ON THE ALMOST SURE ASYMPTOTIC STABILITY OF LINEAR DYNAMIC SYSTEMS WITH STOCHASTIC PARAMETERS

by

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I. INTRODUCTION

The behavior of real-life systems are generally probabilistic in nature, so it seems natural to study the motion of random systems such as systems subjected to random loads, containing random material properties, etc. A special class of random systems can be defined as stochastic systems which is the focus of the present investigation.

Since the linear system plays the fundamental role in the modeling of a dynamical process, the attention of the research is directed to the stability of the linear stochastic systems. It is assumed that the stochastic processes are known. The problem then becomes that of determining the conditions for stability of the system in terms of its parameters and the properties of the stochastic process.

The study of stochastic systems has taken a long time in its development. Originally, the theory of stochastic differential equations was developed by mathematicians as a tool for explicit construction of the trajectories of diffusion processes for given coefficients of drift and diffusion. Stochastic differential equations describing systems, such as one in which "white noise" acts, occur quite naturally in physical and engineering sciences.

The study of stochastic systems can be grouped into two branches: stochastic differential equations with Gaussian white noise coefficients (the Itô Differential Equations) and stochastic differential equations with non-white noise coefficients. The greatest advances in the stability of stochastic systems have been achieved with the use of the Itô differential equations. However, this branch requires the reader to have great understanding of stochastic process theory which has slowed, to a certain extent, its development as a practical tool. The techniques employed in the other branch are somewhat more direct, but the advances are not as great. In both cases, most of the results, that have been obtained, yield only sufficient conditions for stability which often have been too conservative for practical use.

Applications of linear stochastic differential equations cover a wide spectrum of problems. Such an application for a structural engineer and an applied mechanist would be the linearized equations of beams subjected to random forces at the boundaries. Studies of the dynamics of satellites in orbit generate an application of linear stochastic differential equations for the control engineer. The electrical engineer has applications in the studies of parametric amplifier dynamics. The stability of time varying channels described by the stochastic equations are a concern for the communications engineer. Certain chemical and biological problems also involve stochastic differential equations.

The study of the stability of stochastic systems is relatively new. The recent origins of this study commenced in the early 1950's by Rosenbloom [1] and Stratonovich [2]. They were followed by Bertram and Sarachik [3], Kats and Krasovskii [4], and Samuels [5] in the early 1960's. These researchers were concerned with the stability of the moments of the solution process.

Later, it was realized that sample stability is the more meaningful property to determine. This view is based on the fact that a sample solution is observed when a real system subjected to random excitation is tested. Since then most research has been directed to study the almost sure stability properties. Caughey [6], Caughey and Gray [7], Kushner [8], Khas'minskii [9], Nevel'son and Khas'minskii [10], Infante [11], Kozin [12, 13], Kozin and Wu [14], Parthasarathy and Evan-Iwanowski [15], Kozin and Milstead [16], as well as many others have made fine contributions to the study of the almost sure asymptotic stability of linear stochastic systems. Except for Parthasarathy and Evan-Iwanowski [15] and Kozin and Milstead [16], the examples presented so far have all apparently been confined to scalar second-order systems. In these studies, the system equations were written in the form

$$\dot{x} = [A + F(t)]x, \tag{1-1}$$

where x is an n vector, A is an n x n constant stability matrix and F(t) is an n x n matrix whose nonidentically zero elements are stationary ergodic stochastic processes.

Kozin's results [12] were found to be too conservative when applied to a second order scalar equation. Caughey and Gray [7] were able to obtain better results through a Liapunov-type approach. Assuming that A is a stability matrix, in reference [7], the symmetric, positive definite, Liapunov matrix P was generated from the matrix equation,

$$A^T P + PA = -I \tag{1-2}$$

where I is an n x n identity matrix. Then using matrix P and two quadratic norms of the forms

$$||x|| = \sum_{i=1}^{n} |x_i| \tag{1-3}$$

and

$$||x||_P = (x^T P x)^{\frac{1}{2}} = (\sum_{i,j}^{n} P_{ij} x_i x_j)^{\frac{1}{2}}, \tag{1-4}$$

conditions for the almost sure stability were found for the system given by equation (1-1). Later, Infante [11] extended these stability theorems so that they could be applicable for any quadratic norm.

Assuming a Liapunov function of the form $V = x^T P x$, Infante utilized the properties of pencils of quadratic forms and obtained sufficient conditions for the almost sure stability. He also used a brute force approach to determine the optimum Liapunov matrix associated with the norm. Because of the ad hoc nature of the optimization procedure, he applied the procedure to a few second order scalar equations only. For the two specific examples of second order scalar systems, Infante's theorem provided the sharpest stability bounds. However, he also demonstrated that in many cases an optimum matrix may not exist. The procedure for finding an optimum matrix for relatively large systems still remains an open problem. Infante himself writes in his paper: "This technique is very time-consuming for high-order systems."

These studies suggested a definite need for the development of a systematic computational procedure which could be applied to relatively large systems. Parthasarathy and Evan-Iwanowski [15] documented a study aimed at meeting this need which does not yield stability bounds that are too conservative. In their work, the stability bounds were obtained through a generalization of the theorem presented in reference [7]. However, the computation scheme is rather lengthy and very expensive in terms of computer time. To be precise, first, one has to perform the modal analysis to express the equations of motion in terms of the "quasi-normal" coordinates which in turn can be written in the form of equation (1-1). The Liapunov matrix is then generated from

$$A^T P + PA = -Q, \tag{1-5}$$

where Q is an arbitrary, symmetric, positive definite matrix. Equation (1-5) is then solved in conjunction with the Fletcher-Powell-Davidson optimization technique in order to achieve the optimum stability bounds on the elements of F(t). A time-scale parameter is also selected in reference [15] which may or may not have a tangible effect on the stability bounds. If the matrix Q is suitably optimized, the time-scale parameter has little influence on the bounds but does facilitate in locating the optimal Q in the computer search. Although a systematic computational procedure has been sketched in reference [15], it is definitely time consuming as expressed by the authors themselves: "Admittedly, there yet remains much to be accomplished. The ultimate goal may well be the establishment of a theorem that would render the search for the optimum superfluous."

More recently, another optimization scheme has been suggested by Kozin and Milstead [16] for higher-order systems. However, once again the technique is rather complicated and time consuming.

What seems to be needed is a criterion which can be applied directly to the mass, damping, and stiffness matrices of the second order vectorial differential equation because the application of finite elements or other discretization techniques to continuous problems naturally yield such a set of equations. This should also show the effects of qualitative and quantitative changes in these matrices on the stability conditions. The method should be simple and straightforward in terms of computation and still obtain results that are not unduly conservative for large systems. In this study, a theorem directly applicable to a set of second order differential equations is presented which may prove to be useful for certain problems. A Liapunov matrix, in terms of the mass, damping, and stiffness matrices constituting the constant part of the system equations, is also proposed. Several examples are presented to show applications.

In chapter II, a Liapunov function is selected for the study of the almost sure asymptotic stability of linear discrete systems described by a set of second order equations with stochastic parameters. Using this function, a theorem and related corollaries are obtained through an extension of the approach suggested by Infante [11].

Several examples are presented in chapter III to demonstrate the procedure suggested in chapter II. This includes the application of the proposed technique to continuous systems through the use of Galerkin and finite element approximations, as well as to the discrete systems. On the basis of corollaries I and II, a general computer program is developed without placing any restrictions on the dimensions of the system matrices.

In chapter IV, the theorem is extended to yield response and stability bounds for systems for which there exist no equilibrium positions due to the forcing terms appearing in the equations of motion. Some examples are also included.

II. EQUATION OF MOTION AND STABILITY THEOREMS

In this chapter, the stability theroem and related corollaries guaranteeing the almost sure asymptotic stability are presented.

2.1 Preliminaries

Before dealing with the actual formulations, it is convenient to introduce some definitions and important lemmas which will prove useful throughout the analysis.

Definition 1. A stochastic process is a family of random variables $\{X(t)\}$ defined on a probability space, where t varies in a real interval I (I is open, closed, or half-closed).

Definition 2. The ergodic property used in this investigation is one insuring the equality of time averages and ensemble averages of a stochastic process. The mathematical description is given by

$$E\{G[X(t)]\} = E\{G[X(0)]\} = \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} G[X(\tau)]d\tau \qquad (2\text{-}1)$$

exists with probability one for a measurable, integrable, function G defined on the stochastic process X(t). This is often referred to as the expected value of $G[X(t)]$.

Definition 3. [7] The trivial solution, $||x|| = 0$, is almost surely asymptotically stable in the large, if for all solutions of the system we have the property that

$$\lim_{t\to\infty} ||x(t;\, x_0,\, t_0)|| = 0 \qquad (2\text{-}2)$$

holds with probability one for all x_o.

Lemma 1. [24] The characteristic equation of the regular pencil $x^T Dx - x^T Bx$ has n real roots λ_s, where D and B are n x n real symmetric matrices, B is positive definite and x is an n vector. The matrix DB^{-1} has the same eigenvalues as the pencil and, if these eigenvalues are ordered in magnitude as

$$\min_s\{\lambda_s[DB^{-1}]\} = \lambda_1[DB^{-1}] \leq . . . \leq \lambda_n[DB^{-1}] = \max_s\{\lambda_s[DB^{-1}]\}, \quad (2\text{-}3)$$

then

$$\lambda_1[DB^{-1}] = \min_x \frac{x^T Dx}{x^T Bx}, \quad (2\text{-}4a)$$

$$\lambda_n[DB^{-1}] = \max_x \frac{x^T Dx}{x^T Bx} . \quad (2\text{-}4b)$$

Lemma 2. [17] (Schwarz inequality)

$$E\{|XY|\} \leq [E\{X^2\}E\{Y^2\}]^{\frac{1}{2}} \quad (2\text{-}5a)$$

for both random and deterministic functions X and Y.

Lemma 3. [17] (Minkowski inequality)

$$[E\{|X + Y|^P\}]^{\frac{1}{P}} \leq [E\{|X|^P\}]^{\frac{1}{P}} + [E\{|Y|^P\}]^{\frac{1}{P}}, \; p > 1 \quad (2\text{-}5b)$$

for both random and deterministic functions X and Y.

2.2 Equation of Motion and Selection of Liapunov Function

Consider the system described by

$$M\ddot{x} + [C_o + C(t)]\dot{x} + [K_o + K(t)]x = 0, \quad (2\text{-}6a)$$

where x is an n vector; M, C_0, and K_0 are nonsingular $n \times n$ constant matrices; C(t) and K(t) are $n \times n$ matrices whose nonzero elements $c_{ij}(t)$ and $k_{ij}(t)$ are measurable, strictly stationary, stochastic processes which satisfy an ergodic property guaranteeing the equality of time averages and ensemble averages. The corresponding mathematical description is given by definition 2 recorded in section 2.1. For simplicity, let $E\{C(t)\} = E\{K(t)\} = 0$.

Before investigating the almost sure stability of equation (2-6a), consider the constant counterpart of equation (2-6a) given by

$$M\ddot{x} + C_0\dot{x} + K_0 x = 0. \tag{2-7a}$$

From previous studies, it is known that equation (2-7) must be asymptotically stable before the stability bounds for equation (2-6) can be found. It is also evident from earlier studies that the sharpness of the stability bound for equation (2-6a) is heavily dependent on the choice of the Liapunov matrix associated with equation (2-7a).

For further considerations, equations (2-6a) and (2-7a) are rewritten as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}[K_0 + K(t)] & -M^{-1}[C_0 + C(t)] \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{2-6b}$$

and

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K_0 & -M^{-1}C_0 \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{2-7b}$$

where $x_1 = x$ and $x_2 = \dot{x}_1$. For equation (2-7), consider the Liapunov function given by

$$V(x) = x_1^T P_1 x_1 + x_2^T P_2 x_2 + x_2^T P_3 x_1, \tag{2-8a}$$

where P_1, P_2, and P_3 are $n \times n$ constant matrices such that $P_1 = P_1^T$ and $P_2 = P_2^T$ and P_1 and P_2 are positive definite. $(\cdot)^T$ means the transpose of the matrix $(\cdot)$. It is observed that V can also be written as

$$V = y^T \left[\begin{array}{c|c} P_1 & \frac{1}{2}P_3^T \\ \hline \frac{1}{2}P_3 & P_2 \end{array}\right] y \equiv y^T P y, \qquad (2\text{-}8b)$$

where the 2n vector y is defined as $y^T = \{x_1^T, x_2^T\}$. Taking the time-derivative of V and evaluating it along the trajectory of (2-7), one obtains

$$\dot{V} = y^T \left[\begin{array}{c|c} -\frac{1}{2}\{P_3^T M^{-1}K_o + (M^{-1}K_o)^T P_3\} & P_1 - (M^{-1}K_o)^T P_2 - \frac{1}{2}P_3^T M^{-1}C_o \\ \hline P_1 - P_2 M^{-1}K_o - \frac{1}{2}(M^{-1}C_o)^T P_3 & -\{P_2 M^{-1}C_o + (M^{-1}C_o)^T P_2 - \frac{1}{2}(P_3 + P_3^T)\} \end{array}\right] y \equiv$$

$$\equiv y^T A_o y. \qquad (2\text{-}9)$$

It is known [18, 19] that if the system represented by equation (2-7) is asymptotically stable, then for a given negative definite A_o, a unique positive definite P exists and can be found by solving the resulting Liapunov equation.

Without any loss of generality, the off diagonal blocks of the partitioned matrix A_o can be set to zero. However, one must keep in mind that P_1 and P_2 must both be symmetric and positive definite. Following reference [20], this can be achieved by selecting

$$P_1 = \frac{1}{2}P_3^T M^{-1}C_o + (M^{-1}K_o)^T P_2$$

which in turn implies that

$$P_2 M^{-1}K_o - \frac{1}{2}P_3^T M^{-1}C_o$$

should remain symmetric in addition to P_2. It is immediately observed that this would require construction of two matrices, P_2 and P_3, which involves n^2+n unknowns. This idea has been pursued by Walker [20] to find the necessary and sufficient conditions for asymptotic stability of equation (2-7). It is obvious that the use of such a Liapunov function P would still not be a simple choice from the viewpoint of computation. A simple and natural choice seems to be

$$P_3 = P_2 M^{-1} C_o, \tag{2-10}$$

thereby implying,

$$P_1 = P_2 M^{-1} K_o + \tfrac{1}{2}(M^{-1}C_o)^T P_2 (M^{-1}C_o), \tag{2-11}$$

which is symmetric and positive definite if P_2 is symmetric, symmetrizes $M^{-1}K_o$ and both P_2 and $P_2M^{-1}K_o$ are positive definite. For this selection of P_1 and P_3, matrices A_o and P become

$$A_o = -\frac{1}{2}\left[\begin{array}{c|c} \{(M^{-1}C_o)^T P_2 M^{-1} K_o\} + \{(M^{-1}C_o)^T P_2 M^{-1} K_o\}^T & 0 \\ \hline 0 & P_2 M^{-1} C_o + (P_2 M^{-1} C_o)^T \end{array}\right] \tag{2-12}$$

and

$$P = \left[\begin{array}{c|c} P_2 M^{-1} K_o + \tfrac{1}{2}(M^{-1}C_o)^T P_2 (M^{-1}C_o) & \tfrac{1}{2}(P_2 M^{-1} C_o)^T \\ \hline \tfrac{1}{2} P_2 M^{-1} C_o & P_2 \end{array}\right] . \tag{2-13}$$

Thus, if the symmetric parts of $\{(M^{-1}C_o)^T P_2 M^{-1} K_o\}$ and $P_2 M^{-1} C_o$ remain positive definite, then A_o is obviously negative definite. Hence, if system (2-7) is asymptotically stable, then P can be guaranteed to be positive definite [18, 19]. However, a priori knowledge of system (2-7) being asymptotically stable is not needed because it can be shown that the Liapunov matrix P given by equation (2-13) is always positive definite as shown in the following.

