USEFUL METHODS FOR THE DISTRIBUTIONS OF PRODUCTS OF RANDOM VARIABLES

by

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I. INTRODUCTION

In the field of statistics it is often of interest to examine the distribution of a random variable which is some combination of other random variables with known distributions. Many papers in the literature are devoted to finding the distribution of sums of random variables, while very few are concerned with the products or quotients of random variables. This paper is a consolidation and review of the literature in the area of products of random variables and their distributions.

Some of the more important techniques which have been used to find the distribution function of a product or quotient of two or more random variables are, what we call, the density transform technique, the distribution function technique, the characteristic function technique, and the Mellin transform technique. We first examine each of these techniques in a completely general setting in Section II, which we call "Methods and Mathematics." That section is included to review, or introduce, the techniques in the most general terms, so that when we use them to derive distribution functions and probability density functions for products of random variables in Section III, entitled "Products of Random Variables", the basic concepts will be well in mind. Although not a new technique for statisticians, Epstein (1948), the Mellin transform has not been incorporated into textbooks and therefore is receiving the most emphasis.

We should note at this time that approximation methods do exist, in contrast to the exact methods mentioned above. Only two such methods are briefly discussed here, as our main concern is a study of the exact methods. First, suppose X_i, $i=1,2,3,\ldots$ is a collection of independent identically distributed positive valued random variables to be multiplied "n" at a time. If we can assume that the random variable formed by taking the logarithm of the X_i has finite mean and variance, then we can apply the Central Limit Theorem to this new set of random variables. Now, since the logarithm of the product is equal to the sum of the logarithms of the components, the Central Limit Theorem states that the logarithm of the product approaches a Normal distribution as n increases under quite general conditions. This idea is further expounded in both Shellard (1952) and Broadbent (1956). Second, Arojan (1947) has given a limiting distribution for the product of two normals, not necessarily independent, for the case when the coefficients of variation $(\frac{\sigma_x}{\mu_x}, \frac{\sigma_y}{\mu_y})$ both approach infinity. No more is made of either of these methods.

II. Methods and Mathematics

In this section we first provide a short review of transformation techniques similar to those found in most intermediate and advanced textbooks in statistics (e.g. Kendall and Stuart, 1969; Fisz, 1963; Hogg and Craig, 1970). We mention only the general characteristics here, since the conditions which must be met for the application of each method are easily accessible. This section concludes with a short study of the Mellin transform which provides background information in the use of that technique. Throughout this section and the remainder of this paper we restrict our attention to continuous distributions with tractable probability density functions. We do this solely for ease of notation, as insertion of the Lebesgue integral in place of the Riemann integral allows the techniques to be used on all types of distributions with certain regularity conditions.

Within this section we make several assumptions. For all techniques we assume that the random variable Y is a function of the random vector $\underline{X}$, $Y = f(\underline{X})$. It is also convenient to assume that the vector ($\underline{X} = [X_1, X_2, \ldots, X_n]$) has the probability density function $g(\underline{x})$ and distribution function $G(\underline{x})$. Further assumptions will be added when necessary.

In order to treat the density transform technique in all its generality, we begin the discussion of this technique with some additional assumptions (Hogg and Craig, 1970). Let us first assume that $Y = f(\underline{X})$, $Z_2 = f_2(\underline{X})$, $Z_3 = f_3(\underline{X})$, ..., $Z_n = f_n(\underline{X})$ is a transformation from the n-dimensional space where $g(\underline{x}) > 0$, onto a subset of the $[Y, Z_2, Z_3, \ldots, Z_n]$ space. In addition, let us assume that the set $A = \{\underline{x} | g(\underline{x}) > 0\}$ can be partitioned into a finite number, say k, of sets such that $Y = f(\underline{X})$, $Z_2 = f_2(\underline{X})$, $Z_3 = f_3(\underline{X})$, ..., $Z_n = f_n(\underline{X})$ defines a one-to-one transformation from each set of the partition onto the subset of the $[Y, Z_2, Z_3, \ldots, Z_n]$ space previously mentioned. Now let

$$\text{(II.1)} \quad \begin{aligned} X_1 &= u_{1j}(Y, Z_2, Z_3, \ldots, Z_n) \\ X_2 &= u_{2j}(Y, Z_2, Z_3, \ldots, Z_n) \\ &\vdots \\ X_n &= u_{nj}(Y, Z_2, Z_3, \ldots, Z_n) \end{aligned} \qquad j = 1,2, \ldots, k$$

denote the k sets of inverse functions, one set for each element of the partition.

