DIGITAL FILTERS

by

SHAN C. SUN

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Approved by:

Charles A. Halijak

Major Professor

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INTRODUCTION

This report analyzes Butterworth filters in the time domain by using a digital method for finding their transient performances. Digital filter compensation is effected by multiplication in the Z-transform domain by $(1+z^m)/2$. For the sake of comparison, posicast compensation (6) of a compensator $\left[(1-P)+Pz^m\right]$ is employed in each corresponding digital filter compensation case. The analytical digital process used in this report is an approximate method of solving differential equations, the Naumov program (4), which is an improved accuracy trapezoidal convolution program (3).

Previous work on Butterworth filters, pseudo-Butterworth filters, posicast control and Naumov program are reviewed.

Practical examples of analyzing and improving the performances of Butterworth filters from second order to fourth order with unit step inputs are presented.

It is necessary to note that the posicast compensation method results in larger undershoots and time delay when the order of the Butterworth filter being operated becomes higher. These defects are absent in digital filter compensation. Detailed observations are presented in the Discussion section.

BUTTERWORTH FUNCTIONS

A transfer function with spectrum

$$G_n(j\omega) = \frac{1}{\sqrt{1+\omega^{2n}}} \tag{1}$$

is the n^{th} order Butterworth filter or maximally flat low-pass response and it is an approximation to the ideal low-pass filter. The nature of the approximate function is seen from two observations:

(1) From the binomial series expansion

$$(1+X)^{-n} = \sum_{k=0}^{\infty} \binom{n-k+1}{k} (-X)^k \qquad X^2 < 1 \tag{2}$$

it is seen that the Butterworth response when ω is near zero is

$$(1+\omega^{2n})^{-1/2} = 1-(1/2)\omega^{2n} + (3/8)\omega^{4n} - (5/16)\omega^{6n} + \ldots. \tag{3}$$

and that the first (2n-1) derivatives are zero at $\omega = 0$.

(2) $|G_n(j\omega)|$ has fixed points ($|G_{n+1}| = |G_n|$ for all n) at $\omega = 0$, $\omega = 1$ and $\omega = \infty$ for all integers n. Corresponding values for $|G_n(j\omega)|$ are 1, 0.707, and 0.

The Butterworth transfer functions are now determined. The poles of this function are defined by the equation,

$$1 + (-s^2)^n = 0$$

Then the pole locations are

$$s_k = e^{j(2k-1)\pi/2n} \qquad \text{n even}$$

$$s_k = e^{j(2k)\pi/2n} \qquad \text{n odd}$$

or

$$s_k = e^{j(2k+n-1)\pi/2n} \qquad k = 1,2,3,\ldots\ldots,2n$$

The poles so defined are evenly spaced on the unit circle in the s-plane and have symmetry with respect to both the real and the imaginary axes. For n odd, a pair of poles are located on the real axis, but the poles are not located on the imaginary axis for either n even or n odd. These properties accrue because the poles are separated by π/n radians, and are located $\pi/2n$ radians from the real axis for n even and on the real axis for n odd.

Straightforward calculation of

$$G_n(s) = \frac{1}{a_0+a_1s+a_2s^2+a_3s^3+\ldots\ldots+a_ns^n}$$

$$= \frac{1}{\prod_{k=0}^{n/2}(s-s_k)(s-\bar{s}_k)} \qquad (4)$$

yields the coefficients a_k of the denominator polynomial of $G_n(s)$. These Butterworth polynomial coefficients are given in Table 1 and were found in reference (7). Here, the definition

$$G_n(s) \equiv 1/B_n(s) \qquad (4a)$$

is made.

PSEUDO-BUTTERWORTH FUNCTIONS

The pseudo-Butterworth function was first introduced by Marcel J. E. Golay (2) and developed by K. A. Pullen, Jr. Pullen (5) extracted the basic polynomials from Golay's paper and worked out the coefficient relations.

The pseudo-Butterworth polynomial of degree n is

$$B_p(s) = b_0+b_1s+b_2s^2+\ldots\ldots+b_{n-1}s^{n-1}+b_ns^n \qquad (5)$$

$$= \sum_{k=0}^{n} b_ks^k \qquad (6)$$

This polynomial can be approximately converted into a true Butterworth type by replacing s with $s/(b_n)^{1/n}$. The approximate Butterworth polynomial is

$$B_n'(s) \equiv \sum_{k=0}^{n} a_ks^k \qquad (7)$$

where $$a_k = b_k/(b_n)^{k/n} \qquad (8)$$

Table 1. Coefficients of Butterworth Polynomials

$$B_n(s) = \prod_{k=0}^{n/2} (s-s^k)(s-\bar{s}_k)$$

n \ k	0	1	2	3	4	5	6	7	8
0	1.0								
1	1.0	1.0000							
2	1.0	1.4142	1.0000						
3	1.0	2.0000	2.0000	1.0000					
4	1.0	2.6131	3.4142	2.6131	1.0000				
5	1.0	3.2361	5.2361	5.2361	3.2361	1.0000			
6	1.0	3.8637	7.4641	9.1416	7.4641	3.8637	1.0000		
7	1.0	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	1.0000	
8	1.0	5.1258	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258	1.0000

The coefficients of the first ten orders of pseudo-Butterworth polynomial and their corresponding normalized coefficients of the first eight orders of approximate Butterworth polynomial are presented in Table 2 and Table 3 respectively.

The basic coefficient recurrence relation of an n^{th} order pseudo-Butterworth polynomial is

$$C_j^n = (n+j-2)\, C_{j-1}^{n-1} + C_j^{n-1} \qquad 0 \leq j \leq n \qquad (9)$$

where $\qquad n \geq 2$

The coefficients of zero and first order are

$$C_0^0 = 1$$

$$C_0^1 = 1, \qquad C_1^1 = 1$$

Note that the superscripts are not powers.

Table 2. Coefficients of Approximate Butterworth Polynomials

$$B'_n(s) = \sum_{k=0}^{n} s_k s^k \ , \ a_k = b_k/(b_n)^{k/n}$$

When b_k's are the coefficients of pseudo-Butterworth polynomial of order n.

n \ k	0	1	2	3	4	5	6	7	8
0	1.0								
1	1.0	1,0000							
2	1.0	1.4142	1.0000						
3	1.0	2.0000	2.0000	1.0000					
4	1.0	2.6596	3.4687	2.6392	1.0000				
5	1.0	3.3454	5.4833	5.4388	3.3684	1.0000			
6	1.0	4.0438	8.0134	9.9837	7.8102	4.0000	1.0000		
7	1.0	4.7662	11.0079	16.0660	15.9415	10.7663	4.6125	1.0000	
8	1.0	5.4694	14.7004	25.2384	29.6392	24.8341	14.4298	5.4718	1.0000

The additional relations are

$$C_0^n = 1$$
$$C_1^n = (1/2)(n)^{(2)}+1$$
$$C_2^n = (1/8)(n+1)^{(4)}+(1/2)(n+1)^{(2)}+1$$
$$C_3^n = (1/48)(n+2)^{(6)}+(1/8)(n+2)^{(4)}-(1/2)(n+2)^{(2)}+3$$
$$C_4^n = (1/384)(n+3)^{(8)}+(1/48)(n+3)^{(6)}-(1/8)(n+3)^{(4)}+(3/2)(n+3)^{(2)}-15$$
$$C_5^n = (1/3840)(n+4)^{(10)}+(1/384)(n+4)^{(8)}+(1/48)(n+4)^{(6)}+(3/8)(n+4)^{(4)} -(15/2)(n+4)^{(2)}+105$$

etc.

where $n^{(k)} \equiv n(n-1)(n-2).....(n-k+1)$

Table 3. Coefficients of Pseudo-Butterworth Polynomials

$$C_j^n = (n+j-2)\, C_{j-1}^{n-1} + C_j^{n-1}, \quad 0 \leqslant j \geqslant n, \quad n \geqslant 2$$

n \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	2								
3	1	4	8	8							
4	1	7	24	48	48						
5	1	11	59	192	384	384					
6	1	16	125	605	1920	3840	3840				
7	1	22	237	1605	7365	23040	46080	46080			
8	1	29	413	3738	23415	104055	322560	645120	645120		
9	1	37	674	7868	64533	385035	1675275	5160960	10321920	10321920	
10	1	46	1044	15282	158949	1223964	7065765	31290085	92897280	185794560	185794560

The closed form for C_j^n is

$$C_j^n = \frac{(n+j-1)^{(2j)}}{2^j \cdot j!} + \frac{(n+j-1)^{(2j-2)}}{2^{j-1} \cdot (j-1)!} + \sum_{u=2}^{n} \frac{(-1)^{u-1}(2u-3)(2u-5)\ldots 5\cdot 3\cdot 1(n+j-1)^{(2n-2u)}}{2^{n-u}\cdot(n-u)!} \quad (10)$$

and the pseudo-Butterworth polynomial can be written as

$$B_p(s) = \sum_{j=0}^{n} C_j^n s^j \quad (11)$$

POSICAST CONTROLS

The "posicast" or "positive-cast" control method was first introduced by Otto J. M. Smith (6). This method can be applied only when the damping of the complex poles is small. The secret of this method is the division of the input function into two times and two amplitudes. The second input time must follow the first input time by exactly one-half cycle of the transient if the superposition of the results is to be zero for all subsequent times. Any other input-spacing time would excite two oscillations whose vectors would not cancel, because they would always have some phase-angle difference. The first quantity at the first input time must have the value that can bring the maximum overshoot to reach the desired final output value at the second input time. The second quantity must have the value which can maintain that peak value produced by the first quantity.

Figure 1 shows a natural damped transient with amplitude A at one peak and amplitude B at the following peak. The total motion is A+B. Therefore a step A+B should be broken into A initially and B later. The control mechanism should have the transfer function as

$$P_c = \frac{A + Be^{-sT_n/2}}{A + B} \qquad (12)$$

The principle of vector cancellation with posicast control can also be expressed by Fig. 1. In Fig. 1,

(a) Decrement of a damped oscillation. Vector A which starts the oscillation decreases to magnitude B after time $T_{n/2}$. (one-half cycle later).

