SIGNAL DETECTION AND ENHANCEMENT OF

INFRASOUND EVENTS

by

MARK LOUIS SCHUETTE

B.S., KANSAS STATE UNIVERSITY, 1979

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1983

Approved by:

Nasir Ahmed

Major Professor

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THIS IS AS RECEIVED FROM THE CUSTOMER.

CHAPTER I

Introduction

Infrasound is the portion of the acoustic spectrum which lies between 0.001 Hz and 50 Hz, and is generally below the acoustic threshold of man. There are several types of waves in the infrasound range. These waves are "acoustic", "gravity" and "acoustic gravity" waves. Infrasound consists primarily of acoustic waves. The terminology depends on what is the restoring force to an initial displacement. If the restoring force depends on the incompressibility of air, it is an acoustic wave; and if it depends on a gravitational field, it is a gravity wave. Acoustic gravity waves are those in which both restoring forces are approximately equal. Acoustic waves are associated with the higher end of the spectrum, gravity waves with the lower end. Acoustic gravity waves reside between these two extremes.

There are a variety of sources for infrasound, both natural and man-made. The natural sources are occurrences such as volcanic eruptions, earthquakes, exploding meteors, aurora, severe storms, the surface of the sea in stormy conditions and air flow over rugged mountainous areas. Man-made sources of infrasound include atomic/nuclear explosions, large chemical explosions and sonic booms. Each of these various sources has its own distinguishing features. Thus differentiation of the events based on the different features should be possible. Also, one would like to distinguish events from the ambient noise. The two above problems of detection are obviously intertwined. Several types of filtering have been used in the past to solve the problem [Gossard, 1973] and are as follows: (1) Frequency filtering. Infrasound occurs in the high-frequency band of the atmospheric spectrum so that it must compete with turbulence "noise".

(2) Directional filtering. Infrasonic signals are usually distant from the detector system and so approximate nearly plane waves from a source.

(3) Wavelength filtering. Because sound travels at high speeds compared with local winds and turbulence, acoustic waves are relatively large in scale compared with the corresponding scales at the same frequency in the wind and turbulence spectrum.

(4) Velocity filtering. If the apparent velocity of the signal wave is known, those disturbances travelling at different speeds may be rejected.

This thesis deals with some aspects of signal enhancement and detection of infrasound events. An initial theoretical discussion of the propagation of infrasound will be followed by a discussion of a class of signal enhancement and detection schemes. Since the data statistics change continually with time, adaptive digital filtering techniques are considered.

CHAPTER II

Some Theoretical Considerations

In the study of wave propagation problems, there are in general two methods of attack. One is the normal mode approach and the other is ray tracing. The ray tracing method is an approximate approach that can be used to give answers with a minimum of computation, [Tolstoy, 1973, Budden, 1961]. This approximation is valid if the characteristics of the medium remain constant over the distance of a wavelength [Pearson, 1966, Tolstoy, 1959]. Our discussion will center on the "normal" mode approach, since it appears to hold the most promise for a filtering scheme.

A simple example will illustrate the approach used previously in studying infrasound. We consider the vibrating string problem. A problem that is an illustrative analogy of the methods used in acoustive wave propagation is that of a semi-infinite string, which has a continuous, variable density. One end is fixed, while after a certain length, the density of the string is constantly increasing. The "normal" modes are calculated numerically after the string has been subdivided into many small segments of constant density. The constant density distribution approximates the continuous distribution. The term "normal" is used advisedly since the frequency of oscillation for each sub-segment will be different due to the different densities. Hence the modes so determined will not necessarily form a closed and complete set of functions by which any given source function can be represented in the form of an expansion. However, the modes are characterized by relationships that exist between frequency and phase velocity and between frequency and group velocity.

Assuming an impulse displacement of the string takes place at a particular time, the response is the Green's function, that may be synthesized from the "normal" modes, with the amplitude of each normal mode being determined by a Fourier transform relationship. This part of the infrasound analysis will not be examined here.

The actual theoretical development is as follows: we start with the linearized hydrodynamic equations of motion. These have been derived elsewhere [Lamb, 1945, Pierce, 1967] and are the following:

$$\rho_o[D_t\vec{u} + (\vec{u}\cdot\nabla)\vec{v}] = -\nabla p - g\rho\vec{e}_z$$

$$D_t\rho + \vec{u}\nabla(\rho_o) + \rho_o\nabla\cdot(\vec{u}) = 4\pi f(t)\delta(\vec{r}-\vec{r}_o)$$

$$D_t p + \vec{u}\cdot\nabla(p_o) = C^2(D_t\rho + \vec{u}\cdot\nabla\rho_o)$$

$$\frac{dp_o}{dz} = -g\rho_o$$

The symbols are defined as follows:

D_t = Stokes operator $D_t = \frac{\partial}{\partial t} + (\vec{v}\cdot\text{grad})$

C^2 = speed of sound squared $C^2 = \gamma p_o/\rho_o$

γ = ratio of specific heats

ρ_o = initial unperturbed density distribution

ρ = first-order pertubation to ρ_o

p_o = initial unperturbed pressure distribution

p = first-order perturbation to p_o

$\vec{v}$ = initial unperturbed velocity

$\vec{u}$ = first-order perturbation to $\vec{v}$

$\vec{r}_o$ = source location

$\vec{r}$ = radius vector

$f(t)$ = time model of the blast waveform

$\vec{e}_z$ = unit vector in the z direction (up)

The first equation represents the conservation of momentum, the second the conservation of mass and the third the conservation of energy. The last equation is the hydrostatic equation. Note that the source term appears on the right in the equation for the conservation of mass. Thus the source is modelled as an instantaneous introduction of mass into the atmosphere. An alternative formulation would be an instantaneous introduction of energy into the atmosphere. Pierce [Pierce, 1968] has compared the two formulations. They are equivalent if the source is at ground level.

The homogeneous equations without the source term will be considered first. The governing equations can be simplified from five equations with five unknowns to two equations with two unknowns. The five unknowns are p, ρ, $\vec{u}$. The two unknowns are "potentials" Q_1 and Q_2 and are given by the following formulas:

$$u_z = p_o^{-\frac{1}{2}} D_t Q_1$$

$$\nabla \cdot \vec{u} = p_o^{\frac{1}{2}} D_t Q_2$$

In analogy with the vibrating string problem the atmosphere is divided into a number of sublayers of constant temperature, wind direction and speed.

The mathematical manipulations necessary to obtain the system of two equations, with two unknowns (also known as the residual equations) are given in Appendix 1. The initial wind distribution shows up in terms besides the Doppler-shifted frequency, in contrast to what had previously been reported by Hines [Hines, 1974].

The Fourier transform of the residual equation is taken, i.e., solutions of the form.

$$Q_1 = V_1 \exp[-i(wt - k_x x - k_y y)]$$

$$Q_2 = V_2 \exp[-i(wt - k_x x - k_y y)]$$

are assumed. V_1 and V_2 are functions of z only. We now are at the point of determining the normal mode solutions. A discussion of how the normal mode dispersion function is obtained starting from the residual equations is given in Appendix 2. Knowledge of the group velocity versus frequency curves from the dispersion function provides an estimate of the frequencies to be expected in the signal.

The next step is to determine the Green's function for an impulsive source, and to synthesize this function using the "normal" mode solutions. This part of the analysis has been carried to the full by Pierce and Posey [Pierce and Posey, 1971] and so will not be considered further. An alternative formulation has been given by Weston [Weston, 1961].

Thus the final goal of much of the previous theoretical analysis was to synthesize a waveform as close as possible to the observed waveform that emanated from an event.

Another line of investigation may be pursued however. This approach would seek to improve the parameters of the theoretical formulation from data gathered experimentally, while the experimental data would be interpreted more accurately using better theoretical models. The basic tools needed would be a computer program that produces the dispersion relation, a computer program for calculating the group velocity and bearing and a program for bandpass and LMS filtering (to be described in the next chapter). The approach centers on the concept that large amplitude waves are associated with stationary values of the group velocity (i.e., $\frac{\partial V}{\partial k_x} = \frac{\partial V}{\partial k_y} = 0$, and $\frac{\partial^2 V}{\partial k_x^2} \cdot \frac{\partial^2 V}{\partial k_y^2} - \left(\frac{\partial^2 V}{\partial k_x \, \partial k_y}\right)^2 > 0$) [Tolstoy and Clay, 1966]. Thus one would expect the frequencies determined experimentally to coincide with the frequencies at the extrema of the group velocity. This will be the case if what is detected is actually an event and if the theoretical model and parameters are correct. Thus if one knows an event has occurred and the frequencies do not coincide with the frequencies from the theoretical model, suitable adjustments can be made. Another check would be to examine the velocity and bearing of the event and determine if they are in the expected range.

CHAPTER III

Signal Identification Using Prediction-Error Filtering and Power Ratios

Prediction-error filtering is based on the assumption that successive noise samples in each channel are correlated. An adaptive filter can be used to filter out the correlated noise, which is equivalent to saying that the filter adapts to the noise spectrum and attempts to make the resulting output "white." The signal of interest is considered to be a transient which stands out with little change when the input is passed through the adaptive filter. Prediction-error filtering has achieved a level of success in seismic work [Claerbout, 1964]. Our approach will be based on Widrow's LMS (least mean square) algorithm for Wiener filters [Widrow, 1971, Widrow, et. al, 1976]. The LMS filter output is then examined for a signal based on a power ratio between the "signal" window and the "background" window.

The actual processing of the received waveforms is as follows: an initial bandpass filter is applied which passes frequencies from 0.5 Hz to 5 Hz. The output of the bandpass filter is then filtered using an individual LMS filter for each channel. A power ratio using a "signal" and "background" window is computed. The final power ratio is the arithmetic average of the individual power ratios.

The bandpass filter is an FIR (finite impulse response) filter with 768 weights, and linear phase function. The large number of weights is necessary to insure a sharp cutoff of 0.5 Hz, due to the contamination of the received waveform by microbaroms. Microbaroms are caused by stormy conditions at sea. The surface of the sea acts as a giant piston moving against the atmosphere, propagating acoustic waves. Microbaroms can have large amplitudes and their frequency is centered at around 0.18 Hz. The other reason that the received waveform is heavily filtered is that in this intitial analysis we want to focus on acoustic waves and remove gravity waves from the signal. Figure 3.1 shows the amplitude response for the bandpass filter.

The bandpass filtered waveform is then filtered with an individual LMS filter for each channel; 64 filter weights are used and the relevant parameters are 5.0×10^{-5} for convergence parameter alpha, and 5×10^{-4} for the minimum variance. If the variance of the signal falls below the minimum variance there is no update of the filter coefficients. The relevant equations are as follows:

$$\bar{W}(j+1) = \begin{cases} \bar{W}(j) + \dfrac{\alpha}{\sigma_x^2}\, e(j)\, \bar{X}, & \text{if } \sigma_x^2 > \text{min. var.} \\ \bar{W}(j) & \text{if } \sigma_x^2 \leq \text{min. var.} \end{cases}$$

$$\sigma_x^2 = \frac{\bar{X} \cdot \bar{X}}{N-1}$$

$\bar{W}(j+1)$: the $j+1^{th}$ update of the LMS filter vector.

α: convergence parameter

$e(j)$: prediction error

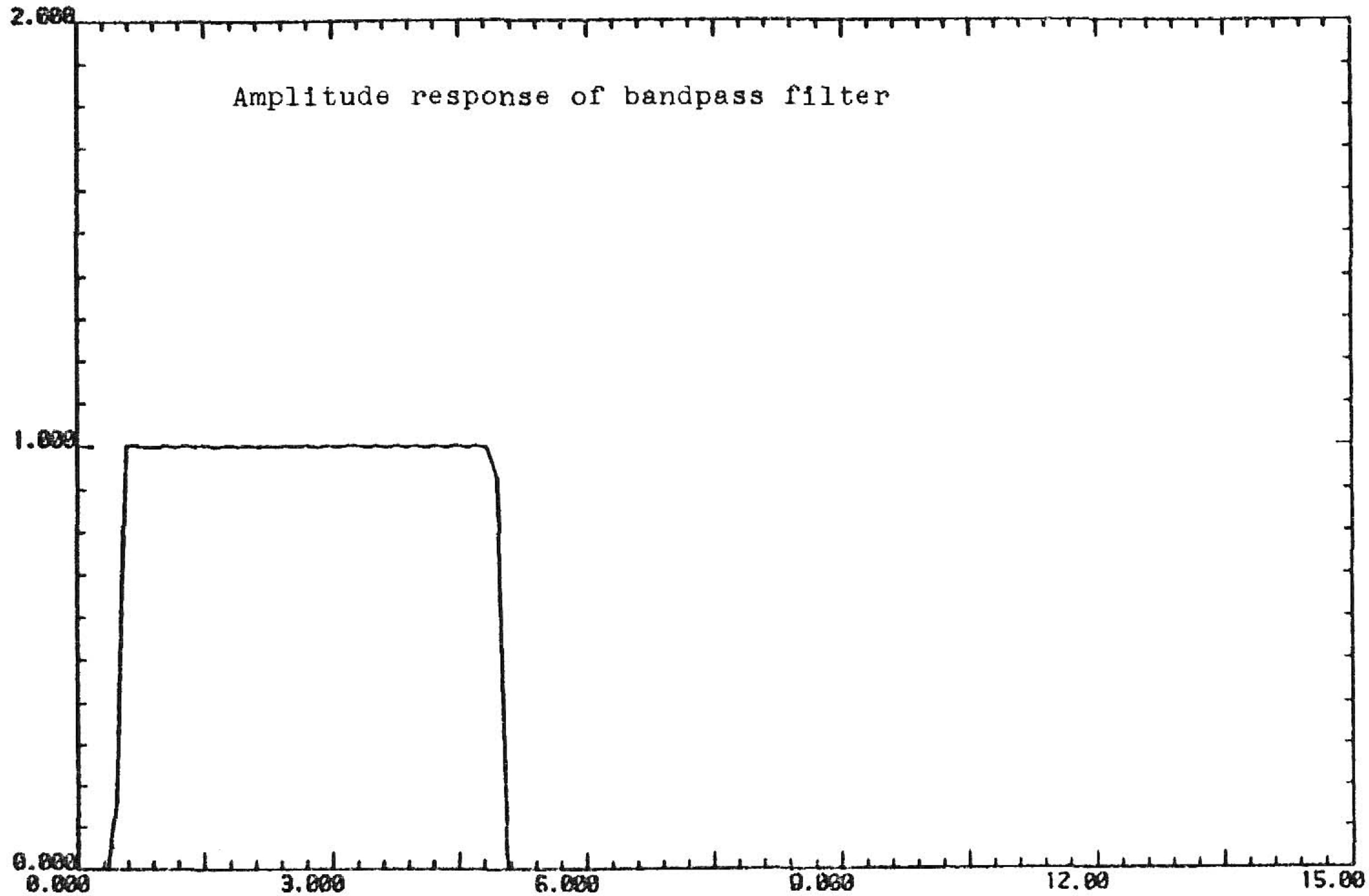
$\bar{X}$: data vector

σ_x^2: variance of $\bar{X}$ vector (in this case the power since the mean is assumed to be zero)

