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FUNDAMENTAL SOLUTIONS TO SOME ELLIPTIC EQUATIONS WITH DISCONTINUOUS SENIOR COEFFICIENTS AND AN INEQUALITY FOR THESE SOLUTIONS

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Abstract. Let $Lu := \nabla \cdot (a(x)\nabla u) = -\delta(x-y)$ in $\mathbb{R}^3$, $0 < c_1 \leqslant a(x) \leqslant c_2$, $a(x)$ is a piecewise-smooth function with the discontinuity surface S which is smooth. It is proved that in an neighborhood of S the behavior of the function u is given by the formula:

$$u(x,y) = \begin{cases} (4\pi a_+)^{-1}[r_{xy}^{-1} + bR^{-1}], & y_3 > 0, \\ (4\pi a_-)^{-1}[r_{xy}^{-1} - bR^{-1}], & y_3 < 0. \end{cases} \qquad (*)$$

Here the local coordinate system is chosen in which the origin lies on S, the plane $x_3 = 0$ is tangent to S, $a_+(a_-)$ is the limiting value of $a(x)$ on S from the half-space $x_3 > 0$, $(x_3 < 0)$, $r_{xy} := |x-y|$, $R := \sqrt{\rho^2 + (|x_3| + |y_3|)^2}$, $\rho := \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, $b := (a_+ - a_-)/(a_+ + a_-)$. If S is the plane $x_3 = 0$ and $a(x) = a_+$ in $x_3 > 0$, $a(x) = a_-$ in $x_3 < 0$, then $(*)$ is the global formula for u in $\mathbb{R}^3$. Inequality for the fundamental solution for small and large $|x-y|$ follows from formula $(*)$.

1. Introduction

There are many papers on the behavior, as $x \to y$, of the fundamental solutions to the elliptic equations of the form

$$Lu := \sum_{i,j=1}^{n} \partial_j \left[a_{ij}(x)u_j(x,y)\right] = -\delta(x-y) \text{ in } \mathbb{R}^n,\ u_j := \frac{\partial u}{\partial x_j} = \partial_j u \qquad (1.1)$$

for smooth coefficients a_{ij}. Methods of pseudo-differential operators theory give expansion in smoothness of the solution to (1.1). In [LSW] existence of the unique solution to (1.1) with the properties

$$0 < c_1 r^{2-n} \leqslant u(x,y) \leqslant c_2 r^{2-n},\ u \in H^1_{\mathrm{loc}}(\mathbb{R}^n \backslash y),\ r := |x-y|, \qquad (1.2)$$

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is obtained under the assumption that a_{ij} are bounded real-valued measurable functions such that

$$a_1 \sum_{i=1}^{n} t_i^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x) t_i t_j \leqslant a_2 \sum_{i=1}^{n} t_i^2,\ a_1, a_2 = \text{const} > 0. \tag{1.3}$$

Our purpose is to give an analytical formula for the fundamental solution of the basic model operator (1.1), namely the operator with

$$a_{ij}(x) = \delta_{ij} a(x), \qquad a(x) = \begin{cases} a_+, & x_3 > 0, \\ a_-, & x_3 < 0. \end{cases} \tag{1.4}$$

Here a_+ and a_- are positive constants,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and $u(x,y)$ is the unique solution of the problem:

$$Lu := \sum_{i=1}^{n} \partial_i \left(a(x) u_i \right) = -\delta(x-y) \text{ in } \mathbb{R}^n\ , \tag{1.5}$$

$$[u]|_S = 0, \quad [a(x) u_N]\,|_S = 0, \tag{1.6}$$

the symbol $[u]|_S$ denotes the jump of u across S, that is,

$$[u] = u_+ - u_- \quad , u_\pm := \lim_{\varepsilon \to 0} u(s \pm \varepsilon N)\ ,$$

s is a point on S, N is the unit normal to S directed along x_3, u_N is the normal derivative on S, S is the plane $x_3 = 0$, $[au_N] := a_+ u_N^+ - a_- u_N^-$.

