International Review of Mechanical Engineering (I.RE.M.E.), Vol. 7, N. 5
ISSN 1970 - 8734
July 2013

Transient Wave Propagation in Non-Homogeneous Viscoelastic Media

Shahin Nayyeri Amiri, Asad Esmaeily

Abstract – *Propagation of transient pressure waves in nonhomogeneous viscoelastic media with a cylindrical hole of circular cross section is investigated by employing the theory of propagating surfaces of discontinuities. The non-homogeneities are assumed to depend on the radial distance from the axis of the cylindrical hole. The solutions for the normal stress components and the radial particle velocity are expressed as Taylor series expansions about the time of arrival of the wave front. Two types of boundary conditions are considered. The wall of the cylindrical hole is either subjected to uniform pressure or to uniform radial particle velocity both of which have arbitrary dependence on time. Then the solutions are reduced to the special case of homogeneous viscoelastic media. Numerical computations are carried out for a homogeneous standard linear solid and for a uniform pressure with a step distribution in time applied at the wall of the hole. These numerical results are compared with those obtained previously by other investigators who have employed the method of characteristics.*

Keywords: *Transient Wave Propagation, Non Homogeneous, Viscoelastic Media*

Nomenclature

u	Displacement vector
ε	Infinitesimal strain
σ	Cauchy stress tensor
$G_1(r,t)$	Shear relaxation functions
$G_2(r,t)$	Bulk relaxation functions
σ'_{ij}	Stress deviators
ε'_{ij}	Strain deviators
δ_{ij}	Kronecker delta
ρ	Mass density material
t	Time

I. Introduction

The propagation of transient waves in nonhomogeneous elastic media has recently attracted the attention of many researchers. Sternberg and Chakravorty [1] investigated the propagation of shock waves in a nonhomogeneous isotropic plate of infinite extent with a cylindrical hole at the wall of which uniform shearing tractions were suddenly applied and there, after steadily maintained.

The Laplace transform technique was employed in their research in obtaining the solution. Later, Chou and Schaller [2] applied the method of characteristics to solve the same problem. The latter method, of course, involved numerical integration and was suitable for wider class of non-homogeneities.

Reddy and Marietta [3] investigated an analogous problem of radial propagation of axial shear waves in a nonhomogeneous elastic medium with a cylindrical cavity by employing the theory of propagating surfaces of discontinuity. Longitudinal wave propagation in non-homogenous elastic rods of finite and semi-infinite lengths was also investigated in considerable detail by several authors [4]-[7].

Transient wave propagation in nonhomogeneous viscoelastic media has received less attention. An important contribution in this field was made by Sun [8] who examined axi-symmetrical transient rotary shear waves in nonhomogeneous viscoelastic media with a cylindrical hole.

Sun [8] employed the theory of propagating surfaces of discontinuity which, previously, was applied by Achenbach and Reddy [9], Reddy and Marietta [3] to investigate the longitudinal wave propagation in a homogeneous semi-infinite viscoelastic rod, in a nonhomogeneous semi-infinite elastic rod and the axial shear wave propagation in a nonhomogeneous elastic medium, respectively.

The work of Avtar Singh and Kishan Chand Gupt [10] who used studied the propagation of one – dimensional stress discontinuities in one-dimensional propagation of discontinuities in non-homogeneous linear viscoelastic semi-infinite media should also be mentioned.

In this study, the propagation of transient cylindrical pressure waves in a nonhomogeneous viscoelastic medium with a cylindrical cavity is investigated.

The theory of propagating surfaces of discontinuity, as used by Sun [8] and others such as Achenbach and

Manuscript received and revised June 2013, accepted July 2013

Reddy [9] is employed. The solutions for the radial and circumferential stresses and the radial particle velocity are expanded as Taylor series about the time of arrival of the wave front. Two types of boundary conditions are considered. The wall of the cylindrical hole is either subjected to uniform pressure or to uniform radial particle velocity both of which have arbitrary dependence on time. By employing the equation of motion, the stress-strain relations and the kinematical condition of compatibility, the coefficients of the Taylor series for the radial particle velocity are obtained as the solutions of linear ordinary differential equations of the first order.

Using the solution for the radial particle velocity together with the constitutive equations and the kinematical condition of compatibility, the solutions for the radial and circumferential stresses are computed.

The solutions are reduced to the special case of homogeneous viscoelastic media by disregarding the nonhomogeneous. In the numerical examples, for purposes of comparison, stress prescribed boundary condition is considered and the time variation of the pressure applied on the wall of the cylindrical cavity is assumed to be stepwise. Furthermore, the homogeneous viscoelastic medium is modeled as a standard linear solid.

In the special case of the homogeneous viscoelastic medium, the wave profiles for radial particle velocity and radial stress distributions at two stations are obtained and are compared with those of MeNiven and Mengi [11] who employed the method of characteristics in their analyses. Good agreement is found over the time range that is close to the time of arrival of the wave front by taking only a few terms in the Taylor series expansions.

II. Formulation of the Problem

We consider a nonhomogeneous, linearly viscoelastic infinite medium with an infinitely long circular cylindrical cavity of radius a. The medium is initially at rest and the material properties are assumed to depend solely on the radial distance from the center of the cavity. A uniform pressure or a uniform radial particle velocity with arbitrary dependence on time is applied on the wall of the cylindrical hole.

The body is referred to a cylindrical coordinate system (r,θ,z) in which the z axis coincides with the axis of the hole. In the development that follows, indicial notation and all of the rules associated with it are used when it is appropriate. Due to the axi -symmetry of the problem, the displacement field can be written as:

$$u_r = u_r(r,t); \quad u_\theta = 0, \quad u_z = 0 \tag{1}$$

Using the strain – displacement relations in a cylindrical coordinate system, we get:

$$\varepsilon_{rr} = u_{r,r}; \ \varepsilon_{\theta\theta} = \frac{u_r}{r}; \ \varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0 \tag{2}$$

If the medium is at rest prior to $t=0$, the stress-strain relations for a linear isotropic non-homogeneous viscoelastic material can be expressed as

$$\sigma'_{ij}(r,t) = G_{10}\varepsilon'_{ij}(r,t) + \int_0^t G_1^{(1)}(r,t-\tau)\varepsilon'_{ij}(r,\tau)\,d\tau \tag{3}$$

$$\sigma_{kk} = G_{20}\varepsilon_{kk}(r,t) + \int_0^t G_2^{(1)}(r,t-\tau)\varepsilon_{kk}(r,\tau)\,d\tau \tag{4}$$

where $G_1(r,t)$ and $G_2(r,t)$ are the shear and bulk relaxation functions, respectively, and $\sigma'_{ij},\varepsilon'_{ij}$ are the components of the stress and strain deviators defined as:

$$\begin{aligned} \sigma'_{ij} &= \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \\ \varepsilon'_{ij} &= \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \end{aligned} \tag{5}$$

where δ_{ij} is the Kronecker delta. From Eq. (2) the second of Eqs. (5), we can write:

$$\begin{aligned} \varepsilon'_{rr} &= \frac{2}{3}\frac{\partial u_r}{\partial r} - \frac{1}{3}\frac{u_r}{r} \\ \varepsilon'_{\theta\theta} &= \frac{2}{3}\frac{u_r}{r} - \frac{1}{3}\frac{\partial u_r}{\partial r} \\ \varepsilon_{kk} &= \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \end{aligned} \tag{6}$$

