GEOMETRIC PROGRAMMING: METHODS FOR DEALING

WITH DEGREES OF DIFFICULTY

by

PARVIZ PARCHAMI-GHAFEROKHI

B.S., Abadan Institute of Technology, Iran, 1967

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1973

Approved by:

Ching-Lai Hwang

Major Professor

LD
2668
R4
1973
P3
C.2
Doc.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Geometric programming is a branch of nonlinear programming dealing with the problem of minimizing posynomials (polynomials with positive coefficients) subject to certain posynomial inequality constraints.

The general theory of geometric programming was initially developed by Duffin, Peterson and Zener in 1966 [6]. A serious limitation in the application of this theory has been that all the functions involved in the problem are to be posynomials. This shortcoming was overcome by Wilde and Beightler [9] in 1967 when they generalized the theory to allow the use of negative coefficients in both objective function and constraints, and also to permit reversed inequality constraints.

The most recent development of geometric programming has been made by Avriel and Williams in 1971 which is called complementary geometric programming [1,2]. Complementary geometric programming removes the restriction of positive coefficient and solves any rational function of posynomials.

Although the theory of geometric programming has many potential application, it lacks provisions for dealing with "degree of difficulty". A geometric programming has "degree of difficulty" if the number of terms appearing in the objective functions and constraints is more than the number of variables plus one. The method is extremely desirable when the degree of difficulty is zero. In this case the optimal solution is determined by solving a set of simultanous linear equations. For problems with one degree of difficulty Duffin et al. [6] has shown that the optimal value of the objective function can be found by expressing the dual variables in terms of one of the variables, substituting them into the dual function and maximizing the dual function by one of the one dimensional search techniques.

When the degree of difficulty is more than one the solution to the problem has to be found through optimizing the dual objective function subject to linear constraints. In 1969 Kochenberger [8] used the method of Lagrange multipliers for maximizing the dual function subject to linear constraints. He used Newton-Raphson iterative technique for locating the maximum point. Williams in 1972 used separable programming for solving the dual problem. The method was based on making the dual objective function separable by a simple linear transformation. Complementary geometric programming can also be used in problems having more than one degree of difficulty [1,2].

The purpose of this report is to present a summary of techniques for extending the applicability of geometric programming to problems with degrees of difficulty. The following chapter presents some of the Duffin's original work. Chapter III presents the Wilde and Beightler's Formulation of generalized geometric programming. Chapter IV presents complementary geometric programming. Chapter V presents some of the solution techniques for optimizing geometric programming problems with one or more degrees of difficulty. Finally, Chapter VI presents some conclusions and the limitations involved in methods described in Chapter V. Examples are solved for each method to illustrate the algorithm.

CHAPTER II

GEOMETRIC PROGRAMMING

1. INTRODUCTION

Geometric programming is defined as the problem of minimizing posynomials (polynomials with positive coefficient) subject to certain inequality constraints.

The basic theory of geometric programming is based on the arithmetic mean geometric mean inequality which states that the arithmetic mean is at least as great as the geometric mean.

The primal problem of any geometric programming is defined as minimizing a posynomial S subject to certain posynomial inequality constraints. Let M denote the constrained minimum value of the primal function, S, then there is a related maximization problem concerning a function v which is called the dual function.

The problem of maximizing the dual function v subject to certain linear constraints is called the dual program. It has been shown that M is the constrained maximum value of v as well as being the constrained minimum value of S [6].

2. POSYNOMIALS

A posynomial is a function of real value consisting of finite sum of positive terms given as

$$S = U_1 + U_2 + \ldots + U_n \tag{1}$$

where

$$U_j = C_j \prod_{i=1}^{m} x_i^{a_{ij}}, \qquad j = 1, 2, \ldots, n \tag{2}$$

or

$$S = \sum_{j=1}^{n} C_j \prod_{i=1}^{m} x_i^{a_{ij}} \tag{3}$$

where C_j are positive constants and the a_{ij} are arbitrary real constants. The design variables $x_1, x_2, \ldots, x_n$ are assumed to be positive variables.

3. ARITHMETIC MEANS AND GEOMETRIC MEANS

Geometric programming as mentioned before is based on the arithmetic mean geometric mean inequality which states that the arithmetic mean is at least as great as the geometric mean.

For any number of posynomial terms U_n,

$$\delta_1 U_1 + \delta_2 U_2 + \ldots + \delta_n U_n \geq U_1^{\delta_1} U_2^{\delta_2} \ldots U_n^{\delta_n} \tag{4}$$

where

$$\delta_1 + \delta_2 + \ldots + \delta_n = 1 \tag{5}$$

δ_i are the weights which must sum to unity, that is the normality condition. The equality sign in (4) is satisfied only if all the U_i, $i = 1, \ldots, m$, are equal.

4. THE PRIMAL FUNCTION

Optimization problems are often concerned with the problems of minimizing an objective function of the form

$$S = U_1 + U_2 + \dots + U_n \tag{6}$$

or

$$S = \sum_{j=1}^{n} C_j \prod_{i=1}^{m} x_i^{a_{ij}} \tag{7}$$

$$x_i \geq 0 \tag{8}$$

where S is posynomial if C_j are positive. S is called the primal function and $x_1, x_2, \dots, x_m$ are called primal variables. The conditions (8) are called natural constraints. The matrix (a_{ij}) is called the exponent matrix.

5. THE DUAL FUNCTION

The arithmetic mean geometric mean inequality (4) can be changed to

$$u_1 + u_2 + \dots + u_n \geq \left(\frac{u_1}{\delta_1}\right)^{\delta_1} \left(\frac{u_2}{\delta_2}\right)^{\delta_2} \dots \left(\frac{u_n}{\delta_n}\right)^{\delta_n} \tag{9}$$

if we let $u_i = \delta_i U_i$ for $i = 1, 2, \dots, n$. The left hand side of (9) is the primal function S to be minimized. The right hand side of (9) is called the predual function and is denoted by V.

$$S \geq V \tag{10}$$

Substituting (2) into the predual function gives

$$V = \left(\frac{C_1}{\delta_1}\right)^{\delta_1} \left(\frac{C_2}{\delta_2}\right)^{\delta_2} \cdots \left(\frac{C_n}{\delta_n}\right)^{\delta_n} x_1^{D_1} x_2^{D_2} \cdots x_m^{D_m} \tag{11}$$

where

$$D_j = \sum_{i=1}^{n} \delta_i \cdot a_{ij}, \quad j = 1,2, \ldots, m \tag{12}$$

$$\delta_i \geq 0 \tag{13}$$

δ_i are called dual variables. Relation (13) is called the positivity condition.

If we choose the weights δ_i so that D_i are zero, then the predual function does not depend on the variable x_i, and it is called the dual function, denoted by v.

$$v = \left(\frac{C_1}{\delta_1}\right)^{\delta_1} \left(\frac{C_2}{\delta_2}\right)^{\delta_2} \cdots \left(\frac{C_n}{\delta_n}\right)^{\delta_n} \tag{14}$$

It follows that

$$S \geq M \geq v \tag{15}$$

6. THE MAXIMUM OF THE DUAL FUNCTION

It can be shown that the minimum value of the primal function S is equal to the maximum value of the dual function v subject to the normality and orthogonality conditions [6], or

$$S_{min} = M = v_{max} \tag{16}$$

subject to

$$\sum_{i=1}^{n} \delta_i = 1 \quad \text{(normality condition)} \tag{17}$$

and

$$D_j = \sum_{i=1}^{n} \delta_i \cdot a_{ij} = 0 \quad \text{(orthogonality condition)} \tag{18}$$

$$\delta_i \geq 0 \qquad i = 1, \ldots, n \tag{19}$$

The proof of duality theory is given in [6].

EXAMPLE 1:

Minimize

$$S = \frac{x_2^2}{x_1^2} + x_2 + 2\,\frac{x_1}{x_2}$$

The normality condition is

$$\delta_1 + \delta_2 + \delta_3 = 1$$

and orthogonality conditions are

$$-2\delta_1 + \delta_3 = 0$$

$$\delta_1 + \delta_2 - \delta_3 = 0$$

From these equations, we obtain

$$\delta_1 = \frac{1}{4}$$

$$\delta_2 = \frac{1}{4}$$

and

$$\delta_3 = \frac{1}{2}$$

From equation (14)

$$v = \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{2}{\delta_3}\right)^{\delta_3}$$

or

$$v = (4)^{1/4} \ (4)^{1/4} \ (4)^{1/2} = 4$$

and the minimum value of S is 4.

7. DETERMINATION OF THE MINIMUM POINT

Geometric programming differs from the other optimization techniques in that it gives the minimum value of $S(\bar{x})$ of a posynomial S without first locating the point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ which makes S a minimum. In order to find the optimum point the following relation is used

$$u_j(\bar{x}) = \bar{\delta}_j \ S(\bar{x}), \qquad j = 1,2, \ldots, n \tag{20}$$

where

$$u_j = C_j \prod_{i=1}^{m} x_i^{a_{ji}}, \qquad j = 1,2, \ldots, n \tag{21}$$

In general there are n equations and m primal variables which can be solved to find the minimum point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$.

EXAMPLE 2:

In the previous example we found that $S(\bar{x}) = 4$, $\delta_1 = 1/4$, $\delta_2 = 1/4$ and $\delta_3 = 1/2$. To find x_1 and x_2 at the minimum point equations (20) and (21) can be used,

or

$$\frac{x_2}{x_1^2} = 1/4 \times 4 = 1$$

$$x_2 = 1/4 \times 4 = 1$$

$$2\frac{x_1}{x_2} = 1/2 \times 4 = 2$$

from which, $\bar{x}_1 = 1$ and $\bar{x}_2 = 1$.

8. CONSTRAINED MINIMA

The general problem of geometric programming is to minimize a posynomial S subject to inequality constraints of the form

$$g_k(x_1, x_2, \ldots x_m) \leq 1, \quad k = 1, \ldots, p \tag{22}$$

where the $g_k(x)$ are also posynomials. The primal problem can now be stated as

minimize

$$S = \sum_{i=1}^{n_o} u_i \tag{23}$$

subject to

$$g_k = \sum_{i=m_k}^{n_k} u_i \leq 1, \qquad k = 1,2, \ldots, p \tag{24}$$

where

$$m_k = n_{k-1} + 1, \qquad k = 1, \ldots, p \tag{25}$$

To be able to handle geometric programming problems with inequality constraints we must express the geometric inequality (9) in more general form in which weights are no longer normalized.

The dual problem that corresponds to the primal problem is then maximize

$$v = \left(\prod_{i=1}^{n} \left(\frac{C_i}{\delta_i} \right)^{\delta_i} \right) \prod_{k=1}^{p} \lambda_k^{\lambda_k} \tag{26}$$

where

$$\lambda_k = \sum_{i=m_k}^{n_k} \delta_i, \qquad k = 1,2, \ldots, p \tag{27}$$

and

$$m_1 = n_o + 1,\ m_2 = n_1 + 1,\ \ldots,\ m_p = n_p + 1 \tag{28}$$

$$n_p = n$$

subject to linear constaints of the form

$$\sum_{i=1}^{n_o} \delta_i = 1 \tag{29}$$

$$\sum_{i=1}^{n} a_{ij} \cdot \delta_i = 0, \qquad j = 1, \ldots, m \tag{30}$$

$$\delta_i \geq 0 \tag{31}$$

Equations (29), (30), and (31) are called normality condition, orthogonality condition and positivity condition respectively.

EXAMPLE 3:

Minimize

$$S = x_1^2 + 2x_1x_2$$

subject to

$$\frac{1}{x_1^2 x_2} \leq 2$$

The normality and orthogonality conditions are

$$\delta_1 + \delta_2 = 1$$

$$2\delta_1 + \delta_2 - 2\delta_3 = 0$$

$$\delta_2 - \delta_3 = 0$$

Therefore $\delta_1 = 1/3$, $\delta_2 = 2/3$, and $\delta_3 = 1/3$ and the dual function becomes

$$v = (3)^{1/3}\ (3)^{2/3}\ (3)^{1/3}\quad (1/3)^{1/3} = 3$$

or

$$S^* = 3$$

To find the optimum primal variables $\bar{x}_1$ and $\bar{x}_2$, we have

$$x_1^2 = 1/3 \times 3 = 1$$

$$2x_1x_2 = 2/3 \times 3 = 2$$

Therefore, $\bar{x}_1 = 1$ and $\bar{x}_2 = 1$.

9. THE DUALITY THEORY

A program (either primal or dual) is said to be consistent if there is at least one point that satisfies its constraints. A primal problem is said to be superconsistent if there is at least one vector x^* that has positive components and satisfies the following strict inequality constraints [6]

$$g_k(x^*) < 1, \qquad k = 1, \ldots, p$$

In terms of the above concepts the main theorem of this formulation of geometric programming is stated as:

THEOREM 1. Suppose that the primal problem is superconsistent and that the primal function S attains its constrained minimum value at a point that satisfies the primal constraints.

Then

(i) The corresponding dual problem is consistent and the dual function $v(\delta)$ attains its constrained maximum value at a point which satisfies the dual constraints.

(ii) The constrained maximum value of the dual function is equal to the constrained minimum value of the primal function.