Consider the quadratic form

$$z^T[P_2M^{-1}K_o + \tfrac{1}{4}(M^{-1}C_o)^TP_2(M^{-1}C_o)]z \tag{2-14}$$

which is positive definite because $P_2M^{-1}K_o$ and P_2 are positive definite. This can also be written as

$$z^T[P_2M^{-1}K_o + \tfrac{1}{2}(M^{-1}C_o)^TP_2(M^{-1}C_o) - \tfrac{1}{4}(M^{-1}C_o)^TP_2(M^{-1}C_o)]z. \tag{2-15}$$

From the form of P_3, given by equation (2-10), (2-14) now takes the form

$$z^T[P_1 - \tfrac{1}{4}P_3{}^TP_2^{-1}\ P_3]z\quad . \tag{2-16}$$

The above quadratic form can be easily written as

$$\left\{\begin{array}{c} z \\ \hline -\tfrac{1}{2}P_2^{-1}\ P_3z \end{array}\right\}^T \left[\begin{array}{c|c} P_1 & \tfrac{1}{2}P_3{}^T \\ \hline \tfrac{1}{2}P_3 & P_2 \end{array}\right] \left\{\begin{array}{c} z \\ \hline -\tfrac{1}{2}P_2^{-1}\ P_3z \end{array}\right\} \quad . \tag{2-17}$$

Since the quadratic form (2-14) is positive definite, it follows that the partitioned matrix in (2-17), which is P, is positive definite. Note that this is always true as long as both $P_2M^{-1}K_o$ and P_2 remain symmetric and positive definite.

For the special case of Rayleigh damping, i.e., when

$$C_o = \alpha M + \beta K_o \quad ; \alpha + \beta > 0,\ \alpha\beta \geq 0, \tag{2-18a}$$

A_o is clearly negative definite. The assumption of this form of damping is very common in the field of structural analysis since the actual damping mechanism is generally not known [21].

Yet another practical way of introducing damping is to assume C_o proportional to the critical damping matrix associated with the system. For this purpose, first the modal analysis is performed on the equation of motion (2-7a) with $C_0 \equiv 0$. This results in a set of decoupled equations in terms of the normal coordinates where the mass matrix simply becomes the identity matrix I and the stiffness matrix takes the diagonal form containing the n natural frequencies ω_n. Since the equations are decoupled, the critical damping in each mode is given by $c_n = 2\omega_n$ because $m_n = 1$. Generally, the C_0 matrix is assumed to be proportional to the diagonal matrix consisting of elements c_n.

Hence, it is seen that if x are the normal coordinates in equation (2-7a), then M=I, $K_0 = \left[\diagdown \omega^2 \diagdown\right]$ and C_0 can be written as

$$C_0 = \zeta K_0^{\frac{1}{2}}, \quad \zeta > 0 \tag{2-18b}$$

Substituting this form of damping in equation (2-12) and selecting $P_2 = M = I$, it is easily seen that A_0 is negative definite. Notice that if K_0 is symmetric, P_2 can always be selected as M. However, one need not do so. The following important observations are also recorded.

Remarks

1. Consider the choice $P_3 \equiv 0$ and assume that M and K_0 are symmetric. For this case $P_2 \equiv M$ and if the symmetric part of C_0 is positive definite, then system (2-7) is asymptotically stable [18]. This essentially implies that the approach reduces to the well-known Kelvin-Tait-Chetaev theorem [22].

2. If $P_3 \equiv 0$ and M, K_0 and C_0 are not necessarily symmetric, the Liapunov function V has the same form as the one suggested by Walker [23]. It was shown by Walker that for $C_0 \equiv 0$, the application of the method produced necessary and sufficient conditions for systems loaded by follower forces.

3. In general, when K_0 is not symmetric, matrix P_2 has to be con-

structured through the symmetry and definiteness requirements on P_2 and $P_2M^{-1}K_o$. This indicates that there are n independent matrices P_2 which satisfy such conditions. For systems with two or three degrees of freedom, P_2 can be found analytically without much effort. However for relatively large systems, it may be desirable to use a suitable scheme on a computer. Such a general approach is indicated in the following [23].

i) Define diag $[P_{2i}] = e_i$, where $e_1 = (1, 0, \ldots)$, $e_2 = (0,1,0,\ldots)$, etc.

ii) Determine P_{2i} by the symmetry of P_{2i} and $P_{2i}M^{-1}K_o$.

iii) Set

$$P_2 = \sum_{i=1}^{n} \alpha_i P_{2i} \quad . \tag{2-19}$$

iv) Note that

$$P_2M^{-1}K_o = \sum_{i=1}^{n} \alpha_i P_{2i}M^{-1}K_o \tag{2-20}$$

$$P_2M^{-1}C_o = \sum_{i=1}^{n} \alpha_i P_{2i}M^{-1}C_o \quad . \tag{2-21}$$

v) Then apply the definiteness conditions on P_2 and $P_2M^{-1}K_o$.

4. Since a Liapunov function yields only sufficient conditions for stability; for certain asymptotically stable systems (2-7), matrix A_o may not remain negative definite although P always remains positive definite. In all such cases, the suggested form of the Liapunov matrix P cannot be used to investigate the stability of equation (2-6).

2.3 Stability Theorem and Related Corollaries

Theorem. If there exists a symmetric positive definite matrix

P_2 such that

i) $P_2M^{-1}K_o$ is symmetric and positive definite,

ii) the symmetric parts of $P_2M^{-1}C_o$ and $(M^{-1}C_o)^TP_2(M^{-1}K_o)$ are positive definite, and

iii) $$E\{\lambda_{max}[(A_o + C_t + K_t)P^{-1}]\} < 0, \tag{2-22}$$

then the system described by equation (2-6) is almost surely asymptotically stable in the large, where $\lambda_{max}[\cdot]$ represents the maximum eigenvalues of the matrix $[\cdot]$. The 2n x 2n matrices C_t and K_t are given by

$$C_t = -\left[\begin{array}{c|c} 0 & \frac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}C(t)\} \\ \hline \frac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}C(t)\}^T & \{P_2M^{-1}C(t)\} + \{P_2M^{-1}C(t)\}^T \end{array}\right], \tag{2-23}$$

$$K_t = -\left[\begin{array}{c|c} \frac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}K(t)\}+\frac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}K(t)\}^T & \{P_2M^{-1}K(t)\}^T \\ \hline P_2M^{-1}K(t) & 0 \end{array}\right], \tag{2-24}$$

and matrices P and A_o have the forms given by equations (2-13) and (2-12), respectively.

Proof. The proofs for the theorem and the related corollaries are obtained in a straightforward manner through the application of an approach suggested by Infante [11].

Consider the Liapunov function discussed in section 2.2, i.e.,

$$V(x) = x_1^TP_1x_1 + x_2^TP_2x_2 + x_2^TP_3x_1 \quad . \tag{2-8a}$$

Evaluating the first time derivative along the trajectory of equation (2-6a) yields

$$\dot{V}(x) = -x_1^T[M^{-1}(K_o + K(t))]^T P_3 x_1 -$$

$$- x_2^T[P_2 M^{-1}(C_o + C(t)) + [M^{-1}(C_o + C(t))]^T P_2 - P_3]x_2 +$$

$$+ x_2^T[P_1 - P_2 M^{-1}(K_o + K(t)) - [M^{-1}(C_o + C(t))]^T P_3]x_1 +$$

$$+ x_1^T[P_1 - [M^{-1}(K_o + K(t))]^T P_2]x_2 \quad . \tag{2-25}$$

Since for quadratic forms,the skew symmetric parts of the $x_1^T[\cdot]x_1$ and $x_2^T[\cdot]x_2$ terms are equal to zero, the quadratic, $\dot{V}(x)$, can be written in the following form.

$$\dot{V}(x) = \begin{Bmatrix} x_1 \\ \hline x_2 \end{Bmatrix}^T \left[\begin{array}{c|c} -\tfrac{1}{2}\{P_3^T[M^{-1}(K_o + K(t))] + [M^{-1}(K_o + K(t))]^T P_3\} & \{P_1 - [M^{-1}(K_o + K(t))]^T P_2 - \tfrac{1}{2}P_3^T M^{-1}(C_o + C(t))\} \\ \hline \{P_1 - P_2 M^{-1}(K_o + K(t)) - \tfrac{1}{2}[M^{-1}(C_o + C(t))]^T P_3\} & -\{P_2 M^{-1}(C_o + C(t)) + [M^{-1}(C_o + C(t))]^T P_2 - \tfrac{1}{2}(P_3 + P_3^T)\} \end{array}\right] \begin{Bmatrix} x_1 \\ \hline x_2 \end{Bmatrix} \tag{2-26}$$

Substituting for P_1 and P_3 from equations (2-11) and (2-10), respectively, equations (2-26) and (2-13) yield

$$\frac{\dot{V}(y)}{V(y)} = \frac{y^T[A_o + C_t + K_t]y}{y^T P y} \equiv \lambda(t) \quad , \tag{2-27}$$

where A_o, C_t, K_t, and P are given by equations (2-12), (2-23), (2-24), and (2-13), respectively. Since A_o, C_t and K_t are symmetric and P is symmetric positive definite; from lemma 1, the following inequality holds.

$$\lambda_{min}[(A_o + C_t + K_t)P^{-1}] \leq \lambda(t) \leq \lambda_{max}[(A_o + C_t + K_t)P^{-1}], \tag{2-28}$$

where λ_{max} and λ_{min}, the maximum and the minimum eigenvalues of a pencil, are real. Using inequality (2-28) with (2-27) and solving for $V[y(t)]$, one obtains

$$V[y(t)] = V[y(t_o)]\exp\left[\int_{t_o}^{t}\lambda(\tau)d\tau\right] = V[y(t_o)]\exp[(t-t_o)(\frac{1}{t-t_o}\int_{t_o}^{t}\lambda(\tau)d\tau)]. \tag{2-29}$$

In the limit when $t \to \infty$, by definition 2, the exponent in equation (2-29) exists with probability one. Thus if $E\{\lambda(t)\}$ is negative, it follows that

$$\lim_{t\to\infty} V(t) \equiv 0 \tag{2-30}$$

independent of $V[y(t_o)]$. But

$$V(t) = y^T P y \equiv ||y|| > 0, \tag{2-31}$$

therefore, equation (2-30) implies that

$$\lim_{t\to\infty} (x \;\&\; \dot{x}) \equiv 0 \tag{2-32}$$

with probability one, independent of the initial conditions. If all the conditions of the theorem are satisfied, then conditions (2-30) and (2-32) hold, which proves the theorem.

In the following, two corollaries are also recorded which are more useful for computation purposes.

Let

$$C(t) = \sum_{i=1}^{R} g_i(t)C_i \text{ and } K(t) = \sum_{j=1}^{L} f_j(t)K_j \;;\; R,L \leq n^2, \tag{2-33}$$

where C_i and K_j are constant matrices. Recalling that $E\{g_i(t)\} = E\{f_j(t)\} = 0$, the following corollary can be obtained.

Corollary I. If the first two conditions stated in the theorem hold and if in addition

$$\sum_{i=1}^{R} \tfrac{1}{2}E\{|g_i(t)|\}(\lambda_{max}[C_{ti}P^{-1}] - \lambda_{min}[C_{ti}P^{-1}]) +$$

$$+ \sum_{j=1}^{L} \tfrac{1}{2}E\{|f_j(t)|\}(\lambda_{max}[K_{tj}P^{-1}] - \lambda_{min}[K_{tj}P^{-1}]) < - \lambda_{max}[A_oP^{-1}] \qquad (2\text{-}34)$$

then (2-6) is almost surely asymptotically stable in the large. The 2n x 2n constant matrices C_{ti} and K_{tj} are given by

$$C_{ti} = -\left[\begin{array}{c|c} 0 & \tfrac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}C_i\} \\ \hline \tfrac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}C_i\}^T & \{P_2M^{-1}C_i\} + \{P_2M^{-1}C_i\}^T \end{array}\right] \qquad (2\text{-}35)$$

and

$$K_{tj} = -\left[\begin{array}{c|c} \tfrac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}K_j\} + \tfrac{1}{2}\{(M^{-1}C_o)^TP_2M^{-1}K_j\}^T & \{P_2M^{-1}K_j\}^T \\ \hline P_2M^{-1}K_j & 0 \end{array}\right] . \qquad (2\text{-}36)$$

Proof. In this case, equation (2-27) of the theorem takes the form

$$\lambda(t) = \frac{y^TA_oy}{y^TPy} + \sum_{i=1}^{R} g_i(t)\frac{y^TC_{ti}y}{y^TPy} + \sum_{j=1}^{L} f_j(t)\frac{y^TK_{tj}y}{y^TPy} . \qquad (2\text{-}37)$$

Since $E\{g_i(t)\} = E\{f_j(t)\} = 0$ by assumption, define the following four functions:

$$g_i^+(t) = \begin{cases} g_i(t) & \text{if } g_i(t) \geq 0 \\ 0 & \text{if } g_i(t) \leq 0 \end{cases}, \quad f_j^+(t) = \begin{cases} f_j(t) & \text{if } f_j(t) \geq 0 \\ 0 & \text{if } f_j(t) \leq 0 \end{cases}, \qquad (2\text{-}38a)$$

$$g_i^-(t) = \begin{cases} g_i(t) & \text{if } g_i(t) \leq 0 \\ 0 & \text{if } g_i(t) \geq 0 \end{cases}, \quad f_j^-(t) = \begin{cases} f_j(t) & \text{if } f_j(t) \leq 0 \\ 0 & \text{if } f_j(t) \geq 0 \end{cases}. \tag{2-38b}$$

It then follows that

$$E\{g_i^+(t)\} = - E\{g_i^-(t)\} = \tfrac{1}{2}E\{|g_i(t)|\} \tag{2-39}$$

and

$$E\{f_j^+(t)\} = - E\{f_j^-(t)\} = \tfrac{1}{2}E\{|f_j(t)|\} \quad . \tag{2-40}$$

Applying lemma 1 and condition (iii) of the theorem, one obtains from equation (2-37)

$$E\{\lambda(t)\} \leq \lambda_{max}[A_o P^{-1}] + \sum_{i=1}^{R} \tfrac{1}{2}E\{|g_i(t)|\}\{\lambda_{max}[C_{ti}P^{-1}] - \lambda_{min}[C_{ti}P^{-1}]\} + $$
$$+ \sum_{j=1}^{L} \tfrac{1}{2}E\{|f_j(t)|\}\{\lambda_{max}[K_{tj}P^{-1}] - \lambda_{min}[K_{tj}P^{-1}]\} < 0 \tag{2-41}$$

as a stability condition which can be written in the form of inequality (2-34). This proves corollary I.