In Eqs. (3)-(4) we also define:

$$\begin{aligned} & G_{10} = G_1(r,0)\ ;\ G_1^{(n)}(r,t) = \frac{\partial^n G_1(r,t)}{\partial t^n} \\ & G_{10}^{(n)} = \left.\frac{\partial^n G_1(r,t)}{\partial t^n}\right|_{t=0} \\ & G_{20} = G_2(r,0)\ ;\ G_2^{(n)}(r,t) = \frac{\partial^n G_2(r,t)}{\partial t^n} \\ & G_{20}^{(n)} = \left.\frac{\partial^n G_2(r,t)}{\partial t^n}\right|_{t=0} \end{aligned} \tag{7}$$

In view of Eqs. (1)-(4), it is clear that only $\sigma_{rr},\sigma_{\theta\theta}$ and σ_{zz} are non-vanishing and that they are functions of r and t only. Then, the stress equation of motion becomes:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho\frac{\partial^2 u_r}{\partial t^2} \tag{8}$$

where $\rho(r)$ is the mass density of the medium. In terms of stress deviators Eq. (8) can be written as:

$$\frac{\partial \sigma'_{rr}}{\partial r}+\frac{1}{3}\frac{\partial \sigma_{kk}}{\partial r}+\frac{\sigma'_{rr}-\sigma'_{\theta\theta}}{r}=\rho\frac{\partial^2 u_r}{\partial t^2} \tag{9}$$

The other two stress equations of motion are satisfied identically. In this study, two types of boundary conditions are considered:

a) At time $t=0$ a time – dependent uniform pressure is applied at $r=a$, i.e. on the surface of the cylindrical hole, and it is assumed that this pressure can be represented by a Maclaurin series:

$$\sigma_{rr}\left(a,t\right)=\sum_{n=0}^{\infty}\sigma_n\frac{t_n}{n!} \tag{10a}$$

b) The radial particle velocity at $r=a$, $v\left(a,t\right)$, is prescribed and can be expanded as:

$$v\left(a,t\right)=\sum_{n=0}^{\infty}v_n\frac{t_n}{n!} \tag{10b}$$

In Eq. (10b), we have used the notation $v=\partial u_r/\partial t$. As for the initial condition, the medium is assumed to be at rest prior to $t=0$.

Thus, the problem, which is completely described now, is to determine the solutions of Eqs. (8) and (2)-(4) subject to quiescent initial conditions and boundary condition as give by Eqs. (10a), (10b). It should be noted that the constitutive Eqs. (3)-(4) are expressed in terms of the components of the stress and strain deviators for simplicity.

The solutions, however, will be found for the stress components $\sigma_{rr},\sigma_{\theta\theta}$ and σ_{zz}.

III. Solution of the Problem

The theory of propagating surfaces of discontinuity will be employed here in obtaining the solutions. A basic equation in the study of propagating discontinuities is the kinematical condition of compatibility which is discussed in general by Thomas [12]. Consider a function $f\left(r,t\right)$ which is discontinuous and has radial discontinuous derivatives across the wave front that moves in the radial direction with velocity c.

The kinematical condition of compatibility for this function takes the form:

$$\frac{d_D}{dt}\left[f\right]=\left[\frac{\partial f}{\partial t}\right]+c\left[\frac{\partial f}{\partial r}\right] \tag{11}$$

where finite jumps across the wave front are denoted by square brackets and the notation d_D/dt is introduced for the time – rate of change of a quantity as observed by an observer who moves with the propagating surface.

In this paper, it is assumed that the displacement remains continuous throughout the process and therefore:

$$\left[u_r\right]=0 \tag{12}$$

Across the wave front. If the kinematical condition of compatibility given by Eq. (11) is applied to the displacement u_r, with Eq. (12) in consideration, we obtain:

$$\left[\frac{\partial u_r}{\partial r}\right]=-\frac{1}{c}\left[\frac{\partial u_r}{\partial t}\right] \tag{13}$$

Conservation of liner momentum, which is discussed in a general form by Thomas [12], can be expressed for the present problem in the form:

$$\left[\sigma_{rr}\right]=-\rho c\left[\frac{\partial u_r}{\partial t}\right] \tag{14}$$

In terms of the components of the stress deviator, Eq. (14) can be written as:

$$\left[\sigma'_{rr}\right]+\frac{1}{3}\left[\sigma_{kk}\right]=-\rho c\left[\frac{\partial u_r}{\partial t}\right] \tag{15}$$

Since the integrals in Eqs. (3) and (4) are continuous at the wave front, we have the relations:

$$\begin{aligned}&\left[\sigma'_{rr}\right]=G_{10}\left[\varepsilon'_{rr}\right];\left[\sigma'_{\theta\theta}\right]=G_{10}\left[\varepsilon'_{\theta\theta}\right]\\&\left[\sigma_{kk}\right]=G_{20}\left[\varepsilon_{kk}\right]\end{aligned} \tag{16}$$

Now, substituting Eqs. (16) together with Eqs. (6) into Eq. (15) and taking into account Eq. (12), we get:

$$\frac{\left(2G_{10}+G_{20}\right)}{3}\left[\frac{\partial u_r}{\partial r}\right]=-\rho c\left[\frac{\partial u_r}{\partial t}\right] \tag{17}$$

Comparing Eqs. (13) and (17), we obtain:

$$c^2=\frac{2G_{10}+G_{20}}{3\rho} \tag{18}$$

Thus, the wave front propagates with a velocity which, depends on G_{10},G_{20} and the mass density, and may very as it penetrates into the medium.

At a cylindrical surface defined by a fixed r, the material is at rest until the wave front arrives. The time it takes for the wave front to arrive at this position can be computed as:

$$t=\phi\left(r\right)=\int_a^r\frac{dr}{c} \tag{19}$$

where a is the radius of the cylindrical cavity and c is the wave velocity given by Eq. (18).