(iii) If $\bar{x}$ is the minimizing point for primal problem, there are non-negative Lagrangian multipliers $\bar{\mu}_k$, $k = 1,2, \ldots, p$, such that the Lagrangian function

$$L(x,\mu) = S(x) + \sum_{i=1}^{p} \mu_k \, [g_k(x) - 1] \tag{33}$$

has the saddle point property

$$L(\bar{x},\mu) \leq S(\bar{x}) = L(\bar{x},\bar{\mu}) \leq L(x,\bar{\mu}) \tag{34}$$

for arbitrary $x_j > 0$ and arbitrary $\mu_k \geq 0$. Moreover, there is a maximizing vector $\bar{\delta}$, for the dual problem whose components are

$$\delta_i = \frac{C_i \sum_{j=1}^{m} x_j^{a_{ij}}}{S(x)}, \qquad i = 1,2, \ldots, n_o \tag{35}$$

$$\delta_i = \frac{\mu_k \, C_i \sum_{j=1}^{m} x_j^{a_{ij}}}{S(x)}, \qquad i = m_k, \ldots, n_k \tag{36}$$

$$k = 1,2, \ldots, p$$

where $x = \bar{x}$ and $\mu = \bar{\mu}$. Furthermore

$$\lambda_k(\bar{\delta}) = \frac{\bar{\mu}_k}{S(\bar{x})}, \qquad k = 1,2, \ldots, p \tag{37}$$

(iv) If $\bar{\delta}$ is a maximizing point for dual problem, each minimizing point $\bar{x}$ for primal problem satisfies the system of equations

$$C_i \prod_{j=1}^{m} x_j^{a_{ij}} = \bar{\delta}_i \cdot v(\bar{\delta}), \qquad i = 1, \ldots, n_o \tag{38}$$

$$C_i \prod_{j=1}^{m} x_j^{a_{ij}} = \bar{\delta}_i/\lambda_k, \qquad i = m_k, \ldots, n_k \tag{39}$$

where k ranges over all possible values for which $\lambda_k(\bar{\delta}) > 0$.

Equations (38) and (39) provide a method for finding a minimum point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ from the knowledge of maximizing point $\bar{\delta}$. Equation (38) and (39) can be solved by taking the logarithm of both sides first and then solving for $\ell n(x_1)$, $\ell n(x_2)$, etc. [6]. From equation (37), we see that the numbers λ_k, aside from a constant factor, are the Lagrangian multipliers for the primal problem.

THEOREM 2. If the primal problem is consistent and there is a point δ^* with positive components which satsifies the constraints of the dual problem, the primal function $S(x)$ attains its constrained minimum value at a point $\bar{x}$ which satisfies the constraints of the primal problem.

The proof of the above theorems are given in [6].

10. DEGREE OF DIFFICULTY

The degree of difficulty of a geometric programming problem is defined as

$$\text{degree of difficulty} = n - (m + 1) \tag{40}$$

where n is the total number of terms in the primal function and primal constraints (forced constraints) and m is the number of primal variables.

If degree of difficulty is zero, the solution is determined by solving the linear constraints without reference to the objective function. If the degree of difficulty is more than zero, then one of the optimization techniques should be used to maximize the non-linear dual objective function subject to the linear dual constraints.

CHAPTER III

GENERALIZED GEOMETRIC PROGRAMMING

Wilde and Beightler generalized the theory of geometric programming to include negative coefficients in both objective function and constraints, and also to permit reversed inequality constraints [9].

The generalized geometric programming problems can be stated as minimize

$$y_0 = \sum_{t=1}^{T_0} \sigma_{0t}\, c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}}, \qquad \sigma_{0t} \equiv \pm 1, \qquad c_{0t} > 0 \tag{1}$$

subject to the constraints

$$y_m = \sum_{t=1}^{T_m} \sigma_{mt}\, c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \leq \sigma_m \equiv \pm 1 \tag{2}$$

$$\sigma_{mt} \equiv \pm 1, \qquad m = 1, \ldots, M \tag{3}$$

$$c_{mt} > 0, \qquad x_n > 0 \tag{4}$$

The dual problem corresponding to this primal problem is then maximize

$$V(\bar{w}) = \sigma \left(\prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}\, w_{m0}}{w_{mt}} \right)^{\sigma_{mt}\, w_{mt}} \right)^{\sigma} \tag{5}$$

subject to the linear constraints

$$\sum_{t=1}^{T_0} \sigma_{0t}\, w_{0t} = \sigma \equiv \pm 1 \qquad \text{(normality condition)} \qquad (6)$$

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt}\, a_{mtn}\, w_{mt} = 0, \qquad n = 1, \ldots, N \quad \text{(orthogonality condition)} \qquad (7)$$

$$w_{m0} \equiv \sigma_m \sum_{t=1}^{T_m} \sigma_{mt}\, w_{mt} \geq 0, \qquad m = 1, \ldots, M \qquad (8)$$

$$w_{mt} \geq 0 \qquad (9)$$

The following conditions are assumed.

$$w_{00} \equiv 1 \qquad (10)$$

$$\lim_{w_{mt} \to 0} \left(\frac{c_{mt}\, w_{m0}}{w_{mt}}\right)^{\sigma_{mt}\, w_{mt}} = 1 \qquad (11)$$

The relationship between the primal and the dual variables at the optimum solution are

$$c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} = w_{0t}\, \sigma\, V(\bar{w}^*), \qquad t = 1, \ldots, T_0 \qquad (12)$$

and

$$c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} = \frac{w_{mt}}{w_{m0}}, \qquad t = 1, \ldots, T_m, \qquad m = 1, \ldots, M \qquad (13)$$

The equations (12) and (13) are linear in $\ell n(x)$ and give the values of $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ at optimum.

Substituting the value of w_{mo} from equation (8) into the dual problem (5)

$$V(\bar{w}) = \sigma \left\{ \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}}{w_{mt}}\right)^{\sigma_{mt} w_{mt}} \prod_{m=0}^{M} \prod_{t=1}^{T_m} (w_{m0})^{\sigma_{mt} w_{mt}} \right\}^{\sigma} \tag{14}$$

$$= \sigma \left\{ \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}}{w_{mt}}\right)^{\sigma_{mt} w_{mt}} \prod_{m=0}^{M} w_{m0}^{\sum_{t=1}^{T_m} \sigma_{mt} w_{mt}} \right\}^{\sigma} \tag{15}$$

Since $\sigma_m = \pm 1$,

$$\sigma_m w_{m0} = \frac{w_{m0}}{\sigma_m} = \sum_{t=1}^{T_m} \sigma_{mt} w_{mt}$$

$$w_{00} \equiv 1$$

The final result is

$$V(\bar{w}) = \sigma \left\{ \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}}{w_{mt}}\right)^{\sigma_{mt} w_{mt}} \prod_{m=0}^{M} w_{m0}^{\sigma_m w_{m0}} \right\}^{\sigma} \tag{16}$$

which is similar to equation (26) in Chapter II.

Generalized geometric programming is more applicable to optimization problems because of the signum function σ_{mt} in equations (1), (2), and

(3). The negative sign of σ_m helps to solve those problems having reversed inquality constraints. The disadvantage of the generalized geometric programming is that the guarantee of global optimality is lost in the optimum solution.

The Degrees of Difficulty

The beauty of geometric programming occurs when the total number of terms is one greater than the number of variables. In this case the solution is determined by solving the linear constraints without reference to the objective function.

The total number of terms is equivalent to

$$T = \sum_{m=0}^{M} T_m \tag{17}$$

The total degrees of difficulty is defined as

$$D = T - (N + 1) \tag{18}$$

When D is greater than zero the dual problem is not so easily optimized.

EXAMPLE 1:

Minimize

$$S = x_1^2 + x_2$$

subject to

$$x_1^2 x_2 \geq 2$$

Arranging the above constraint into the general form given by (2)

$$2x_1^{-2}\ x_1^{-1} \leq 1$$

Here

$$\sigma_{01} = +1, \quad \sigma_{02} = +1, \quad \sigma_{11} = +1 \quad \text{and} \quad \sigma_1 = +1$$

The dual constraints are

$$w_{01} + w_{02} = 1$$

$$2w_{01} - 2w_{11} = 0$$

$$w_{02} - w_{11} = 0$$

Solving for w_{01}, w_{02}, and w_{11}, we obtain

$$w_{01} = w_{02} = w_{11} = 1/2$$

Therefore

$$w_{00} = 1$$

and $$w_{10} = 1/2$$

The value of dual function at this point is

$$v(\bar{w}) = \left(\frac{1x1}{1/2}\right)^{1/2} \left(\frac{1x1}{1/2}\right)^{1/2} \left(\frac{2x1/2}{1/2}\right)^{1/2}$$

$$= 2.83$$

The primal variable are found from equation (12) and (13) as

$$x_1^2 = 1/2 \times 1 \times 2.83 = 1.415$$

or $$x_1 = 1.183$$

$$x_2 = 1/2 \times 1 \times 2.83 = 1.415$$

The minimum value of the objective function at this point is

$$S_{min} = 2.83$$

CHAPTER IV

COMPLEMENTARY GEOMETRIC PROGRAMMING

1. INTRODUCTION

A serious limitation in the successful application of geometric programming to optimizing engineering design has been that all the functions involved in the problem are to be posynomials, i.e., generalized polynomials with positive coefficients. Avriel and Williams [1,2] extended the theory of geometric programming to include any rational function of posynomial terms. The case in which some of the terms may be negative is then a special case of the theory. A program formulated in terms of a rational function of posynomial terms is called a complementary geometric program.

2. PROBLEM STATEMENT

The most general form of a complementary geometric program (CGP) is:

Minimize

$$R_0(x) \tag{1}$$

subject to

$$R_k(x) \leq 1, \qquad k = 1, \ldots, p \tag{2}$$

$$x > 0 \tag{3}$$

where x is a m-dimensional vector, and

$$R_k(x) = \frac{A_k(x) - B_k(x)}{C_k(x) - D_k(x)}, \qquad k = 0,1, \ldots, p \tag{4}$$

where the A, B, C and D are all posynomials and possibly some of them may be absent. It must be assumed, however, that $R_0(x) > 0$ for all feasible x. This can, in principle, always be achieved by adding a sufficiently large constant to R_0.

To treat the problem, a new variable $x_0 > 0$, constrained to satisfy $x_0 \geq R_0(x)$, i.e., $R_0(x)/x_0 \leq 1$, is introduced, so that the problem is reduced to minimizing x_0 subject to constraints of the type

$$\frac{A(x) - B(x)}{C(x) - D(x)} \leq 1 \tag{5}$$

The constraints have meaning only if $C(x) - D(x)$ has constant sign throughout the region of interest. Accordingly as $C(x) - D(x)$ is positive or negative, equation (5) can be written as

$$\frac{A(x) + D(x)}{B(x) + C(x)} \leq 1 \tag{6}$$

if $C(x) - D(x)$ is positive or

$$\frac{B(x) + C(x)}{A(x) + D(x)} \leq 1 \tag{7}$$

if $C(x) - D(x)$ is negative.

Then the standard complementary geometric program (CGP) is

Minimize

$$x_0 \tag{8}$$

subject to

$$P_k(x)/Q_k(x) \leq 1, \qquad k = 0, 1, \ldots, p \tag{9}$$

and

$$x = (x_0, x_1, \ldots, x_m) > 0 \tag{10}$$

where $P_k(x)$ and $Q_k(x)$ are posynomials of the form

$$P_k(x) = \sum_j p_{jk}(x) = \sum_j c_{jk} \prod_i x_i^{a_{ijk}} \tag{11}$$

$$Q_k(x) = \sum_j q_{jk}(x) = \sum_j d_{jk} \prod_i x_i^{a_{ijk}} \tag{12}$$

Complementary geometric programming enables us to handle much larger family of engineering optimization problems than ordinary geometric programming. However, complementary programming does not have the property of the ordinary geometric programming that every constrained local minimum is also a global minimum. Complementary geometric programs can have local minima which are not global minima, however, in many practical situations it is sufficient to find a local minimum.

The algorithm described in the next section obtains a local minimum.

3. ALGORITHM

The algorithm for solving complementary geometric programs is based on the fact that a posynomial divided by a posynomial consisting of only one term is again a posynomial. Therefore, if each of the $Q_k(x)$ in equation (9) are approximated by one term posynomials, we obtain an ordinary geometric programming problem. The algorithm consists of successively approximating the $Q_k(x)$ by one term posynomials so as to produce a sequence of approximating geometric programming problems whose solutions converge to a local minimum of the original CGP program.

The approximation of $Q_k(x)$ to one term is based on the arithmetic-geometric mean inequality (equation (9), Chapter II).

$$\sum_j u_j \geq \prod_j \left(\frac{u_j}{\delta_j}\right)^{\delta_j} \tag{13}$$

Since the $Q_k(x)$ are of the form (omitting the k subscript)

$$Q(x) = \sum_j q_j(x) = \sum_j d_j \prod_i x_i^{a_{ij}} \tag{14}$$

we may take any $x > 0$, and put

$$u_j = q_j(x) \tag{15}$$

and recalling the definition of δ_j

$$\delta_j = \frac{q_j(\bar{x})}{Q(\bar{x})} \tag{16}$$

From equation (13) we have

$$Q(x) \geq \prod_j \left(\frac{Q(\bar{x})}{q_j(\bar{x})} q_j(x)\right)^{q_j(\bar{x})/Q(\bar{x})} \equiv Q(x,\bar{x}) \tag{17}$$

The right-hand side of (17) is a one term posynomial and it is the approximation for $Q(x)$ at $\bar{x}$ and will be denoted by $Q(x,\bar{x})$.

The first step in solving a complementary geometric program then is to select some feasible point, call it $x^{(1)}$, and replace the $Q_k(x)$ by $Q_k(x,x^{(1)})$. Thus (8), (9) and (10) become:

Minimize

$$x_0 \tag{18}$$

subject to

$$\frac{P_k(x)}{Q_k(x,\, x^{(1)})} \le 1, \qquad k = 0,\, 1,\, \ldots,\, p \tag{19}$$

$$x > 0 \tag{20}$$

This is an ordinary geometric programming problem which is solved for some optimal solution; call it $x^{(2)}$. $Q_k(x)$ is then replaced by $Q(x,\, x^{(2)})$ and a new optimal solution $x^{(3)}$ is obtained, etc.

We can see that if $x^{(1)}$ is feasible in (17), then so is $x^{(2)}$, since

$$1 \ge \frac{P_k(x^{(2)})}{Q_k(x^{(2)},\, x^{(1)})} \ge \frac{P_k(x^{(2)})}{Q_k(x^{(2)})} \tag{21}$$

The sequence $x^{(\alpha)}$, therefore, is feasible and will converge to a local minimum.