N: number of filter coefficients

The reason for the large number of weights is to provide adequate frequency resolution [Griffiths, 1975]. The signal we are interested in occurs at 0.6 Hz, while there is a component at 0.75 Hz present. The 0.75 Hz component may or may not be considered part of the signal depending on

Figure 3.1


Amplitude response of bandpass filter
2.000
1.000
0.000
0.000
3.000
6.000
9.000
12.00
15.00

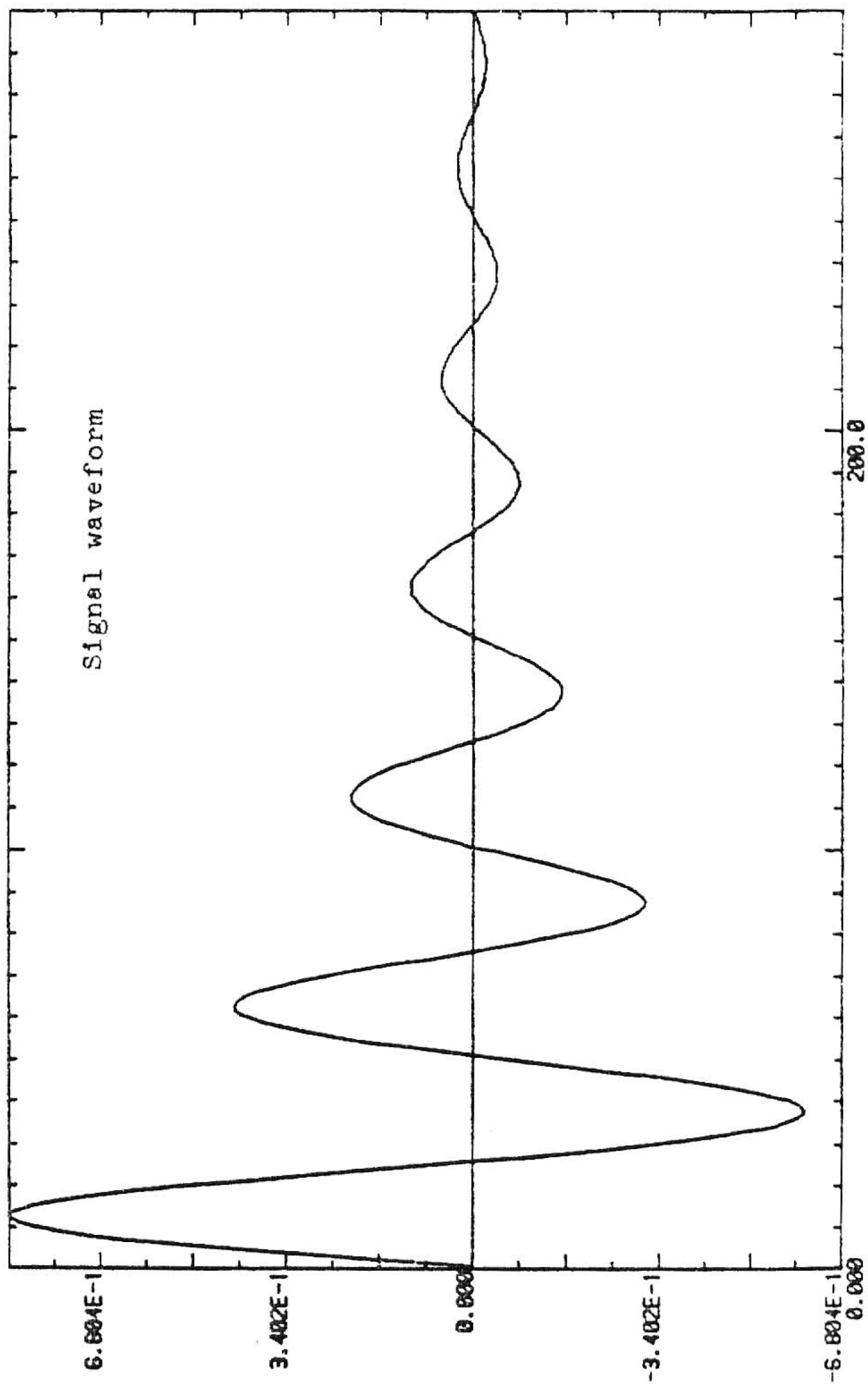
what physical process is under consideration. A basic property of the LMS filter is its ability to adjust itself to the background noise, which will be commented upon later in connection with the experimental results. Once the filter has adjusted to the background noise (i.e., "steady-state" has been obtained) then any event with different frequencies from the background noise will stand out sharply upon being passed through the filter. Of course, an event must have a large enough amplitude to be "seen" by the filter. How large the amplitude must be (or equivalently, the signal-to-noise ratio (SNR)) is a criterion that has been examined by several workers for narrow-band signals in white gaussian noise [Anderson, et. al, 1983].

The power ratio attempts to magnify any sharp increases in power from the LMS filter output. Such a sharp increase would be indicative of a transient signal. A power ratio is formed by dividing the power in a "signal" window by the power in a "noise" window, where the noise window lags the signal window for each channel. The length of the signal window is 100 points, the noise window is 200 points long and the delay between the two windows is 100 points. A threshold power setting determines the amount of power necessary for an event to be labelled as such.

A set of data files for several SNR's was constructed.[1] The sampling rate for all the data was 30 samples per second. The signal is a damped, truncated sine wave 300 points long (Figure 3.2). The signal arrives at point 1990 for channel 1, at point 2000 for channel 4 and at point 2010 for channel 2 (Figure 3.3). The time delays used insure that the signal has acoustic speed. An initial SNR of 0.0208 was chosen, and the amplitudes were scaled appropriately. Other SNR's used were 0.0 (noise only), 0.0416 and 0.0832. For each SNR (and for each channel) the following information

[1] The SNR was calculated by dividing the variance of the scaled signal (300 points long) by the variance of the noise (3,000 points long).

Figure 3.2


Signal waveform
6.804E-1
3.402E-1
0.000
-3.402E-1
-6.804E-1
0.000
200.0

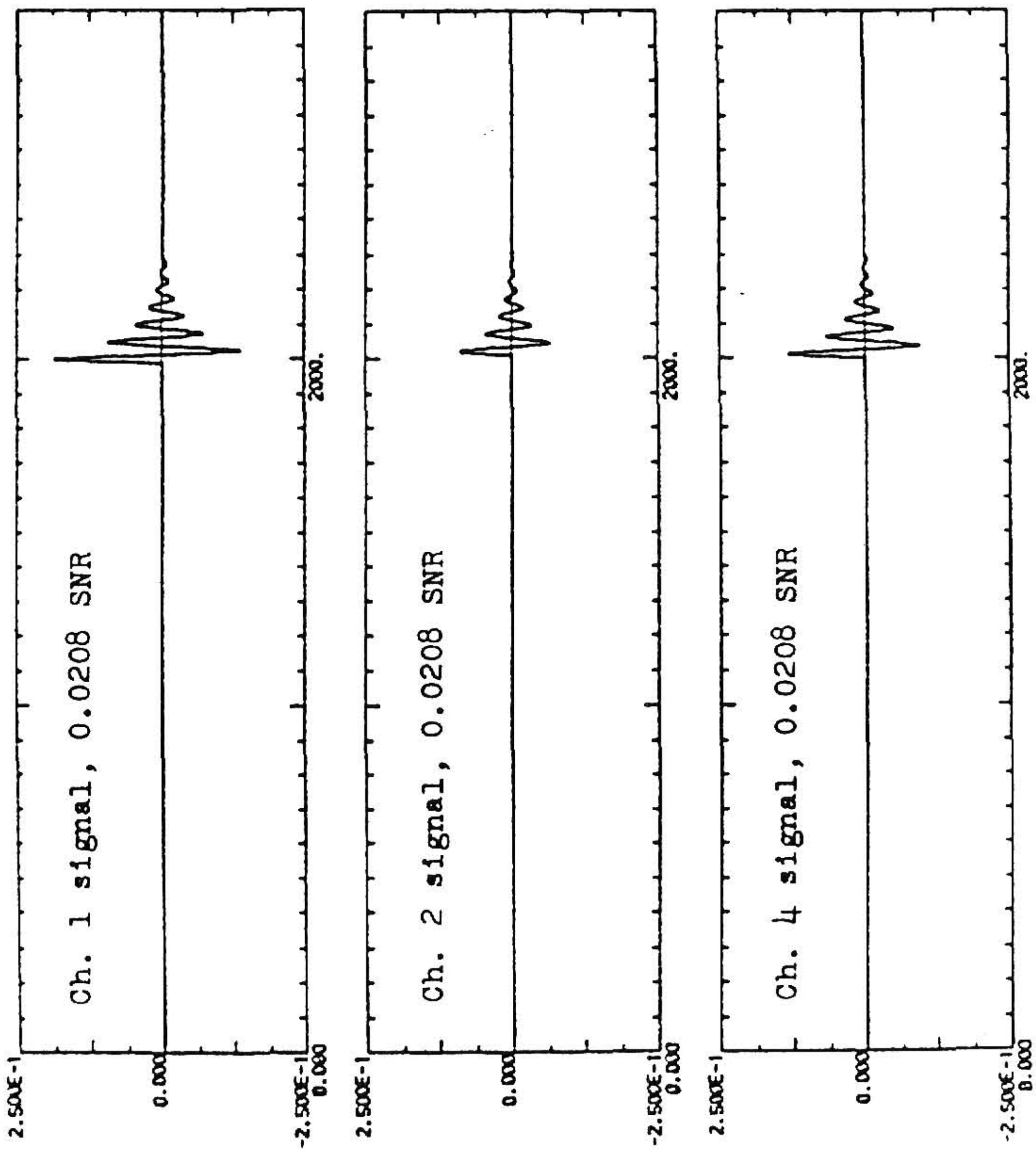

Ch. 1 signal, 0.0208 SNR
Ch. 2 signal, 0.0208 SNR
Ch. 4 signal, 0.0208 SNR
2.500E-1
0.000
-2.500E-1
0.000
2000.

Figure 3.3

was plotted: the received waveform, the bandpass filter output , the Widrow LMS filter output and the LMS transfer function at point 2100. A power ratio plot for each SNR was made using the Widrow LMS filter output for all three channels, as has been previously described. Figure 3.4 shows, for the 0.0 SNR case, the received waveforms and the bandpass filter outputs. Channel one's received waveform and bandpass filter output are denoted by a) and b), respectively. Channel two's received waveform and bandpass filter output are denoted by c) and d), respectively. e) and f) are channel three's received waveform and bandpass filter output, respectively. The same format is used for figures 3.5, 3.6, and 3.7, only the SNR's are 0.028, 0.0416 and 0.0832, respectively. Figure 3.8 shows the LMS filter output, the amplitude response of the LMS filter at point 2100 and finally the power ratio plot using the LMS filter output from all three channels for the 0.0 SNR case. The following system of labelling is used for figure 3.8; a) is channel one's LMS filter output, b) is the amplitude response of the LMS filter at point 2100 for channel one, c) is channel two's LMS filter output, d) is the amplitude response of the LMS filter at point 2100 for channel two, e) is channel four's LMS filter output, f) is the amplitude response of the LMS filter at point 2100 for channel four and g) is the power ratio plot. The same labelling system is used for figures 3.9, 3.10 and 3.11 where the SNR's are 0.0208, 0.0416 and 0.0832, respectively.