Now following Sun [8], and also [9], we seek the solutions for the radial particle and the stress components $\sigma_{rr}, \sigma_{\theta\theta}$ and σ_{zz} as Taylor's expansions about the time of arrival of the wave front:

$$v(r,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\{t-\phi(r)\}^n\left[\frac{\partial^n v}{\partial t^n}\right]_{t=\phi(r)} \quad t\ge\phi(r) \quad (20)$$

$$\sigma_{rr}(r,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\{t-\phi(r)\}^n\left[\frac{\partial^n \sigma_{rr}}{\partial t^n}\right]_{t=\phi(r)} \quad t\ge\phi(r) \quad (21)$$

$$\sigma_{\theta\theta}(r,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\{t-\phi(r)\}^n\left[\frac{\partial^n \sigma_{\theta\theta}}{\partial t^n}\right]_{t=\phi(r)} \quad t\ge\phi(r) \quad (22)$$

$$\sigma_{zz}(r,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\{t-\phi(r)\}^n\left[\frac{\partial^n \sigma_{zz}}{\partial t^n}\right]_{t=\phi(r)} \quad t\ge\phi(r) \quad (23)$$

The coefficients of expansions in Eqs. (20)-(23) represent propagating discontinuities and the problem reduces to the determination of these coefficients.

We shall first find the solutions for the coefficients in the expansion for radial particle velocity given by Eq. (20).

Then, using the solutions for these coefficients together with the constitutive equations and the kinematical condition of compatibility, the coefficients of the expansions given in Eqs. (21)-(23) will be computed.

For this purpose, we first differentiate Eqs. (3) and (4) $n+p$ times with respect to time to obtain:

$$\frac{\partial^{n+p}\sigma'_{rr}}{\partial t^{n+p}}=G_{10}\frac{\partial^{n+p}\varepsilon'_{rr}}{\partial t^{n+p}}+\sum_{i=1}^{n+p}G_{10}^{(i)}\frac{\partial^{n+p-i}\varepsilon'_{rr}}{\partial t^{n+p-i}}+ +\int_0^t G_1^{(n+p)}(r,t-\tau)\varepsilon'_{rr}d\tau \quad (24)$$

$$\frac{\partial^{n+p}\sigma'_{\theta\theta}}{\partial t^{n+p}}=G_{10}\frac{\partial^{n+p}\varepsilon'_{\theta\theta}}{\partial t^{n+p}}+\sum_{i=1}^{n+p}G_{10}^{(i)}\frac{\partial^{n+p-i}\varepsilon'_{\theta\theta}}{\partial t^{n+p-i}}+ +\int_0^t G_1^{(n+p)}(r,t-\tau)\varepsilon'_{\theta\theta}d\tau \quad (25)$$

$$\frac{\partial^{n+p}\sigma_{kk}}{\partial t^{n+p}}=G_{20}\frac{\partial^{n+p}\varepsilon_{kk}}{\partial t^{n+p}}+\sum_{i=1}^{n+p}G_{20}^{(i)}\frac{\partial^{n+p-i}\varepsilon_{kk}}{\partial t^{n+p-i}}+ +\int_0^t G_2^{(n+p)}(r,t-\tau)\varepsilon'_{kk}d\tau, \quad (26)$$

where for p we shall consider only the values 0 and 1.

Since the integrals in Eqs. (24)-(26) are continuous at the wave front, we have the following relations between the discontinuities:

$$\left[\frac{\partial^{n+p}\sigma'_{rr}}{\partial t^{n+p}}\right]=G_{10}\left[\frac{\partial^{n+p}\varepsilon'_{rr}}{\partial t^{n+p}}\right]+\sum_{i=1}^{n+p}G_{10}^{(i)}\left[\frac{\partial^{n+p-i}\varepsilon'_{rr}}{\partial t^{n+p-i}}\right] \quad (27)$$

$$\left[\frac{\partial^{n+p}\sigma'_{\theta\theta}}{\partial t^{n+p}}\right]=G_{10}\left[\frac{\partial^{n+p}\varepsilon'_{\theta\theta}}{\partial t^{n+p}}\right]+\sum_{i=1}^{n+p}G_{10}^{(i)}\left[\frac{\partial^{n+p-i}\varepsilon'_{\theta\theta}}{\partial t^{n+p-i}}\right] \quad (28)$$

$$\left[\frac{\partial^{n+p}\sigma_{kk}}{\partial t^{n+p}}\right]=G_{20}\left[\frac{\partial^{n+p}\varepsilon_{kk}}{\partial t^{n+p}}\right]+\sum_{i=1}^{n+p}G_{20}^{(i)}\left[\frac{\partial^{n+p-i}\varepsilon_{kk}}{\partial t^{n+p-i}}\right] \quad (29)$$

From Eq. (9) we have:

$$\left[\frac{\partial\sigma'_{rr}}{\partial r}\right]+\frac{1}{3}\left[\frac{\partial\sigma_{kk}}{\partial r}\right]+\frac{1}{r}[\sigma'_{rr}]-\frac{1}{r}[\sigma'_{\theta\theta}]=\rho\left[\frac{\partial^2 u_r}{\partial t^2}\right] \quad (30)$$

By writing the kinematical condition of compatibility given by Eq. (11) for $\sigma'_{rr}, \sigma_{kk}$ and $\partial u_r/\partial t$, we get:

$$\frac{d_D}{dt}[\sigma'_{rr}]=\left[\frac{\partial\sigma'_{rr}}{\partial t}\right]+c\left[\frac{\partial\sigma'_{rr}}{\partial r}\right] \quad (31)$$

$$\frac{d_D}{dt}[\sigma_{kk}]=\left[\frac{\partial\sigma_{kk}}{\partial t}\right]+c\left[\frac{\partial\sigma_{kk}}{\partial r}\right] \quad (32)$$

$$\frac{d_D}{dt}\left[\frac{\partial u_r}{\partial t}\right]=\left[\frac{\partial^2 u_r}{\partial t^2}\right]+c\left[\frac{\partial^2 u_r}{\partial r\partial t}\right] \quad (33)$$

Employing relations (31)-(32) in Eq. (30) and rearranging the terms, we get:

$$\frac{1}{c}\frac{d_D}{dt}\left\{[\sigma'_{rr}]+\frac{1}{3}[\sigma_{kk}]\right\}-\frac{1}{c}\left[\frac{\partial\sigma'_{rr}}{\partial t}\right]-\frac{1}{3c}\left[\frac{\partial\sigma_{kk}}{\partial t}\right]+ +\frac{1}{r}[\sigma'_{rr}]-\frac{1}{r}[\sigma'_{\theta\theta}]=\rho\left[\frac{\partial^2 u_r}{\partial t^2}\right] \quad (34)$$