4. DEGREE OF DIFFICULTY OF A CGP PROBLEM

The degree of difficulty of a complementary geometric programming problem is defined as [1,2] the total number of posynomial terms in the numerators of the inequality constraints (9) less $(m + 1)$, where $(m + 1)$ is the number of primal variables $x_0,\, x_1,\, \ldots,\, x_m$.

In other words, the degree of difficulty of a complementary goemetric programming problem is equal to the degree of difficulty of the approximating ordinary geometric programming problem, solved at each iteration. This means that the degree of diffiuclty of a CGP problem is independent of the number of terms appearing in the denominators of the constraints.

5. EXAMPLE (extracted from reference 2)

Minimize

$$x_0$$

subject to

$$8x_0^2 + 8x_1 \geq 11$$

$$-x_0 + 8x_1 \leq 2$$

$$x_0 > 0,\ x_1 > 0$$

Rearranging the inequality constraints into the standard form of equation (9), we obtain

Minimize

$$x_0$$

subject to

$$\frac{11/8}{x_0^2 + x_1} \leq 1$$

$$\frac{8x_1}{2+x_0} \leq 1$$

If we let

$$\varepsilon_j = \frac{q_j(\bar{x})}{Q(\bar{x})}$$

in equation (17), the problem reduces to

Minimize

$$x_0$$

subject to

$$\frac{11}{8}\left(\frac{x_0^2}{\varepsilon_1}\right)^{-\varepsilon_1}\left(\frac{x_1}{\varepsilon_2}\right)^{-\varepsilon_2} \leq 1$$

$$8x_1\left(\frac{2}{\varepsilon_3}\right)^{-\varepsilon_3}\left(\frac{x_0}{\varepsilon_4}\right)^{-\varepsilon_4} \leq 1$$

or

Minimize

$$x_0$$

subject to

$$C_1\, x_0^{-2\varepsilon_1}\, x_1^{-\varepsilon_1} \leq 1$$

$$C_2\, x_0^{-\varepsilon_4}\, x_1 \leq 1$$

where

$$\varepsilon_1 = x_0^2/(x_0^2 + x_1)$$

$$\varepsilon_2 = x_1/(x_0^2 + x_1)$$

$$\varepsilon_3 = 2/(2 + x_0)$$

$$\varepsilon_4 = x_0/(2 + x_0)$$

and

$$C_{11} = \frac{11}{8} (\varepsilon_1)^{\varepsilon_1} (\varepsilon_2)^{\varepsilon_2}$$

$$C_{21} = 8 (\varepsilon_3/2)^{\varepsilon_3} (\varepsilon_4)^{\varepsilon_4}$$

The normality and orthogonality conditions are

$$\delta_{01} = 1$$

$$\delta_{01} - 2\varepsilon_1\delta_{11} - \varepsilon_4\delta_{21} = 0$$

$$- \varepsilon_2\delta_{11} + \delta_{21} = 0$$

which give the solution

$$\delta_{01} = 1$$

$$\delta_{11} = 1/(2\varepsilon_1 + \varepsilon_2 \cdot \varepsilon_4)$$

$$\delta_{21} = \varepsilon_2/(2\varepsilon_1 + \varepsilon_2 \cdot \varepsilon_4)$$

The dual Function can be written as

$$v(\delta) = \left(\frac{C_{11}}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{C_{21}}{\delta_{21}}\right)^{\delta} (\lambda_1)^{\lambda_1} (\lambda_2)^{\lambda_2}$$

where

$$\lambda_1 = \delta_{11}$$

and

$$\lambda_2 = \delta_{21}$$

The term $\left(\frac{C_{01}}{\delta_{01}}\right)^{\delta_{01}}$ in the dual function is considered as one, since $\delta_{01} = C_{01} = 1$. The dual function can be further simplified to

$$v(\delta) = (C_{11})^{\delta_{11}} (C_{21})^{\delta_{21}}$$

Starting with $x^{(1)} = (4, 1/4)$ which is a feasible point, the optimal value of the dual function of the first approximating geometric programming problem would be

$$v\left(\delta^{(1)}\right) = 1.139$$

By solving equations (38) and (39) in chapter II the primal variables are found to be

$$x_0 = \delta_{01}\, v(\delta) = v(\delta)$$

and

$$x_1 = \frac{\left(v(\delta)\right)^{\varepsilon_4}}{C_{21}}$$

which yield the values of

$$x_0 = 1.139$$

and

$$x_1 = 0.325$$

at the first iteration.

This point will be the next trial point, $x^{(2)} = (1.139, 0.325)$, and will be used in the next iteration. The convergence of the algorithm to the desired minimum is presented in Table 1. The optimal solution is

$$\bar{x}_0 = 1.000$$

$$\bar{x}_1 = 0.375$$

and the minimum value of the objective function is

$$S(\bar{x}) = v(\bar{\delta}) = 1.000$$

TABLE 1

Convergence to optimum in example 1

Iteration	x_0	x_1	$v(\delta)$
1	4.000	0.250	1.139
2	1.139	0.325	1.009
3	1.009	0.375	1.000

CHAPTER V

OPTIMIZATION OF THE GEOMETRIC PROGRAMMING PROBLEMS CONTAINING DEGREE OF DIFFICULTY

Geometric programming is a method for solving a class of nonlinear optimization problems. The method is very desireable when the degree of difficulty is zero. In this case the optimal solution is obtained by solving a system of linear equations. If the problem has degree of difficulty greater than zero, the corresponding system of linear equations has no single solution and the optimal solution has to be found by optimizing the dual objective function subject to linear constraints. In this chapter some of the solution techniques to the problems with one or more degrees of difficulty are presented.

1. GEOMETRIC PROGRAMMING PROBLEMS WITH ONE DEGREE OF DIFFICULTY

Method 1. This method provides a quick estimate of the upper and lower bounds of the optimal value of the objective function.

$$S(x_1, \ldots, x_m) \geq S(\bar{x}) = v(\bar{\delta}) \geq v(\delta_1, \ldots, \delta_n) \qquad (1)$$

This is accomplished by neglecting one of the terms in the primal objective function and hence reducing the problem to zero degree of difficulty. The problem with zero degrees of difficulty can now be solved for the weights, which are then substituted into the dual function to obtain a lower bound on the true optimal value of the objective function. The upper bound is obtained by substituting the corresponding values of the primal variables $x_1, \ldots, x_m$ into the original objective function.

The range on which the optimal value of the objective function or $S(\bar{x})$ is bounded can be small or large depending on the size of the weight neglected. Hence by choosing the smallest weight, a good estimate of the optimal value of the objective function can be found.

The smallest weight or δ may be found by writting the relations between the weights and finding the range each δ is bounded. The one with the smallest upper bound is the one to be neglected.

EXAMPLE 1: (extracted from reference 3)

Consider the problem of minimizing the cost function

$$S = 1000\ x_1 + 4 \times 10^9\ x_1^{-1} x_2^{-1} + 2.5 \times 10^5\ x_2 + 9000\ x_1 x_2$$

The normality and orthogonality conditions are

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$$

$$\delta_1 - \delta_2 \qquad + \delta_4 = 0$$

$$- \delta_2 + \delta_3 + \delta_4 = 0$$

This problem has $4 - 3 = 1$ degree of difficulty. Solving δ_1, δ_2, and δ_3 in terms of δ_4 yields

$$\delta_1 = \frac{1}{3}(1 - 2\delta_4)$$

$$\delta_2 = \frac{1}{3}(1 + \delta_4)$$

$$\delta_3 = \frac{1}{3}(1 - 2\delta_4)$$

Since all the δ should be positive, the following bounds can be put on δ.

$$0 \leq \delta_1 \leq 1/3$$

$$1/3 \leq \delta_2 \leq 1/2$$

$$0 \leq \delta_3 \leq 1/3$$

$$0 \leq \delta_4 \leq 1/2$$

As can be seen either δ_1 or δ_3 has the smallest upper bound on it. Neglecting δ_3 or letting

$$\delta_3 = 0$$

we obtain

$$\delta_1 = 0$$

$$\delta_2 = 1/2$$

$$\delta_3 = 0$$

and

$$\delta_4 = 1/2$$

the value of the dual function at this point is

$$v(0,\tfrac{1}{2},0,\tfrac{1}{2}) = \left(\frac{1000}{\delta_1}\right)^{\delta_1} \left(\frac{4\text{x}10^9}{\delta_2}\right)^{\delta_2} \left(\frac{2.5\text{x}10^5}{\delta_3}\right)^{\delta_3} \left(\frac{900}{\delta_4}\right)^{\delta_4}$$

$$= \left(\frac{4\text{x}10^9}{1/2}\right)^{1/2} \left(\frac{900}{1/2}\right)^{1/2}$$

$$= 12.0 \text{ x } 10^6$$

$$x_1 = 411 \quad \text{and} \quad x_2 = 1.63$$

and the value of the primal function at this point is

$$S(411,\ 1.63) = 12.8 \text{ x } 10^6$$

Therefore, the optimum value of the objective function, $S(\bar{x})$, is estimated to be

$$12.0 \text{ x } 10^6 \leq S(\bar{x}) \leq 12.8 \text{ x } 10^6$$

Method 2. In this method weights δ_i are to be expressed in terms of one of the weights called the basic variable (the number of basic variables is equal to the degree of difficulty and in this case is one). The dual problem (function) can then be reformulated in terms of the basic variable (or variables). This problem can now be solved by one of the one dimensional search techniques. Since the dual function is nonlinear with respect to δ_i, it is simpler to take the logarithm of the dual function and then search for the optimum point [6].

EXAMPLE 2:

In the previous example we expressed δ_1, δ_2, and δ_3 in terms of δ_4 as

$$\delta_1 = \frac{1}{3}\,(1 - 2\delta_4)$$

$$\delta_2 = \frac{1}{3}\,(1 + \delta_4)$$

$$\delta_3 = \frac{1}{3}\,(1 - 2\delta_4)$$

Substituting these values into the dual function yields

$$v(\delta) = v(\delta_4) = \left(\frac{100}{\frac{1}{3}(1-2\delta_4)}\right)^{\frac{1}{3}(1-2\delta_4)} \left(\frac{4\text{x}10^9}{\frac{1}{3}(1+\delta_4)}\right)^{\frac{1}{3}(1+\delta_4)} \times$$

$$\cdot \left(\frac{2.3\text{x}10^3}{\frac{1}{3}(1-2\delta_4)}\right)^{\frac{1}{3}(1-2\delta_4)} \left(\frac{9000}{\delta_4}\right)^{\delta_4}$$

which is one dimensional with respect to δ_4. Taking the logarithm of both sides, differentiating with respect to δ_4, and equating to zero gives

$$\ell n(v(\delta_4)) = \frac{1}{3}(1-2\delta_4)\,\ell n\left(\frac{1000}{\frac{1}{3}(1-2\delta_4)}\right) + \frac{1}{3}(1+\delta_4)\,\ell n\left(\frac{4\text{x}10^9}{\frac{1}{3}(1+\delta_4)}\right)$$

$$+ \frac{1}{3}(1-2\delta_4)\,\ell n\left(\frac{2.3\text{x}10^3}{\frac{1}{3}(1-2\delta_4)}\right) + \delta_4\,\ell n\left(\frac{9000}{\delta_4}\right)$$

$$\frac{d\ell n(v(\delta_4))}{d\delta_4} = \frac{dv(\delta_4)}{d\delta_4} \times \frac{1}{v(\delta_4)} = 0$$

or

$$0 = \frac{1}{3}\left(-2\ell n\left(\frac{1000}{\frac{1}{3}(1-2\delta_4)}\right) +2\right) + \frac{1}{3}\left(\ell n\left(\frac{4x10^9}{\frac{1}{3}(1+\delta_4)}\right) - 1\right)$$

$$+ \frac{1}{3}\left(-2\ell n\left(\frac{2.3x10^3}{\frac{1}{3}(1-2\delta_4)}\right) + 2\right) + \left(\ell n\left(\frac{9000}{\delta_4}\right) - 1\right)$$

Further simplification gives

$$\frac{(1+\delta_4)(\delta_4)^3}{(1-2\delta_4)^4} = 2.045x10^3$$

which gives the solution, by one-dimensional search, as

$$\delta_4^* = 0.453$$

and therefore the other weights are

$$\delta_1 = 0.031$$

$$\delta_2 = 0.484$$

and

$$\delta_3 = 0.031$$

The optimum value of the objective function at this point is

$$S(\bar{x}) = 12.6\text{x}10^6$$

2. GEOMETRIC PROGRAMMING PROBLEMS WITH TWO OR MORE DEGREES OF DIFFICULTY

When two or more degrees of difficulty is involved in geometric programming problems the solution is not easily found. In this case the optimal solution is obtained by optimizing the nonlinear dual objective function subject to linear equality constraints.

In this section some of the methods for dealing with the geometric programming problems with two or more degrees of difficulty are presented.

Method 1. Solution by Separable Programming

Separable programming can be used for maximizing the dual function under the linear constraints. The method provides an approximate solution for the dual problem.

Since the constraints are linear there is no problem in applying a separable programming algorithm to them. The nonlinear dual function can be made separable by taking the logarithm of the function.

From equation (16) Chapter III it is apparent that

$$\ell n(V(\bar{w})) = \sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} w_{mt} \ell n\left(\frac{C_{mt}}{w_{mt}}\right) + \sum_{m=0}^{M} \sigma_m w_{m0} \ell n(w_{m0}) \tag{1}$$

which is separable in the dual variables.

The constraints of this modified function are

$$w_{m0} - \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} w_{mt} = 0 \tag{2}$$

$$\sum_{t=1}^{T_0} \sigma_{0t} w_{0t} = \sigma \tag{3}$$

and

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} a_{mtn} w_{mt} = 0, \qquad \text{for} \quad n = 1,2, \ldots, N \tag{4}$$

A FORTRAN program is available for generating data to solve the dual geometric programming problems using MPS/360 [10].

