TORQUE LOAD DISTURBANCES IN SERVOMECHANISMS

by

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INTRODUCTION

The methods of analysis and design procedures used in the study and synthesis of servomechanisms have been fully discussed in numerous books and periodicals. However, in the majority of these discussions it is assumed that the only torques acting on the output member of the servomechanism are the torque generated in the servo motor and the torque derived from the moment of inertia and viscous friction associated with the output member when the output member is in motion. However in many applications of servomechanisms such is not the case. For example, in the case of a servo controlled radar antenna there is also a torque acting to turn the antenna which is produced by the action of the wind on the antenna itself. In this case the total torque which acts to determine the position of the antenna is a function of the wind velocity as well as a function of the setting of the antenna positioning indicator and the moment of inertia and viscous friction associated with the antenna mount, gear train and drive motor. It is the purpose of this discussion to analyze the effects of such a torque which is produced in the output member of the servomechanism and to discuss various methods of minimizing this effect.

In this discussion only those servomechanisms whose operating characteristics satisfy the following conditions will be considered.

(a) The power supplied to the output member by the servo motor is a continuous function of the system error.

(b) The system, in the absence of a torque load output disturbance, will exhibit no steady state error for a constant value of input.

(c) The system is stable.

(d) All elements of the system operate within their linear range.

For purposes of analysis the concept of the system transfer function will be used (James, Nichols and Phillips, 1947, p. 58; Brown and Campbell, 1948, p. 84). In general, for any stable system having only one input and one output the transfer function may be defined as the ratio of the Laplace transform of the input to the transform of the output when the input and output are expressed as functions of time and when the magnitude of the input is zero for time less than zero. The transfer function diagram of a typical servomechanism is shown in Fig. 1. where

$\Theta_i(s)$ is the Laplace transform of the servomechanism input function $\Theta_i(t)$,

$\Theta_o(s)$ is the Laplace transform of the servomechanism output function $\Theta_o(t)$

$E_e(s)$ is the transform of the error function $E_e(t)$ between the system input and output, i.e. $E_e(t) = \Theta_i(t) - \Theta_o(t)$

$Y(s)$ is the transfer function of the system elements between the differential element and the point of appearance of the output. $Y(s)$ is defined in such a way that $\Theta_o(s) = E_e(s)\, Y(s)$

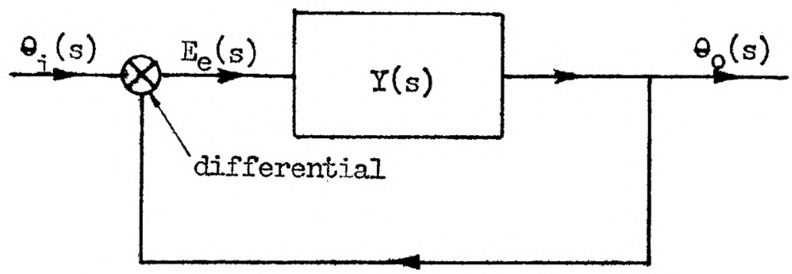


Fig. 1

The symbol noted as a differential in Fig. 1 will be used throughout the following discussion to symbolize any element whose function is the combining of two inputs such that the output from the element is equal to the difference between the inputs.

Since $\theta_o(s) = E_e(s)$ and $E_e(s) = \theta_i(s) - \theta_o(s)$

$$\theta_o(s) = \frac{Y(s)}{1 + Y(s)} \theta_i(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the overall input to output transfer function of the system becomes

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{Y(s)}{1 + Y(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

DERIVATION OF THE FORM OF THE TRANSFER FUNCTION DIAGRAM TO INCLUDE THE EFFECT OF A TORQUE LOAD DISTURBANCE

For purposes of analysis it is convenient to first examine the operating characteristics and to then derive the transfer function of the servomechanism power output member or servo motor. In a typical servomechanism the net torque applied to the output shaft is balanced at any instant by an equal and opposite torque made up of two components. The first component is that resulting from the viscous friction associated with the output member. It is assumed that all dry friction has been eliminated. This relationship may be expressed mathematically as shown in (3)

$$T_m(t) + T_L(t) = J \frac{d^2\theta_o(t)}{dt^2} + f \frac{d\theta_o(t)}{dt} \quad . \quad . \quad . \quad (3)$$

where $T_m(t)$ is the torque developed by the servo motor and expressed as a function of time,

$T_L(t)$ is any extraneous torque which is developed in the output member itself and which may be expressed as a function of time alone.

J is the moment of inertia of the output member

f is the coefficient of viscous friction acting on the output member

$\theta_o(t)$ is the angular displacement of the output member expressed as a function of time.

Taking the Laplace transform of (3) and setting $\theta_o(t = 0)$ and $\frac{d\theta_o(t = 0)}{dt}$ equal to zero

$$T_m(s) + T_L(s) = J\, s^2\, \theta_o(s) + f\, s\, \theta_o(s) \quad . \quad . \quad . \quad (4)$$

and thus

$$\Theta_o(s) = \frac{1}{s(Js + f)} T_m(s) + T_L(s) \quad . \quad . \quad . \quad . \quad . \qquad (4a)$$

An equivalent transfer function diagram representing the expression of (4a) is shown in Fig. 2.

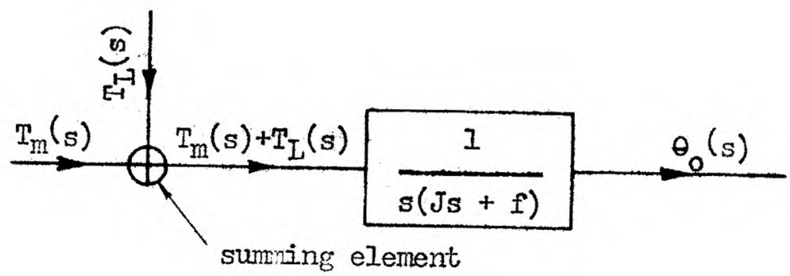


Fig. 2

Thus a servomechanism output member which is subjected to load torque disturbances may be represented by the transfer function, $\frac{1}{s(Js + f)}$, a summing element and two input torques, Ta(s) and $T_L(s)$. The symbol noted as a "summing element" in Fig. 2 will henceforth be understood to refer to an element which will produce a single output equal to the sum of two inputs applied to the elements. It is apparent that $T_m(s)$ is some function of $E_m(s)$, the input to the servomotor. It is also apparent that the form of the servomotor transfer function $\frac{T_m(s)}{E_m(s)}$ will not be the same for all the various types of servomotors (e.g. armature voltage controlled DC motor, field voltage controlled DC motor, hydraulic motor, etc.) that might be used. Moreover, as will be shown, it is in many cases impossible to represent the servomotor input versus torque relationship by a simple transfer function of the form $\frac{T_m(s)}{E_m(s)}$.

For purposes of illustration the transfer function and equivalent transfer function diagram of a DC motor with constant field current and a variable armature voltage will be derived. The input to the motor is assumed to be a voltage, $E_m(t)$, which is applied to the armature terminals. If the operating characteristics of the motor are linear then

$$T_m(t) = K\phi \, I_a(t) \quad \ldots\ldots\ldots \quad (5)$$

where $I_a(t)$ is the armature current, expressed as a function of time,

ϕ is the magnitude of the constant field flux

K is a constant of proportionality.

The equivalent circuit of the motor as viewed from the armature terminals is shown in Fig. 3 where L_a and R_a are the armature inductance and resistance and $E_b(t)$ is the back e.m.f. of the motor expressed as a function of time.

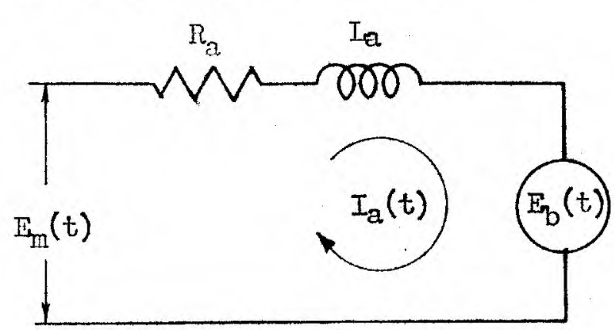


Fig. 3

Thus

$$E_m(t) = R_a\, I_a(t) + L_a \frac{dI_a(t)}{dt} + E_b(t) \quad . \quad . \quad . \quad . \quad (6)$$

But, assuming linearity, $E_b(t)$ is proportional to the magnitude of the flux ϕ and the angular speed of rotation of the motor $\frac{d\,\Theta_o(t)}{dt}$.

Therefore

$$E_b(t) = K\,\phi \frac{d\,\Theta_o(t)}{dt}$$

or combining constants

$$E_b(t) = K'_b \frac{d\,\Theta_o(t)}{dt}.$$

Substituting this expression for $E_b(t)$ in (6)

$$E_m(t) = R_a\, I_a(t) + L_a \frac{dI_a(t)}{dt} + K'_b \frac{d\,\Theta_o(t)}{dt} \quad . \quad . \quad (7)$$

Taking the Laplace transform of the equation and solving for $I_a(s)$ (7a) is obtained.

$$I_a(s) = \frac{E_m(s) - K'_b s\, \Theta_o(s)}{L_a s + R_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

By combining the constants (5) may be written in the form

$$T_m(t) = K'_m I_a(t)$$

and taking the Laplace transform

$$T_m(s) = K'_m\, I_a(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

Substituting (7a) in (5a) and rearranging terms the following expression is obtained.

$$T_m(s) = \frac{K'_m}{L_a S + R_a} \left[E_m(s) - K'_b s\, \Theta_o(s) \right] \quad . \quad . \quad . \quad (8)$$

The relationship expressed in (8) can be represented by the transfer function diagram shown in Fig. 4.

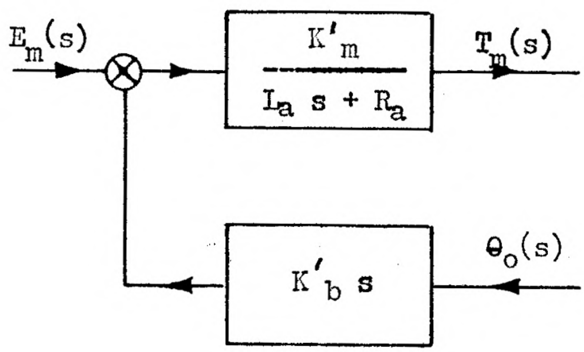


Fig. 4

Combining Fig. 4 with Fig. 2, Fig. 5 is obtained as the complete transfer function diagram for an armature voltage controlled DC servomotor and load with torque load disturbances acting on the output member.

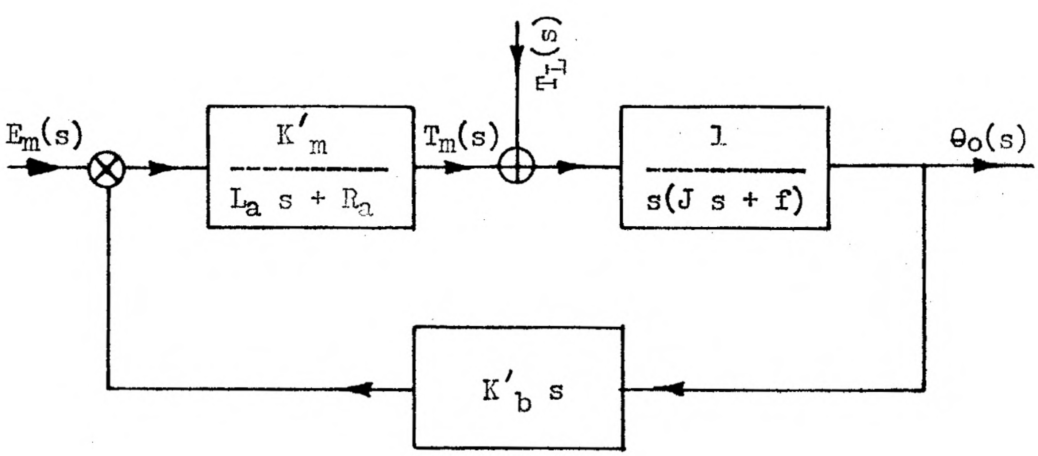


Fig. 5

It is of particular interest to note that, in this case, it was necessary to represent a DC motor by a feedback system. Examination of (7) shows that this configuration is dictated by the fact that the motor armature current $I_a(t)$ is not only a function of the input voltage, E_m, but is also a function of the speed of rotation of the motor, $\frac{d\Theta_o(t)}{dt}$. If T_m is defined as the developed torque of the motor it is impossible to formulate an input torque transfer function of the form $\frac{T_m(s)}{E_m(s)} = f(s)$, when f(s) is a function of the system parameters and 3 alone. Of course, in the case where the torque load T_L is negligible, a single overall servomotor transfer function $\frac{\Theta_o(s)}{E_m(s)}$ may be obtained by combining the transfer function blocks of Fig. 5.

In general, if a servomotor with a sloping speed torque characteristic, such as is shown in Fig. 6 curves (a) and (b), is to be studied with reference to torque load effects in a servomechanism, it must be broken down into an equivalent transfer function diagram of the general type shown in Fig. 5.