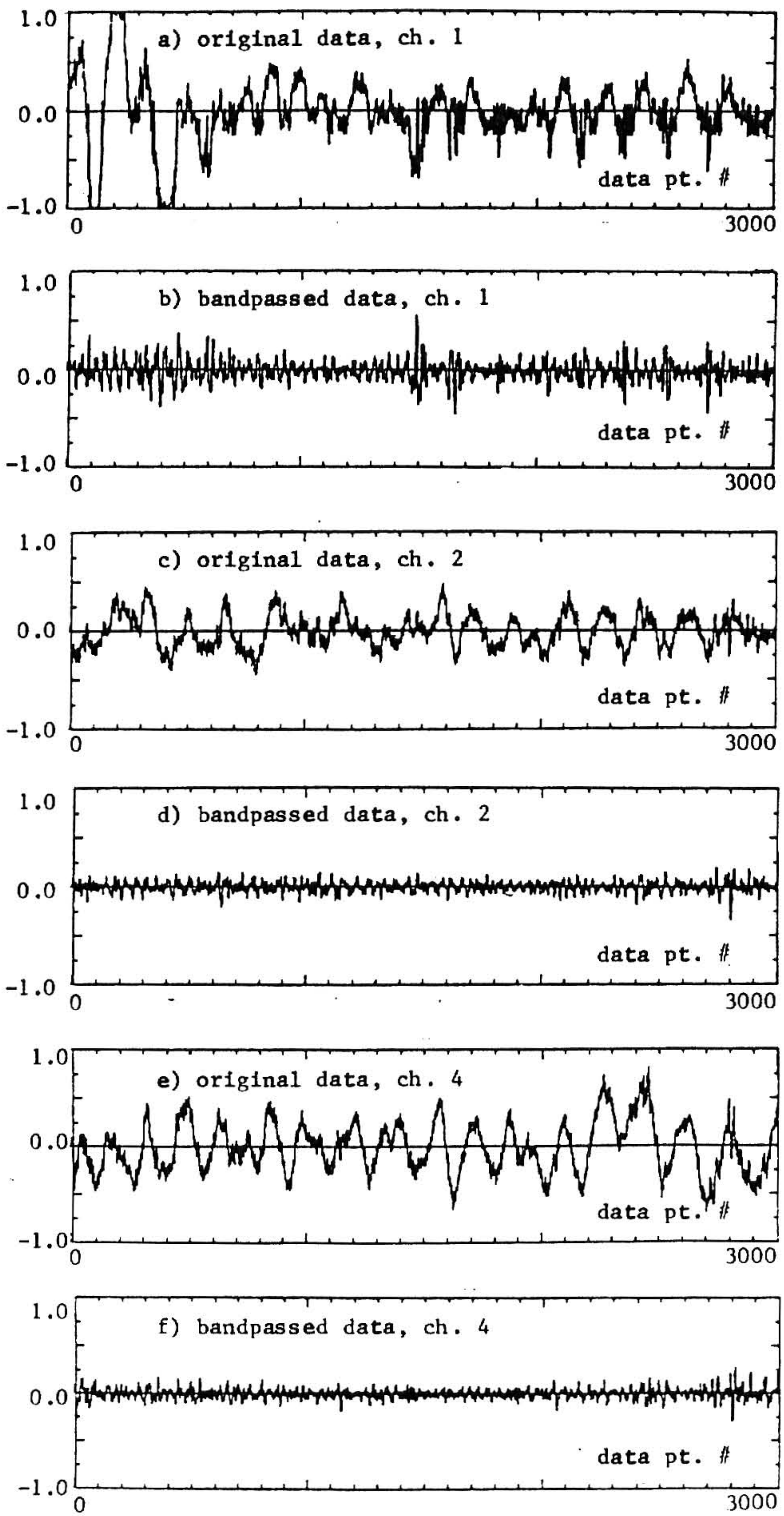

a) original data, ch. 1
b) bandpassed data, ch. 1
c) original data, ch. 2
d) bandpassed data, ch. 2
e) original data, ch. 4
f) bandpassed data, ch. 4
data pt. #
1.0
0.0
-1.0
0
3000

Figure 3.4

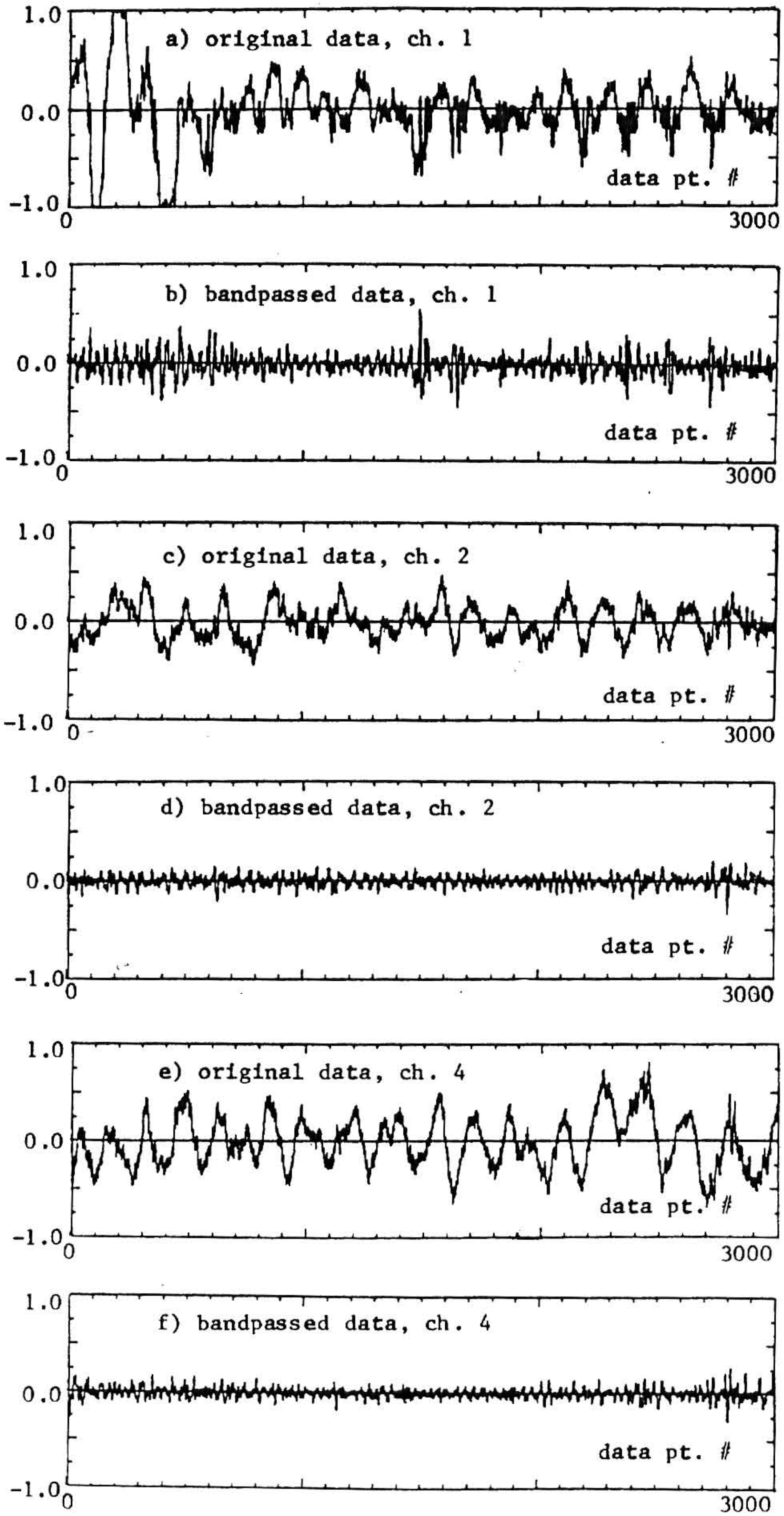

1.0
a) original data, ch. 1
0.0
data pt. #
-1.0
0
3000
1.0
b) bandpassed data, ch. 1
0.0
data pt. #
-1.0
0
3000
1.0
c) original data, ch. 2
0.0
data pt. #
-1.0
0
3000
1.0
d) bandpassed data, ch. 2
0.0
data pt. #
-1.0
0
3000
1.0
e) original data, ch. 4
0.0
data pt. #
-1.0
0
3000
1.0
f) bandpassed data, ch. 4
0.0
data pt. #
-1.0
0
3000

Figure 3.5

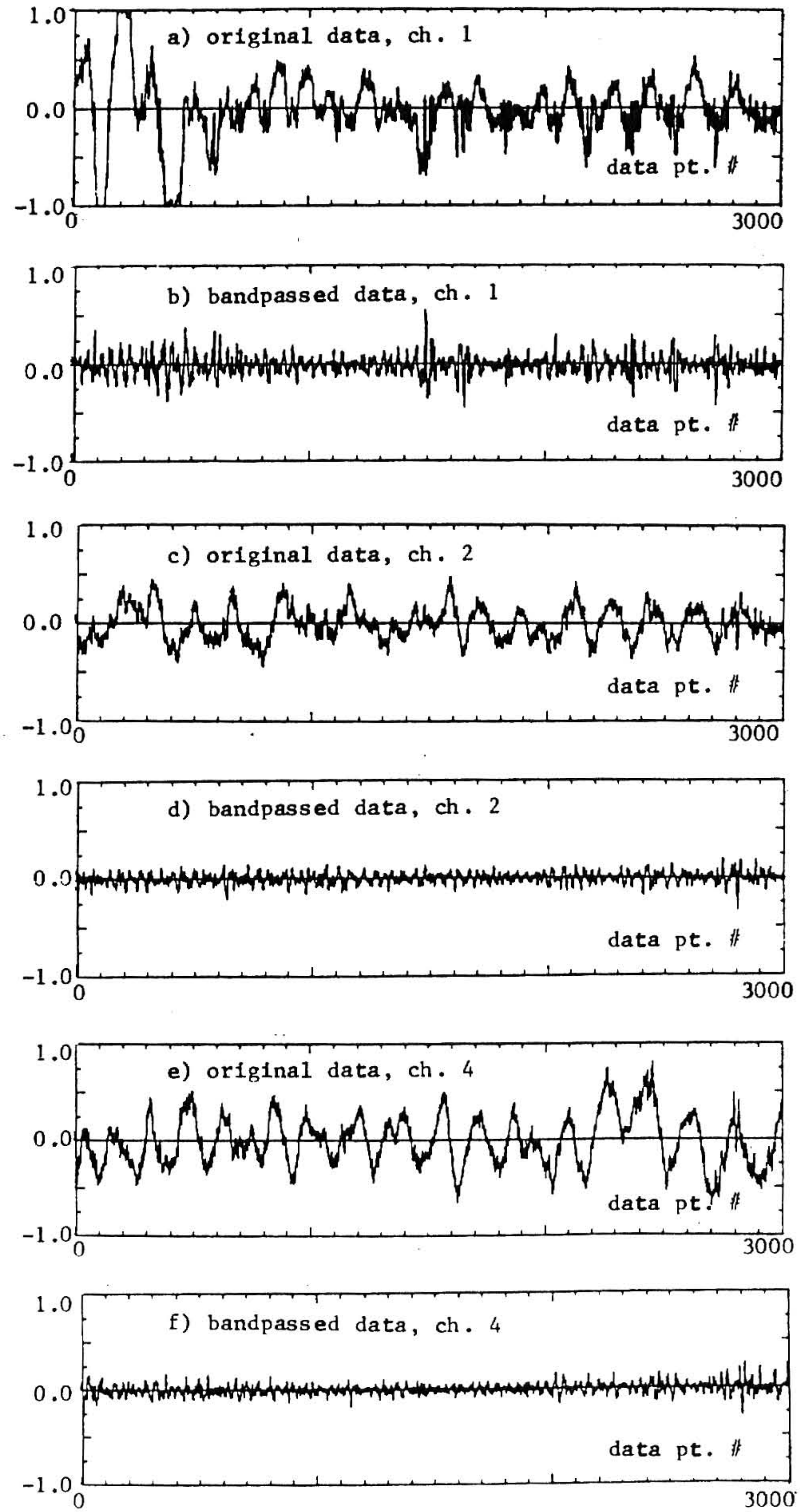

a) original data, ch. 1
b) bandpassed data, ch. 1
c) original data, ch. 2
d) bandpassed data, ch. 2
e) original data, ch. 4
f) bandpassed data, ch. 4
data pt. #
1.0
0.0
-1.0
0
3000