(b) Required excitation function. Vector B is instantaneously equal to vector A and 180° out of phase with it.

(c) Response of the oscillatory system when driven by the excitation function in Fig. 1(b), is

$$A \cdot u(t) + B \cdot u(t - T_{n/2})$$

the Laplace transform of which is $(A + Be^{-sT_n/2})/s$.

THE NAUMOV PROGRAM

Naumov (4) solved differential equations in the time domain employing trapezoidal integration on a multiple integration formula,

$$\frac{1}{n!}\int_0^t (t-\tau)^n f(\tau)d\tau \qquad (13)$$

A look-up table for $t^n/n!$ for various times t and exponents n is placed in computer storage. A method is then given for employing the look-up table in conjunction with the differential equation to obtain the solution.

Recently Criswell (1) devised an algorithm that easily computes exact $Z(1/s^n)$. The algorithm readily generates tables of Z-transforms of $t^n/n!$ for n much larger than those in presently available tables. Such a large table of Z-transforms can be used for checking and improving approximate

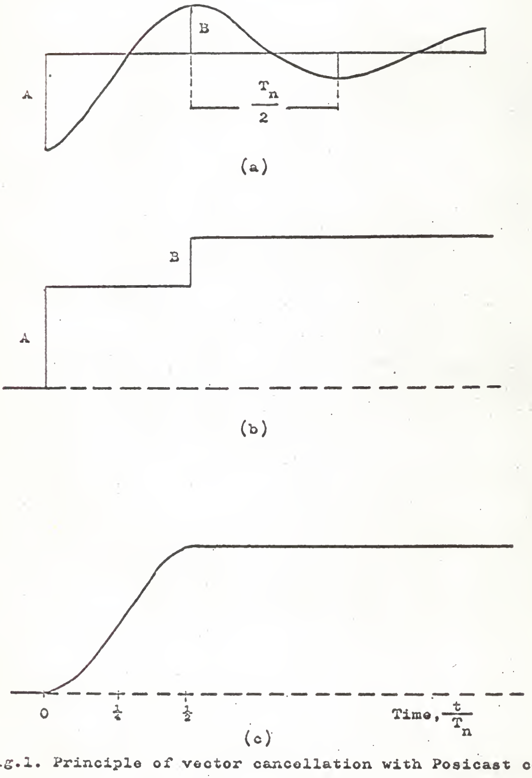

Fig.1. Principle of vector cancellation with Posicast control

integration formulas generated by trapezoidal convolution (3). Replacement of approximate Z-transforms of $1/s^n$ by their exact Z-transforms yields a trapezoidal convolution integrator substitution program identical to Naumov's method but dispensing with digital computer memory for storing look-up table of $t^n/n!$ required in Naumov's method.

A brief derivation of Criswell's algorithm is now presented. Since it is known that

$$Z(1/s^n) = \left[Tz/(n-1)\right]\left[\frac{d}{dz} Z(1/s^{n-1})\right], \quad z \equiv e^{-Ts} \qquad (14)$$

then the first few calculations of $Z(1/s^n)$ can be easily carried out. These calculations yield

$$Z(1/s) = \frac{T^o z}{1} \cdot \frac{z^{-1}}{1-z} \qquad (15a)$$

$$Z(1/s^2) = \frac{Tz}{1} \cdot \frac{1}{(1-z)^2} \qquad (15b)$$

$$Z(1/s^3) = \frac{T^2 z}{2} \cdot \frac{(1+z)}{(1-z)^3} \qquad (15c)$$

$$Z(1/s^4) = \frac{T^3 z}{6} \cdot \frac{z^2+4z+1}{(1-z)^4} \qquad (15d)$$

Assume that the previous results can be written for any integer n in the canonical form

$$Z(1/s^n) = \frac{T^{n-1} z}{(n-1)!} \; \frac{A_n(z)}{(1-z)^n} \qquad (16)$$

Where $A_n(z)$ is a polynomial of degree n-2. Differentiation of equation (16) shows that $Z(1/s^{n+1})$ is also in the canonical form. Hence by mathematical induction the canonical form holds for any positive n. And, the differential-recurrence relation which can generate the desired algorithm is

$$A_n(z) = (1-z)\,\frac{d}{dz}\,(zA_{n-1}) + (n-1)zA_{n-1} \text{ for } n \geq 2 \qquad (17)$$

Recall that the starting polynomial is $A_1(z) = z^{-1}$.

Define $A(n,p)$ to be the coefficient of z^p in $A_n(z)$. The differential-recurrence relation for $A_n(z)$ induces the coefficient relation

$$A(n,p) = (p+1)A(n-1,p) + (n-p-1)A(n-1,p-1) \qquad (18)$$

The coefficients are presented in Table 4 with their corresponding n's and p's.

DIGITAL FILTERS

Conversion of a linear continuous filter function into digital form can be accomplished by the Naumov program in the Z-transform domain. The linear continuous filter functions are Butterworth functions. Since the Naumov program gives exact $Z(1/s^n)$, the transient response of a digital filter can approach the exact response of the Butterworth filter by choosing the smaller sampling interval sizes, T. The numerical calculations are carried out by a digital computer. Conversion of the Butterworth filter into a digital filter with unit step function input is shown for the second, third and fourth order Butterworth filters.

Second Order Digital Butterworth Filter

If the second order Butterworth filter is

$$G_2(s) = \frac{1}{1 + \sqrt{2}s + s^2} \qquad (19)$$

Table 4

COEFFICIENTS $A(n,p)$ IN $A_n(z) = \sum_{p=-1}^{n-2} A(n,p)\, z^p$

FOR $Z(1/s^n) = \frac{T^{n-1}z}{(n-1)!} \frac{A_n(z)}{(1-z)^n}$

n \ p	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1														
2		1													
3		1	1												
4		1	4	1											
5		1	11	11	1										
6		1	26	66	26	1									
7		1	57	302	302	57	1								
8		1	120	1191	2416	1191	120	1							
9		1	247	4293	15619	15619	4293	247	1						
10		1	502	14,608	88,234	156,190	88234	14608	502	1					
11		1	1013	47,840	455,192	1,310,354	1,310,354	455192	47840	1013	1				
12		1	2036	152,637	2,203,488	9,738,114	15,724,248	9738114	2203488	152637	2036	1			
13		1	4083	478,271	10,187,685	66,318,474	162,512,286	162512286	66318474	10187685	478271	4083	1		
14		1	8178	1479726	45533450	423281535	1505621508	2275172004	1505621508	423281535	45533450	1479726	8178	1	
15		1	16369	4537314	198410786	2571742175	12843262863	27971176092	27971176092	12843262863	2571742175	198410786	4537314	16369	1

then the output function with unit step input is

$$\bar{y} = \frac{1}{s}\,\frac{1}{1 + \sqrt{2}s + s^2} \tag{20}$$

$$y(0) = \dot{y}(0) = 0$$

Rearranging this equation and dividing by s^2 gives

$$(1 + \frac{\sqrt{2}}{s} + \frac{1}{s^2})\bar{y} = \frac{1}{s^3} \tag{21}$$

Taking the Z-transform of both sides yields

$$Z\left[(1 + \frac{2}{s} + \frac{1}{s^2})\bar{y}\right] = Z(\frac{1}{s^3}) \tag{22}$$

Then, by employing both the Naumov program and the approximate Z-transform (3) of convolution, $(\frac{1}{s^n}\bar{y})$, gives

$$Z(\frac{\sqrt{2}}{s}\bar{y}) = \frac{\sqrt{2}(1+z)}{2(1-z)}\,Z\bar{y} \tag{23a}$$

$$Z(\frac{1}{s^2}\bar{y}) = \frac{T^2 z}{(1-z)^2}\,Z\bar{y} \tag{23b}$$

$$Z(\frac{1}{s^3}) = \frac{T^2 z(1+z)}{2(1-z)^3} \tag{23c}$$

Substituting equation (23) into equation (22) and then multiplying by $(1-z)^2$ yields

$$\left[z^2(1 - \frac{\sqrt{2}}{2}) + z(T^2 - 2) + (\frac{\sqrt{2}}{2}\,T + 1)\right] Z\bar{y}$$

$$= \frac{T^2 z(1+z)}{2(1-z)} \tag{24}$$

The corresponding recurrence relation is

$$y_n = \frac{2(2-T^2)}{\sqrt{2}T + 2} y_{n-1} + \frac{\sqrt{2}T-2}{\sqrt{2}T+2} y_{n-2} + \frac{T^2 x_n}{\sqrt{2}T+2} \tag{25}$$

$$x_n = \{0, 1, 2, 2, 2, 2, 2, \ldots\ldots\ldots\ldots, 2, \ldots\ldots\}$$

Let $A=2(2 - T^2)$

$B=\sqrt{2}T - 2$

$C=T^2$

$D=\sqrt{2}T - 2$

Then $y_n=\left[(Ay_{n-1} + By_{n-2})/D\right] + (C/D)x_n$ (26)

When n=0 $\quad y_0=0;$

$n=1 \quad y_1=C/D;$

$n\geq 2 \quad y_n=(Ay_{n-1} + By_{n-2} + 2C)/D.$

If the sampling interval T is chosen as 0.1 second, then the graphical and numerical solutions of y(t) are shown in Fig. 2 and Table 5 respectively. It is noted that the graphical solution is based on numerical calculations.