EXAMPLE 1: (extracted from reference 10)

Minimize

$$Y_0 = 4x_1 + 10x_2 + 4x_3 + 2(x_1^2 + x_2^2)^{1/2}$$

subject to the constraint

$$x_1 x_2 x_3 \geq 100$$

$$x_i \geq 0$$

The objective function is not a posynomial, but geometric programming can be applied to the equivalent problem of minimizing the posynomial

$$Y_0 = 4x_1 + 10x_2 + 4x_3 + 2x_4$$

subject to the constraints

$$x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \leq 1$$

$$\frac{100}{x_1 x_2 x_3} \leq 1$$

where

$$x_4 = (x_1^2 + x_2^2)^{1/2}$$

The problem has 7 - (4+1) = 2 degrees of difficulty. The dual function is

$$v(\bar{w}) = \left(\frac{4}{w_{01}}\right)^{w_{01}} \left(\frac{10}{w_{02}}\right)^{w_{02}} \left(\frac{4}{w_{03}}\right)^{w_{03}} \left(\frac{2}{w_{04}}\right)^{w_{04}} \left(\frac{1}{w_{11}}\right)^{w_{11}} \left(\frac{1}{w_{12}}\right)^{w_{12}} \times$$

$$\left(\frac{100}{w_{21}}\right)^{w_{21}} (w_{10})^{w_{10}} (w_{20})^{w_{20}}$$

From equation (1) the dual problem becomes

maximize

$$\ln(v(\bar{w})) = w_{01}\ln\left(\frac{4}{w_{01}}\right) + w_{02}\ln\left(\frac{10}{w_{02}}\right) + w_{03}\ln\left(\frac{4}{w_{03}}\right) + w_{04}\ln\left(\frac{2}{w_{04}}\right)$$

$$+ w_{11}\ln\left(\frac{1}{w_{11}}\right) + w_{12}\ln\left(\frac{1}{w_{12}}\right) + w_{21}\ln\left(\frac{100}{w_{21}}\right) + w_{10}\ln(w_{10})$$

$$+ w_{20}\ln(w_{20})$$

subject to the constraints

$$
\begin{array}{lllllll}
w_{01} + w_{02} + w_{03} + w_{04} & & & & & = 1 \\
w_{01} & & + 2w_{11} & & - w_{21} & = 0 \\
\quad w_{02} & & & + 2w_{12} & - w_{21} & = 0 \\
\qquad w_{03} & & & & - w_{21} & = 0 \\
\qquad\quad w_{04} & - 2w_{11} & - 2w_{12} & & & = 0
\end{array}
$$

and

$$w_{10} - w_{11} - w_{12} = 0$$

$$w_{20} \qquad - w_{21} = 0$$

By using the FORTRAN program for generating the data and using MPS/360 the problem converges to the following results.

The dual variables are

$$w_{01} = 0.23117$$

$$w_{02} = 0.30500$$

$$w_{03} = 0.33333$$

$$w_{04} = 0.13050$$

$$w_{11} = 0.05108$$

$$w_{12} = 0.01417$$

$$w_{21} = 0.33333$$

The value of the dual function is then determined from

$$v(\bar{w}) = \exp(4.47719) = 87.98708$$

The primal variables are found from equations (12) and (13) to be

$$4x_1 = (0.23117)(87.98708)$$

$$10x_2 = (0.30500)(87.98708)$$

$$4x_3 = (0.33333)(87.98708)$$

$$2x_4 = (0.13050)(87.98708)$$

or

$$x_1 = 5.085$$

$$x_2 = 2.684$$

$$x_3 = 7.332$$

$$x_4 = 5.741$$

Substituting these values into the primal objective function yields

$$Y_0 = 87.990$$

Since this is a posynomial problem

$$87.98708 < v(\bar{w}) = Y_0(\bar{x}) < 87.990$$

The given solution is apparantly very close to the true optimum.

Method 2. Solution by Complementary Geometric Programming.

This method can be utilized to remove the degree of difficulty and solve a sequence of ordinary geometric programming problems. Consider the general complementary geometric programming problem of the form

Minimize

$$R_0(x) \tag{1}$$

subject to

$$P_k(x)/Q_k(x) \leq 1, \qquad k = 1, \ldots, p \tag{2}$$

$$x > 0$$

where $P_k(x)$ and $Q_k(x)$ are posynomials. The term $Q_k(x)$ can be approximated by the monomial

$$\bar{Q}_k(x) = \prod_{i=1}^{I(k)} \left(\frac{q_{ik}(x)}{\varepsilon_{ik}}\right)^{\varepsilon_{ik}}, \qquad k = 1, \ldots, p \tag{3}$$

where

$$\varepsilon_{ik} = \frac{q_{ik}(\bar{x})}{Q_k(\bar{x})} \qquad \begin{array}{l} i = 1, \ldots, I(k) \\ k = 1, \ldots, p \end{array} \tag{4}$$

$$Q_k(x) = \sum_{i=1}^{I(k)} q_{ik}(x), \qquad k = 1, \ldots, p \tag{5}$$

The resultant approximating geometric programming problem is then

Minimize

$$x_0 \tag{6}$$

subject to

$$P_k(x)[\bar{Q}(x)]^{-1} \leq 1, \qquad k = 0,1, \ldots, p \tag{7}$$

where

$$P_k(x) = \sum_{j=1}^{J(k)} p_{jk}(x), \qquad k = 0,1, \ldots, p \tag{8}$$

The degrees of difficulty of this CGP problem is n-(m+1) and is independent of the number of terms appearing in the denominator of equation (2). Here n is the total number of terms appearing in $P_k(x)$ and m is the number of primal variables (x_1, x_2, ..., x_m). To reduce the problem to zero degree of difficulty, we have to continue to "condense" until the approximating program given by (6) and (7) has zero degree of difficulty. Condensation is done by approximating (n-m) terms of the posynomials $P_k(x)$ in equation (7) to one term posynomial using equation

$$\bar{P}_k(x) = \prod_{j=1}^{J'(k)} \left(\frac{p_{jk}(x)}{\varepsilon_{jk}}\right)^{\varepsilon_{jk}}, \qquad k = 0,1, \ldots, p \tag{9}$$

where J'(k) is the number of terms condensed in the constraint $P_k(x)$ and

$$\varepsilon_{jk} = \frac{p_{jk}(\bar{x})}{\sum_{j=1}^{J'(k)} p_{jk}(\bar{x})}, \qquad \begin{array}{l} j = 1, \ldots, J'(k) \\ k = 0,1, \ldots, p \end{array} \tag{10}$$

The associated dual geometric program of the approximated problem is maximize

$$v(\delta) = \prod_{k=0}^{p} \left[\prod_{j=1}^{n} \left(\frac{\bar{C}_j}{\delta_j}\right)^{\delta_j} \right] \prod_{k=0}^{p} (\lambda_k)^{\lambda_k} \tag{11}$$

subject to

$$\delta_0 = 1 \qquad \text{(normality condition)} \tag{12}$$

$$\sum_{j=1}^{J''(k)} \bar{a}_{\ell j}\delta_j = 0 \qquad \ell = 1, \ldots, m \text{ (orthogonality condition)} \quad k = 0,1, \ldots, p \tag{13}$$

$$\delta_j \geq 0 \qquad j = 1, \ldots, J''(k) \quad \text{(positivity condition)} \tag{14}$$

where

$$J''(k) = J(k) - J'(k) + 1 \tag{15}$$

$$\lambda_k = \sum_{j=1}^{J''(k)} \delta_j, \qquad k = 0,1, \ldots, p \tag{16}$$

$$\bar{C}_j = C_j \prod_{i=1}^{I(k)} \left(\frac{C_i'}{\varepsilon_{ik}}\right)^{-\varepsilon_{ik}}, \qquad k = 0,1, \ldots, p, \quad j = 1, \ldots, J''(k) \tag{17}$$

C_j and C'_j refer to the terms $P_k(x)$ and $Q_k(x)$ after condensing the problem to zero degree of difficulty

$$\bar{a}_{\ell j} = a_{\ell j} - \sum_{i=1}^{I(k)} a_{\ell i}\varepsilon_{ik}, \qquad \begin{array}{l} \ell = 1, \ldots, m \\ j = 1, \ldots, J''(k) \end{array} \tag{18}$$

The algorithm for solving the sequence of approximating problems can be described in the following steps:

(i) Choose a feasible starting point $x^{(1)}$.

(ii) Determine the ε_{ik} and ε_{jk} from equations (4) and (10).

(iii) Solve the dual program associated with the approximating program expressed by (11), (12), (13) and (14).

(iv) Find the corresponding primal variables from the optimum values of the dual variables and the dual function using equations (38) and (39) in Chapter II. Call the new point $x^{(2)}$.

(v) If the new solution, $x^{(2)}$, is within some specified tolerence level of the old one, $x^{(1)}$, terminate the algorithm. If not return to step (i) using the new solution, $x^{(2)}$, as the second starting point.

Figure 1 shows the general flow diagram for the solution of geometric programming problems by CGP.

EXAMPLE 4: Multistage Heat Exchanger Design by Complementary Geometric Programming

This problem was solved previously by Boas [4] via dynamic programming and also by Fan and Wang [8] via the discrete maximum principle

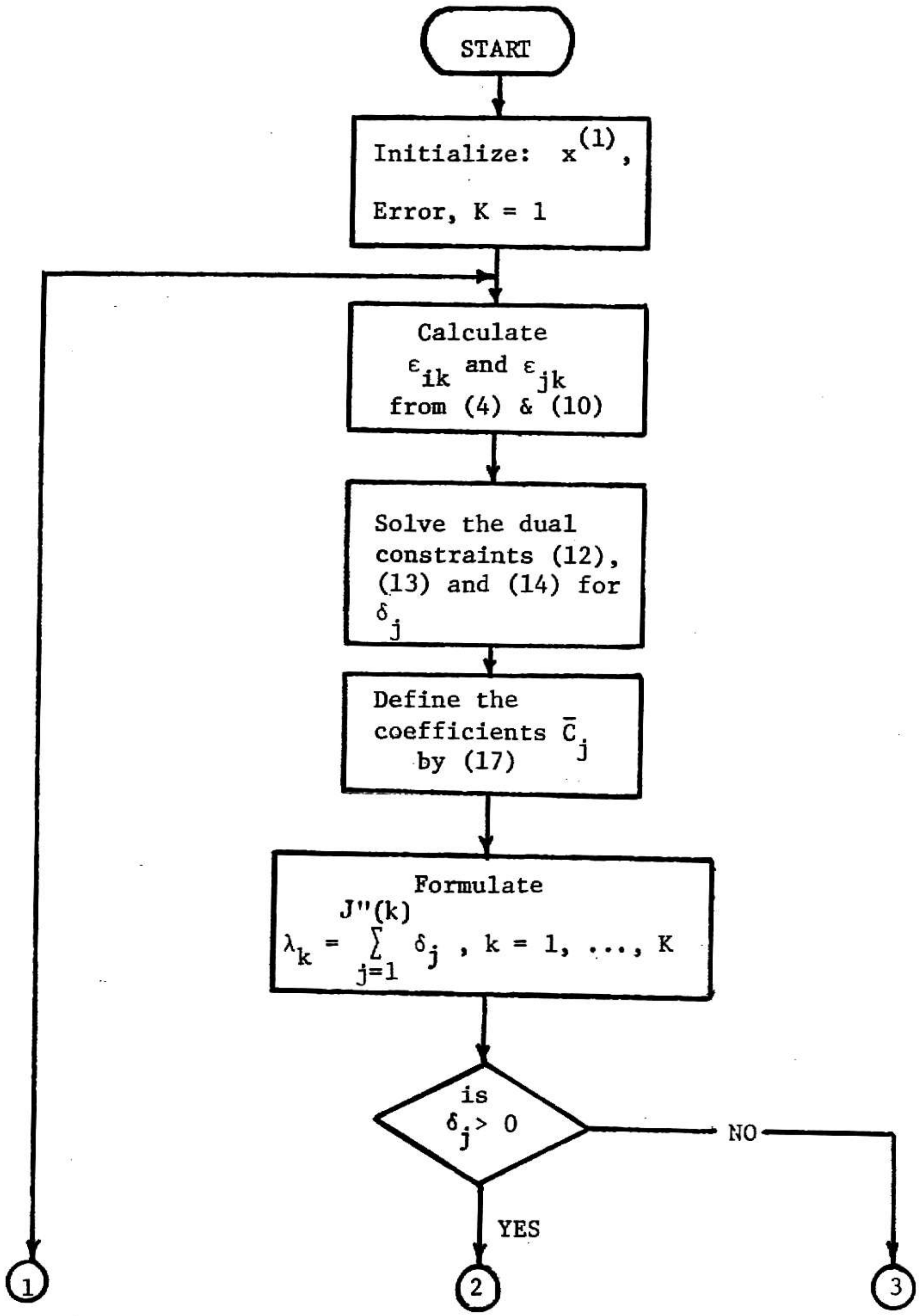


Fig. 1. The Flow Diagram for Solving Problems with Degree of Difficulty by the Method of Complementary Geometric Programming