Problem (1.5)–(1.6) is important in many applications and is called a transmission problem. The solution to (1.5)–(1.6) is sought in the class $H^1_{\text{loc}}(\mathbb{R}^n \backslash y) \cap W^{1,1}_{\text{loc}}(R^n)$, where $H^1 := W^{2,1}$ and $W^{\ell,p}_{\text{loc}}$ is the Sobolev space of functions whose distributional derivatives up to the order ℓ belong to L^p_{loc}. If

$$a_{ij}(x) = \begin{cases} a_{ij}^+, & x_3 > 0, \\ a_{ij}^-, & x_3 < 0, \end{cases} \tag{1.4$'$}$$

and the constant matrices $a_{ij}^{\pm}$ are positive definite, then there exists an orthogonal coordinate transformation which reduces a_{ij}^+ to δ_{ij} and a_{ij}^- to $\lambda_j \delta_{ij}$, $\lambda_j > 0$. We do not give the formula for $u(x,y)$ in this more general case.

Finally note that for discontinuous coefficients equation (1.5) is understood in the weak sense, namely as the identity:

$$\int_{\mathbb{R}^n} a(x) u_i(x,y) \phi_i(x)\, dx = \phi(y), \tag{1.7}$$

The identity (1.7) for $u \in H^1_{\text{loc}}(\mathbb{R}^n \backslash y) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ implies conditions (1.6).

Let $n = 3$. The formula for the solution to problem (1.4)–(1.6), or the equivalent problem (1.4), (1.7) is given in Theorem 1.1.

THEOREM 1.1. *The unique solution to problem* (1.4)–(1.5) *is:*

$$u(x,y) = \begin{cases} \frac{1}{4\pi a_+}\left[\frac{1}{r} + \frac{b}{R}\right], & y_3 > 0, \quad b := \frac{a_+ - a_-}{a_+ + a_-}, \\ \frac{1}{4\pi a_-}\left[\frac{1}{r} - \frac{b}{R}\right], & y_3 < 0, \quad r := |x-y|, \end{cases} \tag{1.8}$$

where $R := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (|x_3| + |y_3|)^2}$.

COROLLARY 1.1. *The following inequality holds for* $r \to 0$*:*

$$|u(x,y)| < c|x-y|^{-1}, \tag{1.9}$$

where the constant $c > 0$ *does not depend on* x *and* y.

Thus, the fundamental solution of the equation (1.1) with discontinuous senior coefficients has a different representation than the fundamental equation for the similar operator with continuous coefficients, but satisfies similar inequality for small $|x-y|$.

A formula, similar to (1.8) can be derived by the same method for $n > 3$ as well. Formula (1.8) allows one to get asymptotics of $u(x,y)$ and of $\nabla_x u(x,y)$ as $|x-y| \to 0$. Such asymptotics are useful in the study of inverse problems for discontinuous media [3].

In section 2 we prove Theorem 1.1. In section 3 various generalizations and applications are discussed.

2. Proof of Theorem 1

The proof is given for $n = 3$, but it holds with obvious small changes for $n > 3$.

The idea of the proof is to take the Fourier transform of equation (1.5) with respect to the variables $\hat{x} := (x_1, x_2)$, to solve the resulting problem for an ordinary differential equation analytically, and then to Fourier-invert the solution of this problem.

Let us go through the steps.

Step 1. Let $y_1 = y_2 = 0$ without loss of generality (since u is translation-invariant in the plane (x_1, x_2)). Denote

$$w(\xi, x_3, y_3) := \int_{\mathbb{R}^2} e^{i\xi\cdot\hat{x}} u(\hat{x}, x_3; y)\, d\hat{x}; \; u = \frac{1}{(2\pi)^2}\int_{R^2} w e^{-i\xi\cdot x}\, d\xi. \tag{2.1}$$

$\xi := (\xi_1, \xi_2)$, $\xi^2 = |\xi|^2 = \xi_1^2 + \xi_2^2$. Denote $w' := \frac{\partial w}{\partial x_3}$.