For $n=0$ and $p=1$, Eqs. (27) and (29) become:

$$\left[\frac{\partial\sigma'_{rr}}{\partial t}\right]=G_{10}\left[\frac{\partial\varepsilon'_{rr}}{\partial t}\right]+G_{10}^{(1)}[\varepsilon'_{rr}] \quad (35)$$

$$\left[\frac{\partial\sigma_{kk}}{\partial t}\right]=G_{20}\left[\frac{\partial\varepsilon_{kk}}{\partial t}\right]+G_{20}^{(1)}[\varepsilon_{kk}] \quad (36)$$

From Eqs. (6) and their time derivatives, the following relations between the discontinuities can be written:

$$\left[\varepsilon'_{rr}\right]=\frac{2}{3}\left[\frac{\partial u_r}{\partial r}\right];\ \left[\varepsilon'_{\theta\theta}\right]=-\frac{1}{3}\left[\frac{\partial u_r}{\partial r}\right] \quad (37)$$
$$\left[\varepsilon_{kk}\right]=\left[\frac{\partial u_r}{\partial r}\right]$$

$$\left[\frac{\partial\varepsilon'_{rr}}{\partial t}\right]=\frac{2}{3}\left[\frac{\partial^2 u_r}{\partial r\partial t}\right]-\frac{1}{3r}\left[\frac{\partial u_r}{\partial t}\right] \quad (38)$$

$$\left[\frac{\partial\varepsilon_{kk}}{\partial t}\right]=\left[\frac{\partial^2 u_r}{\partial r\partial t}\right]+\frac{1}{r}\left[\frac{\partial u_r}{\partial t}\right] \quad (39)$$

In writing Eqs. (37), Eq. (12) is taken into account.

Now substituting Eqs. (15), (16), (35)-(36) into Eq. (34) and then employing Eqs. (37)-(39), we obtain:

$$\begin{aligned}&-\rho\frac{d_D}{dt}\left[\frac{\partial u_r}{\partial t}\right]-\frac{1}{c}\frac{d_D}{dt}(\rho c)\left[\frac{\partial u_r}{\partial t}\right]+\\&-\frac{2G_{10}+G_{20}}{3c}\left[\frac{\partial^2 u_r}{\partial r\partial t}\right]-\frac{2G_{10}^{(1)}+G_{20}^{(1)}}{3c}\left[\frac{\partial u_r}{\partial r}\right]+\\&+\frac{G_{10}}{3cr}\left[\frac{\partial u_r}{\partial t}\right]-\frac{G_{20}}{3cr}\left[\frac{\partial u_r}{\partial t}\right]+\frac{G_{10}}{r}\left[\frac{\partial u_r}{\partial r}\right]=\rho\left[\frac{\partial^2 u_r}{\partial t^2}\right]\end{aligned} \quad (40)$$

Now, using Eqs. (13), (33) and (18) in Eq. (40), we get, after some manipulation:

$$\frac{d_D}{dt}\left[\frac{\partial u_r}{\partial t}\right]+\alpha(t)\left[\frac{\partial u_r}{\partial t}\right]=0 \quad (41)$$

where:

$$\alpha(t)=\left\{\frac{c}{2r}+\frac{1}{2\rho c}\frac{d_D}{dt}(\rho c)-\frac{m_1}{2}\right\} \quad (42)$$

and:

$$m_1=\left\{\frac{2G_{10}^{(1)}+G_{20}^{(1)}}{2G_{10}+G_{20}}\right\} \quad (43)$$

Eq. (41) is a liner ordinary differential equation of the first order whose solution give $[v]=[\partial u_r/\partial t]$, the coefficient corresponding to $n=0$ in the Taylor's expansion in Eq. (20). To determine the coefficients for $n\geq 1$, we differentiate the equation of motion (9) with respect to time for n times to obtain:

$$\begin{aligned}&\left[\frac{\partial^{n+1}\sigma'_{rr}}{\partial r\partial t^n}\right]+\frac{1}{3}\left[\frac{\partial^{n+1}\sigma_{kk}}{\partial r\partial t^n}\right]+\frac{1}{r}\left[\frac{\partial^n\sigma'_{rr}}{\partial t^n}\right]+\\&-\frac{1}{r}\left[\frac{\partial^n\sigma'_{\theta\theta}}{\partial t^n}\right]=\rho\left[\frac{\partial^{n+2}u_r}{\partial t^{n+2}}\right]\end{aligned} \quad (44)$$

By writing the kinematical condition of compatibility for $\partial^n\sigma'_{rr}/\partial t^n$ and $\partial^n\sigma_{kk}/\partial t^n$, and employing these relations in Eq. (44), we obtain:

$$\begin{aligned}&\frac{1}{c}\frac{d_D}{dt}\left[\frac{\partial^n\sigma'_{rr}}{\partial t^n}\right]-\frac{1}{c}\left[\frac{\partial^{n+1}\sigma'_{rr}}{\partial t^{n+1}}\right]+\frac{1}{3c}\frac{d_D}{dt}\left[\frac{\partial^n\sigma_{kk}}{\partial t^n}\right]+\\&-\frac{1}{3c}\left[\frac{\partial^{n+1}\sigma_{kk}}{\partial t^{n+1}}\right]+\frac{1}{r}\left[\frac{\partial^n\sigma'_{rr}}{\partial t^n}\right]-\frac{1}{r}\left[\frac{\partial^n\sigma'_{\theta\theta}}{\partial t^n}\right]=\\&=\rho\left[\frac{\partial^{n+2}u_r}{\partial t^{n+2}}\right]\end{aligned} \quad (45)$$

Differentiation of the strain – displacement relations (6) with respect to time for $n+p$ times yields between the discontinuities the following relations:

$$\left[\frac{\partial^{n+p}\varepsilon'_{rr}}{\partial t^{n+p}}\right]=\frac{2}{3}\left[\frac{\partial^{n+p+1}u_r}{\partial r\partial t^{n+p}}\right]-\frac{1}{3r}\left[\frac{\partial^{n+p}u_r}{\partial t^{n+p}}\right] \quad (46)$$

$$\left[\frac{\partial^{n+p}\varepsilon'_{\theta\theta}}{\partial t^{n+p}}\right]=\frac{2}{3r}\left[\frac{\partial^{n+p}u_r}{\partial t^{n+p}}\right]-\frac{1}{3}\left[\frac{\partial^{n+p+1}u_r}{\partial r\partial t^{n+p}}\right] \quad (47)$$

$$\left[\frac{\partial^{n+p}\varepsilon_{kk}}{\partial t^{n+p}}\right]=\left[\frac{\partial^{n+p+1}u_r}{\partial r\partial t^{n+p}}\right]+\frac{1}{r}\left[\frac{\partial^{n+p}u_r}{\partial t^{n+p}}\right] \quad (48)$$

where, again, only the values 0 and 1 will be considered for p.

