_X$, $f_2 := f|_Y$, and $f_3 := f|_Z$ are embeddings of S^3 in $\mathbb{R}^5$. Let $f_\# : S^3 \looparrowright \mathbb{R}^5$ denote the immersion obtained by tubing X, Y, Z via two tubes (in general position). Then, $E_1(f_\#) = -3 \cdot lk_X(L_{XY}, L_{XZ}) = -3 \cdot lk_Y(L_{YX}, L_{YZ}) = -3 \cdot lk_Z(L_{ZX}, L_{ZY})$.*

Note that Lemma 4.2.2 follows from Lemma 4.2.1 by adding an inclusion $Z \subset \mathbb{R}^4 \subset \mathbb{R}^5$ disjointly from $f(X \sqcup Y)$.

Proof. Note that the self-intersection of $f_{\#}$ in $\mathbb{R}^5$ is comprised of three parts: $f_1(X) \cap f_2(Y)$, $f_1(X) \cap f_3(Z)$ and $f_2(Y) \cap f_3(Z)$. Without loss of generality, let $A = A_1 \sqcup \ldots \sqcup A_m$ be the intersection of $f_1(X)$ and $f_2(Y)$, with preimages $L_{XY} = \overset{m}{\underset{i=1}{\sqcup}} L_{XY,i}$ and $L_{YX} = \overset{m}{\underset{i=1}{\sqcup}} L_{YX,i}$ in X and Y, respectively, where each $A_i, L_{XY,i}, L_{YX,i}$ is a circle. For each A_i, let $\widetilde{A_i}$ be the shift of A_i along the sum of the shifts corresponding to $\widetilde{L_{XY,i}}$ and $\widetilde{L_{YX,i}}$, using the framing fr_1. Moreover, let $\widetilde{A_{X,i}}$ (respectively, $\widetilde{A_{Y,i}}$) be the shift of A_i only along the framing inside X (respectively, Y). By the same argument given in the previous lemma, one can show that $lk_{\mathbb{R}^5}(\widetilde{A_i}, f_2(Y)) = lk_{\mathbb{R}^5}(\widetilde{A_{X,i}}, f_2(Y)) = lk_X(\widetilde{L_{XY,i}}, L_{XY})$ and $lk_{\mathbb{R}^5}(\widetilde{A_i}, f_1(X)) = lk_{\mathbb{R}^5}(\widetilde{A_{Y,i}}, f_1(X)) = lk_Y(\widetilde{L_{YX,i}}, L_{YX})$. Moreover, we get $lk_X(L_{XY,i}, L_{XZ}) = lk_Y(L_{YX,i}, L_{YZ})$ since both can be interpreted as $lk_{\mathbb{R}^5}(A_i, f_3(Z))$. Then

$$\begin{aligned}
lk_{\mathbb{R}^5}(\widetilde{A}, f_{\#}(S^3)) &= \sum_{i=1}^{m} lk_{\mathbb{R}^5}(\widetilde{A_i}, f_{\#}(S^3)) \\
&= \sum_{i=1}^{m} \Big(lk_{\mathbb{R}^5}(\widetilde{A_{X,i}}, f_2(Y)) + lk_{\mathbb{R}^5}(\widetilde{A_{Y,i}}, f_1(X)) + lk_{\mathbb{R}^5}(A_i, f_3(Z)) \Big) \\
&= \sum_{i=1}^{m} \Big(lk_X(\widetilde{L_{XY,i}}, L_{XY}) + lk_Y(\widetilde{L_{YX,i}}, L_{YX}) + lk_X(L_{XY,i}, L_{XZ}) \Big) \\
&\qquad \Big(\text{note that the last term can also be written as } lk_Y(L_{YX,i}, L_{YZ}) \Big) \\
&= \underbrace{\sum_{i=1}^{m} \Big(lk_X(\widetilde{L_{XY,i}}, L_{XY}) + lk_X(L_{XY,i}, L_{XZ}) \Big)}_{=0 \text{ (using (4.1.2))}} + \sum_{i=1}^{m} lk_Y(\widetilde{L_{YX,i}}, L_{YX}) \\
&= \sum_{i=1}^{m} - lk_Y(L_{YX,i}, L_{YZ}) \qquad \text{(using (4.1.2) one more time)} \\
&= - lk_Y(L_{YX}, L_{YZ}) \qquad \Big(\text{or } - lk_X(L_{XY}, L_{XZ}) \Big)
\end{aligned}$$

Thus, for A as an intersection of $f_1(X)$ and $f_2(Y)$, we get $lk_{\mathbb{R}^5}(\widetilde{A}, f_{\#}(S^3)) = -lk_Y(L_{YX}, L_{YZ}) = -lk_X(L_{XY}, L_{XZ})$. We get similar results by repeating the argument for the other two intersections. Thus, $E_1(f_{\#}) = -3lk_X(L_{XY}, L_{XZ}) = -3lk_Y(L_{YX}, L_{YZ}) = -3lk_Z(L_{ZX}, L_{ZY})$.

$\square$

Now we discuss the examples.

Example 1: We claim that $\mathcal{H}_1(\mathcal{T}) = 1$, where $\mathcal{T}$ denotes the Haefliger trefoil knot, the generator of $\pi_0 Emb(S^3, \mathbb{R}^6)$. By definition, the Haefliger trefoil is constructed by tubing the three components of the Borromean link $\mathcal{B} = X \sqcup Y \sqcup Z \subset \mathbb{R}^6$, as shown in the figure below.

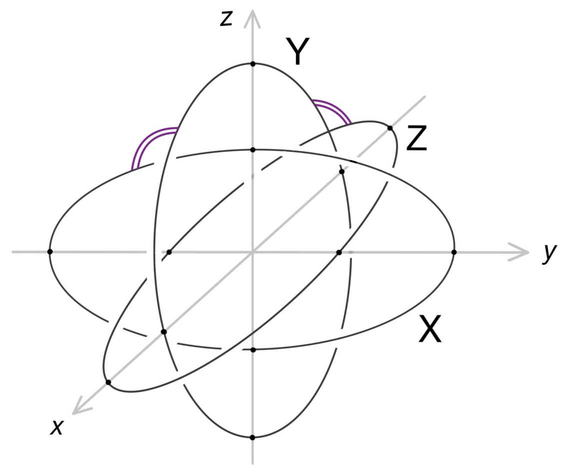


Figure 4.1: Haefliger trefoil knot $\mathcal{T}$, the generator of $\pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) \simeq \mathbb{Z}$.

Recall that for computations we used the second description of the Borromean link as given in the Subsection 3.5.3 (now we have $\ell = 2$). Here, $\mathcal{T}$ is defined as the connected sum $\mathcal{T} := X \# Y \# Z \# F_0$, where $F_0 : \mathbb{R}^3 \subset \mathbb{R}^6$ is the standard inclusion. If we choose the two tubes in general position, then under the projection p (as given in Theorem 3.5.2), the self-intersection of $p \circ \mathcal{T}$ is the same as that of $p \circ \mathcal{B}$ i.e, consisting of six disjoint circles in $\mathbb{R}^5$. The set of double points L of $p \circ \mathcal{T}$ is also the same, given by six disjoint Hopf links as depicted in the picture below.

