

THE EFFECTS OF DAMPING ON THE LATERAL VIBRATIONS OF A FREE-FREE BEAM

by 1264

Carol Ann Rubin

B. S., Columbia University, 1966

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

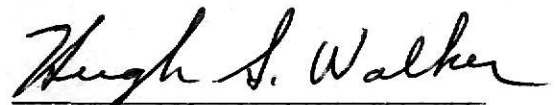
Department of Mechanical Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1969

Approved by:


Major Professor

LD
2668
R4
1969
R811

TABLE OF CONTENTS

	Page
NOMENCLATURE	iii
INTRODUCTION	1
DERIVATION OF THE EQUATION OF MOTION	2
THE EIGENVALUE PROBLEM	7
DERIVATION OF THE CHARACTERISTIC EQUATION FOR A BEAM OF PARABOLIC PROFILE	9
COMPUTER SOLUTION OF THE CHARACTERISTIC EQUATION	14
ANALYSIS OF RESULTS	19
The Effects of Damping on the Natural Frequencies	37
The Effects of Damping on the Mode Shapes	39
BIBLIOGRAPHY	40
ACKNOWLEDGEMENT	41

NOMENCLATURE

w	Lateral displacement of the beam
z	Horizontal coordinate of a differential beam element
ℓ	Length of beam
$M(z,t)$	Moment acting on beam
$V(z,t)$	Vertical component of thrust
$P(z,t)$	Horizontal component of thrust
α	Angle of beam axis with horizontal
q	Constant axial load
t	Time
u	Longitudinal displacement of beam
ρ	Mass per unit length of beam (constant)
$A(z)$	Cross sectional area of beam
$I(z)$	Moment of inertia of beam
A_o	Cross sectional area of beam at $z=\ell$
I_o	Moment of inertia of beam at $z=\ell$
n,m	Real numbers
T_c	Thrust exerted on beam at $z=\ell$
E	Modulus of elasticity (constant)
ξ	z/ℓ
τ	$t \sqrt{\frac{EI_o}{\rho A_o \ell^4}}$
$\bar{T}$	$\frac{\ell^2 T_c}{EI_o}$
$\bar{Q}$	$\frac{\ell^3 q}{EI_o}$

ψ	Function of ξ
F	Function of τ
λ^2	Eigenvalue of differential equation
k	Directional control parameter
T	Differential operator

INTRODUCTION

The problem of the influence of transverse vibrations on the dynamic stability of rockets has been dealt with recently in a paper by Huang and Walker [1]. This study idealized the rocket as a free-free elastic beam with a constant end thrust and a constantly varying cross-sectional area. It is the purpose of the present study to introduce an additional variable and, using the Huang-Walker paper as a basis for comparison, see what the effects are on the vibration and stability properties. This new variable provides an approximation of the effect that frictional drag will have on rockets. A constant axial load is assumed to act along the axis of the beam and in a direction opposite to the motion of the beam. The simplifying assumptions made here for the form of the drag force allow for the derivation and solution of the governing differential equations, and at the same time provide a basis for discussion of the effects of a motion-opposing force on a rocket.

The governing differential equation of motion is derived using simple beam theory; and the characteristic equation for the special case of a beam with a parabolic surface of revolution is obtained by means of Frobenius' method. In addition, a variable indicating the angle of applied thrust is introduced to further generalize the results.

DERIVATION OF THE EQUATION OF MOTION

Simple beam theory is used to obtain the governing differential equation. The effects of longitudinal vibrations are neglected and the motion is assumed to be planar. Fig. 1 shows a differential beam element with the relevant forces and couples acting. The plane of motion is the w, z -plane, as shown, where w is the distance from the z -axis to the beam axis and z is the distance along the beam (the beam begins at $z=0$ and ends at $z=l$ where the thrust is considered to be applied). $M(z,t)$ is the moment acting on the beam at a distance z and a time t . $V(z,t)$ and $P(z,t)$ are the vertical and horizontal components, respectively, of the thrust at a given position and time. The angle α indicates the angle that the beam axis makes with the horizontal axis at any given point. A constant axial load representing the drag force is denoted by q and is acting in the direction α . D'Alembert forces are assumed to be acting as shown in Fig. 1, where u is the longitudinal displacement of a material element. It is assumed that u is independent of z , that is, that the rocket material is so stiff that vibrations due to bending occur long before longitudinal vibrations. Therefore, u will be considered dependent only upon t .

D'Alembert's principle yields the following equations of motion:

$$\rho A(z) \frac{\partial^2 u(t)}{\partial t^2} = \frac{\partial}{\partial z} [P(z,t)] - q \quad , \quad (1)$$

$$\rho A(z) \frac{\partial^2 w(z,t)}{\partial t^2} = - \frac{\partial}{\partial z} [V(z,t)] + q \frac{\partial w}{\partial z} \quad , \quad (2)$$

and

$$\begin{aligned} P \frac{\partial w}{\partial z} dz + \left(\frac{\partial P}{\partial z} dz \right) \left(\frac{\partial w}{\partial z} \frac{dz}{2} \right) - V dz - \left(\frac{\partial V}{\partial z} dz \right) \left(\frac{dz}{2} \right) \\ + \frac{\partial M}{\partial z} dz = 0 \quad , \end{aligned} \quad (3a)$$

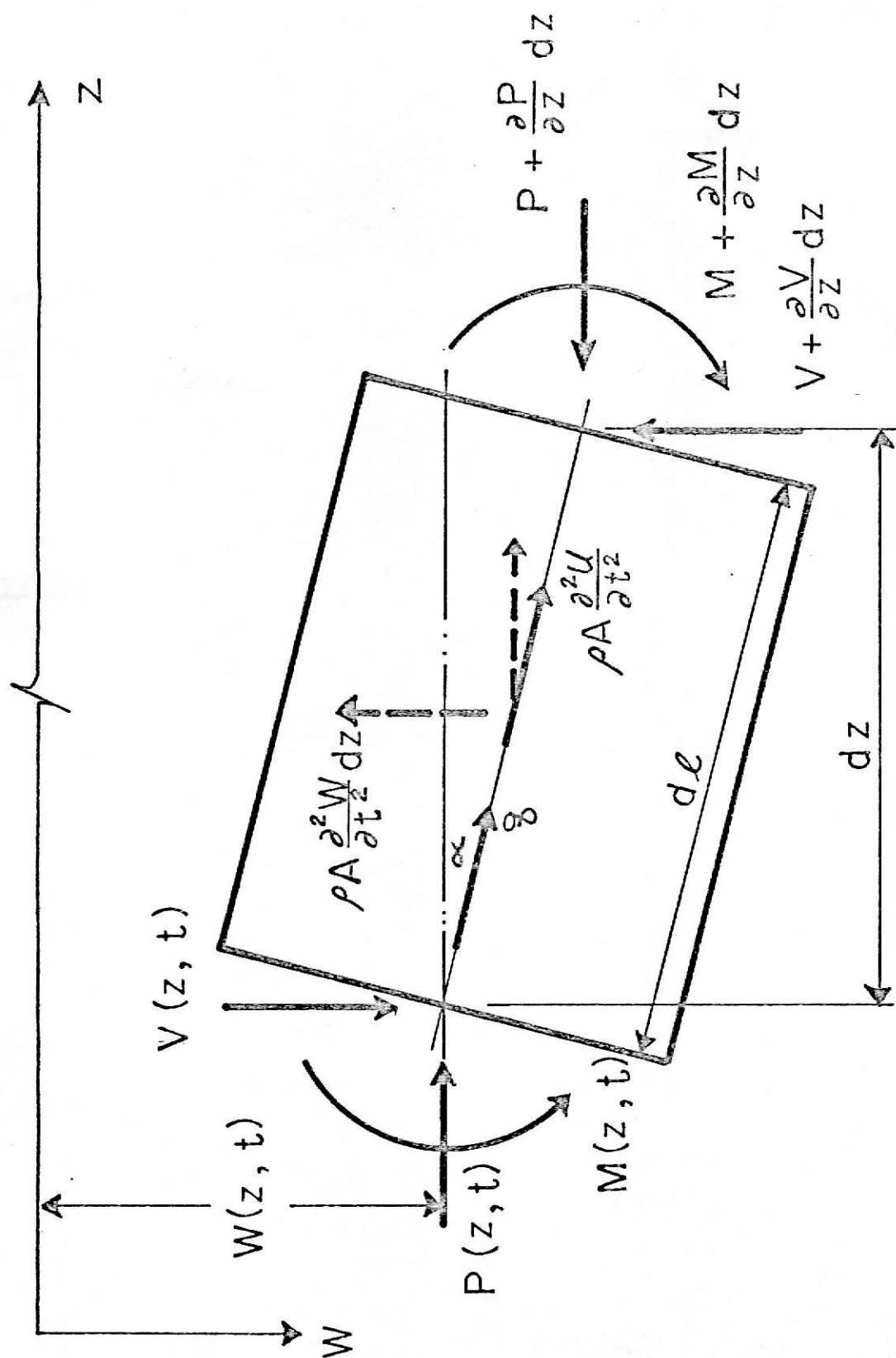


FIG. 1. DIFFERENTIAL ELEMENT OF THE
TAPERED BEAM

where a small angle assumption has been made for α so that $\cos \alpha = \frac{dz}{d\ell} \approx 1$, and $\sin \alpha = \frac{dw}{dz} \approx \alpha$. Neglecting terms of higher order, Eq. (3a) becomes:

$$P(z,t) \frac{\partial}{\partial z}[w(z,t)] - V(z,t) + \frac{\partial}{\partial z}[M(z,t)] = 0 \quad (3)$$

The cross sectional area and the moment of inertia of the beam at a point z are denoted, respectively, as

$$\begin{aligned} A(z) &= A_0 \left(\frac{z}{\ell}\right)^n, \\ I(z) &= I_0 \left(\frac{z}{\ell}\right)^m, \end{aligned} \quad (4)$$

where A_0 and I_0 are the respective cross sectional area and moment of inertia of the beam at $z=\ell$. The constants m and n are real numbers and depend upon the choice of rocket shape; for a constant cross sectional area, $m=n=0$, and a conical cross section has $m=n=1$.

The following boundary conditions are compatible with the physical system and are assumed to be:

$$P(z,t) \Big|_{z=0} = 0 \quad \text{and} \quad P(z,t) \Big|_{z=\ell} = T_c,$$

where T_c is the thrust exerted on the rocket at $z=\ell$. Integrating Eq. (1) with respect to z yields

$$P(z,t) = \frac{\rho \ddot{u} A \ell}{n+1} \left(\frac{z}{\ell}\right)^{n+1} + qz + C_1 \quad (5a)$$

Evaluating this equation at the boundary points and substituting the boundary conditions gives

$$P(0,t) = C_1 = 0,$$

and
$$P(\ell, t) = \frac{\rho \ddot{u} A \ell}{n+1} + q \ell = T_c .$$

We can now substitute $\ddot{u} = \frac{(T_c - q \ell)(n+1)}{\rho A \ell}$ into Eq. (5a). Thus,

$$P(z, t) = (T_c - q \ell) \left(\frac{z}{\ell}\right)^{n+1} + qz . \quad (5)$$

To derive the differential equation of motion, Eq. (3) is differentiated with respect to z giving

$$\frac{\partial}{\partial z} [P(z, t) \frac{\partial w}{\partial z}] - \frac{\partial V}{\partial z} + \frac{\partial^2 M}{\partial z^2} = 0 . \quad (6a)$$

Using Eq. (2) for $\frac{\partial V}{\partial z}$, Eq. (5) for $P(z, t)$, and

$$M = EI \frac{\partial^2 w}{\partial z^2} = EI_0 \left(\frac{z}{\ell}\right)^m \frac{\partial^2 w}{\partial z^2}$$

and substituting into Eq. (6a) gives

$$\begin{aligned} \rho A_0 \left(\frac{z}{\ell}\right)^n \frac{\partial^2 w}{\partial t^2} - q \frac{\partial w}{\partial z} + \frac{\partial}{\partial z} \left\{ [T_c - q \ell] \left(\frac{z}{\ell}\right)^{n+1} \frac{\partial w}{\partial z} + qz \frac{\partial w}{\partial z} \right\} \\ + \frac{\partial^2}{\partial z^2} \left\{ [EI_0 \left(\frac{z}{\ell}\right)^m] \frac{\partial^2 w}{\partial z^2} \right\} = 0 . \end{aligned} \quad (6)$$