The condition in inequality (2-34) is evidently satisfied if this inequality is majorized further by noting that

$$\tfrac{1}{2}\{\lambda_{max}[C_{ti}P^{-1}] - \lambda_{min}[C_{ti}P^{-1}]\} \leq |\mu_i|_{max} \tag{2-42}$$

and

$$\tfrac{1}{2}\{\lambda_{max}[K_{tj}P^{-1}] - \lambda_{min}[K_{tj}P^{-1}]\} \leq |\alpha_j|_{max} \quad , \tag{2-43}$$

where $|\mu_i|_{max}$ is the largest eigenvalue, in absolute value, of $[C_{ti}P^{-1}]$ and $|\alpha_j|_{max}$ is the largest eigenvalue, in absolute value, of $[K_{tj}P^{-1}]$. Therefore, the following corollary can be stated.

Corollary II. If the first two conditions stated in the theorem are satisfied and if

$$\sum_{i=1}^{R} E\{|g_i(t)|\}|u_i|_{max} + \sum_{j=1}^{L} E\{|f_j(t)|\}|\alpha_j|_{max} < -\lambda_{max}[A_o P^{-1}] \qquad (2\text{-}44)$$

then (2-6) is almost surely asymptotically stable in the large.

Although the results obtained from the corollaries are not expected to be as sharp as those obtained from the theorem, the corollaries are more useful from the viewpoint of computation, specially for relatively large systems.

It is observed that only matrix P_2 (which is only $n \times n$) needs to be determined in the case when K_o is not symmetric. For all cases where the static part of the loading results in a symmetric K_o, the Liapunov matrix P is predetermined in terms of M, C_o and K_o by setting $P_2 \equiv M$.

The stability conditions of the theorem and the corollaries can be expressed in terms of first and second moments of the random processes. If the distributional properties of the coefficient processes are known, the stability conditions can be represented in terms of the various process parameters as long as the ergodicity property holds. For example, if the coefficient process is known to be a Gaussian one, then it can be shown that [17]

$$E\{z^2(t)\} = E\{z^2(0)\} = \int_{-\infty}^{\infty} z^2 dF(z) = \int_{-\infty}^{\infty} \frac{z^2}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(z-m)^2}{2\sigma^2}\right] dz \qquad (2\text{-}45)$$

and

$$E\{|z(t)|\} = E\{|z(0)|\} = \int_{-\infty}^{\infty} |z| dF(z) = \int_{-\infty}^{\infty} \frac{|z|}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{(z-m)^2}{2\sigma^2}\right] dz \qquad (2\text{-}46)$$

for $-\infty < z < \infty$ where $z(t)$ is the process, $F(z)$ is its distribution function, m is the mean, and σ is the standard deviation. The integrals in equations (2-45) and (2-46) can be easily evaluated for $m = 0$ to yield

$$E\{z^2(t)\} = E\{z^2(0)\} = \sigma^2 \tag{2-47}$$

and

$$E\{|z(t)|\} = E\{|z(0)|\} = \sigma(2/\pi)^{\frac{1}{2}} \quad . \tag{2-48}$$

Another example of a common coefficient process is what is called the Rice noise. The simplified version can be represented by

$$z(t) = a\cos(\omega t + \phi), \tag{2-49}$$

where a and ω are real constants and ϕ is a random variable uniformly distributed in the interval $[0, 2\pi]$. From equation (2-49) the following quantities are easily obtained.

$$E\{z^2(t)\} = E\{z^2(0)\} = \frac{a^2}{2\pi}\int_0^{2\pi}\cos^2(\omega t + \phi)d\phi = \frac{a^2}{2} \tag{2-50}$$

$$E\{|z(t)|\} = E\{|z(0)|\} = \frac{|a|}{2\pi}\int_0^{2\pi}|\cos(\omega t + \phi)|d\phi = \frac{2|a|}{\pi} \tag{2-51}$$

The coefficient processes do not necessarily need to be stochastic in nature, they can be deterministic, as well. Take for example, the function

$$z(t) = a\sin(\omega t). \tag{2-52}$$

where a and ω are real positive constants. The $E\{z^2(t)\}$ and $E\{|z(t)|\}$ are obtained as follows

$$E\{z^2(t)\} = E\{z^2(0)\} = \frac{\omega a^2}{2\pi}\int_0^{2\pi/\omega} \sin^2(\omega\tau)d\tau = \frac{a^2}{2} \tag{2-53}$$

and

$$E\{|z(t)|\} = E\{|z(0)|\} = \frac{\omega a}{2\pi}\int_0^{2\pi/\omega} |\sin(\omega\tau)|d\tau = \frac{2a}{\pi} . \tag{2-54}$$

Note that the means of the coefficient processes do not need to be zero for the application of the theorem and corollaries. In this study, the mean is taken equal to zero for the matter of convenience.

In the next chapter, the stability theorem and corollaries are applied to some well known problems to demonstrate the effectiveness of the suggested method.

III. APPLICATIONS

In this chapter, the theorem and corollaries are applied to several examples in order to demonstrate the simplicity and usefulness of the suggested technique. The examples are divided into two separate groups: discrete systems and continuous systems.

The first three examples in the discrete section are scalar second-order differential equations which can be used to model many systems with single degree of freedom. The final example of this section is a discrete model of an elastic column bearing a "follower" type stochastic load.

In the continuous section, the stability theorem and corollaries are first applied to a simply supported beam-column. In the second example, a clamped-clamped column subjected to an axial stochastic load is considered. For this problem, a finite element model is used to demonstrate the application of the method to relatively large systems. Following this example is the application of the suggested approach to the viscoelastic "Beck's" column which involves follower forces.

3.1 Examples of Discrete Systems

First, the stability conditions for some well-known scalar equations are presented. These equations have been considered by several authors [7, 11, 12].

Example 3.1.1. Consider the scalar differential equation

$$\ddot{x} + 2\zeta\dot{x} + [\omega^2 + f(t)]x = 0. \tag{3-1}$$

With $P_2 = 1$, simple computation yields

$$P = \begin{bmatrix} \omega^2 + 2\zeta^2 & \zeta \\ \zeta & 1 \end{bmatrix}, \tag{3-2}$$

$$A_0 = \begin{bmatrix} -2\zeta\omega^2 & 0 \\ 0 & -2\zeta \end{bmatrix}, \tag{3-3}$$

and

$$K_t = \begin{bmatrix} -2\zeta f(t) & -f(t) \\ -f(t) & 0 \end{bmatrix}. \tag{3-4}$$

It is seen immediately that P is positive definite and A_0 is negative definite. With further computation, one obtains

$$[A_0 + K_t]P^{-1} = \frac{1}{\omega^2 + \zeta^2}\begin{bmatrix} -\zeta(2\omega^2 + f(t)) & \omega^2(2\zeta^2 - f(t)) \\ 2\zeta^2 - f(t) & -4\zeta^3 - 2\zeta\omega^2 + \zeta f(t) \end{bmatrix}. \tag{3-5}$$

The two eigenvalues of the matrix (3-5) are

$$\lambda_{1,2} = -2\zeta \pm \frac{2\zeta^2}{\omega^2 + \zeta^2}\left[(\omega^2 + \zeta^2)(1 - \frac{f(t)}{\zeta^2} + \frac{f^2(t)}{4\zeta^4})\right]^{\frac{1}{2}} \tag{3-6}$$

Taking the expected value of the maximum eigenvalue of equation (3-6), setting it less than zero, and recalling that $E\{f(t)\} \equiv 0$, the theorem from section 2.3 yields the following stability condition for equation (3-1).

$$E\{f^2(t)\} < 4\zeta^2\omega^2 . \tag{3-7}$$

Inequality (3-7) is the best possible result as shown by Infante [11]. Note that, unlike reference [11], no optimization procedure has been used.

By setting $f(t) = 0$ in equation (3-6), the maximum eigenvalue of matrix A_oP^{-1} is obtained as

$$\lambda_{max}\left[A_oP^{-1}\right] = -2\zeta + 2\zeta^2\left[\omega^2 + \zeta^2\right]^{-\frac{1}{2}} . \tag{3-8}$$

Next, evaluating the eigenvalues of matrix $K_{tj}P^{-1}$ where

$$K_{tj}P^{-1} = \frac{1}{\omega^2 + \zeta^2}\begin{bmatrix} -\zeta & -\omega^2 \\ -1 & \zeta \end{bmatrix} , \tag{3-9}$$

yield

$$\lambda_{1,2}\left[K_{tj}P^{-1}\right] = \pm\left[\omega^2 + \zeta^2\right]^{-\frac{1}{2}} . \tag{3-10}$$

Then application of corollary I results in the following stability condition for equation (3-1).

$$E\{|f(t)|\} < 2\zeta\left[\omega^2 + \zeta^2\right]^{\frac{1}{2}} - 2\zeta^2 \tag{3-11a}$$

This is similar to the result obtained by Caughey and Gray [7]. The result of inequality (3-11a) can also be expressed in terms of $E\{f^2(t)\}$ through a direct application of lemma 2 as

$$E\{f^2(t)\} < \left[2\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - 2\zeta^2\right]^2 . \tag{3-11b}$$

Corollary II yields the same stability conditions as corollary I. Figure 3.1 displays the results of the theorem and the corollaries given by inequalities (3-7) and (3-11b), respectively. Whereas the theorem yields the optimum result for the problem, the results from the corollaries are not so sharp.

Example 3.1.2. Now, consider the equation

$$\ddot{x} + \left[2\zeta + g(t)\right]\dot{x} + \omega^2 x = 0 \tag{3-12}$$

with $E\{g(t)\} \equiv 0$.

Following the same approach as in example 3.1.1 by selecting $P_2 = 1$, the P and A_0 matrices are the same as those given by equations (3-2) and (3-3), respectively. C_t is given by

$$C_t = \begin{bmatrix} 0 & -\zeta g(t) \\ -\zeta g(t) & -2g(t) \end{bmatrix} . \qquad (3\text{-}13)$$

Simple computation then yields

$$[A_0 + C_t]P^{-1} = \frac{1}{\omega^2 + \zeta^2}\begin{bmatrix} -2\zeta\omega^2 + \zeta^2 g(t) & 2\zeta^2\omega^2 - \zeta g(t)(\omega^2 + 2\zeta^2) \\ 2\zeta^2 + \zeta g(t) & -g(t)(3\zeta^2 + 2\omega^2) - 2\zeta\omega^2 - 4\zeta^3 \end{bmatrix} . \qquad (3\text{-}14)$$

The two eigenvalues of this matrix are

$$\lambda_{1,2} = -2\zeta - g(t) \pm (\omega^2 + \zeta^2)^{-\frac{1}{2}}[4\zeta^4 + 4\zeta^3 g(t) + g^2(t)(2\zeta^2 + \omega^2)]^{\frac{1}{2}} . \qquad (3\text{-}15)$$

Taking the expected value of the maximum eigenvalue of equation (3-15) and setting it less than zero, the theorem yields the stability condition

$$E\{g^2(t)\} < \frac{4\zeta^2\omega^2}{\omega^2 + 2\zeta^2} \qquad (3\text{-}16)$$

for equation (3-12).

Inequality (3-16) does not provide the optimum stability bound for equation (3-12), unlike the case in example 3.1.1. In the limit when $\zeta \to \infty$, inequality (3-16) yields $E\{g^2(t)\} < 2$ for $\omega^2 = 1$. While according to reference [11], the optimum condition for $\omega^2 = 1$ is $E\{g^2(t)\} < 4$. The result of the present investigation is quite reasonable since no optimization procedure has been used.

Setting $g(t) \equiv 0$ in equation (3-15), the maximum eigenvalue of matrix A_0P^{-1} is once again given by equation (3-8). Matrix $C_{ti}P^{-1}$ is computed as

$$C_{ti}P^{-1} = \frac{1}{\omega^2 + \zeta^2}\begin{bmatrix} \zeta^2 & -\zeta(\omega^2 + 2\zeta^2) \\ \zeta & -2\omega^2 - 3\zeta^2 \end{bmatrix} . \tag{3-17}$$

The eigenvalues are

$$\lambda_{1,2}\left[C_{ti}P^{-1}\right] = -1 \pm\left[(\omega^2 + 2\zeta^2)/(\omega^2 + \zeta^2)\right]^{\frac{1}{2}} . \tag{3-18}$$

Applying corollary I; from equations (3-8) and (3-18), the following stability condition is obtained for equation (3-12).

$$E\{|g(t)|\} < \left[2\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - 2\zeta^2\right]\left[\omega^2 + 2\zeta^2\right]^{-\frac{1}{2}} \tag{3-19}$$

This result is slightly more restrictive than the optimized result acquired using a corollary in reference [11] which is similar to corollary I.

In this case, corollary II yields a different result than corollary I. Using equation (3-8) and the eigenvalues given by equation (3-18), the following stability condition is obtained from corollary II.

$$E\{|g(t)|\} < \left[2\zeta - 2\zeta^2(\omega^2 + \zeta^2)^{-\frac{1}{2}}\right]/\left[1 + ((\omega^2 + 2\zeta^2)/(\omega^2 + \zeta^2))^{\frac{1}{2}}\right] \tag{3-20}$$

As expected, this condition is not as sharp as that obtained by corollary I. The results are displayed in figure 3.2 along with the optimized results of reference [11].