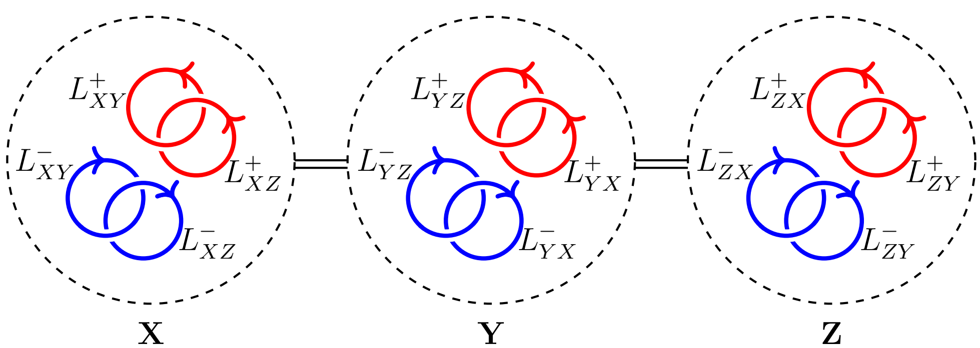


Figure 4.2: The set of double points of $p \circ \mathcal{T}$

By computations in the proof of Theorem 3.5.2, the linking number of each of these Hopf links is -1. Note that none of the Hopf links have a linking between an overcrossing component with an undercrossing component, thus $lk_{S^3}(L^+, L^-) = 0$. Moreover, $\Omega(p \circ \mathcal{T}) = 0$ since the projection of $\mathcal{T}$ to $\mathbb{R}^5$ is still trivial as an immersion (essentially one can pull apart the three spheres S^3 in $\mathbb{R}^5$ and rotate them so that they all are a parallel copy of the standard inclusion $S^3 \subset \mathbb{R}^4 \subset \mathbb{R}^5$). Thus, the only non-trivial term in our formula is $E_1(p \circ \mathcal{T})$. From Lemma 4.2.2, $E_1(p \circ \mathcal{T}) = -3 \cdot lk_X(L_{XY}, L_{XZ}) = -3 \cdot \big(lk_X(L^+_{XY}, L^+_{XZ}) + lk_X(L^-_{XY}, L^-_{XZ})\big) = -3 \cdot (-1 - 1) = 6$. Therefore,

$$\mathcal{H}_1(\mathcal{T}) = \frac{1}{2} \cdot 0 + \frac{1}{6} \cdot 6 - \frac{1}{12} \cdot 0 = 1.$$

Example 2: From Theorem 3.7.2, under the graphing map G_* the generator $[\alpha_{12}, \alpha_{23}] \in \pi_3^{br} C(3, \mathbb{R}^3)$ gets mapped to the (negative) generator $-\mathcal{B} \in \pi_0^{br} Emb_\partial(\bigsqcup_3 \mathbb{R}^3, \mathbb{R}^6)$. Since the tubing of the Borromean link $\mathcal{B}$ results in $\mathcal{T}$, the tubing of the embedding corresponding to $G_*([\alpha_{12}, \alpha_{23}])$ must result in a knot $\mathcal{F}$ isotopic to the inverse of $\mathcal{T}$. This is compatible with our formula. Indeed, by the same argument as given in the proof of Theorem 3.7.2, the self-intersection of $p \circ \mathcal{F}$ in $\mathbb{R}^5$ consists of four circles with preimage $L \subset S^3$ given as follows:

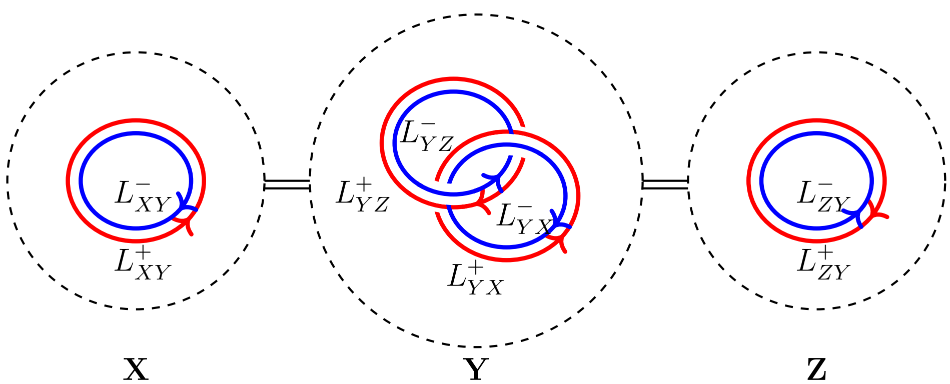


Figure 4.3: The set of double points of $p \circ \mathcal{F}$

The first term in our formula is then $lk_{S^3}(L^+, L^-) = lk(L^+_{YZ}, L^-_{YX}) + lk(L^-_{YZ}, L^+_{YX}) = -1-1 = -2$, using the computations done in the proof of Theorem 3.7.2. Since the projection of any braid is trivial as an immersion, we have $\Omega(p \circ \mathcal{F}) = 0$. From Lemma 4.2.2, we get

$E_1(p \circ \mathcal{F}) = -3 \cdot lk_X(L_{XY}, L_{XZ}) = -3 \cdot 0$. Hence, our formula gives

$$\mathcal{H}_1(\mathcal{F}) = \frac{1}{2} \cdot (-2) + \frac{1}{6} \cdot 0 - \frac{1}{12} \cdot 0 = -1 = \mathcal{H}_1(-\mathcal{T}).$$

Example 3: In the proof of Theorem 3.7.3, we have shown that $[\alpha_{12}, \alpha_{12}] \in \pi_3^{br} C(2, \mathbb{R}^3)$ gets mapped to $-\mathcal{B}_1 - \mathcal{B}_2 \in \pi_0^{br} Emb_\partial(\sqcup_2 \mathbb{R}^3, \mathbb{R}^6)$ under the graphing map G_*. Note that tubing the two components in $\mathcal{B}_1 + \mathcal{B}_2$ results in $2\mathcal{T} \in \pi_0^{br} Emb_\partial(\sqcup_2 \mathbb{R}^3, \mathbb{R}^6)$. Thus, after tubing the components of the embedding representing $G_*([\alpha_{12}, \alpha_{12}])$ (where the tube is in general position), we obtain a knot $\mathcal{J}$ isotopic to the inverse of $2\mathcal{T}$. The self-intersection of $p \circ \mathcal{J}$ and its preimage are the same as in the proof of Theorem 3.7.3 given in (3.7.5). In particular, the set of double points L in S^3 is given as follows:

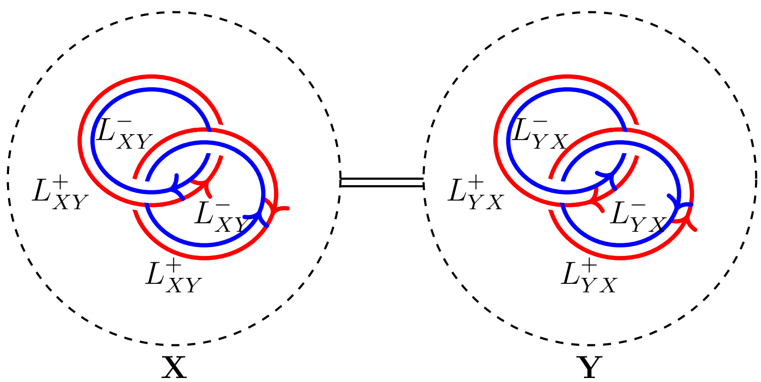


Figure 4.4: The set of double points of $p \circ \mathcal{J}$

The first term in our formula is $lk(L^+, L^-) = lk(L^+_{XY}, L^-_{XY}) + lk(L^+_{YX}, L^-_{YX}) = -2 - 2 = -4$, see the proof of Theorem 3.7.3 for computations. From Lemma 4.2.1 and the fact that the projection of a braid is trivial as immersion, we get $E_1(p \circ \mathcal{J}) = 0$ and $\Omega(p \circ \mathcal{J}) = 0$, respectively. Thus, our formula gives

$$\mathcal{H}_1(\mathcal{J}) = \frac{1}{2} \cdot (-4) + \frac{1}{6} \cdot 0 - \frac{1}{12} \cdot 0 = -2 = \mathcal{H}_1(-2\mathcal{T}).$$

Example 3': From Remark 3.7.1 (since $k = 1$), there exist an element $[Hopf] = \frac{1}{2}[\alpha_{12}, \alpha_{12}] \in \pi_3 S^2$ corresponding to the Hopf fibration, which gets mapped to $\frac{1}{2}(-\mathcal{B}_1 - \mathcal{B}_2)$. Let $\mathcal{P}$ denote the knot obtained by tubing the components of the embedding representing $G_*([Hopf])$. The set of double points of $p \circ \mathcal{P}$ is as follows:

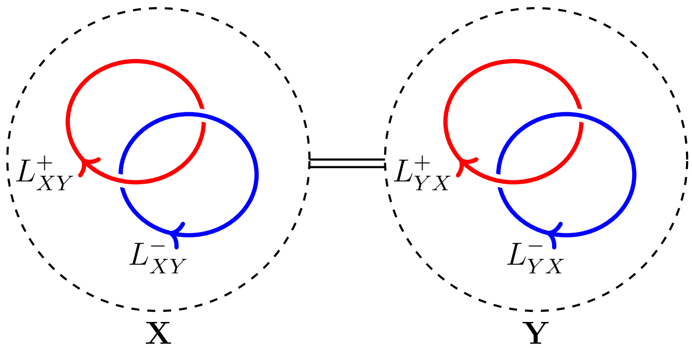


Figure 4.5: The set of double points of $p \circ \mathcal{P}$

By Remark 3.7.1, both linking numbers $lk_X(L^+_{XY}, L^-_{XY})$ and $lk_Y(L^+_{YX}, L^-_{YX})$ are -1. Thus, the first term in our formula is $lk_{S^3}(L^+, L^-) = lk_X(L^+_{XY}, L^-_{XY}) + lk_Y(L^+_{YX}, L^-_{YX}) = -2$. Moreover, $E_1(p \circ \mathcal{P}) = 0$ and $\Omega(p \circ \mathcal{P}) = 0$ by the same reasoning as for $p \circ \mathcal{J}$. Hence, $\mathcal{H}_1(\mathcal{P}) = \mathcal{H}_1(\frac{1}{2}\mathcal{J}) = \frac{1}{2} \cdot (-2) + \frac{1}{6} \cdot 0 - \frac{1}{12} \cdot 0 = -1 = \mathcal{H}_1(-\mathcal{T})$.

Example 4: Ekholm in [13, §8] constructed immersions $S^3 \looparrowright \mathbb{R}^5$ with arbitrary Smale invariant by rotating a 2-disk with a kink in $\mathbb{R}^5$. For every half integer $m \in \frac{1}{2}\mathbb{Z}$, he gives $f_m : S^3 \looparrowright \mathbb{R}^5$ that runs through regular homotopy classes. The self-intersection of f_m is one circle, such that f_m on the set of double points L is a 2-sheeted covering. The preimage L of this circle is connected if m is not an integer, and is a two component link with unknotted components that are linked with linking number m, if m is an integer. Moreover, Ekholm has shown that $\Omega(f_m) = -2m$ and $E_1(f_m) = -4m$. When m is an integer then the immersion f_m can be lifted to an embedding $F_m : S^3 \hookrightarrow \mathbb{R}^6$, where we choose one component in the preimage to be over and the other one to be undercrossing i.e., $L = L^+ \sqcup L^-$ such that $lk_{S^3}(L^+, L^-) = m$. We claim that the resulting embedding is trivial. Indeed Ekholm considers a kink $K : D^2 \looparrowright \mathbb{R}^4$ (depending on θ) so that rotating it along θ yields f_m (see [13, § 8.1 and §8.2] for more details). When m is an integer, the self-intersection of this kink can be resolved in one dimension higher, thus it can be lifted to an embedding $\bar{K} : D^2 \hookrightarrow \mathbb{R}^5$. Since any embedding $D^2 \hookrightarrow \mathbb{R}^5$ is trivial by Haefliger unknotting result [21, Corollary 6.6], there exists an isotopy between $\bar{K}$ and the trivial one. Now rotating this isotopy gives another isotopy between F_m and the trivial embedding $S^3 \hookrightarrow \mathbb{R}^6$. This is in accordance with our formula,

$$\mathcal{H}_1(F_m) = \frac{1}{2} \cdot m + \frac{1}{6} \cdot (-4m) - \frac{1}{12} \cdot (-2m) = 0.$$

4.2.2 Invariance under cobordisms and third Reidemeister moves

Let $\{F_t\}$ be the 1-parameter family of embeddings $S^3 \hookrightarrow \mathbb{R}^6$ for $t \in (-\epsilon, \epsilon)$ such that its projection to $\mathbb{R}^5$ along the last coordinate is a generic 1-parameter family $\{f_t\}$ of immersions $S^3 \looparrowright \mathbb{R}^5$. None of the terms in our formula change under the cobordism moves (note that because of the specific type of isotopy we consider, Whitney Umbrella points do not appear). There are two such cobordisms: one appearing as the analogue of the second Reidemeister move and the other one where two saddles occur simultaneously, see Section 3.3.1. In both moves, $lk_{S^3}(L^+, L^-)$ is preserved as there are no new linkings between L^+ and L^-. Moreover, Ekholm in [13, Theorem 6.2.1] has shown that $E_1(f)$ is preserved under these two moves. We now show that our formula is unchanged under the analogue of the third Reidemeister move. To be precise, we show that the first two terms compensate each other under this move (note that $\Omega(f)$ is an invariant of regular homotopy). Assume that f_0 has a triple point, say $0 \in \mathbb{R}^5$. Let the three sheets coming together at the triple point be X, Y and Z such that X is above Y, and Y is above Z, with respect to the last coordinate in $\mathbb{R}^6$. One can find the following (positively oriented) local coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), and (z_1, z_2, z_3) on X, Y, and Z respectively such that near the triple point $0 \in \mathbb{R}^5$ we have

$$f_t(X) = (x_1, 0, 0, x_2, x_3),$$

$$f_t(Y) = (t, y_1, y_3, 0, y_2),$$

$$f_t(Z) = (z_1, z_1, z_2, z_3, 0).$$

(The coordinates in $\mathbb{R}^5$ must also be adjusted). The oriented self-intersections of f_t are the lines

$$f_t(X) \cap f_t(Y) = (t, 0, 0, 0, -s),$$

$$f_t(X) \cap f_t(Z) = (0, 0, 0, \tilde{s}, 0),$$

$$f_t(Y) \cap f_t(Z) = (t, t, \tilde{\tilde{s}}, 0, 0)$$

(see the proof of Theorem 3.5.1 for orientations and computations that we use in this subsection). The oriented preimages of the self-intersections of f_t are given by

$$\begin{aligned} L^+_{XY}(t) &= (t, 0, -s), \\ L^+_{XZ} &= (0, \tilde{s}, 0), \\ L^-_{YX} &= (0, -s, 0), \\ L^+_{YZ}(t) &= (t, 0, \tilde{\tilde{s}}), \\ L^-_{ZX} &= (0, 0, \tilde{s}), \\ L^-_{ZY}(t) &= (t, \tilde{\tilde{s}}, 0). \end{aligned}$$

Let us see how the first two terms in our formula change when t goes from $-\delta$ to δ. We have computed in the proof of Theorem 3.5.1 that the only non-trivial change is in the linking numbers $lk(L^+_{XY}, L^+_{XZ}), lk(L^+_{YZ}, L^-_{YX})$, and $lk(L^-_{ZX}, L^-_{ZY})$, which is given by $+1$ in each case. However, only $\Delta lk(L^+_{YZ}, L^-_{YX})$ contributes to the change in $lk_{S^3}(L^+, L^-)$, thus $\Delta lk_{S^3}(L^+, L^-) = 1$. Note that $\Delta E_1(f)$ depends on two things: the motion of Y and the changes in the framing that this motion causes. Let us choose one (out of three) of the self-intersections, say $A(X, Z) = f_t(X) \cap f_t(Z) = (0, 0, 0, \tilde{s}, 0)$, with preimages L^+_{XZ} and L^-_{ZX}. Both preimages are shifted using fr_1, thus $\Delta lk(L^+_{XZ}, \widetilde{L^+_{XZ}}) = -\Delta lk(L^+_{XZ}, L^+_{XY}) = -1$ and $\Delta lk(L^-_{ZX}, \widetilde{L^-_{ZX}}) = -\Delta lk(L^-_{ZX}, L^-_{ZY}) = -1$. We assume that the shifting distance is very small compared to δ.