Figure 3.6

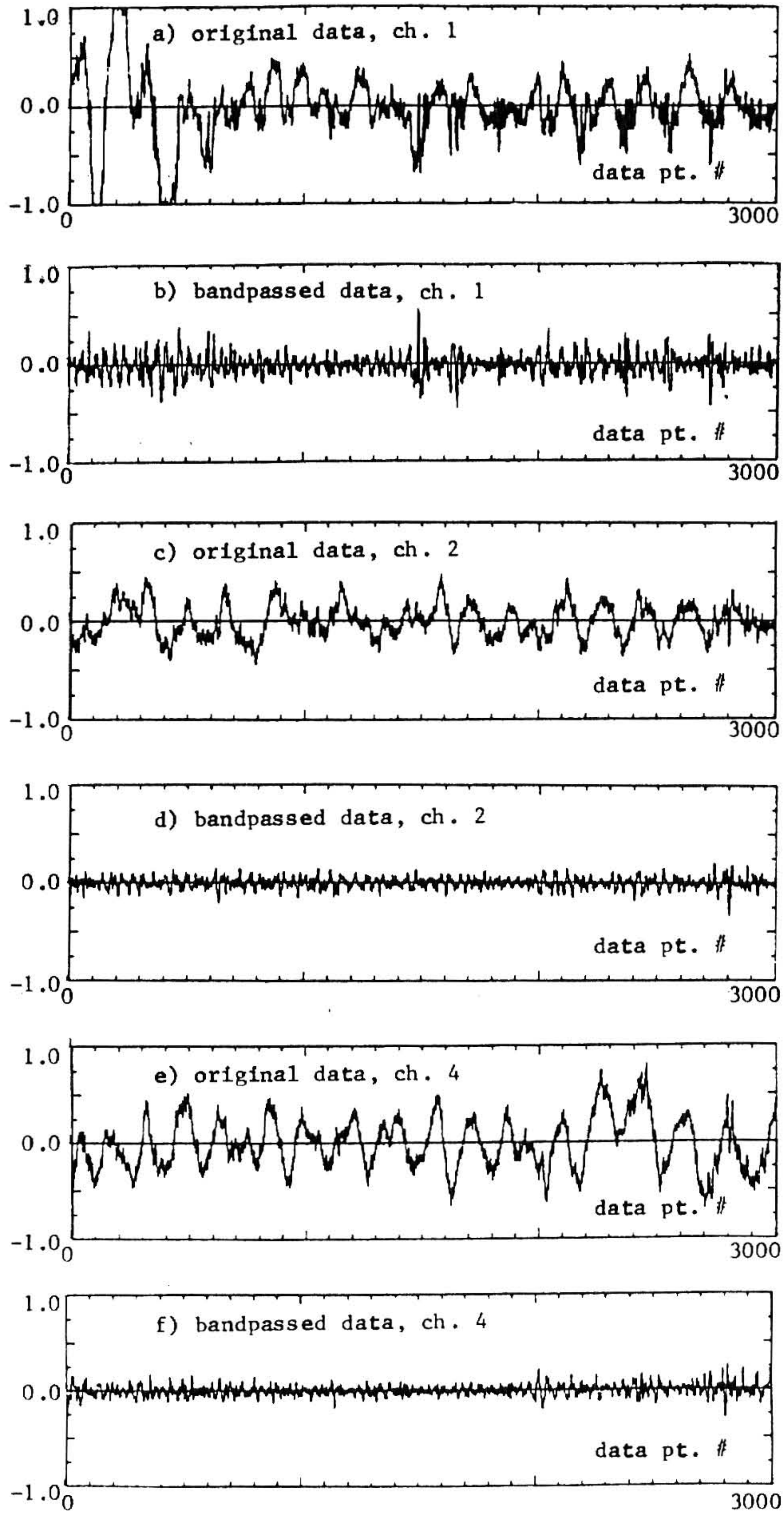

1.0
0.0
-1.0
a) original data, ch. 1
data pt. #
0
3000
b) bandpassed data, ch. 1
c) original data, ch. 2
d) bandpassed data, ch. 2
e) original data, ch. 4
f) bandpassed data, ch. 4

Figure 3.7

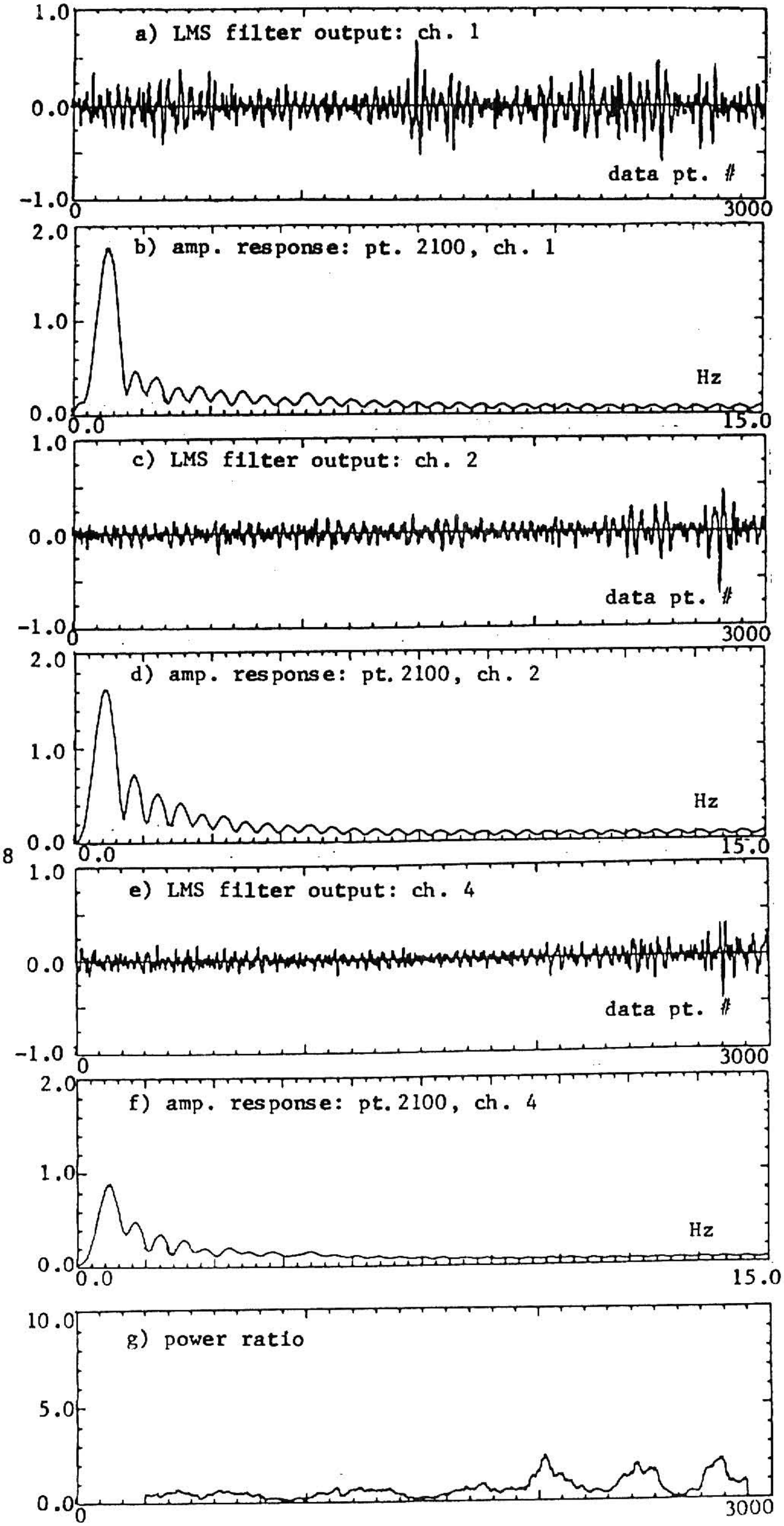

a) LMS filter output: ch. 1
data pt. #
b) amp. response: pt. 2100, ch. 1
Hz
c) LMS filter output: ch. 2
data pt. #
d) amp. response: pt. 2100, ch. 2
Hz
e) LMS filter output: ch. 4
data pt. #
f) amp. response: pt. 2100, ch. 4
Hz
g) power ratio
1.0
0.0
-1.0
2.0
0
3000
0.0
15.0
10.0
5.0

Figure 3.8

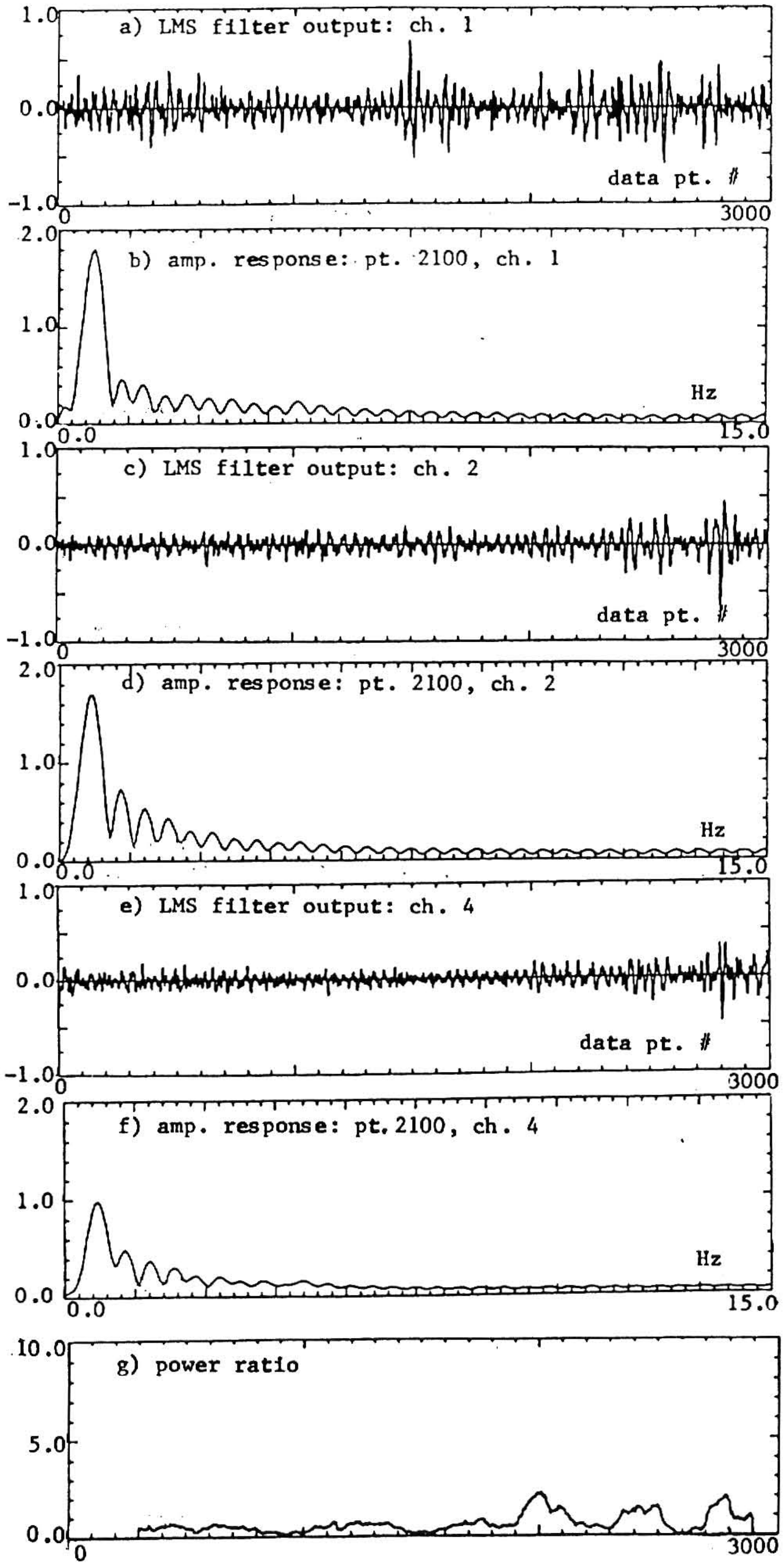

1.0
a) LMS filter output: ch. 1
0.0
-1.0
data pt. #
0
3000
2.0
b) amp. response: pt. 2100, ch. 1
1.0
0.0
Hz
0.0
15.0
1.0
c) LMS filter output: ch. 2
0.0
-1.0
data pt. #
0
3000
2.0
d) amp. response: pt. 2100, ch. 2
1.0
0.0
Hz
0.0
15.0
1.0
e) LMS filter output: ch. 4
0.0
-1.0
data pt. #
0
3000
2.0
f) amp. response: pt. 2100, ch. 4
1.0
0.0
Hz
0.0
15.0
10.0
g) power ratio
5.0
0.0
0
3000