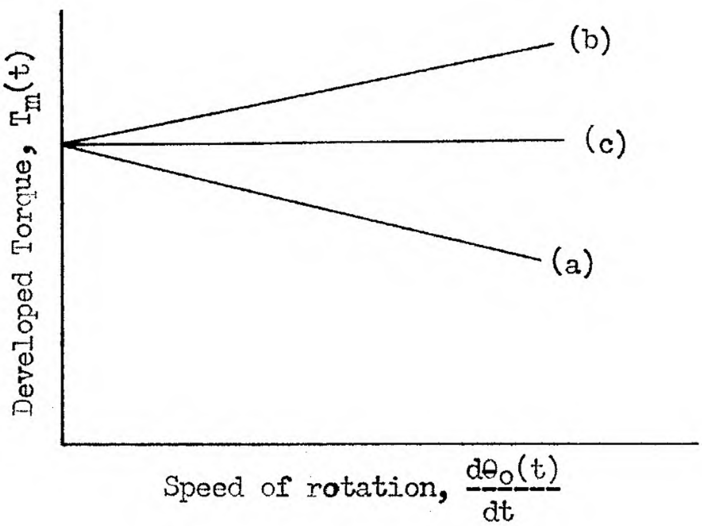


Fig. 6

Only in the case of a characteristic as shown in curve (c) of Fig. 6 would the simpler diagram which neglects feedback effects be a satisfactory representation of the servomotor with torque load disturbances.

It is possible to simplify the form of the transfer function diagram by suitable manipulation of the transfer function blocks shown in Fig. 5. However, when this simplification has been made, it is impossible to locate on the diagram the exact input voltage to armature current transfer function block or the actual current to torque block in the order in which they would exist in the block diagram of the motor itself. This, of course, is no real disadvantage once the fundamental form of the diagram is understood.

Referring to the Fig. 5

$$T_m(s) = \left[E_m(s) - K'_b s\,\theta_o(s)\right] \frac{K'_m}{L_a s + R_a}$$

This relationship is maintained in the diagram of Fig. 7 where the position of the differential has been shifted but the feedback transfer function has been changed so that the overall transfer function remains unchanged.

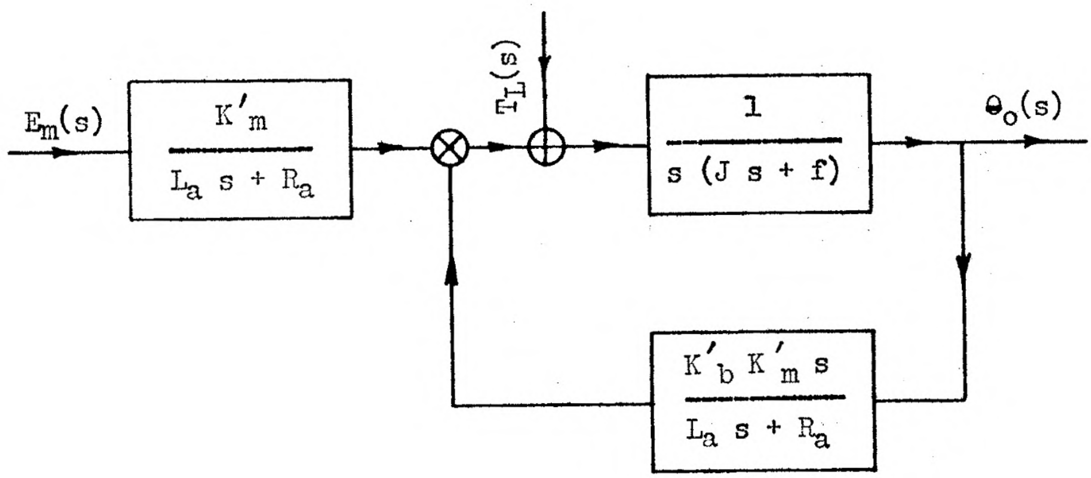


Fig. 7

Since the functions of the summing and differential elements are linear it is possible to reverse their relative positions as is shown in Fig. 8.

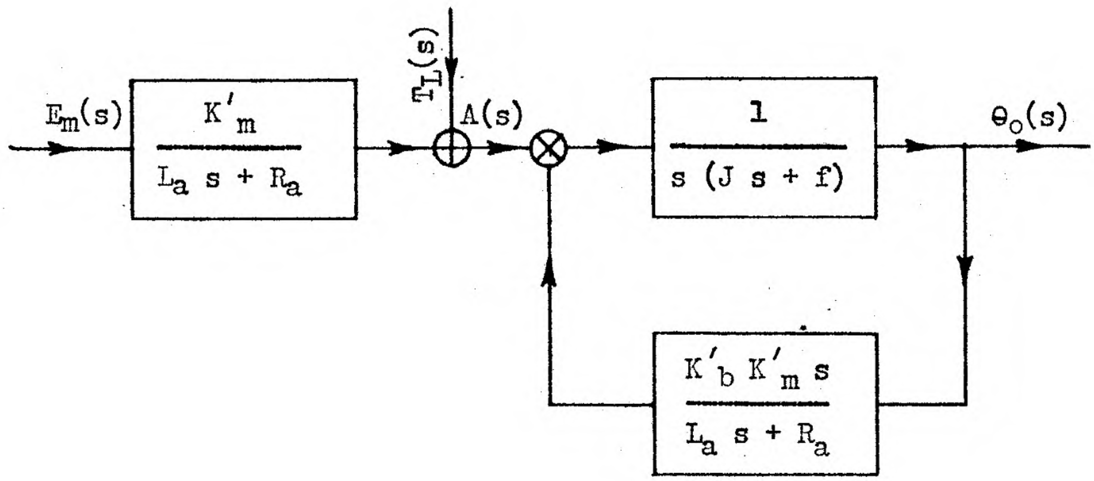


Fig. 8

The transfer function $\frac{\Theta_o(s)}{A(s)}$ of the entire feedback section of Fig. 8 may now be written, where the function A(s) has no real physical significance except as a combination of certain other functions and parameters.

$$\frac{\Theta_o(s)}{A(s)} = \frac{L_a s + R_a}{s\left[(J s + f)(L_a s + R_a) + K_b' K_m'\right]} \quad . \quad . \qquad (9)$$

Thus the transfer function of the servo motor operating under load disturbance conditions may be redrawn as shown in Fig. 9 and the new expression for $\Theta_o(s)$ as a function of $E_m(s)$ and $T_L(s)$ is

$$\Theta_o(s) = \frac{\left[\frac{K_m' E_m'(s)}{L_a s + R_a} \quad T_L(s)\right]\left[L_a s + R_a\right]}{s\left[(J s + f)(L_a s + R_a) + K_b' K_m'\right]} \quad . \qquad (10)$$

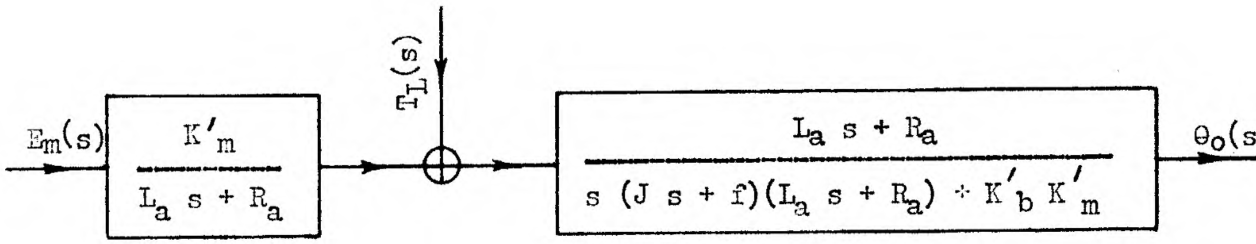


Fig. 9

Equation 10 could, of course, have been derived directly by combining (3), (5), and (7). However, the method of approach illustrated will allow a closer correlation between the transfer function diagram and the actual system component block diagram. In the case of more complex servo motor transfer functions an analysis of the type shown is a particularly valuable aid to understanding the manner in which the individual motor parameters enter into the overall motor transfer function.

In practical servomechanism studies the time constant of the armature circuit, i.e. $\frac{L_a}{R_a}$, is often so small as to have negligible effect on the overall system performance. In this case the term $L_a \frac{dI_a(t)}{dt}$ in (6) may be neglected and (10) becomes

$$\Theta_o(s) = \frac{\left[\dfrac{K_m'}{R_a} E_m(s) + T_L(s)\right] R_a}{s\left[(J s + f) R_a + K_b' K_m'\right]} \quad . \quad . \quad . \quad . \quad (10a)$$

which upon dividing both numerator and denominator by R_a becomes

$$\Theta_o(s) = \frac{\dfrac{K_m'}{R_a} E_m(s) + T_L(s)}{s (J s + f) + \dfrac{K_b' K_m'}{R_a} s} \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

After making the substitutions $\dfrac{K_m'}{R_a} = K_m$ and $\dfrac{K_b' K_m'}{R_a} = K_b$ and rearranging terms the following expression for $\Theta_o(s)$ is obtained.

$$\Theta_o(s) = \frac{K_m E_m(s) + T_L(s)}{s\left[\dfrac{J}{f + K_b} s + 1\right](f + K_b)} \quad . \quad . \quad . \quad . \quad (11)$$

Equation 11 may be represented by the transfer function diagram of Fig. 10.

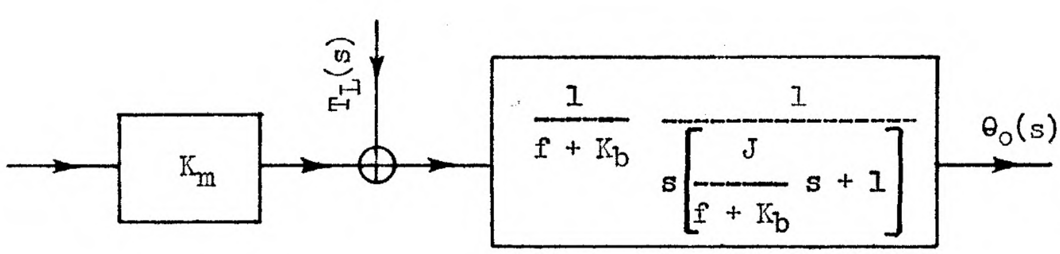


Fig. 10

It is of interest to note that K_b is the slope of the speed vs. developed torque characteristic shown in Fig. 6. Also K_m' is the slope of the stall torque vs. input voltage characteristic of the motor. In fact, if K_m and K_b are defined in this manner and load torque disturbances are assumed to be negligible, the transfer function

$$\frac{E_m(s)}{\theta_o(s)} = \frac{K_m}{(f + K_b)\ s\left[\dfrac{J}{f + K_b}\ s + 1\right]}$$

for the entire servo motor and load may be derived directly.

There are, of course, many different special transfer function diagrams which could be derived for different types of servo motors which operate in conjunction with torque load disturbances. However, the general method of attack for the analysis of such a system has been illustrated and by following the procedure given it is possible to develop the diagram for any linear servo motor when the motor parameters are known either directly or with reference to the speed versus torque curves for the motor. Additional information on the various types of servomotors may be found in James, Nichols and Phillips (1947, p. 103); Brown and Campbell (1948, p. 128).

Generally the servo motors encountered in practice can be represented by a transfer function diagram of the type shown in Fig. 10. For this reason it will henceforth be assumed that this transfer function diagram is a satisfactory representation of the particular servo motors to be considered in this discussion.

THE EXPRESSION FOR THE SERVOMECHANISM OUTPUT IN TERMS OF THE TORQUE LOAD DISTURBANCE FUNCTION

It will be assumed that the servo motor with torque loading may be represented by two transfer function blocks Y_m (s) and $Y_o(s)$ of the general type shown in Fig. 9 or Fig. 10.

$$\text{If } Y_a(s) = \frac{E_m(s)}{E_e(s)}$$

where $Y_a(s)$ is the transfer function of all the elements in the system between the point of appearance of the error function, $E_e(s)$ and the servo motor, then the transfer function diagram of the servomechanism which is subjected to torque load disturbances is as shown in Fig. 11.

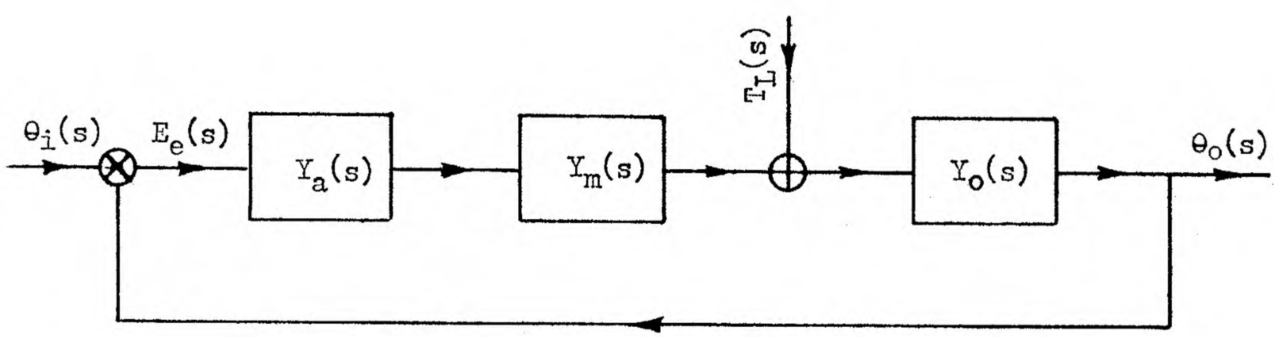


Fig. 11

As shown in Fig. 11, the torque load disturbance, $T_L(s)$, may be handled as an additional input to the servomechanism.