The variables may be non-dimensionalized by taking $\xi = \frac{z}{\ell}$, $\frac{\partial}{\partial z} = \frac{1}{\ell} \frac{\partial}{\partial \xi}$, and

$$\frac{\partial^2}{\partial z^2} = \frac{1}{\ell^2} \frac{\partial^2}{\partial \xi^2} ; \text{ and we let } \tau = t \sqrt{\frac{EI_0}{\rho A_0 \ell^4}}, \quad \frac{\partial}{\partial t} = \sqrt{\frac{EI_0}{\rho A_0 \ell^4}} \frac{\partial}{\partial \tau}, \text{ and } \frac{\partial^2}{\partial t^2}$$

$$= \frac{EI_0}{\rho A_0 \ell^4} \frac{\partial^2}{\partial \tau^2} . \text{ If these values are substituted into Eq. (6), the non-}$$

dimensionalized parameters may then be taken as

$$\bar{T} = \frac{\ell^2 T_c}{EI_0} ,$$

and

$$\bar{Q} = \frac{\ell^2 q}{EI_0} .$$

The differential equation of motion now becomes

$$\begin{aligned} \xi^n \frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^2}{\partial \xi^2} [\xi^m \frac{\partial^2 w}{\partial \xi^2}] + \bar{T} \frac{\partial}{\partial \xi} [\xi^{n+1} \frac{\partial w}{\partial \xi}] - \bar{Q} \frac{\partial}{\partial \xi} [\xi^{n+1} \frac{\partial w}{\partial \xi}] \\ + \bar{Q} \xi \frac{\partial^2 w}{\partial \xi^2} = 0 \quad . \end{aligned} \quad (7)$$

THE EIGENVALUE PROBLEM

The above partial differential equation may be solved by separation of the variables ξ and τ . This will result in two ordinary differential equations in the form of an eigenvalue problem. Assume,

$$w(\xi, \tau) = \psi(\xi)F(\tau) \quad . \quad (8)$$

Substituting $w(\xi, \tau)$ into Eq. (7) gives

$$\xi^n F'''\psi + F \frac{d^2}{d\xi^2} [\xi^m \psi''] + F(\bar{T}-\bar{Q}) \frac{d}{d\xi} [\xi^{n+1} \psi'] + \bar{Q} \xi \psi'' F = 0 \quad .$$

Separating the variables, using λ^2 as the constant of separation, yields the following pair of equations:

$$F'' + \lambda^2 F = 0 \quad ,$$

$$\text{and} \quad \frac{d^2}{d\xi^2} [\xi^m \frac{d^2 \psi}{d\xi^2}] + (\bar{T}-\bar{Q}) \frac{d}{d\xi} [\xi^{n+1} \frac{d\psi}{d\xi}] + \bar{Q} \xi \frac{d^2 \psi}{d\xi^2} - \xi^n \lambda^2 \psi = 0 \quad . \quad (9)$$

The boundary conditions for a free-free tapered beam with end thrust, $\bar{T}$, are such that at $z=0$ and at $z=l$ the moment and shear are equal to zero. In terms of our non-dimensionalized parameters:

$$\begin{aligned} \text{at} \quad \xi=0 : \quad \xi^m \frac{d^2 \psi}{d\xi^2} &= 0 \quad \text{and} \quad \frac{d}{d\xi} [\xi^m \frac{d^2 \psi}{d\xi^2}] = 0 \quad , \\ \text{and at} \quad \xi=1 : \quad \frac{d^2 \psi}{d\xi^2} &= 0 \quad \text{and} \quad \frac{d^3 \psi}{d\xi^3} = -\bar{T}k \left[\frac{d\psi}{d\xi} \right]_{\xi=1} \quad . \end{aligned} \quad (10)$$

The parameter k indicates the angle of thrust and ranges from zero to unity. When $k=0$, there is no shear due to $\bar{T}$, that is, $\bar{T}$ acts in an axial direction; and when $k=1$, $\bar{T}$ acts at a slope of $\frac{d\psi}{d\xi}$.

The force $\bar{Q}$ has no effect on either moment or shear since it is acting in an axial direction.

Eqs. (9) and (10) are the complete formulation of an eigenvalue problem in terms of the variables ξ , $\bar{T}$, and $\bar{Q}$. Setting T equal to the entire differential operator, that is,

$$T = \frac{d^2}{d\xi^2} \left[\xi^m \frac{d^2}{d\xi^2} \right] + (\bar{T} - \bar{Q}) \frac{d}{d\xi} \left[\xi^{n+1} \frac{d}{d\xi} \right] + \bar{Q} \xi \frac{d^2}{d\xi^2} ,$$

Eq. (9) becomes:

$$(T - \xi^n \lambda^2) \psi = 0 \quad . \quad (11)$$

DERIVATION OF THE CHARACTERISTIC EQUATION
FOR A BEAM OF PARABOLIC PROFILE

The special case of a beam with a parabolic surface of revolution is now considered. The generating line has an equation of the form

$$x(z) = \sqrt{\frac{A_0}{\pi}} \left(\frac{z}{\ell}\right)^{\frac{1}{2}}. \quad (12)$$

The beam cross sectional area and moment of inertia are represented, respectively as

$$A = A_0 \left(\frac{z}{\ell}\right) \quad \text{and} \quad I = I_0 \left(\frac{z}{\ell}\right)^2. \quad (13)$$

Thus, the constants used in the previous calculations are $n=1$ and $m=2$.

A desired form of the differential equation is obtained by performing all possible differentiations in Eq. (9), rearranging terms, and multiplying through by ξ^2 . If the above values for n and m are then substituted, the following equation results:

$$\xi^4 \frac{d^4 \psi}{d\xi^4} + 4\xi^3 \frac{d^3 \psi}{d\xi^3} + [2\xi^2 + \bar{Q}\xi^3 + (\bar{T}-\bar{Q})\xi^4] \frac{d^2 \psi}{d\xi^2} + 2(\bar{T}-\bar{Q})\xi^3 \frac{d\psi}{d\xi} - \xi^3 \lambda^2 \psi = 0. \quad (14)$$

The boundary conditions are

$$\text{at } \xi=1; \quad \frac{d^2 \psi}{d\xi^2} = 0 \quad \text{and} \quad \frac{d^3 \psi}{d\xi^3} = -\bar{T}k \left[\frac{d\psi}{d\xi} \right]_{\xi=1}. \quad (15)$$

It is noted that the boundary conditions at $\xi=0$ are satisfied identically.

Frobenius' method is used to solve the above boundary value problem.

A solution of the form

$$\psi(\xi) = \sum_{r=0}^{\infty} a_r \xi^{r+s} \quad (16)$$

is assumed, where s and the a_r 's must be determined so that Eq. (14) is satisfied. The following notation is used:

$$(s)_n = s(s-1)(s-2)\dots(s-(n-1)) \quad ,$$

and
$$(s+r)_n = (s+r)(s+r-1)\dots(s+r-(n-1)) \quad .$$

Substituting Eq. (16) into Eq. (14) yields

$$\begin{aligned} & \sum_{r=0}^{\infty} a_r (r+s)_4 \xi^{r+s} + 4 \sum_{r=0}^{\infty} a_r (r+s)_3 \xi^{r+s} + \sum_{r=0}^{\infty} a_r (r+s)_2 2\xi^{r+s} \\ & + \sum_{r=0}^{\infty} a_r \bar{Q}(r+s)_2 \xi^{r+s+1} + \sum_{r=0}^{\infty} a_r (\bar{T}-\bar{Q})(r+s)_2 \xi^{r+s+2} \\ & + \sum_{r=0}^{\infty} 2a_r (\bar{T}-\bar{Q})(r+s)_1 \xi^{r+s+2} - \sum_{r=0}^{\infty} a_r \lambda^2 \xi^{r+s+3} = 0 \quad . \end{aligned} \quad (17)$$

Collecting like powers of ξ gives:

$$\begin{aligned} & \sum_{r=0}^{\infty} a_r [\xi^{r+s} \{(r+s)_4 + 4(r+s)_3 + 2(r+s)_2\} + \xi^{r+s+1} \{\bar{Q}(r+s)_2\} \\ & + \xi^{r+s+2} \{(\bar{T}-\bar{Q})(r+s)_2 + 2(r+s)_1\} - \xi^{r+s+3} \lambda^2] = 0 \quad . \end{aligned}$$

For Eq. (17) to represent an equality, coefficients of like powers of ξ must be equal to zero. Therefore,

$$a_0[(s)_4 + 4(s)_3 + 2(s)_2] = 0 \quad (18a)$$

$$a_1[(s+1)_4 + 4(s+1)_3 + 2(s+1)_2] + a_0[\bar{Q}(s)_2] = 0 \quad (18b)$$

$$\begin{aligned} & a_2[(s+2)_4 + 4(s+2)_3 + 2(s+2)_2] + a_1[\bar{Q}(s+1)_2] \\ & + a_0(\bar{T}-\bar{Q})[(s)_2 + 2(s)_1] = 0 \end{aligned} \quad (18c)$$

•
•
•

$$a_j[(s+j)_4 + 4(s+j)_3 + 2(s+j)_2] + a_{j-1}[\bar{Q}(s+j-1)_2] \\ + a_{j-2}(\bar{T}-\bar{Q})[(s+j-2)_2 + 2(s+j-2)_1] - a_{j-3}\lambda^2 = 0 \quad ,$$

where $j \geq 3$.

For an arbitrary a_0 , Eq. (18a) is a fourth order polynomial in s .

Solution of this equation yields four roots:

$$s_0=0, \quad s_1=1, \quad s_2=0, \quad \text{and} \quad s_3=1 \quad .$$

Eq. (18b) becomes:

$$a_1 = \frac{-\bar{Q}(s)_2 a_0}{(s+1)_4 + 4(s+1)_3 + 2(s+1)_2} \quad , \quad (20a)$$

and Eq. (18c) becomes:

$$a_2 = \frac{-a_0(\bar{T}-\bar{Q})[(s)_2 + 2(s)_1] - a_1[\bar{Q}(s+1)_2]}{(s+2)_4 + 4(s+2)_3 + 2(s+2)_2} \quad . \quad (20)$$

Rearranging terms in Eq. (18d) gives the general expression for $j \geq 3$:

$$a_j = \frac{a_{j-3}\lambda^2 - (s+j-1)_2[a_{j-2}(\bar{T}-\bar{Q}) + a_{j-1}\bar{Q}]}{[(s+j)_2]^2} \quad .$$

We are now in a position to solve for the coefficients, a_r , of the solutions to the fourth order differential equation. Eq. (16) represents the form of our solution. From Eq. (19), $s_0=0$, therefore $\psi_0 \Big|_{s=0} = \sum_{r=0}^{\infty} a_r \xi^r$. Since a_0 is assumed arbitrary, let $a_0=1$; then $a_1=0$, and $a_2=0$. The general term ($j \geq 3$) is

$$a_j = \frac{a_{j-3}\lambda^2 - (j-1)_2[a_{j-2}(\bar{T}-\bar{Q}) + a_{j-1}\bar{Q}]}{[(j)_2]^2} \quad .$$

For $s_1=1$ we have $\psi_1 = \xi \sum_{r=0}^{\infty} a_r \xi^r$. If a_0 is again assumed equal to unity, then $a_1=0$, $a_2 = \frac{-2(\bar{T}-\bar{Q})}{(3!)^2}$, and for $j \geq 3$:

$$a_j = \frac{a_{j-3} \lambda^2 - (j)_2 [a_{j-2} (\bar{T}-\bar{Q}) + a_{j-1} \bar{Q}]}{[(j+1)_2]^2}$$

The remaining two solutions are:

$$\psi_2 \Big|_{s=0} = \psi_0 \ln \xi + \sum_{r=0}^{\infty} b_r \xi^r \quad (22)$$

$$\text{and } \psi_3 \Big|_{s=1} = \psi_1 \ln \xi + \xi \sum_{r=0}^{\infty} b_r \xi^r .$$

The complete solution will be

$$\psi(\xi) = \sum_{i=0}^3 A_i \psi_i . \quad (23)$$

In order to obtain a finite solution at $\xi=0$, we must set $A_2=A_3=0$. Therefore, the complete solution is

$$\psi(\xi) = A_0 \psi_0 + A_1 \psi_1 . \quad (24)$$

Substituting the boundary conditions of Eq. (15) into Eq. (24) gives

$$\frac{d^2 \psi}{d\xi^2} \Big|_{\xi=1} = 0 = (A_0 \frac{d^2 \psi_0}{d\xi^2} + A_1 \frac{d^2 \psi_1}{d\xi^2}) \Big|_{\xi=1} \quad (24a)$$

$$\text{and } \frac{d^3 \psi}{d\xi^3} \Big|_{\xi=1} + \bar{T}k \left[\frac{d\psi}{d\xi} \right]_{\xi=1} = 0 = A_0 \left[\frac{d^3 \psi_0}{d\xi^3} + \bar{T}k \frac{d\psi_0}{d\xi} \right]_{\xi=1} + A_1 \left[\frac{d^3 \psi_1}{d\xi^3} + \bar{T}k \frac{d\psi_1}{d\xi} \right]_{\xi=1} .$$

Setting the determinant of the coefficients of A_0 and A_1 equal to zero and rearranging terms, results in the characteristic equation:

$$\left[\frac{d^2 \psi_0}{d\xi^2} \frac{d^3 \psi_1}{d\xi^3} - \frac{d^2 \psi_1}{d\xi^2} \frac{d^3 \psi_0}{d\xi^3} \right]_{\xi=1} + \bar{T}k \left[\frac{d^2 \psi_0}{d\xi^2} \frac{d\psi_1}{d\xi} - \frac{d^2 \psi_1}{d\xi^2} \frac{d\psi_0}{d\xi} \right]_{\xi=1} = 0 . \quad (25)$$

The variables in this equation are ψ_0 , ψ_1 , $\bar{T}$, $\bar{Q}$, k , and λ^2 . ψ_0 and ψ_1 may be evaluated at $\xi=1$ by computations of the coefficients a_r ; and $\bar{T}$, $\bar{Q}$ and k are given for a specific system. Therefore, the characteristic equation may be solved for the eigenvalues, λ^2 , where λ is the natural frequency of the lateral vibrations.