Applying the kinematical condition of compatibility given by Eq. (11) to $\partial^{n+p}u_r/\partial t^{n+p}$ and $\partial^{n+p-i}u_r/\partial t^{n+p-i}$, we obtain:

$$\frac{d_D}{dt}\left[\frac{\partial^{n+p}u_r}{\partial t^{n+p}}\right]=\left[\frac{\partial^{n+p+1}u_r}{\partial t^{n+p+1}}\right]+c\left[\frac{\partial^{n+p+1}u_r}{\partial r\partial t^{n+p}}\right] \quad (49)$$

$$\frac{d_D}{dt}\left[\frac{\partial^{n+p-i}u_r}{\partial t^{n+p-i}}\right]=\left[\frac{\partial^{n+p+1-i}u_r}{\partial t^{n+p+1-i}}\right]+c\left[\frac{\partial^{n+p+1-i}u_r}{\partial r\partial t^{n+p-i}}\right] \quad (50)$$

Now, substituting Eqs. (27-29) into Eq. (45) and then employing first Eqs. (46)-(48) then Eqs. (49)-(50) and (18) with the values 0 or 1 chosen appropriately for p, we get, after some manipulations:

$$\frac{d_D}{dt}\left[\frac{\partial^n v}{\partial t^n}\right]+\alpha(t)\left[\frac{\partial^n v}{\partial t^n}\right]=F_n(t) \quad (51)$$
$$n\geq 1$$

where $\alpha(t)$ is defined by Eq. (42) and:

$$F_n(t)=\frac{1}{2}\frac{d^2{}_D}{dt^2}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\alpha(t)\frac{d_D}{dt}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+$$
$$+\begin{Bmatrix}\frac{1}{2r}\left(\frac{c}{2G_{10}+G_{20}}\right)\left(\frac{d_D}{dt}(G_{20}-G_{10})\right)+\\-\frac{c^2}{2r^2}-\frac{m_1c}{2r}\end{Bmatrix}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+$$
$$+\sum_{i=1}^{n}\begin{Bmatrix}\frac{m_i}{2}\frac{d^2{}_D}{dt^2}\left[\frac{\partial^{n-i-1}v}{\partial t^{n-1-i}}\right]+\\+\left(\frac{1}{2\rho c}\frac{d_D}{dt}(m_ic\rho)+\frac{m_ic}{2r}\right)\frac{d_D}{dt}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]+\\-\begin{pmatrix}\frac{1}{2}\frac{m_ic^2}{r^2}-\frac{1}{2r}\left(\frac{c}{2G_{10}+G_{20}}\right)\\\left(\frac{d_D}{dt}\left(G_{20}^{(i)}-G_{10}^{(i)}\right)\right)\end{pmatrix}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]\end{Bmatrix}+\tag{52}$$
$$-\sum_{i=2}^{n+1}\begin{Bmatrix}m_i\frac{d_D}{dt}\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]+\left(\frac{1}{2\rho c}\frac{d_D}{dt}(c\rho)+\frac{m_ic}{2r}\right)\\\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]-\frac{m_i}{2}\left[\frac{\partial^{n+1-i}v}{\partial t^{n+1-i}}\right]\end{Bmatrix}$$

In eq. (52), m_i is defined as:

$$m_i=\left\{\frac{2G_{10}^{(i)}+G_{20}^{(i)}}{2G_{10}+G_{20}}\right\}\tag{53}$$

and m_1 is given by Eq. (43). It should be noted once again that in writing Eqs. (51) and (52), the relation $v=\partial u_r/\partial t$ is used. If we define $F_n(t)=0$ for $n=0$, the general solutions for the linear ordinary differential Eqs. (41) and (51) can be written in a single expression as:

$$\left[\frac{\partial^n v}{\partial t^n}\right]=e^{-\beta(t)}\left\{\int_0^t F_n(s)e^{\beta(s)}ds+A_n\right\}\qquad n\geq 0\tag{54}$$

where:

$$\beta(t)=\int_0^t\alpha(s)ds\tag{55}$$

and $A_n(n\geq 0,)$ are integration constants to be determined from the boundary conditions.

Before we determine the integration constants, A_n, we shall proceed to determine, $\left[\partial^n\sigma_{rr}/\partial t^n\right]$, the coefficients of the Taylor's expansion for radial as give by Eq. (21).

Since the solutions for the coefficients, $\left[\partial^n v/\partial t^n\right]$, have been obtained, Eq. (54), we shall attempt to express $\left[\partial^n\sigma_{rr}/\partial t^n\right]$ in terms of $\left[\partial^n v/\partial t^n\right]$ and their time derivatives. Form Eq. (14), we have:

$$[\sigma_{rr}]=-\rho c[v]\tag{56}$$

where, of course, the relation $v=\partial u_r/\partial t$ is used.

For $n\geq 1$, we make use of the constitutive equation for the radial stress written at the wave front, Eq. (27). Employing Eqs. (46)-(48) in Eq. (27), with p taken as 0, and considering Eqs. (5) together with eq. (29), we find:

$$\begin{aligned}\left[\frac{\partial^n\sigma_{rr}}{\partial t^n}\right]&=\frac{G_{20}}{3}\left\{\left[\frac{\partial^{n+1}u_r}{\partial r\partial t^n}\right]+\frac{1}{r}\left[\frac{\partial^n u_r}{\partial t^n}\right]\right\}+\\&+\sum_{i=1}^{n}\frac{G_{20}^{(i)}}{3}\left\{\left[\frac{\partial^{n+1-i}u_r}{\partial r\partial t^{n-i}}\right]+\frac{1}{r}\left[\frac{\partial^{n-i}u_r}{\partial t^{n-i}}\right]\right\}+\\&+G_{10}\left[\left[\frac{\partial^{n+1-i}u_r}{\partial r\partial t^n}\right]\right]-\frac{G_{10}}{3}\left\{\left[\frac{\partial^{n+1}u_r}{\partial r\partial t^n}\right]+\frac{1}{r}\left[\frac{\partial^n u_r}{\partial t^n}\right]\right\}+\\&+\sum_{i=1}^{n}G_{10}^{(i)}\left\{\left[\frac{\partial^{n+1-i}u_r}{\partial r\partial t^{n-i}}\right]-\frac{1}{3}\left(\left[\frac{\partial^{n+1-i}u_r}{\partial r\partial t^{n-i}}\right]+\frac{1}{r}\left[\frac{\partial^{n-i}u_r}{\partial t^{n-i}}\right]\right)\right\}\end{aligned}\tag{57}$$