Let us further assume, for the purposes of the density transform technique, that each of the u_{ij} $(i=1,2, \ldots, n;\ j=1,2, \ldots, k)$ have continuous first partial derivatives and that each

$$\text{(II.2)} \quad J_j = \begin{vmatrix} \frac{\partial u_{1j}}{\partial y} & \frac{\partial u_{1j}}{\partial z_2} & \cdots & \frac{\partial u_{1j}}{\partial z_n} \\ \frac{\partial u_{2j}}{\partial y} & \frac{\partial u_{2j}}{\partial z_2} & \cdots & \frac{\partial u_{2j}}{\partial z_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_{nj}}{\partial y} & \frac{\partial u_{nj}}{\partial z_2} & \cdots & \frac{\partial u_{nj}}{\partial z_n} \end{vmatrix} \qquad j = 1,2, \ldots, k$$

is not identically zero in our subset of the $[Y, Z_2, Z_3, \ldots, Z_n]$ space. Utilizing the above assumptions and the probability characteristics of disjoint sets the density transform technique gives the probability density function of $\underline{Z} = [Y, Z_2, Z_3, \ldots, Z_n]$ to be,

$$(II.3) \qquad h(\underline{z}) = \sum_{j=1}^{k} |J_j| \; g(u_{1j}(\underline{z}), u_{2j}(\underline{z}), \ldots, u_{nj}(\underline{z})).$$

To find the probability density function of Y, we need only to integrate over the ranges of the extraneous variables $(Z_2, Z_3, \ldots, Z_n)$, i.e.

$$(II.4) \qquad h_1(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\underline{z}) dz_2 dz_3 \ldots dz_n.$$

Integrating $h_1(y)$ from $-\infty$ to y will then give us the distribution function of Y.

In using the distribution function technique, we make a direct appeal to the probability function $P(Y \leq y) = P(f(\underline{X}) \leq y)$. The object is to reduce $P(f(\underline{X}) \leq y)$ to an integral of a product of conditional and marginal probability statements which can be evaluated from $G(\underline{x})$ (or $g(\underline{x})$). For example, suppose $\underline{X} = [X_1, X_2]$ and $Y = X_1 + X_2$, then

$$(II.5) \qquad P(Y \leq y) = P(X_1 + X_2 \leq y) = \int_{-\infty}^{\infty} P(X_1 \leq y - x_2 | X_2 = x_2) \; dP_2(x_2),$$

where,

$$(II.6) \qquad dP_2(x_2) = \int_{-\infty}^{\infty} g(x_1, x_2) dx_1 dx_2 = g_2(x_2) dx_2.$$

Since

$$\text{(II.7)} \qquad g_1(x_1|x_2) = \frac{g(x_1,x_2)}{g_2(x_2)} \qquad \text{whenever } g_2(x_2) \neq 0,$$

and

$$\text{(II.8)} \qquad P(X_1 \leq y-x_2 | X_2=x_2) = \int_{-\infty}^{y-x_2} g_1(x_1|x_2)dx_1,$$

we conclude that

$$\text{(II.9)} \qquad H(y) = P(Y \leq y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y-x_2} g(x_1,x_2)dx_1dx_2.$$

We now see that, at least for reasonably well-behaved functions, we can find $H(y) = P(Y \leq y)$ by this method.

Another method for finding the distribution of $Y = f(\underline{X})$ is the characteristic function method. The characteristic function, $\phi(t)$, of Y is given by

$$\text{(II.10)} \qquad \phi(t) = E(e^{itY}) = E(e^{itf(\underline{X})}).$$

This last expected value can be found by the use of the probability density function of $\underline{X}$, $g(\underline{x})$. Since distribution functions and characteristic functions exhibit a one-to-one relationship it follows that we can find $\phi(t)$ by the above and apply an inversion theorem to get the required distribution function or probability density function. If for now we assume that the distribution function of Y, $H(y)$, is continuous everywhere and $dH(y) = h(y)dy$, then Kendall and Stuart (1969) give one form of the inversion formula as:

(II.11) $$h(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)e^{-iyt}dt = \lim_{c\to\infty}\frac{1}{2\pi}\int_{-c}^{c} \phi(t)e^{-iyt}dt$$

The characteristic function approach is particularly useful when summing independent random variables. For example, consider the case where $Y = X_1+X_2+\cdots+X_n$ and the X_j are independent. The independence and (II.10) imply that

(II.12) $$\phi(t) = E(e^{itY}) = E\left(e^{it(X_1+X_2+\cdots+X_n)}\right) = \prod_{j=1}^{n} E\left(e^{itX_j}\right) = \prod_{j=1}^{n} \phi_j(t).$$

If, in addition, the X_j are identically distributed with characteristic function $\phi_1(t)$, (II.12) reduces to:

(II.13) $$\phi(t) = [\phi_1(t)]^n.$$

The final transformation technique, which we present, is the Mellin transform. Titchmarsh (1937) defined the Mellin transform of g(x), under certain restrictions to be

(II.14) $$M_x(s) = \int_0^{\infty} x^{s-1}g(x)dx,$$

where s is a complex variable. It can be shown that $M_x(s)$ is analytic in a strip parallel to the imaginary axis and that it has an inverse formula, similar to that of the characteristic function, which allows us to find g(x) from $M_x(s)$. The inversion formula (Titchmarsh, 1937) is given by

$$\text{(II.15)} \qquad g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \, M_x(s) ds,$$

where the path of integration is any line parallel to the imaginary axis and within the strip of analyticity of $M_x(s)$.