To find the mode shapes of the system, we rearrange Eq. (24a) so that

$$\frac{A_1}{A_0} = - \frac{\left(\frac{d^2 \psi_0}{d\xi^2} \right)}{\left(\frac{d^2 \psi_1}{d\xi^2} \right)} \bigg|_{\xi=1}.$$

Since from Eq. (24) it may be seen that $\psi = A_0 \psi_0 + A_1 \psi_1$, we get,

$$\psi = A_0 \left[\psi_0 - \frac{d^2 \psi_0 / d\xi^2}{d^2 \psi_1 / d\xi^2} \bigg|_{\xi=1} \psi_1 \right] \quad (26)$$

for the mode shape.

COMPUTER SOLUTION OF THE CHARACTERISTIC EQUATION

The Huang-Walker paper consisted of a solution of this problem with $\bar{Q}=0$. The purpose of this paper was to vary the parameter $\bar{Q}$ in order to see how vibration and stability properties would be affected. Therefore, the objective of the computer work was to solve the characteristic equation for various values of $\bar{Q}$ at different values of $\bar{T}$ and k so that a thorough comparison of the results of the two studies would be possible.

The program used included a main program, a subroutine consisting of a root finding technique based on straight line interpolation and a mode shape evaluation, and a function subroutine which evaluated the characteristic equation by finding the required number of coefficients necessary for the convergence of the infinite series. To insure that there were no mistakes in the root finding technique another type of method in which a step by step operation which searches for sign changes was used [3]. The results checked. Included below is a copy of the program used on the 360/50 computer in Fortran IV language.

```
      IMPLICIT REAL*8 (A-H,C-Z)
      COMMON RS,R2PRIM,S2PRIM,M
C01  FORMAT(4F12.5)
C02  FORMAT(2I10)
C    X=LAMDA SQAURED
      READ(1,2)IPRINT,ILAST
      WRITE(3,2)IPRINT,ILAST
      READ(1,1)X,DELI,DDEL,DELX
      READ(1,1)XMAX
      WRITE(3,1)X,DELI,DDEL,DELX
      WRITE(3,1)XMAX
      CALL RCOCT(X,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
      STOP
      END
```

```

      SUBROUTINE ROOT(X,DELI,CDEL,DELX,XMAX,ILAST,XS,XL,YS,Y
11,IPRINT)
      IMPLICIT REAL*8 (A-H,O-Z)
C      THIS SUBROUTINE CALLS FUN(X)
C      IPRINT=1 PRINTS XS,XL,YS,YL.  IPRINT=0 DOES NOT PRINT.
C      ROOTS OF THE TRANSCENDENTAL EQUATION FUN(X)
C      DELI IS THE INITIAL INCREMENT
C      CDEL IS THE FACTOR FOR DIVIDING THE INTERVAL. CDEL<1.0
C      DELX IS THE ACCURACY ON THE SOLUTION
C      XMAX IS THE MAXIMUM THAT THE ARGUMENT X MAY BECOME
C      ILAST IS THE NUMBER OF ROOTS DESIRED
      COMMON RS,R2PRIM,S2PRIM,M
      4  FORMAT(4E20.8)
200  FORMAT(4E16.8,15)
201  FORMAT(3F12.5)
203  FORMAT(11C)
401  FORMAT(5E16.8)
      DIMENSION TBAR(11),GBAR(11),AK(11)
      DIMENSION F(200),G(200)
      DO 202 I=1,11
      X=-5000.0
      READ(1,201)TBAR(I),GBAR(I),AK(I)
      WRITE(3,201)TBAR(I),GBAR(I),AK(I)
      IUI=0
      X=X-DELI
10  DEL = DELI
      X = X + DEL
      Y=FUN(X,TBAR(I),GBAR(I),AK(I))
      IF(Y) 98,500,99
98  ZAZ=-1.0
      GO TO 100
99  ZAZ=+1.0
100 XS = X
      YS = Y
      IF(X.GT.XMAX) GO TO 510
      X = X + DEL
      Y=FUN(X,TBAR(I),GBAR(I),AK(I))
      IF(ZAZ*Y) 110,500,100
110 XL = X
      YL = Y
120 X = XS - YS*((XL-XS)/(YL-YS))
      IF(XL-XS-DELX) 500,130,130
130 Y=FUN(X,TBAR(I),GBAR(I),AK(I))
      DEL = DEL*CDEL
      IF(ZAZ*Y) 140,500,100
140 XL = X
      YL = Y
      X = X - DEL
      Y=FUN(X,TBAR(I),GBAR(I),AK(I))
      IF(ZAZ*Y) 140,500,150

```

```

150 XS = X
    YS = Y
    GO TO 120
500 IUI=IUI+1
    IF(IPRINT.EQ.0) GO TO 500
    WRITE(3,4) XS,XL,YS,YL
    WRITE(3,200)TBAR(I),QBAR(I),X,RS,M
    DO 400 J=1,26
        D=J
        N=4
        CSI=(D-1.0)/25.0
        F(1)=1.0
        F(2)=0.0
        F(3)=0.0
        F(4)=X/36.0
        G(1)=1.0
        G(2)=0.0
        G(3)=(QBAR(I)-TBAR(I))/18.0
        G(4)=(X+(QBAR(I)*(TBAR(I)-QBAR(I)))/3.0)/144.0
        R=F(1)+F(4)*(CSI**3)
        S=G(1)*CSI+G(3)*(CSI**3)+G(4)*(CSI**4)
403 N=N+1
        D=FLOAT(N)
        F(N)=(F(N-3)*X-(D-2.0)*(D-3.0)*(F(N-2)*(TBAR(I)-QBAR(I)
1)))+F(N-1)*QBAR(I))/((D-1.0)*(D-2.0))**2
        G(N)=(G(N-3)*X-(D-1.0)*(D-2.0)*(G(N-2)*(TBAR(I)-QBAR(I)
2)))+G(N-1)*QBAR(I))/((D*(D-1.0))**2
        R=R+F(N)*(CSI**(N-1))
        S=S+G(N)*(CSI**N)
        IF(N.LE.M) GO TO 403
404 AMODE=R-((R2PRIM/S2PRIM)*S)
    IF(J.EQ.1) DIV=AMODE
    ANCRM=AMODE/DIV
    WRITE(3,201)ANCRM
400 CONTINUE
500 CONTINUE
502 IF(ILAST-IUI) 510,510,10
510 CONTINUE
    WRITE(3,203)IUI
202 CONTINUE
    RETURN
    END

```

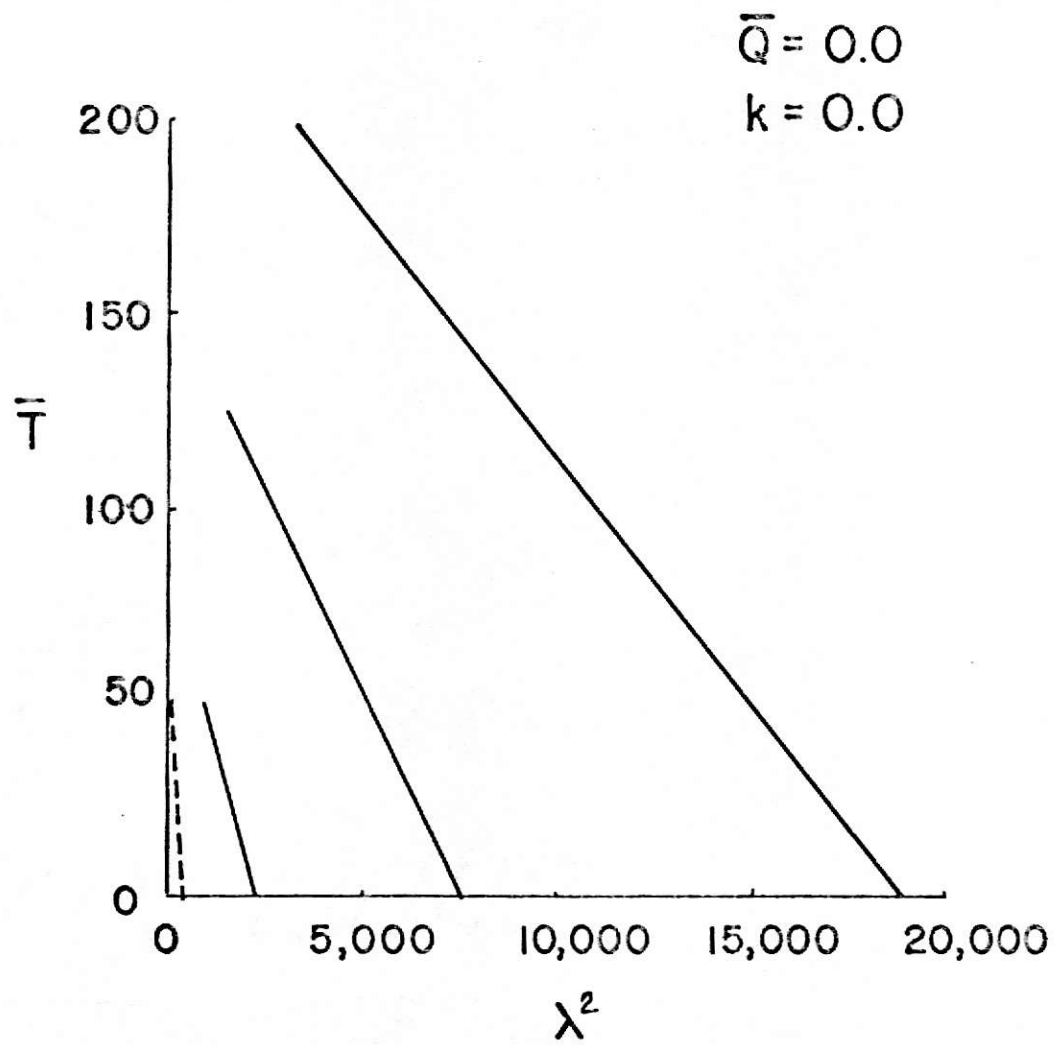
```

      DOUBLE PRECISION FUNCTION FUN(X,TBAR,QBAR,AK)
      IMPLICIT REAL*8 (A-H,C-Z)
      COMMON RS,R2PRIM,S2PRIM,M
003  FORMAT(4E20.8)
      DIMENSION A(200),B(200)
      NCDEF=200
      M=4
      TEST=1.0
      RS=1.0
      A(1)=1.0
      A(2)=0.0
      A(3)=0.0
      A(4)=X/36.0
      B(1)=1.0
      B(2)=0.0
      B(3)=(QBAR-TBAR)/18.0
      B(4)=(X+(QBAR*(TBAR-QBAR))/3.0)/144.0
      RPRIME=3.0*A(4)
      R2PRIM=6.0*A(4)
      R3PRIM=6.0*A(4)
      SPRIME=1.0+3.0*B(3)+4.0*B(4)
      S2PRIM=6.0*B(3)+12.0*B(4)
      S3PRIM=6.0*B(3)+24.0*B(4)
300  M=M+1
      FSAVE=RS
      IF(TBAR.NE.0.0)GO TO 10
      IF(QBAR.EQ.0.0)M=M+2
010  C=FLOAT(M)
      A(M)=(A(M-3)*X-(C-2.0)*(C-3.0)*(A(M-2)*(TBAR-QBAR)+A(M-
2-1)*QBAR)))/((C-1.0)*(C-2.0))**2
      B(M)=(B(M-3)*X-(C-1.0)*(C-2.0)*(B(M-2)*(TBAR-QBAR)+B(M-
1-1)*QBAR)))/(C*(C-1.0))**2
      RPRIME=RPRIME+(C-1.0)*A(M)
      R2PRIM=R2PRIM+(C-1.0)*(C-2.0)*A(M)
      R3PRIM=R3PRIM+(C-1.0)*(C-2.0)*(C-3.0)*A(M)
      SPRIME=SPRIME+C*B(M)
      S2PRIM=S2PRIM+C*(C-1.0)*B(M)
      S3PRIM=S3PRIM+C*(C-1.0)*(C-2.0)*B(M)
      RS=R2PRIM*S3PRIM-S2PRIM*R3PRIM+TBAR*AK*(R2PRIM*SPRIME-
3RPRIME*S2PRIM)
      IF(M.GE.NCDEF)GO TO 303
      IF(CABS(RS-FSAVE).GT.TEST)GO TO 300
303  FUN=RS
      RETURN
      END