Let us Fourier-transform equation (1.5), with $a(x)$ given in (1.4), and get

$$w''(\xi, x_3, y_3) - \xi^2 w(\xi, x_3, y_3) = \begin{cases} -\frac{1}{a_+}\delta(x_3 - y_3), & x_3 > 0, \\ -\frac{1}{a_-}\delta(x_3 - y_3), & x_3 < 0, \end{cases} \tag{2.2}$$

$$w(\xi, +0, y_3) - w(\xi, -0, y_3) = 0, \quad a_+ w'(\xi, +0, y_3) - a_- w'(\xi, -0, y_3) = 0. \tag{2.3}$$

In what follows we omit ξ in the variables of w, and write $w(x_3, y_3)$ for brevity. Thus, w solves problem (2.2)–(2.3) and satisfies the condition

$$w(\pm\infty, y_3) = 0\,. \tag{2.4}$$

Assume that $y_3 \neq 0$. Then problem (2.2)–(2.4) has a solution and this solution is unique. A lengthy but straightforward calculation yields the formula for w:

$$w = \begin{cases} \dfrac{\exp(-|\xi||x_3-y_3|)}{2|\xi|a_+} + b\dfrac{\exp[-|\xi|(|x_3|+|y_3|)]}{2|\xi|a_+}, & y_3 > 0 \\ \dfrac{\exp(-|\xi||x_3-y_3|)}{2|\xi|a_-} - b\dfrac{\exp[-|\xi|(|x_3|+|y_3|)]}{2|\xi|a_-}, & y_3 < 0 \end{cases} \tag{2.5}$$

$$\text{where} \quad b := \frac{a_+ - a_-}{a_+ + a_-}, \qquad a_-, a_+ > 0. \tag{2.6}$$

Step 2. The function $u(x,y)$ is obtained from w by the second formula (2.1). Let us denote $|\xi| := v$, $\rho := |\hat{x}| = \sqrt{x_1^2 + x_2^2}$, and remember that $y_1 = y_2 = 0$. Since w depends on $|\xi|$ and does not depend on the angular variable, $|\xi| := v$, we have

$$u = \frac{1}{(2\pi)^2}\int_0^\infty dv\, v \int_0^{2\pi} e^{-iv\rho\cos\varphi} w\, d\varphi = \frac{1}{2\pi}\int_0^\infty dv\, vwJ_0(v\rho) \tag{2.7}$$

where $J_0(x)$ is the Bessel function and we have used the known formula:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{iv\rho\cos\varphi} d\varphi = J_0(v\rho).$$

We need another well-known formula:

$$\int_0^\infty e^{-vt} J_0(v\rho) dv = \frac{1}{\sqrt{\rho^2 + t^2}}, \qquad t > 0 \tag{2.8}$$

From (2.5), (2.7) and (2.8) we get (1.8) with $y_1 = y_2 = 0$. Therefore, recalling the translation invariance of u in the horizontal directions, we get (1.8).

Theorem 1.1 is proved. □

REMARK 2.1. Note that the limits of $u(x,y)$ as $y_3 \to \pm 0$ exist and are equal:

$$u(x,\hat{y},+0) = u(x,\hat{y},-0) = \frac{1}{2\pi r(a_+ + a_-)}. \tag{2.9}$$

A result similar to (2.9) is mentioned in [K, p.318], however the argument [K] is not clear: the differentiation is done in the classical sense but the functions involved have no classical derivatives: they have a jump.

3. Generalizations, applications

This section contains some remarks.