In accordance with Fig. 11

$$\theta_o(s) = \left[Y_a(s)\, Y_m(s)\, E_e(s) + T_L(s)\right] Y_o(s)$$

and since

$$E_e(s) = \Theta_i(s) - \Theta_o(s)$$

$$\Theta_o(s) = \frac{Y_a(s)\,Y_m(s)\,Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s)}\,\Theta_i(s) + \frac{Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s)}\,T_L(s) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It will be noted that the servomechanism output is made up of two independent components, one of which results from the input Θ_i and the other from the "unwanted input", T_L. This separation of the response into two independent parts is in accordance with the "superposition theorem" which states that "In any linear system, the output is the sum of the outputs due to each of the individual inputs taken separately." Equation 12 may also be written in the form

$$\Theta_o(s) = \frac{Y_a(s)\,Y_o(s)\,Y_m(s)}{1 + Y_a(s)\,Y_o(s)\,Y_m(s)}\,\Theta_i(s) + \frac{Y_a(s)\,Y_m(s)\,Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s)}\;\frac{1}{Y_a(s)\,Y_m(s)}\,T_L(s) \quad . \quad . \quad (12a)$$

Since the overall input to output transfer function of the system is

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{Y_a(s)\,Y_m(s)\,Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s)}$$

The torque load to output transfer function may be written in the form

$$\frac{\Theta_o(s)}{T_L(s)} = \left[\frac{\Theta_o(s)}{\Theta_i(s)}\right]\frac{1}{Y_a(s)\,Y_m(s)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In accordance with (13) it is seen that, whenever the servomechanism which is being acted upon by a torque load disturbance, $T_L(t)$, is represented by a transfer function diagram of the type shown in Fig. 11, the transfer function $\frac{\theta_o(s)}{T_L(s)}$ may be obtained directly by dividing the transfer function $\frac{\theta_o(s)}{\theta_i(s)}$ by the transfer function of the elements in the system which occurs ahead of the $T_L(s)$ inputs. It will be noted that the sign of the right side of (13) is determined only by the arbitrary selection of the sign of $T_L(t)$ in (3). Had $T_L(t)$ been defined as $-T_L(t)$, the summing element in Fig. 2 and Fig. 4 would have been shown as a differential. This in turn would have resulted in a negative sign in place of the positive sign on the second term in (12) and the right side of (13).

It is of interest to note that, even though the system is further complicated by the inclusion of stabilizing feedback loops around the point of load torque input, the form of (13) is unchanged. To illustrate, the transfer function diagram of Fig. 12 is the same as that of Fig. 11 except for the addition of the feedback element with a transfer function, $Y_f(s)$.

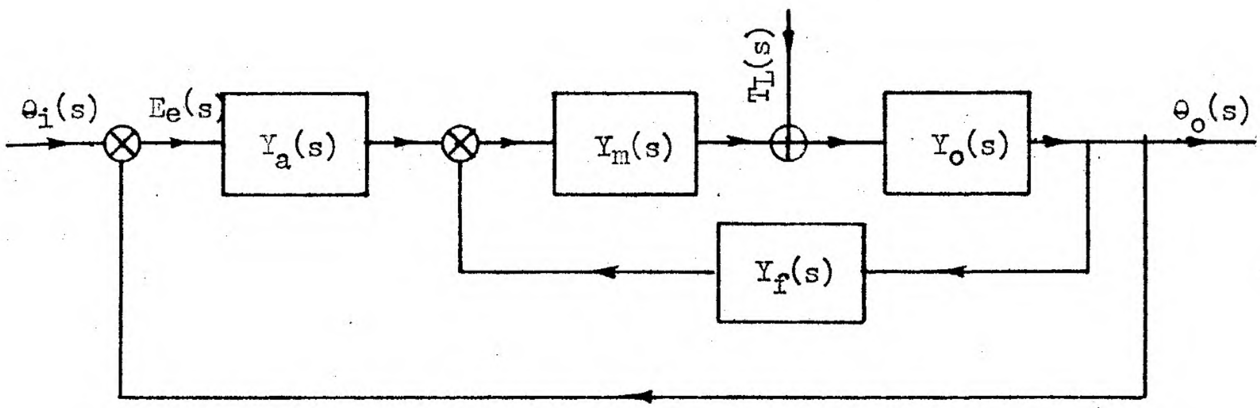


Fig. 12

In accordance with Fig. 12

$$\theta_o(s) = \left\{\left[\left[\theta_i(s) - \theta_o(s)\right] Y_a(s) - Y_f(s)\,\theta_o(s)\right] Y_m(s) + T_L(s)\right\} Y_o(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

on rearranging terms

$$\theta_o(s) = \frac{Y_a(s)\,Y_m(s)\,Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s) + Y_f(s)\,Y_m(s)\,Y_o(s)}\,\theta_i(s)$$

$$\frac{Y_o(s)}{1 + Y_a(s)\,Y_m(s)\,Y_o(s) + Y_f(s)\,Y_m(s)\,Y_o(s)}\,T_L(s). \quad . \qquad (14a)$$

The first part of (14a) is in accord with the results derived elsewhere James, Nichols and Phillips (1947, p. 134); i.e., the transfer function, $\frac{\theta_o(s)}{\theta_i(s)}$, is equal to the straight through transfer function of the system divided by one plus the sum of the transfer functions taken completely around each of the various feedback loops. Thus, here again,

$$\frac{\Theta_o(s)}{T_L(s)} = \left[\frac{\Theta_o(s)}{\Theta_i(s)}\right] \frac{1}{Y_a(s)\,Y_m(s)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

In the case of a system containing even another loop the transfer function, $\dfrac{\Theta_o(s)}{T_L(s)}$, will still be of the form shown in (13).

THE EFFECT OF THE STEADY STATE COMPONENT OF THE TORQUE LOAD DISTURBANCE

In some particular servomechanism applications the torque load acting on the output member may be made up of a non varying component plus additional components of higher frequency. A torque load of this type is shown in Fig. 13 where the torque load, T_L, is plotted as a function of time.

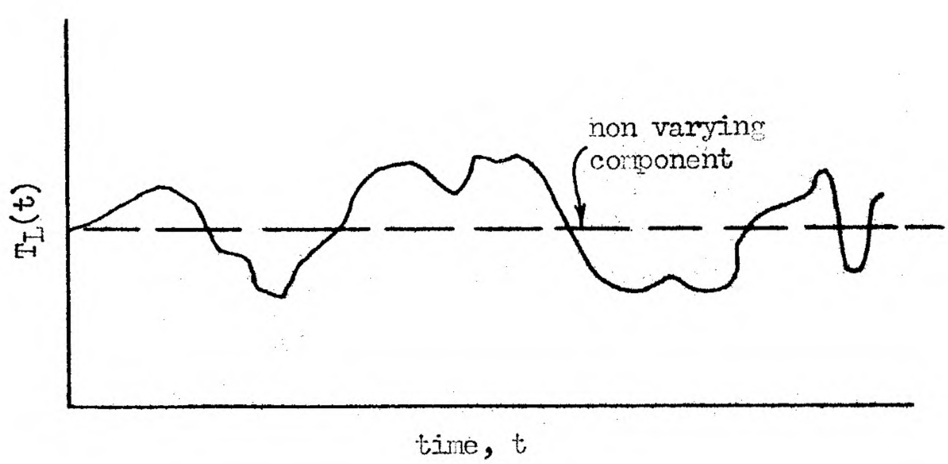


Fig. 13

In this case if the non varying component were of large magnitude, it would be desirable for the servomechanism control characteristic to be of such a nature as to completely eliminate the error resulting from the non varying component of the load torque. The form of the servomechanism transfer function required to completely eliminate the effect of this component of torque will now be derived.

Since all elements in the servomechanism are assumed to be linear, the operating characteristics of each element can be expressed in the form of a linear differential equation with constant coefficients of the form

$$a \frac{d^n E_i(t)}{dt^n} + \frac{d^{n-1}E_i(t)}{dt^{n-1}} + - - - m \frac{dE_i(t)}{dt} + k = E_o(t)$$

where $E_i(t)$ and $E_o(t)$ are respectively the input to the element and the output from the element expressed as a function of time. Thus the transfer function of any element in the servomechanism will be of the form $K \frac{f(s)}{s^r g(s)}$ where $f(s)$ and $g(s)$ are polyomials in s of the form $f_m s^m + f_{m-1}s^{m-1} + - - - f_1 s + 1$ and $g_n s^n + g_{n-1}s^{n-1} + - - - - g_1 s + 1$, r is an integer and k is a constant. It is apparent that in the case of two elements in cascade, the overall transfer function would be the product of the two individual transfer functions and thus would also be of the general form

$$K \frac{f(s)}{s^r g(s)}.$$

If the substitutions

$$Y_a(s)\, Y_m(s) = K_1 \frac{f_1(s)}{s^p g_1(s)} \text{ and } Y_o(s) = K_2 \frac{f_2(s)}{s^q g_2(s)}$$

are made in (12) the expression for $\Theta_o(s)$ in terms of $\Theta_i(s)$ and $T_L(s)$ may be written in the form

$$\Theta_o(s) = \frac{K_1K_2 \dfrac{f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s)}}{1 + K_1K_2 \dfrac{f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s)}}\, \Theta_i(s) + \frac{K_2 \dfrac{f_2(s)}{s^q\, g_2(s)}}{1 + K_1K_2 \dfrac{f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s)}}\, T_L(s) \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

If the numerator and denominator of both terms on the right side of this expression are multiplied by s^{p+q}, the following expression is obtained.

$$\Theta_o(s) = \frac{K_1K_2\, h_1(s)\, h_2(s)}{s^{p+q} + K_1K_2\, h_1(s)\, h_2(s)}\, \Theta_i(s) + \frac{K_2\, s^p\, h_2(s)}{s^{p+q} + K_1K_2\, h_1(s)\, h_2(s)}\, T_L(s) \quad . \quad . \quad . \quad . \quad . \qquad (15a)$$

where $h_1(s) = \dfrac{f_1(s)}{g_1(s)}$ and $h_2(s) = \dfrac{f_2(s)}{g_2(s)}$

Referring to the term in 15a which gives the component of $\Theta_o(s)$ resulting from the input, $\Theta_i(s)$, and assuming $\Theta_i(t)$ to be a step function of magnitude Θ_i at $t = 0$, then, for $T_L(s) = 0$.[1]

$$\Theta_o(s) = \frac{K_1K_2\, h_1(s)\, h_2(s)}{s^{p+q} + K_1K_2\, h_1(s)\, h_2(s)}\, \frac{|\Theta_i|}{s} \quad . \quad . \quad . \qquad (16)$$

---

[1]A unit step function, $U(t)$, is a function defined in a manner such that $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t > 0$. The Laplace transform of the unit step function is $1/s$.

It may be shown that, when the indicated limits exist, the following expression is valid (James, Nichols, and Phillips, 1947, p. 55).

$$\lim_{s \to 0} \left[s\ f(s) \right] = \lim_{t \to \infty} \left[F(t) \right]$$

where f(s) is the Laplace transform of F(t).

Thus, from (16)

$$\lim_{t \to \infty} \Theta_o(t) = \lim_{s \to 0} \left[s \ \frac{K_1 K_2\ h_1(s)\ h_2(s)}{s^{p+q} + K_1 K_2\ h_1(s)\ h_2(s)} \ \frac{|\Theta_i|}{s} \right] \quad . \ (17)$$

Since $\lim_{s \to 0} h_1(s) = 1$ and $\lim_{s \to 0} h_2(s) = 1$, (17) may be reduced to the form

$$\lim_{t \to \infty} \Theta_o(t) = \lim_{s \to 0} \left[\frac{K_1 K_2}{s^{p+q} + K_1 K_2} \ |\Theta_i| \right] \quad . \quad . \quad . \quad . \ (17a)$$

In order that the system exhibit zero steady state error $\lim_{t \to \infty} \Theta_o(t) = \Theta_i$, therefore the coefficient of Θ_i in 17a must approach unity as s approaches zero as a limit. In order that this condition be satisfied

$$\lim_{s \to 0} s^{p+q} = 0 \text{ and } p+q > 0$$

Similarly, if $\Theta_i(s)$ is set equal to zero and $T_L(t)$ is taken as a step function of magnitude T_L at time t = 0, (18) is obtained.

$$\lim_{t \to \infty} \Theta_o(t) = \lim_{s \to 0} \left[s \ \frac{K_2\ s^p\ h_2(s)}{s^{p+q} + K_1 K_2\ h_1(s)\ h_2(s)} \ \frac{|T_L|}{s} \right] \quad . \ (18)$$

If the steady state component of $T_L(t)$ is to have no effect on the system output, $\Theta_o(t)$, the right side of (18) must vanish as s approaches zero as a limit. Therefore, since $\lim_{s \to 0} s^{p+q} = 0$,

$$\lim_{s \to 0} s^p = 0 \text{ and } p > 0$$

In the case where the system contains an internal feedback loop as shown in Fig. 12 a similar procedure may be followed in determining the requirements for zero steady state torque load effect. Here again, it is found that assuming $Y_f(s)$ is of the form required for zero steady state system error with $T_L = 0$, p must be a positive integer.