```

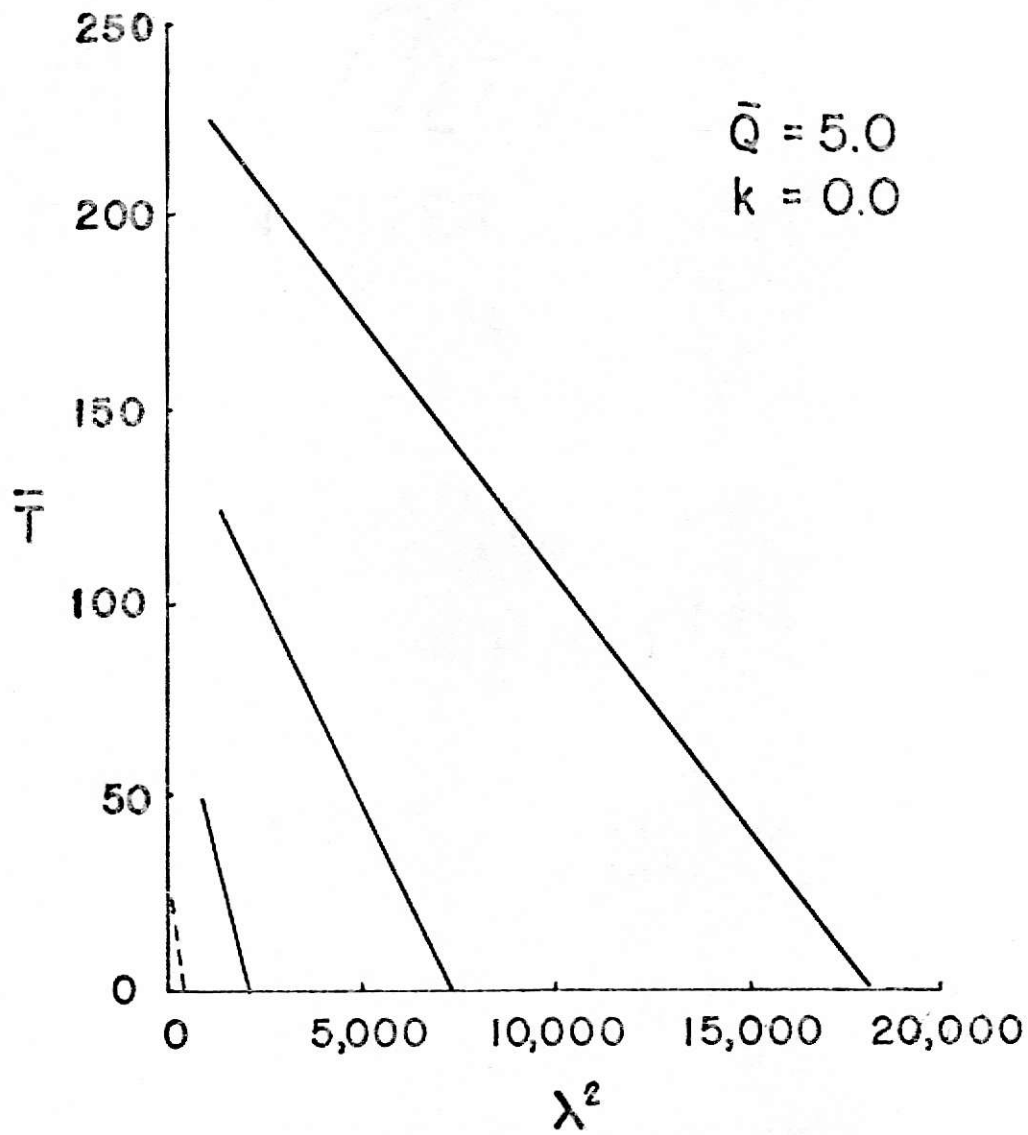
ANALYSIS OF RESULTS

The results obtained from the preceding program are shown on the following graphs. Some difficulty was encountered in obtaining values near the natural frequency origin, because at $\lambda^2=0$ the differential equation is satisfied identically. Therefore the value of the characteristic equation becomes very small as λ^2 approaches zero in the function subroutine and the computer does not have a high enough accuracy to clearly distinguish lower roots. When double precision was introduced there was only a slight improvement. In some cases several roots very close together resulted. Because of this some of the values shown on the graphs have been indicated by dotted lines. These are the values obtained which seemed the most reasonable in terms of the previous study by Huang and Walker, and the expected results.