Third Order Digital Butterworth Filter

The third order Butterworth filter is

$$B_3(s) = \frac{1}{1 + 2s + 2s^2 + s^3} \tag{27}$$

Following the procedures and assumptions of the second order case yields

$$Z\left[\left(\frac{1}{s^3} + \frac{2}{s^2} + \frac{2}{s} + 1\right)\bar{y}\right] = Z\left(\frac{1}{s^4}\right) \tag{28}$$

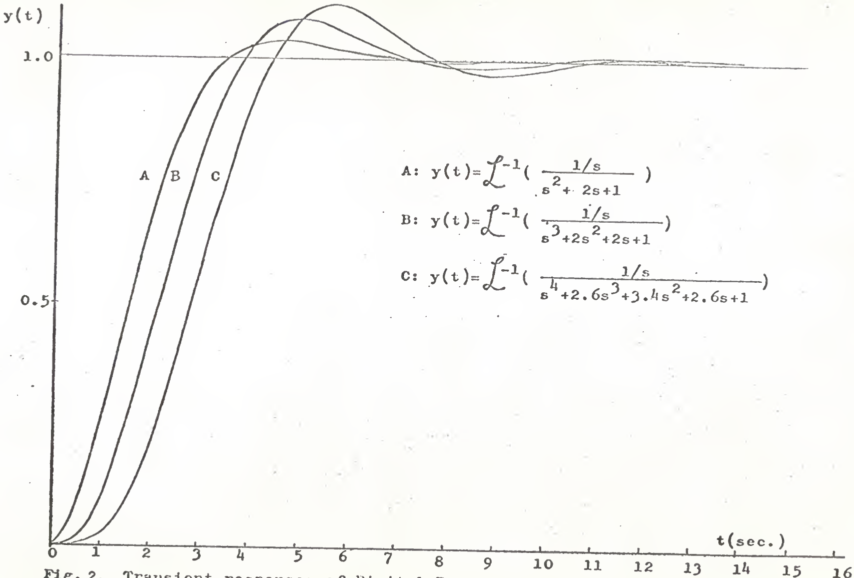

Fig. 2. Transient responses of Digital Butterworth filters with unit step input.

Table 5. $y(t) = \mathcal{L}^{-1}\left(\frac{1/s}{s^2+\sqrt{2}s + 1} \right), y(0) = 0, T = 0.1$

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	2.7	0.9096	5.4	1.0307
0.1	0.0047	2.8	0.9385	5.5	1.0288
0.2	0.0180	2.9	0.9455	5.6	1.0268
0.3	0.0338	3.0	0.9607	5.7	1.0248
0.4	0.0685	3.1	0.9743	5.8	1.0229
0.5	0.0979	3.2	0.9864	5.9	1.0210
0.6	0.1343	3.3	0.9970	6.0	1.0192
0.7	0.1739	3.4	1.0062	6.1	1.0174
0.8	0.2160	3.5	1.0141	6.2	1.0157
0.9	0.2598	3.6	1.0209	6.3	1.0141
1.0	0.3048	3.7	1.0266	6.4	1.0126
1.1	0.3504	3.8	1.0313	6.5	1.0111
1.2	0.3959	3.9	1.0350	6.6	1.0097
1.3	0.4411	4.0	1.0380	6.7	1.0085
1.4	0.4856	4.1	1.0402	6.8	1.0073
1.5	0.5290	4.2	1.0417	6.9	1.0062
1.6	0.5710	4.3	1.0426	7.0	1.0052
1.7	0.6116	4.4	1.0431	7.1	1.0042
1.8	0.6513	4.5	1.0430	7.2	1.0034
1.9	0.6873	4.6	1.0426	7.3	1.0026
2.0	0.7222	4.7	1.0418	7.4	1.0019
2.1	0.7552	4.8	1.0408	7.5	1.0013
2.2	0.7861	4.9	1.0394	7.6	1.0008
2.3	0.8149	5.0	1.0379	7.7	1.0003
2.4	0.8416	5.1	1.0363	7.8	0.9999
2.5	0.8663	5.2	1.0345	7.9	0.9995
2.6	0.8889	5.3	1.0326		

The Naumov program gives

$$Z(\tfrac{2}{s}\bar{y}) = \frac{(1+z)T}{(1-z)}Z\bar{y} \tag{29a}$$

$$Z(\tfrac{2}{s^2}\bar{y}) = \frac{2T^2z}{(1-z)^2}Z\bar{y} \tag{29b}$$

$$Z(\tfrac{1}{s^3}\bar{y}) = \frac{T^3z(1+z)}{2(1-z)^3}Z\bar{y} \tag{29c}$$

$$Z(\tfrac{1}{s^4}) = \frac{T^3z}{6}\,\frac{z\ +4z+1}{(1-z)^4} \tag{29d}$$

Then

$$Z\bar{y}\left[(T-1)z^3+(3-T-2T^2+0.5T^3)z^2+(0.5T^3+2T^2-T-3)z+(T+1)\right]$$
$$= \frac{T^3}{6}\,\frac{z(z^2+4z+1)}{(1-z)} \tag{30}$$

Let

$$A = (3+T-2T^2-0.5T^3)$$

$$B = (T+2T^2-0.5T^3-3)$$

$$C = 1-T$$

$$D = T^3$$

$$R = 1+T$$

Then the recurrence relation is

$$y_n = \left[(Ay_{n-1}+By_{n-2}+Cy_{n-3})/R\right] + (\tfrac{D}{6R})x_n \tag{31}$$

$$x = \left\{0,\ 1,\ 5,\ 6,\ 6,\ 6,\ \ldots\ldots\ 6,\ \ldots\right\}$$

When

$n = 0 \qquad y_0 = 0;$

$n = 1 \qquad y_1 = \frac{D}{6R};$

$n = 2 \qquad y_2 = \frac{A}{R}y_1 + \frac{5D}{6R};$

$$n \geq 3 \qquad y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}D)/R.$$

Results are presented in Fig. 2 and Table 6.

Fourth Order Digital Butterworth Filter

The fourth order Butterworth filter is

$$G_4(s) = \frac{1}{s^4+2.6s^3+3.4s^2+2.6s+1} \tag{32}$$

Calculation yields

$$Z\left[\left(\frac{1}{s^4}+\frac{2.6}{s^3}+\frac{3.4}{s^2}+\frac{2.6}{s}+1\right)\bar{y}\right] = Z\left(\frac{1}{s^5}\right) \tag{33}$$

The Naumov program gives

$$Z\left(\frac{2.6}{s^2}\bar{y}\right) = \frac{1.3T(1+z)}{(1-z)}\, Z\bar{y} \tag{34a}$$

$$Z\left(\frac{3.4}{s^3}\bar{y}\right) = \frac{3.4T^2z}{(1-z)^2}\, Z\bar{y} \tag{34b}$$

$$Z\left(\frac{2.6}{s^3}\bar{y}\right) = \frac{1.3T^3z(1+z)}{(1-z)^3}\, Z\bar{y} \tag{34c}$$

$$Z\left(\frac{1}{s^4}\bar{y}\right) = \frac{T^4z(z^2+4z+1)}{6(1-z)^4}\, Z\bar{y} \tag{34d}$$

$$Z\left(\frac{1}{s^5}\right) = \frac{T^4z(z^3+11z^2+11z+1)}{24(1-z)^5} \tag{34e}$$

Let $\quad W = \dfrac{T^4}{6}$

$$E = \frac{T^4}{24}$$

Table 6. $y(t) = \mathcal{L}^{-1}\left(\frac{1}{s(s^3+2s^2+2s+1)}\right)$, $y(0)=0$, $T=0.1$ sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	3.2	0.8750	6.4	1.0300
0.1	0.0002	3.3	0.9012	6.5	1.0256
0.2	0.0012	3.4	0.9254	6.6	1.0214
0.3	0.0038	3.5	0.9477	6.7	1.0174
0.4	0.0086	3.6	0.9681	6.8	1.0137
0.5	0.0160	3.7	0.9866	6.9	1.0101
0.6	0.0263	3.8	1.0033	7.0	1.0067
0.7	0.0397	3.9	1.0181	7.1	1.0036
0.8	0.0562	4.0	1.0311	7.2	1.0008
0.9	0.0758	4.1	1.0424	7.3	0.9982
1.0	0.0985	4.2	1.0521	7.4	0.9959
1.1	0.1242	4.3	1.0602	7.5	0.9938
1.2	0.1525	4.4	1.0668	7.6	0.9920
1.3	0.1834	4.5	1.0721	7.7	0.9904
1.4	0.2165	4.6	1.0760	7.8	0.9891
1.5	0.2515	4.7	1.0788	7.9	0.9890
1.6	0.2882	4.8	1.0805	8.0	0.9871
1.7	0.3263	4.9	1.0811	8.1	0.9864
1.8	0.3654	5.0	1.0809	8.2	0.9859
1.9	0.4053	5.1	1.0798	8.3	0.9856
2.0	0.4465	5.2	1.0781	8.4	0.9854
2.1	0.4861	5.3	1.0757	8.5	0.9854
2.2	0.5264	5.4	1.0727	8.6	0.9856
2.3	0.5663	5.5	1.0693	8.7	0.9858
2.4	0.6056	5.6	1.0656	8.8	0.9862
2.5	0.6441	5.7	1.0615	8.9	0.9867
2.6	0.6815	5.8	1.0572	9.0	0.9872
2.7	0.7176	5.9	1.0528	9.1	0.9878
2.8	0.7524	6.0	1.0482	9.2	0.9885
2.9	0.7856	6.1	1.0436	9.3	0.9892
3.0	0.8172	6.2	1.0390	9.4	0.9900
3.1	0.8470	6.3	1.0344	9.5	0.9907

Substituting equation (34) into equation (33) yields

$$\bar{Zy}\left[(1+1.3T)+z(-4-2.6T+3.4T^2+1.3T^3+W) + z^2(6-6.8T^2+4W) + z^3(-4+2.6T+3.4T^2-1.3T^3+W) + z^4(1-1.3T)\right] = \frac{Ez(z^3+11z^2+11z+1)}{(1-z)} \tag{35}$$

Let

$$A = 4+2.6T-3.4T^2-1.3T^3-W$$

$$B = 6.8T^2-4W-6$$

$$C = 4-2.6T-3.4T^2+1.3T^3-W$$

$$D = 1.3T-1$$

$$F = 1+1.3T$$

Then

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4})}{F} + \left(\frac{Ex_n}{F}\right) \tag{36}$$

$$x_n = \left\{0,\ 1,\ 12,\ 23,\ 24,\ 24,\ 24,\ \ldots\ldots\ 24,\ \ldots\ldots\right\}$$

When

$n = 0$ $\quad y_0 = 0;$

$n = 1$ $\quad y_1 = \frac{E}{F};$

$n = 2$ $\quad y_2 = (Ay_1+12E)/F;$

$n = 3$ $\quad y_3 = (Ay_2+By_1+23E)/F;$

$n \geqslant 4$ $\quad y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+24E)/F.$

Results are shown both in Table 7 and Fig. 2.