$$
\begin{aligned}
\Delta\mathcal{H}_1(F) &= \mathcal{H}_1(F_\delta) - \mathcal{H}_1(F_{-\delta}) \\
&= \frac{1}{2}\Delta lk_{S^3}(L^+, L^-) + \frac{1}{6}\Delta E_1(f) - \frac{1}{12}\Delta\Omega(f) \\
&= \frac{1}{2}\cdot 1 + \frac{1}{6}\cdot(-3) - \frac{1}{12}\cdot 0 \\
&= 0.
\end{aligned}
$$

4.2.3 Compatibility with Sakai's Lin-Wang type formula

In [39], Sakai studied the behavior of the Haefliger invariant for long embeddings $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$ whose projection to $\mathbb{R}^5$ are generic immersions. He derived a Lin-Wang type formula that describes the change in the Haefliger invariant under crossing changes. This result is based on his earlier work in [37], where he expressed the Haefliger invariant in terms of configuration space integrals. We show that our formula aligns with Sakai's result in [39, Theorem 2.5]. Let $F : \mathbb{R}^3 \hookrightarrow \mathbb{R}^6$ be a long embedding such that its projection $p \circ F : \mathbb{R}^3 \looparrowright \mathbb{R}^5$ is a generic immersion, where $p : \mathbb{R}^6 \to \mathbb{R}^5$ denotes the projection $(x_1, \dots, x_6) \mapsto (x_1, \dots, x_5)$. Let $A = A_1 \sqcup A_2$ be the self-intersection of $p \circ F$ with preimage $(p\circ F)^{-1}(A_i) = L_i = L_i^+ \sqcup (-L_i^-)$ for $i = 1, 2$, taken with the adjusted Ekholm orientation (i.e. undercrossing is taken with the opposite orientation). Here, both A_1 and A_2 can be disconnected or empty. Suppose F_{A_2} is an embedding obtained from F by crossing change at A_2, i.e., we swap overcrossing L_2^+ and undercrossing L_2^-. Then $(p \circ F_{A_2})^{-1}(A_1) = L_1^+ \sqcup -L_1^-$ and $(p\circ F_{A_2})^{-1}(A_2) = L_2^- \sqcup -L_2^+$ (now L_2^+ is undercrossing and thus oppositely oriented). Note that the invariants E_1 and Ω in our formula depend only on the immersion and thus are preserved under crossing changes. Then, we get the following

$$
\begin{aligned}
\mathcal{H}_1(F) - \mathcal{H}_1(F_{A_2}) &= \frac{1}{2} lk(L_1^+ \sqcup L_2^+, L_1^- \sqcup L_2^-) - \frac{1}{2} lk(L_1^+ \sqcup L_2^-, L_1^- \sqcup L_2^+) \\
&= \frac{1}{2}\big(lk(L_1^+, L_2^-) - lk(L_1^+, L_2^+) - lk(L_1^-, L_2^-) + lk(L_1^-, L_2^+)\big) \\
&= -\frac{1}{2} lk(L_1, L_2),
\end{aligned}
$$

which is compatible with [39, Theorem 2.5], up to a minus sign. Note however, that Sakai determined his integral invariant on the Haefliger trefoil to be ± 1 (see [39, § 3]), while with our conventions, our conjectural formula produced $+1$ on the Haefliger trefoil.

4.2.4 Compatibility with Cabling

In knot theory, the cabling operation involves doubling the knot along a chosen normal vector field, resulting in a two-component link. Let $Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ denote the space of long embeddings $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$ along with a non-vanishing normal vector field (both with the standard behavior outside a compact set). To be precise, outside a compact set, the normal field is chosen to be $\mathbf{0} := e_5 + e_6 = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}$ (since we require it to be non-tangent to any of the components of the Borromean link). Each element in $Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ is given by a pair (F, n), where F is a long embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$ with a nonzero normal vector field n.

We define the cabling by a map

$$C : Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6) \to Emb_\partial(\bigsqcup_2 \mathbb{R}^3, \mathbb{R}^6)$$

$$(F, n) \mapsto F \sqcup \widetilde{F}_n$$

where $\widetilde{F}_n$ is a shifted copy of F along the normal field n. We use cabling C and our invariant $\mathcal{W}$ for two-component links (as given in Theorem 3.6.1), to justify our choice of the coefficient of $\Omega(f)$ in (4.1.5). Note that this can also be seen from Example 4. However, the argument below allows us to determine the coefficient of $\Omega(f)$ for high dimensional immersions (as discussed in Section 4.4).

By [21, Theorem 5.17], we know that twice Haefliger trefoil knot given by $2\mathcal{T} = \mathcal{T}\#\mathcal{T} : \mathbb{R}^3 \hookrightarrow \mathbb{R}^6$, is compressible to $\mathbb{R}^5$ i.e., it is isotopic to a knot inside $\mathbb{R}^5 \times 0$. Let $\mathcal{X} \in Emb_\partial(\mathbb{R}^3, \mathbb{R}^5)$ denote the compressed version of $2\mathcal{T}$. We want to understand $\mathcal{X}$ as a non-trivial immersion in $Imm_\partial(\mathbb{R}^3, \mathbb{R}^5)$ by computing $\Omega(\mathcal{X})$. By Ekholm [13, Proposition 4.1.2], the set of regular homotopy classes which contain embeddings form a subgroup of index 24 in $\pi_0 Imm_\partial(\mathbb{R}^3, \mathbb{R}^5)$. Thus, under the natural map $i_* : \pi_0 Emb_\partial(\mathbb{R}^3, \mathbb{R}^5) \to \pi_0 Imm_\partial(\mathbb{R}^3, \mathbb{R}^5)$, the element $\mathcal{X}$ must be mapped to ± 24 times the generator $[\sigma]$ in $\pi_3 V_{5,3} = \pi_0 Imm_\partial(\mathbb{R}^3, \mathbb{R}^5)$, see Section 4.1 for notation. We claim that $\Omega(i(\mathcal{X})) = -24 \cdot [\sigma]$.

Let $Imm_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ be the space of long immersions $\mathbb{R}^3 \looparrowright \mathbb{R}^6$ with a normal vector field. Consider the inclusion $j : Imm_\partial(\mathbb{R}^3, \mathbb{R}^5) \hookrightarrow Imm_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ such that for $f \in Imm_\partial(\mathbb{R}^3, \mathbb{R}^5)$, $j(f)$ is the immersion f with the constant normal vector field $\mathbf{0} = e_5 + e_6$. Let $j' : Emb_\partial(\mathbb{R}^3, \mathbb{R}^5) \hookrightarrow Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ be the restriction of j to embeddings. We have the following commutative diagram:

$$\begin{tikzcd}
Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6) \arrow[r, "i^+"] & Imm_\partial^+(\mathbb{R}^3, \mathbb{R}^6) \arrow[r, "\Omega"] & \pi_3 V_{6,4} = \mathbb{Z} \\
Emb_\partial(\mathbb{R}^3, \mathbb{R}^5) \arrow[r, "i"] \arrow[u, "j'"] & Imm_\partial(\mathbb{R}^3, \mathbb{R}^5) \arrow[r, "\Omega"] \arrow[u, "j"] & \pi_3 V_{5,3} = \mathbb{Z} \arrow[u, "\approx"]
\end{tikzcd}$$