Example 3.1.3. Consider a generalization of the two previous examples represented by the differential equation,

$$\ddot{x} + [2\zeta + g(t)]\dot{x} + [\omega^2 + f(t)]x = 0. \tag{3-21}$$

Using $P_2 = 1$ as before; the P, A_o, C_t, and K_t matrices are given by equations (3-2), (3-3), (3-13), and (3-4), respectively. Further compu-

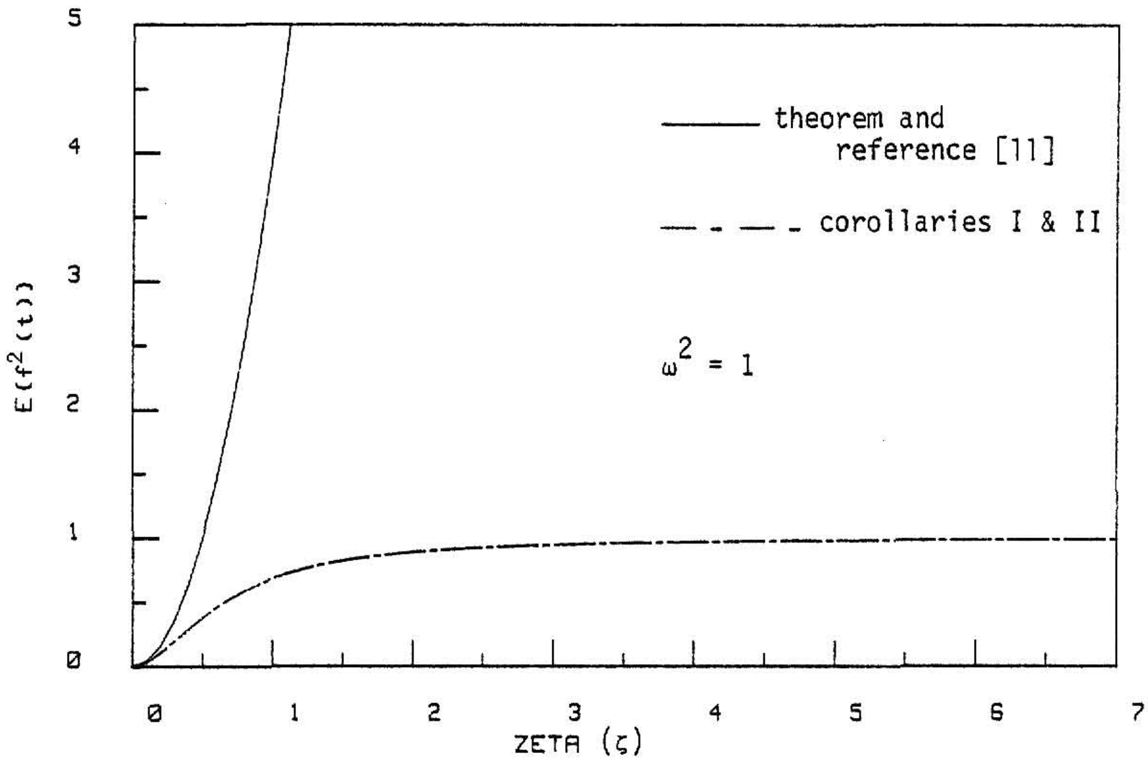

Figure 3.1: $E\{f^2(t)\}$ vs. Zeta (ζ)

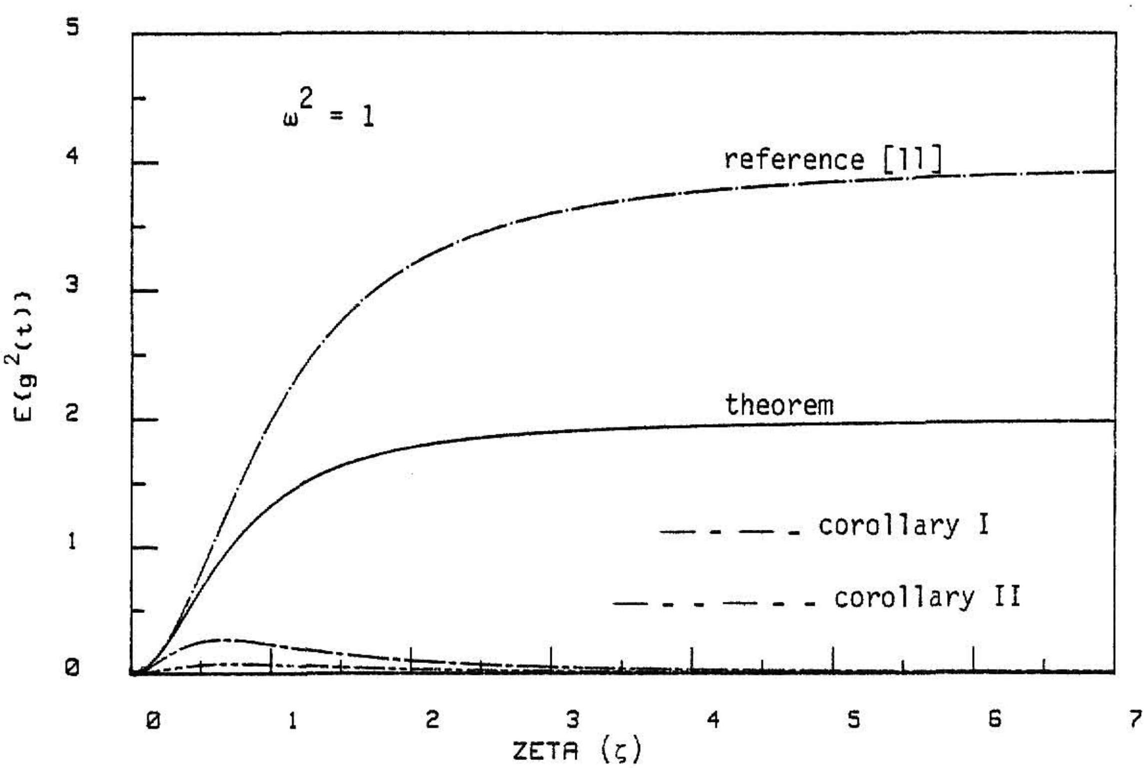

Figure 3.2: $E\{g^2(t)\}$ vs. Zeta (ζ)

tation yields

$$[A_0 + C_t + K_t]P^{-1} =$$

$$= \frac{1}{\omega^2+\zeta^2}\begin{bmatrix} -2\zeta\omega^2 - \zeta f(t) + \zeta^2 g(t) & 2\zeta^2\omega^2 - \omega^2 f(t) - \zeta g(t)(\omega^2+2\zeta^2) \\ 2\zeta^2 - f(t) + \zeta g(t) & -4\zeta^3 - 2\zeta\omega^2 + \zeta f(t) - g(t)(2\omega^2+3\zeta^2) \end{bmatrix}. \quad (3\text{-}22)$$

The eigenvalues of this expression are computed as

$$\lambda_{1,2} = -2\zeta - g(t) \pm (\omega^2+\zeta^2)^{-\frac{1}{2}}[4\zeta^4 + f^2(t) - 4\zeta^2 f(t) - 2\zeta f(t)g(t) + g^2(t)(\omega^2 + 2\zeta^2) + 4\zeta^3 g(t)]^{\frac{1}{2}} \quad (3\text{-}23)$$

Taking the expected value of the maximum eigenvalue of equation (3-23) and setting it less than zero, the following stability condition is obtained.

$$E\{[\zeta g(t) - f(t)]^2 + g^2(t)(\omega^2 + \zeta^2)\} < 4\zeta^2\omega^2 \quad (3\text{-}24)$$

Note, that if $g(t) \equiv 0$, inequality (3-24) yields the almost sure asymptotic condition given by inequality (3-7). In the event when $f(t) \equiv 0$, the stability condition for equation (3-12) is recovered from inequality (3-24). The stability condition of inequality (3-24) can be represented in terms of $E\{g^2(t)\}$ and $E\{f^2(t)\}$ through an application of lemma 3 as

$$(\omega^2 + \zeta^2)\, E\{g^2(t)\} + [\zeta E\{g^2(t)\}^{\frac{1}{2}} + E\{f^2(t)\}^{\frac{1}{2}}]^2 < 4\zeta^2\omega^2. \quad (3\text{-}25)$$

Infante [11] pointed out that an optimum quadratic norm does not exist for this problem, unless $g(t)$ and $f(t)$ are related.

Corollary I can be applied by using equations (3-8), (3-10), and (3-18). Simple calculation yields

$$(\omega^2 + 2\zeta^2)^{\frac{1}{2}}E\{|g(t)|\} + E\{|f(t)|\} < 2\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - 2\zeta^2 \quad . \quad (3\text{-}26)$$

Further majorization results in

$$\left[(\omega^2 + \zeta^2)^{\frac{1}{2}} + (\omega^2 + 2\zeta^2)^{\frac{1}{2}}\right] E\{|g(t)|\} + E\{|f(t)|\} < 2\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - 2\zeta^2 \qquad (3\text{-}27)$$

as the stability condition from corollary II.

Example 3.1.4. As an example of a system with two degrees of freedom, consider the inverted double pendulum problem bearing a directional stochastic load as shown in figure 3.3. In the figure, ϕ_1 and ϕ_2 are angular displacements, k is torsional spring constant, m is the mass, and ℓ is the length of each segment of the pendulum. For the proportional constant, $\alpha < 1$, p_t is a subtangential load. For $\alpha > 1$, p_t is a supertangential load. And for $\alpha = 0$, p_o is a conservative load. The special case of this problem without damping and with $p_t \equiv p_o$ has been considered by Herrmann and Bungay [25] and Walker [23].

The equations of motion for the system can be written as

$$M\ddot{\phi} + C_o\dot{\phi} + [K_o + K(t)]\phi = 0 \qquad (3\text{-}28)$$

where

$$M = m\ell^2 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \qquad (3\text{-}29)$$

$$K_o = k \begin{bmatrix} 2 - \gamma_o & \alpha\gamma_o - 1 \\ -1 & 1 - (1 - \alpha)\gamma_o \end{bmatrix}, \qquad (3\text{-}30)$$

C_o is taken to be proportional to the K_o or $[M + K_o]$ matrix, $m\ell^2 > 0$, $k > 0$, and $\gamma_o = p_o\ell/k$. The matrix K(t) due to the stochastic part of the load, p(t), is given as

$$K(t) = \frac{\ell p(t)}{k} \begin{bmatrix} -1 & \alpha \\ 0 & -(1 - \alpha) \end{bmatrix}. \qquad (3\text{-}31)$$

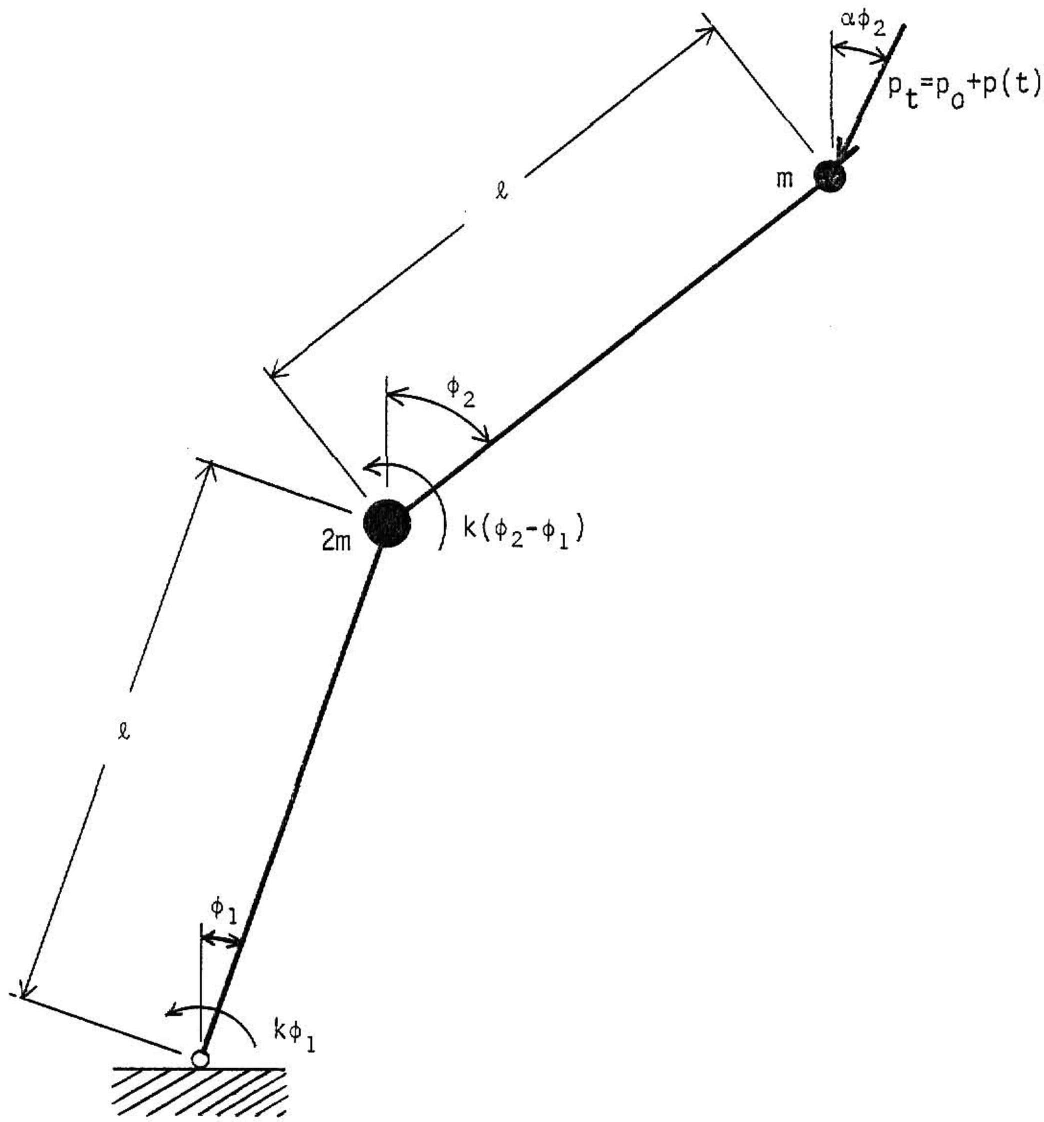

Figure 3.3: Inverted Double Pendulum Subjected to a Follower-Type Stochastic Load

Let $\gamma(t) = \ell p(t)/k$. It is observed that for $\alpha \neq 0$, the stiffness matrix K_0 is not symmetric and a symmetric positive definite matrix P_2 must be found in order to be able to apply the theorem or the corollaries.

Choosing

$$P_2 = \begin{bmatrix} 5 - \gamma_0 & 0 \\ 0 & 2 - \gamma_0 \end{bmatrix}, \tag{3-32}$$

results in

$$P_2 M^{-1} K_0 = \frac{km\ell^2}{2} \begin{bmatrix} (5 - \gamma_0)(3 - \gamma_0) & -(5 - \gamma_0)(2 - \gamma_0) \\ -(5 - \gamma_0)(2 - \gamma_0) & \gamma_0(2 - \gamma_0)[4 - (3 - 2\alpha)] \end{bmatrix}. \tag{3-33}$$

It can be shown that P_2 is positive definite for $\gamma_0 < 2$ and $P_2 M^{-1} K_0$ is positive definite provided

$$1 + (1 - \alpha)(\gamma_0^2 - 3\gamma_0) > 0 . \tag{3-34}$$

For $p(t) \equiv 0$, this selection of P_2 provides a necessary and sufficient asymptotic stability condition for $\gamma_0 < 2$ as shown in reference [23]. For $p(t) \neq 0$, the stability is investigated in presence of proportional damping.

Since the application of the theorem would be somewhat tedious for this example, corollary I will be utilized instead. The bounds for the almost sure asymptotic stability are found by using a computer program based on corollary I. The details of the program are included in Appendix A. The results are illustrated in figures 3.4 - 3.9 for various values of γ_0 and α.

Figures 3.4 and 3.5 show the stability bounds with $p_0 = 0$ under symmetric loading condition, i.e., $\alpha = 0$. Both figures show the effect of the selection of P_2. Since for symmetric loading the stiffness matrix is symmetric, the P_2 matrix can be selected as M. For this case, it was found that the selection, $P_2 = M$, gave higher stability bounds on $E\{|\gamma(t)|\}$ than the selection of P_2 given by equation (3-22). Figure 3.4 shows this effect for C_0 proportional to the constant stiffness matrix K_0, while in figure 3.5, C_0 is assumed to be of the form $C_0 = \zeta(M + K_0)$. For $P_2 = M$, as ζ becomes large, $E\{|\gamma(t)|\}$ approaches the maximum constant load γ_0 allowed for the non-stochastic system with no damping. This indicates that the present technique does not yield unduly conservative results.