Note that the Mellin transform, as defined in (II.14), is only applicable to positive valued random variables. Epstein (1948) has developed a method for handling random variables which are both positive and negative valued. That extension, as it relates to the distributions of products of random variables, is discussed in the next section.

Epstein (1948) presents several properties of the Mellin transform. For example, if g(x) is considered to be the probability density function of a positive valued random variable, X, then we can consider the Mellin transform of g(x) to be $E(X^{s-1})$, i.e.

$$\text{(II.16)} \qquad M_x(s) = \int_0^\infty x^{s-1} \, g(x) dx = E(X^{s-1}).$$

From Epstein (1948) we obtain the following relationships. First, if $Y = aX$ $(a>0)$, the Mellin transform of Y is $a^{s-1} \, M_x(s)$, i.e.

$$\text{(II.17)} \qquad M_y(s) = E(Y^{s-1}) = E[(aX)^{s-1}] = a^{s-1} \, M_x(s).$$

Second, if $Y = X^a$, $M_y(s) = M_x(as-a+1)$, i.e.

$$\text{(II.18)} \qquad M_y(s) = E(Y^{s-1}) = E(X^{a(s-1)}) = M_x(as-a+1).$$

For example, $a = -1$ $(Y = \frac{1}{X})$ implies that $M_y(s) = M_x(-s+2)$. Third, if $Y = X_1 \cdot X_2$, and X_1 and X_2 are independent, then the Mellin transform of Y is $M_{x_1}(s) \cdot M_{x_2}(s)$, i.e.

$$\text{(II.19)} \quad M_y(s) = E(Y^{s-1}) = E[(X_1X_2)^{s-1}] = E(X_1^{s-1})E(X_2^{s-1}) = M_{x_1}(s)M_{x_2}(s).$$

The Mellin transform shares with the characteristic function the property of a one-to-one correspondence between the probability density function and its transform. From this, it is obvious that the probability density function of any of the new random variables produced above can be found from the inversion formula once $M_x(s)$ is known.

We believe that this provides an adequate background in Mellin transforms for those who wish to use this transform in statistics. The complete theory, along with the theory of Fourier transforms, is given in Titchmarsh (1937). Of general interest is a reasonable sized table of Mellin transforms in Bateman (1954). This table can be used to either find a Mellin transform or the function corresponding to a particular transform.

III. Products of Random Variables

In this section we examine each of the four techniques in respect to products of random variables. For the first two techniques we display the method for the case of two dependent variables, then progress to two independent variables, and finally generalize to n independent variables. For the Mellin transform and the characteristic function techniques we confine our interest to the two independent cases.

We return now to the density transform technique, for which we know an inverse transformation is necessary. Suppose we are looking for the probability density function of Y, where $Y = X_1X_2$ and $[X_1,X_2]$ has the probability density function $g(x_1,x_2)$. Then a reasonable transformation is to let $Y = X_1X_2$ and $Z_2=X_2$, which is one-to-one from the $[X_1,X_2]$ space onto the $[Y,Z_2]$ space. This gives an inverse transformation of $X_1 = \frac{Y}{Z_2}$, $X_2=Z_2$. The Jacobian of this transformation is

$$\text{(III.1)} \qquad J = \begin{vmatrix} \frac{1}{z_2} & \frac{-y}{z_2^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{z_2} .$$

Therefore, the density function of $[Y,Z_2]$ is

$$\text{(III.2)} \qquad h(y,z) = g(\tfrac{y}{z},z) \frac{1}{|z|} ,$$

and the density function of Y is given by

$$\text{(III.3)} \qquad h_1(y) = \int_{-\infty}^{\infty} \frac{1}{|z|} g(\tfrac{y}{z},z)\,dz.$$

We can easily generalize this situation to the case of the independent random variables. Let X_1 have the probability density function $g_1(x_1)$ and X_2 have the probability density function $g_2(x_2)$. Then,

(II.4) $$g(x_1,x_2) = g_1(x_1)g_2(x_2),$$

and the density (III.3) can be written

(III.5) $$h_1(y) = \int_{-\infty}^{\infty} \frac{1}{|z|} g_1(\tfrac{y}{z})g_2(z)dz.$$