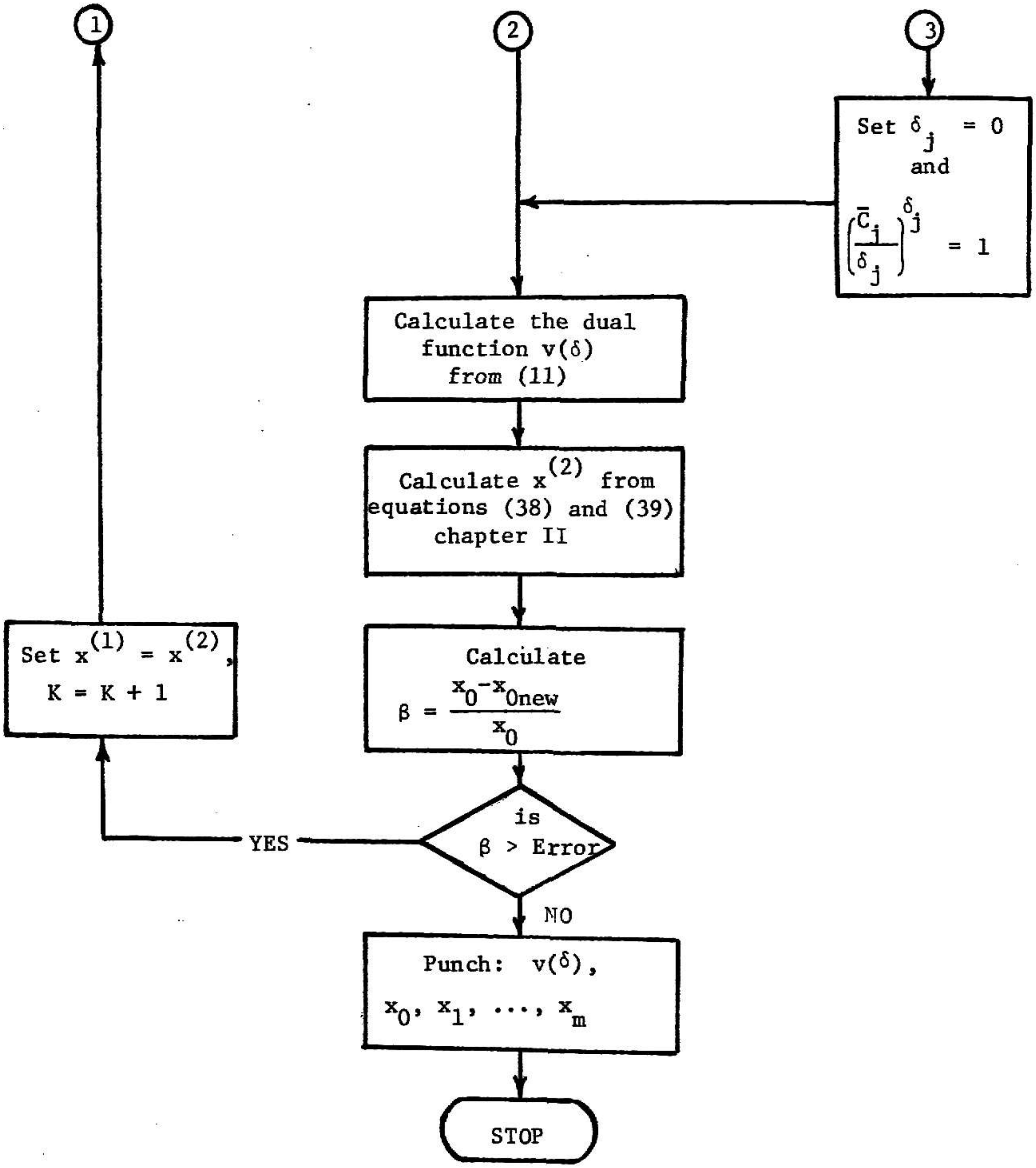


Fig. 1. Continued

and lately by Avriel and Williams [1,2] via complementary geometric programming.

There are three heat exchangers as shown in Fig. 2 each with the flow rate of W and specific heat C_p. The fluid is heated from temperature T_o to T_3 by passing through three heat exchangers in series. At each stage, the cold stream is heated by a hot fluid having the same flow rate W and specific heat C_p as the cold stream. The temperature of the hot fluid entering the heat exchangers, t_{11}, t_{21}, and t_{31} and the overall heat transfer coefficients U_1, U_2, and U_3 of the heat exchangers are known constants. Optimum design involves minimizing the sum of the heat transfer areas of the three heat exchangers, $A_T = A_1 + A_2 + A_3$.

There are three heat balance equations expressing the fact that the rate of heat transferred to the cold fluid is less than or equal to the rate of heat loss by the hot stream;

$$W\,C_p(T_i - T_{i-1}) \leq W\,C_p(t_{i1} - t_{i2}), \qquad i = 1, 2, 3$$

or

$$T_i + t_{i2} \leq t_{i1} + T_{i-1}, \qquad t = 1, 2, 3$$

or

$$\frac{T_i + t_{i2}}{t_{i1} + T_{i-1}} \leq 1, \qquad i = 1, 2, 3$$

Similarly, the heat gain of the cold stream at the i^{th} stage is equal to or less than the heat transfered at the same stage, that is

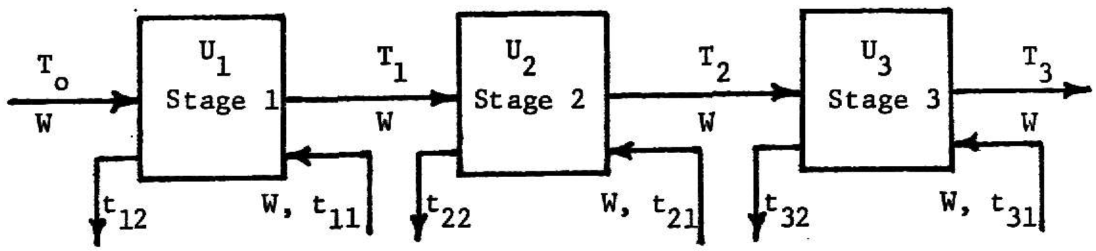
$$W\,C_p(T_i - T_{i-1}) \leq U_iA_i(t_{i2} - T_{i-1}), \qquad i = 1, 2, 3$$

Let $\quad \hat{U}_i = U_i/WC_p, \qquad i = 1, 2, 3$

$$T_i + \hat{U}_iA_iT_{i-1} \leq T_{i-1} + \hat{U}_iA_it_{i2} \qquad i = 1, 2, 3$$

FIGURE 2

Multistage Heat Exchanger System



or

$$\frac{T_i + U_i A_i T_{i-1}}{T_{i-1} + U_i A_i t_{i2}} \leq 1, \qquad i = 1, 2, 3$$

Rearranging all inequalities and the objective function, we will have minimize

$$A_T$$

subject to

$$(A_1 + A_2 + A_3)/A_T \leq 1$$

$$\frac{T_1 + t_{12}}{t_{11} + T_o} \leq 1$$

$$\frac{T_2 + t_{22}}{t_{21} + T_1} \leq 1$$

$$\frac{T_3 + t_{32}}{t_{31} + T_2} \leq 1$$

$$\frac{T_1 + \hat{U}_1 A_1 T_o}{T_o + \hat{U}_1 A_1 t_{12}} \leq 1$$

$$\frac{T_2 + \hat{U}_2 A_2 T_1}{T_1 + \hat{U}_2 A_2 t_{22}} \leq 1$$

$$\frac{T_3 + \hat{U}_3 A_3 T_2}{T_2 + \hat{U}_3 A_3 t_{32}} \leq 1$$

The following data was given for this example:

$T_o = 100^oF$, $T_3 = 500^oF$, $WC_p = 10^5$ B.t.u./hr ^oF, and

i	t_{i1}	U_i (B.t.u./hr-ft^2-^oF)
1	300	120
2	400	80
3	600	40

Substituting the constants into the inequalities we will have

minimize

$$A_T$$

subject to

$$A_1/A_T + A_2/A_T + A_3/A_T \leq 1$$

$$\frac{1}{400}(T_1 + t_{12}) \leq 1$$

$$\frac{T_2 + t_{22}}{400 + T_1} \leq 1$$

$$\frac{500 + t_{32}}{600 + T_2} \leq 1$$

$$\frac{T_1 + 0.12\ A_1}{100 + 0.0012\ A_1 \cdot t_{12}} \leq 1$$

$$\frac{T_2 + 0.0008\ A_2T_1}{T_1 + 0.0008A_2t_{22}} \leq 1$$

$$\frac{500 + 0.0004\ A_3T_2}{T_2 + 0.0004\ A_3t_{32}} \leq 1$$

There are 15 - 9 = 6 degrees of difficulty in this problem. Using the method of condensation, we have to condense twelve terms into six terms to reduce degrees of difficulty to zero. Condensing the last six inequalities and also approximating the denominators by equation (3), the problem reduces to

minimize

$$A_T$$

subject to

$$A_1/A_T + A_2/A_T + A_3/A_T \leq 1$$

$$\left(\frac{1}{400}\right)\left(\frac{T_1}{\varepsilon_1}\right)^{\varepsilon_1}\left(\frac{t_{12}}{\varepsilon_2}\right)^{\varepsilon_2} \leq 1$$

$$\left(\frac{T_2}{\varepsilon_3}\right)^{\varepsilon_3}\left(\frac{t_{22}}{\varepsilon_4}\right)^{\varepsilon_4}\left(\frac{400}{\varepsilon_5}\right)^{-\varepsilon_5}\left(\frac{T_1}{\varepsilon_6}\right)^{-\varepsilon_6} \leq 1$$

$$\left(\frac{500}{\varepsilon_7}\right)^{\varepsilon_7}\left(\frac{t_{32}}{\varepsilon_8}\right)^{\varepsilon_8}\left(\frac{600}{\varepsilon_9}\right)^{-\varepsilon_9}\left(\frac{T_2}{\varepsilon_{10}}\right)^{-\varepsilon_{10}} \leq 1$$

$$\left(\frac{T_1}{\varepsilon_{11}}\right)^{\varepsilon_{11}}\left(\frac{0.12\ A_1}{\varepsilon_{12}}\right)^{\varepsilon_{12}}\left(\frac{100}{\varepsilon_{13}}\right)^{-\varepsilon_{13}}\left(\frac{0.0012\ A_1t_{12}}{\varepsilon_{14}}\right)^{-\varepsilon_{14}} \leq 1$$

$$\left(\frac{T_2}{\varepsilon_{15}}\right)^{\varepsilon_{15}} \left(\frac{0.008\ A_2T_1}{\varepsilon_{16}}\right)^{\varepsilon_{16}} \left(\frac{T_1}{\varepsilon_{17}}\right)^{-\varepsilon_{17}} \left(\frac{0.008\ A_2t_{22}}{\varepsilon_{18}}\right)^{-\varepsilon_{18}} \leq 1$$

$$\left(\frac{500}{\varepsilon_{19}}\right)^{\varepsilon_{19}} \left(\frac{0.0004\ A_3T_2}{\varepsilon_{20}}\right)^{\varepsilon_{20}} \left(\frac{T_2}{\varepsilon_{21}}\right)^{-\varepsilon_{21}} \left(\frac{0.0004\ A_3t_{32}}{\varepsilon_{22}}\right)^{-\varepsilon_{22}} \leq 1$$

where

$$\varepsilon_1 = T_1/(T_1 + t_{12})$$

$$\varepsilon_2 = t_{12}/(T_1 + t_{12})$$

$$\varepsilon_3 = T_2/(T_2 + t_{22})$$

$$\varepsilon_4 = t_{22}/(T_2 + t_{22})$$

$$\varepsilon_5 = 400/(400 + T_1)$$

$$\varepsilon_6 = T_1/(400 + T_1)$$

$$\varepsilon_7 = 500/(500 + t_{32})$$

$$\varepsilon_8 = t_{32}/(500 + t_{32})$$

$$\varepsilon_9 = 600/(600 + T_2)$$

$$\varepsilon_{10} = T_2/(600 + T_2)$$

$$\varepsilon_{11} = T_1/(T_1 + 0.12\ A_1)$$

$$\varepsilon_{12} = 0.12\ A_1/(T_1 + 0.12\ A_1)$$

$$\varepsilon_{13} = 100/(100 + 0.0012\ A_1 t_{12})$$

$$\varepsilon_{14} = 0.0012\ A_1 t_{12}/(100 + 0.0012\ A_1 t_{12})$$

$$\varepsilon_{15} = T_2/(T_2 + 0.0008\ A_2 T_1)$$

$$\varepsilon_{16} = 0.0008\ A_2\ T_1/(T_2 + 0.0008\ A_2 T_1)$$

$$\varepsilon_{17} = T_1/(T_1 + 0.0008\ A_2 t_{22})$$

$$\varepsilon_{18} = 0.0008\ A_2\ t_{22}/(T_1 + 0.0008\ A_2 t_{22})$$

$$\varepsilon_{19} = 500/(500 + 0.0004\ A_3 T_2)$$

$$\varepsilon_{20} = 0.0004\ A_3\ T_2/(500 + 0.0004\ A_3 T_2)$$

$$\varepsilon_{21} = T_2/(T_2 + 0.0004\ A_3\ t_{32})$$

$$\varepsilon_{22} = 0.0004\ A_3\ t_{32}/(T_2 + 0.0004\ A_3\ t_{32})$$

The normality and orthogonality conditions are

$$
\begin{array}{llllr}
\delta_0 & & & & = 1 \\
\delta_0 - \delta_1 - \delta_2 - \delta_3 & & & & = 0 \\
\quad \delta_1 & & + (\varepsilon_{12} - \varepsilon_{14})\delta_7 & & = 0 \\
\quad\quad \delta_2 & & & + (\varepsilon_{16} - \varepsilon_{18})\delta_8 & = 0 \\
\quad\quad\quad \varepsilon_3 & & & + (\varepsilon_{20} - \varepsilon_{22})\delta_9 & = 0 \\
\quad\quad\quad \varepsilon_1\delta_4 - \varepsilon_6\delta_5 & & + \varepsilon_{11}\delta_7 + (\varepsilon_{16} - \varepsilon_{17})\delta_8 & & = 0 \\
\quad\quad\quad\quad \varepsilon_3\delta_5 - \varepsilon_{10}\delta_6 & & + \varepsilon_{15}\delta_8 + (\varepsilon_{20} - \varepsilon_{21})\delta_9 & & = 0 \\
\quad\quad\quad \varepsilon_2\delta_4 & & -\varepsilon_{14}\delta_7 & & = 0 \\
\quad\quad\quad\quad \varepsilon_4\delta_3 & & - \varepsilon_{18}\delta_8 & & = 0 \\
\quad\quad\quad\quad\quad \varepsilon_8\delta_6 & & - \varepsilon_{22}\delta_9 & & = 0
\end{array}
$$

There are ten simultaneous linear equations which can be solved for $\delta_0, \delta_1, \delta_2, \ldots, \delta_9$. It is evident that $\delta_0 = 1$.