In accordance with the foregoing analysis the general criterion for zero steady state torque load effect may be now stated in the following form.

For a servomechanism to exhibit no system error due to the steady state component of load torque, the overall transfer function of the elements shown ahead of the point of application of the load torque on the system transfer function diagram must be of the form $\dfrac{f(s)}{s^r \; g(s)}$, where r is an integer greater than zero.

As was shown, $Y_o(s)$ will in general be of the form $\dfrac{f(s)}{s \; g(s)}$; i.e., the exponent q in (15) will be unity. Since for zero steady state torque load effect the exponent p in $Y_a(s)\, Y_m(s)$ must be an integer greater than zero, p+q in $Y_a(s)\, Y_m(s)\, Y_o(s)$ must be at least two. That is, referring to Fig. 11 or Fig. 12,

$$Y_a(s)\, Y_m(s)\, Y_o(s) = K \frac{f(s)}{s^2 \; g(s)} \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

where $f(s) = f_m s^m + f_{m-1} s^{m-1} + \ldots \ldots f_1 s + 1$

$g(s) = g_n s^n + g_{n-1} s^{n-1} + \ldots \ldots g_1 s + 1$

and K is a constant, generally referred to as the frequency invariant gain or simply the gain of the elements. This result is in accordance with the statement often heard in connection with servo-mechanism work, namely that, "An infinite velocity constant system is unaffected by a load torque of constant magnitude."[2]

Referring to (15) the stability of such a system may be investigated by setting $T_L(s) = 0$ and taking $\Theta_i(t)$ as a unit impulse occurring at time $t = 0$.[3] Upon making this substitution

$$\Theta_o(s) = \frac{K_1 K_2 \dfrac{f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s)}}{1 + K_1 K_2 \dfrac{f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s)}}$$

or, upon rearranging terms,

$$\Theta_o(s) = \frac{K_1 K_2\, f_1(s)\, f_2(s)}{s^{p+q}\, g_1(s)\, g_2(s) + K_1 K_2\, f_1(s)\, f_2(s)} \quad . \quad . \qquad (20)$$

Since $f_1(s)$, $f_2(s)$, $g_1(s)$ and $g_2(s)$ are general polynomials in s as shown in (19), the products $f_1(s)\, f_2(s)$ and $g_1(s)\, g_2(s)$ will also be polynomials of the same general form. That is

$$f_1(s)\, f_2(s) = f_u s^u + f_{u-1} s^{u-1} + \ldots + f_{u-w} s^{u-w} + \ldots + f_1\, s + 1$$

---

[2]The velocity constant of a servomechanism is defined as the reciprocal of the steady state error which results from the application of an input of constant velocity. (James, Nichols and Phillips, 1947, p. 145).

[3]A unit impulse is a function of infinite amplitude and infinitesmal duration defined such that the time integral of the function from minus to plus infinity is unity. An impulse of strength other than unity is defined such that its time integral is equal to the strength of the impulse (Goldman, 1949, p. 100.)

$$g_1(s)\ g_2(s) = g_v s^v + g_{v-1}s^{v-1} + \ldots + g_{v-x}s^{v-x} + \ldots + g_1 s + 1$$

where f_{u-w} and g_{v-x} are constants.

In accordance with "Routh's Stability Criterion" the system defined by (20) will under no conditions be stable if the coefficient of any power of s in the denominator of (20) is zero (Routh, 1877). That is, for the system to be stable a term in each power of s from s^{p+q} to s^0 must be present when the denominator of (20) is expanded into polynomial form. This in turn requires that the exponent u in the expansion of $f_1(s)\ f_2(s)$ be not less than p+q-1.

As was shown previously p+q will generally be equal to two in systems designed for zero steady state torque load effect. Thus, in this case, $f_1(s)\ f_2(s)$ would have to contain only a term in the first power of s. Therefore

$$Y_a(s)\ Y_m(s)\ Y_o(s) = K_1\ K_2 \frac{as + 1}{s\ g_1(s)\ g_2(s)} \quad . \quad . \quad . \quad (21)$$

where $f_1(s)\ f_2(s) = as + 1$ and $a \neq 0$.

It will be noted that the form of $g_1(s)\ g_2(s)$ is unimportant except that if any term in the expansion of $g_1(s)\ g_2(s)$ is missing, then $f_1(s)\ f_2(s)$ must contain a term of the form $f_{p+q+r}\ s^{p+q+r}$ where r is the power of s in the missing term of $g_1(s)\ g_2(s)$. This is however a trivial case since, in a practical network f(s) and g(s) are always made up of terms of the form $(a_1 s + 1)(a_2 s + 1) \ldots\ldots (a_n s + 1)$ where all the coefficients $a_1 \ldots a_n$ are real numbers.

Actually the mere existence of a term as + 1 in the numerator of (21) is a necessary rather than a sufficient condition for stability of the system. In addition certain requirements must be placed on the magnitude of the constant (a) in relation to the magnitude of the other constants in (21). This relationship may best be studied by plotting the transfer function $\frac{\Theta_o(s)}{E_e(s)}$ on the s plane and applying the Nyquist Stability Criterion (James, Nichols and Phillips, 1947, p. 70; Brown and Campbell, 1948, p. 170). In reference to this plot it will be noted that the term (as + 1) is needed to produce the positive phase shift in the high frequency region required in order that the plot of $\frac{\Theta_o(s)}{E_e(s)}$ will not encircle the point (-1 + j0).

The design and construction of an element having a transfer function of the form $Y(s) = \frac{as + 1}{s}$ presents a rather complex problem when this element must be included ahead of the servo motor. When an element has such a transfer function it has the properties of a true integrator since, if f(s) is the Laplace transform of F(t) then $\frac{1}{s}$ f(s) is the transform of $\int_0^t F(t)\,dt$ (Churchill, 1944, p. 39). In the case of the servo motor this integration is inherent in the action of the motor. That is, integration occurs when the shaft torque produced in the motor produces a rotation of the motor shaft. However, when it is desired to include such an element at a point in the system where, for example, both the input and output must be time variations of a voltage, a rather complex circuit is required (Brown and

Campbell, 1947, p. 206; Korn, 1948, p. 124).

In many cases it is possible to obtain a satisfactory servomechanism without completely eliminating the effect of the steady state load torque even though such a torque does act on the output member of the system. In particular, if the magnitude of the steady state component is not too great it may be possible to construct a servomechanism which will reduce the error resulting from the non varying torque load component to a value below the maximum allowable steady state error which may exist in the system. Whenever possible this type of system is used since it does not necessitate the inclusion of the additional integrator element.

An expression for the error resulting from the application of a steady state load torque to a servomechanism without the additional integrating element will now be derived.

In the system shown in Fig. 11 if $\theta_i(s) = 0$ then

$$\theta_o(s) = \frac{Y_o(s)}{1 + Y_a(s)\, Y_m(s)\, Y_o(s)}\, T_L(s) \quad . \quad . \quad . \quad . \qquad (22)$$

Setting $Y_a(s)\, Y_m(s) = K_1 \dfrac{f_1(s)}{g_1(s)}$, $Y_o(s) = K_2 \dfrac{f_2(s)}{s\, g_2(s)}$ and taking $T_L(t)$ to be a step function of magnitude T_L occurring at time $t = 0$, the following expression is obtained.

$$\theta_o(s) = \frac{K_2 \dfrac{f_2(s)}{s\, g_2(s)}}{1 + K_1 K_2 \dfrac{f_1(s)\, f_2(s)}{s\, g_1(s)\, g_2(s)}}\, \frac{|T_L|}{s} \quad . \quad . \quad . \qquad (22a)$$

If this expression is multiplied by s and the limit taken as s approaches zero, the limit of $\Theta_o(t)$ as t approaches infinity is obtained. Therefore, since $\lim_{s \to 0} f(s)$ and $\lim_{s \to 0} g(s)$ are unity,

$$\lim_{t \to \infty} \Theta_o(t) = \lim_{s \to 0} \left[\frac{K_2 \frac{1}{s}}{1 + K_1 K_2 \frac{1}{s}} \left| T_L \right| \right]$$

which, upon multiplying both numerator and denominator by s, becomes

$$\lim_{t \to \infty} \Theta_o(t) = \lim_{s \to 0} \left[\frac{K_2}{s + K_1 K_2} \left| T_L \right| \right] = \frac{\left| T_L \right|_{ss}}{K_1}$$

where $\left| T_L \right|_{ss}$ is the magnitude of the steady state component of the torque load. But for $\Theta_i(s) = 0$ the magnitude of the steady state error in the system, $E_{e\ ss}$ is equal to the steady state value of the output $\lim_{t \to \infty} \Theta_o(t)$. Therefore

$$E_{e\ ss} = \frac{\left| T_L \right|_{ss}}{K_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Thus, in accordance with (23), in a servomechanism which does not have a true integrating element ahead of the point of load torque application, the error resulting from the action of a steady state component of load torque on the output member of a servomechanism is inversely proportional to the gain of all the elements in the system which are represented as being ahead of the point of torque application on the transfer function diagram. In the case of the system with an internal feedback loop the result given in (23) is also obtained.

In the majority of servomechanisms the load is connected to the motor through a gear train. Since for a given magnitude of system error, the speed of rotation of the motor shaft is generally much greater than the desired load speed, a speed reduction gear train is generally included in the system between the servo motor and the load. A servomechanism utilizing such a gear train is generally represented by a transfer function diagram such as that shown in Fig. 14 where $Y_a(s)$ is the transfer function of all the elements in the system ahead of the servomotor, $Y_{mo}(s)$ is the combined transfer function of the motor and load and N is the speed reduction gear ratio. In this representation the moment of inertia, J, and the friction coefficient, f, of the load and the gear train are referred to the motor shaft where the corresponding terms for the motor, gear train and the load may be lumped into two constants J and f.

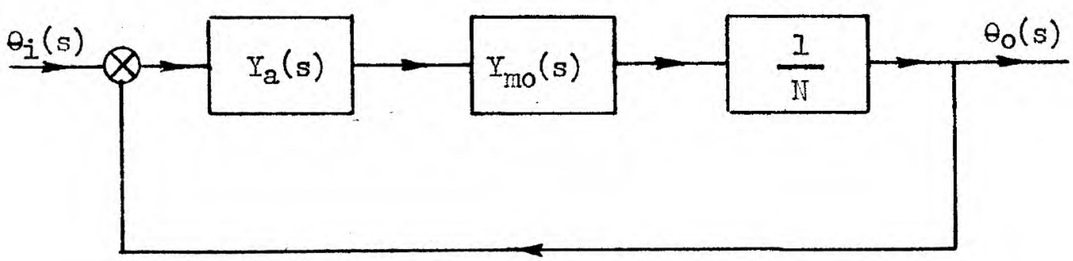


Fig. 14

If the effect of torque load disturbances is to be considered, $Y_{mo}(s)$ must be divided into two parts and, for the case of the DC motor with negligible armature inductance, may be represented as shown in Fig. 10. However, in this case the load torque acting on the motor shaft is $\frac{T_L(s)}{N}$ rather than $T_L(s)$ since the torque when referred to the motor shaft is reduced by a factor of N, the gear train ratio. If the motor with its associated gear train and load were included in a simple servomechanism which had only an amplifier of gain K_a and negligible time delay ahead of the motor, the overall servomechanism transfer function diagram would be as shown in Fig. 15.