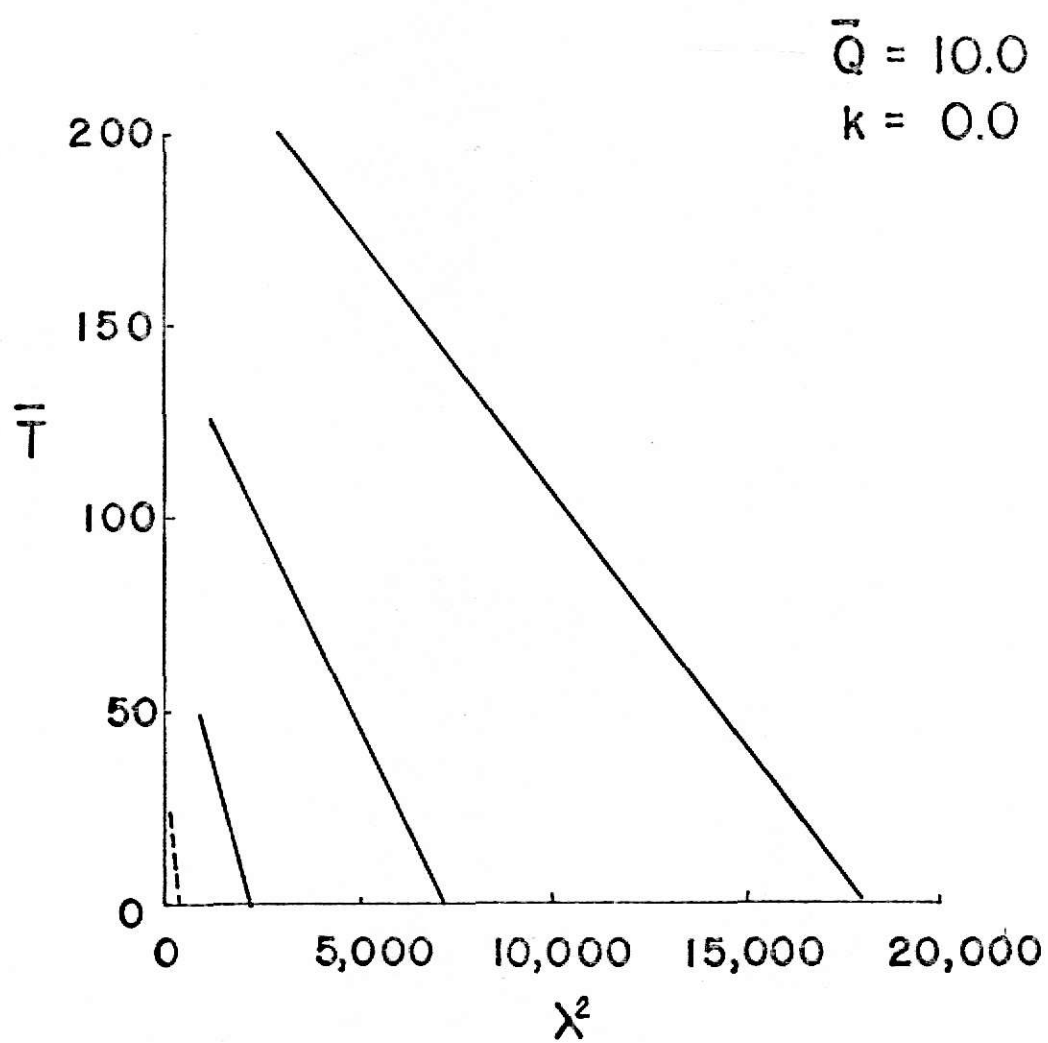
Thrust vs λ^2

Fig. 2



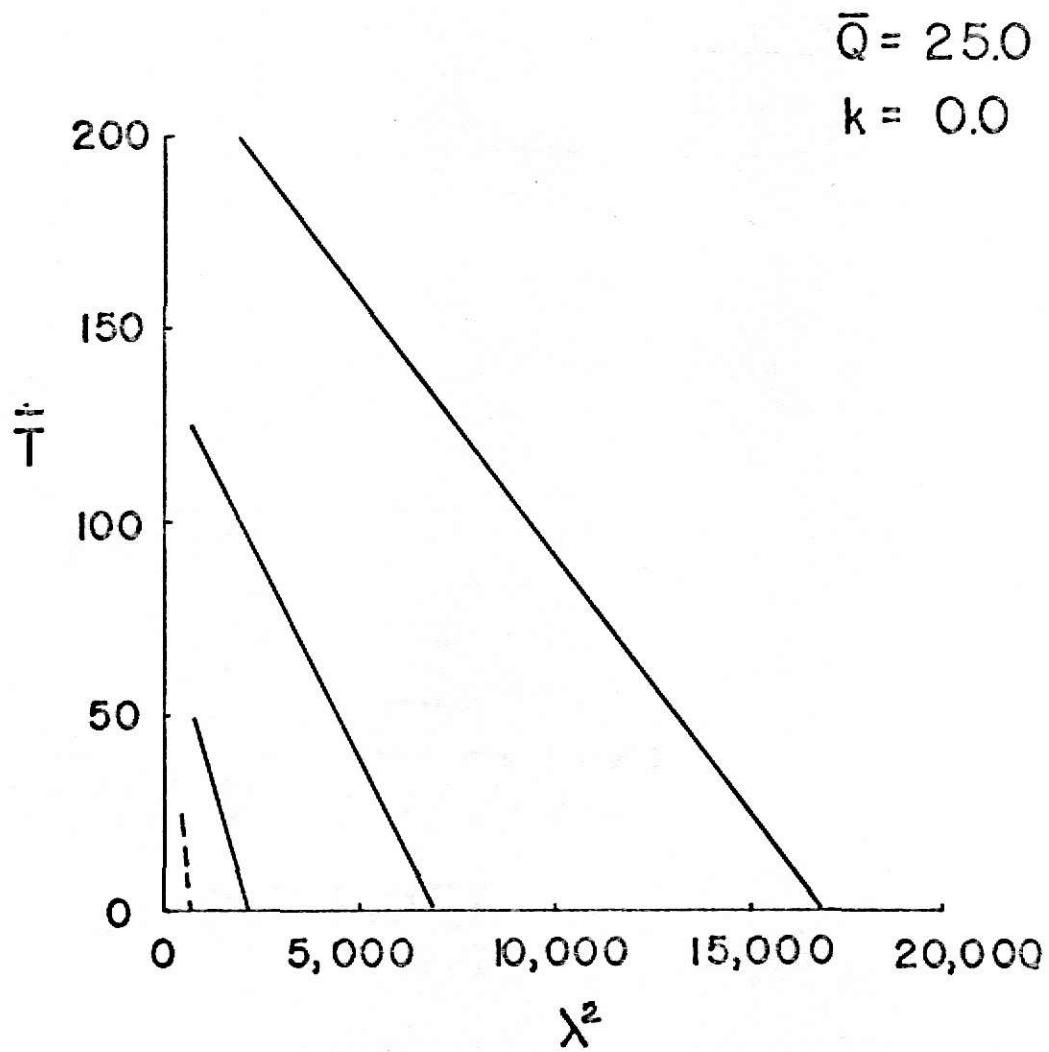
Thrust vs. λ^2

Fig. 3



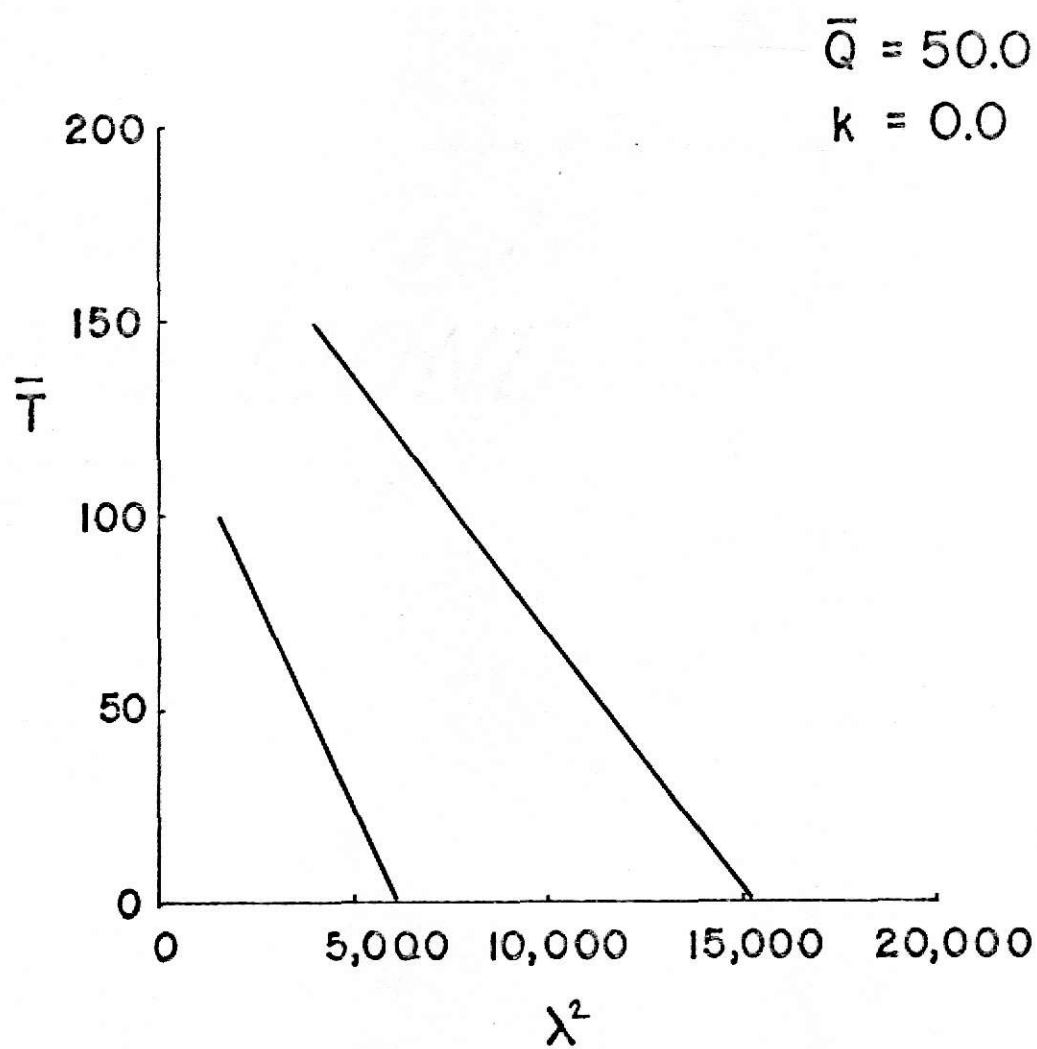
Thrust vs. X^2

Fig. 4



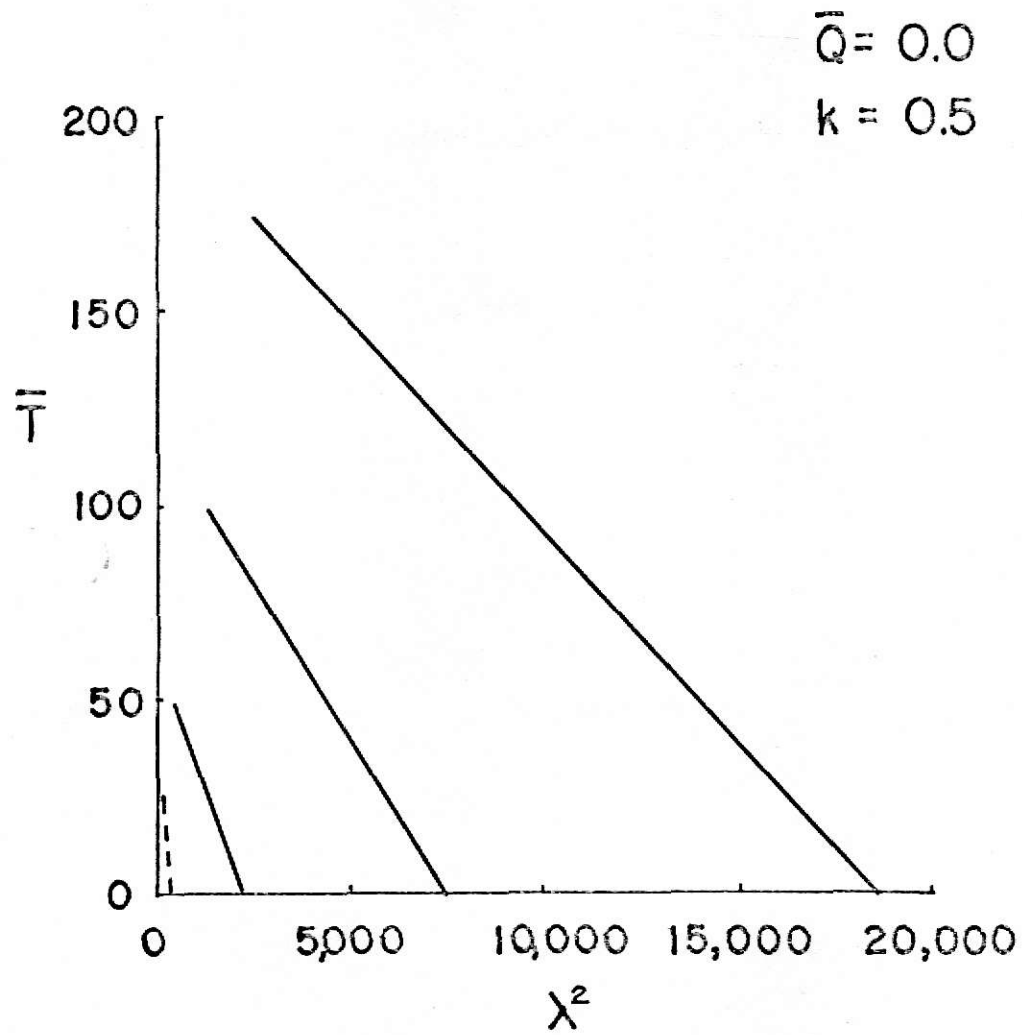
Thrust vs λ^2

Fig. 5



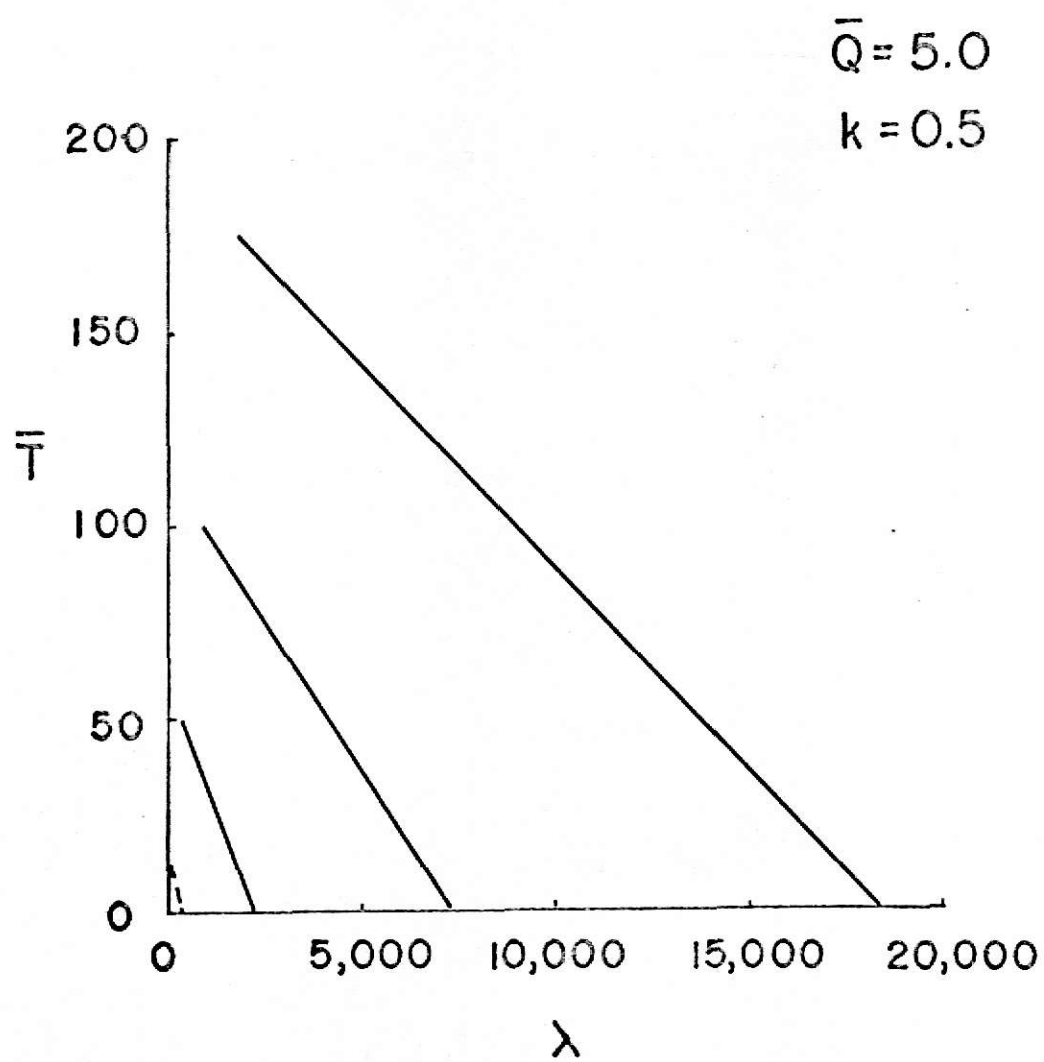
Thrust vs. λ^2

Fig. 6



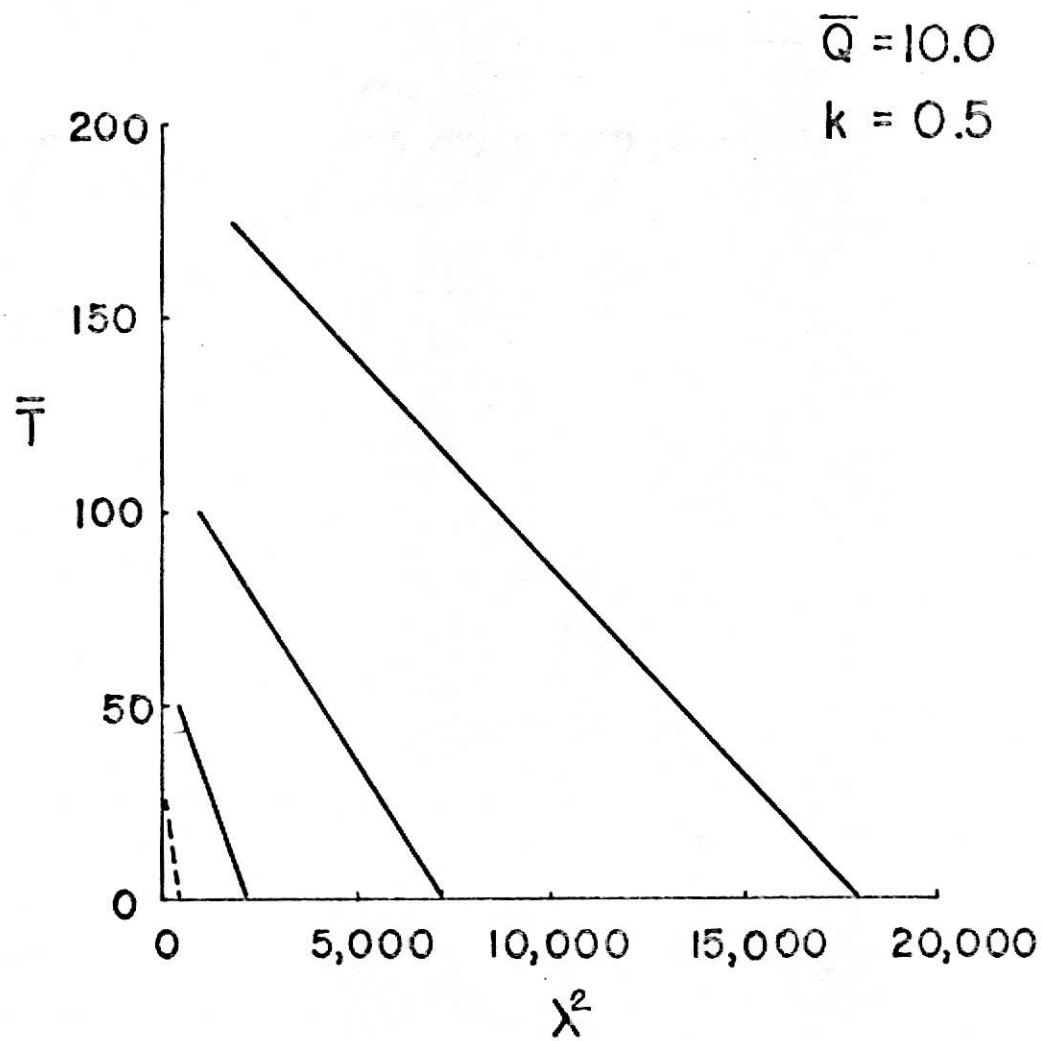
Thrust vs. X^2

Fig. 7



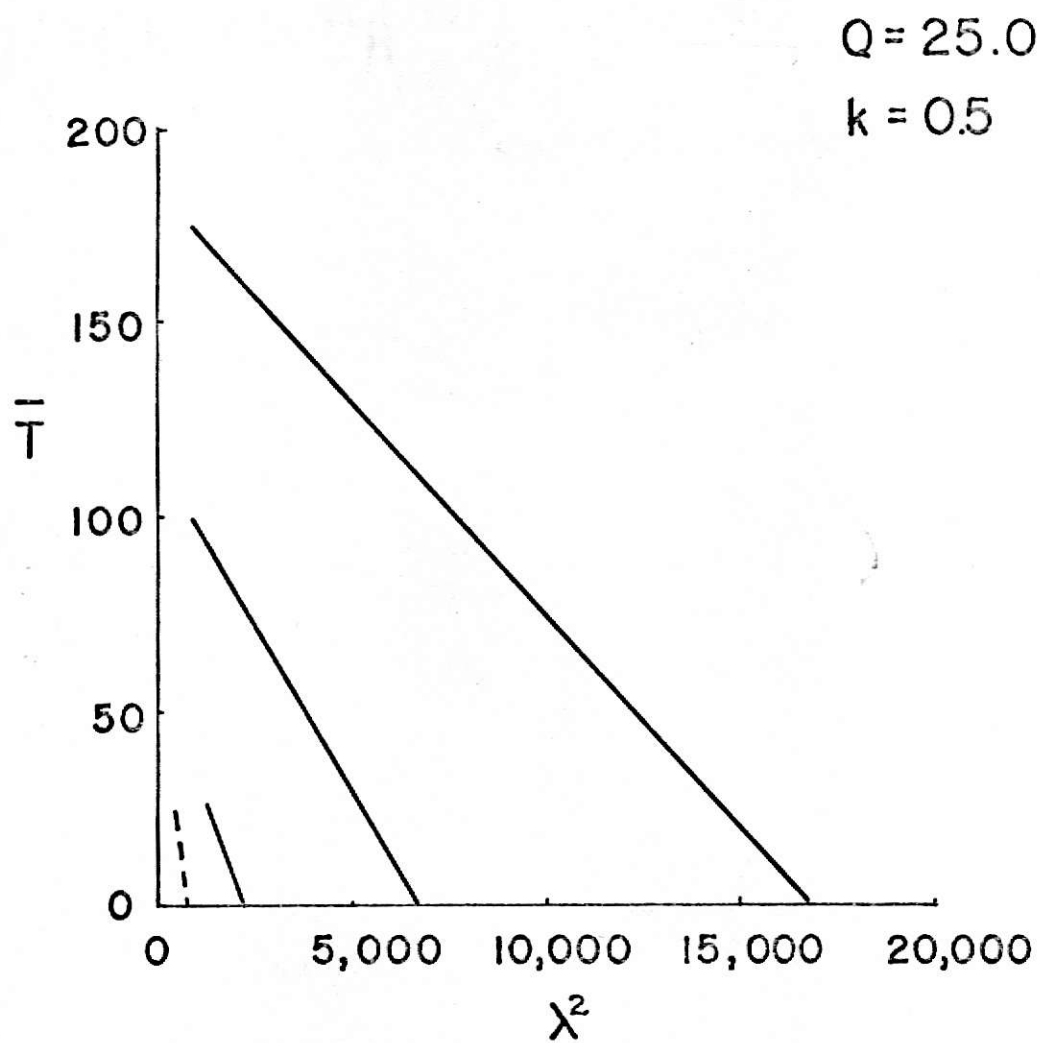
Thrust vs. λ

Fig. 8



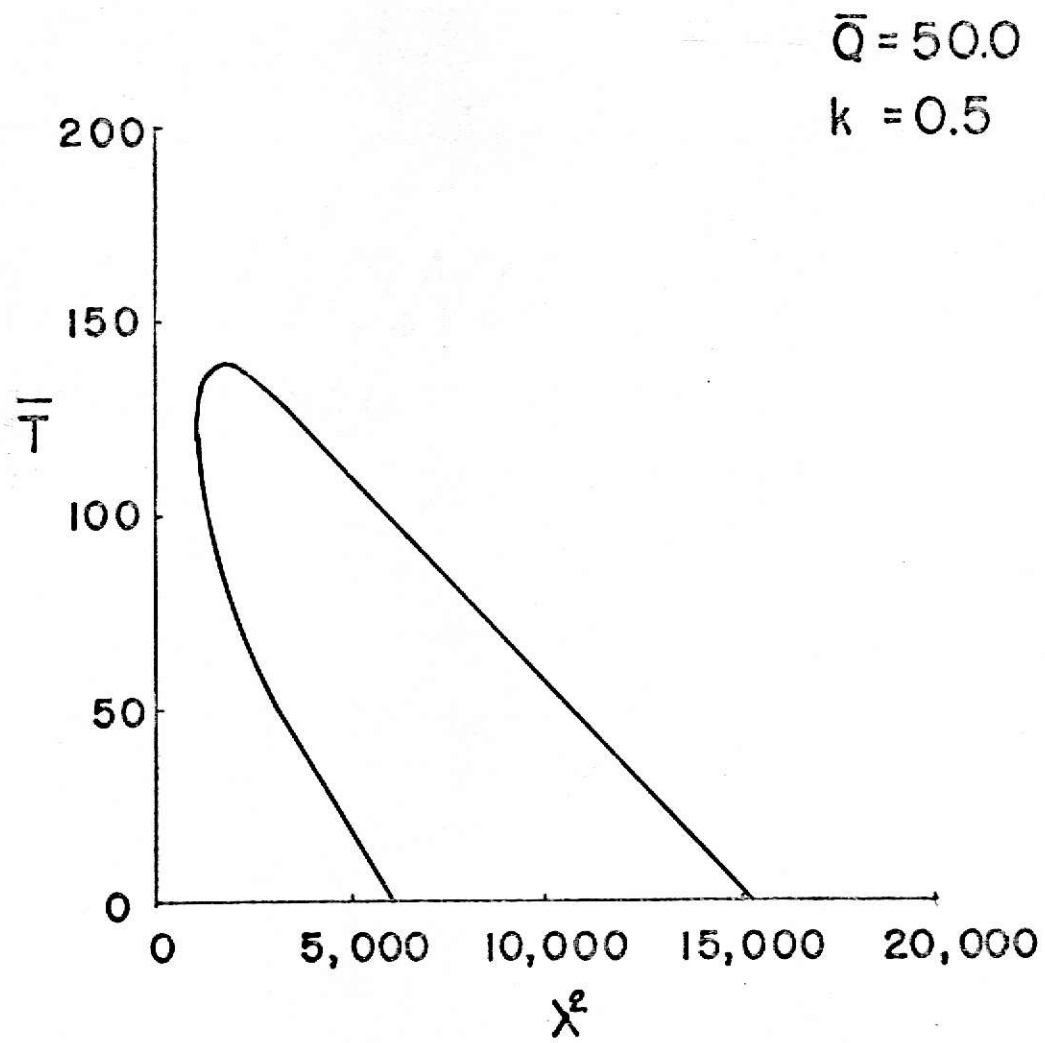
Thrust vs X^2

Fig. 9



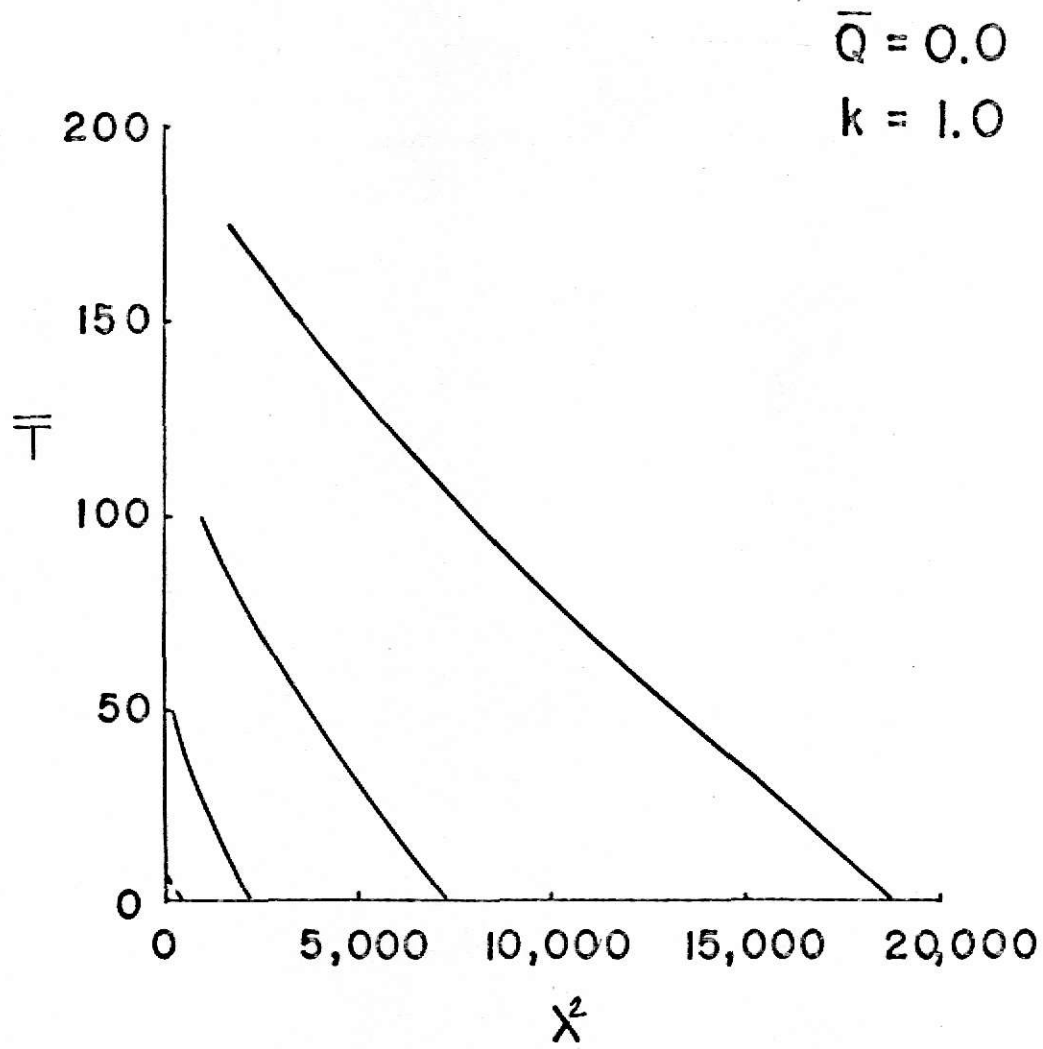
Thrust vs X^2

Fig. 10



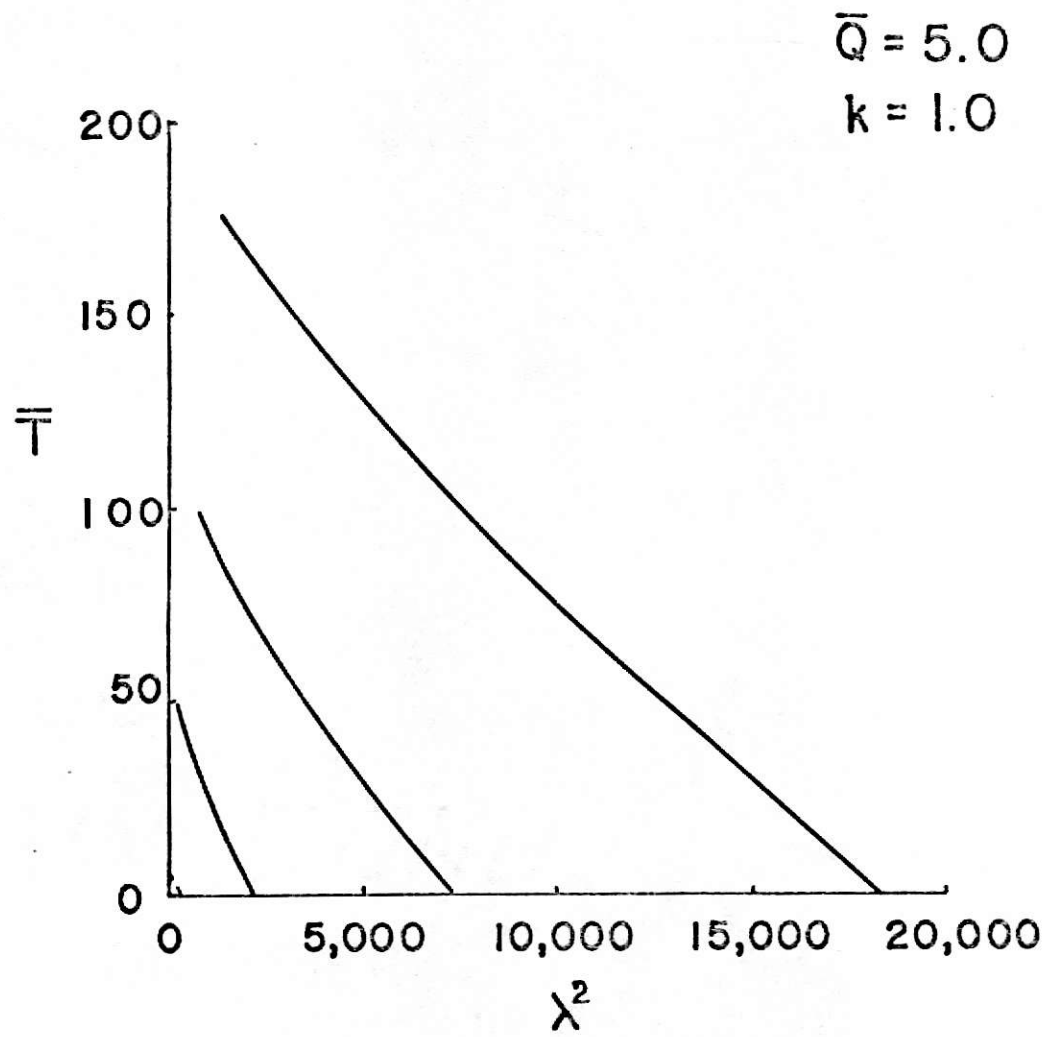
Thrust vs X^2

Fig. 11



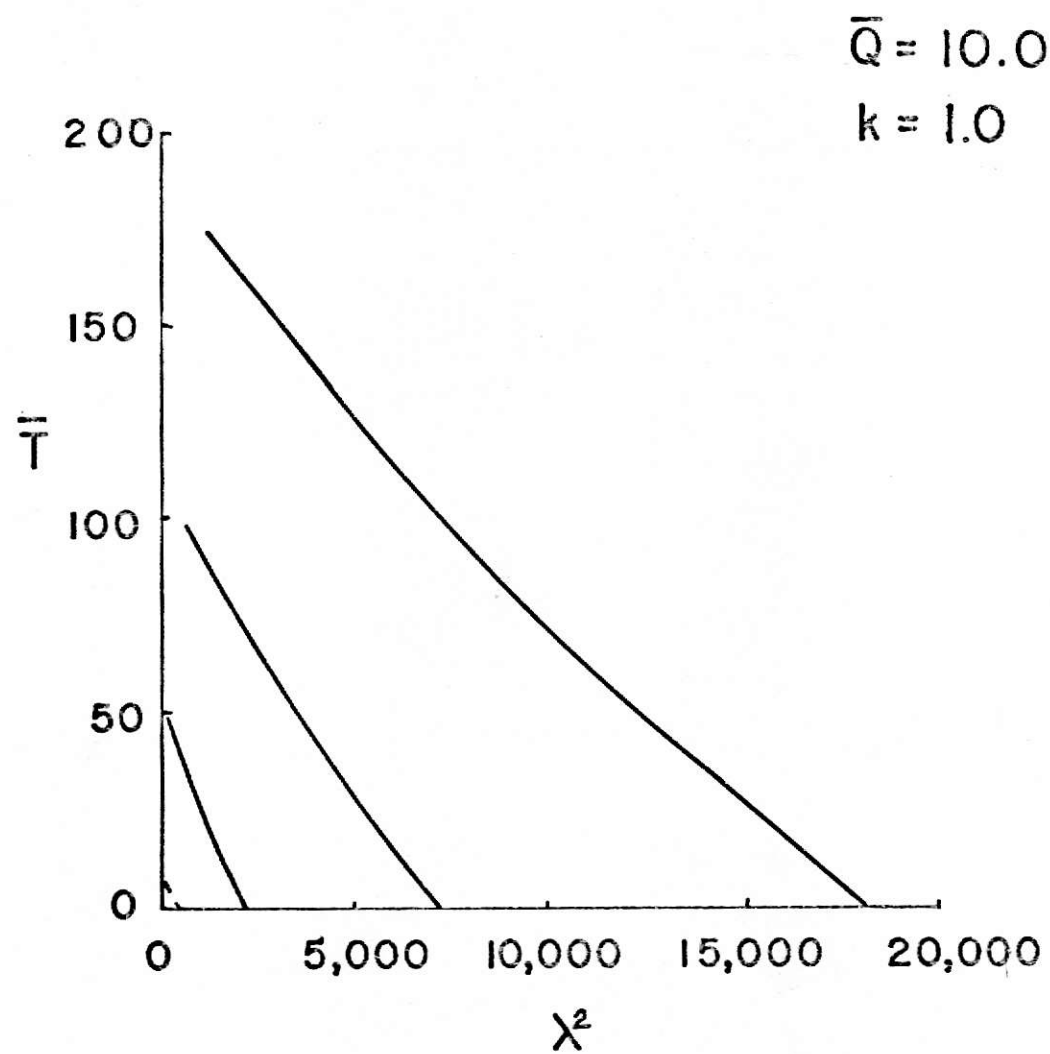
Thrust vs. X^2

Fig. 12



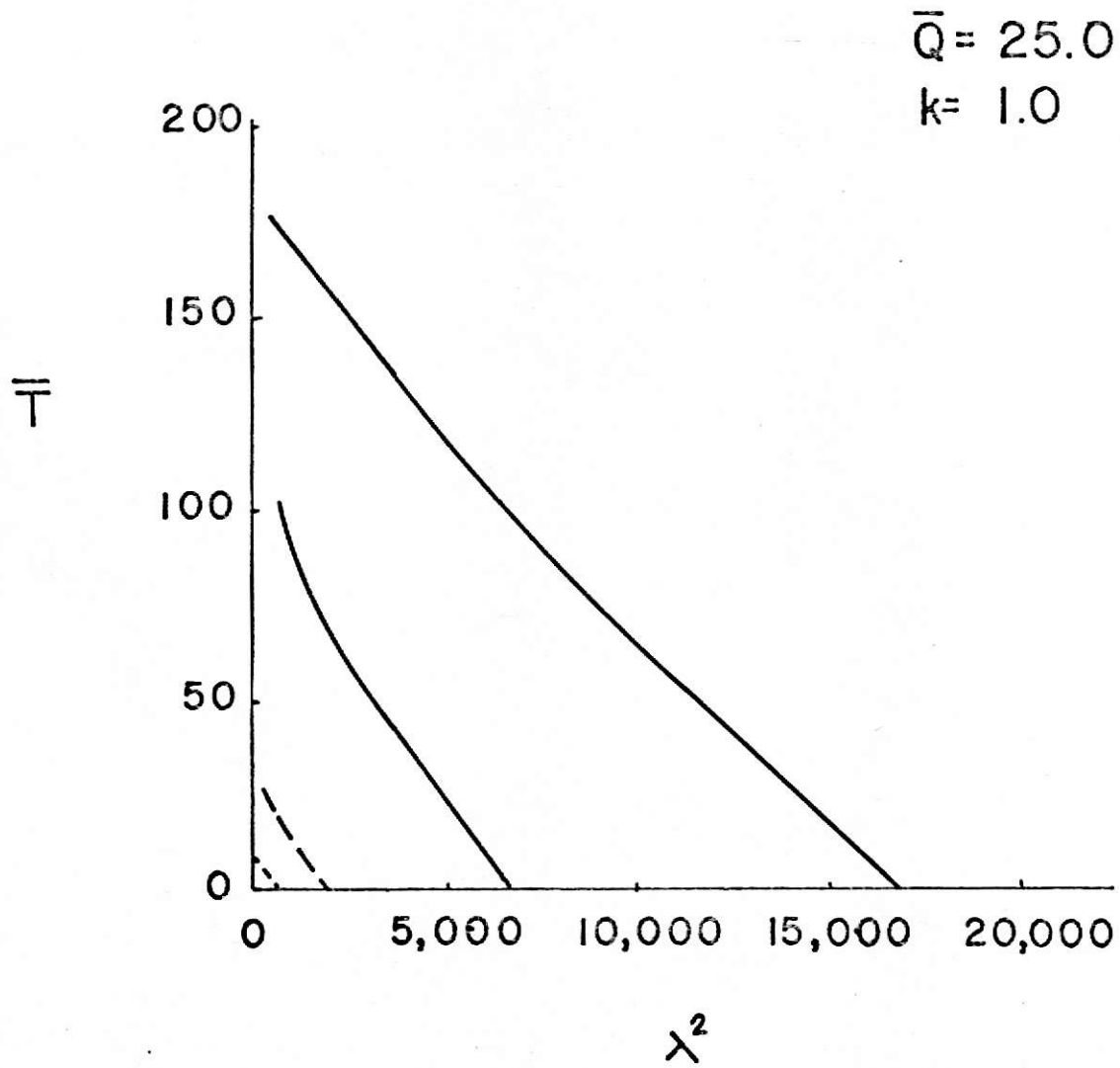
Thrust vs. λ^2

Fig. 13



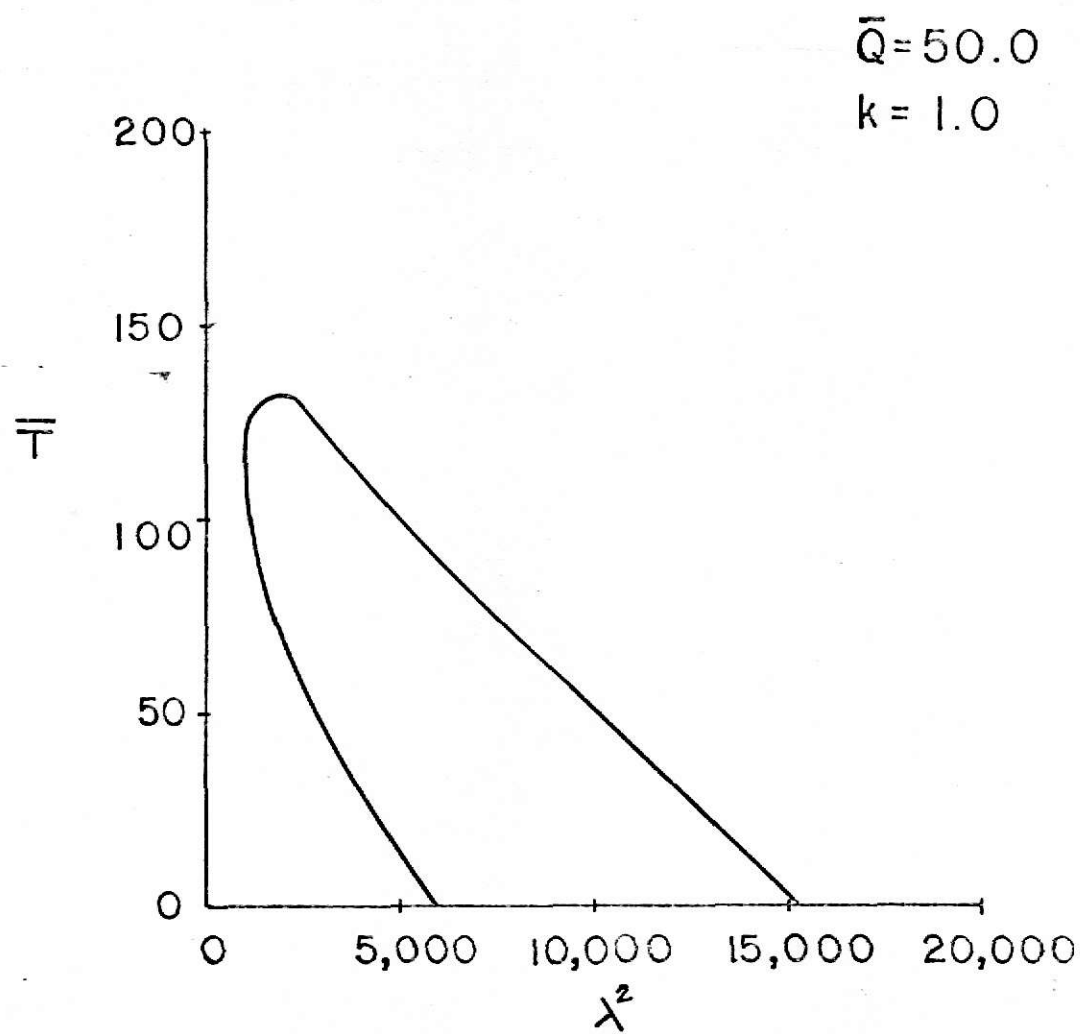
Thrust vs. λ^2

Fig. 14



Thrust vs. λ^2

Fig. 15



Thrust vs. λ^2

Fig. 16

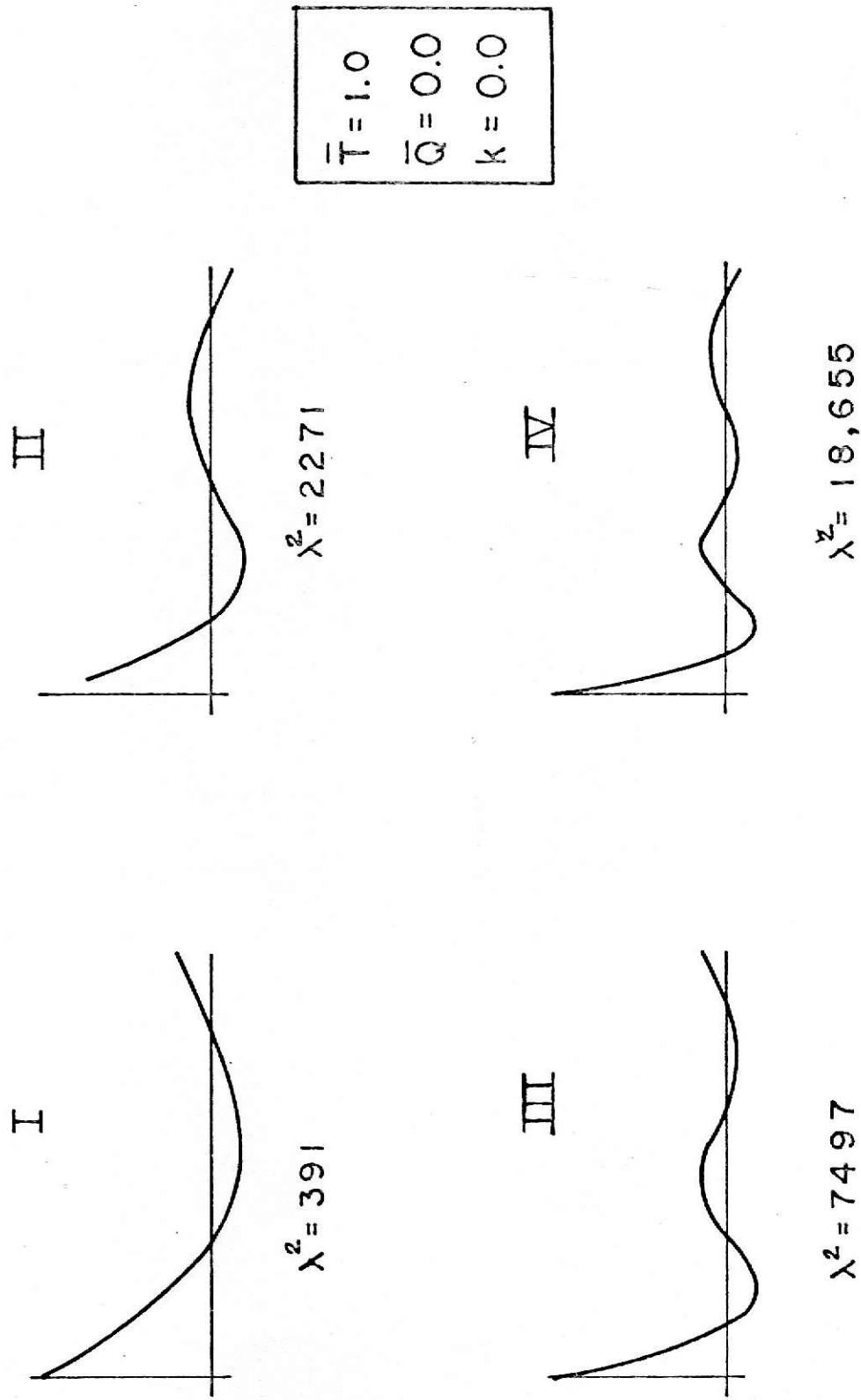
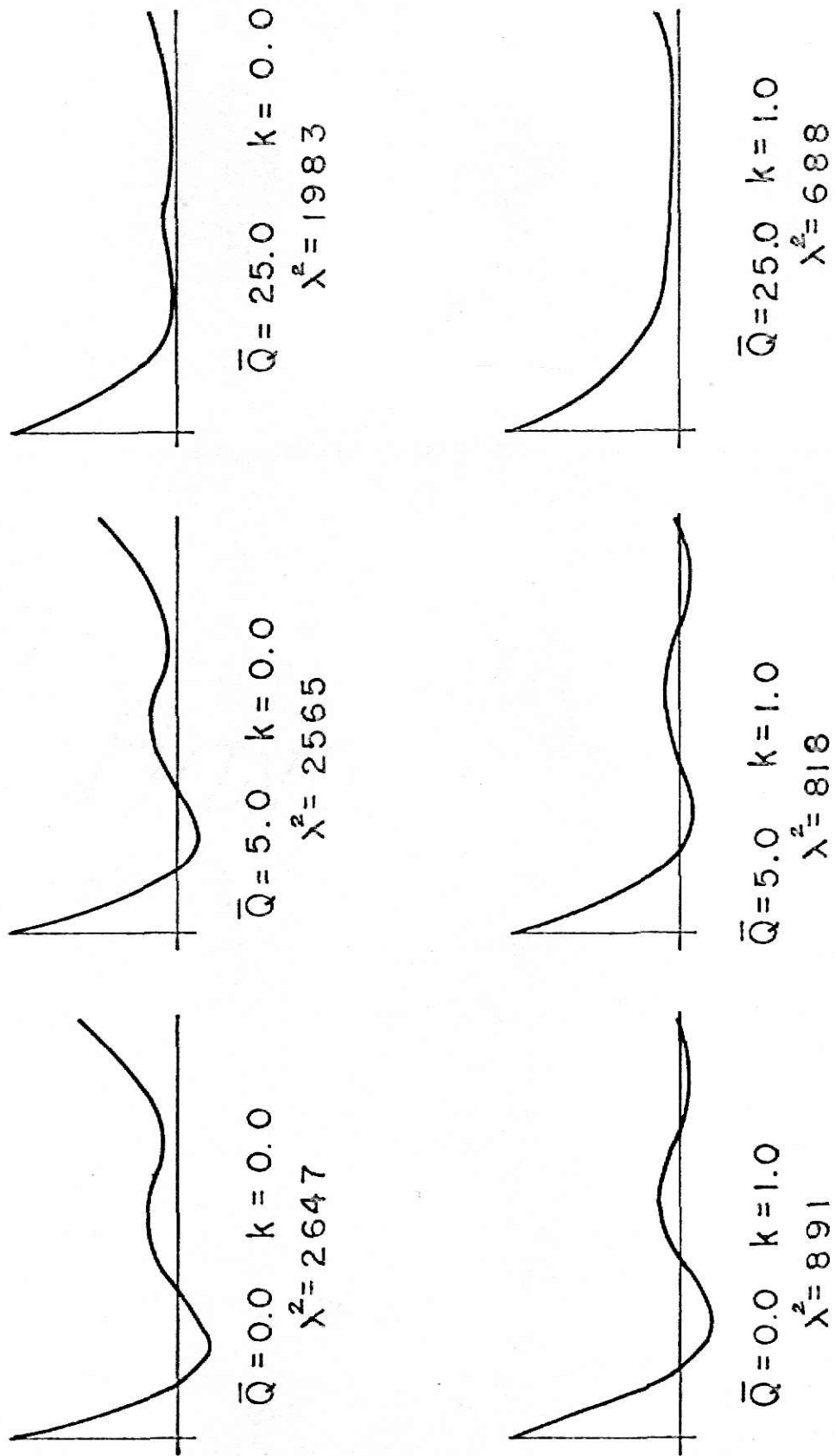


Fig. 17. FIRST FOUR MODES OF VIBRATING BEAM



$$\bar{T} = 100$$

Fig. 18. THE EFFECT OF DAMPING ON THE THIRD MODE

THE EFFECTS OF DAMPING ON THE NATURAL FREQUENCIES

The results obtained in the Huang-Walker paper are shown on Figs. 2, 7 and 12. It may be seen from these that as the thrust goes from zero to $\frac{d\psi}{d\xi}$, the lines of constant mode rotate counterclockwise about their lowest point. That is, the thrust angle has a greater effect on the natural frequencies at high thrusts than it has at relatively low thrusts. The same is true when shear is applied to the system. A comparison of Figs. 6, 11 and 16 shows this clearly.