To generalize to n independent variables, we must expand our transformation and compute a new Jacobian. The problem is to find the probability density function of $Y = X_1X_2...X_n$, where X_i, $i = 1,2, ..., n$, has the probability density function $g_i(x_i)$. Let $Z_i = X_i$, $i = 2,3, ..., n$, where the corresponding inverse transformation is $X_1 = \frac{Y}{Z_2Z_3...Z_n}$, $X_i = Z_i$, $i = 2,3, ..., n$. The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{1}{z_2\cdots z_n} & \frac{-y}{z_2^2z_3\cdots z_n} & \frac{-y}{z_2z_3^2z_4\cdots z_n} & \cdots & \frac{-y}{z_2\cdots z_{n-1}z_n^2} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot & \cdot\ \cdot & 1 \end{vmatrix}$$

(III.6)

$$J = \frac{1}{z_2z_3\cdots z_n}.$$

Therefore, $h(y,z_2,\ldots,z_n)$ is given by

$$\text{(III.7)}\qquad h(y,z_2,\ldots,z_n) = g_1\left(\frac{y}{z_2\ldots z_n}\right)g_2(z_2)\ldots g_n(z_n)\frac{1}{|z_2\ldots z_n|}$$

To find $h_1(y)$ we integrate the extraneous variables ever the proper ranges,

$$\text{(III.8)}\qquad h_1(y) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\frac{1}{|z_2\ldots z_n|}\, g_1\left(\frac{y}{z_2\ldots z_n}\right)g_2(z_2)\ldots g_n(z_n)\,dz_2\ldots dz_n.$$

We now return to the case of two variables $[X_1, X_2]$ with probability density function $g(x_1,x_2)$ and also suppose that the marginal densities are $g_1(x_1)$ and $g_2(x_2)$. The distribution function technique to find the distribution function of $Y = X_1X_2$, $H(y)$, is applied as follows:

$$\text{(III.9)}\qquad H(y)=P(Y\leq y)=P(X_1X_2\leq y) = \int_0^{\infty} P(X_1\leq\frac{y}{x_2}|X_2=x_2)\,dP_2(x_2)$$

$$+\int_{-\infty}^{0}(P(X_1\geq\frac{y}{x_2}|X_2=x_2)\,dP_2(x_2)$$

or equivalently,

$$H(y) = \int_0^{\infty}\int_{-\infty}^{\frac{y}{x_2}} g(x_1|X_2=x_2)\,dx_1 g_2(x_2)\,dx_2$$

$$+\int_{-\infty}^{0}\int_{\frac{y}{x_2}}^{\infty} g(x_1|X_2=x_2)\,dx_1 g_2(x_2)\,dx_2$$

$$= \int_0^{\infty} \int_{-\infty}^{\frac{y}{x_2}} \frac{g(x_1,x_2)}{g_2(x_2)} dx_1 g_2(x_2) dx_2$$

$$\text{(III.10)} \qquad + \int_{-\infty}^{0} \int_{\frac{y}{x_2}}^{\infty} \frac{g(x_1,x_2)}{g_2(x_2)} dx_1 g_2(x_2) dx_2$$

$$= \int_0^{\infty} \int_{-\infty}^{\frac{y}{x_2}} g(x_1,x_2) dx_1 dx_2 + \int_{-\infty}^{0} \int_{\frac{y}{x_2}}^{\infty} g(x_1,x_2) dx_1 dx_2$$

The above equations (III.10) are given for fairly well behaved functions. Many functions do not behave well enough to make all of the above substitutions and often the correct limits of integration cannot be easily determined.

If X_1 and X_2 are independent (i.e. $g_1(x_1)g_2(x_2) = g(x_1,x_2)$) the final equation of (III.10) can be written

$$\text{(III.11)} \qquad H(y) = \int_0^{\infty} G_1(\tfrac{y}{x_2}) g_2(x_2) dx_2 + \int_{-\infty}^{0} \left[1 - G_1(\tfrac{y}{x_2})\right] g_2(x_2) dx_2$$

where

$$\text{(III.12)} \qquad G_1(z) = \int_{-\infty}^{z} g_1(x_1) dx_1 .$$

For n independent random variables and $Y = X_1X_2...X_n$ with all X_i positive valued, we get,

$$H(y)=P(X_1X_2\ldots X_n\leq y) = \int_0^\infty P(X_1X_2\ldots X_{n-1}\leq\frac{y}{x_n}\mid X_n=x_n)\,dP_n(x_n)$$

$$= \int_0^\infty P(X_1X_2\ldots X_{n-1}\leq\frac{y}{x_n})\,dP_n(x_n)$$

(III.13)

$$= \int_0^\infty \cdots \int_0^\infty P(X_1\leq \frac{y}{x_2x_3\ldots x_n})\; dP_2(x_2)dP_3(x_3)\ldots dP_n(x_n)$$

$$= \int_0^\infty \cdots \int_0^\infty G_1(\frac{y}{x_2x_3\ldots x_n})g_2(x_2)g_3(x_3)\ldots g_n(x_n)dx_2dx_3\ldots dx_n$$

where $G_1(z)$ is defined by (III.12).