where the right vertical map is a bijection (since the map $V_{5,3} \to V_{6,4}$ is a fiber inclusion over S^5 and thus is 4-connected). Thus, j induces an isomorphism on π_0. To find $\Omega(i(\mathcal{X}))$ we use the above commutative square along with the cabling C and the invariant $\mathcal{W}$. Firstly, we get that $j'(\mathcal{X})$ is isotopic to $(2\mathcal{T}, v) \in Emb_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$, where v is some nowhere zero normal field. Note that $(2\mathcal{T}, v)$ can be chosen to be knot $2\mathcal{T}$ with trivial framing $\mathbf{0}$, connected sum with the trivial knot $\mathcal{U}$ with some non-trivial framing (given by v) i.e., $(2\mathcal{T}, v) \simeq (2\mathcal{T}, \mathbf{0}) \#(\mathcal{U}, v)$. Since $2\mathcal{T}$ is trivial as immersion, the image $i^+(2\mathcal{T}, v) \in Imm_\partial^+(\mathbb{R}^3, \mathbb{R}^6)$ is determined only by this non-trivial framing v. To find this non-zero vector field, we first note that the freedom of choice of normal field for the trivial knot $\mathcal{U}$ is measured by $\pi_3 S^2 = \mathbb{Z}$. Since the generator of $\pi_3 S^2$ is given by $[\rho]$, where ρ is the Hopf fibration (see Section 4.1), the framing adjustment is done by adding some multiple of $[\rho]$. Now we consider $j'(\mathcal{X})$ with the choice of trivial framing $\mathbf{0}$, we claim that $\mathcal{W}(C(j'(\mathcal{X}), \mathbf{0})) = (0,0) = \mathcal{W}(C(2\mathcal{T}, v))$. Indeed, the cabling $C(j'(\mathcal{X}), \mathbf{0})$ results in a two-component link, say $\mathcal{X} \sqcup \mathcal{X}'$, where the components $\mathcal{X}$ and $\mathcal{X}'$ lie inside $\mathbb{R}^5 \times 0$ and $(\mathbb{R}^5 \times 0) + \epsilon \cdot \mathbf{0}$, respectively. Note that under the projection to $\mathbb{R}^5$ along the vector $(1, \ldots, 1) \in \mathbb{R}^6$, there is no self-intersection for each component of $C(j'(\mathcal{X}), \mathbf{0})$ (since both $\mathcal{X}$ and $\mathcal{X}'$ are embeddings in $\mathbb{R}^5$), but there can be some non-trivial intersection between the projection of $\mathcal{X}$ and $\mathcal{X}'$. However, we can isotope $\mathcal{X}'$ in $(\mathbb{R}^5 \times 0) + \epsilon \cdot \mathbf{0}$ so that its support lies far away from that of $\mathcal{X}$, as shown in Figure 4.6.

As a result, there is an empty intersection between the projected components and we get

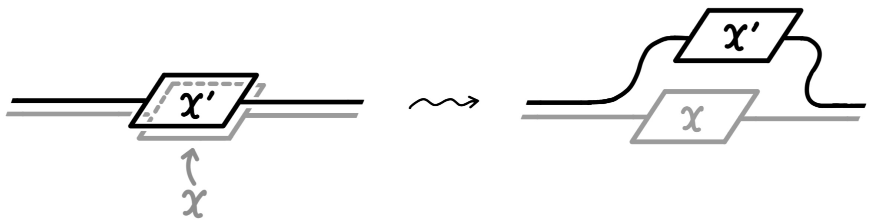


Figure 4.6: Isotoped $\mathcal{X}'$ in $(\mathbb{R}^5 \times 0) + \epsilon \cdot \mathbf{0}$

$\mathcal{W}(C(j'(\mathcal{X}), e_6)) = (0, 0)$. Moreover, from Remark 3.7.1, we have $\mathcal{W}(G_*([\rho])) = (1, 1)$. Thus, if $v = m \cdot [\rho]$ for some integer m, we get $\mathcal{W}(C(\mathcal{U}, v)) = (m, m)$. As a corollary, our invariant $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ on any two-component link resulting from cabling must satisfy $\mathcal{W}_1 = \mathcal{W}_2$.

Furthermore, it is easy to compute that $\mathcal{W}(C(\mathcal{T}, \mathbf{0})) = \mathcal{W}(\mathcal{T} \sqcup \mathcal{T}') = (-6, -6)$, where $\mathcal{T}'$ is a slightly shifted copy of $\mathcal{T}$ along $\mathbf{0}$. Indeed, using the notation from Example 1 in Subsection 4.2.1, let $\mathcal{T} = X\#Y\#Z\#F_0$ and $\mathcal{T}' := X'\#Y'\#Z'\#F_0'$, where $F_0 : \mathbb{R}^3 \subset \mathbb{R}^6$ is the standard inclusion and F_0' is its shifted copy. Let L_{ij} denote the preimage of the intersection for $i, j = \mathcal{T}, \mathcal{T}'$ lying in the component i. Recall (from Example 1) that there were six circles in the self-intersection of $p \circ \mathcal{T}$, with their preimages given by six disjoint Hopf links (two in each X, Y, and Z). We now get four circles in the intersection of $p \circ (\mathcal{T} \sqcup \mathcal{T}')$ corresponding to any circle in the self-intersection of $p \circ \mathcal{T}$. For example, given a circle A_1 lying in $p(X) \cap p(Z)$ in the self-intersection of $p \circ \mathcal{T}$, now we get four (shifted) copies of it in the double intersection of $p \circ (\mathcal{T} \sqcup \mathcal{T}')$ lying in $p(X) \cap p(Z)$, $p(X') \cap p(Z)$, $p(X) \cap p(Z')$, and $p(X') \cap p(Z')$, as shown in the figure below.

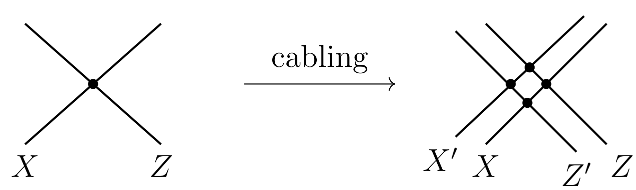


Figure 4.7: Double intersection under cabling

Moreover, it is easy to check that the set of double points consists of two double Hopf links in each X, Y, Z and X', Y', Z'. For example, here is the link that we get inside Z:

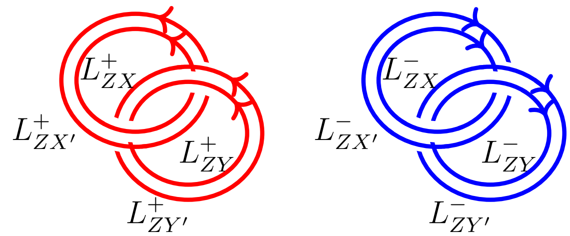


Figure 4.8: The preimages of the double intersection inside Z

For concreteness, we check below that $L^{\pm}_{ZX'}$ is a parallel copy of $L^{\pm}_{ZX}$ and thus $lk(L^{\pm}_{ZX}, L^{\pm}_{ZX'}) = 0$. Following our conventions from the proof of Theorem 3.5.2, the projections of components X, X' and Z are given by the following respective maps

$$(0, 0, y_1, y_2, z_1, z_2) \mapsto (-z_2, -z_2, y_1 - z_2, y_2 - z_2, z_1 - z_2)$$

$$(0, 0, y_1, y_2, z_1 + \epsilon, z_2 + \epsilon) \mapsto (-(z_2 + \epsilon), -(z_2 + \epsilon), y_1 - (z_2 + \epsilon), y_2 - (z_2 + \epsilon), z_1 - z_2)$$

$$(\tilde{\tilde{\mathrm{x}}}_1, \tilde{\tilde{\mathrm{x}}}_2, \tilde{\tilde{\mathrm{y}}}_1, \tilde{\tilde{\mathrm{y}}}_2, 0, 0) \mapsto (\tilde{\tilde{\mathrm{x}}}_1, \tilde{\tilde{\mathrm{x}}}_2, \tilde{\tilde{\mathrm{y}}}_1, \tilde{\tilde{\mathrm{y}}}_2, 0)$$

Note that the intersection $p(X) \cap p(Z)$ consists of two circles:

$$\Big\{\Big(\pm \frac{\beta}{\sqrt{2}}, \pm\frac{\beta}{\sqrt{2}}, w_1, w_2, 0\Big) \in \mathbb{R}^5 | w_1^2 + w_2^2 = \beta^2\Big\},$$