An interesting result of the addition of damping may be observed on Figs. 11 and 16. The damping causes the constant mode shape curves to come together on the real side of the thrust axis; this is exactly the way a beam of uniform circular cross section behaves [1]. From this it would seem that damping might tend to decrease the effects of the parabolic shape of the beam. Further analysis would be necessary to see if there were any numerical correlation between the case of the undamped prismatic beam and the case under consideration.

The three series of Figs. 2 through 6, 7 through 11, and 12 through 16, indicate the effects of air friction on natural frequencies for constant thrust angle. The lines of constant mode maintain the same slope and move to the left as the damping increases. Since the lines remain parallel it appears that the amount of thrust has little influence on the effects of damping for constant k , and increased damping will lower the natural frequencies independent of thrust.

In Figs. 6, 11, and 16, where there is high damping, the first two lines of constant mode do not appear. This would indicate that at high damping the first mode shape to be reached is the third mode shape. This result, along

with the discussion of mode shapes which follows, leaves certain questions open for further work in this area, since the problem of damped mode shapes is a complicated one.

THE EFFECTS OF DAMPING ON THE MODE SHAPES

Fig. 17 indicates the first four mode shapes of the parabolic beam with no damping, zero thrust angle, and very small thrust. Fig. 18 shows the effect of damping on the third mode shape^{*} for a constant thrust and two different thrust angles. These graphs are typical of those for the other mode shapes. It may be seen that as the damping increases the mode shape becomes less clearly defined.