Next we consider the characteristic function technique as applied to two independent random variables, X_1 and X_2. Let $\phi_1(t)$ be the characteristic function of X_1 and let $g_1(x_1)$ and $g_2(x_2)$ be the probability density functions of X_1 and X_2. Kendall and Stuart (1969) show that the characteristic function of $Y = X_1X_2$, $\phi(t)$, is

$$\text{(III.14)}\quad \phi(t) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} e^{itx_2x_1}g_1(x_1)dx_1\right\}g_2(x_2)dx_2 = \int_{-\infty}^{\infty} \phi_1(tx_2)g_2(x_2)dx_2.$$

Once the characteristic function is known, an inversion integral, such as (II.11), can be applied to find the distribution function or the probability density function. The above result can easily be generalized to the case of n independent random variables $X_1,X_2,\ldots,X_n$. For $Y = X_1X_2\ldots X_n$,

$$\phi_y(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} e^{itx_1 \ldots x_n} g_1(x_1)dx_1\right\} g_2(x_2)\ldots g_n(x_n)dx_2\ldots dx_n$$

(III.15)

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi_1(tx_2x_3\ldots x_n)g_2(x_2)g_3(x_3)\ldots g_n(x_n)dx_2dx_3\ldots dx_n.$$

The remainder of this section is devoted to the use of the Mellin transform to find the distribution of a product of independent random variables. In Section II, the expected value concept was used to show that the Mellin transform of a product of independent positive valued variables is the product of the Mellin transforms. A more complete proof is given in Titchmarsh (1937), by showing that

(III.16) $$h(y) = \int_0^{\infty} \frac{1}{z} g_1\left(\frac{y}{z}\right) g_2(z)dz$$

is the Mellin convolution of $g_1(x_1)$ and $g_2(x_2)$. We have shown earlier that, for independent positive valued random variables, (III.16) is the probability density function of $Y = X_1X_2$.

Corresponding to the previous work, for the product of two independent positive valued random variables, we have

(III.17) $$M_y(s) = M_{x_1}(s)M_{x_2}(s),$$

and for the product of n independent positive valued random variables, we have

(III.18) $$M_y(s) = M_{x_1}(s)M_{x_2}(s)\ldots M_{x_n}(s).$$

To either one we can apply the inverse transformation,

$$\text{(III.19)} \qquad h(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} M_y(s)\, ds$$

to get the probability density function of Y.

Up to now, the Mellin transform has been defined only for positive valued random variables. A method, due to Epstein (1948), has made it possible to apply the Mellin transform to random variables which take on both positive and negative values. Let X_1 and X_2 be two independent random variables with probability density functions $g_1(x_1)$ and $g_2(x_2)$. Let us decompose these density functions into

$$\text{(III.20)} \qquad \begin{aligned} g_{11}(x_1) &= g_1(x_1) \quad x_1 \geq 0 & \qquad g_{12}(x_1) &= 0 \quad x_1 \geq 0 \\ &= 0 \quad x_1 < 0 & &= g_1(x_1) \quad x_1 < 0 \\ g_{21}(x_2) &= g_2(x_2) \quad x_2 \geq 0 & \qquad g_{22}(x_2) &= 0 \quad x_2 \geq 0 \\ &= 0 \quad x_2 < 0 & &= g_2(x_2) \quad x_2 < 0.^* \end{aligned}$$

*Equation (III.20) is written to conform with equation (III.21). Epstein (1948), when discussing this method, does not include the points $x_1=0$ or $x_2=0$ in the ranges of either part of the decompositions, while Springer and Thompson (1966) on the other hand include these points in both parts of the decomposition. We consider the question of inclusion of the zero points to be a matter of convergence and beyond the scope of this report.

Then,

$$g_1(x_1) = g_{11}(x_1) + g_{12}(x_1)$$

(III.21)

$$g_2(x_2) = g_{21}(x_2) + g_{22}(x_2).$$

Now we operate on the pairs $[g_{11}(x_1), g_{21}(x_2)]$, $[g_{11}(x_1), g_{22}(x_2)]$, $[g_{12}(x_1), g_{21}(x_2)]$, and $[g_{12}(x_1), g_{22}(x_2)]$. The density function of $Y = X_1X_2$ is composed of four parts, $h_1(y)$, $h_2(y)$, $h_3(y)$, and $h_4(y)$ where each part is derived from one of the pairs and

(III.22) $$h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y).$$

For $h_1(y)$, corresponding to the pair $[g_{11}(x_1), g_{21}(x_2)]$, we can apply the Mellin transform to obtain

(III.23) $$h_1(y) = \int_0^\infty \frac{1}{z}\, g_{11}\left(\frac{y}{z}\right) g_{21}(z)dz$$

since $g_{11}(x_1)$ and $g_{21}(x_2)$ are both zero for negative values of their arguments and thus $h_1(y) = 0$, $y<0$. To compute $h_2(y)$ we first use the Mellin transform to compute $h_2^*(y)$, where

(III.24) $$h_2^*(y) = \int_0^\infty \frac{1}{z}\, g_{11}\left(\frac{y}{z}\right) g_{22}(-z)dz.$$

This is possible since $g_{11}(x_1)$ and $g_{22}(-x_2)$ are zero for negative values of x_1 and x_2. Now $h_2^*(y) = 0$, $y\leq 0$, so $h_2(y) = h_2^*(-y) = 0$ for $y\geq 0$. We can similarily find $h_3(y)$ and $h_4(y)$, where $h_3(y) = 0$, $y\geq 0$, and $h_4(y) = 0$, $y\leq 0$. Combining the four sections by use of (III.22), we get the desired probability density function.