Table 7. $y(t) = \mathcal{L}^{-1}\left(\frac{1/s}{s^4+2.6s^3+3.4s^2+2.6s+1}\right)$, y(0)=0, T=0.1 sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.5	1.0226	9.0	0.9714
0.1	0.0000	4.6	1.0394	9.1	0.9717
0.2	0.0000	4.7	1.0543	9.2	0.9723
0.3	0.0003	4.8	1.0673	9.3	0.9732
0.4	0.0009	4.9	1.0785	9.4	0.9742
0.5	0.0020	5.0	1.0880	9.5	0.9754
0.6	0.0039	5.1	1.0957	9.6	0.9768
0.7	0.0068	5.2	1.1019	9.7	0.9784
0.8	0.0111	5.3	1.1064	9.8	0.9800
0.9	0.0168	5.4	1.1096	9.9	0.9817
1.0	0.0242	5.5	1.1113	10.0	0.9835
1.1	0.0334	5.6	1.1118	10.1	0.9854
1.2	0.0447	5.7	1.1111	10.2	0.9872
1.3	0.0581	5.8	1.1093	10.3	0.9891
1.4	0.0738	5.9	1.1066	10.4	0.9909
1.5	0.0917	6.0	1.1030	10.5	0.9927
1.6	0.1118	6.1	1.0987	10.6	0.9954
1.7	0.1342	6.2	1.0937	10.7	0.9962
1.8	0.1588	6.3	1.0882	10.8	0.9978
1.9	0.1855	6.4	1.0822	10.9	0.9994
2.0	0.2142	6.5	1.0759	11.0	1.0008
2.1	0.2447	6.6	1.0693	11.1	1.0022
2.2	0.2770	6.7	1.0626	11.2	1.0034
2.3	0.3107	6.8	1.0557	11.3	1.0045
2.4	0.3457	6.9	1.0488	11.4	1.0056
2.5	0.3818	7.0	1.0420	11.5	1.0065
2.6	0.4189	7.1	1.0353	11.6	1.0073
2.7	0.4565	7.2	1.0287	11.7	1.0079
2.8	0.4946	7.3	1.0224	11.8	1.0085
2.9	0.5330	7.4	1.0163	11.9	1.0090
3.0	0.5713	7.5	1.0106	12.0	1.0093
3.1	0.6093	7.6	1.0052	12.1	1.0095
3.2	0.6469	7.7	1.0001	12.2	1.0097
3.3	0.6839	7.8	0.9955	12.3	1.0098
3.4	0.7200	7.9	0.9912	12.4	1.0097
3.5	0.7551	8.0	0.9874	12.5	1.0096
3.6	0.7891	8.1	0.9840	12.6	1.0094
3.7	0.8217	8.2	0.9810	12.7	1.0092
3.8	0.8528	8.3	0.9785	12.8	1.0089
3.9	0.8824	8.4	0.9763	12.9	1.0085
4.0	0.9103	8.5	0.9746	13.0	1.0081
4.1	0.9365	8.6	0.9732	13.1	1.0077
4.2	0.9608	8.7	0.9723	13.2	1.0072
4.3	0.9833	8.8	0.9717	13.3	1.0068
4.4	1.0039	8.9	0.9714	13.4	1.0063

DIGITAL FILTER COMPENSATION

Posicast control compensation can only be applied when the damping is small and introduces large time delay in the transient response; as the order of the system becomes higher, posicast compensation always results in large undershoots in the transient response. For the sake of reducing these defects, "digital filter compensation" is employed.

The basic idea of digital filter compensation is conversion the unit step function input into the double half-step function displayed in Fig. 3(b). The second half-step input follows the first half-step function after a time mT seconds, where mT is the time duration of the first overshoot of the Butterworth response to a unit step function. This is clarified in Fig. 3(a).

The symbol T is the duration between sampling times, and m is a constant to be determined for each Butterworth filter. Therefore a pre-compensating operater

$$\frac{1 - e^{-msT}}{2} = \frac{1 - z^{m}}{2} \tag{37}$$

can be constructed. The block diagram of the system with this compensator is shown in Fig. 7(a).

Where X, Y and G(s) are the input, output and transfer (Butterworth filter) functions respectively. This completes the first step of compensation. This step reduces most of the overshoot of the original transient response but as happens in posicast control compensation, there now exists larger time delay which is much greater than that in the original system. Hence, the next step is to reduce time delay.

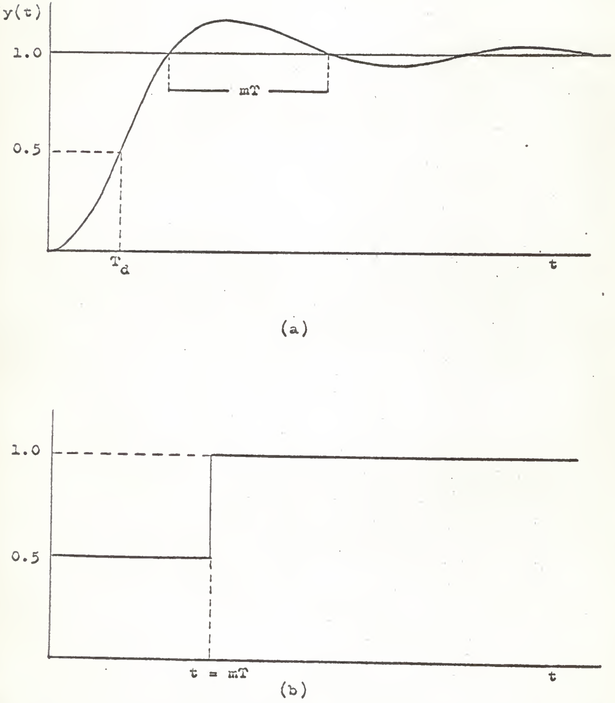

Fig. 3. Principle of double step input function of digital filter compensation

Assume that the delay time in the original system is T_{do} and after first step compensation is T_{dl}. Since T_{dl} is greater than T_{do}, the ratio T_{do}/T_{dl} is some value less than 1. The procedure is to replace s by $(T_{do}/T_{dl})s$ in the compensated Butterworth polynomial. This replacement of s will renormalize time delay since T_{do}/T_{dl} is less than 1, this meets the second requirement; so, the goal of compensation is accomplished.

Transient behaviors of several orders of Butterworth filter by applying digital filter compensation will be investigated in the next section.

Compensation of Second Order Digital Butterworth Filter

A. First step of compensation

The time duration mT of the first overshoot of $G_2(s)$ with unit step input should be known; this can be found from Table 5 where

$$T = 0.1 \text{ sec.}$$

$$mT \doteq 7.8-3.4 = 4.4 \text{ sec.}$$

Therefore m = 44.

Then the system response with unit step input is shown in Fig. 7(b).

Thus

$$\bar{y} = \left(\frac{1}{s}\right)\left(\frac{1 + z^{44}}{2}\right)\left(\frac{1}{s^2 + \sqrt{2}s + 1}\right) \tag{38}$$

Rearranging equation (38) by leaving $\left(\frac{1}{s}\right)\left(\frac{1 + z^{44}}{2}\right)$ at one side yields

$$(s^2 + \sqrt{2}s + 1)\bar{y} = \left(\frac{1}{s}\right)\frac{1 + z^{44}}{2} \tag{39}$$

Taking the Z-transform yields after division by s^2 at both sides

$$Z\left(1 + \frac{\sqrt{2}}{s} + \frac{1}{s^2}\right)\bar{y} = Z\ \frac{1}{s^3}\left(\frac{1 + z^{44}}{2}\right) \tag{40}$$

After applying the Naumov program this equation yields

$$z\bar{y}\left[(1-\frac{\sqrt{2}}{2}T)z^2+(T^2-2)z+(1+\frac{\sqrt{2}}{2}T)\right]$$

$$=\frac{T^2z(1+z)(1+z^{44})}{4(1-z)} \qquad (41)$$

Let $A = 2(2-T^2)$

$B = \sqrt{2}T-2$

$C = T^2$

$D = \sqrt{2}T+2$

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2})}{D} + (\frac{C}{2D})x_n \qquad (42)$$

$$x_n = 0,\ 1,\ 2,\ \ldots\ 2,\ 3,\ 4,\ 4,\ \ldots\ 4,\ \ldots$$

This equation yields

$y_0 = 0$ $\qquad n = 0$

$y_1 = \frac{C}{2D}$ $\qquad n = 1$

$y_n = (Ay_{n-1}+By_{n-2}+C)/D$ $\qquad 2 \leq n \leq 44$

$y_{45} = (Ay_{44}+By_{43}+1.5C)/D$ $\qquad n = 45$

$y_n = (Ay_{n-1}+By_{n-2}+2C)/D$ $\qquad n \geq 46$

The numerical and graphical results of $y(t)$ are presented in Table 8 and Fig. 4 respectively.

B. The second step of compensation

From Table 5 and Table 8, the delay times which are found before and after the first step compensation are

$T_{do} = 1.43$ sec.

$T_{d1} = 3.35$ sec.