* A damped mode shape has been defined here as one which has the same general shape as the undamped mode shape, or a different shape which may be clearly shown to have progressed from the undamped shape (such as for high damping in Fig. 17). The question of whether these can be considered mode shapes at all is an interesting one since they do represent solutions of the characteristic equation, and yet they do not look like what is generally considered to be a mode shape; however, for the purposes of the present study the above definition has been used.

BIBLIOGRAPHY

1. Huang, C. L., and H. S. Walker. "Vibration and Stability of Rockets." American Astronautical Society, Advances in Astronautical Sciences. 24:5-8. 1967.
2. Wayland, Harold. Differential Equations Applied in Science and Engineering. Princeton: D. Van Nostrand Company, Inc., 1959.
3. James, M. L., G. M. Smith and J. C. Wolford. Applied Numerical Methods for Digital Computation with FORTRAN. Scranton: International Textbook Company, 1967.
4. Sun, Sai-Mu. "The Eigenvalue Problem for Natural Frequency of Uniform Beam with Linearly Varying Axial Load." Master's Report, Kansas State University, 1967.
5. Fauconneau, Guy, and William M. Laird. "The Eigenvalue Problem for Beams and Rectangular Plates with Linearly Varying In-Plane and Axial Load." NASA CR-549, 1966.
6. Sharp, George R. "An Investigation of the Natural and Forced Lateral Vibration of a Free-Free Beam (Missile)." The American Society of Mechanical Engineers, Vibrations Conference. March, 1967.
7. Tong, Kin N. Theory of Mechanical Vibration. New York: John Wiley & Sons, Inc., 1963.