The next two figures, 3.6 and 3.7, show the stability bounds for $p_0 = 0$ and p(t) being applied in the tangential direction, i.e., $\alpha = 1$. As expected, $E\{|\gamma(t)|\}$ is much higher for the tangential loading as compared to the symmetric loading. Once again the selection of $P_2 = M$ yields higher bounds when $C_0 = \zeta K_0$. However, if $C_0 = \zeta(M + K_0)$, figure 3.7 shows that the choice of P_2 has very little effect on the stability bounds when the stochastic loading is non-symmetric. Also note, that for $P_2 = M$, as ζ becomes large, $E\{|\gamma(t)|\}$ approaches about 66% of the maximum allowable constant load, $p_0 = 2k/\ell$, for the system [23].

In figure 3.8, the stability bounds are shown for the case when the stochastic loading is non-symmetric and the constant load $p_0 = k/\ell$, $(\gamma_0 = 1)$. Since $\gamma_0 = 1$ makes the stiffness matrix non-symmetric, P_2 can not be taken as M and the form given by equation (3-32) must be used. It is seen that for small ζ, the stability bounds for $C_0 = \zeta(M + K_0)$ increases more rapidly to its peak value as compared to the case

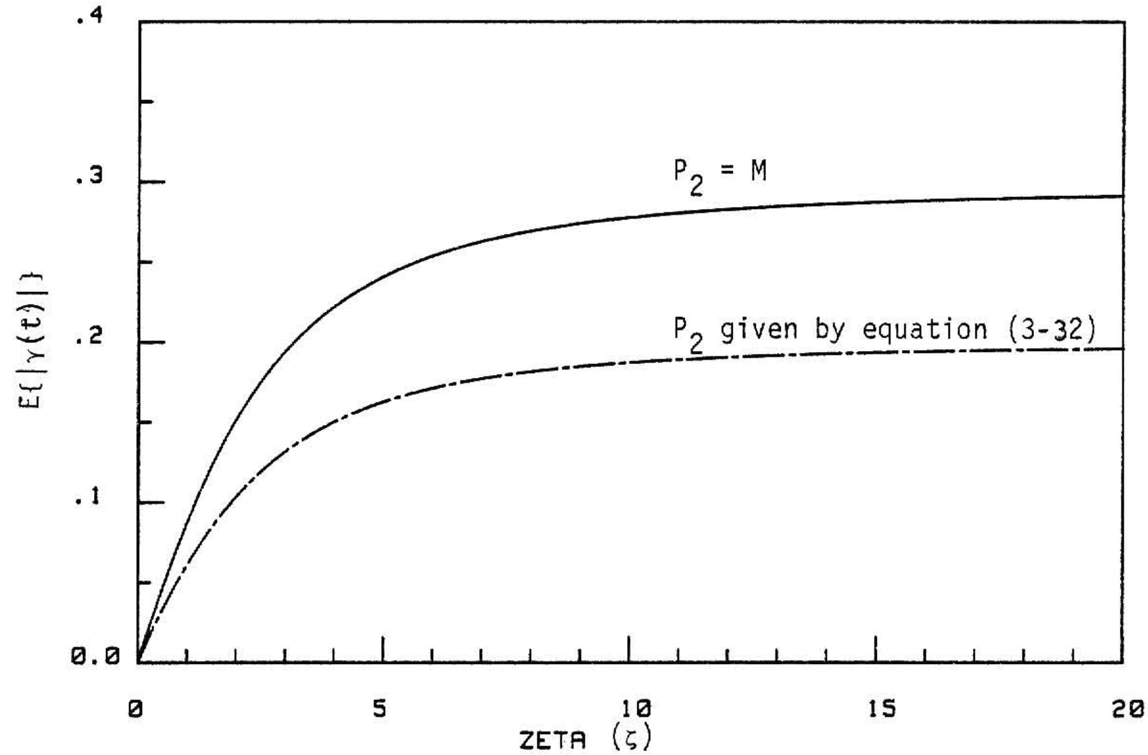

Figure 3.4: E{|γ(t)|} vs. Zeta (ζ) for the Case $p_o = 0$, $\alpha = 0$, and $C_o = \zeta K_o$

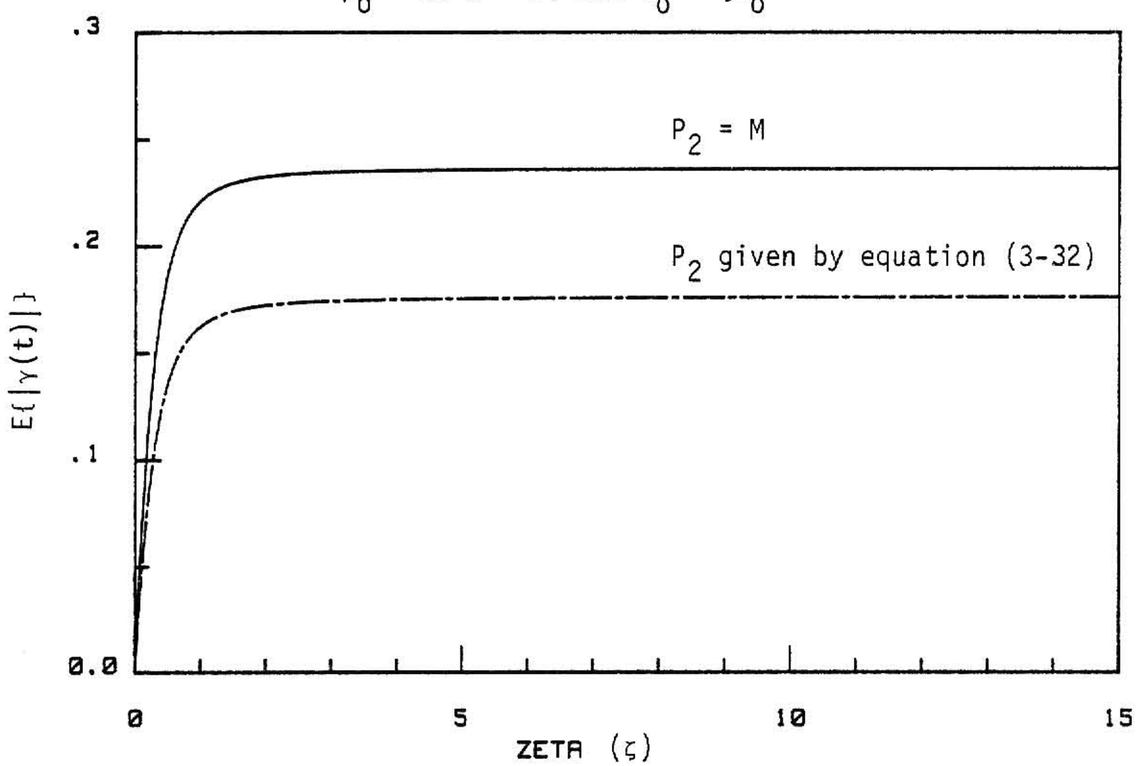

Figure 3.5: E{|γ(t)|} vs. Zeta (ζ) for the Case $p_o = 0$, $\alpha = 0$, and $C_o = \zeta(M + K_o)$

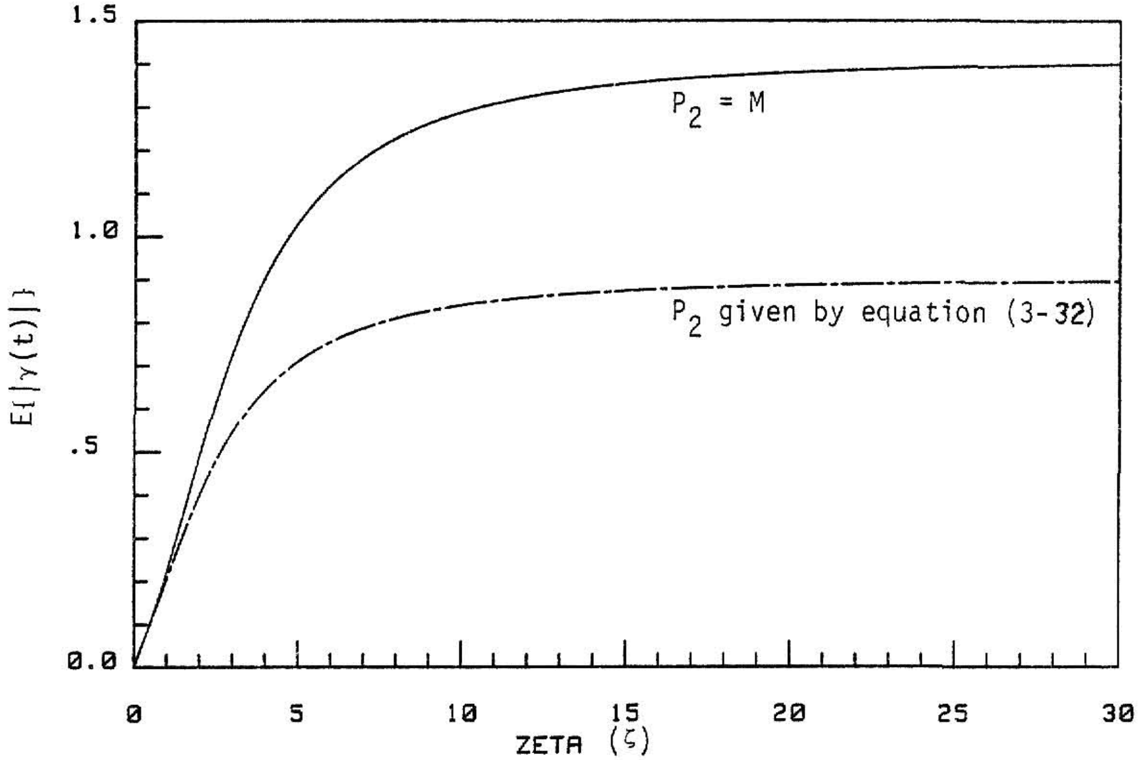

Figure 3.6: E{|γ(t)|} vs. Zeta (ζ) for the Case $p_o = 0$, $\alpha = 1$, and $C_o = \zeta K_o$

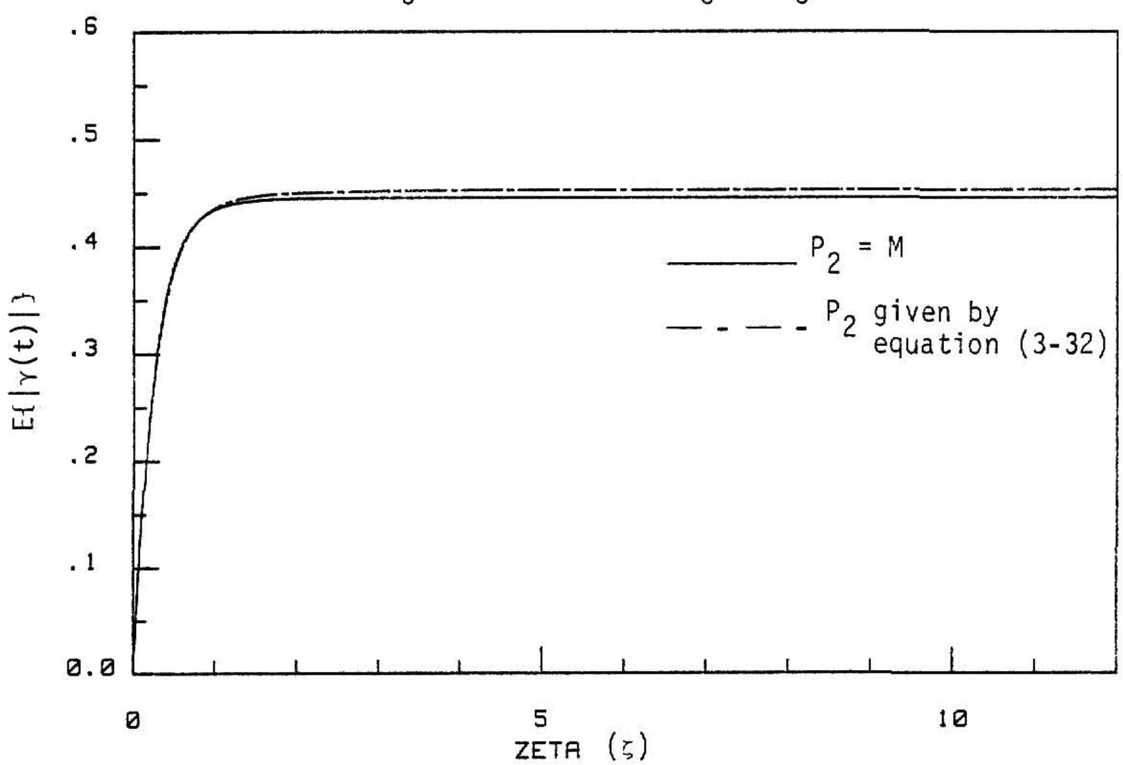

Figure 3.7: E{|γ(t)|} vs. Zeta (ζ) for the Case $p_o = 0$, $\alpha = 1$, and $C_o = \zeta(M + K_o)$

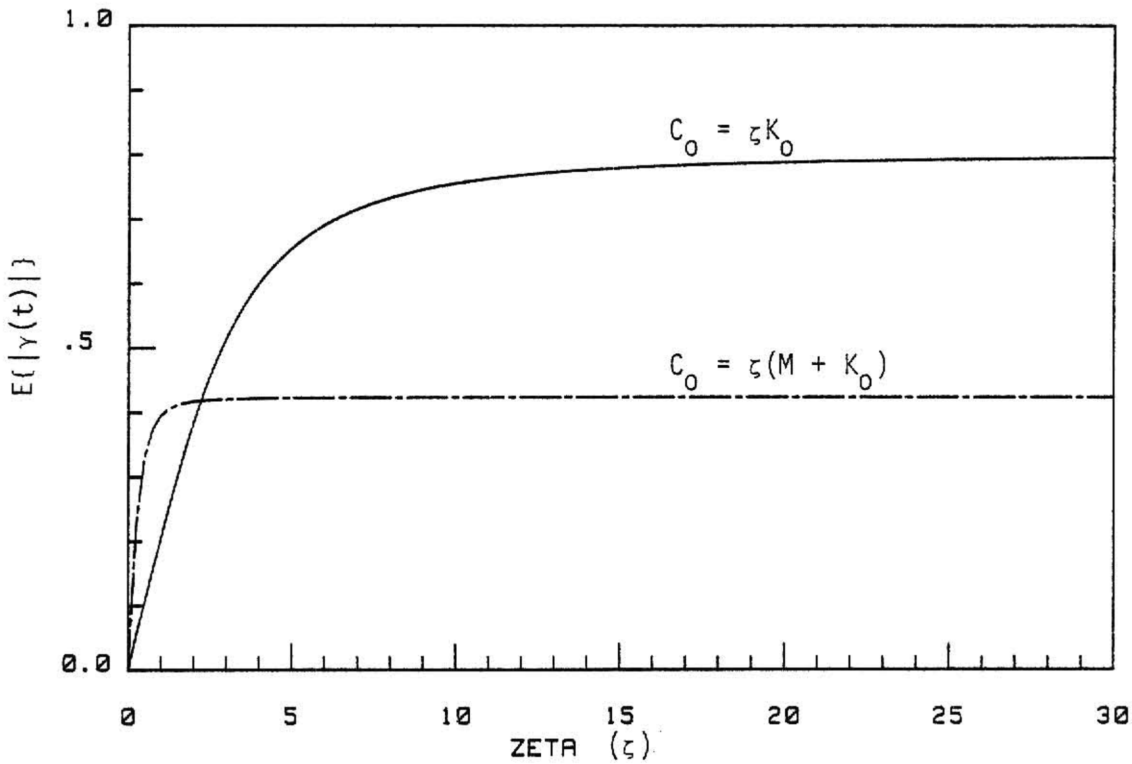

Figure 3.8: $E\{|\gamma(t)|\}$ vs. Zeta (ζ) for the Case $p_0 = k/\ell$, $\alpha = 1$, and P_2 has the Form of Equation (3-32)