Table 8. $y(t)=\mathcal{L}^{-1}\left(\frac{1+z^{44}}{2}\cdot\frac{1/s}{s^2+2s+1}\right)$, y(0)=0, T=0.1 sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.2	0.5209	8.5	1.0192
0.1	0.0023	4.3	0.5213	8.6	1.0200
0.2	0.0090	4.4	0.5215	8.7	1.0204
0.3	0.0194	4.5	0.5239	8.8	1.0206
0.4	0.0329	4.6	0.5303	8.9	1.0206
0.5	0.0490	4.7	0.5403	9.0	1.0204
0.6	0.0671	4.8	0.5533	9.1	1.0200
0.7	0.0896	4.9	0.5687	9.2	1.0195
0.8	0.1080	5.0	0.5861	9.3	1.0189
0.9	0.1299	5.1	0.6051	9.4	1.0181
1.0	0.1524	5.2	0.6252	9.5	1.0173
1.1	0.1752	5.3	0.6462	9.6	1.0165
1.2	0.1980	5.4	0.6678	9.7	1.0156
1.3	0.2206	5.5	0.6896	9.8	1.0147
1.4	0.2428	5.6	0.7114	9.9	1.0137
1.5	0.2645	5.7	0.7330	10.0	1.0128
1.6	0.2855	5.8	0.7543	10.1	1.0119
1.7	0.3056	5.9	0.7750	10.2	1.0109
1.8	0.3252	6.0	0.7951	10.3	1.0100
1.9	0.3436	6.1	0.8145	10.4	1.0092
2.0	0.3611	6.2	0.8330	10.5	1.0083
2.1	0.3776	6.3	0.8507	10.6	1.0075
2.2	0.3930	6.4	0.8674	10.7	1.0067
2.3	0.4074	6.5	0.8831	10.8	1.0060
2.4	0.4208	6.6	0.8979	10.9	1.0053
2.5	0.4331	6.7	0.9117	11.0	1.0046
2.6	0.4445	6.8	0.9244	11.1	1.0040
2.7	0.4548	6.9	0.9362	11.2	1.0035
2.8	0.4642	7.0	0.9470	11.3	1.0029
2.9	0.4727	7.1	0.9569	11.4	1.0025
3.0	0.4804	7.2	0.9659	11.5	1.0020
3.1	0.4872	7.3	0.9740	11.6	1.0016
3.2	0.4932	7.4	0.9813	11.7	1.0013
3.3	0.4985	7.5	0.9878	11.8	1.0009
3.4	0.5031	7.6	0.9936	11.9	1.0006
3.5	0.5071	7.7	0.9986	12.0	1.0004
3.6	0.5105	7.8	1.0030	12.1	1.0001
3.7	0.5133	7.9	1.0068	12.2	0.9999
3.8	0.5156	8.0	1.0101	12.3	0.9998
3.9	0.5175	8.1	1.0128	12.4	0.9996
4.0	0.5190	8.2	1.0150	12.5	0.9995
4.1	0.5201	8.3	1.0168		

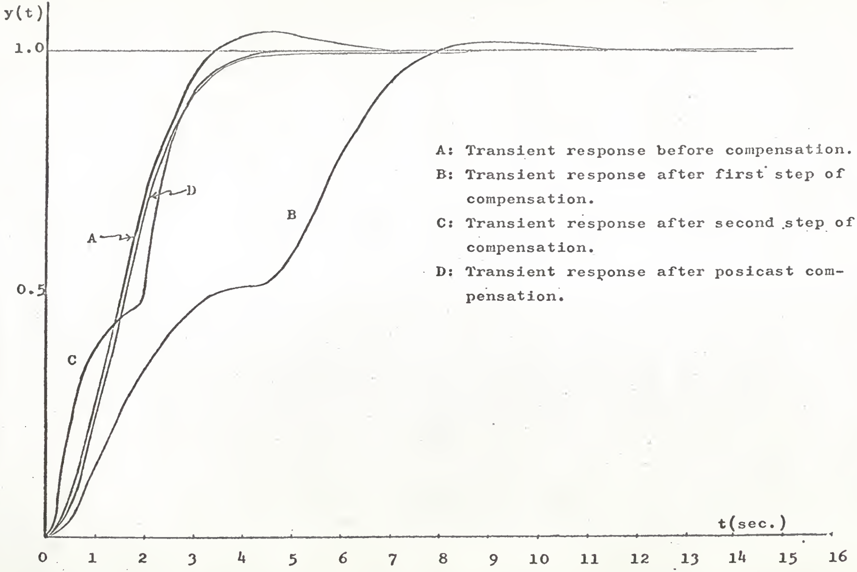

Fig. 4. Compensation of second order digital Butterworth filter with unit step function input.

Then

$$\frac{T_{do}}{T_{d1}} = \frac{1.43}{3.35} = 0.427$$

Replacing s by 0.427s in the transfer function $(\frac{1 + z^{44}}{2})(\frac{1}{1 + \sqrt{2}s + s^2})$ of equation (38) yields

$$\bar{y} = (\frac{1}{s})(\frac{1 + z^{19}}{2})(\frac{1}{1+0.605+0.0181 s^2}) \tag{43}$$

Rearranging and taking the Z-transform of both sides yields

$$Z\left[(\frac{1}{s^2} + \frac{0.605}{s} + 0.0181)\bar{y}\right] = Z\left[\frac{1}{s^3}(\frac{1 + z^{19}}{2})\right] \tag{44}$$

Application of the Naumov program yields

$$Z\bar{y}\left[(0.0181 + 0.303T) + (T^2-0.0362)z + (0.0181-0.303T)z^2\right]$$

$$= \frac{T^2 z(1+z)(1+z^{19})}{4(1-z)} \tag{45}$$

Let $A = 0.0362-T^2$

$B = 0.303T-0.0181$

$C = T^2$

$D = 0.303T + 0.0181$

Then, the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-1})}{D} + (\frac{0.25C}{D})x_n \tag{46}$$

$$x_n = \left\{0,\ 1,\ 2,\ 2,\ \ldots\ 2,\ 3,\ 4,\ 4,\ \ldots,\ 4\right\}$$

This equation yields

$y_0 = 0$ $\quad$ $n = 0$

$y_1 = 0.25C/D$ $\quad$ $n = 1$

$y_n = (Ay_{n-1}+By_{n-2}+0.5C)/D$ $\quad$ $2 \leq n \leq 19$

$y_{20} = (Ay_{19}+By_{18}+0.75C)/D$ $\quad$ $n = 20$

$$y_n = (Ay_{n-1}+By_{n-2}+C)/D \qquad n \geq 21$$

The graphical and numerical results are shown in Fig. 4 and Table 9 respectively. In Fig. 4, we can see that the transient response after the second step of compensation is better than the original Butterworth filter response. The maximum overshoot is reduced from 0.0431 to none but the delay time increases to 0.52 seconds.

Compensation of Third Order Digital Butterworth Filter

A. First step of compensation

Table 6 gives $m = 35$

Thus

$$\bar{y} = \left(\frac{1}{s}\right)\left(\frac{1+z^{35}}{2}\right)\left(\frac{1}{s^3+2s^2+2s+1}\right) \tag{47}$$

The block diagram of this system is shown in Fig. 7(c).

Calculation yields

$$Z\bar{y}\left[z^3(T-1) + z^2(3-T-2T^2+0.5T^3) + z(0.5T^3+2T^2-T-3) + (T+1)\right]$$
$$= \frac{T^3 z(z^2+4z+1)(1+z^{35})}{12(1-z)} \tag{48}$$

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3})}{R} + \frac{D}{12R}x_n \tag{49}$$

$$x_n = \left\{0,\ 1,\ 5,\ 6,\ 6,\ \ldots\ldots,\ 6,\ 7,\ 11,\ 12,\ 12,\ \ldots\ldots,\ 12,\ \ldots\ldots\right\}$$

Where

$$A = 3+T-2T^2-0.5T^3$$

$$B = T+2T^2-0.5T^3-3$$

$$C = 1-T$$

$$D = T^3$$

$$R = 1+T$$

Table 9. $y(t)=\mathcal{L}^{-1}\left(\frac{1+z^{19}}{2}\cdot\frac{1/s}{1+2(0.427)s+(0.427)^2s^2}\right)$, $y(0)=0$, $T=0.1$ sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	2.7	0.8634	5.4	0.9987
0.1	0.0025	2.8	0.8851	5.5	0.9989
0.2	0.1047	2.9	0.9034	5.6	0.9991
0.3	0.1606	3.0	0.9188	5.7	0.9992
0.4	0.2166	3.1	0.9317	5.8	0.9994
0.5	0.2610	3.2	0.9425	5.9	0.9995
0.6	0.2992	3.3	0.9517	6.0	0.9995
0.7	0.3311	3.4	0.9594	6.1	0.9996
0.8	0.3579	3.5	0.9658	6.2	0.9997
0.9	0.3805	3.6	0.9712	6.3	0.9997
1.0	0.3995	3.7	0.9758	6.4	0.9998
1.1	0.4155	3.8	0.9797	6.5	0.9998
1.2	0.4289	3.9	0.9829	6.6	0.9999
1.3	0.4402	4.0	0.9856	6.7	0.9999
1.4	0.4497	4.1	0.9879	6.8	0.9999
1.5	0.4577	4.2	0.9898	6.9	0.9999
1.6	0.4644	4.3	0.9914	7.0	0.9999
1.7	0.4701	4.4	0.9928	7.1	0.9999
1.8	0.4748	4.5	0.9939	7.2	0.9999
1.9	0.4788	4.6	0.9949	7.3	1.0000
2.0	0.5339	4.7	0.9957	7.4	1.0000
2.1	0.6163	4.8	0.9964	7.5	1.0000
2.2	0.6748	4.9	0.9970	7.6	1.0000
2.3	0.7272	5.0	0.9975	7.7	1.0000
2.4	0.7704	5.1	0.9979	7.8	1.0000
2.5	0.8069	5.2	0.9982	7.9	1.0000
2.6	0.8376	5.3	0.9985	8.0	1.0000

Equation (49) yields

$$y_0 = 0 \qquad n = 0$$

$$y_1 = D/12R \qquad n = 1$$

$$y_2 = (Ay_1/R) + (5D/12R) \qquad n = 2$$

$$y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+0.5D)/R \qquad 3 \leqslant n \leqslant 35$$

$$y_{36} = (Ay_{35}+By_{34}+Cy_{33})/R + 7R/12R \qquad n = 36$$

$$y_{37} = (Ay_{36}+By_{35}+Cy_{34})/R + 11D/12R \qquad n = 37$$

$$y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+D)/R \qquad n \geqslant 38$$

Results are presented in Table 10 and Fig. 5.