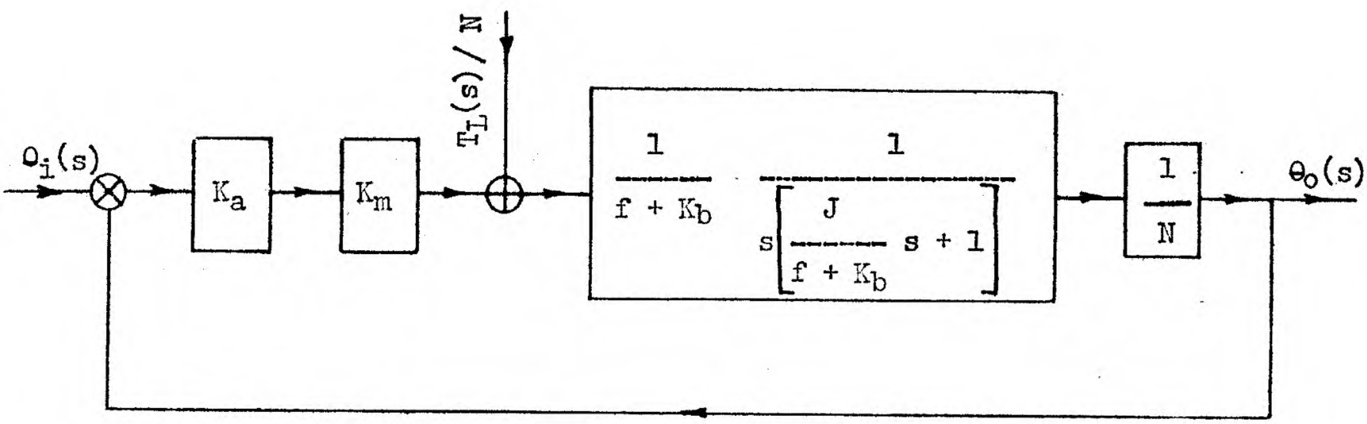


Fig. 15

From (23) the expression for the error due to the steady state component of load torque is obtained as

$$\left|E_e\right|_{ss} = \frac{\left|T_L\right|_{ss}}{K_a K_m N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

Referring to the diagram of Fig. 15 it is apparent that for a given servomechanism input to output response two conditions must be met regardless of the gear train ratio N. These are, first, that the ratio $\frac{J}{f + K_b}$ must be held constant and second, that the overall system gain $\frac{K_a\ K_m}{N\ (f + K_b)}$ must be held constant. Since J is the moment of inertia of the entire output member referred to the motor shaft, this term will vary approximately as the inverse square of the gear train ratio if it is assumed that the moment of inertia of the load when referred to the motor shaft is much greater than the corresponding terms associated with the gear train and the motor itself. Thus the term $(f + K_b)$ must also vary as the inverse square of N if the ratio is to be held constant. That is, if N_1 and N_2 are two different gear ratios being investigated for use in a particular case and f_1, K_{b_1}, f_2 and K_{b_2} are the values of f and K_b associated with the two cases then

$$\frac{f_1 + K_{b_1}}{f_2 + K_{b_2}} = \frac{N_2^2}{N_1^2}$$

or

$$f_1 + K_{b_1} = \frac{N_2^2}{N_1^2}\ (f_2 + K_{b_2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Also, since the overall gain of the system is to be held constant, the gain for the two cases must be equal. Thus

$$\frac{K_{a_1}\ K_{m_1}}{N_1\ (f_1 + K_{b_1})} = \frac{K_{a2}\ K_{m2}}{N_2\ (f_2 + K_{b_2})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

After substituting (25) for $(f_1 + K_{b_1})$ in (26) and rearranging terms

$$K_{a1}\ K_{m1} = \frac{N_2}{N_1}\ K_{a2}\ K_{m2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From (24) it is seen that, for a given value of steady state torque load, the ratio of the steady state errors for the two cases is

$$\frac{|E_e|_{ss1}}{|E_e|_{ss2}} = \frac{K_{a2}\, K_{m2}\, N_2}{K_{a1}\, K_{m1}\, N_1} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

If (27) is substituted for $K_{a1}\, K_{m1}\, N_1$ in this expression the final expression for the ratio of the steady state error is obtained as

$$\frac{|E_e|_{ss1}}{|E_e|_{ss2}} = \frac{K_{a2}\, K_{m2}\, N_2}{\frac{N_2}{N_1}\, K_{a2}\, K_{m2}\, N_1} = 1 \quad . \; . \; . \; . \; . \; . \quad (29)$$

Thus in accordance with (29) the error due to a steady state load torque is, to a first approximation, independent of the gear ratio for a given servomechanism response characteristic.

In a practical case of this type it will be noted that, as the gear ratio N is raised the decrease in the value of J would probably be somewhat less than predicted by the inverse square law assumed in this analysis. This difference would be due to the increased inertia associated with the higher ratio gear train. However, the higher ratio gear train would also require a higher speed motor. For a given motor power rating, the moment of inertia of the motor armature is found to vary approximately as the inverse cube of the speed for motors with power ratings above about one horse power and at a slower rate for motors in the fractional horsepower class. Thus it is seen that if the inertia of the load as referred to the motor shaft were not much greater than the inertia of the gear train and motor itself the assumption of an inverse square law variation in J might be considerably in error. It, therefore, becomes apparent that, in the practical case, the effect of each of the components making up the term J must be carefully investigated in order to determine the actual variation in J for a change of the gear train ratio. No further analysis of this particular problem will be made here. However, in view of (29), it is clear that the steady state error resulting from a steady state component of load torque is at least not directly affected by changes in the gear ratio.

THE EFFECT OF THE COMPONENT OF THE TORQUE LOAD DISTURBANCE WHICH VARIES AS A RANDOM FUNCTION IN TIME

Thus far the effect of the torque load disturbance has been analyzed only with reference to the steady state component of torque. In practice a torque load of this type is seldom encountered and the load torque will in general be composed of one or more periodic components and a randomly varying component as well as the steady state component. Such a torque load disturbance plotted as a function of time might be as shown in Fig. 13.

For the present it will be assumed that it is possible to represent the torque load disturbance by some general function in time $T_L(t)$, and further, that by means of the Fourier integral, $T_L(t)$ may be expressed as a combination of periodic functions of frequency $\frac{\omega}{2\pi}$. When expressed in this form the general torque disturbance will be written $T_L(j\omega)$ where

$$T_L(j\omega) = \int_{-\infty}^{\infty} T_L(t)\, e^{-j\omega t} dt.$$

Actually it is immediately apparent that this integral does not converge unless $T_L(t)$ approaches 0 as t approaches infinity.

In general such will not be the case. However, for purposes of first analysis it will be assumed that the integral is taken over a finite range so large that, for all practical purposes, this integral does give a satisfactory approximation to the true spectrum. This particular problem will be treated at length in a later section of this discussion.

The function of $T_L(j\omega)$ is said to be the Fourier transform of $T_L(t)$ and could be described as the frequency spectrum of the torque load disturbance in that it defines $T_L(t)$ particular complex function of ω. The amplitude of the fun any value of $\omega = \omega_1$ is the magnitude of the component of the frequency spectrum having a frequency $\frac{\omega 1}{2\pi}$ and the phase angle associated with each component is given by the arc tangent of the ratio of the imaginary to the real component of $T_L(j\omega_1)$.

It can be shown that, if the variable s in the transfer function of any sustem is replaced by the variable $j\omega$, the resulting expression is the frequency response function of the system (Brown and Campbell, 1948, p. 96). In particular, if $Y_p(s)$ is the transfer function of a system, then $Y_p(j\omega)$ is the frequency response function of the system. It will be noted that $Y_p(j\omega)$ is really the ratio of the steady state output to input of the system when the impedances of the circuit are expressed in the conventional form used in standard alternating current circuit theory. Thus if a sinusoidal input of amplitude E_q and frequency f_q were applied to the system having a transfer function $Y_p(s)$, the steady state output of the system would be $E_q\ Y_p(j\omega_q)$

where $\omega_q = 2\pi f_q$. In fact it can be shown that the general expression for the spectrum or "Fourier transform" of the output is given by

$$E_o(j\omega) = Y_p(j\omega)\, E_i(j\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $E_i(j\omega)$ and $E_o(j\omega)$ are the Fourier transforms of the input and output functions $E_i(t)$ and $E_o(t)$, respectively (James, Nichols and Phillips, 1947, p. 48). Of course this expression is seen to be valid only when $Y_p(s)$ defines a stable system.

In the study of any particular servomechanism problem it is convenient to deal almost exclusively with the frequency response functions of the system rather than with the transfer functions. Generally the relationship between the various system parameters and the input to output response of the system can best be studied by a graphical manipulation of the plots of the frequency response functions of the various system components when the logarithm of the amplitude (or its equivalent in decibels) and the phase of the function are plotted on a linear vertical scale and ω is plotted on a logarithmic horizontal scale (Brown and Campbell, 1948, p. 236; James, Nichols and Phillips, 1947, p. 169). If the product of two functions is to be taken the result is given by the graphical sum of the decibel amplitude plots and the phase plots for the two functions. Thus, referring to (30), if the decibel magnitude and phase of the function $E_i(j\omega)$ were plotted versus log ω and these plots were added to the corresponding plots for the function $Y_p(j\omega)$ in (30) then the resultant would be the decibel amplitude and phase plots of the output function $E_o(j\omega)$.

It was previously shown that

$$\frac{\Theta_o(s)}{T_L(s)} = \left[\frac{\Theta_o(s)}{\Theta_i(s)}\right] \frac{1}{Y_a(s)\,Y_m(s)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

where $Y_a(s)\,Y_m(s)$ is the transfer function of all the elements ahead of the point of application of the load torque disturbance as shown on the transfer function diagram. Thus, from (13 and (30)

$$\Theta_o(j\omega) = \left[\left[\frac{\Theta_o(j\omega)}{\Theta_i(j\omega)}\right] \frac{1}{Y_a(j\omega)\,Y_m(j\omega)}\right] T_L\,(j\omega) \quad . \qquad (31)$$

In any servomechanism design study the input to output response function $\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)}$ is obtained by a graphical manipulation of the type described above. This function is the overall input to output response function of the servomechanism. Since $Y_a(s)\,Y_m(s)$ is of the general form $K_1 \dfrac{f_1(s)}{s^p\,g_1(s)}$, (31) may be written in the form

$$\Theta_o(j\omega) = \left[\left[\frac{\Theta_o(j\omega)}{\Theta_i(j\omega)}\right] \frac{(j\omega)^p\,g_1(j\omega)}{K_1\,f_1(j\omega)}\right] T_L(j\omega) \quad . \quad . \qquad (31a)$$

From (31a) it is apparent that the magnitude of the output response $\Theta_o(j\omega)$ is inversely proportional to K_1 for any particular set of functions $\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)}$, $T_L(j\omega)$, and $\dfrac{f_1(j\omega)}{(j\omega)^p\,g_1(j\omega)}$. It will be noted that this variation with respect to K_1 is the same as was derived for the case of the steady state component of torque when the value of the exponent p in (31a) was taken as zero. This is as would be expected since, if $p = 0$, and, in the case of the steady state component of torque $\omega = 0$, then

$$\frac{f_1(j\omega)}{(j\omega)^p\,g_1(j\omega)} = 1$$

Also, in view of condition (b) of the introduction, for $\omega = 0$

$$\left| \frac{\theta_o(j\omega)}{\theta_i(j\omega)} \right| = 1$$

Therefore, for $\omega = 0$

$$\theta_o(j\omega) = \frac{T_L(j\omega)}{K_1}$$

Since $\omega = 0$ this expression is equivalent to the expression given in (23). Actually the value of $T_L(j\omega)$ for $\omega = 0$ is implicitly expressed in the general function $T_L(j\omega)$. Thus it is seen that the result given in (23) is really a special case of the more general expression (31).

As was mentioned previously the representation of the torque load disturbance $T_L(t)$ by the function $T_L(j\omega)$ will not always be possible since, if $T_L(j\omega)$ is defined by the integral

$$T_L(j\omega) = \int_{-\infty}^{\infty} T_L(t)\, e^{-j\omega t} dt \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

$T_L(t)$ must vanish as t approaches infinity in order that the integral converge. It is apparent that, in practice, the function $T_L(t)$ is generally of the type which would not vanish as t approaches infinity. In this case it has been found convenient to treat the function $T_L(t)$ as a power distribution function in ω rather than as an amplitude function and phase function as defined by the Fourier integral. This power distribution function, here written as $T_L(\omega)_{sd}$, is generally referred to as the spectral density of the function $T_L(t)$.

This function may be defined in the following manner. If a voltage E(t) is applied to a unit resistance, and the spectral density function associated with the voltage $E(t)$ is $E(\omega)_{sd}$, then the average power dissipated in the resistance over the angular frequency interval ω to $\omega + d\omega$ is $E(\omega)_{sd}d\omega$. On the basis of this definition it may be shown that

$$E(\omega)_{sd} = \lim_{t \to \infty} \left[\frac{\left|E(j\omega)\right|^2}{2\pi t} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $E(j\omega)$ is the Fourier transform of the function $E(t)$.
A detailed mathematical analysis of this function as related to the study of the response of a servomechanism to a general input is given in the following reference (James, Nichols and Phillips, 1947, Chapters 6, 7, and 8). The method of calculating the spectral density function for any particular set of data is also given and for purposes of this discussion it will be assumed that the form of the function has already been determined for the particular problem to be considered. In particular this discussion will be concerned with the methods of minimizing the effect of a general torque load disturbance after having previously determined the spectral density function for the torque load disturbance.

There are however certain important features of the spectral density function as related to the study of torque load disturbances which should be noted. Referring to (33) it is immediately apparent that the spectral density function $E(\omega)_{sd}$ is only a magnitude function and is independent of the phase characteristics of the function $E(j\omega)$. This would of course be expected since the spectral density function was defined as a power relationship.

A second feature may also be noted by referring to the original definition given for the function $E(\omega)_{sd}$. For purposes of illustration it will be assumed that the entire spectrum of $E(t)$ is made up of only one sinusoidal component of angular frequency ω_1. That is $E(t) = E_1 \sin \omega_1 t$, where E_1 is a constant. In this case the average power dissipated in the unit resistance would be a constant independent of ω. But, by definition, the average power dissipated over the interval ω to $\omega + d\omega$ is $E(\omega)_{sd}\, d\omega$. Thus the total average power dissipated over all values of ω must be

$$\overline{P} = \int_0^\infty E(\omega)_{sd} d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

However in the particular case being considered $E(\omega)_{sd}$ is zero for all values of ω except $\omega = \omega_1$. Thus, in order that $\overline{P}$ be a constant greater than zero, $E(\omega)_{sd}$ must be of the form

$$E(\omega)_{sd} = \infty \text{ for } \omega = \omega_1$$

$$E(\omega)_{sd} = 0 \quad \text{for } \omega \neq \omega_1$$

In particular, in accordance with (34) $E(\omega)_{sd}$ must be an impulse function occurring at $\omega = \omega_1$ and having a strength of $\overline{P}$ (Goldman, 1949, p. 101).