Now, applying Eqs. (49) and (50), with $p=0$, in Eq. (57) and using the relation $v=\partial u_r/\partial t$, we get:

$$\begin{aligned}\left[\frac{\partial^n\sigma_{rr}}{\partial t^n}\right]&=c\rho\frac{d_D}{dt}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]-c\rho\left[\frac{\partial^n v}{\partial t^n}\right]+\\&+\frac{1}{3r}(G_{20}-G_{10})\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\\&+\sum_{i=1}^{n}\begin{Bmatrix}c\rho m_i\left(\frac{d_D}{dt}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]-\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]\right)+\\+\frac{1}{3r}\left(G_{20}^{(i)}-G_{10}^{(i)}\right)\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]\end{Bmatrix}\end{aligned}\tag{58}$$

In view of Eqs. (56) and (58), the Taylor's expansion for the radial stress σ_{rr} can be written as:

$$\begin{aligned}\sigma_{rr}(r,t)&=\sum_{n=0}^{\infty}\frac{1}{n!}\{t-\phi(r)\}^n\left\{c\rho\frac{d_D}{dt}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\right.\\&-c\rho\left[\frac{\partial^n v}{\partial t^n}\right]+\frac{1}{3r}(G_{20}-G_{10})\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\\&+\sum_{i=1}^{n}\left\{c\rho m_i\left(\frac{d_D}{dt}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]-\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]\right)+\right.\\&\left.\left.+\frac{1}{3r}\left(G_{20}^{(i)}-G_{10}^{(i)}\right)\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]\right\}\right\}_{t=\phi(r)}\qquad t\geq\phi(r)\end{aligned}\tag{59}$$

In which we set:

$$\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] \equiv 0 \quad \text{if} \quad n = 0 \tag{60}$$

The two types of boundary conditions as described by Eqs. (10a) and (10b) are now employed to determine the integration constants A_n which appear in the solutions through Eq. (54).

For type A of the boundary conditions the radial stress is prescribed at $r = a$.

Noting that the relation $t = \phi(r)$ at the wave front becomes $t = \phi(r) = 0$ at the surface of the cylindrical cavity, we compare in this case Eq. (59) with Eq. (10a) at $r = a$ to obtain:

$$\begin{aligned}\sigma_n = &\left\{ c\rho \frac{d_D}{dt}\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] - c\rho\left[\frac{\partial^n v}{\partial t^n}\right] + \right.\\ &+ \frac{1}{3r}\left(G_{20} - G_{10}\right)\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] + \\ &+ \sum_{i=1}^{n}\left\{ c\rho m_i \left(\frac{d_D}{dt}\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right] - \left[\frac{\partial^{n-i} v}{\partial t^{n-i}}\right]\right) + \right.\\ &\left.\left. + \frac{1}{3r}\left(G_{20}^{(i)} - G_{10}^{(i)}\right)\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right]\right\}\right\}_{t=0}\end{aligned} \tag{61}$$

From Eq. (54), we have:

$$A_n = \left[\frac{\partial^n v}{\partial t^n}\right]_{t=0} \tag{62}$$

Substituting Eq. (62) in Eq. (61) give:

$$\begin{aligned}A_n = &\left\{ \frac{d_D}{dt}\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] - \frac{\sigma_n}{c\rho} + \frac{c}{r}\left(\frac{G_{20} - G_{10}}{2G_{10} + G_{20}}\right)\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] + \right.\\ &+ \sum_{i=1}^{n}\left\{ m_i \left\{\frac{d_D}{dt}\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right] - \left[\frac{\partial^{n-i} v}{\partial t^{n-i}}\right]\right. + \right.\\ &\left.\left. + \frac{c}{r}\left(\frac{G_{20}^{(i)} - G_{10}^{(i)}}{2G_{10} + G_{20}}\right)\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right]\right\}\right\}_{t=0} \qquad n \geq 0\end{aligned} \tag{63}$$

It is noted that Eq. (60) should be considered when Eq. (63) is used to compute A_0. It is observed from eq. (54) together with Eq. (52) that the solution for $\left[\partial^{n-1} v / \partial t^{n-1}\right]$ involves only the constants $A_i\left(i \leq n-1\right)$.

Therefore, all the constants A_n can be determined successively from Eq. (63) once the first constant A_0 is known. This completes the solutions for the radial particle velocity and radial stress when boundary conditions of type A are considered.

For type B of the boundary conditions the radial particle velocity is prescribed at the cylindrical surface $r = a$ as given by Eq. (10b).

A comparison between Eq. (10b) and Eq. (20) in conjunction with Eq. (54) yields:

$$A_n = v_n \tag{64}$$

The coefficients of the Taylor's expansions for the stress components $\sigma_{\theta\theta}$ and σ_{zz} as given by Eqs. (22)-(23) can be computed in terms of $\left[\partial^n v / \partial t^n\right]$ in an exactly analogous manner by employing the constitutive equations, the strain-displacement relations and the kinematical condition of compatibility. We give here the result for the circumferential stress $\sigma_{\theta\theta}$:

$$\begin{aligned}\sigma_{\theta\theta}(r,t) = &\sum_{n=0}^{\infty}\frac{1}{n!}\left\{t - \phi(r)\right\}^n\\ &\left\{\left(\frac{2G_{10} + G_{20}}{3r}\right)\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] + \right.\\ &+ \frac{1}{3c}\left(G_{20} - G_{10}\right)\left(\frac{d_D}{dt}\left[\frac{\partial^{n-1} v}{\partial t^{n-1}}\right] - \left[\frac{\partial^n v}{\partial t^n}\right]\right) + \\ &+ \sum_{i=1}^{n}\left\{\frac{1}{3c}\left(G_{20}^{(i)} - G_{10}^{(i)}\right)\left(\frac{d_D}{dt}\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right] - \left[\frac{\partial^{n-i} v}{\partial t^{n-i}}\right]\right) + \right.\\ &\left.\left. + \frac{2\left(G_{10}^{(i)} - G_{20}^{(i)}\right)}{3r}\left[\frac{\partial^{n-1-i} v}{\partial t^{n-1-i}}\right]\right\}\right\}_{t=\phi(r)}\end{aligned} \tag{65}$$

For the special case of homogeneous viscoelastic medium the shear and bulk relaxation functions and the mass density are independent of the spatial coordinate r, i.e.:

$$G_1 = G_1(t)\ ;\ G_2 = G_2(t)\ ;\ \rho = const \tag{66}$$

The solutions for this special case can be derived from the solutions obtained above for the nonhomogeneous viscoelastic medium by considering relations (66). The propagation velocity becomes $c^2 = \left(2G_{10} + G_{20}\right)/3\rho =$ constant and the time it takes for the disturbance to arrive at a position r is computed as $t = \phi(r) = (r - a)/c$ in this case.

The radial particle velocity, the radial stress and the circumferential stress are computed Eqs. (20), (59) and (65), respectively, in view of Eqs (66). The coefficients $\left[\partial^n v / \partial t^n\right]$ have the same form as in equation (54), but $\beta(t)$ and $F_n(t)$ for this case are obtained from Eqs.