B. Second step of compensation

From Table 6 and Table 10, we get

$$T_{do} = 2.157 \text{ sec.}$$

$$T_{dl} = 3.75 \text{ sec.}$$

Then

$$\frac{T_{do}}{T_{dl}} = \frac{2.157}{3.75} = 0.573$$

Replacing s by 0.573s in the transfer function of equation (47) yields

$$\bar{y} = \left(\frac{1}{s}\right)\left(\frac{1+z^{20}}{2}\right)\left(\frac{1}{0.188s^3+0.564s^2+1.146s+1}\right) \tag{50}$$

Applying the same process as in second order case yields

$$Z\bar{y}\Big[(0.188+0.327T)+(1.146T^2-0.327T-0.564+0.5T^3)z$$
$$+(0.106-0.327T-1.146T^2+0.5T^3)z^2+(0.327T-0.188)z^3\Big]$$
$$= \frac{T^3z(z^2+4z+1)(1+z^{20})}{12(1-z)} \tag{51}$$

Table 10. $y(t)=\mathcal{L}^{-1}\left(\frac{1+z^{35}}{2}\cdot\frac{1/s}{s^3+2s^2+2s+1}\right)$, $y(0)=0$, $T=0.1$ sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.5	0.5853	9.0	1.0282
0.1	0.0000	4.6	0.6001	9.1	1.0267
0.2	0.0006	4.7	0.6157	9.2	1.0250
0.3	0.0019	4.8	0.6319	9.3	1.0232
0.4	0.0043	4.9	0.6488	9.4	1.0213
0.5	0.0080	5.0	0.6662	9.5	1.0194
0.6	0.0132	5.1	0.6840	9.6	1.0175
0.7	0.0199	5.2	0.7022	9.7	1.0156
0.8	0.0281	5.3	0.7206	9.8	1.0137
0.9	0.0379	5.4	0.7390	9.9	1.0118
1.0	0.0432	5.5	0.7575	10.0	1.0101
1.1	0.0621	5.6	0.7758	10.1	1.0083
1.2	0.0763	5.7	0.7939	10.2	1.0067
1.3	0.0917	5.8	0.8118	10.3	1.0051
1.4	0.1082	5.9	0.8292	10.4	1.0037
1.5	0.1258	6.0	0.8461	10.5	1.0023
1.6	0.1441	6.1	0.8625	10.6	1.0010
1.7	0.1632	6.2	0.8783	10.7	0.9999
1.8	0.1827	6.3	0.8934	10.8	0.9988
1.9	0.2027	6.4	0.9078	10.9	0.9979
2.0	0.2228	6.5	0.9214	11.0	0.9970
2.1	0.2430	6.6	0.9342	11.1	0.9963
2.2	0.2632	6.7	0.9462	11.2	0.9957
2.3	0.2832	6.8	0.9574	11.3	0.9952
2.4	0.3028	6.9	0.9677	11.4	0.9947
2.5	0.3220	7.0	0.9772	11.5	0.9944
2.6	0.3407	7.1	0.9859	11.6	0.9941
2.7	0.3588	7.2	0.9937	11.7	0.9939
2.8	0.3762	7.3	1.0007	11.8	0.9938
2.9	0.3928	7.4	1.0070	11.9	0.9938
3.0	0.4086	7.5	1.0125	12.0	0.9938
3.1	0.4235	7.6	1.0172	12.1	0.9939
3.2	0.4375	7.7	1.0212	12.2	0.9940
3.3	0.4506	7.8	1.0246	12.3	0.9942
3.4	0.4627	7.9	1.0274	12.4	0.9944
3.5	0.4739	8.0	1.0296	12.5	0.9946
3.6	0.4841	8.1	1.0312	12.6	0.9949
3.7	0.4939	8.2	1.0323	12.7	0.9952
3.8	0.5035	8.3	1.0330	12.8	0.9955
3.9	0.5134	8.4	1.0333	12.9	0.9958
4.0	0.5236	8.5	1.0331	13.0	0.9962
4.1	0.5344	8.6	1.0327	13.1	0.9965
4.2	0.5459	8.7	1.0319	13.2	0.9968
4.3	0.5582	8.8	1.0309	13.3	0.9972
4.4	0.5713	8.9	1.0297	13.4	0.9975

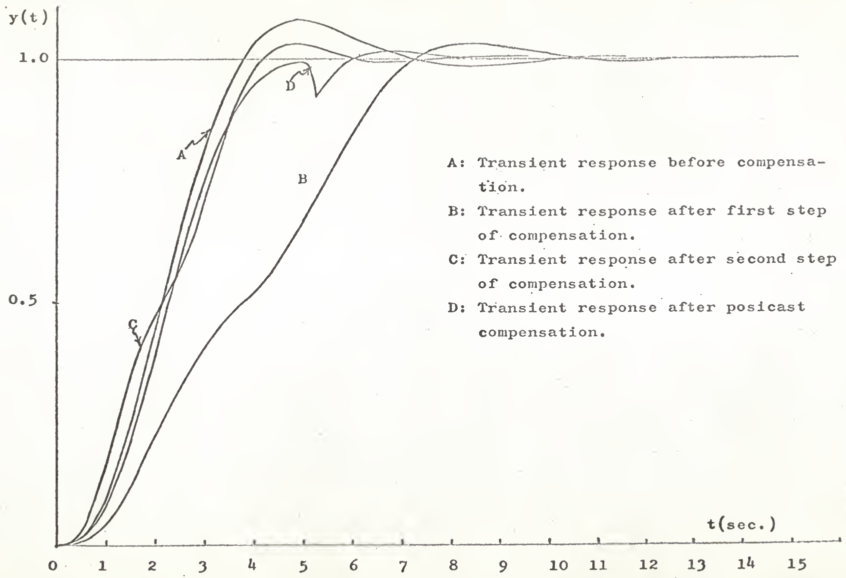

Fig. 5. Compensation of third order digital Butterworth filter with unit step function input.

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3})}{R} + \frac{DX_n}{12R} \qquad (52)$$

$$x_n = \left\{0,1,5,6,6,\ldots\ldots,6,7,11,12,12,\ldots\ldots 12,\ldots\ldots\right\}$$

Where

$$A = 0.564 + 0.327T-1.146T^2-0.5T^3$$

$$B = 0.327T + 1.146T^2-0.5T^3-0.564$$

$$C = 0.188-0.327T$$

$$D = T^3$$

$$R = 0.188 + 0.327T$$

Equation (52) yields

$$y_0 = 0 \qquad n = 0$$

$$y_1 = D/12R \qquad n = 1$$

$$y_2 = (Ay_{1/R}) + (5D/12R) \qquad n = 2$$

$$y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+0.5D)/R \qquad 3 \leqslant n \leqslant 20$$

$$y_{21} = (Ay_{20}+By_{19}+Cy_{18})/R \ +(7D/12R) \qquad n = 21$$

$$y_{22} = (Ay_{21}+By_{20}+Cy_{19})/R \ +(11D/12R) \qquad n = 22$$

$$y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+D)/R \qquad n \geqslant 23$$

The graphical and numerical results are presented in Fig. 5 and Table 11 respectively. The maximum overshoot reduces from 0.0811 to 0.0328 and the time delay reduces from 2.157 seconds to 2.15 seconds.

Compensation of Fourth Order Digital Butterworth Filter

A. First step of compensation

Table 7 gives $m = 33$.

Table 11. $y(t)=\mathcal{L}^{-1}\left(\frac{1+z^{20}}{2}\cdot\frac{1}{0.188s^3+0.654s^2+1.146s+1}\right)$,

$y(0)=0$, $T=0.1$ sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.2	1.0033	8.4	1.0005
0.1	0.0004	4.3	1.0131	8.5	1.0007
0.2	0.0029	4.4	1.0207	8.6	1.0008
0.3	0.0090	4.5	1.0262	8.7	1.0009
0.4	0.0196	4.6	1.0299	8.8	1.0009
0.5	0.0350	4.7	1.0320	8.9	1.0010
0.6	0.0552	4.8	1.0328	9.0	1.0009
0.7	0.0796	4.9	1.0324	9.1	1.0009
0.8	0.1078	5.0	1.0310	9.2	1.0008
0.9	0.1390	5.1	1.0289	9.3	1.0008
1.0	0.1725	5.2	1.0263	9.4	1.0007
1.1	0.2072	5.3	1.0233	9.5	1.0006
1.2	0.2426	5.4	1.0201	9.6	1.0005
1.3	0.2778	5.5	1.0168	9.7	1.0004
1.4	0.3120	5.6	1.0135	9.8	1.0003
1.5	0.3449	5.7	1.0104	9.9	1.0002
1.6	0.3757	5.8	1.0074	10.0	1.0002
1.7	0.4043	5.9	1.0047	10.1	1.0001
1.8	0.4301	6.0	1.0023	10.2	1.0000
1.9	0.4532	6.1	1.0002	10.3	1.0000
2.0	0.4733	6.2	0.9984	10.4	0.9999
2.1	0.4909	6.3	0.9969	10.5	0.9999
2.2	0.5078	6.4	0.9958	10.6	0.9999
2.3	0.5255	6.5	0.9949	10.7	0.9999
2.4	0.5451	6.6	0.9944	10.8	0.9998
2.5	0.5671	6.7	0.9940	10.9	0.9998
2.6	0.5917	6.8	0.9939	11.0	0.9998
2.7	0.6187	6.9	0.9940	11.1	0.9998
2.8	0.6478	7.0	0.9942	11.2	0.9999
2.9	0.6785	7.1	0.9946	11.3	0.9999
3.0	0.7103	7.2	0.9950	11.4	0.9999
3.1	0.7427	7.3	0.9956	11.5	0.9999
3.2	0.7749	7.4	0.9961	11.6	0.9999
3.3	0.8064	7.5	0.9967	11.7	0.9999
3.4	0.8369	7.6	0.9973	11.8	0.9999
3.5	0.8657	7.7	0.9978	11.9	1.0000
3.6	0.8926	7.8	0.9983	12.0	1.0000
3.7	0.9173	7.9	0.9988	12.1	1.0000
3.8	0.9396	8.0	0.9993	12.2	1.0000
3.9	0.9594	8.1	0.9997	12.3	1.0000
4.0	0.9766	8.2	1.0000	12.4	1.0000
4.1	0.9912	8.3	1.0003	12.5	1.0000