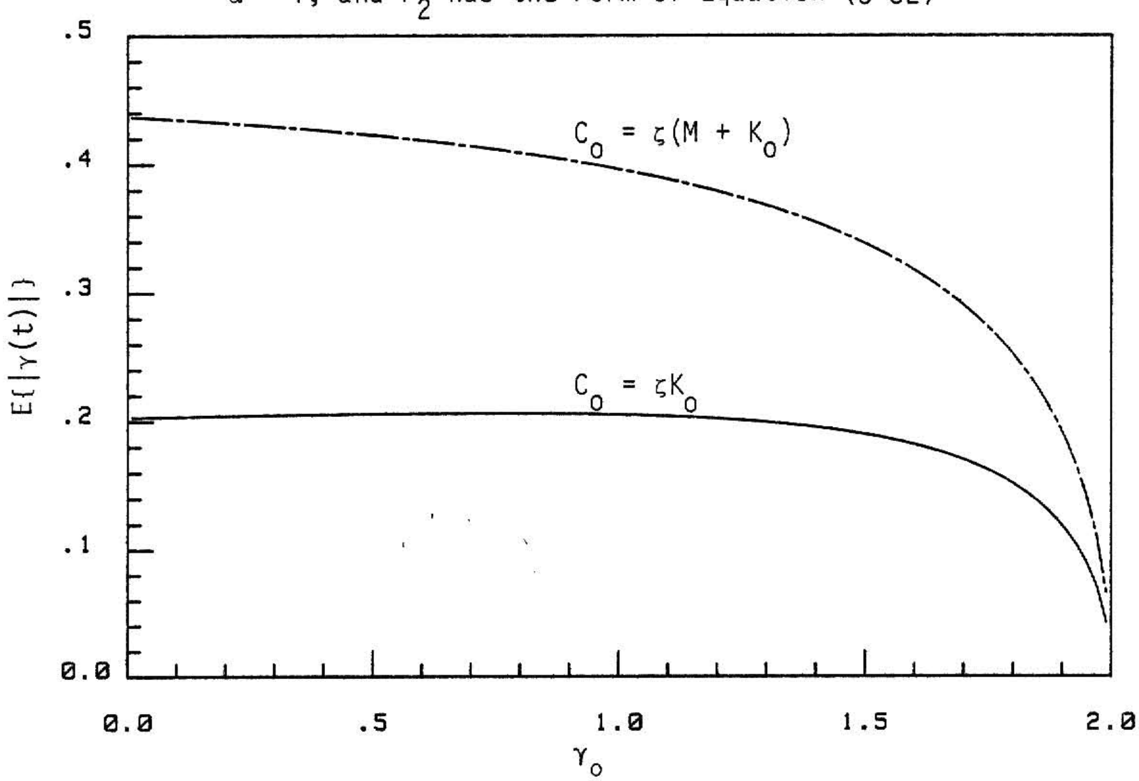

Figure 3.9: $E\{|\gamma(t)|\}$ vs. γ_0 for the Case $\alpha = 1$, $\zeta = 1$, and P_2 has the Form of Equation (3-32)

$C_0 = \zeta K_0$. The bounds for $C_0 = \zeta(M + K_0)$ are much higher for low ζ than for the case $C_0 = \zeta K_0$. As ζ gets large the reverse is true. For both cases the stability bounds reach some asymptotic values as ζ becomes large. This trend is also seen in the previous figures. Such a behavior clearly indicates that the results of the corollaries are not very sharp for large values of ζ.

Finally, figure 3.9 shows the relationship of the allowable values of $E\{|\gamma(t)|\}$ and the constant load γ_0. For this illustration, $\alpha = 1$, $\zeta = 1$, and P_2 is given by (3-32). The relationship is shown for both cases of damping. Note, that at $\gamma_0 = 0$, there are two possible stability bounds depending on which P_2 is selected.

As seen from figure 3.9, $E\{|\gamma(t)|\} \to 0$ as $\gamma_0 \to 2$ since P_2 no longer remains positive definite at $\gamma_0 = 2$. A similar trend is expected for all ζ.

3.2 Examples of Continuous Systems

In this section, the application of the theorem and corollaries to continuous systems is shown. Various approximation schemes are used to discretize the continuous systems.

Example 3.2.1. First, consider a simply supported column subjected to a stochastic axial load p(t). The nondimensional equation of motion is given as [7]

$$\frac{\partial^2 w}{\partial t^2} + 2\zeta\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + p(t)\frac{\partial^2 w}{\partial x^2} = 0, \; 0 \le x \le 1, \qquad (3\text{-}35)$$

where p(t) is the axial load, ζ is the damping coefficient ($\zeta > 0$) and w is the transverse displacement. The boundary conditions are

$$w(0,\ t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(1,\ t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0 \quad . \tag{3-36}$$

Satisfying equation (3-36), the displacement $w(x,\ t)$ can be assumed of the form

$$w(x,\ t) = \sum_{m=1}^{\infty} a_m(t)\sin m\pi x \quad . \tag{3-37}$$

Upon substitution of equation (3-37) in (3-35), the following set of ordinary differential equations are obtained.

$$\ddot{a}_m + 2\zeta\dot{a}_m + \left[m^4\pi^4 - m^2\pi^2 p(t)\right]a_m = 0,\ m = 1,\ 2,\ldots \tag{3-38}$$

This is of the same form as example 3.1.1 for each m. The terms $m^4\pi^4$ and $(-m^2\pi^2 p(t))$ are equivalent to ω^2 and $f(t)$ in equation (3-1), respectively. Therefore, it follows that the almost sure asymptotic stability condition for each m from inequality (3-7) is

$$E\{p^2(t)\} < 4\zeta^2 \quad . \tag{3-39}$$

For corollaries I and II, the stability condition from inequality (3-11a) becomes

$$E\{|p(t)|\} < \frac{1}{m^2\pi^2}\left[2\zeta(m^4\pi^4 + \zeta^2)^{\frac{1}{2}} - 2\zeta^2\right] \tag{3-40}$$

for each m. This implies that as long as condition (3-39) or (3-40) is satisfied, motion in all modes will be almost surely stable in the large.

Example 3.2.2. In this example, the stability of a clamped-clamped uniform column subjected to an axial stochastic load $p_0 + p(t)$, as shown in figure 3.10, is investigated. The presence of external viscous damping is assumed. The equation of motion and boundary conditions are

$$m\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} + EI\frac{\partial^4 v}{\partial x^4} + [P_0 + p(t)]\frac{\partial^2 v}{\partial x^2} = 0 \qquad (3\text{-}41)$$

with

$$v(x,\ t) = \frac{\partial v(x,\ t)}{\partial x} = 0 \quad \text{at } x = 0,\ L, \qquad (3\text{-}42)$$

where m, c and EI are mass density, coefficient of viscous damping and flexural rigidity, respectively.

Using Hamilton's principle and discretizing the spatial properties, equation (3-41) can be replaced with a system of ordinary differential equations

$$M\ \ddot{q} + C\ \dot{q} + [K + K_G(t)]q = 0 \quad . \qquad (3\text{-}43)$$

The axial load is introduced through finite element method using the concept of geometric stiffness matrix, $K_G(t)$, associated with uniform elements of length ℓ (figure 3.11). Mass, elastic and geometric stiffness matrices for the element are given in Appendix B. The global system (3-43) is found from the element properties and the transformation matrix discussed in reference [26].

For this example, the finite element matrix is obtained by dividing the column into four elements of equal length. The transformation matrix for this particular problem takes the form of an identity matrix because all the elements are in alignment with each other, and the global and element coordinate systems are all parallel to each other. The global matrices, in general, are given by

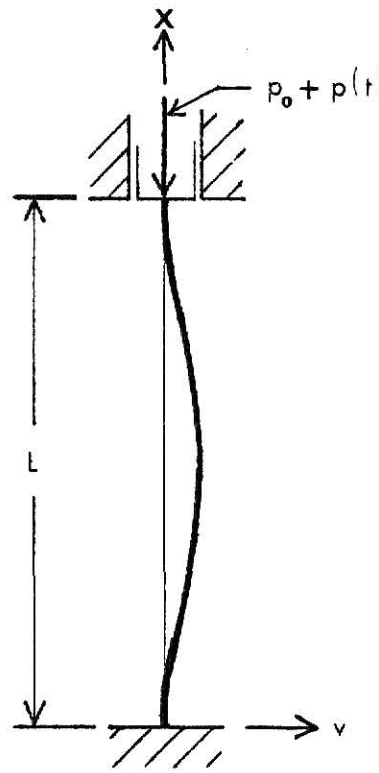

Figure 3.10: Clamped-Clamped Uniform Column Subjected to an Axial Stochastic Load

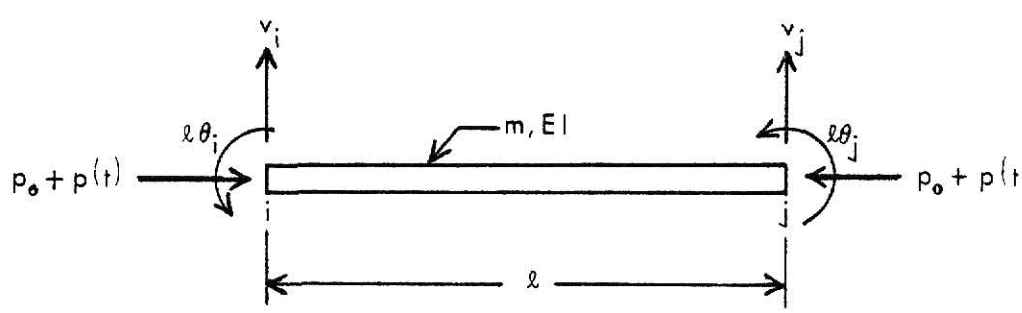

Figure 3.11: Element Geometry and Degrees-of-Freedom

$$K = \frac{EI}{\ell^3}\;\begin{array}{c} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ \end{array} \begin{array}{c} \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \\ \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \end{array},$$

(3-44)

$$K_G(t) = \frac{p_0 + p(t)}{\ell}\;\begin{array}{c} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ \end{array} \begin{array}{c} \begin{bmatrix} \frac{6}{5} & \frac{1}{10} & -\frac{6}{5} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & \frac{2}{15} & -\frac{1}{10} & -\frac{1}{30} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} & 0 & 0 & 0 & 0 \\ \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6}{5} & -\frac{1}{10} & \frac{6}{5} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{1}{30} & -\frac{1}{10} & \frac{2}{15} \end{bmatrix} \\ \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \end{array},$$

(3-45)

and

$$M = \frac{m\ell}{420}\;\begin{array}{c} \begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\end{matrix} \end{array}\begin{array}{c} \begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 & 0 & 0 & 0 & 0 \\ -13 & -3 & 0 & 8 & 13 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 54 & 13 & 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 0 & -13 & -3 & 0 & 8 & 13 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 54 & 13 & 312 & 0 & 54 & -13 \\ 0 & 0 & 0 & 0 & -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & 0 & 0 & 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \end{array} . \tag{3-46}$$

Applying the boundary conditions given by (3-42), the global matrices $K, K_G(t)$, and M reduce to the following forms because the rows and columns corresponding to the fixed ends (1,2,9,10) must be eliminated.

$$K = \frac{EI}{\ell^3}\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 24 & 0 \\ 0 & 0 & 6 & 2 & 0 & 8 \end{bmatrix} \tag{3-47}$$

$$K_G(t) = \frac{p_0 + p(t)}{\ell} \begin{bmatrix} \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} & 0 & 0 \\ -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 & -\frac{6}{5} & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} & -\frac{1}{10} & -\frac{1}{30} \\ 0 & 0 & -\frac{6}{5} & -\frac{1}{10} & \frac{12}{5} & 0 \\ 0 & 0 & \frac{1}{10} & -\frac{1}{30} & 0 & \frac{4}{15} \end{bmatrix} \tag{3-48}$$

$$M = \frac{m\ell}{420} \begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 312 & 0 \\ 0 & 0 & -13 & -3 & 0 & 8 \end{bmatrix} \tag{3-49}$$

As seen, with 4 elements and the given boundary conditions, a 6 x 6 order system for equation (3-43) is obtained. The damping matrix C is taken proportional to either the mass matrix M, or the stiffness matrix K_0, for which the element matrix k_0 is defined in Appendix B.

If $p(t) = c \equiv 0$, the static criterion of stability can be used to find the critical load p_{cr}. The static criterion of stability [27] states that for a system of the form

$$M\ddot{x} + Kx = 0, \tag{3-50}$$

where M and K are both constant matrices, the critical value of the parameter in matrix K can be determined from

$$\det(M^{-1}K) = 0 \quad . \tag{3-51}$$

Applying this criterion to the system at hand yields a critical load of $p_{cr} = 2.486(\frac{EI}{\ell^2})$ which is in error of less than 1% when compared with the exact critical value of $(\frac{\pi^2}{4})(\frac{EI}{\ell^2})$. This indicates that a four element model is reasonably accurate for the study.

The Liapunov matrix P, for this problem, has a dimension of 12 x 12 and hence the possibility of application of the theorem is virtually ruled out. The stability conditions of the corollaries can be obtained rather easily from the general computer program given in Appendix A. Since the stiffness matrices K and K_G are symmetric for all p_0, matrix P_2 is taken as M. The results are shown in figures 3.12 and 3.13.

Figure 3.12 shows how the allowable expected value of the stochastic load $|\bar{p}(t)|$ varies as the constant load p_0 increases to p_{cr}. The results of this figure are based on a damping matrix C proportional to the mass matrix, i.e., $C = \zeta M$. Three different values of ζ are selected for illustration.

As seen from figure 3.13, an increase in ζ does not always increase the bound on $E\{|\bar{p}(t)|\}$. This figure also shows that $E\{|\bar{p}(t)|\}$ approaches an asymptotic value as ζ becomes large. The solid curve is for the case when $C = \zeta K_0$, while the lower curve is for the case when $C = \zeta M$. Such behavior of $E\{|\bar{p}(t)|\}$ for large ζ is attributed to the inherent characteristics of the corollaries. Note that corollaries I and II give the same results.