Thus it is seen that any hidden periodicities in the function $E(t)$ will exhibit themselves as impulse functions in the spectral density function $E(\omega)_{sd}$. It is of interest to note that, in the case of a torque load disturbance having a steady state component, the spectral density function of the torque load would exhibit an impulse function at $\omega = 0$.

Referring again to the spectral density function as defined by (33), it can be shown that, if $Y(s)$ is the transfer function of a particular system, the output spectral density function

$E_o(\omega)_{sd}$ which results from an input having a spectral density $E_i(\omega)_{sd}$ is given by

$$E_o(\omega)_{sd} = \left|Y(j\omega)\right|^2 E_i(\omega)_{sd} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

where $Y(j\omega)$ is the frequency response function of the system (James, Nichols and Phillips, 1947, p. 289). This relationship affords a rather simple means of evaluating the spectral density of the output which results from the application of a torque load disturbance having a given spectral density function.

With reference to (31) it is seen that for a given torque load spectral density $T_L(\omega)_{sd}$ the output spectral density is given by

$$\Theta_o(\omega)_{sd} = \left|\left[\frac{\Theta_o(j\omega)}{\Theta_i(j\omega)}\right]\frac{1}{Y_a(j\omega)\,Y_m(j\omega)}\right|^2 T_L(\omega)_{sd} \quad . \qquad (36)$$

As was mentioned previously, in a servomechanism design study the function $\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)}$ is generally obtained in a graphical form as a plot of decibel magnitude and phase versus log ω. If the decibel magnitude plot of $Y_a(j\omega)\,Y_m(j\omega)$, i.e. the transfer function of all the elements ahead of the point of load torque application, were subtracted from the plot of $\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)}$ and the resultant decibel magnitude plot were multiplied by a factor of two, a decibel magnitude plot of the squared term in (36) would be obtained. This plot could then be added graphically to a plot of the decibel magnitude of $T_L(\omega)_{sd}$ versus log ω to obtain the decibel magnitude versus log ω plot of $\Theta_o(\omega)_{sd}$. In an analysis of this type the impulse functions in $T_L(\omega)_{sd}$ can be ignored since the effect of the components of $T_L(t)$ which appear as the impulse functions can be analyzed separately by methods previously described. If $T_L(t)$ contains some periodic component of angular frequency ω_1 the response of the system to this component may be obtained by either substituting the value of ω_1 in (31) or by a study of the graphical relationships representing (31).

In obtaining the decibel magnitude versus ω plot as given by (36) only the relative magnitude of the function above or below some reference level need be considered in the actual plotting. It is often convenient to select as a reference level some particular value of $T_L(\omega)_{sd}$, for example, the average value of the function. For purposes of this discussion the function will be plotted with reference to an arbitrarily selected value of $T_L(\omega)_{sd}$ in which case the function plotted will be

$$20 \log \frac{T_L(\omega)_{sd}}{T_L(\omega_1)_{sd}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where $T_L(\omega_1)_{sd}$ is the value of the spectral density function selected as the reference. In any case it is obvious that all plots of $T_L(\omega)_{sd}$ regardless of the reference level will have the same shape so that the actual magnitude of the output spectral density can be obtained by considering the particular zero decibel reference level used. Henceforth in this discussion it will be assumed that, unless otherwise specified, the plot given as the decibel magnitude plot of $T_L(\omega)_{sd}$ will actually be the plot of the function (37).

Referring to (31a) it was shown that the magnitude of the output resulting from the torque load disturbance was inversely proportional to K_1, the gain of all the elements located ahead of the point of load torque application on the transfer function diagram. Thus from (36) it is seen that the output spectral density function magnitude is inversely proportional to the square of this constant. The methods of raising the value of this constant, i.e., the gain, in a particular servomechanism have been discussed in detail in the various publications dealing with the servomechanism design problem. For this reason the remainder of this discussion will deal mainly with the frequency variant part of the output spectral density function. The gain of the system, or more particularly, the value of the constant term in $Y_a(j\omega)\,Y_m(j\omega)$ will be considered only in relation to the magnitude and form of the output spectral density function plot.

With reference to (34) the total average power dissipated in the unit resistance is given by $\overline{P}$ and in the case of the output resulting from a torque load disturbance

$$\overline{P}_o = \int_0^{\infty} \Theta_o(\omega)_{sd}\, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where $\Theta_o(\omega)_{sd}$ is as defined in (36). Since $\overline{P}_o$ is also the square of the r.m.s. value of the output function $\Theta_o(t)$, it is of interest to determine the function $\Theta_o(\omega)_{sd}$ which will result in the minimum value of $\overline{P}_o$ and thus the minimum value of Θ_o(r.m.s.). From (38) it is apparent that $\overline{P}_o$ is the area under the curve $\Theta_o(\omega)_{sd}$ versus ω, and from (36) it is seen that the problem is that of selecting the proper function $Y_a(j\omega)$ which will result in the smallest average amplitude for the function $\Theta_o(\omega)_{sd}$ for a given response function $Y_m(j\omega)\ Y_o(j\omega)$ and a given servomechanism input to output response function $\Theta_o(j\omega)/\Theta_i(j\omega)$. Since the input to output response function is also a function of $Y_a(j\omega)\ Y_m(j\omega)$ the selection of this function will of course be a matter of compromise between that which would produce the minimum output for the particular torque load disturbance and that which would produce the optimum servomechanism response characteristic as determined by the requirements of the particular control problem being studied. From (36) it is also seen that the higher the value of the gain in the function $Y_a(j\omega)\ Y_m(j\omega)$ the smaller will be the overall amplitude of the output function $\Theta_o(\omega)_{sd}$. However, the value of this gain constant is determined by the stability requirements of the servomechanism and, for any particular frequency variant component of the function $Y_a(j\omega)\ Y_m(j\omega)\ Y_o(j\omega)$ only a certain maximum value of gain may be used.

As was mentioned previously, the actual design study for a particular servomechanism would be carried out with the aid of decibel magnitude plots and thus, if the torque load disturbance is handled graphically by the method outlined previously, the plot of $\Theta_o(\omega)_{sd}$ obtained is a plot of the decibel amplitude of the output function. If two plots of $\Theta_o(\omega)_{sd}$ are to be compared with a view toward determining which of the two results in the smaller value of $\overline{P}_o$ the comparison may be accomplished by plotting the two on the same sheet. Since it is generally convenient to neglect the constant terms in the various functions when making such a plot it would of course be necessary to shift one of the plots vertically by an amount equal to twice the difference in the decibel gains of $Y_a(j\omega)$ $Y_m(j\omega)$ for the two cases before comparing the two plots obtained. It will, however, be noted that a comparison of two curves of $\Theta_o(\omega)_{sd}$ when plotted on a decibel versus log ω coordinate system can lead to a somewhat erronious conclusion. In general a much clearer picture of the situation is obtained by replotting the functions to a linear scale of amplitude and ω before a comparison of the two curves is made.

In order to illustrate the effect of the general torque load disturbance and the methods of reducing such an effect, a particular torque load spectral density function will be considered with reference to a particular servomechanism problem. It will be assumed that the major portion of the power associated with the torque load disturbance exists in the lower range of the servomechanism frequency response range. The impulse function components of the torque load spectral density, which result from a steady state periodic components in the torque load disturbance, will be neglected since, as was mentioned previously, the response resulting from these components may be evaluated separately.

A servomechanism with a transfer function diagram as shown in Fig. 16 will be taken as the basis for the study. Specific values have been assumed for the various constants in order to facilitate the plotting of the various functions. In all cases the amplitude plots will be in terms of decibel amplitude versus log ω. The phase angle versus log ω plots will be given only when necessary to illustrate certain points of importance in the consideration of the torque load disturbance, e.g., where the evaluation of the term $\frac{\theta_o(j\omega)}{\theta_i(j\omega)}$ in (36) requires a previous determination of certain phase functions.

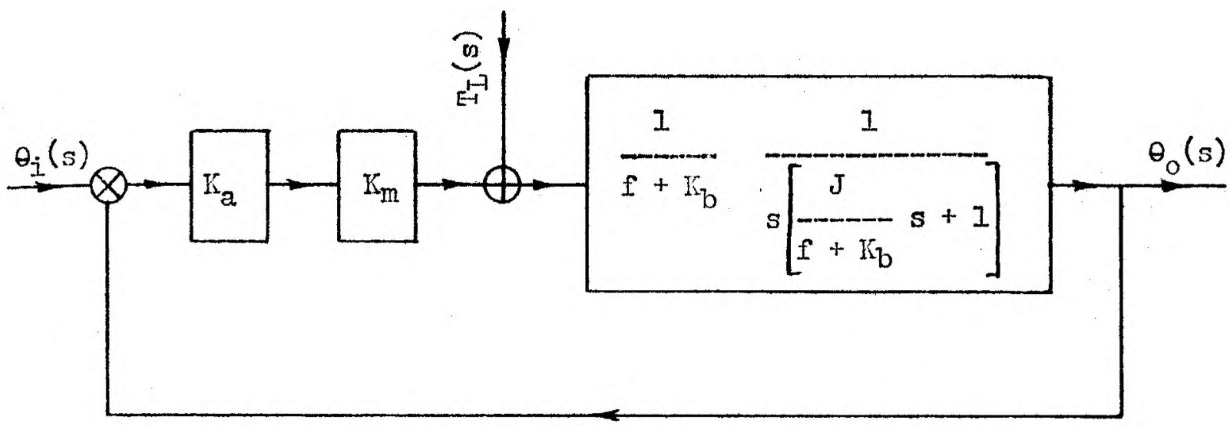


Fig. 16

With reference to Fig. 16 the following constants will be assumed.

$$\frac{J}{f+K_b} = 1 \text{ sec.} \qquad K_m = 1 \text{ in.lb./volt}$$

$$f + K_b = 1 \ \frac{\text{in.lb.}}{\text{rad/sec}}$$

If this system is analyzed by standard methods described in previously cited references it is found that an overall gain of 4 db will result in an input to output amplitude response curve, shown as (a) in Fig. 17. Since the overall gain $\frac{K_a K_m}{f + K_b} = 4$ db and $f + K_b = 1$ and $K_m = 1$, K_a is found to be 4 db. The curve of

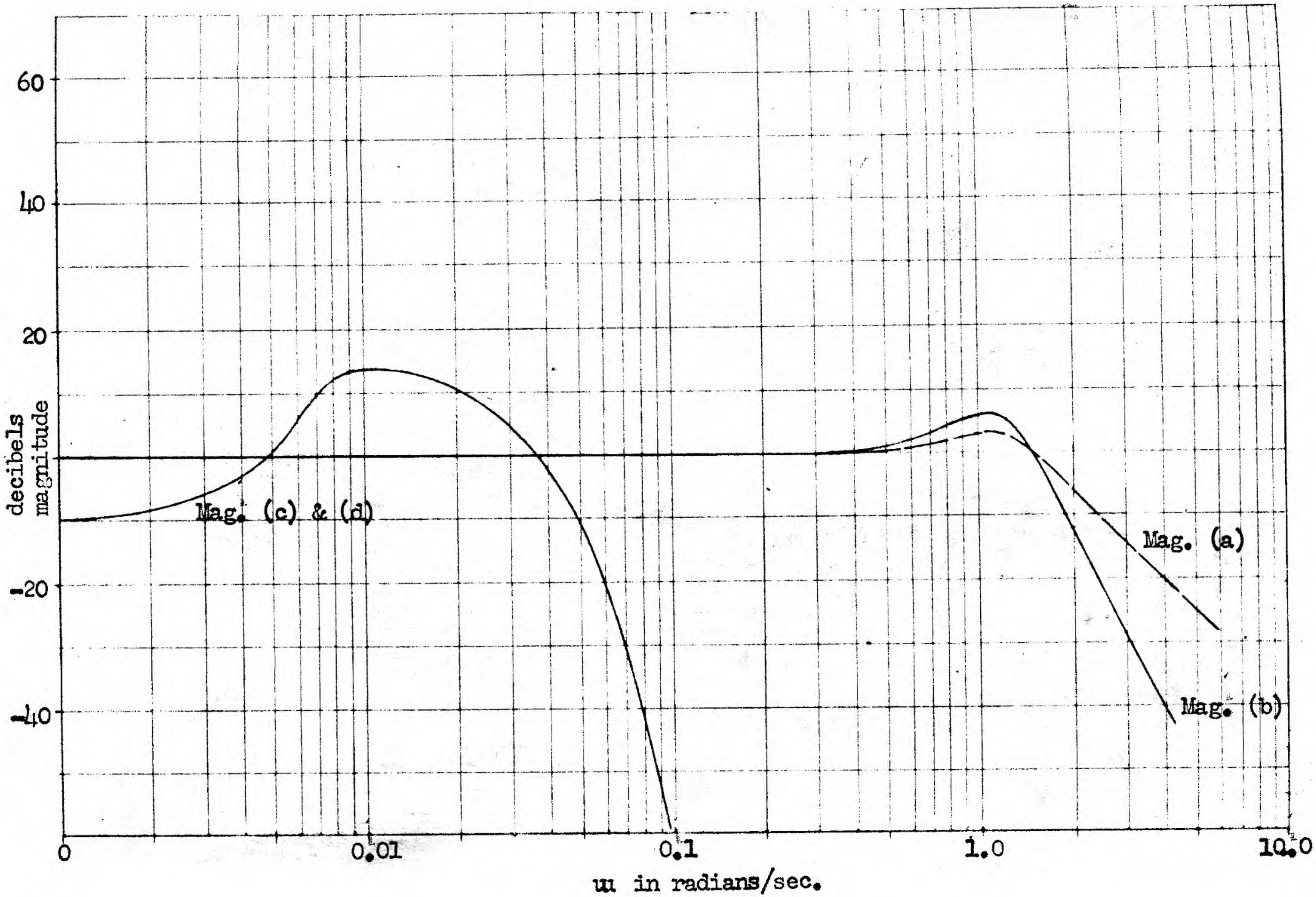


Fig. 17 Graphical analysis used to determine $\theta_o(\omega)_{sd}$ for a simple servomechanism.