Figure 3.9

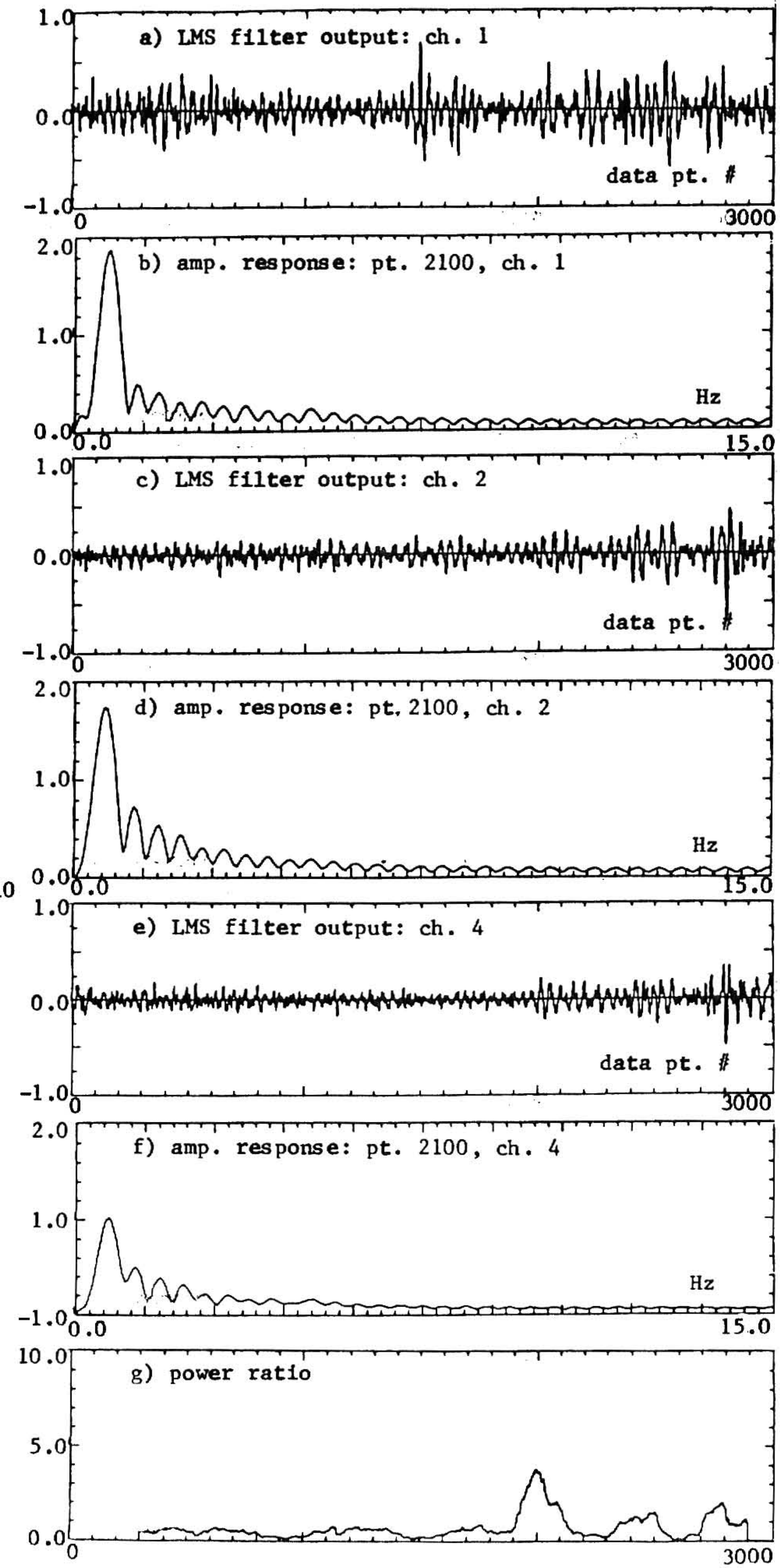

a) LMS filter output: ch. 1
data pt. #
1.0
0.0
-1.0
0
3000
b) amp. response: pt. 2100, ch. 1
Hz
2.0
1.0
0.0
0.0
15.0
c) LMS filter output: ch. 2
data pt. #
1.0
0.0
-1.0
0
3000
d) amp. response: pt. 2100, ch. 2
Hz
2.0
1.0
0.0
0.0
15.0
e) LMS filter output: ch. 4
data pt. #
1.0
0.0
-1.0
0
3000
f) amp. response: pt. 2100, ch. 4
Hz
2.0
1.0
-1.0
0.0
15.0
g) power ratio
10.0
5.0
0.0
0
3000

Figure 3.10

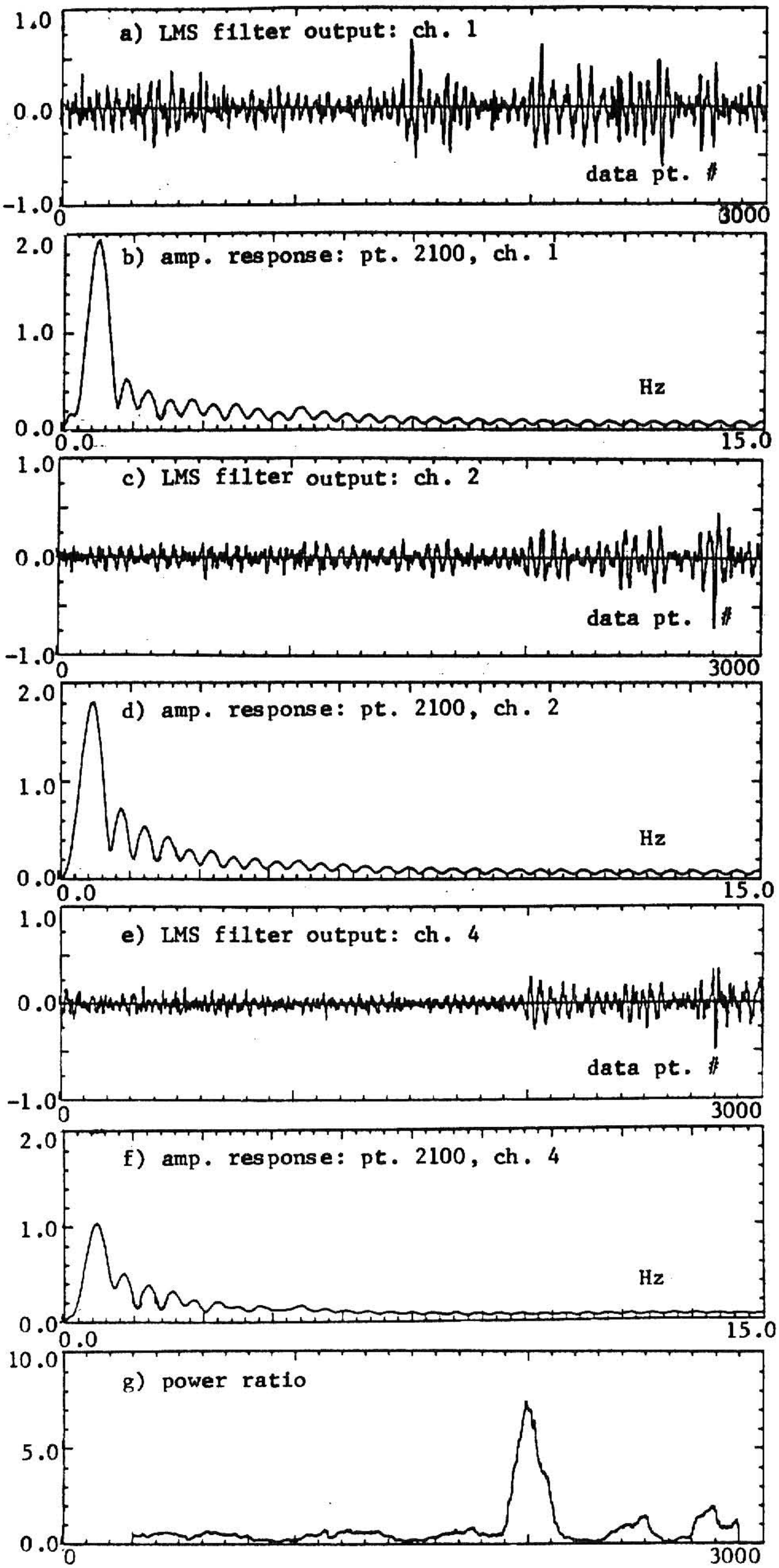

a) LMS filter output: ch. 1
data pt. #
b) amp. response: pt. 2100, ch. 1
Hz
c) LMS filter output: ch. 2
data pt. #
d) amp. response: pt. 2100, ch. 2
Hz
e) LMS filter output: ch. 4
data pt. #
f) amp. response: pt. 2100, ch. 4
Hz
g) power ratio
1.0
0.0
-1.0
2.0
5.0
10.0
0
3000
0.0
15.0

Figure 3.11

CHAPTER IV

Conclusions

A discrepancy in the development of the normal mode solutions due to wind was found. This may alter somewhat previous theoretical results. Adaptive filtering using Widrow's LMS algorithm appears to be justified in its ability to detect a signal superimposed on microbarom noise. A large number of weights was necessary (64) and a SNR of 0.0416 appears to be adequate to detect an event. More detailed conclusions were:

1) The large number of filter weights used was necessary to achieve adequate frequency resolution. Difficulty was experienced with 8 and 16 weight filters since their spectral resolution was poor for this data set, i.e., convergence to a stable filter was difficult to obtain and once convergence was obtained little actual filtering was accomplished since the filter weights were so small.

2) The amount of power at point 2000 actually decreases when the signal is added. (Compare Figures 3.8g and 3.9g). Thus the initial effect is one of destructive interference. However comparing the plots of Figures 3.10g and 3.11g for the 0.0416 SNR case and the 0.0832 SNR case, respectively, we see that the power ratio increases by almost exactly 2 at data point 2,000. This should appear to indicate that the noise "rides" on top of the signal, i.e., the signal predominates very much over the noise for both cases.

3) More than one event is present in the record as shown by the other power peaks in the power ratio plots (Figures 3.8g, 3.9g, 3.10g, 3.11g). Data point 2,000 appears to be just inside such an event. The signal was thus superimposed on top of an already existing event; this gives us a rough idea as to what SNR's are needed for an event to be detected when placed on another event.

Acknowledgement

I would like to thank the members of my committee, Dr. Nasir Ahmed, Dr. Donald R. Hummels, and Dr. Winston Yang for their support. Rod Whitaker and Paul Mutschlecner helped with many insightful discussions, their help is gratefully appreciated. The relevant data from Los Alamos National Laboratories is also appreciated. I would also like to thank my parents who made it all possible.

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Appendix 1

Derivation of the Residual Equations from the Linearized Hydrodynamical Equations of Motion.

We follow Pierce's derivation [Pierce, 1967] and numbering of equations to facilitate comparison, since we obtain a different result from his.

$$\rho_o[D_t\vec{u} + (\vec{u}\cdot\nabla)\vec{v}] = -\nabla p - g\rho\vec{e}_z \tag{1a}$$

$$D_t\rho + \vec{u}\cdot\nabla\rho_o + \rho_o\nabla\cdot\vec{u} = 0 \tag{1b}$$

$$D_t p + \vec{u}\cdot\nabla p_o = C^2(D_t\rho + \vec{u}\cdot\nabla\rho_o) \tag{1c}$$

$$D_t = \frac{\partial}{\partial t} + (\vec{v}\cdot\nabla) \tag{2}$$

D_t is the Stokes operator. The variables are defined on page 4 of Chapter II and so will not be repeated here. A Cartesian coordinate system is used. The effects of the earth's curvature are included in a multiplicative correction factor that can be found in Pierce, Posy, and Iliff's paper [Pierce, Possey, and Iliff, 1971]. Other important equations in the following derivation are the hydrostatic equation and the speed of sound expressed in terms of the ambient pressure and density. The hydrostatic equation is $\frac{dp_o}{dz} = -g\rho_o$ and the speed of sound is given by $C^2 = \gamma p_o/\rho_o$. Pierce [Pierce, 1967] defined two potentials, as follows;

$$u_z = p_o^{-\frac{1}{2}} D_t Q_1 \tag{4a}$$

$$\nabla\cdot\vec{u} = p_o^{-\frac{1}{2}} D_t Q_2 \tag{4b}$$

(Q_1 and Q_2 have been substituted for ψ_1 and ψ_2 for typing ease) where u_z is the vertical component of $\vec{u}$. (The subscript variable indicates the component.)

The following assumptions are made:

A) The ambient variables p_o, ρ_o and $\vec{v}$ are independent of time t and of the horizontal coordinates x and y.

B) The ambient wind $\vec{v}$ is horizontal ($v_z = 0$). Note that the Stokes operator is therefore

$$D_t = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$$

C) The only variation in the ambient wind is with height.