IV. Examples

In this section some examples are presented to illustrate the techniques which have been discussed. We first demonstrate the characteristic function and Mellin transform techniques for the product of two normal variables. Then we use the Mellin transform technique for both the product and quotient of two uniform variables. Finally the density transform and the Mellin transform techniques are applied to gamma and beta variables.

Example 1. Assume that X_1 and X_2 are both distributed normally with mean zero and variance one, i.e.

$$g_i(x_i) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x_i^2}{2}) \qquad i = 1,2. \tag{IV.1}$$

Then for $Y = X_1X_2$, from (III.14) (Kendall and Stuart, 1969)

$$\begin{aligned} \phi_y(t) &= \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(tx_2)^2](2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2})\,dx \\ &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2} x_2^2(1+t^2)]\,dx_2 \\ &= (1+t^2)^{-\frac{1}{2}}, \end{aligned} \tag{IV.2}$$

and if we invert this characteristic function we get

$$h(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+t^2)^{-\frac{1}{2}} \exp(-iyt)\,dt, \tag{IV.3}$$

which, because of symmetry, can be written

(IV.4) $$h(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1+t^2)^{-\frac{1}{2}} \cos(yt) = \frac{1}{\pi} K_0(y).$$

The function K_0 is the Bessel function of the second kind of a purely imaginary argument of order zero. Tables of K_0 can be found in Handbook of Mathematical Functions (Abramowitz and Stegun, Eds; 1964).

Epstein (1948) worked this same problem using Mellin transforms by partitioning the density function as described in equations (III.20) and (III.21). It is sufficient to evaluate only one factor of $h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y)$, since, by the symmetry of $g_1(x_1)$ and $g_2(x_2)$ around zero, $h_1(y)$ and $h_4(y)$ are equal and both $h_1(y)$ and $h_4(y)$ are zero for $y<0$, $h_2(y) = h_3(y)$ and both are equal to zero for $y \geq 0$, also $h_1(-y) = h_2(y)$.

Since we need only find one $h_i(y)$ we will find

(IV.5) $$h_1(y) = \int_0^{\infty} \frac{1}{x} g_{11}(\tfrac{y}{x}) g_{21}(x)\, dx.$$

Since $g_{11}(x) = g_{21}(x)$, we need only find one Mellin transform. For i=1,2

(IV.6) $$M_{g_{i1}}(s) = \int_0^{\infty} x^{s-1} g_{11}(x)\, dx = \int_0^{\infty} x^{s-1} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2} x^2)\, dx$$

$$= \pi^{-\frac{1}{2}} 2^{\frac{1}{2}(s-3)} \Gamma(\tfrac{1}{2}s)$$

and is analytic for $Re(s)>0$. Therefore

(IV.7) $$M_{h_1}(s) = M_{g_{11}}(s) M_{g_{21}}(s) = \pi^{-1} 2^{s-3} [\Gamma(\tfrac{1}{2}s)]^2.$$

Applying the inverse transformation we get

$$h_1(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\,\frac{2^{s-3}}{\pi}\,[\Gamma(\tfrac{s}{2})]^2 \qquad c>0$$

(IV.8)

$$= \frac{1}{2\pi}K_0(y) \qquad y\geq 0$$

Similarly,

$$h_2(y) = \frac{1}{2\pi}K_0(y) \qquad y<0$$

(IV.9) $$h_3(y) = \frac{1}{2\pi}K_0(y) \qquad y<0$$

$$h_4(y) = \frac{1}{2\pi}K_0(y) \qquad y\geq 0.$$

Therefore,

(IV.10) $$h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y) = \frac{1}{\pi}K_0(y) \qquad -\infty<y<\infty$$

which agrees with (IV.4) from the characteristic function technique.

Springer and Thompson (1966) and Lomnicki (1967) have developed methods to find the distribution of the product of more than two normal random variables. Lomnicki's paper is even more general in that his results also apply to several other distributions including the Weibull distribution, which has many engineering applications.