$$\left|\frac{\theta_o(j\omega)}{\theta_i(j\omega)}\right|^2$$ is shown as (b) in Fig. 17. It will be noted that this function has a value of unity over the range where the magnitude of the function is appreciable. Since $Y_a(j\omega)\,Y_m(j\omega)$ is in this case the constant term K_aK_m = 4 db the overall plot of

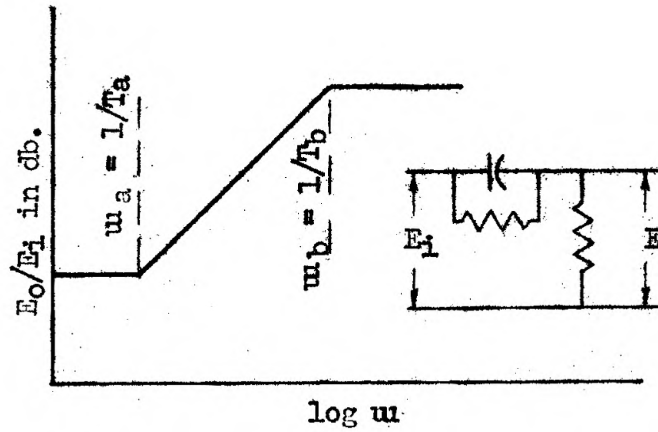
$$\left|\left[\frac{\theta_o(j\omega)}{\theta_i(j\omega)}\right]\frac{1}{K_a\,K_m}\right|^2$$ is also as shown in (b) of Fig. 17

if the constant factor of -8 db due to the term $\frac{1}{(K_aK_m)^2}$ is neglected in the plotting.

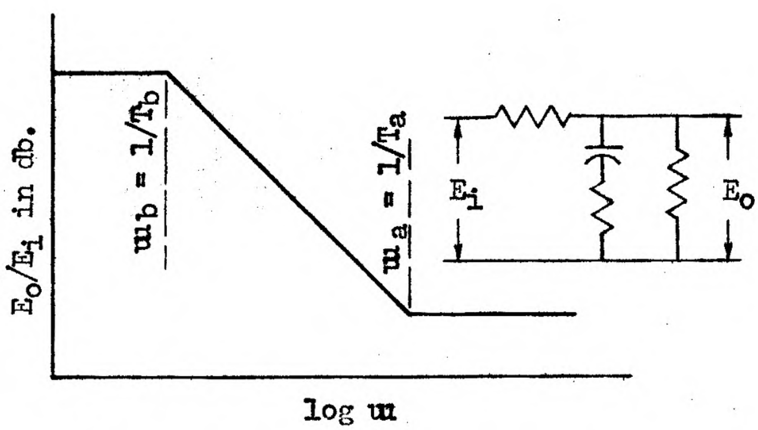
It will be assumed that the spectral density function $T_L(\omega)_{sd}$ for the case to be considered is as shown in curve (c) of Fig. 17 where the function $T_L(\omega)_{sd}$ is plotted as decibels magnitude, with respect to some convenient zero decibel level $T_L(\omega_o)_{sd}$, versus log ω. It is immediately apparent that the plot of the function $\theta_o(\omega)_{sd}$ is identical, at least within the limits of this plot, to the plot of the torque load spectral density function $T_L(\omega)_{sd}$ since within this region the function $\frac{\theta_o(j\omega)}{\theta_i(j\omega)}$ is of unit magnitude. The function $\theta_o(\omega)_{sd}$ is shown as curve (d) of Fig. 17. This function when plotted on a linear coordinate system is as shown in (a) of Fig. 20.

Two possible methods of reducing the value of $\theta_o(t)$ r.m.s. present themselves. First in accordance with (36) it is seen that the magnitude of the function $\theta_o(\omega)_{sd}$ is inversely proportional to the square of the gain of the elements located ahead of the servomotor. Thus a change of the system which aims toward an increase in gain should also result in a decrease in the effect of the torque load disturbance. Second, referring to curves (c) and (d) of Fig. 17 and (a) of Fig. 20 it is also seen that if the peak in the curve could be reduced down to about the value of the function at $\omega = 0$ a considerable reduction in the area under the curve and thus the value of $\theta_o(t)$ r.m.s. would result even though the gain of the system were not changed. Both methods of reducing $\theta_o(t)$ r.m.s. will now be considered with reference to the servomechanism as shown in Fig. 16 and $T_L(\omega)_{sd}$ as shown in (c) of Fig. 17.

The effect of different methods of raising the gain of the system will first be considered. If a network having a transfer function $Y(j\omega) = K \dfrac{T_a\, j\omega + 1}{T_b\, j\omega + 1}$ is inserted in the error channel of the servomechanism and the time constants T_a and T_b of the network are properly selected the maximum allowable gain of the system may be raised considerably (Brown and Campbell, 1948, p. 265). In the case where T_a is greater than T_b the network is referred to as the proportional plus derivative network and when T_a is less than T_b the network is referred to as the proportional plus integral network (James, Nichols and Phillips, 1947, p. 114). The circuit diagrams and the asymptotic plots of the decibel amplitude versus log ω response characteristics associated with these two networks is shown in Fig. 18. In the first case to be considered the effect of using the proportional plus derivative network will be investigated. In connection with the system under consideration if $Y_a(j\omega) = K \dfrac{T_1 j\omega + 1}{T_2 j\omega + 1}$



Proportional plus derivative network



Proportional plus integral network

Fig. 18 Circuit diagrams and asymptotic response characteristics for compensation networks

is the response function of the network where $\omega_1 = \frac{1}{T_1} = 1$ and $\omega_2 = \frac{1}{T_2} = 4$, it is found that a total gain of 16 db will result in a $\frac{\theta_o(j\omega)}{\theta_i(j\omega)}$ response similar in shape to that shown as (a) of Fig. 17 except that the value of ω at the peak of the response function was shifted from $\omega = 1.1$ as shown in (a) to a value of $\omega = 5.0$. It will be noted that, since for this network $\omega_1 = 1.0$, the network would have no appreciable effect on the shape of the plots of $Y_a(j\omega)\, Y_o(j\omega)$ in the range below $\omega = 0.1$.

Thus, over the frequency range where the magnitude of $T_L(\omega)_{sd}$ is appreciable, the function $\left| \left[\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)} \right] \dfrac{1}{Y_a(j\omega)\, Y_m(j\omega)} \right|^2$ is merely a constant equal to twice -16 or -12 decibels. The plot shown as (b) of Fig. 20 is the linear plot of $\Theta_o(\omega)_{sd}$ for this condition. This plot was obtained by reducing the entire function (d) of Fig. 17 by a factor of 32 decibels and then computing the magnitude of the function for each value of ω. A comparison of the area under this curve with that under (a) of Fig. 20 clearly shows the large decrease in $\Theta_o(t)$r.m.s. that has been obtained. It will be noted that (b) is (a) reduced by a factor of 24 db which is twice the difference in the gain for the two cases.

Sometimes the previous method of raising the system gain is not satisfactory in view of the fact that an increase in the response range of the servomechanism always accompanies the increase in gain. In this case the proportional plus integral network is sometimes used since it will result in an increase in the allowable gain without extending the servomechanism response range (Brown and Campbell, 1948, p. 265; James, Nichols and Phillips, 1947, p. 114). The response function for such a network is

$$Y_a(j\omega) = K \frac{T_2 j\omega + 1}{T_1 j\omega + 1} \quad \text{where } T_1 = \frac{1}{\omega_1} \text{ is greater than } T_2 = \frac{1}{\omega_2}.$$

The asymptotic plot of this function is shown in Fig. 18.

If the magnitude and argument of this function are plotted for the arbitrarily selected case where $\omega_2 = 8\omega_1$, it will be noted that an appreciable negative phase shift (about 6.5 degrees) occurs even at frequencies as great as $8\omega_2$ and that the phase shift as $\omega = 4\omega_2$ is considerable.

If the response range of the compensated system is to be held equal to that for the uncompensated system it is necessary that the network not introduce any additional negative phase shift in the overall response function $Y_a(j\omega)\,Y_o(j\omega)$ in the region near the peak of the function $\dfrac{\Theta_o(j\omega)}{\Theta_i(j\omega)}$ as shown in curve (a) of Fig. 17. Thus for $\omega_2 = 8\omega_1$, ω_2 must be made less than $\omega_2 = 0.1$. From (36) it is seen that, for any given value of ω, $\Theta_o(\omega)_{sd}$ varies inversely as the square of the magnitude of $Y_a(j\omega)\,Y_m(j\omega)$. Therefore, with reference to the function $T_L(\omega)_{sd}$ shown as (c) of Fig. 17, it is seen that the rise in $\dfrac{1}{Y_a(j\omega)}$ between ω_1 and ω_2 will tend to accentuate the peak in $\Theta_o(\omega)_{sd}$ at the same time the increase in gain tends to reduce the value of $\Theta_o(\omega)_{sd}$.

In particular, if $\omega_2 = 0.08$ and $\omega_1 = 0.01$ it is found that a gain of 22 decibels will result in a response function almost identical to that shown as (a) of Fig. 17.

The function $\Theta_o(\omega)_{sd}$ which is obtained for this particular set of conditions is plotted as curve (c) of Fig. 20. It will be noted that the area under curve (c) is slightly greater than that under curve (b). Thus even though the gain of the particular integral compensated system here analyzed is 6 decibels greater than the gain for the derivative compensated system, the value of Θ_o(r.m.s.) for the integral system is still slightly greater than that obtained for the derivative compensated system. This reduction in the effectiveness of the increased gain is due solely to the accentuation of the peak in $\Theta_o(\omega)_{sd}$ by the response function of the network used. To cite an extreme example of this situation curve (c) of Fig. 20 is a plot of the function $\Theta_o(\omega)_{sd}$ for the case where ω_1 and ω_2 in the integral network were taken as $\omega_1 = .0005$ and $\omega_2 = .004$ where again the total system gain was 22 decibels.

With reference to the problem of reduction of the effect of a general torque load disturbance by increasing the gain of the system a partial criterion for the selection of the method of compensation may be formulated in view of the previous illustrations. Certainly for a given value of gain the greatest reduction in $\Theta_o(t)$r.m.s. will be obtained with a compensation network which neither accentuates an existing peak nor produces any peak in the function $\Theta_o(\omega)_{sd}$. Since a proportional plus derivative network has a response characteristic which rises over the range ω_1 to ω_2 the use of such a network can never have the effect of accentuating or producing a peak. This may be seen from an examination of (36) for, if $Y_a(j\omega)$ exhibits a rising response characteristic over the range ω_1 to ω_2, $\left|\dfrac{1}{Y_a(j\omega)}\right|^2$ will decrease over this range. Therefore in this case the entire function $\Theta_o(\omega)_{sd}$ will be reduced by at least twice the increase of the system gain in decibels.

Also for a torque load disturbance peak in the upper range of the servomechanism response range the peak would be attenuated by the use of such a network which would result in a further decrease in $\theta_o(t)$r.m.s. However since the proportional plus integral network has a decreasing response characteristic over the range ω_1 to ω_2 the function $\left|\frac{1}{Y_a(j\omega)}\right|^2$ will rise over this range. Thus, such a network should certainly not be used for values of ω_2 less than the value of ω at the peak in $T_L(\omega)_{sd}$. In the limit, if ω_2 is much less than the value of ω at the peak, the increase in gain will be exactly nullified by the variation of the function $\frac{1}{Y_a(j\omega)}$ with frequency. For the case where the peak in $T_L(\omega)_{sd}$ occurs within the range ω_1 to ω_2 (as in the case for curve (b) of Fig. 20) the degree of improvement will vary with the particular function $T_L(\omega)_{sd}$ being considered.

With reference to (13) and (14a) it is seen that, for a system which utilizes an auxiliary feedback loop, the output due to a torque load disturbance for a given $\frac{\theta_o(j\omega)}{\theta_i(j\omega)}$ response function is independent of the characteristics of the feedback path itself. It is often possible to increase the gain of a system by the use of an auxiliary feedback loop without introducing any network having a special response function directly into the error channel of the system itself. This method may often be used in place of the proportional plus integral compensation method where the latter is found to result in a peak in the torque load output function which either nullifies or severely reduces the effectiveness of the intended reduction in the value of the r.m.s. output.