REMARK 3.1. First, note that if $a(x)$ is a piecewise-smooth function with a smooth discontinuity surface S, $s \in S$, $a_\pm = \lim_{\varepsilon\to 0} a(s \pm \varepsilon N)$, where N is the exterior normal to S at the point s, then the main term of the asymptotics of the fundamental solution $u(x,y)$ in a neighborhood U_s of the point $s \in S$ is given by formula (1.8) in which

$x, y \in U_s$. This follows from the fact that the main term in smoothness of the solution to an elliptic equation in U_s is the same as to the equation with constant coefficients which are limits of $a(x)$ as $x \to s$. In our case, this "frozen-coefficients" model problem is given by equations (1.4)–(1.6). This argument shows that the same conclusion holds if the coefficient $a(x)$ in $\mathbb{R}^3_+$ and in $\mathbb{R}^3_-$ is not smooth but just Lipschitz-continuous.

REMARK 3.2. In principle, our method for calculation of $u(x,y)$ for the model problem (1.4)–(1.6) is applicable for the model problem $(1.4')$ with anisotropic matrix.

REMARK 3.3. We only mention that our result concerning asymptotics of $u(x,y)$ as $|x-y| \to 0$ for piecewise-smooth coefficients is applicable to inverse problems of geophysics and inverse scattering problems for acoustic and electromagnetic scattering by layered bodies.

For example, if the governing equation is [R, p.14]:

$$\nabla \cdot [a(x)\nabla u] + k^2 q(x) u = -\delta(x-y) \quad \text{in } \mathbb{R}^3,$$

$k = \text{const} > 0$, say $k = 1$, $q(x) = 1 + p(x)$, where $p(x)$ is a compactly supported real-valued function, $p(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$, $\text{supp}(p(x)) \subset \mathbb{R}^3_- := \{x : x_3 < 0\}$, $a(x) = 1 + A(x)$, where $A(x)$ is compactly supported piecewise-smooth function with finitely many closed compact smooth surfaces $S_j \subset \mathbb{R}^3_-$ of discontinuity. Across these surfaces the transmission conditions (1.6) hold, and at infinity u satisfies the radiation condition. Then $u(x,y)$ is uniquely determined.

An inverse problem is: given $g(x)$ and $u(x,y)$ for all $x, y \in S := \{x : x_3 = 0\}$ and a fixed $k = 1$, can one uniquely determine $a(x)$, in particular, the discontinuity surfaces S_j?

To explain how Theorem 1.1 can be used in this inverse problem, note that if two systems of surfaces $S_j^{(1)}$ and $S_j^{(2)}$ and two functions a_1 and a_2 produce the same surface data on S for all $x, y \in S$, then an orthogonality relation [R, pp. 65, 86] holds:

$$\int v(x)\nabla u_1(x,y)\nabla u_2(x,z)\,dx = 0, \qquad \forall y, z \in D'_{12}, \tag{3.1}$$

where $v(x) = a_1 - a_2$, $u_m(x,y)$, $m = 1, 2$, are the fundamental solutions corresponding to the obstacle D_m (i.e., to a_m and $S_j^{(m)}$), $D'_{12} := \mathbb{R}^3 \setminus D_{12}$, $D_{12} := D_1 \cup D_2$.

Let us prove, e.g., that $\partial D_1 = \partial D_2$, using (3.1). If there is a part of ∂D_1 which lies outside D_2, and s is a point at this part, then, assuming (for simplicity only) that $v(x)$ is piecewise-constant and using for ∇u_1 and ∇u_2 formulas, which follow from (1.8) as $y = z \to s$, we conclude that the left-hand side of (3.1) is an integral which contains a part, unbounded as $y = z \to s \in \partial D_1$: $c \int |x-y|^{-4} dx$, $c = \text{const} \neq 0$. This contradicts to (3.1). Therefore there is no part of ∂D_1 which lies outside D_2. Likewise, there is not part of ∂D_2 which lies outside ∂D_1. Thus, $\partial D_1 = \partial D_2$. Similarly one proves that $S_j^{(1)} = S_j^{(2)}$ for all j, provided (3.1) holds.

A detailed presentation of such an argument is given in the paper by C. Athanasiadis, A. G. Ramm and I. Stratis, *Inverse acoustic scattering by layered obstacle* (in preparation).

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