Thus

$$\bar{y} = \left(\frac{1}{s}\right)\left(\frac{1 + z^{33}}{2}\right)\left(\frac{1}{s^4+2.6s^3+3.4s^2+2.6s+1}\right) \qquad (53)$$

The block diagram is shown in Fig. 7(d). Following the same procedures as in the previous cases yields

$$z\bar{y}\Big[(1+1.3T)+z(-4-2.6T+3.4T^2+1.3T^3+W)+z^2(6-6.8T^2+4W)+ z^3(-4+2.6T+3.4T^2-1.3T^3+W)+z^4(1-1.3T)\Big]$$

$$= \frac{T^4z(z^3+11z^2+11z+1)(1+z^{33})}{48(1-z)} \qquad (54)$$

where $W=T^4/6$

Therefore the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4})}{F} + \left(\frac{E}{F}\right) x_n \qquad (55)$$

$$x_n = \Big\{0,\ 1,\ 12,\ 23,\ 24,\ 24,\ \ldots\ldots,\ 24,\ 25,\ 36,\ 47,\ 48,\ 48,\ \ldots\ldots\ldots\Big\}$$

where

$$A = 4+2.6T+3.4T^2-1.3T^3-W$$

$$B = 6.8T^2-4W-6$$

$$C = 4-2.6T-3.4T^2+1.3T^3-W$$

$$D = 1.3T-1$$

$$E = \frac{T^4}{48}$$

$$F = 1+1.3T$$

Equation (55) yields

$y_0 = 0$ $\qquad n = 0$

$y_1 = E/F$ $\qquad n = 1$

$y_2 = (Ay_1+12E)/F$ $\qquad n = 2$

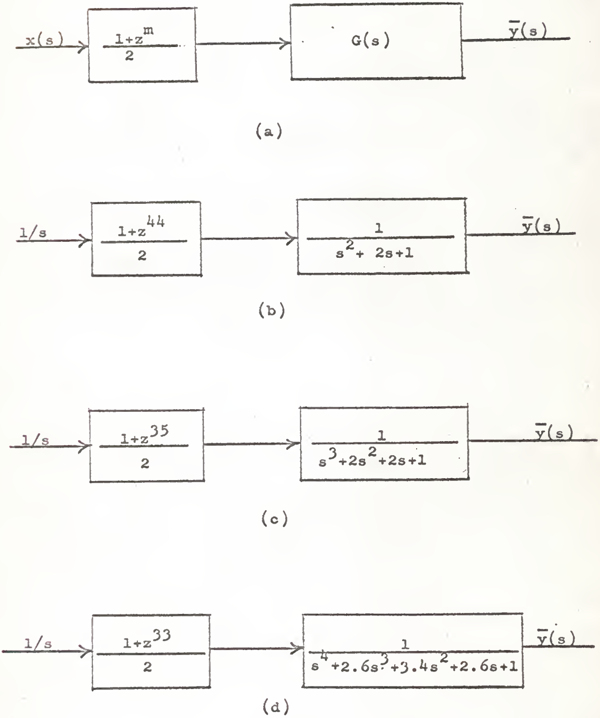

Fig. 7 . Digital Butterworth filters with pre-compensators

$$y_3 = \frac{Ay_2+By_1+23E}{F} \qquad n = 3$$

$$y_n = \frac{Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+24E}{F} \qquad 4 \leqslant n \leqslant 33$$

$$y_{34} = \frac{Ay_{33}+By_{32}+Cy_{31}+Dy_{30}+25E}{F} \qquad n = 34$$

$$y_{35} = \frac{Ay_{34}+By_{33}+Cy_{32}+Dy_{31}+36E}{F} \qquad n = 35$$

$$y_{36} = \frac{Ay_{35}+By_{34}+Cy_{33}+Dy_{32}+47E}{F} \qquad n = 36$$

$$y_n = \frac{Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+48E}{F} \qquad n \geqslant 37$$

Results are presented in Table 12 and Fig. 6.

B. Second step of compensation

From Table 7 and Table 12 it is found,

$$T_{do} = 2.85 \text{ sec.}$$

$$T_{d1} = 4.25 \text{ sec.}$$

Then

$$\frac{T_{do}}{T_{d1}} = \frac{2.85}{4.25} = 0.67$$

Replacing s by 0.67s in the transfer function of equation (53) yields

$$\bar{y} = \left(\frac{1}{s}\right)\left(\frac{1 + z^{22}}{2}\right)\left(\frac{1}{0.202s^4+0.782s^3+1.527s^2+1.742s+1}\right) \qquad (56)$$

Applying the same process as in the previous cases yields

$$Z\bar{y}\Big[(0.202+0.391T)+z(0.871T^3+1.527T^2-0.782T-0.808+W)$$
$$+z^2(1.212-3.045T^2+4W)+z^3(W-0.871T^3+1.527T^2+0.782T-0.808)$$
$$z^4(0.202-0.391T)\Big]$$

Table 12. $y(t)=\mathcal{L}^{-1}\left(\frac{1+z^{33}}{2}\cdot\frac{1/s}{s^4+2.6s^3+3.4s^2+2.6s+1}\right)$,

$y(0)=0$, $T=0.1$ sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.5	0.5336	9.0	1.0410
0.1	0.0000	4.6	0.5487	9.1	1.0403
0.2	0.0000	4.7	0.5639	9.2	1.0392
0.3	0.0001	4.8	0.5794	9.3	1.0378
0.4	0.0004	4.9	0.5951	9.4	1.0361
0.5	0.0010	5.0	0.6110	9.5	1.0342
0.6	0.0019	5.1	0.6271	9.6	1.0321
0.7	0.0034	5.2	0.6435	9.7	1.0298
0.8	0.0055	5.3	0.6601	9.8	1.0274
0.9	0.0084	5.4	0.6770	9.9	1.0250
1.0	0.0121	5.5	0.6939	10.0	1.0225
1.1	0.0167	5.6	0.7110	10.1	1.0199
1.2	0.0224	5.7	0.7282	10.2	1.0174
1.3	0.0291	5.8	0.7454	10.3	1.0149
1.4	0.0369	5.9	0.7625	10.4	1.0124
1.5	0.0458	6.0	0.7796	10.5	1.0100
1.6	0.0559	6.1	0.7965	10.6	1.0077
1.7	0.0671	6.2	0.8132	10.7	1.0055
1.8	0.0794	6.3	0.8296	10.8	1.0034
1.9	0.0928	6.4	0.8456	10.9	1.0015
2.0	0.1071	6.5	0.8613	11.0	0.9996
2.1	0.1224	6.6	0.8765	11.1	0.9980
2.2	0.1385	6.7	0.8912	11.2	0.9965
2.3	0.1553	6.8	0.9053	11.3	0.9951
2.4	0.1729	6.9	0.9189	11.4	0.9939
2.5	0.1909	7.0	0.9318	11.5	0.9928
2.6	0.2094	7.1	0.9440	11.6	0.9920
2.7	0.2283	7.2	0.9556	11.7	0.9912
2.8	0.2473	7.3	0.9664	11.8	0.9906
2.9	0.2665	7.4	0.9765	11.9	0.9902
3.0	0.2856	7.5	0.9585	12.0	0.9899
3.1	0.3047	7.6	0.9943	12.1	0.9897
3.2	0.3235	7.7	1.0021	12.2	0.9896
3.3	0.3419	7.8	1.0091	12.3	0.9896
3.4	0.3600	7.9	1.0154	12.4	0.9898
3.5	0.3776	8.0	1.0209	12.5	0.9900
3.6	0.3947	8.1	1.0257	12.6	0.9904
3.7	0.4113	8.2	1.0298	12.7	0.9908
3.8	0.4274	8.3	1.0332	12.8	0.9912
3.9	0.4431	8.4	1.0360	12.9	0.9917
4.0	0.4585	8.5	1.0382	13.0	0.9923
4.1	0.4737	8.6	1.0398	13.1	0.9929
4.2	0.4887	8.7	1.0408	13.2	0.9935
4.3	0.5037	8.8	1.0413	13.3	0.9942
4.4	0.5186	8.9	1.0414	13.4	0.9949