(55) and (42), and Eq. (52), respectively, in the forms:

$$\beta(t)=\int_0^t \frac{1}{2}\left(\frac{c}{r}-m_1\right)ds \tag{67}$$

$$\begin{aligned}F_n(t)&=\frac{1}{2}\frac{d^2{}_D}{dt^2}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\left(\frac{c}{2r}-m_1\right)\frac{d_D}{dt}\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\\&-\left(\frac{c^2}{2r^2}+\frac{m_1c}{2r}\right)\left[\frac{\partial^{n-1}v}{\partial t^{n-1}}\right]+\sum_{i=1}^{n}\left\{\frac{m_i}{2}\frac{d^2{}_D}{dt^2}\left[\frac{\partial^{n-i-1}v}{\partial t^{n-1-i}}\right]+\right.\\&\left.+\frac{m_ic}{2r}\frac{d_D}{dt}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]-\frac{m_ic^2}{2r^2}\left[\frac{\partial^{n-1-i}v}{\partial t^{n-1-i}}\right]\right\}+\\&+\sum_{i=2}^{n}\left\{-m_i\frac{d_D}{dt}\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]-\frac{m_ic}{2r}\left[\frac{\partial^{n-i}v}{\partial t^{n-i}}\right]+\right.\\&\left.+\frac{m_i}{2}\left[\frac{\partial^{n+1-i}v}{\partial t^{n+1-i}}\right]\right\}\end{aligned} \tag{68}$$

where m_1 and m_i are given by Eqs. (43) and (53), respectively, and they, of course, should be interpreted in view of Eqs. (66).

The integration constants A_n are determined from Eq. (63) or (64) depending upon whether stress or particle velocity is prescribed at the boundary surface with necessary considerations given for homogeneity.

For simplicity and for purposes of comparison with McNiven and Mengi [11], standard liner solid is chosen as the specific viscoelastic model and radial stress is prescribed at the wall the cylindrical cavity $r=a$.

The shear and bulk moduli for the standard liner solid are:

$$\begin{aligned}G_1(t)&=G_{1F}+(G_{10}-G_{1F})e^{-t/\tau_1}\\G_2(t)&=G_{2F}+(G_{20}-G_{2F})e^{-t/\tau_2}\end{aligned} \tag{69}$$

respectively. In Eqs. (69), the constants τ_1 and τ_2 are the relaxation times of the shear and bulk moduli, respectively, and:

$$\begin{aligned}&G_{1F}=G_1(\infty)\ ;\ G_{2F}=G_2(\infty)\ ;\ G_{10}=G_1(0)\\&G_{20}=G_2(0)\end{aligned} \tag{70}$$

For material constants and relaxation times, we take the same numerical values as considered by McNiven and Mengi [11] for what they called "material one".

These are:

$$\begin{aligned}&\frac{G_{1F}}{G_{10}}=0.40\ ;\ \frac{G_{20}}{G_{10}}=2.28571\\&\frac{G_{2F}}{G_{10}}=1.142852\ ;\ \frac{c\tau_1}{a}=3.0\ ;\ \frac{c\tau_2}{a}=5.0\end{aligned} \tag{71}$$

The coefficients $\left[\partial^n v/\partial t^n\right]$ of the Taylor's expansion for radial particle velocity as given by Eq. (20) can be obtained from Eq. (54) together with Eqs. (66)-(68). The integration constants A_n can be computed from eq. (63) with Eq. (66) taken in to account.

As for the boundary condition, we assume specifically that a radial pressure of magnitude P_0 is suddenly applied at $r=a$ and maintained constant thereafter.

The boundary condition, then, is given by eq. (10a) with:

$$\begin{aligned}&\sigma_0=-P_0\ ;\ \sigma^n=0\\&\text{for}\quad n\ge 1\end{aligned} \tag{72}$$

Now, to compute the coefficient $[v]$, we carry out the integration of Eq. (67) to obtain:

$$\beta(t)=ln\left(\frac{r}{a}\right)^{1/2}-\frac{m_1}{2c}(r-a) \tag{73}$$

where m_1 is given by Eq. (43), and the wave front $t=\phi(r)=(r-a)/c$ From Eq. (63) we obtain, in view of Eq. (72):

$$A_0=P_0/(\rho c) \tag{74}$$

For $n=0$, Eq. (54) gives:

$$[v]=A_0e^{-\beta(t)} \tag{75}$$

Substituting Eqs. (73) and (74) into Eq. (75) and using the numerical values given by Eq. (71), we get:

$$[v]=\left(\frac{a}{r}\right)^{1/2}\frac{P_0}{\rho c}e^{-0.0733\left(\frac{r}{a}-1\right)} \tag{76}$$

The integrations to be carried out in obtaining $\left[\partial^n v/\partial t^n\right]$ for $n\ge 1$ are simple and these coefficients can be calculated as described above in a straightforward manner without difficulty. We give here only the next two coefficients:

$$\left[\frac{\partial v}{\partial t}\right]=\left\{0.375\left(\frac{r}{a}\right)^{-3/2}-0.012826\left(\frac{r}{a}\right)^{1/2}+\right.\\\left.-0.4888\left(\frac{r}{a}\right)^{-1/2}\right\}\frac{cP_0}{a\rho c}e^{-0.0733\left(\frac{r}{a}-1\right)} \tag{77}$$

$$\left[\frac{\partial^2 v}{\partial t^2}\right] = \left\{ -0.117712\left(\frac{r}{a}\right)^{-5/2} - 0.21032\left(\frac{r}{a}\right)^{-3/2} + \right.$$
$$-0.00604\left(\frac{r}{a}\right)^{-1/2} ln\left(\frac{r}{a}\right) + 0.003376\left(\frac{r}{a}\right)^{1/2} +$$
$$+0.00018\left(\frac{r}{a}\right)^{3/2} + \tag{78}$$
$$\left. -0.011623\left(\frac{r}{a}\right)^{-1/2} \right\} \frac{c^2}{a^2}\frac{P_0}{\rho c} e^{-0.0733\left(\frac{r}{a}-1\right)}$$

In obtaining Eqs. (77) and (78), the numerical values given in Eq. (71) are employed.

The first seven coefficients thus computed, we have plotted the dimensionless radial velocity $\frac{v}{P_0/\rho c}$ against the dimensionless time $\frac{ct}{a}$ for two stations $\frac{r}{a}=1$ and $\frac{r}{a}=2.5$ in Figs. 1 and 2. For comparison, the solution obtained by MeNiven and Mengi [11] is also plotted. In the same figures. It is seen from the figures that the present solution with seven terms taken compares well with the solution of McNiven [11] over the time range that is close to the time of arrival of wave front.