The dual function $v(\delta)$ is

$$
v(\delta) = \left(\frac{C_1}{\delta_1}\right)^{\delta_1} \left(\frac{C_2}{\delta_2}\right)^{\delta_2} \left(\frac{C_3}{\delta_3}\right)^{\delta_3} \left(\frac{C_4}{\delta_4}\right)^{\delta_4} \left(\frac{C_5}{\delta_5}\right)^{\delta_5} \left(\frac{C_6}{\delta_6}\right)^{\delta_6} \left(\frac{C_7}{\delta_7}\right)^{\delta_7} \left(\frac{C_8}{\delta_8}\right)^{\delta_8} \left(\frac{C_9}{\delta_9}\right)^{\delta_9} \times
$$

$$
(\lambda_1)^{\lambda_1} (\lambda_2)^{\lambda_2} (\lambda_3)^{\lambda_3} (\lambda_4)^{\lambda_4} (\lambda_5)^{\lambda_5} (\lambda_6)^{\lambda_6} (\lambda_7)^{\lambda_7}
$$

where

$$\lambda_1 = \delta_1 + \delta_2 + \delta_3$$

$$\lambda_2 = \delta_4$$

$$\lambda_3 = \delta_5$$

$$\lambda_4 = \delta_6$$

$$\lambda_5 = \delta_7$$

$$\lambda_6 = \delta_8$$

$$\lambda_7 = \delta_9$$

and the coefficients are

$$C_1 = 1$$

$$C_2 = 1$$

$$C_3 = 1$$

$$C_4 = \frac{1}{400}\left(\frac{1}{\varepsilon_1}\right)^{\varepsilon_1}\left(\frac{1}{\varepsilon_2}\right)^{\varepsilon_2}$$

$$C_5 = \left(\frac{1}{\varepsilon_3}\right)^{\varepsilon_3}\left(\frac{1}{\varepsilon_4}\right)^{\varepsilon_4}\left(\frac{\varepsilon_s}{400}\right)^{\varepsilon_5}(\varepsilon_6)^{\varepsilon_6}$$

$$C_6 = \left(\frac{500}{\varepsilon_7}\right)^{\varepsilon_7}\left(\frac{1}{\varepsilon_8}\right)^{\varepsilon_8}\left(\frac{\varepsilon_9}{600}\right)^{\varepsilon_9}(\varepsilon_{10})^{\varepsilon_{10}}$$

$$C_7 = \left(\frac{1}{\varepsilon_{11}}\right)^{\varepsilon_{11}}\left(\frac{0.12}{\varepsilon_{12}}\right)^{\varepsilon_{12}}\left(\frac{\varepsilon_{13}}{100}\right)^{\varepsilon_{13}}\left(\frac{\varepsilon_{14}}{0.0012}\right)^{\varepsilon_{14}}$$

$$C_8 = \left(\frac{1}{\varepsilon_{15}}\right)^{\varepsilon_{15}} \left(\frac{0.0008}{\varepsilon_{16}}\right)^{\varepsilon_{16}} (\varepsilon_{17})^{\varepsilon_{17}} \left(\frac{\varepsilon_{18}}{0.0008}\right)^{\varepsilon_{18}}$$

$$C_9 = \left(\frac{500}{\varepsilon_{19}}\right)^{\varepsilon_{19}} \left(\frac{0.0004}{\varepsilon_{20}}\right)^{\varepsilon_{20}} (\varepsilon_{21})^{\varepsilon_{21}} \left(\frac{\varepsilon_{22}}{0.0004}\right)^{\varepsilon_{22}}$$

Starting with point (A_T = 15,000, A_1 = 5,000, A_2 = 5,000, A_3 = 5,000, T_1 = 200, T_2 = 350, t_{12} = 150, t_{22} = 225 and t_{32} = 425) which is in the feasible region, the problem converges to the solution after 8 iterations. Table 2 shows the convergence of the algorithm to the desired solution.

Table 2

Convergence to Optimum in Example 4

ITERATION	A_T	A_1	A_2	A_3	T_1	T_2	t_{12}	t_{22}	t_{32}	$v(\delta)$
1	15,000	5,000	5,000	5,000	200	330	150	225	425	5597
2	5,597	124	802	4,670	193	311	214	292	409	7614
3	7,614	608	1536	5,469	163	284	240	276	384	7074
4	7,074	478	1392	5,203	173	292	227	280	392	7031
5	7,051	539	1391	5,120	178	295	221	283	393	7049
6	7,044	563	1376	5,109	180	295	219	285	395	7049
7	7,049	572	1367	5,109	181	295	218	283	395	7049
8	7,049	576	1363	5,109	181	295	218	286	395	7049

Method 3. Solution by Method of Lagrange Multipliers

The method of Lagrange multipliers can be used to maximize the dual function subject to linear equality constraints [7].

The Lagrangian function is

$$L(\bar{w},\bar{\lambda}) = v(\bar{w}) - \sum_{j=1}^{N+1} \lambda_j F_j \tag{1}$$

where the λ_j are Lagrange multipliers, F_j are dual constraints given by equations (6) and (7) of Chapter III, and $v(\bar{w})$ is the dual function.

The optimal values of the dual variables can be found by taking the first partial derivatives of the Lagrangian function $L(\bar{w},\bar{\lambda})$ with respect to the dual variables w_{mt} and setting them equal to zero, or

$$\frac{\partial L(\bar{w},\bar{\lambda})}{\partial w_{mt}} = \frac{\partial v(\bar{w})}{\partial w_{mt}} - \sum_{j=1}^{N+1} \lambda_j \frac{\partial F_j}{\partial w_{mt}} = 0 \tag{2}$$

and

$$\frac{\partial L(\bar{w},\bar{\lambda})}{\partial \lambda_j} = - F_j = 0, \qquad j = 1, \ldots, N+1 \tag{3}$$

The Newton-Raphson iterative technique can be used for finding the solution set to equations (2) and (3).

$$\bar{w}_{i+1} = \bar{w}_i - H^{-1}(\nabla L)^T \tag{4}$$

where H is the Hessian matrix of $L(\bar{w},\bar{\lambda})$ and (∇L) is the gradient of $L(\bar{w},\bar{\lambda})$, both evaluated at $\bar{w}_i$. The gradient of $L(\bar{w},\bar{\lambda})$ is found from (2) and (3) and using

$$\frac{\partial v(\bar{w})}{\partial w_{mt}} = v(\bar{w})\sigma\sigma_{mt}\ell n\left(\frac{C_{mt}w_{m0}}{w_{mt}}\right) \tag{5}$$

The Hessian matrix H can be written as

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & & \vdots \\ h_{n1} & h_{n2} & & h_{nn} \end{bmatrix} \tag{6}$$

where

$$h_{ij} = \left.\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_i \partial w_j}\right|_{\bar{w}} \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, n \end{array} \tag{7}$$

w_i and w_j are the general expressions for w_{mt} and λ_j and n is the total number of dual variables and the Lagrange multipliers.

The elements of the Hessian matrix, h_{ij}, can be divided into five groups:

The first group consists of the second partial derivatives of $L(\bar{w},\bar{\lambda})$ with respect to w_{mt}, or

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{mt}^2} = \frac{\partial^2 v(\bar{w})}{\partial w_{mt}^2} = \frac{1}{v(\bar{w})}\left(\frac{\partial v(\bar{w})}{\partial w_{mt}}\right)^2 + \tag{8}$$

$$\sigma\sigma_{mt} v(\bar{w})\left(\frac{\sigma_m \sigma_{mt}}{w_{m0}} - \frac{1}{w_{mt}}\right)$$

and

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial \lambda_j^2} = 0, \qquad j = 1, \ldots, N+1 \tag{9}$$

The second group consists of the second partial derivative of $L(\bar{w},\bar{\lambda})$ with respect to w_{mt} and $w_{mt'}$, or

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{mt}\partial w_{mt'}} = \frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{mt'}\partial w_{mt}} = \tag{10}$$

$$= \frac{1}{v(\bar{w})}\left(\frac{\partial v(\bar{w})}{\partial w_{mt}}\right)\left(\frac{\partial v(\bar{w})}{\partial w_{mt'}}\right) + \frac{\sigma\sigma_{mt}\sigma_{mt'}\sigma_m v(\bar{w})}{w_{m0}}$$

The third group consists of the second partial derivative of $L(\bar{w},\bar{\lambda})$ with respect to w_{mt} and $w_{m't'}$, or

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{mt}\partial w_{m't'}} = \frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{m't'}\partial w_{mt}} = \tag{11}$$

$$= \frac{1}{v(\bar{w})}\left(\frac{\partial v(\bar{w})}{\partial w_{mt}}\right)\left(\frac{\partial v(\bar{w})}{\partial w_{m't'}}\right)$$

The fourth group consists of the second partial derivative of $L(\bar{w},\bar{\lambda})$ with respect to w_{mt} and λ_j, or

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial w_{mt}\partial\lambda_j} = \frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial\lambda_j\partial w_{mt}} = -\sum_{j=1}^{N+1} \frac{\partial F_j}{\partial w_{mt}} \quad (12)$$

The fifth group is the second partial derivative of $L(\bar{w},\bar{\lambda})$ with respect to λ_j and λ_k, or

$$\frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial\lambda_j\partial\lambda_k} = \frac{\partial^2 L(\bar{w},\bar{\lambda})}{\partial\lambda_k\partial\lambda_j} = 0, \qquad \begin{matrix} j = 1, \ldots, N+1 \\ k = 1, \ldots, N+1 \end{matrix} \quad (13)$$

A subroutine may be used for inverting the matrix H and its product by the transpose matrix $(\nabla L)^T$. Figure 3 shows the flow diagram for the method of Lagrange multipliers.

EXAMPLE 5:

Consider the following problem

minimize

$$x_0$$

subject to

$$8x_0^2 + 8x_1 \geq 11$$

$$-x_0 + 8x_1 \leq 2$$

Arranging the constraints according to the generalized geometric programming, we obtain

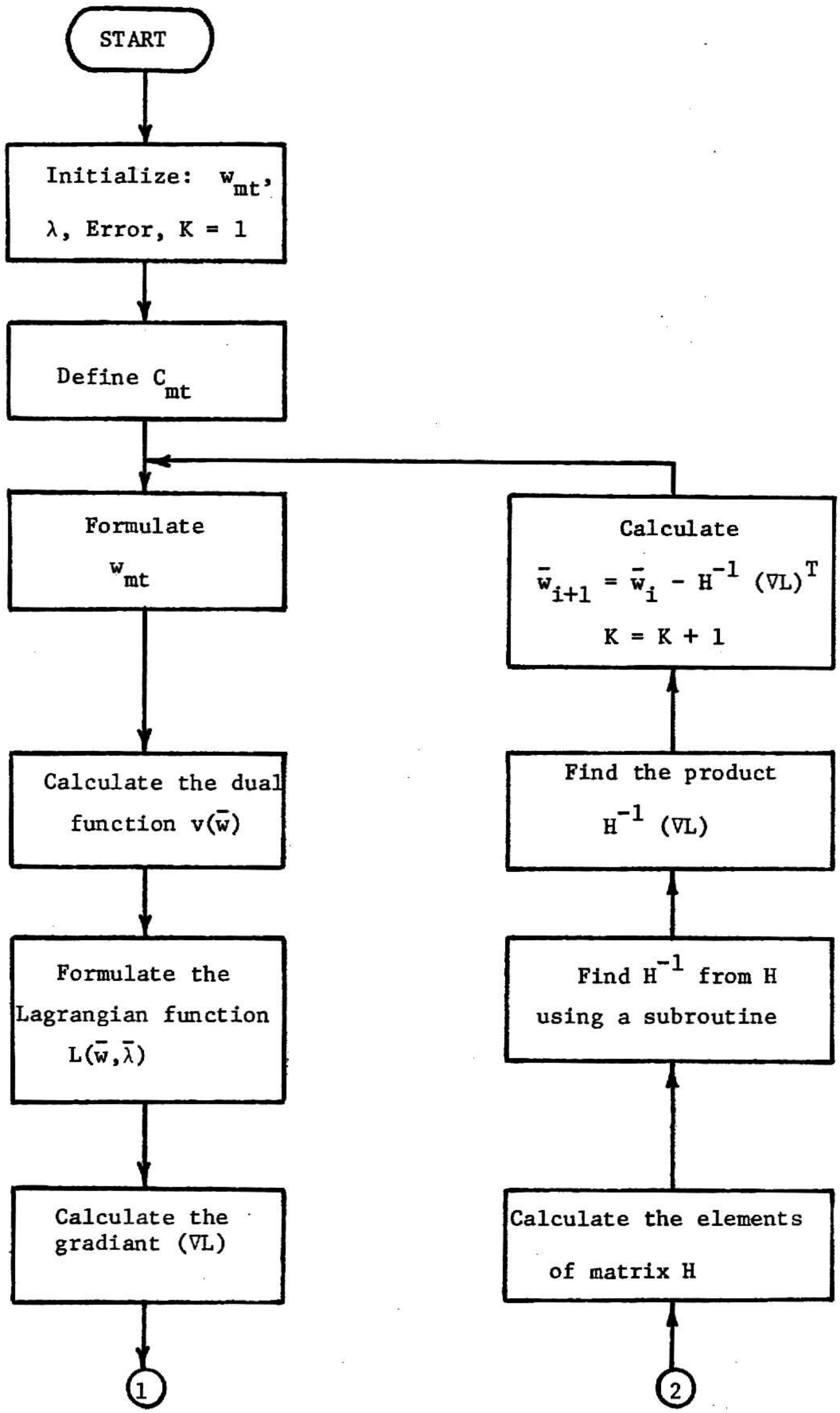


Fig. 3. The Flow Diagram for Solution by the Method of Lagrange Multipliers.