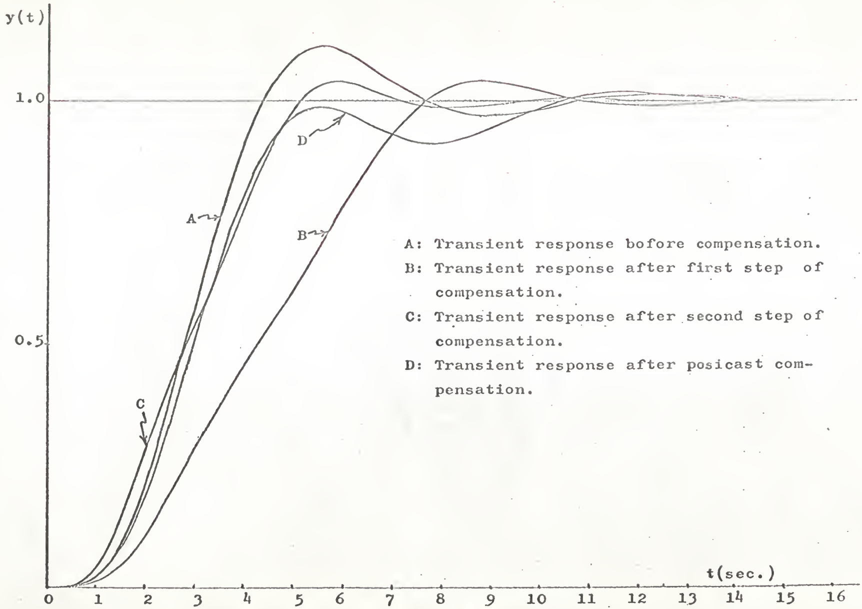

Fig. 6. Compensation of fourth order digital Butterworth filter with unit step function input.

$$= \frac{T^4 z(z^3+11z^2+11z+1)(1+z^{22})}{48(1-z)} \tag{57}$$

The recurrence relation is

$$y_n = \frac{Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}}{F} + \left(\frac{E}{F}\right) x_n \tag{58}$$

$$x_n = \left\{0,\ 1,\ 12,\ 23,\ 24,\ \ldots\ldots,\ 24,\ 25,\ 36,\ 47,\ 48,\ 48,\ \ldots \right.$$
$$\left. 48,\ \ldots\ldots\ldots \right\}$$

where

$$A = 0.808+0.782T-1.527T^2-0.871T^3-W$$

$$B = 3.054T^2-4W-1.212$$

$$C = 0.808-0.782T-1.527T^2+0.871T^3-W$$

$$D = 0.391T-0.202$$

$$E = T^4/48$$

$$W = T^4/6$$

$$F = 0.391T+0.202$$

Equation (58) yields

$$y_0 = 0 \qquad n = 0$$

$$y_1 = \frac{E}{F} \qquad n = 1$$

$$y_2 = \frac{Ay_1+12E}{F} \qquad n = 2$$

$$y_3 = \frac{Ay_2+By_1+23E}{F} \qquad n = 3$$

$$y_n = \frac{Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+24E}{F} \qquad 4 \leq n \leq 22$$

$$y_{23} = \frac{Ay_{22}+By_{21}+Cy_{20}+Dy_{19}+25E}{F} \qquad n = 23$$

$$y_{24} = \frac{Ay_{23}+By_{22}+Cy_{21}+Dy_{20}+36E}{F} \qquad n = 24$$

$$y_{25} = \frac{Ay_{24}+By_{23}+Cy_{22}+Dy_{21}+47E}{F} \qquad n = 25$$

$$y_n = \frac{Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+48E}{F} \qquad n \geqslant 26$$

Results are presented in Table 13 and Fig. 6. The maximum overshoot reduces from 0.1118 to 0.0401 and the time delay reduces from 2.85 seconds to 2.83 seconds.

Table 13. $y(t)=\mathcal{L}^{-1}\left(\frac{(1+z^{22})/2 \quad (1/s)}{0.202s^4+0.782s^3+1.527s^2+1.742s+1}\right), y(0)=0,$

T=0.1 sec.

t	y(t)	t	y(t)	t	y(t)
0.0	0.0000	4.4	0.8740	8.8	0.9920
0.1	0.0000	4.5	0.8958	8.9	0.9930
0.2	0.0001	4.6	0.9164	9.0	0.9940
0.3	0.0006	4.7	0.9355	9.1	0.9950
0.4	0.0019	4.8	0.9531	9.2	0.9960
0.5	0.0043	4.9	0.9691	9.3	0.9969
0.6	0.0082	5.0	0.9834	9.4	0.9978
0.7	0.0140	5.1	0.9960	9.5	0.9986
0.8	0.0219	5.2	1.0069	9.6	0.9994
0.9	0.0323	5.3	1.0161	9.7	1.0001
1.0	0.0451	5.4	1.0237	9.8	1.0006
1.1	0.0604	5.5	1.0297	9.9	1.0011
1.2	0.0782	5.6	1.0343	10.0	1.0015
1.3	0.0984	5.7	1.0374	10.1	1.0018
1.4	0.1207	5.8	1.0393	10.2	1.0020
1.5	0.1450	5.9	1.0401	10.3	1.0021
1.6	0.1708	6.0	1.0398	10.4	1.0022
1.7	0.1978	6.1	1.0387	10.5	1.0022
1.8	0.2258	6.2	1.0368	10.6	1.0021
1.9	0.2542	6.3	1.0342	10.7	1.0020
2.0	0.2828	6.4	1.0312	10.8	1.0018
2.1	0.3112	6.5	1.0279	10.9	1.0016
2.2	0.3390	6.6	1.0243	11.0	1.0013
2.3	0.3658	6.7	1.0205	11.1	1.0011
2.4	0.3916	6.8	1.0167	11.2	1.0008
2.5	0.4164	6.9	1.0130	11.3	1.0005
2.6	0.4402	7.0	1.0094	11.4	1.0002
2.7	0.4633	7.1	1.0059	11.5	1.0000
2.8	0.4860	7.2	1.0027	11.6	0.9997
2.9	0.5084	7.3	0.9998	11.7	0.9995
3.0	0.5310	7.4	0.9972	11.8	0.9992
3.1	0.5537	7.5	0.9949	11.9	0.9990
3.2	0.5769	7.6	0.9930	12.0	0.9989
3.3	0.6006	7.7	0.9914	12.1	0.9987
3.4	0.6248	7.8	0.9902	12.2	0.9986
3.5	0.6495	7.9	0.9893	12.3	0.9985
3.6	0.6746	8.0	0.9887	12.4	0.9984
3.7	0.7001	8.1	0.9884	12.5	0.9984
3.8	0.7259	8.2	0.9883	12.6	0.9983
3.9	0.7516	8.3	0.9885	12.7	0.9983
4.0	0.7772	8.4	0.9889	12.8	0.9984
4.1	0.8024	8.5	0.9895	12.9	0.9984
4.2	0.8271	8.6	0.9902	13.0	0.9985
4.3	0.8510	8.7	0.9911	13.1	0.9985

DISCUSSION

Investigation of the graphs and numerical results in the uncompensated cases shows that as the order of the Butterworth filter increases the ratio of rise time to half-cycle overshoot duration, mT, increases; this behavior is shown in Fig. 8. Thus digital filter compensation is most effective on higher order Butterworth filters and a smoother transient response curve results as seen in Fig.'s 4, 5, 6.

The transient response of the higher order Butterworth filter with unit step input always has larger maximum overshoot and time delay which cannot be extensively reduced, the first time when the digital filter compensation is introduced. However, the process can be repeated until the desired reduction in transient overshoot is obtained; this is the goal of digital filter compensation.

For comparison, posicast control compensation is applied to each case and the resulting transient responses are presented in Fig.'s 4, 5, 6. In the second order case, the two compensating methods give almost the same result, although there is inflection in the digital filter compensation case. In the third order case large undershoot is produced with posicast compensation and digital filter compensation gives better results. In the fourth order case, oscillation with much larger undershoot occurs under posicast control compensation; however, digital filter compensation results in much better behavior.

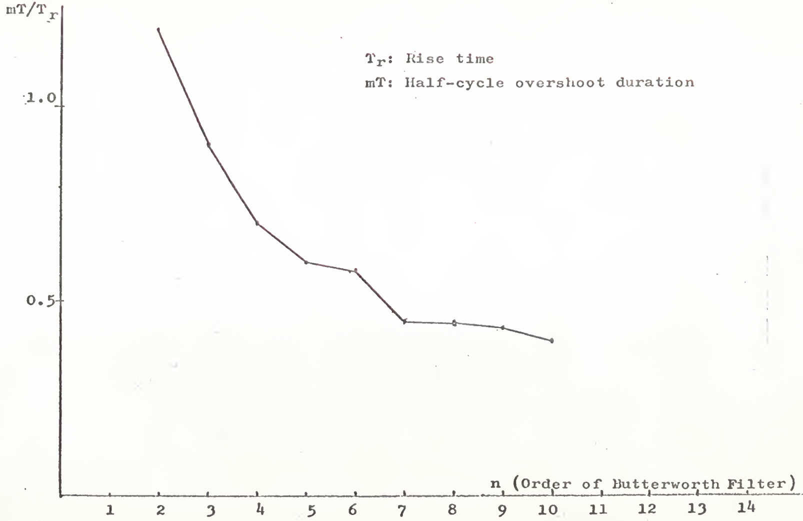

Fig. 8. Behavior of the ratio of rise time to half-cycle overshoot duration in the transient responses of Butterworth filters with unit step function inputs.

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DIGITAL FILTERS

by

SHAN C. SUN

B. S., National Taiwan University, 1961

AN ABSTRACT OF A MASTER'S REPORT

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The essential content of this report is to analyze Butterworth filters in the time domain by using a digital method for finding their transient performances. Digital filter compensation is effected by multiplication in the Z-transform domain by $(1+z^m)/2$. For comparison, posicast compensation of a compensator $(1-P)+Pz^m$ is employed in each digital filter compensation case. The approximate digital analysis is accomplished by the Naumov program which is an improved accuracy trapezoidal convolution program.

Transient responses of higher order digital Butterworth filters employing digital filter compensation are much better than those employing posicast compensation. Digital filter compensation can be repeated until desired reduction in transient overshoot is obtained--the goal of digital filter compensation.

This report introduces an interesting subject, the pseudo-Butterworth function. Pseudo-Butterworth functions can be converted into approximate Butterworth functions by replacing s with $s/(b_n)^{1/n}$, where b_n is the coefficient of the n^{th} term of pseudo-Butterworth polynomial of order n.