In an analogous manner, the coefficients $\left[\partial^n \sigma_{rr} / \partial t^n\right]$ of the Taylor's expansion for radial stress can be calculated in terms of the already computed coefficients $\left[\partial^n v / \partial t^n\right]$ according to Eq. (58) which should be interpreted in view of Eqs. (66). The first seven coefficients thus found, the radial stress σ_{rr} is computed from Eq. (59), with Eqs. (66) taken into consideration, and the dimensionless radial stress σ_{rr}/P_0 is plotted against the dimensionless time $\frac{ct}{a}$ at station $\frac{r}{a}=2.5$, together with the solution obtained by McNiven and Mengi [11], in Fig. 3. It is found that the present 7-term solution agrees favorably with the solution of McNiven [11] up till the dimensionless time $\frac{ct}{a}=4$. As $\frac{ct}{a}$ increases, the discrepancy between the two solutions increases and the convergence of the series solution presented here slows down considerably. Thus for long times after the time of arrival of the wave front the present method loses its effectiveness and advantages.

IV. Conclusion

The theory of propagating surfaces of discontinuity was employed to study the propagation of transient pressure waves in an extended non-homogenous viscoelastic medium with a cylindrical cavity.

The solutions were obtained for viscoelastic materials with in homogeneities depending arbitrarily on the radial coordinate r, only, and satisfying, otherwise, the most general linear stress-strain relations. Both stress-prescribed and velocity-prescribed boundary conditions were considered. By disregarding the non-homogeneities, the solutions for the homogeneous viscoelastic medium were, then, obtained. The solutions for the special case of non-homogenous elastic media can also be found easily by neglecting the viscoelastic effects in the solutions already obtained for the non-homogenous viscoelastic media. It was shown that the method presented here yielded good results for short times after the time of arrival of the wave front.

This makes the method useful for transient wave propagation problems. However, over the time range that is far to the time of arrival of the wave front, the method loses its advantages.

The present method may also prove to be useful for solving nonlinear wave propagation problems in nonhomogeneous viscoelastic materials.

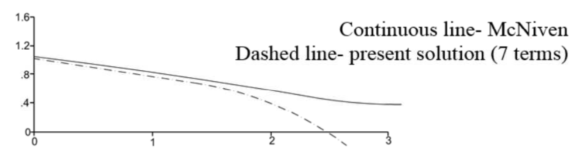

Fig. 1. Radial velocity as a function of a time at the station $r=a$

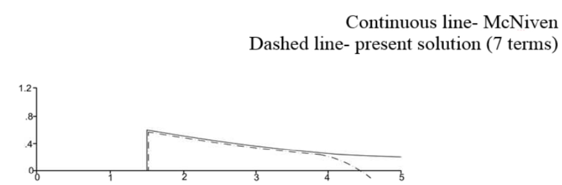

Fig. 2. Radial velocity as a function of time at the station $r=2.5a$

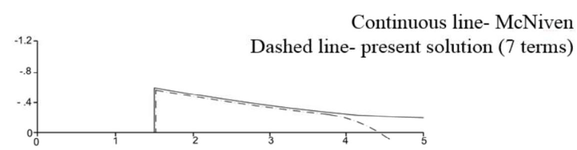

Fig. 3. Radial stress as a function of time at the station $r=2.5a$

References

[1] Sternberg, E., Chakravorty, J. G." On the propagation of shock waves in a non- homogeneous elastic medium". J. Appl. Mech. 26, , 528, 1959, 528.

[2] Chou, P. C. and Schaller, R. J.," Propagation of cylindrical waves in a non-homogeneous elastic medium" AIAA Journal, 6, 4 (1966).

[3] Reddy, D. P. and Marietta, M. G., "Propagation of Shear-Stress Waves in an Infinite Nonhomogeneous Elastic Medium with Cylindrical Cavity "Soc. Amer., 46, 1381 (1969).

[4] Lindholm, U. S. and Doshi, K. D., “Wave propagation in an elastic nonhomogeneous bar of finite length "J. Applied Mech., 32, 135 (1965).

[5] Han Zhao and Gérard Gary, "A three dimensional analytical solution of the longitudinal wave propagation in an infinite linear viscoelastic cylindrical bar. Application to experimental techniques" Journal of the Mechanics and Physics of Solids, Volume 43, Issue 8, (1995).
[6] Zhang, J.; Shen, Y. P.; Du, J. K." The effect of inhomogeneous initial stress on Love wave propagation in layered magneto-electro-elastic structures" Smart Materials and Structures, Volume 17, Issue 2, pp. 250-260 (2008).
[7] Yang, Ke . "A unified solution for longitudinal wave propagation in an elastic rod" Journal of Sound and Vibration, Volume 314, Issue 1-2, p. 307-329 (2008).
[8] S. Chin-Teh. Transient rotary shear waves in nonhomogeneous viscoelastic media. Int. J. Solids Struct, 7:25{37, 1971.
[9] Achenbach, J. D. and Reddy, D. P., Z. Angew. "Wave Propagation in Linearly Viscoelastic Media" Math. Phys., 18, 141 (1967).
[10] Avtar Singh and Kishan Chand Gupt."One-dimensional propagation of discontinuities in non-homogeneous linear viscoelastic semi-infinite media" Sadhana, Volume 9, (1986).
[11] McNiven, H. D. and Mengi, Y.," Propagation of transient waves in elastic laminated composites " int. J. Solids Structures, 15 (1979c), pp. 303–318.
[12] T. Y. Thomas, Plastic Flow and Fracture in Solids, Academic Press, New York, 1961.

Authors' information

Kansas State University.

Shahin Nayyeri Amiri received his Ph.D., M. Phil, and M.Sc degrees in Structural Engineering, second M.Sc. in Geotechnical Engineering, and B.Sc in Civil Engineering. Currently he is lecturer in Civil Engineering department as well as he is research associate in Mechanical and Nuclear Engineering department at Kansas State University. He is teaching theory of Plates and Shells, Vibration, Advanced Dynamics, Design of Tall Buildings for graduate students at Kansas State University. Shahin is a registered professional engineer. His research interest includes Buckling of Plates and Shells, Structural Dynamics, Vibration of Plates and Shells, Beam on Elastic Foundation, Finite Element Method, Biomechanics, Damage detection and Health Monitoring of the Structures, Sustainable Design of Structures, Geotechnical and Earthquake Engineering, Numerical Modeling of Soil Structure Interaction, Slope Stability, field and laboratory soil testing, Bridge Engineering and air bleeding in jet engines.

Asad Esmaeily is an associate professor of structural engineering at the Kansas State University, Civil Engineering Department. He earned his PhD in civil engineering from the University of Southern California (USC) in 2001. He also holds an MS in structural engineering and an MS in electrical engineering from the same university. He had an MSc and BS in civil engineering from Tehran University before joining USC graduate school.