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} = \frac{\partial v_y}{\partial y} = 0$$

The actual derivation is as follows: first we eliminate $D_t\rho$ from 1b) and 1c) and substitute in 4a) and 4b).

$$D_t\rho = -\,(\vec{u}\cdot\nabla\rho_o + \rho_o\nabla\cdot\vec{u})$$

$$D_t p + \vec{u}\cdot\nabla p_o = C^2(-\vec{u}\cdot\nabla\rho_o - \rho_o\nabla\cdot\vec{u} + \vec{u}\cdot\nabla\rho_o)$$

$$D_t p = -\vec{u}\cdot\nabla p_o - C^2\,\rho_o\nabla\cdot\vec{u}$$

also, we need $\qquad \nabla(p_o) = \dfrac{dp_o}{dz}\vec{e}_z = -g\rho_o\vec{e}_z$

$$D_t p = g\rho_o u_z - C^2\,\rho_o\nabla\cdot\vec{u}$$

$$= g\rho_o p_o^{-\frac{1}{2}} D_t Q_1 - C^2\,\rho_o p_o^{-\frac{1}{2}} D_t Q_2$$

$$= \rho_o p_o^{-\frac{1}{2}} (g\,D_t Q_1 - C^2 D_t Q_2)$$

$$= -\rho_o\,p_o^{-\frac{1}{2}} D_t(C^2 Q_2 - gQ_1)$$

$$D_t p = -D_t[\rho_o p_o^{-\frac{1}{2}} (C^2 Q_2 - gQ_1)]$$

since $\qquad D_t = \dfrac{\partial}{\partial t} + v_x \dfrac{\partial}{\partial x} + v_y \dfrac{\partial}{\partial y}$

Since equation 4a) and 4b) define Q_1 and Q_2 only to within an additive constant, we can choose that constant to be 0 and thus obtain

$$p = -\rho_o p_o^{-\frac{1}{2}} (C^2 Q_2 - gQ_1) \qquad 5)$$

Next, ρ, u_x and u_y are eliminated from equation 1). Both sides of 1a) are operated on with the horizontal divergence, $\nabla_h = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y$

$$\left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot\rho_o[D_t\vec{u} + (\vec{u}\cdot\nabla)\vec{v}] = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot[-\nabla(p) - g\rho\vec{e}_z]$$

We examine the different terms separately, starting with the left hand side.

$$\left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot D_t\vec{u} = D_t\left[\left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot\vec{u}\right]$$

$$= D_t\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)$$

$$= D_t\left[\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \frac{\partial u_z}{\partial z}\right]$$

$$= D_t\left[\nabla\cdot\vec{u} - \frac{\partial u_z}{\partial z}\right]$$

The next term on the left hand side is

$$\nabla_h\cdot((\vec{u}\cdot\nabla)\vec{v}) = \nabla_h\cdot(\vec{u}\cdot\nabla)v_x\vec{e}_x + \nabla_h\cdot(\vec{u}\cdot\nabla)v_y\vec{e}_y$$

$$= \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot(\vec{u}\cdot\nabla)v_x\vec{e}_x + \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot(\vec{u}\cdot\nabla)v_y\vec{e}_y$$

$$= \frac{\partial}{\partial x}(\vec{u}\cdot\nabla)v_x + \frac{\partial}{\partial y}(\vec{u}\cdot\nabla)v_y$$

Now $\vec{u}\cdot\nabla = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$. Therefore

$$\nabla_h\cdot((\vec{u}\cdot\nabla)\vec{v}) = \frac{\partial}{\partial x}\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)v_x + \frac{\partial}{\partial y}\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)v_y$$

$$= \frac{\partial}{\partial x}\left(u_x\frac{\partial v_x}{\partial x} + u_y\frac{\partial v_x}{\partial y} + u_z\frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(u_x\frac{\partial v_y}{\partial x} + u_y\frac{\partial v_y}{\partial y} + u_z\frac{\partial v_y}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(u_z\frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(u_z\frac{\partial v_y}{\partial z}\right)$$

$$= \frac{\partial u_z}{\partial x}\frac{\partial v_x}{\partial z} + u_z\frac{\partial^2 v_x}{\partial x\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial v_y}{\partial z} + u_z\frac{\partial^2 v_y}{\partial y\partial z}$$

In general, we are considering atmospheres where $\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z}\right)$ and $\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial z}\right)$. Since $\frac{\partial v_y}{\partial y} = \frac{\partial v_x}{\partial x} = 0$, we have:

$$\nabla_h\cdot(\vec{u}\cdot\nabla)\vec{v} = \frac{\partial u_z}{\partial x}\frac{\partial v_x}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial v_y}{\partial z} = \frac{\partial\vec{v}}{\partial z}\cdot\nabla u_z.$$

On the right hand side we have:

$$\nabla_h\cdot[-\nabla(p) - g\rho\vec{e}_z] = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y\right)\cdot\left[-\left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right)p - g\rho\vec{e}_z\right]$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p$$

$$= -\nabla_2^2 p \text{ where } \nabla_2^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

We now have $\rho_o[D_t(\nabla\cdot\vec{u} - \frac{\partial u_z}{\partial z}) + \frac{\partial\vec{v}}{\partial z}\cdot\nabla u_z] = -\nabla_2^2\, p$ which is Pierce's [Pierce, 1967] equation 6).

We follow Pierce by "operating on both sides of the z-component of (1a) with D_t and then eliminating $D_t\rho$ by use of (1b)." The z-component of (1a) is:

$$\rho_o[D_t\, u_z + (\vec{u}\cdot\nabla)\, v_z] = -\frac{\partial p}{\partial z} - g\rho\ .$$

Since $v_z = 0$, we have $\rho_o D_t u_z = -\frac{\partial p}{\partial z} - g\rho$. This leads to

$$\rho_o\, D_t^2\, u_z = -\frac{\partial}{\partial z} D_t\, p - g\, D_t\, \rho.$$

But $D_t\,\rho = -\vec{u}\cdot\nabla\,\rho_o - \rho_o\,\nabla\cdot\vec{u}$.

Thus we have $\rho_o\, D_t^2\, u_z = -D_t\frac{\partial p}{\partial z} + g(\vec{u}\cdot\nabla\,\rho_o + \rho_o\,\nabla\cdot\vec{u})$ 7)

which is Pierce's equation 7). Pierce goes on and substitutes 4a), 4b) and 5) into 6) and 7). For 6) we have

$$\rho_o[D_t(\nabla\cdot\vec{u} - \frac{\partial u_z}{\partial z}) + \frac{\partial\vec{v}}{\partial z}\cdot\nabla u_z] = -\nabla_2^2\, p$$

$$\rho_o[D_t(p_o^{-\frac{1}{2}}\, D_tQ_2) - D_t\frac{\partial}{\partial z}(p_o^{-\frac{1}{2}}\, D_tQ_1) + \frac{\partial\vec{v}}{\partial z}\cdot\nabla(p_o^{-\frac{1}{2}}\, D_tQ_1)]$$
$$= \nabla_2^2\,\rho_o p_o^{-\frac{1}{2}}(C^2Q_2 - gQ_1)$$

Now $\frac{\partial}{\partial z}(p_o^{-\frac{1}{2}}\, D_tQ_1) = -\frac{1}{2}\, p_o^{-\frac{3}{2}}\frac{dp_o}{dz}\, D_tQ_1 + p_o^{-\frac{1}{2}}\frac{\partial}{\partial z} D_tQ_1$.

We therefore have:

$$[p_o^{-\frac{1}{2}}\, D_t^2Q_2 - D_t(-\frac{1}{2}\, p_o^{-\frac{3}{2}}\frac{dp_o}{dz} D_tQ_1 + p_o^{-\frac{1}{2}}\frac{\partial}{\partial z} D_tQ_1) + p_o^{-\frac{1}{2}}\frac{\partial\vec{v}}{\partial z}\cdot\nabla(D_tQ_1)]$$
$$= \nabla_2^2\, p_o^{-\frac{1}{2}}\,(C^2Q_2 - gQ_1)$$

Since $\frac{dp_o}{dz} = -g\rho_o$ we have:

$$p_o^{-\frac{1}{2}}\,[D_t^2Q_2 - \frac{1}{2}\frac{g\rho_o}{p_o} D_t^2Q_1 - \frac{\partial}{\partial z} D_t^2Q_1 + \frac{\partial\vec{v}}{\partial z}\cdot\nabla(D_tQ_1)] = p_o^{-\frac{1}{2}}\,\nabla_2^2(C^2Q_2 - gQ_1)$$

$$[D_t^2Q_2 - \frac{1}{2}\frac{g\rho_o}{p_o} D_t^2Q_1 - \frac{\partial}{\partial z} D_t^2Q_1 + \frac{\partial\vec{v}}{\partial z}\cdot\nabla(D_tQ_1)] = \nabla_2^2(C^2Q_2 - gQ_1)$$

Rearranging terms, we have:

$$[D_t^2 Q_2 - \frac{1}{2}\frac{g\rho_o}{p_o} D_t^2 Q_1 + \frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1)] - \nabla_2^2(C^2 Q_2 - gQ_1) = \frac{\partial}{\partial z} D_t^2 Q_1$$

$$\frac{\partial}{\partial z} D_t^2 Q_1 = (g\nabla_2^2 - \frac{1}{2}\frac{g\rho_o}{p_o} D_t^2)Q_1 + (D_t^2 - C^2\ \nabla_2^2)Q_2 + \frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1).$$

The above equation corresponds to Pierce's equation 8), [Pierce, 1967], if $\frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1) = 0$. However, this is the "wind" term and setting this term equal to zero would defeat much of the purpose of the preceeding analysis. Pierce [Pierce, 1967] takes the wind into account by using a Doppler shifted frequency, $\Omega = \omega - \vec{k} \cdot \vec{v}$, which will be developed in Appendix 2.

Continuing with the derivation of the residual equations, we substitute 4a), 4b) and 5) into 7). Equation 7) is shown below.

$$\rho_o\ D_t^2\ u_z = -\ D_t\ (\frac{\partial p}{\partial z}) + g[\vec{u}\cdot\nabla\rho_o + \rho_o\nabla\cdot\vec{u}] \qquad 7)$$

Substituting the potentials defined above, we have

$$\rho_o D_t^2(p_o^{-\frac{1}{2}}\ D_t Q_1) = -D_t\frac{\partial}{\partial z}[-\rho_o p_o^{-\frac{1}{2}}(C^2 Q_2 - gQ_1)] + g[u_z\frac{d\rho_o}{dz} + \rho_o(p_o^{-\frac{1}{2}}\ D_t Q_2)]$$

$$\rho_o p_o^{-\frac{1}{2}}\ D_t^3 Q_1 = D_t(\rho_o p_o^{-\frac{1}{2}})\ \frac{\partial}{\partial z}\ (C^2 Q_2 - gQ_1) + D_t[(\frac{\partial}{\partial z}(\rho_o p_o^{-\frac{1}{2}}))(C^2 Q_2 - gQ_1)] +$$

$$g[(p_o^{-\frac{1}{2}}\ D_t Q_1)(\frac{d\rho_o}{dz} + \rho_o p_o^{-\frac{1}{2}}\ D_t Q_2)]$$

$$\text{Now } \frac{\partial}{\partial z}\ (\rho_o p_o^{-\frac{1}{2}}) = \rho_o(-\ \frac{1}{2}\ p_o^{-\frac{3}{2}}\frac{dp_o}{dz}) + p_o^{-\frac{1}{2}}\frac{d\rho_o}{dz}$$

$$= -\ \frac{1}{2}\ \rho_o p_o^{-\frac{1}{2}}\ (-g\rho_o) + p_o^{-\frac{1}{2}}\frac{d\rho_o}{dz},\ \text{since}\ \frac{dp_o}{dz} = -\ g\rho_o$$

$$\rho_o p_o^{-\frac{1}{2}}\ D_t^3 Q_1 = \rho_o p_o^{-\frac{1}{2}}\ D_t\ \frac{\partial}{\partial z}\ (C^2 Q_2 - gQ_1) + (\frac{1}{2}\ g\rho_o^2\ p_o^{-\frac{3}{2}} + p_o^{-\frac{1}{2}}\frac{d\rho_o}{dz})D_t(C^2 Q_2 - gQ_1)$$

$$+\ g[p_o^{-\frac{1}{2}}\ D_t Q_1\ \frac{d\rho_o}{dz} + \rho_o p_o^{-\frac{1}{2}}\ D_t Q_2]$$

$$\rho_o P_o^{-\frac{1}{2}} D_t^3 Q_1 = \rho_o P_o^{-\frac{1}{2}} D_t \frac{\partial}{\partial z} (C^2 Q_2 - gQ_1) + (\frac{1}{2} g\rho_o^2 P_o^{-\frac{3}{2}}) D_t (C^2 Q_2 - gQ_1)$$

$$+ P_o^{-\frac{1}{2}} \frac{d\rho_o}{dz} D_t C^2 Q_2 + g\rho_o P_o^{-\frac{1}{2}} D_t Q_2$$

$$D_t^3 Q_1 = D_t \frac{\partial}{\partial z} (C^2 Q_2 - gQ_1) + (\frac{1}{2} g\rho_o P_o^{-1}) D_t (C^2 Q_2 - gQ_1) + \frac{1}{\rho_o} \frac{d\rho_o}{dz} C^2 D_t Q_2 + gD_t Q_2$$

$$= C^2 D_t \frac{\partial}{\partial z} (Q_2) - gD_t \frac{\partial}{\partial z} (Q_1) + \frac{1}{2} gC^2 \rho_o P_o^{-1} D_t Q_2 - \frac{1}{2} g^2 \rho_o P_o^{-1} D_t Q_1$$

$$+ \frac{1}{\rho_o} \frac{d\rho_o}{dz} C^2 D_t Q_2 + gD_t Q_2$$

$$D_t \frac{\partial}{\partial z} Q_2 = \frac{1}{C^2} D_t^3 Q_1 + \frac{g}{C^2} D_t \frac{\partial}{\partial z} Q_1 - \frac{1}{2} g\rho_o P_o^{-1} D_t Q_2 + \frac{1}{2} \frac{g^2}{C^2} \rho_o P_o^{-1} D_t Q_1$$

$$- \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t Q_2 - \frac{g^2}{C^2} D_t Q_2$$

$$= (\frac{1}{C^2} D_t^3 + \frac{g}{C^2} D_t \frac{\partial}{\partial z} + \frac{1}{2} \frac{g^2}{C^2} \rho_o P_o^{-1} D_t) Q_1$$

$$+ (- \frac{g}{C^2} D_t - \frac{1}{2} g\rho_o P_o^{-1} D_t - \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t) Q_2$$

$$D_t^2 \frac{\partial}{\partial z} Q_2 = (\frac{1}{C^2} D_t^4 + \frac{1}{2} \frac{g^2}{C^2} \rho_o P_o^{-1} D_t^2) Q_1 + \frac{g}{C^2} D_t^2 \frac{\partial}{\partial z} Q_1$$

$$- (\frac{1}{2} g\rho_o P_o^{-1} D_t^2 + \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t^2 + \frac{g}{2} D_t^2) Q_2$$

We now use 7) to substitute for $D_t^2 \frac{\partial}{\partial z} Q_1$

$$D_t^2 \frac{\partial}{\partial z} Q_2 = (\frac{1}{C^2} D_t^4 + \frac{1}{2} \frac{g^2}{C^2} \rho_o P_o^{-1} D_t^2) Q_1$$

$$+ \frac{g}{C^2} [(g(\nabla_2^2) - \frac{1}{2} \frac{g\rho_o}{P_o} D_t^2) Q_1 + (D_t^2 - C^2 \nabla_2^2) Q_2 + \frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1)]$$

$$- (\frac{1}{2} g\rho_o P_o^{-1} D_t^2 + \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t^2 + \frac{g}{C^2} D_t^2) Q_2$$

$$= \left(\frac{1}{C^2} D_t^4 + \frac{g^2}{C^2} \left(\frac{\vec{v}}{z} \cdot \nabla D_t\right)\right) Q_1$$

$$- \left(g \nabla_2^2 + \frac{1}{2} g\rho_o P_o^{-1} D_t^2 + \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t^2\right) Q_2 \qquad 8)$$

This result is also different from Pierce's equation 8) [Pierce, 1967]. The further substitution of $C^2 = \gamma p_o/\rho_o$ has been made in Pierce's result, but the discrepancy is in the number of terms.