Example 3.2.3. As the final example, a viscoelastic cantilever column subjected to a stochastic follower load, as shown in figure

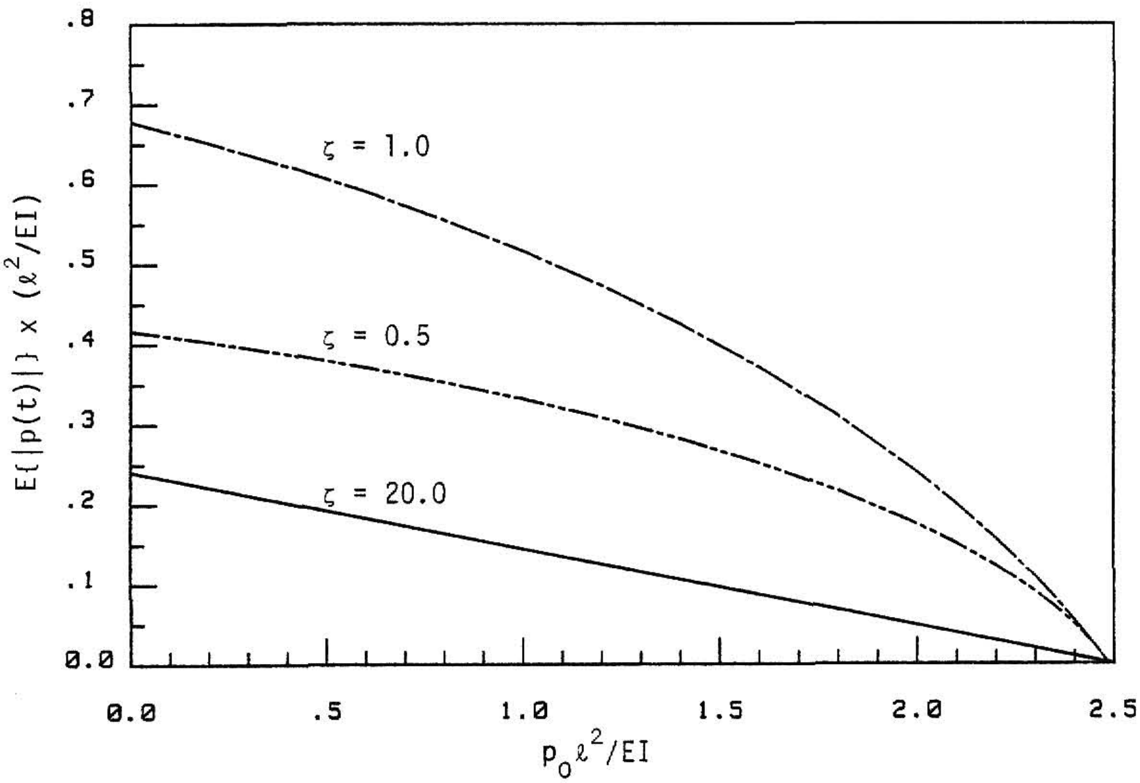

Figure 3.12: $E\{|p(t)|\} \times (\ell^2/EI)$ vs. $(p_o\ell^2/EI)$ for the Case $C = \zeta M$

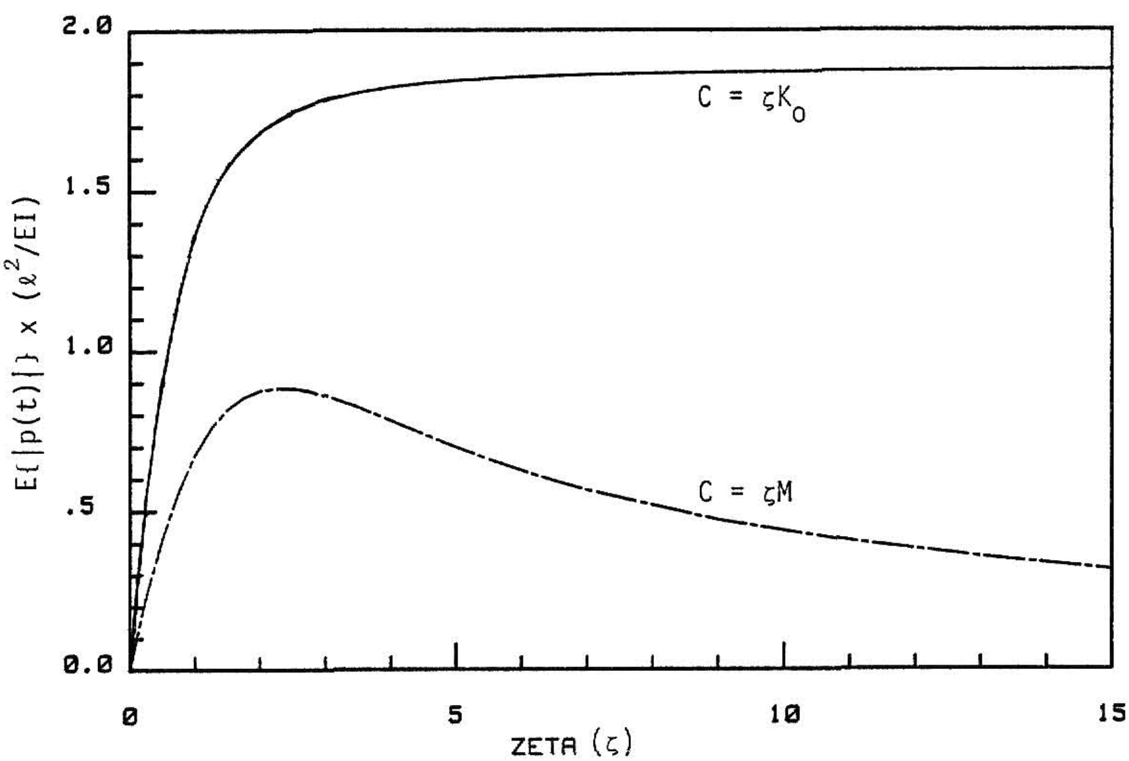

Figure 3.13: $E\{|p(t)|\} \times (\ell^2/EI)$ vs. Zeta (ζ) for the Case $p_o = 0$

(3.14), is considered. Presence of external viscous damping is assumed. The nondimensional equation of motion and the boundary conditions are [28]

$$\frac{\partial^2 w}{\partial t^2} + G\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + R\frac{\partial^5 w}{\partial t \partial x^4} + [p + k(t)]\frac{\partial^2 w}{\partial x^2} = 0 \tag{3-52}$$

and

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0; \ \frac{\partial^2 w}{\partial x^2}(1,t) = \frac{\partial^3 w}{\partial x^3}(1,t) = 0, \tag{3-53}$$

where the follower force consists of a static and dynamic component, p and k(t), respectively. k(t) is a scalar stochastic process. The damping on the system is represented by an external damping factor G and a Kelvin-type material dissipation factor R. This problem has been considered by Parthasarathy and Evan-Iwanowski [15].

Following reference [28], a two term Galerkin solution is introduced in order to reduce equations (3-52) and (3-53) to a system of two ordinary temporal differential equations. For this purpose, let

$$w(x,t) = u_1(t)w_1(x) + u_2(t)w_2(x), \tag{3-54}$$

where $w_1(x)$ and $w_2(x)$ are chosen to be the first two eigenfunctions of the free vibration of a uniform elastic cantilever beam. These are given by

$$\begin{aligned} w_1(x) = {} & 4.148(\cos 1.875\,x - \cosh 1.875\,x) - \\ & - 3.037(\sin 1.875\,x - \sinh 1.875x) \end{aligned} \tag{3-55}$$

and

$$\begin{aligned} w_2(x) = {} & 53.640(\cos 4.694\,x - \cosh 4.694x) - \\ & - 54.631(\sin 4.694\,x - \sinh 4.694x) \quad . \end{aligned} \tag{3-56}$$

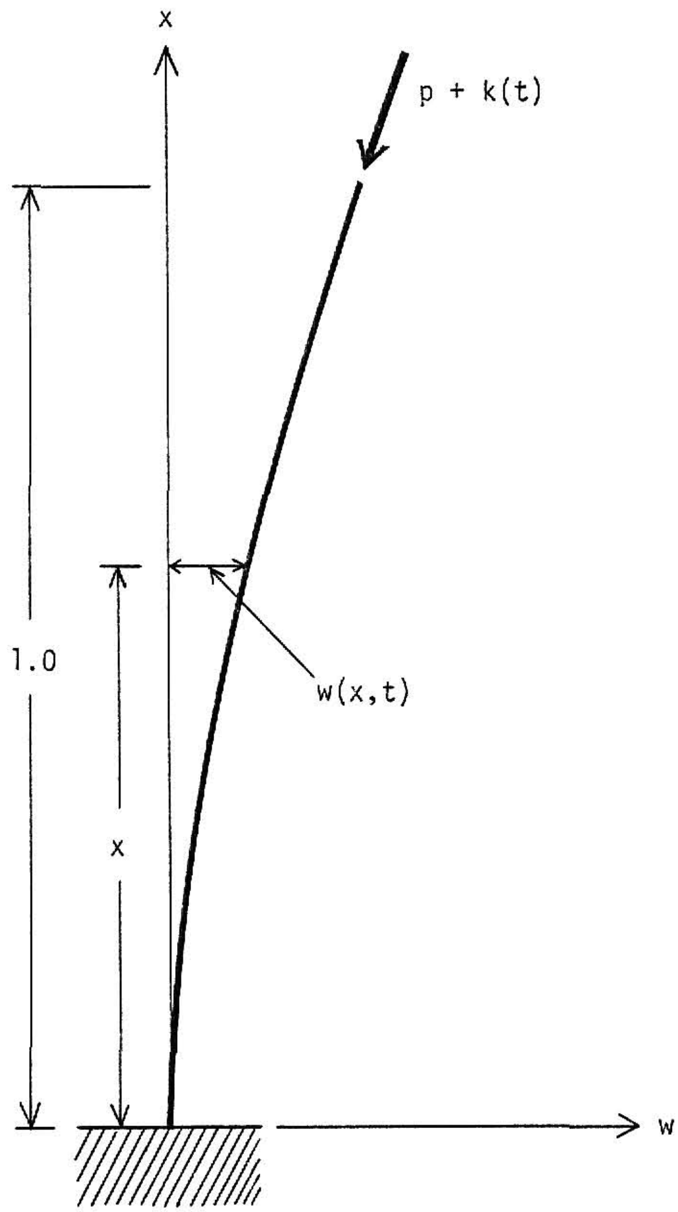

Figure 3.14: Viscoelastic Cantilever Column Subjected to a Stochastic Follower Load

Galerkin's method requires

$$\int_0^1 L(w)w_j(x)dx = 0; \quad j = 1, 2, \tag{3-57}$$

where L( ) is the linear operator,

$$L(\ ) = \frac{\partial^2(\ )}{\partial t^2} + G\frac{\partial(\ )}{\partial t} + \frac{\partial^4(\ )}{\partial x^4} + R\,\frac{\partial^5(\ )}{\partial t\partial x^4} + [p + k(t)]\frac{\partial^2(\ )}{\partial x^2} \tag{3-58}$$

and w is given by equation (3-54). After the integrations in equation (3-57) are carried out, the following system of two ordinary temporal differential equations is obtained in terms of the vector $u^T = \{u_1, u_2\}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{u} + \begin{bmatrix} G + a_4R & 0 \\ 0 & G + b_4R \end{bmatrix}\dot{u} + \begin{bmatrix} a_1 + a_2p + a_2k(t) & a_3p + a_3k(t) \\ b_3p + b_3k(t) & b_1 + b_2p + b_2k(t) \end{bmatrix}u = 0, \tag{3-59}$$

where $a_1 = 12.3596$, $a_2 = 0.8716$, $a_3 = -151.599$, $a_4 = a_1$, $b_1 = 485.4811$, $b_2 = -13.2919$, $b_3 = 0.145$, and $b_4 = b_1$.

It is observed that the constant part of the stiffness matrix is not symmetric and a symmetric positive definite matrix P_2 must be found such that $P_2M^{-1}K_o$ also remains symmetric and positive definite.

Let

$$P_2 = \begin{bmatrix} \alpha_1 & g \\ g & \alpha_2 \end{bmatrix}, \tag{3-60}$$

then

$$P_2M^{-1}K_o = \begin{bmatrix} \alpha_1(a_1 + a_2p) + gb_3p & \alpha_1a_3p + g(b_1 + b_2p) \\ g(a_1 + a_2p) + \alpha_2b_3p & ga_3p + \alpha_2(b_1 + b_2p) \end{bmatrix}. \tag{3-61}$$

From the symmetry requirement of (3-61), the condition on g is

$$g = \frac{(\alpha_1a_3 - \alpha_2b_3)p}{(a_1 - b_1) + (a_2 - b_2)p}\,. \tag{3-62}$$

At this point, P_2 is normalized such that $\alpha_1\alpha_2 = 1$ and α_1 and α_2 satisfy

$$-a_3\alpha_1 = b_3\alpha_2 \quad . \tag{3-63}$$

From the conditions of equations (3-62) and (3-63),

$$P_2 = \begin{bmatrix} \dfrac{1}{32.333} & -\dfrac{9.377p}{14.164p - 473.120} \\ -\dfrac{9.377p}{14.164p - 473.120} & 32.333 \end{bmatrix} \quad . \tag{3-64}$$

P_2 can be shown to be positive definite for $p < 20.095$. The requirement of $P_2M^{-1}K_0$ to be positive definite is satisfied as long as the following conditions hold:

$$\alpha_1(a_1 + a_2p) + gb_3p > 0 \tag{3-65}$$

and

$$(1 - g^2)\left[(a_1 + a_2p)(b_1 + b_2p) - a_3b_3p^2\right] > 0 \quad . \tag{3-66}$$

It can be verified that these inequalities are satisfied for $p < 20.095$. Recall that $p = 20.095$ is precisely the critical flutter load for an elastic column without any external damping.

For this selection of P_2, the results from either of the corollaries are shown in figures 3.15-3.20 for some typical values of system parameters. The range of damping parameter G in figures 3.15 and 3.16 is purposefully kept relatively small so that an effective comparison can be made between the present results and those obtained in reference [15] through an iterative optimization scheme. It is seen that $E\{|k(t)|\}$ appears to be increasing linearly with increase in G. However, it is clear from figures 3.17 and 3.18 that such is not the case as G becomes large. As seen, the trend is the same for both selections of p. The graphical results presented in reference [15] are rather misleading in this respect. Some sample results are also presented in the following table along with results available in literature.

Sample Results for E{|k(t)|}

System Parameters			$E\{\lvert k(t)\rvert\}(10^{-2})$		
G	R (10^{-3})	p	Corollaries I and II	Optimized Results of [15]	Caughey & Gray [7]*
0.1	0.0	6	6.38	8.55	0.01
0.1	0.0	8	6.21	7.95	0.02
0.1	1.5	2	7.63	11.39	0.01
0.1	1.5	6	6.29	9.85	0.04
0.1	1.5	14	0.80	3.99	0.04

* As reported in reference [15].