Example 2. In this example we will find the probability density function of the quotient as well as the product of two independent uniform random variables. This is done to illustrate the use of relationship (II.19), which says that if $M_x(s)$ is the Mellin transform of X, then $M_x(-s+2)$ is the Mellin transform of $Y = \frac{1}{X}$. Let X_i, $i = 1,2$, have the probability density function

$$\text{(IV.11)} \qquad \begin{aligned} f(x) &= 1 \qquad 0 \leq x \leq 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then its Mellin transform is

$$\text{(IV.12)} \qquad M_{x_i}(s) = \frac{1}{s} .$$

Therefore, if $Y = X_1 X_2$, the Mellin transform of Y is

$$\text{(IV.13)} \qquad M_y(s) = \frac{1}{s^2} .$$

Then the probability density function is

$$\text{(IV.14)} \qquad h(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} s^{-2} \, ds,$$

or

$$\text{(IV.15)} \qquad \begin{aligned} h(y) &= \log_e\left(\frac{1}{y}\right) = -\log_e y \qquad 0 \leq y \leq 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Suppose that we are now interested in finding the distribution of the quotient $Y = \frac{X_1}{X_2}$, where X_1 and X_2 are defined by (IV.11). The Mellin transform of $\frac{1}{X_2}$ is

$$\text{(IV.16)} \qquad M_{\frac{1}{x_2}}(s) = M_{x_2}(-s+2) = (-s+2)^{-1}.$$

Therefore, the Mellin transform of $Y = \frac{X_1}{X_2}$ is

(IV.17) $$M_y(s) = \frac{1}{s(-s+2)},$$

and the probability density function of Y is

(IV.18) $$h(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^{-s}}{s(-s+2)}\, ds.$$

This last integral requires either a table or the theory of residuals for evaluation. We choose here to give only the result given in Springer and Thompson (1966),

(IV.19) $$\begin{aligned} h(y) &= \tfrac{1}{2} & 0 \le y < 1 \\ &= \tfrac{1}{2}\, y^{-2} & 1 \le y < \infty. \end{aligned}$$

Springer and Thompson (1966) give a more general result for the product of n and quotient of 2 independent identically distributed random variables, each with the distribution

(IV.20) $$\begin{aligned} g(x) &= (a+1)x^{a} & 0 \le x \le 1 \quad & a > -1 \\ &= 0 & \text{otherwise.} \end{aligned}$$

The uniform distribution is a special case of this distribution with $a=0$. For the product of n such variables, Springer and Thompson (1966) show that the distribution function of $Y = X_1X_2 \ldots X_n$ is given by

$$h(y) = \frac{(a+1)^n}{(n-1)!} y^a (\log \frac{1}{y})^{n-1} \qquad 0\le y\le 1$$

(IV.21)

$$= 0 \qquad \text{otherwise.}$$

For the quotient $Y = \frac{X_1}{X_2}$ of two independent random variables, the distribution is given by

$$h(y) = \frac{1}{2}(a+1)y^a \qquad 0\le y<1$$

(IV.22)

$$= \frac{1}{2}(a+1)y^{-a-2} \qquad 1\le y<\infty.$$

Example 3. For our final example, we assume that X_1 is a gamma random variable with parameter p and X_2 is a beta random variable with parameters q and p-q. Then

$$g_1(x_1) = \frac{1}{\Gamma(p)} x_1^{p-1}\exp(-x_1) \qquad x_1\ge 0$$

$$= 0 \qquad \text{otherwise,}$$

(IV.23)

$$g_2(x_2) = \frac{\Gamma(p)}{\Gamma(q)\Gamma(p-q)} x_2^{q-1}(1-x_2)^{p-q-1} \qquad 0\le x_2\le 1$$

$$= 0 \qquad \text{otherwise}$$

We show, first by the density transform technique, then by the Mellin transform technique, that if X_1 and X_2 are independent, then $Y = X_1X_2$ is a gamma random variable with parameter q. The joint distribution of $[X_1, X_2]$ is $g_1(x_1)g_2(x_2)$ and letting $Y = X_1X_2$, $Z = X_2$ the Jacobian is $\frac{1}{z}$, from equation (III.1). The probability density function of Y is then

$$h(y) = \int_0^1 \frac{\Gamma(p)}{\Gamma(p)\Gamma(q)\Gamma(p-q)} \left(\frac{y}{z}\right)^{p-1} z^{q-1} (1-z)^{p-q-1} \exp\left(-\frac{y}{z}\right) \frac{1}{z}\, dz$$

(IV.24)

$$= \frac{y^{p-1}}{\Gamma(q)\Gamma(p-q)} \int_0^1 \left(\frac{(1-z)}{z}\right)^{p-q-1} \exp\left(-\frac{y}{z}\right) \frac{dz}{z^2}$$

Let $w = \frac{1-z}{z}$; then $z = \frac{1}{w+1}$ and $dw = \frac{-1}{z^2}\, dz$. The limits of integration are infinity to zero, therefore

$$h(y) = \frac{y^{p-1}}{\Gamma(q)\Gamma(p-q)} \int_0^\infty w^{p-q-1} \exp[-y(w+1)]dw$$