The matix formulation of the residual equations (Equations 7) and 8)) is as follows:

$$D_t^2 \frac{\partial}{\partial z} Q_1 = \left(g \nabla_2^2 - \frac{1}{2} \frac{g\rho_o}{P_o} D_t^2 + \frac{\partial \vec{v}}{\partial z} \cdot \nabla D_t\right) Q_1 + \left(D_t^2 - C^2 \nabla_2^2\right) Q_2$$

$$D_t^2 \frac{\partial}{\partial z} Q_2 = \left(\frac{1}{C^2} D_t^4 + \frac{g^2}{C^2} \nabla_2^2 + \frac{g}{C^2} \left(\frac{\partial v}{\partial z} \cdot \nabla D_t\right)\right) Q_1 + \left(-g \nabla_2^2 - \frac{1}{2} g\rho_o P_o^{-1} D_t^2 - \frac{1}{\rho_o} \frac{d\rho_o}{dz} D_t^2\right) Q_2$$

$$D_t^2 \frac{\partial}{\partial z} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

A_{11}, A_{12}, A_{21}, A_{22} are apparent from the above equations.

Appendix 2

Derivation of the Normal Mode Dispersion Function from the Residual Equations

If we assume that $Q_1 = V_1 \exp[-i(\omega t - k_x x - k_y y)]$

$$Q_2 = V_2 \exp[-i(\omega t - k_x x - k_y y)]$$

Then D_t^2 becomes $-(\omega - v_x k_x - v_y k_y)^2$ and ∇_2^2 becomes $-(k_x^2 + k_y^2)$. We want to express the residual equations (Appendix 1) using the above definitions. To that end we have the following intermediate caculations:

$$\frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1) = \frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t V_1 \exp[-i(\omega t - k_x x - k_y y)])$$

But $D_t V_1 \exp[-i(\omega t - k_x x - k_y y)] = V_1[-i(\omega - k_x v_x - k_y v_y)]\exp[-i(\omega t - k_x x - k_y y)]$

$$\nabla(D_t Q_1) = [ik_x(-i(\omega - k_x v_x - k_y v_y))]V_1 \exp[-i(\omega t - k_x x - k_y y)]\vec{e}_x + [ik_y(-i(\omega - k_x v_x - k_y v_y))]V_1 \exp[-i(\omega t - k_x x - k_y y)]\vec{e}_y + \frac{dV_1}{dz} \exp[-i(\omega t - k_x x - k_y y)]\vec{e}_z$$

Then $\frac{\partial \vec{v}}{\partial z} \cdot \nabla(D_t Q_1) = \frac{\partial v_x}{\partial z} (\nabla(D_t Q_1))_x + \frac{\partial v_y}{\partial z} (\nabla(D_t Q_1))_y$

$$= [\frac{\partial v_x}{\partial z} \cdot k_x(\omega - k_x v_x - k_y v_y) + \frac{\partial v_y}{\partial z} \cdot k_y(\omega - k_x v_x - k_y v_y)] \cdot V_1 \exp[-i(\omega t - k_x x - k_y y)]$$

The residual equations can be written as (after the additional substitution of $\Omega = \omega - k_x v_x - k_y v_y$ and $k^2 = k_x^2 + k_y^2$):

$$\frac{\partial}{\partial z} V_1 = [\frac{g}{\Omega^2} k^2 - \frac{1}{2} \frac{g\rho_o}{p_o} + \frac{\partial v_x}{\partial z} \cdot \frac{k_x}{\Omega} + \frac{\partial v_y}{\partial z} \cdot \frac{k_y}{\Omega}]V_1 + [1 - \frac{c^2}{\Omega^2} k^2]V_2$$

$$\frac{\partial}{\partial z} V_2 = \left[-\frac{\Omega^2}{2} + \frac{g^2}{2}\frac{k^2}{2} + \frac{g}{2}\left(\frac{\partial v_x}{\partial z} \cdot \frac{k_x}{\Omega} + \frac{\partial v_y}{\partial z} \cdot \frac{k_y}{\Omega}\right)\right]V_1 + \left[-\frac{gk^2}{\Omega^2} - \frac{1}{2}\frac{g\rho_o}{p_o} - \frac{1}{\rho_o \Omega}\frac{d\rho_o}{dz}\right]V_2$$

The equivalent matrix form is:

$$\frac{d}{dz}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $A_{11} = \frac{g}{\Omega^2} k^2 - \frac{1}{2}\frac{g\rho_o}{p_o} + \frac{\partial v_x}{\partial z}\frac{k_x}{\Omega} + \frac{\partial v_y}{\partial z} \cdot \frac{k_y}{\Omega}$

$$A_{12} = 1 - \frac{c^2}{\Omega^2} k^2$$

$$A_{21} = -\frac{\Omega^2}{c^2} + \frac{g^2}{c^2}\frac{k^2}{\Omega^2} + \frac{g}{c^2}\left(\frac{\partial v_x}{\partial z} \cdot \frac{k_x}{\Omega} + \frac{\partial v_y}{\partial z} \cdot \frac{k_y}{\Omega}\right)$$

$$A_{22} = -\frac{gk^2}{\Omega^2} - \frac{1}{2}\frac{g\rho_o}{p_o} - \frac{1}{\rho_o\Omega}\frac{d\rho_o}{dz}$$

$\frac{\partial}{\partial z}$ becomes $\frac{d}{dz}$ since V_1 and V_2 depend only on z.

The atmosphere is divided up into a number of isothermal layers, where the coefficients in the matrix differential equation are constant for that layer. The topmost layer is taken as extending out to infinity, while the boundary of the lowest layer is the earth's surface. Certain boundary conditions have to be satisfied for the results to be physically meaningful. These conditions are that V_1 and V_2 vanish as z goes to infinity and that V_1 vanish at the earth's surface, which implies no vertical displacement at the earth's surface.

The analysis depends heavily on the concept of the state transition matrix. The solution for the topmost layer is obtained first. The solution in any other layer can then be found since V_1 and V_2 are constant across the interface and the coefficients of the matrix differential equation are constant. The calculations embodying the above statements are as follows:

We start with the matrix differential equation

$$\frac{d}{dz}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where A_{11}, A_{12}, A_{21}, A_{22} are constant for each sublayer. The state transition matrix T is then calculated. (The state transition matrix relates the state within a layer to the values of the variables at the top interface of the layer). Thus

$$\begin{bmatrix} V_1(Z+\delta) \\ V_2(Z+\delta) \end{bmatrix} = T(Z+\delta)\begin{bmatrix} V_1(Z) \\ V_2(Z) \end{bmatrix} \qquad \delta < 0$$

The state transition matrix is defined as

$$T = L^{-1}\left\{(sI-A)^{-1}\right\}$$

where L^{-1} is the inverse Laplace transform and I is the identity matrix.

$$sI-A = \begin{bmatrix} s-A_{11} & -A_{12} \\ -A_{21} & s-A_{22} \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s-A_{11})(s-A_{22}) - A_{12}\, A_{21}}\begin{bmatrix} s-A_{22} & A_{12} \\ A_{21} & s-A_{11} \end{bmatrix}$$

$$\begin{aligned}(s-A_{11})(s-A_{22}) - A_{12}\, A_{21} &= s^2 - (A_{11}+A_{12})s + A_{11}\, A_{22} - A_{12}\, A_{21} \\ &= s^2 - Js + K\end{aligned}$$

where $J = A_{11} + A_{12}$ $\qquad K = A_{11}\, A_{22} - A_{12}\, A_{21}$

$$\begin{aligned}(s-A_{11})(s-A_{22}) - A_{12}\, A_{21} &= \left(s - \frac{J + \sqrt{J^2-4K}}{2}\right)\left(s - \frac{J - \sqrt{J^2-4K}}{2}\right) \\ &= (s - L_1)(s - L_2)\end{aligned}$$

where $L_1 = \frac{J + \sqrt{J^2-4K}}{2}$, $L_2 = \frac{J - \sqrt{J^2-4K}}{2}$

Computing the elements of the T matrix, we have

$$T_{11} = L^{-1}\left[\frac{s-A_{22}}{(s-L_1)(s-L_2)}\right] = L^{-1}\left[\frac{\frac{L_1-A_{22}}{L_1-L_2}}{s-L_1} + \frac{\frac{L_2-A_{22}}{L_2-L_1}}{s-L_2}\right]$$

$$= \frac{(L_1-A_{22})}{L_1-L_2}\, e^{L_1 z} - \frac{(L_2-A_{22})}{L_1-L_2}\, e^{L_2 z} \qquad L_1-L_2 = (J^2-4K)^{1/2}$$

$$= \frac{1}{(J^2-4K)^{1/2}}\left[(L_1-A_{22})\, e^{L_1 z} - (L_2-A_{22}) e^{L_2 z}\right]$$

$$T_{12} = L^{-1}\left[\frac{A_{12}}{(s-L_1)(s-L_2)}\right] = A_{12}\, L^{-1}\left[\frac{\frac{1}{L_1-L_2}}{s-L_1} + \frac{\frac{1}{L_2-L_1}}{s-L_2}\right]$$

$$= \frac{A_{12}}{(J^2-4K)^{1/2}}\left[e^{L_1 z} - e^{L_2 z}\right]$$

Similarly, $T_{21} = \frac{A_{12}}{(J^2-4K)^{1/2}}\left[e^{L_1 z} - e^{L_2 z}\right]$

$$T_{22} = \frac{1}{(J^2-4K)^{1/2}}\left[(L_1-A_{11})\, e^{L_1 z} - (L_2-A_{11})\, e^{L_2 z}\right]$$

For the first boundary condition (exponential decrease of V_1 and V_2 as z goes to infinity) to be fulfilled, we must have $Re(L_1) < 0$, $Re(L_2) < 0$. This condition is necessary to trap the energy from the source near the surface of the earth. Francis [Francis, 1973] has shown that a variety of other conditions will accomplish the same effect.

Following Pierce, [Pierce, 1967] we assume that one exponentially decaying term suffices to describe V_1 in the topmost layer.

$$V_1 = C_1 \exp[-b(z-z_m)] \qquad z \geq z_m$$

$$\frac{dV_1}{dz} = A_{11} V_1 + A_{12} V_2 = -b V_1$$

$$V_2 = \frac{-b-A_{11}}{A_{12}} V_1$$

Therefore, for the top layer extending to infinity, where z_m is the height of the top interface,

$$\begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = C \begin{bmatrix} A_{12} \\ -b-A_{11} \end{bmatrix} \exp[-b(z-z_m)].$$

We want to obtain the solution for V_1 and V_2 at the surface, knowing the values at the top layer. This is easily obtained since

$$\begin{bmatrix} V_1^s \\ V_2^s \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} V_1^u \\ V_2^u \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} C(A_{12}) \\ C(-b-A_{11}) \end{bmatrix}$$

where u stands for the top interface and s stands for surface values. The matrix R is the product of all the transition matrices for the layers. For $V_1^s = 0$ we have $R_{11} V_1^u + R_{12} V_2^u = 0$ or $R_{11}A_{12}^u - R_{12}(b+A_{11}^u) = 0$.
The last equation defines the normal mode dispersion function, i.e.,

$$F(\omega,k_x,k_y) = R_{11}A_{12}^u + R_{12}(b-A_{11}^u) \qquad \text{(A2.1)}$$

$F(\omega,k_x,k_y)$ is also a function of the height profiles of C, v_x, v_y and ρ_o. If the group velocity direction is held constant (as it will be for a single array) then another dispersion function can be obtained, which Pierce [Pierce, 1967] calls the auxiliary dispersion function $P(\omega,k,k_y,\theta)$.

$$P(\omega,k_x,k_y,\theta) = \left(\frac{\partial F}{\partial k_x}\right) \sin\theta - \left(\frac{\partial F}{\partial k_y}\right) \cos\theta.$$

Here θ is the angle between the group velocity vector and the x-axis. The group velocity may be calculated from the equations:

$$V_{gx} = -\left(\frac{\partial F}{\partial k_x}\right) / \left(\frac{\partial F}{\partial \omega}\right) = V_g \cos\theta$$

$$V_{gy} = -\left(\frac{\partial F}{\partial k_y}\right) / \left(\frac{\partial F}{\partial \omega}\right) = V_g \sin\theta$$

The phase velocity may be calculated from the equations $V_p = (\omega^2/k^2)^{1/2}$

and
$$\cos\theta_k = k_x/k$$
$$\sin\theta_k = k_y/k$$

θ_k is the direction of the phase velocity and is defined by the above equations. The "normal" mode solutions are the solutions of the residual equations when $F(\omega,k_x,k_y) = 0$ or $P(\omega,k_x,k_y,\theta) = 0$ for a fixed direction of θ.