Similarly, the intersection $p(X') \cap p(Z)$ consists of two circles:

$$\Big\{\Big(\pm \frac{\beta}{\sqrt{2}} - \epsilon, \pm\frac{\beta}{\sqrt{2}} - \epsilon, w_1, w_2, 0\Big) \in \mathbb{R}^5 | w_1^2 + w_2^2 = \beta^2\Big\}.$$

Clearly, the preimages of these circles inside Z given by

$$L^{\pm}_{ZX} = \Big\{\Big(\pm \frac{\beta}{\sqrt{2}}, \pm\frac{\beta}{\sqrt{2}}, \tilde{\tilde{\mathrm{y}}}_1, \tilde{\tilde{\mathrm{y}}}_2, 0, 0\Big) \in Z | (\tilde{\tilde{\mathrm{y}}}_1)^2 + (\tilde{\tilde{\mathrm{y}}}_2)^2 = \beta^2\Big\}$$

$$L^{\pm}_{ZX'} = \Big\{\Big(\pm \frac{\beta}{\sqrt{2}} - \epsilon, \pm\frac{\beta}{\sqrt{2}} - \epsilon, \tilde{\tilde{\mathrm{y}}}_1, \tilde{\tilde{\mathrm{y}}}_2, 0, 0\Big) \in Z | (\tilde{\tilde{\mathrm{y}}}_1)^2 + (\tilde{\tilde{\mathrm{y}}}_2)^2 = \beta^2\Big\}$$

are parallel. Hence,

$$\mathcal{W}_1(C(\mathcal{T},\mathbf{0}))) = \frac{1}{2}lk(L_{\mathcal{T}\mathcal{T}}, L_{\mathcal{T}\mathcal{T}'}) - \underbrace{lk(L^+_{\mathcal{T}'\mathcal{T}}, L^-_{\mathcal{T}'\mathcal{T}})}_{=0} = \frac{1}{2}(6\cdot(-2)) = -6$$

(note that only $lk(L^+_{ZX}, L^+_{ZY'})$ and $lk(L^+_{ZX'}, L^+_{ZY})$ from the first double Hopf link in the picture above contribute (each by -1) to $\mathcal{W}_1$, and since there are six such double Hopf links in $\mathcal{T}$, we get $6\cdot(-2)$).

Using the additivity of $\mathcal{W}_1$, we have

$$0 = \mathcal{W}_1(C(2\mathcal{T}, v)) = \mathcal{W}_1(C(2\mathcal{T},\mathbf{0})) + \mathcal{W}_1(C(\mathcal{U}, v)) = -12 + m$$

We get $m = 12$ and hence the non-trivial framing on $(2\mathcal{T}, v)$ is given by $12\cdot[\rho]$. Therefore, $i^+(2\mathcal{T}, v)$ is regularly homotopic to the trivial immersion $\mathbb{R}^3 \looparrowright \mathbb{R}^6$ with non-trivial framing given by $12\cdot[\rho] \in \pi_3 S^2$. To see the image under Ω, we consider the inclusions $S^2 \xhookrightarrow{\iota} V_{6,4} \xhookleftarrow{\eta} V_{5,3}$, where ι is obtained by fixing the first three vectors in $V_{6,4}$ and rotating the last one; while the inclusion $V_{5,3} \hookrightarrow V_{6,4}$ is obtained by adding a fixed vector $\frac{1}{\sqrt{2}}(e_5 + e_6)$ as the last vector. Note that there is a copy of S^2 included in $V_{5,3}$ and the composition $S^2 \hookrightarrow V_{5,3} \hookrightarrow V_{6,4}$ is not the same as the inclusion ι. There is a sign action of the symmetric group S_4 on $V_{6,4}$ which permutes the four vectors. However, on π_3, the induced action is trivial and thus the generator $[\rho] = \frac{1}{2}[id, id] \in \pi_3 S^2$ gets mapped to $-2[\sigma'] \in \pi_3 V_{6,4}$, where $[\sigma']$ is the generator of $\pi_3 V_{6,4}$ and is the image of $[\sigma]$ under the induced map (bijection on π_3) corresponding to inclusion η.[2] Under Ω, $i^+(2\mathcal{T}, v)$ gets mapped to $12\cdot(-2[\sigma']) = -24\cdot[\sigma']$ and thus $\Omega(i(\mathcal{X})) = -24\cdot[\sigma]$.

Hence, our formula (4.1.5) evaluated on $2\mathcal{T}$ when it is compressed as an embedding to

[2]More generally, the group $S_4 \wr \mathbb{Z}_2$ acts on $V_{6,4}$ by permuting and changing orientation of the vectors. This induces a sign action of $S_4 \wr \mathbb{Z}_2$ on $\pi_2 V_{6,4} = H_2 V_{6,4} = \mathbb{Z}$. However, the induced (sign) action of this group on $\pi_3 V_{6,4}$ is trivial since given a generator $\alpha \in \pi_2 V_{6,4}$, one has $[-\alpha, -\alpha] = [\alpha, \alpha] \in \pi_3 V_{6,4}$.

$\mathbb{R}^5$ gives

$$\mathcal{H}_1(2\mathcal{T}) := \frac{1}{2}\underbrace{lk_{S^3}(L^+, L^-)}_{=0} + \frac{1}{6}\underbrace{E_1(\mathcal{X})}_{=0} - \frac{1}{12}\Omega(\mathcal{X}) = -\frac{1}{12}\cdot(-24) = 2.$$

We use the following remark in Section 4.4 for the (rational) generalization of Ω to higher dimensional immersions $S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$. In higher dimensions, it is difficult to see the generator of the $\mathbb{Z}$ summand of $\pi_{4k-1}S^{2k}$, thus we express $[f] \in \pi_0 Imm(S^{4k-1}, \mathbb{R}^{6k-1}) = \pi_{4k-1}V_{6k-1,4k-1}$ in terms of the rational generator $[id, id]$ of $\pi_{4k-1}S^{2k}$, where $[id, id]$ is the Whitehead bracket of the identity map on S^{2k} with itself.

Remark 4.2.1. For an immersion $f : S^3 \looparrowright \mathbb{R}^5$, $\pi_0 Imm(S^3, \mathbb{R}^5) = \pi_3 V_{5,3} = \mathbb{Z}$ implies that $[f] = \Omega(f)\cdot[\sigma]$, where $[\sigma]$ is the generator of $\pi_3 V_{5,3}$. Rationally, we define $\Omega_{S^2}(f) \in \mathbb{Q}$ so that $[f] = \Omega_{S^2}(f)\cdot[id, id]$ where $[id, id]$ is the rational generator of $\pi_3 S^2$. Moreover, since $[id, id] = 2[\rho] = -4[\sigma]$, we get $\Omega(f) = -4\cdot\Omega_{S^2}(f)$.

4.3 Second Formula $\mathcal{H}_2$

In this section we provide our second conjectural formula for the Haefliger invariant. We also prove that the second formula is equivalent to the first one. According to [14, 36, 33], there is a different way to define the linking invariant $E_1(f)$. We denote this new invariant as $E_2(f)$ for any generic immersion $f : S^3 \looparrowright \mathbb{R}^5$. It is shown in [33, Proposition 3.1.5] that $E_2(f) = -E_1(f)$. Let $L \subset S^3$ be the set of double points of f and $f(L)$ be its image. Note that the orientable 2-dimensional normal bundle $N(S^3)$ of f is trivial (since the oriented 2-dimensional vector bundles are classified by $\pi_3(BSO(2)) = 0$). Moreover, any two (coorientation preserving) trivializations are homotopic since their difference represents an element in $\pi_4(BSO(2)) = 0$. Let (v_1, v_2) be the (unique up to homotopy) normal framing of f. Then for $p = f(p_1) = f(p_2) \in f(L)$, $u(p) = v_1(p_1) + v_1(p_2)$ defines a normal vector field along $f(L)$. Let $\widehat{f(L)} \subset \mathbb{R}^5$ be obtained by pushing $f(L)$ slightly along u, then $\widehat{f(L)} \cap f(S^3) = \emptyset$. Define

$$E_2(f) := lk_{\mathbb{R}^5}(\widehat{f(L)}, f(S^3)).$$

Note that the normal bundle $N(L)$ over L in S^3 is also 2-dimensional and is naturally identified with $N(S^3)|_L$. Thus, (v_1, v_2) also gives the (unique up to homotopy) normal framing on $L \subset S^3$. Let us call these choices of normal vector fields for L and $f(L)$ as framing fr_2.