(IV.25)

$$= \frac{y^{p-1}\exp(-y)}{\Gamma(q)\Gamma(p-q)} \int_0^\infty w^{p-q-1} \exp(-wy)dw.$$

Using the definition of a gramma function, given in Fisz (1963), we can evaluate the last integral in (IV.25) and find

$$h(y) = \frac{y^{p-1}\exp(-y)}{\Gamma(q)\Gamma(p-q)} \frac{(p-q)}{y^{p-q}} = \frac{1}{\Gamma(q)} y^{q-1} \exp(-y). \tag{IV.26}$$

h(y) is seen to be the probability density function of a gamma random variable with parameter q. Jambunathan (1954) contains several other results for beta and gamma distributions based on the density transform technique.

Kotlarski (1965) came to the same conclusion by the use of Mellin transforms. The Mellin transform of $g_1(x_1)$ is

$$M_{x_1}(s) = \frac{1}{\Gamma(p)} \int_0^\infty x_1^{s-1} x_1^{p-1} \exp(-x_1)\, dx_1$$

$$\text{(IV.27)} \qquad = \frac{1}{\Gamma(p)} \int_0^\infty x_1^{p+s-1-1} \exp(-x_1)\, dx_1$$

$$= \frac{\Gamma(p+s-1)}{\Gamma(p)} \qquad \mathrm{Re}(s) > -p.$$

The Mellin transform of $g_2(x_2)$ is

$$M_{x_2}(s) = \frac{\Gamma(p)}{\Gamma(q)\Gamma(p-q)} \int_0^1 x_2^{s-1} x_2^{q-1} (1-x_2)^{p-q-1} dx_2$$

$$\text{(IV.28)} \qquad = \frac{\Gamma(p)}{\Gamma(q)\Gamma(p-q)} \frac{\Gamma(q+s-1)\Gamma(p-q)}{\Gamma(p+s-1)}$$

$$= \frac{\Gamma(p)\Gamma(q+s-1)}{\Gamma(q)\Gamma(p+s-1)} \qquad \mathrm{Re}(s) > -q, -p.$$

Therefore, the Mellin transform of $Y = X_1 X_2$ is

$$\text{(IV.29)} \qquad M_y(s) = \frac{\Gamma(p+s-1)}{\Gamma(p)} \frac{\Gamma(p)\Gamma(q+s-1)}{\Gamma(q)\Gamma(p+s-1)} = \frac{\Gamma(q+s-1)}{\Gamma(q)} \qquad \mathrm{Re}(s) > -q.$$

which is seen to be of the same form as the Mellin transform of X_1 with q in place of p. Since the Mellin transform has a one-to-one relationship with the distribution functions, we can conclude that $Y = X_1 X_2$ is distributed as a gamma random variable with parameter q.

V. SUMMARY

In Section II we examined several techniques which may be valuable in finding the distribution function of transformed random variables. In Section III we showed how each can be applied when the transformation is a product of two or more random variables. We then in Section IV gave examples using most of the techniques of products of independent random variables.

We feel that in most cases where the product of independent random variables is of interest that the Mellin transform is often the easiest technique to apply, just as the characteristic function is often the easiest technique to apply in the case of sums of independent random variables. Occasionally, further uses of the product, such as addition of another independent set of variables, may cause us to use the characteristic function approach, where the characteristic function of the product can be used with the characteristic functions of the additional independent variables as in (II.12).

In problems involving the product of dependent random variables, for which we did not discuss either the characteristic function or Mellin transform techniques, we suggest using either the density transform or distribution function technique. The choice of which to use will probably depend on the problem at hand and only trial and error will tell which is easier to apply.

We suggest that in the future more textbooks examine the problems associated with products of random variables. We also believe statisticians should become familiar with the uses of the Mellin transform, as they have in the past of characteristic functions.

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USEFUL METHODS FOR THE DISTRIBUTIONS
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by

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ABSTRACT

It is frequently of interest to find the distribution of a random variable which is a function of random variables with known distributions. Some of these transformations have several methods which can be applied effectively. The product of two or more random variables has four such methods.

The density transform technique appears to be a viable tool in most transformation problems when a density function exists for the original random variables. It's use in sums of random variables is described in many textbooks. It is also applicable to product transformations whether the original variables are independent or not.

Another technique which does not require independence is the distribution function technique, which is used by appealing to probability statements. This is also a technique applied in textbooks, often for sums of random variables. Both the distribution function and the density transform are widely applicable in transformation problems.

Two methods are available which are most easily applied when assuming independence of the original random variables. The characteristic function technique for products can be found in Vol. 1 of the textbooks by Kendall and Stuart. Another method which is easily applied under the condition of independence, the Mellin transform technique, is not yet a textbook method. Epstein (1948) seems to be the first author to apply the Mellin transform in statistics. Since that time several authors have used this method for the distribution of products of independent random variables.