As was mentioned previously it is also possible to achieve a reduction in the effect of the torque load disturbance having a peaked spectral density function by attenuating the peak alone while keeping the gain of the system constant. It should, however, be realized that such a method would seldom if ever be used as such since an increase in the gain of the servomechanism is almost never undesirable. Also the network that is required to achieve this peak reduction is generally more complex than that required to achieve an increase in gain which would result in the same reduction in $\theta_o(t)$ r.m.s. This method will however be briefly described since the particular manipulations involved clearly illustrate the effect of variations of the function $Y_a(j\omega)$ alone on the output function $\theta_o(\omega)_{sd}$. Actually it is possible that for an extreme torque load disturbance condition it might be desirable to use this method in conjunction with a suitable gain increase as a means of procuring a satisfactorily small value of $\theta_o(t)$ r.m.s.

The graphical manipulations involved in this particular study are shown on Fig. 19 where curve (a) is the previously shown plot of $T_L(\omega)_{sd}$ and curves (b) are the plots of magnitude and argument of $Y_o(j\omega)$ where this response function is as defined for the other illustrations. It is clearly impossible to reduce the peak for the particular torque load disturbance function being considered by the use of the simple proportional plus derivative network since, in order to reduce the peak, the constant ω_2 of the network would have to be less than the value of ω at the peak and a network of this type would lead to a considerable decrease in the value of the overall available system gain consistent with the stability requirements of the system.

It is, however, possible to use a combination of both the proportional plus derivative and the proportional plus integral networks in such a way that the resulting response function $Y_a(j\omega)$ $Y_o(j\omega)$ is the same as $Y_o(j\omega)$ alone in the critical region near $\omega = 1$. In this particular case the constants for the derivative network are taken as $\omega_{1d} = 0.001$ and $\omega_{2d} = 0.008$ while those for the integral network are taken as $\omega_{1i} = 0.008$ and $\omega_{2i} = 0.064$. Since in each case the intervals ω_1 to ω_2 include the same number of octaves, the net effect of this combination of networks is negligible for values of ω considerably greater than $\omega = 0.064$. Thus it has no effect on the overall error channel response function for values of ω near unity. The transfer function of this particular network is plotted as curves (c) of Fig. 19 where the dashed line is the asymptotic plot of the magnitude function. It will be noted that the argument plot has been reduced by 90 degrees to simplify the plotting procedure. The curves (d) are the magnitude and argument plots for the function $Y_a(j\omega)$ $Y_o(j\omega)$ and the magnitude of the system response function $\dfrac{\theta_o(j\omega)}{\theta_i(j\omega)}$ for a gain of four decibels is shown as (e). It is seen that (e) of Fig. 19 is almost identical to (a) of Fig. 17 and that in each case a gain of four decibels was used.

If in accordance with (36), twice the value of the magnitude plot (c) is subtracted from (a), the torque load spectral density function, a plot of $\theta_o(\omega)_{sd}$ will result. This function, neglecting the constant factor of eight decibels, is shown as (f) of Fig. 19.

The reduction in the peak of $\Theta_o(\omega)_{sd}$ is clearly seen by a comparison of (f) and (a). If for each value of ω the decibel magnitude of the function $\Theta_o(\omega)_{sd}$ here shown as (e) is reduced by a factor equal to twice the gain of the elements $Y_a(j\omega)\ Y_m(j\omega)$ and the resultant function then converted to a linear scale a linear plot of $\Theta_o(\omega)_{sd}$ versus ω will be obtained. This plot is shown as (d) of Fig. 20. It is clearly evident that only the peak of the function was reduced since curve (d) approaches the curve (a) for both low and high values of ω.

In passing it should be noted that a somewhat similar solution might be obtained by a suitable auxiliary feedback loop used in conjunction with a simple proportional plus derivative network inserted in the error channel with the value of ω_2 for the network less than the value of ω at the peak in $T_L(\omega)_{sd}$.

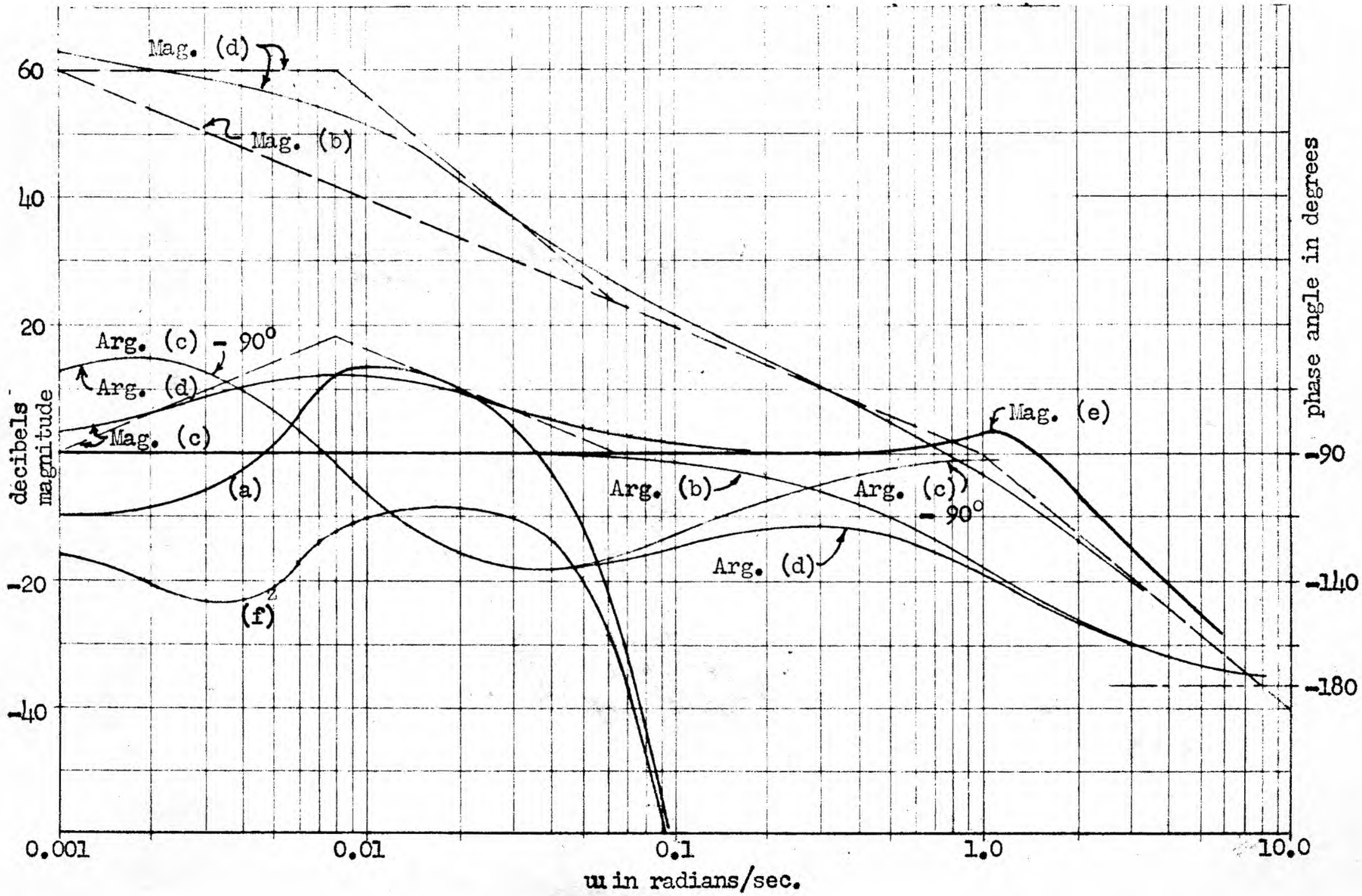


Fig. 19 Graphical analysis used to determine $\theta_o(\omega)_{sd}$ for a servomechanism utilizing special torque load compensation.

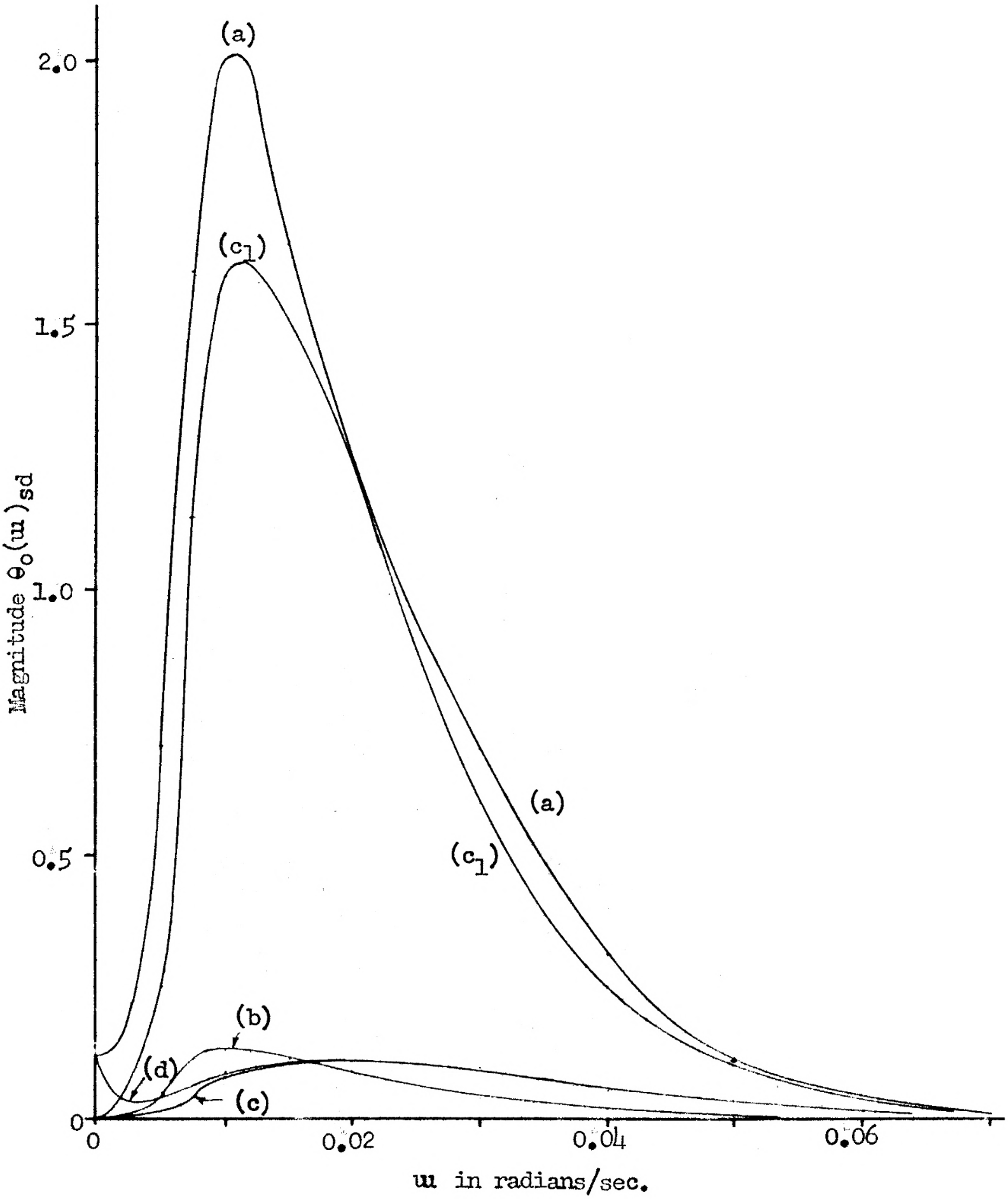


Fig. 20 Linear plots of $\Theta_o(u)_{sd}$ obtained for various types of torque load compensation.

CONCLUSION

In view of the previous discussion it is seen that the output which results from the application of a torque load disturbance to a servomechanism may best be analyzed by first separating the torque load disturbance function into its steady state, periodic and random components. The effect of each component may then be evaluated separately and the overall output obtained as a combination of the separate outputs. In each case the method of analysis was based on equations derived from the special transfer function diagram which is required when the effect of a torque load disturbance is to be considered. In the case of the steady state torque the results may be obtained by a simple substitution of the appropriate values into the formulas derived. For the case of the periodic component of torque load the analysis may be carried out either by direct substitution into the equations or by an equivalent graphical analysis. However, in the case of the random component of torque load only the graphical method of analysis was discussed.

In this particular discussion no consideration was given to the problem of obtaining the expressions for the three components of the torque load disturbance function in the form required from a given set of empirical data which determines the torque load disturbance as a general function of time. In the case of a torque load disturbance function such as was illustrated in Fig. 13 this might constitute a rather complex and lengthy problem. However, the methods of handling such a problem are discussed in the references herein cited, and in any practical servomechanism problem wherein a random varying torque load disturbance is encountered this separation of the function into its various components must be made before the methods illustrated in this discussion may be applied.

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LITERATURE CITED

Brown, Gordon S. and Donald P. Campbell
Principles of servomechanisms. New York: John Wiley and Sons, 400 p. 1948.

Churchill, Ruel V.
Modern operational mathematics in engineering. New York: McGraw-Hill, 306 p. 1944.

Goldman, Stanford
Transformation calculus and electrical transients. New York: Prentice-Hall, 439 p. 1949.

James, H. M., N. B. Nichols and R. S. Phillips
Theory of Servomechanisms. New York: McGraw-Hill, 375 p. 1947.

Korn, Granino A.
Elements of DC analog computers. Electronics. New York: McGraw-Hill. April, 1948.

Routh, E. J.
On the stability of a given state of motion. Adams Prize Essay: 1877.