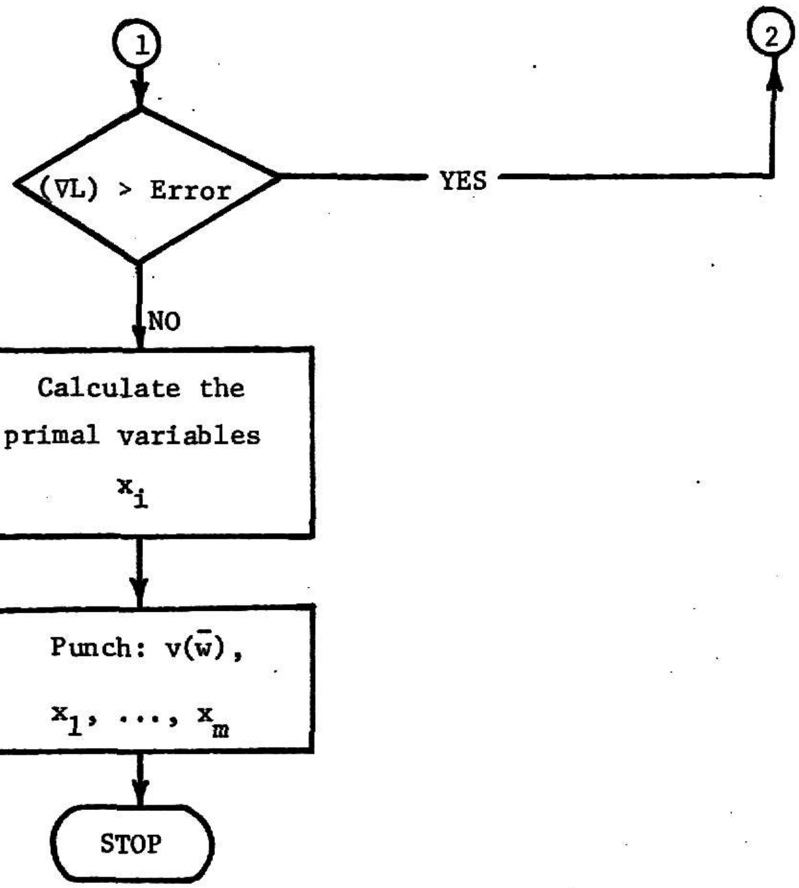


Fig. 3. Continued

minimize

$$x_0$$

subject to

$$-\frac{8}{11}x_0^2 - \frac{8}{11}x_1 \leq -1$$

$$-\frac{1}{2}x_0 + 4x_1 \leq 1$$

The problem has 5 - 3 = 2 degrees of difficulty. From the objective function and the constraints the following values are found

$\sigma_0 = 1,$	$\sigma_{01} = 1,$	$C_{01} = 1,$	$a_{011} = 1,$	$a_{122} = 1$
$\sigma_1 = -1,$	$\sigma_{11} = -1,$	$C_{11} = 8/11,$	$a_{012} = 1,$	$a_{211} = 1$
$\sigma_2 = 1,$	$\sigma_{12} = -1,$	$C_{12} = 8/11,$	$a_{111} = 2,$	$a_{212} = 0$
	$\sigma_{21} = -1,$	$C_{21} = 1/2,$	$a_{112} = 0,$	$a_{221} = 0$
	$\sigma_{22} = 1,$	$C_{22} = 4,$	$a_{121} = 0,$	$a_{222} = 1$

The dual function is

$$v(\bar{w}) = \left(\frac{C_{01}w_{00}}{w_{01}}\right)^{w_{01}}\left(\frac{C_{11}w_{10}}{w_{11}}\right)^{-w_{11}}\left(\frac{C_{12}w_{10}}{w_{12}}\right)^{-w_{12}} \times$$

$$\left(\frac{C_{21}w_{20}}{w_{21}}\right)^{-w_{21}}\left(\frac{C_{22}w_{20}}{w_{22}}\right)^{w_{22}}$$

where

$$w_{00} = w_{01} = 1$$

$$w_{10} = w_{11} + w_{12}$$

$$w_{20} = -w_{21} + w_{22}$$

The dual constraints are

$$\begin{aligned} w_{01} & & & & &= 1 \\ w_{01} &- 2w_{11} & & - w_{21} & &= 0 \\ & & - w_{12} & & + w_{22} &= 0 \end{aligned}$$

The Lagrangian function can be written as

$$L(\bar{w},\bar{\lambda}) = v(\bar{w}) - \lambda_1(2w_{11} + w_{21} - 1)$$

$$- \lambda_2 (w_{12} - w_{22})$$

where λ_1 and λ_2 are the Lagrange multipliers. Using the Newton-Ralphson iterative technique, the problem converges to the desired optimim in 3 iterations. Table 3 shows the convergence to optimum. The optimum values of x_0 and x_1 are

$$\bar{x}_0 = 1.000$$

$$\bar{x}_1 = 0.375$$

and the minimum value of the objective function is

Table 3

Convergence to optimum in Example 5.

ITERATION	w_{11}	w_{12}	w_{21}	w_{22}	λ_1	λ_2	$v(\bar{w})$
1	0.40	0.15	0.10	0.30	0.00	0.00	1.1585
2	0.47	0.18	0.06	0.18	0.00	-0.97	1.0000
3	0.47	0.18	0.06	0.18	0.00	-0.98	1.0000

$$S(\bar{x}) = 1.000$$

which quite agree with the solution obtained by complementary geometric programming in chapter IV.

CHAPTER VI

CONCLUSION

Chapter V illustrates how separable programming, complementary geometric programming and the method of Lagrange multipliers can be used to optimize constrained polynomials when faced with degrees of difficulty. These methods offer a significant extension of the applicability of geometric programming especially when the degrees of difficulty is large. Although the guarantee of global optimality is lost in these methods, in many practical situations it is sufficient to find a local minimum.

Separable programming is a powerful method for optimizing the nonlinear dual objective function of geometric programming problems with degrees of difficulty. Since the constraints of the dual problem are linear, if the dual objective function is concave, the solution will always be global. In general, the separable programming algorithm produces an approximate solution which is a local optimum. One of the advantages of the separable programming technique is that unlike other techniques its success does not depend on having a "good" starting point. This property makes separable programming very attractice when the problem has too many degrees of difficulty.

Complementary geometric programming as described in Chapter V gives an approximate solution to the geometric programming problems with degrees of difficulty. The optimal solution obtained by CGP is a local optimum and is shown to be independent of the starting point around the optimal point. However if the starting point is too far from the optimal point the algorithm does not guarantee the convergence of solution, even if it is in the feasible region (See appendix, example 1).

Complementary geometric programming as well can be used for solving maximization problems. Consider example 2 in the appendix; the problem is to maximize the daily net profit S given by

$$S = [350 - (50 + 0.25P^{1.25}R^{1.1}) - (2000/P) - (8000/P) - (20/P^{.6})]P$$

subject to

$$P/R \leq 1310$$

The problem can be changed into the general complementary geometric programming form by minimizing

$$Y_0 = C - S$$

subject to

$$P/R \leq 1310$$

where C is a large number added to -S to make the modified objective function positive. As can be seen from Table 5 in the appendix, the same results were obtained for different values of C. The results obtained shows that even if $C \leq S$ or $Y_0 < 0$, the algorithm converges to the desired solution.

The last method mentioned in Chapter V is the method of Lagrange multipliers. Like the other two methods, the method of Lagrange multipliers obtains an approximate solution to the dual problem which is mostly a local optimum. Wilde and Beightler [9] have shown that if all the signum functions (σ, σ_m, σ_{mt}) are positive, the logarithm of the dual function will always be concave and the optimal solution to the dual problem will be a global optimum. With negative signums, however, the character of the dual function is uncertain and local optimums may be produced.

In conclusion, all of the methods described in Chapter V produce an approximate solution to the geometric programming problems by giving the optimal solution to the corresponding dual problem. The solution obtained then would be a local optimum. Therefore the local optimal solution is dependent upon the starting point around the local optimal point and the guarantee of convergence to solution is lost for starting points too far from the optimal point.

ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude to his major professor, Dr. C. L. Hwang, for his sincere guidance and assistance during the preparation of this report. The author is also grateful to his committee members, Drs. F. A. Tillman, D. Grosh, and L. T. Fan, for their comments on the work.

REFERENCES

1. Avriel, M., and A. C. Williams, Complementary Geometric Programming, SIAM Journal of Appl. Math., 19, 123-141 (1970).

2. Avriel, M., and A. C. Williams, An Extension of Geometric Programming with Applications in Engineering Optimization, Journal of Engr. Math., 5, 187-194 (1971).

3. Beightler, C. S., R. M. Crisp, Jr., and W. L. Meier, Jr., Optimization by Geometric Programming, J. of Industrial Engineering, Vol. 19, pp. 117-120 (1968).

4. Boas, A. H., Optimization via Linear and Dynamic Programming, Chem. Eng., 70 April 1, 85-88 (1963).

5. Chen, Ning Hsing, Otpimization by Geometric Programming, Chemical and Process Engr., Oct. (1968).

6. Duffin, R. J., E. L. Peterson, and C. Zener, Geometric Programming, Wiley, New York, 1967.

7. Kochenberger, G. A., Geometric Programming: Extension to Deal with Degrees of Difficulty, Ph.D. Thesis, University of Colorado (1969).

8. Fan, L. T., and C. S. Wang, The Discrete Maximum Principle, Wiley New York, 1964.

9. Wilde, D. J. and C. S. Beightler, Foundation of Optimization, Prentice Hall, Englewood Cliffs, New Jersey, 1967.

10. Williams, J. L., Optimization of Industrial Systems with the Separable Programming and the Generalized Reduced Gradiant Method, M.S. Thesis, Kansas State University (1972).

APPENDIX

This section presents some of the important features involved in the solution of the geometric programming problems containing degrees of difficulty by complementary geometric programming and the method of Lagrange multiplier. Examples 1 and 2 are solved by complementary geometric programming to show the effect of different starting points in convergence of the problem to the desired solution. Example 3 is solved by the method of Lagrange multipliers for checking the results given in [7].

EXAMPLE 1: Solution by Complementary Geometric Programming

The following problem with two degrees of difficulty was solved by separable programming (See Chapter V, example 3). The solution by complementary geometric programming is presented here

Minimize

$$Y_0 = 4x_1 + 10x_2 + 4x_3 + 2(x_1^2 + x_2^2)^{1/2}$$

subject to

$$x_1x_2x_3 \geq 100$$

$$x_i \geq 0$$

If we let $x_4 = (x_1^2 + x_2^2)^{1/2}$, then the problem changes to minimize

$$Y_0 = 4x_1 + 10x_2 + 4x_3 + 2x_4$$

subject to

$$x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \leq 1$$

$$100 x_1^{-1} x_2^{-1} x_3^{-1} \leq 1$$

Now by introducing a new variable $x_0 \geq Y_0$, the problem changes to a CGP problem or

minimize

$$x_0$$

subject to

$$x_0^{-1} (4x_1 + 10x_2 + 4x_3 + 2x_4) \leq 1$$

$$x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \leq 1$$

$$100 x_1^{-1} x_2^{-1} x_3^{-1} \leq 1$$

This problem has 7 - 5 = 2 degree of difficulty, therefore the number of terms should be reduced by 2. Condensing the first inequality into two terms reduces the degrees of difficulty to zero and the modified problem is

minimize

$$x_0$$

subject to

$$x_0^{-1}\left(\frac{4x_1}{\varepsilon_1}\right)^{\varepsilon_1}\left(\frac{10x_2}{\varepsilon_2}\right)^{\varepsilon_2} + x_0^{-1}\left(\frac{4x_3}{\varepsilon_3}\right)^{\varepsilon_3}\left(\frac{2x_4}{\varepsilon_4}\right)^{\varepsilon_4} \le 1$$

$$x_1^2x_4^{-2} + x_2^2x_4^{-2} \le 1$$

$$100x_1^{-1}x_2^{-1}x_3^{-1} \le 1$$

where

$$\varepsilon_1 = 4x_1/(4x_1 + 10x_2)$$

$$\varepsilon_2 = 10x_2/(4x_1 + 10x_2)$$

$$\varepsilon_3 = 4x_3/(4x_3 + 2x_4)$$

$$\varepsilon_4 = 2x_4/(4x_3 + 2x_4)$$

The dual constraints are

$$\begin{aligned}
\delta_0 &= 1 \\
\delta_0 - \delta_1 - \delta_2 &= 0 \\
\varepsilon_1\delta_1 + 2\delta_3 - \delta_5 &= 0 \\
\varepsilon_2\delta_1 + 2\delta_4 - \delta_5 &= 0 \\
\varepsilon_3\delta_2 - \delta_5 &= 0 \\
\varepsilon_4\delta_2 - 2\delta_3 - 2\delta_4 &= 0
\end{aligned}$$

Solving for δ_0, δ_1, δ_2, δ_3 and δ_5, we obtain

$$\delta_1 = (\varepsilon_3 - \varepsilon_4)/(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - \varepsilon_4)$$

$$\delta_2 = (\varepsilon_1 + \varepsilon_2)/(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - \varepsilon_4)$$

$$\delta_3 = 1/2[\varepsilon_3(\varepsilon_1 + \varepsilon_2) - \varepsilon_1(2\varepsilon_3 - \varepsilon_4)]/(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - \varepsilon_4)$$

$$\delta_4 = 1/2[(\varepsilon_1 + \varepsilon_2)(\varepsilon_4 - \varepsilon_3) + \varepsilon_1(2(-3 - \varepsilon_4)]/(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - \varepsilon_4)$$

The dual function can be written as

$$v(\delta) = \left(\frac{C_1}{\delta_1}\right)^{\delta_1} \left(\frac{C_2}{\delta_2}\right)^{\delta_2} \left(\frac{C_3}{\delta_3}\right)^{\delta_3} \left(\frac{C_4}{\delta_4}\right)^{\delta_4} \left(\frac{C_5}{\delta_5}\right)^{\delta_5} \cdot (\lambda_1)^{\lambda_1} \cdot (\lambda_2)^{\lambda_2} \cdot (\lambda_3)^{\lambda_3}$$

where

$$\lambda_1 = \delta_1 + \delta_2$$

$$\lambda_2 = \delta_3 + \delta_4$$

and

$$\lambda_3 = \delta_5$$

The coefficients are

$$C_1 = \left(\frac{4}{\varepsilon_1}\right)^{\varepsilon_1} \left(\frac{10}{\varepsilon_2}\right)^{\varepsilon_2}$$

$$C_2 = \left(\frac{4}{\varepsilon_3}\right)^{\varepsilon_3} \left(\frac{2}{\varepsilon_4}\right)^{\varepsilon_4}$$

$$C_3 = 1$$

$$C_4 = 1$$

and

$$C_5 = 100$$

The primal variables x_1, x_2, x_3 and x_4 can be found by taking logarithm of equations (38) and (39) in Chapter II and solving for $\ell n(x_i)$.