$\frac{\partial F}{\partial\omega}$, $\frac{\partial F}{\partial k_x}$, $\frac{\partial F}{\partial k_y}$ are computed as follows (P can be substituted for F and the same general analysis holds). Let q denote ω, k_x or k_y. Then from equation (A2.1) we have

$$\frac{\partial F}{\partial q} = \left(\frac{\partial R_{11}}{\partial q}\right) A^u_{12} - \left(\frac{\partial R_{12}}{\partial q}\right) (b + A^u_{11}) + R_{11}\left(\frac{\partial A^u_{12}}{\partial q}\right) - R_{12}\left(\frac{\partial b}{\partial q} + \frac{\partial A^u_{11}}{\partial q}\right)$$

The derivatives of the matrix terms may be obtained from the equation

$$\left[\frac{\partial R}{\partial q}\right] = \sum_{i=1}^{M} \left[G^{(i)}\right]\left[\partial T^{(i)}/\partial q\right]\left[H^{(i)}\right]$$ (M is the total number of layers)

where
$$G^{(i)} = \left[T^{(1)}\right]\left[T^{(2)}\right] \ldots \left[T^{(i-1)}\right]$$
$$H^{(i)} = \left[T^{(i+1)}\right]\left[T^{(i+2)}\right] \ldots \left[T^{(M)}\right] .$$

The differentials of the transition matrix $\left[\frac{\partial T^{(i)}}{\partial q}\right]$, may be calculated, after some algebra, since the transition matrix is made up of known functions of ω, k_x and k_y. The same holds true for $\frac{\partial b}{\partial q}$, $\frac{\partial A^u_{11}}{\partial q}$ and $\frac{\partial A^u_{12}}{\partial q}$; i.e., b, A^u_{11} and A^u_{12} are known functions of ω, k_x and k_y.

Appendix 3

Listing of Computer Programs

```
C
C   IMPLEMENTATION OF WIENER'S LMS ALGORITHMN
C
        DIMENSION CVSUM(3),A(128,3),VAR(3),V1(3),V(3),SUM(3),SQUARE(3)
        DIMENSION X(32766,3),Y(32766,3)
        CHARACTER*20 IN1,IN2,IN3,IN4,OUT1,OUT2,OUT3,OUT4,OUT5,OUT6
C
C   INPUT PARAMETERS
C
1       FORMAT(A)
2       FORMAT(I2)
3       FORMAT(F9.9)
        TYPE 1, '$NUMBER OF FILTER WEIGHTS: '
        ACCEPT *, NUMFC
        TYPE 1, '$VALUE OF TIME DELAY (DELTA): '
        ACCEPT *, IDELTA
        TYPE 1, '$TYPE IN INITIAL FILTER COEFF FOR 1ST CHANNEL? '
        TYPE 1, '$(1=YES,2=NO) '
        ACCEPT *,INFC
        IF(INFC.NE.1) GOTO 68
        ACCEPT *, (A(J,1),J=1,NUMFC)
68      TYPE 1, '$CONVERGENCE PARAMETER ALPHA :  '
        ACCEPT *, ALPHA
        B=1.0-ALPHA
        TYPE 1, '$MINIMUM VARIANCE:  '
        ACCEPT *, VARM
        TYPE 1, '$ITERATION # FOR COEFF. DUMP OUT:  '
        ACCEPT *, IDP
        TYPE 1, '$INPUT FILENAME #1: '
        ACCEPT 1, IN1
        CALL GETLEN(IN1,ISIZE)
        OPEN (UNIT=1, NAME=IN1, STATUS='OLD', FORM='UNFORMATTED')
        TYPE 1, '$INPUT FILENAME #2: '
        ACCEPT 1, IN2
        OPEN (UNIT=2, NAME=IN2, STATUS='OLD', FORM='UNFORMATTED')
        TYPE 1, '$INPUT FILENAME #3: '
        ACCEPT 1, IN3
        OPEN (UNIT=3, NAME=IN3, STATUS='OLD', FORM='UNFORMATTED')
        NPTS=(ISIZE-((ISIZE+2044)/2048)*4)/4
        TYPE *, NPTS
        TYPE 1,'$OUTPUT FILENAME #1: '
        ACCEPT 1,OUT1
        OPEN (UNIT=10, NAME=OUT1, STATUS='NEW',FORM='UNFORMATTED')
        TYPE 1,'$OUTPUT FILENAME #2: '
        ACCEPT 1,OUT2
        OPEN (UNIT=11, NAME=OUT2, STATUS='NEW',FORM='UNFORMATTED')
        TYPE 1,'$OUTPUT FILENAME #3: '
        ACCEPT 1,OUT3
        OPEN (UNIT=12, NAME=OUT3, STATUS='NEW',FORM='UNFORMATTED')
        READ (1) (X(I,1),I=1,NPTS)
        READ (2) (X(I,2),I=1,NPTS)
        READ (3) (X(I,3),I=1,NPTS)
        TYPE 1,'$OUTPUT FILTER COEFF. TO FILE? (1=YES,2=NO) '
        ACCEPT *, IANS2
        IF (IANS2.NE.1) GOTO 99
        TYPE 1,'$COEFF. FILE NAME? (1ST FILE) '
        ACCEPT 1,OUT4
        TYPE 1,'$COEFF. FILE NAME? (2ND FILE) '
        ACCEPT 1,OUT5
        TYPE 1,'$COEFF. FILE NAME? (3RD FILE) '
        ACCEPT 1,OUT6
```

```
      OPEN (UNIT=13, NAME=OUT4, STATUS='NEW', FORM='UNFORMATTED')
      OPEN (UNIT=14, NAME=OUT5, STATUS='NEW', FORM='UNFORMATTED')
      OPEN (UNIT=15, NAME=OUT6, STATUS='NEW', FORM='UNFORMATTED')
C
C  INITIALIZE VARIABLES
C
      DO 54 L4=1,3
      SUM(L4)=0.0
      SQUARE(L4)=0.0
      DO 66 M6=1,IDELTA
      Y(NPTS-IDELTA+1,L4)=0.0
66    CONTINUE
      DO 55 L5=1,NUMFC
      A(L5,L4)=0.0
55    CONTINUE
54    CONTINUE
C
C  DO LOOP OVER THE ARRAY OF POINTS
C
99    DO 51 L1=1,NPTS-IDELTA
C
C  CONVOLVE DATA WITH FILTER
C
      K=NUMFC
      IF(L1.LT.NUMFC) K=L1
              DO 53 L3=1,3
              CVSUM(L3)=0.0
                      DO 52 L2=1,K
                      CVSUM(L3)=CVSUM(L3)+X(L1-L2+1,L3)*A(L2,L3)
52                    CONTINUE
              Y(L1,L3)=X(L1+IDELTA,L3)-CVSUM(L3)
53            CONTINUE
C
C  DO LOOP FOR OUTPUTTING FILTER COEFFICIENTS
C
      IF(IDP.NE.L1) GOTO 108
      TYPE *, IDP
      WRITE (13) (A(KK,1),KK=1,NUMFC)
      WRITE (14) (A(KK,2),KK=1,NUMFC)
      WRITE (15) (A(KK,3),KK=1,NUMFC)
      DO 108 K8=1,NUMFC
      TYPE *,  K8,A(K8,1)
108   CONTINUE
C
C  ESTIMATE THE VALUE OF THE VARIANCE OF X
C
      DO 109 K9=1,3
      SQUARE(K9)=SQUARE(K9)+X(L1,K9)*X(L1,K9)
      VAR(K9)=SQUARE(K9)
      IF(K.NE.1) VAR(K9)=VAR(K9)/REAL(K-1)
      IF(L1.GT.NUMFC) SQUARE(K9)=SQUARE(K9)-X(L1-K,K9)*X(L1-K,K9)
      V(K9)=0.0
      IF(VAR(K9).GT.VARM) V(K9)=(1.0-B)/VAR(K9)
C
C  UPDATE FILTER COEFFICIENTS
C
      DO 107 K7=1,K
      A(K7,K9)=A(K7,K9)+V(K9)*Y(L1,K9)*X(L1-K+K7,K9)
107   CONTINUE
109   CONTINUE
51    CONTINUE
```

```
WRITE (10) (Y(L,1),L=1,NPTS)
WRITE (11) (Y(L,2),L=1,NPTS)
WRITE (12) (Y(L,3),L=1,NPTS)
CLOSE (1)
CLOSE (2)
CLOSE (3)
CLOSE (10)
CLOSE (11)
CLOSE (12)
CLOSE (13)
CLOSE (14)
CLOSE (15)
STOP 'END OF RUN'
END
```

```
C
C   COMPUTE THE POWER RATIO USING A RECTANGULAR FILTER
C
        REAL NSUM
        DIMENSION PNSUM(3),PSSUM(3)
        DIMENSION X(32766,3),Y(32766)
        CHARACTER*20 IN1,IN2,IN3,IN4,OUT1,OUT2,OUT3,OUT4
C
C   INITIALIZE VARIABLES
C
        SSUM=0.0
        NSUM=0.0
C
C   INPUT PARAMETERS
C
1       FORMAT(A)
2       FORMAT(I2)
3       FORMAT(F9.9)
        TYPE 1, '$LENGTH OF BACKGROUND WINDOW: '
        ACCEPT *, LW1
        TYPE 1, '$LENGTH OF SIGNAL WINDOW:  '
        ACCEPT *, LW2
        TYPE 1, '$DISTANCE BETWEEN WINDOWS:  '
        ACCEPT *, LD
        TYPE 1, '$TOLERANCE FOR NOISE WINDOW: '
        ACCEPT *,TOL
        NN1=LD+LW1
        NN2=NN1+LW2
        DO 102 M2=1,NN1
        Y(M2)=0.0
102     CONTINUE
        TYPE 1, '$INPUT FILENAME #1: '
        ACCEPT 1, IN1
        CALL GETLEN(IN1,ISIZE)
        OPEN (UNIT=1, NAME=IN1, STATUS='OLD', FORM='UNFORMATTED')
        TYPE 1, '$INPUT FILENAME #2: '
        ACCEPT 1, IN2
        OPEN (UNIT=2, NAME=IN2, STATUS='OLD', FORM='UNFORMATTED')
        TYPE 1, '$INPUT FILENAME #3: '
        ACCEPT 1, IN3
        OPEN (UNIT=3, NAME=IN3, STATUS='OLD', FORM='UNFORMATTED')
        NPTS=(ISIZE-((ISIZE+2044)/2048)*4)/4
        TYPE *, NPTS
        TYPE 1,'$OUTPUT FILENAME #1: '
        ACCEPT 1,OUT1
        OPEN (UNIT=10, NAME=OUT1, STATUS='NEW',FORM='UNFORMATTED')
        READ (1) (X(I,1),I=1,NPTS)
        READ (2) (X(I,2),I=1,NPTS)
        READ (3) (X(I,3),I=1,NPTS)
C
C   PROCESS THE DATA
C
                DO 53 L3=1,3
                DO 56 L6=1,LW1
                PNSUM(L3)=PNSUM(L3)+X(L6,L3)*X(L6,L3)
56              CONTINUE
                DO 57 L7=NN1,NN1+LW2
                PSSUM(L3)=PSSUM(L3)+X(L7,L3)*X(L7,L3)
57              CONTINUE
                        DO 52 L2=NN1,NPTS-LW2-1
                        SSUM=SSUM+PSSUM(L3)
```

```
                    NSUM=NSUM+PNSUM(L3)
              IF((L3.EQ.3).AND.(NSUM.GT.TOL)) Y(L2)=SSUM/NSUM
                    IF(L3.EQ.3) SSUM=0.0
                    IF(L3.EQ.3) NSUM=0.0
              PSSUM(L3)=PSSUM(L3)+X(L2+LW2+1,L3)**2-X(L2,L3)**2
              PNSUM(L3)=PNSUM(L3)+X(L2-LD+1,L3)**2-X(L2-NN1+1,L3)**2
52                  CONTINUE
53            CONTINUE
        WRITE (10) (Y(L),L=1,NPTS)
        CLOSE (1)
        CLOSE (10)
        STOP 'END OF RUN'
        END
```

SIGNAL DETECTION AND ENHANCEMENT OF

INFRASOUND EVENTS

by

MARK LOUIS SCHUETTE

B.S., KANSAS STATE UNIVERSITY, 1979

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1983

Abstract

This thesis examines signal enhancement and detection schemes for detecting infrasound events. A theoretical review of the normal mode analysis of infrasound as made by previous authors was examined, with a veiw toward frequency filtering. Actual frequency filtering was accomplished using Widrow's (least mean square) LMS filter. The data sets used for the LMS filter consisted of a signal superimposed on microbarom noise. The signal strength was altered to obtain various signal-to-noise ratios. A power ratio based on the output of the LMS filters was used to examine the effect of changing the signal to noise ratio on detection.