Conjecture 4.3.1. *Given a knot $F : S^3 \hookrightarrow \mathbb{R}^6$ such that its projection $f = p \circ F : S^3 \looparrowright \mathbb{R}^5$ is a generic immersion with $L \subset S^3$ as the set of double points (with the same adjusted Ekholm orientation convention $L := L^+ \sqcup -L^-$ and $\widehat{L} := \widehat{L}^+ \sqcup -\widehat{L}^-$, where $\widehat{L}$ is the result of pushing L using the framing fr_2). Then, the second conjectural formula is given by*

$$\mathcal{H}_2(F) := -\frac{1}{8} lk_{S^3}(L, \widehat{L}) + \frac{1}{12} E_2(f) - \frac{1}{12}\Omega(f), \tag{4.3.1}$$

which produces the Haefliger isomorphism $\mathcal{H}_2 : \ \pi_0 Emb(S^3, \mathbb{R}^6) \xrightarrow{\approx} \mathbb{Z}$.

Remark 4.3.1. Similar to the invariance of $\mathcal{H}_1(F)$ discussed in Subsection 4.2.2, we have that $\mathcal{H}_2(F)$ is invariant under the two cobordisms moves (the analogue of the second Reidemeister move and the other one where two saddles occur simultaneously) and the analogue of the third Reidemeister move. Indeed, $lk_{S^3}(L, \widehat{L})$ is unchanged under both cobordisms moves since these cobordisms can be extended as framed cobordisms (along the unique normal framing). It is shown by Ekholm in [12, Theorem 1] that $E_2(f)$ is unchanged under these two cobordisms moves. Under the triple point move, note that $E_2(f) = -E_1(f)$ and we have proved in Subsection 4.2.2 that $E_1(f_\delta) - E_1(f_{-\delta}) = -3$, thus $E_2(f_\delta) - E_2(f_{-\delta}) = 3$. To see the change in $lk_{S^3}(L, \widehat{L})$, we note that both $lk(L^+_{XY}, \widehat{L^+_{XZ}})$ and $lk(L^+_{XZ}, \widehat{L^+_{XY}})$ have the same contribution as $lk(L^+_{XY}, L^+_{XZ})$ i.e., the framing does not matter, thus $\Delta lk(L^+_{XY}, \widehat{L^+_{XZ}}) = 1 = \Delta lk(L^+_{XZ}, \widehat{L^+_{XY}})$. Similarly, $\Delta lk(L^-_{ZX}, \widehat{L^-_{ZY}}) = 1 = \Delta lk(L^-_{ZY}, \widehat{L^-_{ZX}})$. Note that $\Delta lk(L^+_{YZ}, \widehat{L^-_{YX}}) = 1 = \Delta lk(L^-_{YX}, \widehat{L^+_{YZ}})$, but their contributions are taken with opposite sign in $lk_{S^3}(L, \widehat{L})$. Thus, $\Delta lk_{S^3}(L, \widehat{L}) = 1 + 1 + 1 + 1 - 1 - 1 = 2$.

Hence, the overall change in our formula is given by

$$\begin{aligned}
\Delta\mathcal{H}_2(F) &= \mathcal{H}_2(F_\delta) - \mathcal{H}_2(F_{-\delta}) \\
&= -\frac{1}{8}\Delta lk_{S^3}(L, \widehat{L}) + \frac{1}{12}\Delta E_2(f) - \frac{1}{12}\Delta\Omega(f) \\
&= -\frac{1}{8}\cdot 2 + \frac{1}{12}\cdot 3 - \frac{1}{12}\cdot 0 \\
&= 0.
\end{aligned}$$

Theorem 4.3.1. *For any smooth knot $F : S^3 \hookrightarrow \mathbb{R}^6$ such that its projection $p \circ F : S^3 \looparrowright \mathbb{R}^5$ on $\mathbb{R}^5$ is a generic immersion, we get $\mathcal{H}_1(F) = \mathcal{H}_2(F)$ and thus the two conjectural formulas in (4.1.5) and (4.3.1) are equivalent.*

Proof. We will be using formula (4.1.6) for $\mathcal{H}_1(F)$ discussed in Section 4.1. Recall that $\widetilde{L}$ and $\widetilde{f(L)}$ in (4.1.6) and $\widehat{L}$ and $\widehat{f(L)}$ in (4.3.1) are defined using the framings fr_1 and fr_2, respectively. We have

$$lk_{S^3}(L, \widetilde{L}) - lk_{S^3}(L, \widehat{L}) = \sum_i lk_{S^3}(L_i^+, \widetilde{L_i^+} - \widehat{L_i^+}) + \sum_i lk_{S^3}(L_i^-, \widetilde{L_i^-} - \widehat{L_i^-}).$$

Let us denote this difference by Δ_{fr} since it captures the difference in framing at each component in the preimage. Moreover,

$$E_1(f) - E_2(f) = \sum_i lk_{\mathbb{R}^5}(\widetilde{f(L_i)} - \widehat{f(L_i)}, f(S^3)).$$

By definitions of fr_1 and fr_2, each $lk_{\mathbb{R}^5}(\widetilde{f(L_i)} - \widehat{f(L_i)}, f(S^3))$ is same as the sum $lk_{S^3}(L_i^+, \widetilde{L_i^+} - \widehat{L_i^+}) + lk_{S^3}(L_i^-, \widetilde{L_i^-} - \widehat{L_i^-})$, thus $E_1(f) - E_2(f) = \Delta_{fr}$. Since $E_2(f) = -E_1(f)$, we have

$2E_1(f) = \Delta_{fr}$. Thus, the difference of our two conjectural formulas is given by

$$\begin{aligned}
\mathcal{H}_1(F) - \mathcal{H}_2(F) &= -\frac{1}{8}\Big(lk_{S^3}(L, \widetilde{L}) - lk_{S^3}(L, \widehat{L})\Big) + \frac{1}{6}\Big(E_1(f) - \frac{1}{2}E_2(f)\Big) \\
&= -\frac{1}{8}\Big(lk_{S^3}(L, \widetilde{L}) - lk_{S^3}(L, \widehat{L})\Big) + \frac{1}{4}\big(E_1(f)\big) \\
&= -\frac{1}{8} \cdot \Delta_{fr} + \frac{1}{4} \cdot \frac{1}{2}\Delta_{fr} \\
&= 0.
\end{aligned}$$

□

4.4 Higher dimensional versions of $\mathcal{H}_1(F)$ and $\mathcal{H}_2(F)$

In this section, we give a generalization of our conjectural formulas to higher dimensional embeddings $S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ for any k, when their projection to $\mathbb{R}^{6k-1}$ is a generic immersion.