To test the effect of different starting points in the solution by complementary geometric programming problems, different starting points were used in this example. The results obtained indicated that around the optimum point all the starting points converge to the desired optimum. However the solution could not be obtained with the starting point far from the optimum point. Table 4 shows the effect of different starting points on the optimal solution.

The optimum values of x_1, x_2, x_3 and x_4 are

$$x_1 = 5.09$$

$$x_2 = 2.68$$

$$x_3 = 7.33$$

$$x_4 = 5.74$$

and the minimum value of the objective function is

Table 4

The effect of different starting points in the solution of geometric programming problems by complementary geometric programming in example 1

Run No.	Starting Point					Optimal Solution					
	x_0	x_1	x_2	x_3	x_4	x_0	x_1	x_2	x_3	x_4	Y_0
1	87.990	5.085	2.684	7.332	5.741	87.987	5.085	2.684	7.332	5.741	87.990
2	90.0	5.0	2.5	8.0	6.0	87.608	4.819	2.851	7.276	5.600	88.098
3	110.0	6.0	5.0	4.0	8.0	87.045	5.900	2.947	5.749	5.900	87.878
4	150.0	5.0	5.0	5.0	10.0	no solution was obtained					

$$Y_0^* = 87.99$$

EXAMPLE 2: Solution by Complementary Geometric Programming

This example was solved by Chen [5] via generalized geometric programming. Complementary geometric programming solution is used to show the applicability of CGP in maximizing problems and check the effect of adding a constant to the objective function in optimal dual solution.

A TV manufacturing plant produces sets at the rate of P units per day. The manufacturing costs per set have been found to be \$50 + $0.25\ P^{1.25}\ R^{1.1}$, where R is the research and development expenses in man-hours per day. The unit cost of the research and development per set is $\$20/R^{0.6}$ per man-hour. The total daily fixed charges are \$2,000 and all other expenses are \$8,000/day. If the selling price per set is \$350.00, calculate the maximum daily net profit for a ratio of $P/R \leq 1310$.

Selling price of TV set = 350.00 (in dollar)

manufacturing cost per set = $50 + 0.25\ P^{1.25}\ R^{1.1}$ (in dollar)

fixed daily charges/set = 2,000/P (in dollar)

other daily expenses/set = 8,000/P (in dollar)

research and development expenses/set = $20/R^{0.6}$ (in dollar)

The objective function is then the daily net profit S

$$S = [350 - (50 + 0.25\ P^{1.25}\ R^{1.1}) - (2000/P) - (8000/P) - (20/R^{0.6})]P$$

which has to be maximized subject to

$$P/R \leq 1310$$

This problem can be changed into minimization problem by multiplying the objective function by -1. A constant C has to be added to the objective function since -S is a negative number. The new problem is then to minimize

$$Y_0 = C - S$$

subject to

$$P/R \leq 1310$$

Changing the problem into CGP form yields

minimize

$$x_0$$

subject to

$$(C + 10{,}000 + 0.25\ P^{2.25}\ R^{1.1} + 20\ PR^{-0.6})/(x_0 + 300\ P) \leq 1$$

This problem has 4 - 3 = 1 degree of difficulty. Condensing the last two terms in the numerator of the first inequality into one term gives

minimize

$$x_0$$

subject to

$$(C + 10{,}000)\left(\frac{x_0}{\varepsilon_3}\right)^{-\varepsilon_3}\left(\frac{300\ P}{\varepsilon_4}\right)^{-\varepsilon_4} +$$

$$\left(\frac{0.25P^{2.25}R^{1.1}}{\varepsilon_1}\right)^{\varepsilon_1}\left(\frac{20\ PR^{-0.6}}{\varepsilon_2}\right)^{\varepsilon_2}\left(\frac{x_0}{\varepsilon_3}\right)^{-\varepsilon_3}\left(\frac{300\ P}{\varepsilon_4}\right)^{-\varepsilon_4} \le 1$$

$$\frac{1}{1310}\ PR^{-1} \le 1$$

where

$$\varepsilon_1 = 0.25\ P^{2.25}R^{1.1}/(0.25\ P^{2.25}R^{1.1} + 20\ PR^{-0.6})$$

$$\varepsilon_2 = 20\ PR^{-0.6}/(0.25\ P^{2.25}R^{1.1} + 20\ PR^{-0.6})$$

$$\varepsilon_3 = x_0/(x_0 + 300\ P)$$

$$\varepsilon_4 = 300\ P/(x_0 + 300\ P)$$

The dual constraints are

$$\delta_0$$

$$\delta_0 - \varepsilon_3\delta_1 \qquad\qquad \varepsilon_3\ \delta_2$$

$$- \varepsilon_4\delta_1 + (2.25\varepsilon_1 + \varepsilon_2 - \varepsilon_4)\delta_2 + \delta_3 = 0$$

$$(1.1\ \varepsilon_1 - 0.6\ \varepsilon_2)\delta_2 - \delta_3 = 0$$

The solution to the system of linear equations is

$$\delta_1 = \frac{1}{\varepsilon_3} - \frac{\varepsilon_4}{3.35\varepsilon_1 \cdot \varepsilon_3 + 0.4\varepsilon_2 \cdot \varepsilon_3}$$

$$\delta_2 = \frac{\varepsilon_4}{(3.35\varepsilon_1 \cdot \varepsilon_3 + 0.4\varepsilon_2 \cdot \varepsilon_3)}$$

$$\delta_3 = \frac{(1.1\varepsilon_1 - 0.6\varepsilon_2)(\varepsilon_4)}{(3.35\varepsilon_1 \cdot \varepsilon_3 + 0.4\varepsilon_2 \cdot \varepsilon_3)}$$

The dual function can be written as

$$v(\delta) = \left(\frac{C_1}{\delta_1}\right)^{\delta_1} \left(\frac{C_2}{\delta_2}\right)^{\delta_2} \left(\frac{C_3}{\delta_3}\right)^{\delta_3} \cdot (\lambda_1)^{\lambda_1} \cdot (\lambda_2)^{\lambda_2}$$

where

$$\lambda_1 = \delta_1 + \delta_2$$

and

$$\lambda_2 = \delta_3$$

The primal variables are found by taking logarithm of equations (38) and (39) of Chapter II and then solving for x_0, P and R.

The problem was solved using $x_0 = 60{,}000$, $P = 100$, and $R = 1.00$ as the starting point. To see the effect of size of C, different runs were obtained with different C. The results showed that for all values of C the problem converges to the desired solution. Table 5 shows the effect of different values of C in the optimal solution.

The optimum value of S is \$48,586 and for this optimum value

$$P = 342.5$$

and

$$R = 0.2614$$

EXAMPLE 3: Solution by the method of Lagrange multipliers

Minimize

$$Y_0 = 2x_1 + x_1x_2 + 3x_2$$

subject to

$$x_1^2 + x_2 \geq 3$$

$$x_1 + 2x_2 \geq 4$$

Arranging the problem into the generalized GP form we obtain

minimize

$$Y_0 = 2x_1 + x_1x_2 + 3x_2$$

subject to

$$-\frac{1}{3}x_1^2 - \frac{1}{3}x_2 \leq -1$$

Table 5

The effect of C on the optimal solution by complementary geometric solution in example 2

Run No.	Starting Point				Optimal Solution			
	C	x_0	P	R	x_0	P	R	Y_0
1	40,000	60,000	100	1.0	231143.9	342.8	0.26168	48,585.97
2	60,000	60,000	100	1.0	273607.1	342.5	0.26146	48,586.10
3	100,000	60,000	100	1.0	346674.7	342.5	0.26146	48,586.02

$$-\frac{1}{4}x_1 - \frac{1}{2}x_2 \leq -1$$

This problem has 7 - 3 = 4 degrees of difficulty.

From the objective function and the constraints, we obtain

$\sigma_0 = 1,\quad \sigma_{01} = 1,\quad \sigma_{11} = -1,\quad \sigma_{21} = -1,$

$\sigma_1 = -1,\quad \sigma_{02} = 1,\quad \sigma_{12} = -1,\quad \sigma_{22} = -1,$

$\sigma_2 = -1,\quad \sigma_{03} = 1,$

$C_{01} = 2,\quad C_{11} = 1/3,\quad C_{21} = 1/4,$

$C_{02} = 1,\quad C_{12} = 1/3,\quad C_{22} = 1/2,$

$C_{03} = 3,$

$a_{011} = 1,\quad a_{111} = 2,\quad a_{211} = 1,$

$a_{012} = 0,\quad a_{112} = 0,\quad a_{212} = 0,$

$a_{021} = 1,\quad a_{121} = 0,\quad a_{221} = 0,$

$a_{022} = 1,\quad a_{122} = 1,\quad a_{222} = 1,$

$a_{031} = 0,$

$a_{032} = 1$

The dual constraints are

$$w_{01} + w_{02} + w_{03} = 1$$

$$w_{01} + w_{02} - 2w_{11} - w_{21} = 0$$

$$w_{02} + w_{03} - w_{12} - w_{22} = 0$$

The dual function is

$$v(\bar{w}) = \left(\frac{C_{01}w_{00}}{w_{01}}\right)^{w_{01}} \left(\frac{C_{02}w_{00}}{w_{02}}\right)^{w_{02}} \left(\frac{C_{03}w_{00}}{w_{03}}\right)^{w_{03}} \left(\frac{C_{11}w_{10}}{w_{11}}\right)^{-w_{11}} \times$$

$$\left(\frac{C_{12}w_{10}}{w_{12}}\right)^{-w_{12}} \left(\frac{C_{21}w_{20}}{w_{21}}\right)^{-w_{21}} \left(\frac{C_{22}w_{20}}{w_{22}}\right)^{-w_{22}}$$

where

$$w_{00} = w_{01} + w_{02} + w_{03} = 1$$

$$w_{10} = w_{11} + w_{12}$$

$$w_{20} = w_{21} + w_{22}$$

The Lagrangian function becomes

$$L(\bar{\theta}, \bar{\lambda}) = v(\bar{w}) - \lambda_1 (w_{01} + w_{02} + w_{03} - 1)$$

$$- \lambda_2 (w_{02} + w_{02} - 2w_{11} - w_{21})$$

$$- \lambda_3 (w_{02} + w_{03} - w_{12} - w_{22})$$

Starting with point

$$w_{01} = 0.30$$

$$w_{02} = 0.20$$

$$w_{03} = 0.50$$

$$w_{11} = 0.11$$

$$w_{12} = 0.10$$

$$w_{21} = 0.28$$

$$w_{22} = 0.60$$

$$\lambda_1 = 0.0$$

$$\lambda_2 = 0.0$$

$$\lambda_3 = 0.0$$

the problem converges to the desired solution after six iterations. The optimal solution is

$$x_1 = 1.28277$$

$$x_2 = 1.36063$$

and the minimum value of the objective function Y_0 is 8.39284. Table 6 shows the convergence of the algorithm to the desired solution in example 3.

Table 6

Convergence to optimum in example 3

ITERATION	w_{01}	w_{02}	w_{03}	w_{11}	w_{12}	w_{21}	w_{22}	λ_1	λ_2	λ_3	$v(\bar{w})$
1	0.3000	0.2000	0.5000	0.1100	0.1000	0.2800	0.6000	0.0000	0.0000	0.0000	8.3805
2	0.3398	0.2175	0.4427	0.1183	0.0407	0.3206	0.5602	17.6778	-3.1911	-1.9862	8.3370
3	0.3224	0.2125	0.4650	0.1006	0.0622	0.3338	0.6368	17.9693	-2.6491	-2.3121	8.4534
4	0.3100	0.2092	0.4808	0.1085	0.0809	0.3022	0.6278	17.9392	-2.2400	-2.5225	8.4320
3	0.3066	0.2080	0.4854	0.1123	0.0911	0.2900	0.6125	17.8788	-2.1127	-2.5676	8.4080
6	0.3037	0.2078	0.4865	0.1146	0.0947	0.2843	0.6032	17.8406	-2.0810	-2.5756	8.3911

GEOMETRIC PROGRAMMING: METHODS FOR DEALING

WITH DEGREES OF DIFFICULTY

by

PARVIZ PARCHAMI-GHAFEROKHI

B.S., Abadan Institute of Technology, Iran, 1967

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1973

The objective of this report is to give a summary of methods for solving geometric programming problems with degrees of difficulty. Each method is tested by several numerical examples. Problems considered in this report are: geometric programming problems with only one degree of difficulty, and geometric programming problems with two or more degrees of difficulty..

In type one problem the optimal value of the objective function can be either estimated by assigning an upper and lower bound to the optimal value, or evaluated by expressing the dual variables in terms of one of the variables, substituting the dual variables into the dual function and maximizing the dual function by one of the one dimensional search techniques.

Type two is the more general case of the geometric programming problems and has to handled by optimizing the dual objective function subject to the linear constraints. The optimization techniques employed are separable programming, complementary geometric programming and the method of Lagrange multipliers. Some examples and design problems are solved to illustrate the algorithm.