ACKNOWLEDGEMENT

I wish to thank Dr. H. S. Walker, Professor of Mechanical Engineering, for his invaluable assistance and encouragement in the preparation of this paper.

THE EFFECTS OF DAMPING ON THE LATERAL VIBRATIONS OF A FREE-FREE BEAM

by

Carol Ann Rubin

B. S., Columbia University, 1966

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1969

ABSTRACT

The problem of the influence of transverse vibrations on the dynamic stability of rockets has been dealt with recently in a paper by Huang and Walker. This study idealized the rocket as a free-free elastic beam with a constant end thrust and a constantly varying cross-sectional area. It is the purpose of the present study to introduce an additional variable and, using the Huang - Walker paper as a basis for comparison, see what the effects are on the vibration and stability properties. This new variable provides an approximation of the effect that frictional drag will have on rockets. A constant axial load is assumed to act along the axis of the beam and in a direction opposite to the motion of the beam. The simplifying assumptions made here for the form of the drag force allow for the derivation and solution of the governing differential equations, and at the same time provide a basis for discussion of the effects of a motion-opposing force on a rocket.

The governing differential equation of motion is derived using simple beam theory; and the characteristic equation for the special case of a beam with a parabolic surface of revolution is obtained by means of Frobenius' method. In addition, a variable indicating the angle of applied thrust is introduced to further generalize the results.