By [14], there exist analogous linking invariants E_1 and E_2 for generic immersions $S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$. Let $f : S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ be a generic immersion with the $(2k-1)$-dimensional self-intersection given by $f(L) \subset \mathbb{R}^{6k-1}$. Let $L \subset S^{4k-1}$ denote the set of double points such that $f|_L : L \to f(L)$ is a 2-fold covering. Since the codimension of f is even, there is an induced orientation on $f(L)$ and hence on L (see Remark 3.2.1). Let $N(S^{4k-1})$ denote the $2k$-dimensional normal bundle of $f(S^{4k-1})$ and $N(S^{4k-1})|_L$ be its restriction on the submanifold $L^{2k-1} \subset S^{4k-1}$. Since the embedding $L \hookrightarrow S^{4k-1}$ is a null-homotopic map, the normal bundle $N(S^{4k-1})|_L$ must be trivial i.e., $N(S^{4k-1})|_L \cong L \times \mathbb{R}^{2k}$ and moreover the trivialization is uniquely determined up to homotopy (this is because $\pi_i S^{4k-1} = 0$ for $i \leq 2k$ and therefore the choice of homotopy to the constant map is also homotopically unique). Note that $N(S^{4k-1})|_L$ is naturally identified with the $2k$-dimensional normal bundle $N(L)$ over L in S^{4k-1}. Let $(v_1, \ldots, v_{2k})$ be the framing of this homotopically unique trivialization of $N(L)$. For $p \in f(L)$ such that $f(p_1) = f(p_2) = p$, we define $u(p) = df(v_1(p_1)) + df(v_1(p_2))$ as the normal vector field along $f(L)$ in $\mathbb{R}^{6k-1}$. Let $\widehat{f(L)} \subset \mathbb{R}^{6k-1}$ denote a copy of $f(L)$ shifted slightly along u, then $\widehat{f(L)} \cap f(S^{4k-1}) = \emptyset$. Define

$$E_2(f) := lk(\widehat{f(L)}, f(S^{4k-1}))$$

Let us call these choices of normal vector fields for L and $f(L)$ as framing fr_2.

Now we recall how $E_1(f)$ is defined in higher dimensions. Let $L_i \subset L$ be one of the path components of $L \subset S^{4k-1}$. Recall again that the normal bundle $N(L)$ is naturally trivialized. The set of non-zero sections over $N(L_i)$ is thus in one-to-one correspondence with the homotopy classes of maps from L_i to S^{2k-1}, which are classified by their degree. Therefore, $\pi_0\Gamma(N(L_i)\backslash L_i) = [L_i^{2k-1}, S^{2k-1}] = \mathbb{Z}$. This implies that we can choose a non-zero normal vector field v along $L \subset S^{4k-1}$ such that $[\widetilde{L}] = 0 \in H_1(S^{4k-1} - L)$, where $\widetilde{L}$ denote the slightly shifted copy of L along v. In particular, for every path-component L_i of L, we get

$$lk(L_i, \widetilde{L}) = lk(L_i, \widetilde{L}_i) + \sum_{j \neq i} lk(L_i, L_j) = 0.$$

We define a vector field w along $f(L)$ such that at the point $p = f(p_1) = f(p_2)$ we get $w(p) = df(v(p_1)) + df(v(p_2))$. Let $\widetilde{f(L)} \subset \mathbb{R}^{6k-1}$ be the result of pushing $f(L)$ slightly along w, then $\widetilde{f(L)} \cap f(S^{4k-1}) = \emptyset$. Define

$$E_1(f) := lk(\widetilde{f(L)}, f(S^{4k-1}))$$

Let us call these choices of normal vector fields for L and $f(L)$ as framing fr_1.

By Smale [41], there is a bijection between the set (group) of regular homotopy classes of immersions $S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ and the elements of the group $\pi_{4k-1}V_{6k-1,4k-1}$, where $V_{6k-1,4k-1}$ denotes the Stiefel manifold of $(4k-1)$-frames in $(6k-1)$-space. Given an immersion $f : S^{4k-1} \to \mathbb{R}^{6k-1}$, let $\Omega(f) \in \pi_{4k-1}V_{6k-1,4k-1}$ denote its Smale invariant. In the higher dimensions, it is easier to understand $\Omega(f)$ rationally. Note that there is a natural fiber sequence given by $S^{2k} \to V_{6k-1,4k-1} \to V_{6k-1,4k-2}$ which induces a rational isomorphism between $\pi_{4k-1}S^{2k}$ and $\pi_{4k-1}V_{6k-1,4k-1}$. Thus, any immersion $f : S^{4k-1} \to \mathbb{R}^{6k-1}$ can be related to the rational generator $[id, id]$ of $\pi_{4k-1}S^{2k}$ i.e., $\Omega(f) = \Omega_{S^{2k}}(f) \cdot [id, id] \in \pi_{4k-1}V_{6k-1,4k-1} \otimes \mathbb{Q}$, where $\Omega_{S^{2k}}(f) \in \mathbb{Q}$. In case of $k = 1$, by Remark 4.2.1, we have $\Omega(f) = -4 \cdot \Omega_{S^2}(f)$. Note that $[id, id]$ is also a generator of the $\mathbb{Z}$-summand of $\pi_{4k-1}S^{2k}$ for $k \neq 1, 2, 4$; otherwise the generator corresponds to the Hopf fibration, see Remark 3.7.1.

Let $p : \mathbb{R}^{6k} \to \mathbb{R}^{6k-1}$ be the projection map.

Conjecture 4.4.1. *Consider a knot $F : S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ such that its projection $f = p \circ F : S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ is a generic immersion with $L := L^+ \sqcup L^- \subset S^{4k-1}$ as the set of double points. Then, the formula*

$$\mathcal{H}_1(F) := \frac{1}{2} lk(L^+, L^-) + \frac{1}{6} E_1(f) + \frac{1}{3} \Omega_{S^{2k}}(f), \tag{4.4.1}$$

produces the Haefliger isomorphism $\mathcal{H}_1 : \ \pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) \xrightarrow{\approx} \mathbb{Z}$ (sending the Haefliger trefoil $\mathcal{T}$ to 1).

Moreover, similar to the case $k = 1$, there is an equivalent adjusted version of (4.4.1) given as follows. Let $\widetilde{L}$ be a copy of L sligtly shifted along the framing fr_1. Then

$$\mathcal{H}_1(F) = -\frac{1}{8} lk(L, \widetilde{L}) + \frac{1}{6} E_1(f) + \frac{1}{3} \Omega_{S^{2k}}(f), \tag{4.4.2}$$

where $L := L^+ \sqcup -L^-$ and $\widetilde{L} := \widetilde{L}^+ \sqcup -\widetilde{L}^-$ are defined using the adjusted Ekholm orientation, according to Remark 3.2.1(2).

Conjecture 4.4.2. *Consider a knot $F : S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ such that its projection $f = p \circ F : S^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ is a generic immersion with $L := L^+ \sqcup -L^- \subset S^{4k-1}$ as the set of double points and $\widehat{L} := \widehat{L}^+ \sqcup -\widehat{L}^-$, its shifted copy along the framing fr_2. Then, the formula*

$$\mathcal{H}_2(F) := -\frac{1}{8} lk(L, \widehat{L}) + \frac{1}{12} E_2(f) + \frac{1}{3} \Omega_{S^{2k}}(f), \tag{4.4.3}$$

produces the Haefliger isomorphism $\mathcal{H}_2 : \ \pi_0 Emb(S^{4k-1}, \mathbb{R}^{6k}) \xrightarrow{\approx} \mathbb{Z}$ (sending the Haefliger trefoil $\mathcal{T}$ to 1).

By the same argument as in Theorem 4.3.1, the three formulas (4.4.1), (4.4.2) and (4.4.3) are equivalent. Note that each term in these formulas is additive, thus both $\mathcal{H}_1(F)$ and $\mathcal{H}_2(F)$ are additive. Moreover, we get a similar justification of the formula (4.4.1) as given in Section 4.2 for $k = 1$. To be precise, we get the generalization of examples 1, 2, and 3 by following the same arguments given in Subsection 4.2.1 (but not examples 3' and 4 which are specific to cases $k = 1, 2, 4$ and $k = 1$, respectively). Furthermore, the arguments given in Subsections 4.2.2, 4.2.3 and 4.2.4 for the invariance under cobordisms and the third Reidemeister move, the compatibility with Sakai's Ling-Wang type formula as well as the cabling argument justifying the last term, generalize to higher dimensions with a little or no change. To be precise, in the cabling argument we need to take $[id, id]$ (instead of the Hopf fibration) as a rational generator of $\pi_{4k-1}S^{2k}$.

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