Note that there is no Kelvin-type material dissipation (R=0) in the systems represented in figures 3.15-3.18.

From figure 3.19, it is seen that $E\{|k(t)|\} \to 0$ as the static load p approaches the critical value of 20.095. Partial results available from reference [15] are also represented on the graph. Figure 3.20 shows a similar trend for $R = 1.5 \times 10^{-3}$. In this case, $E\{|k(t)|\} \to 0$ before reaching the critical value. This is probably due to the destabilizing nature of the internal damping coefficient [28].

Figures 3.15 - 3.19 and the data presented in the table for R = 0 show that the present analysis yields values for $E\{|k(t)|\}$ which are

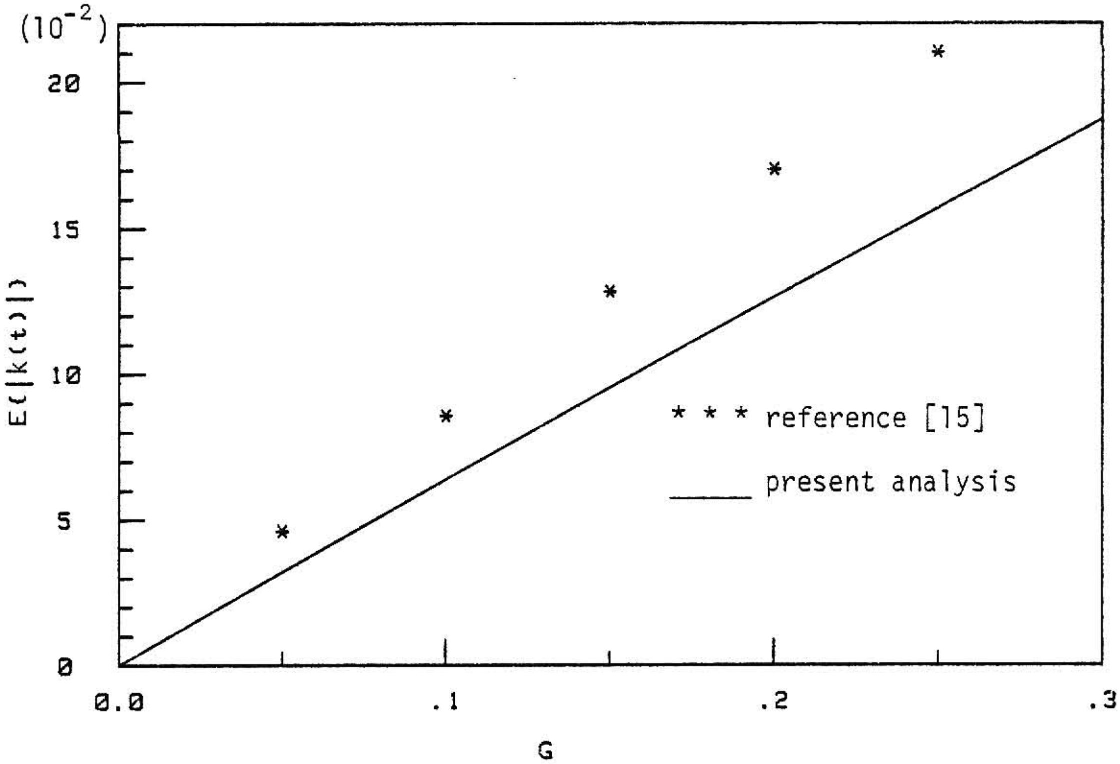

Figure 3.15: E{|k(t)|} vs. G for the Case p = 6 and R = 0

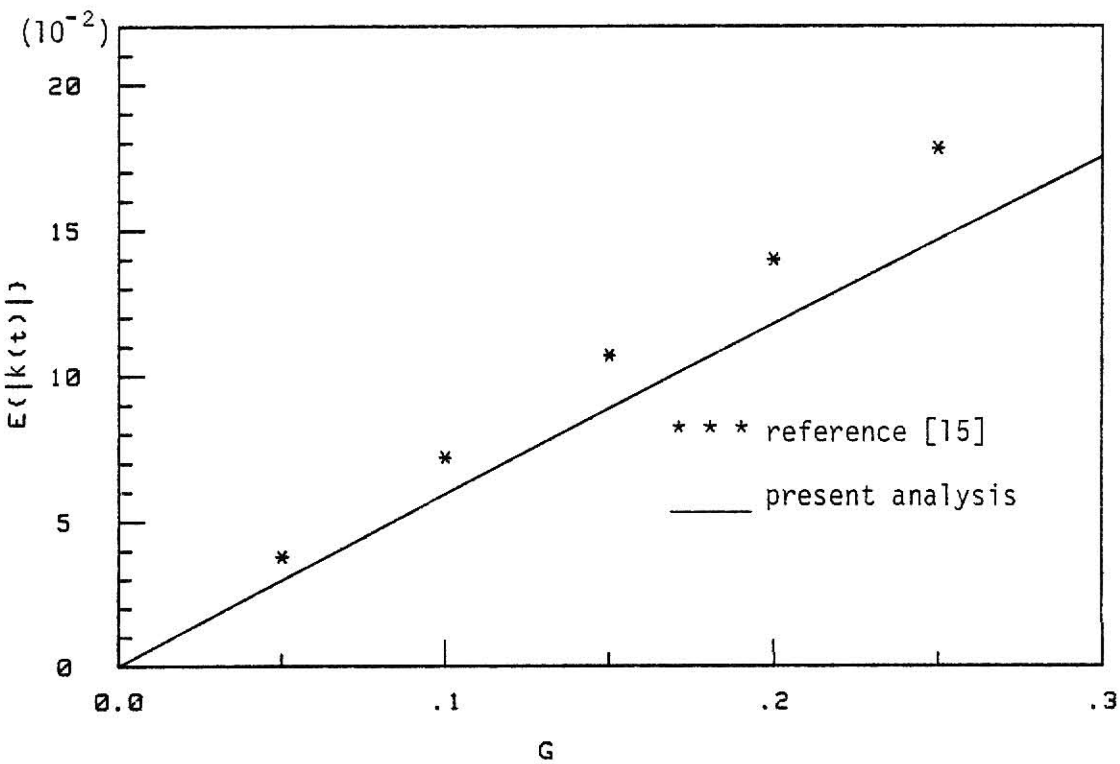

Figure 3.16: E{|k(t)|} vs. G for the Case p = 10 and R = 0

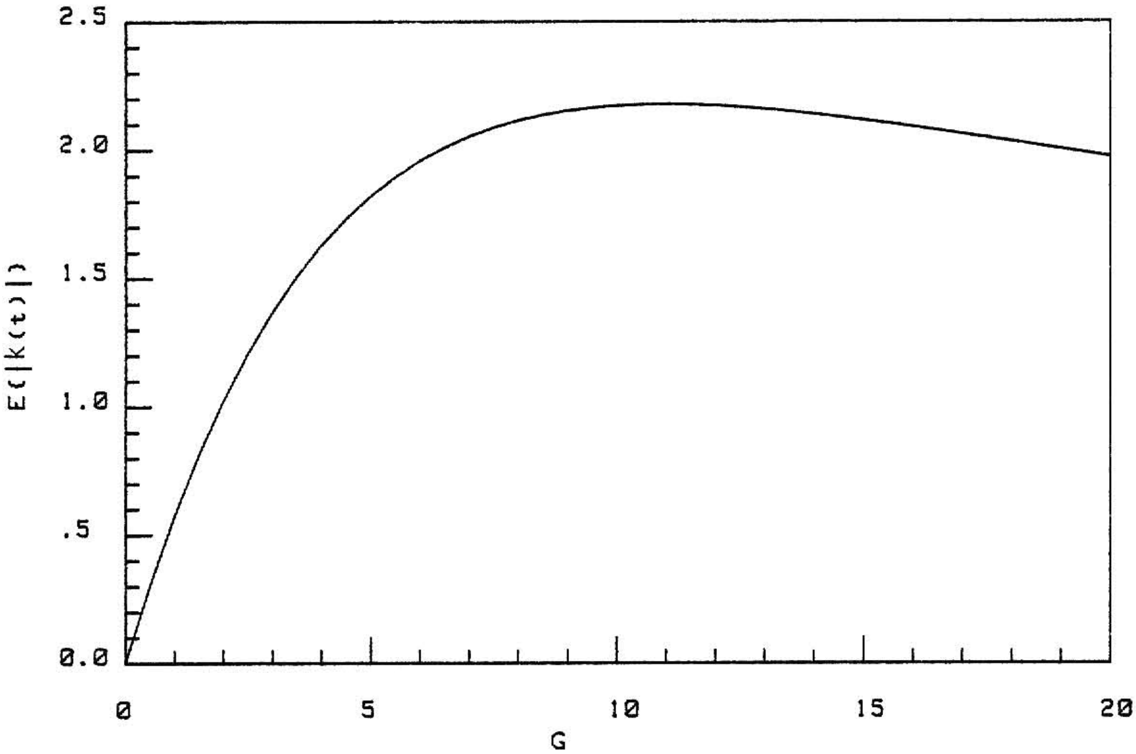

Figure 3.17: E{|k(t)|} vs. G for the Case p = 6 and R = 0

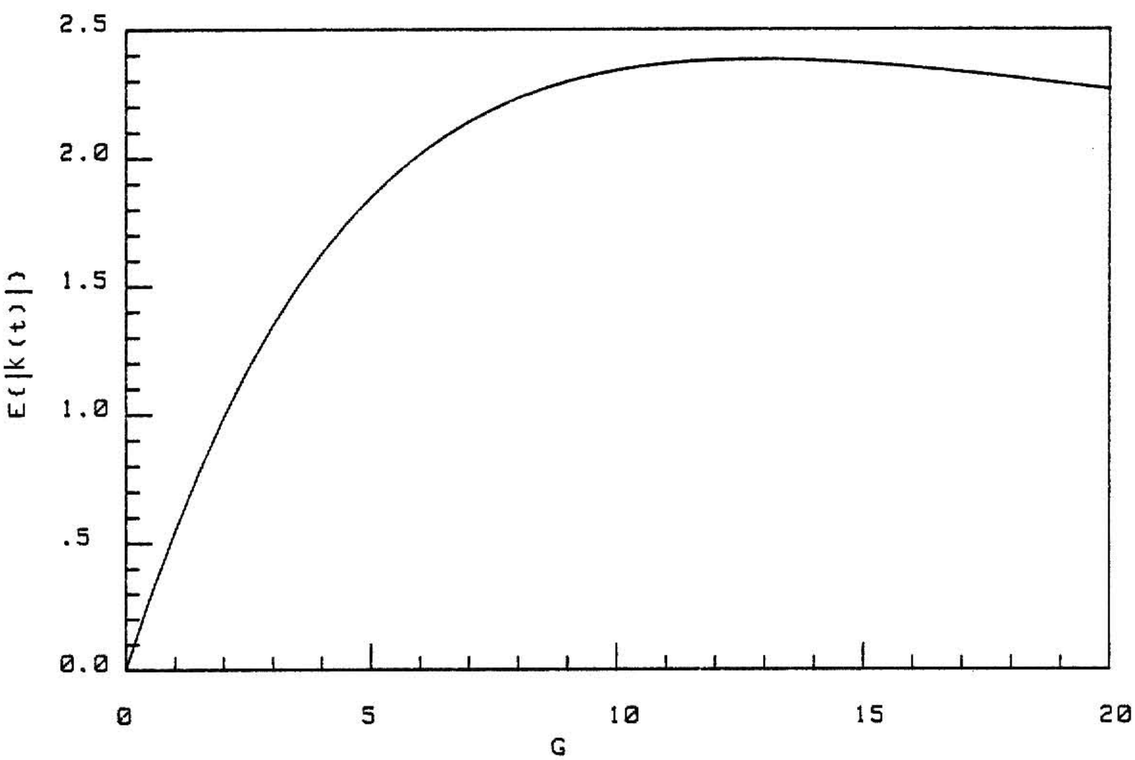

Figure 3.18: E{|k(t)|} vs. G for the Case p = 10 and R = 0

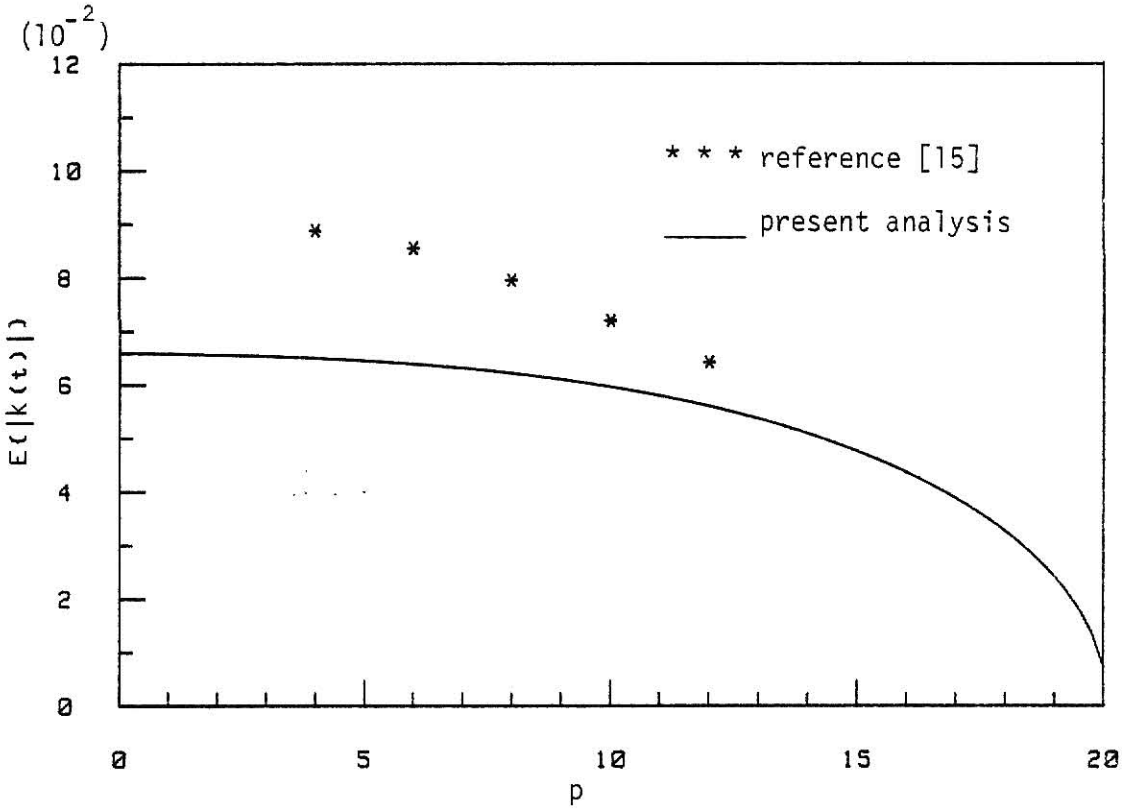

Figure 3.19: E{|k(t)|} vs. p for the Case G = 0.1 and R = 0

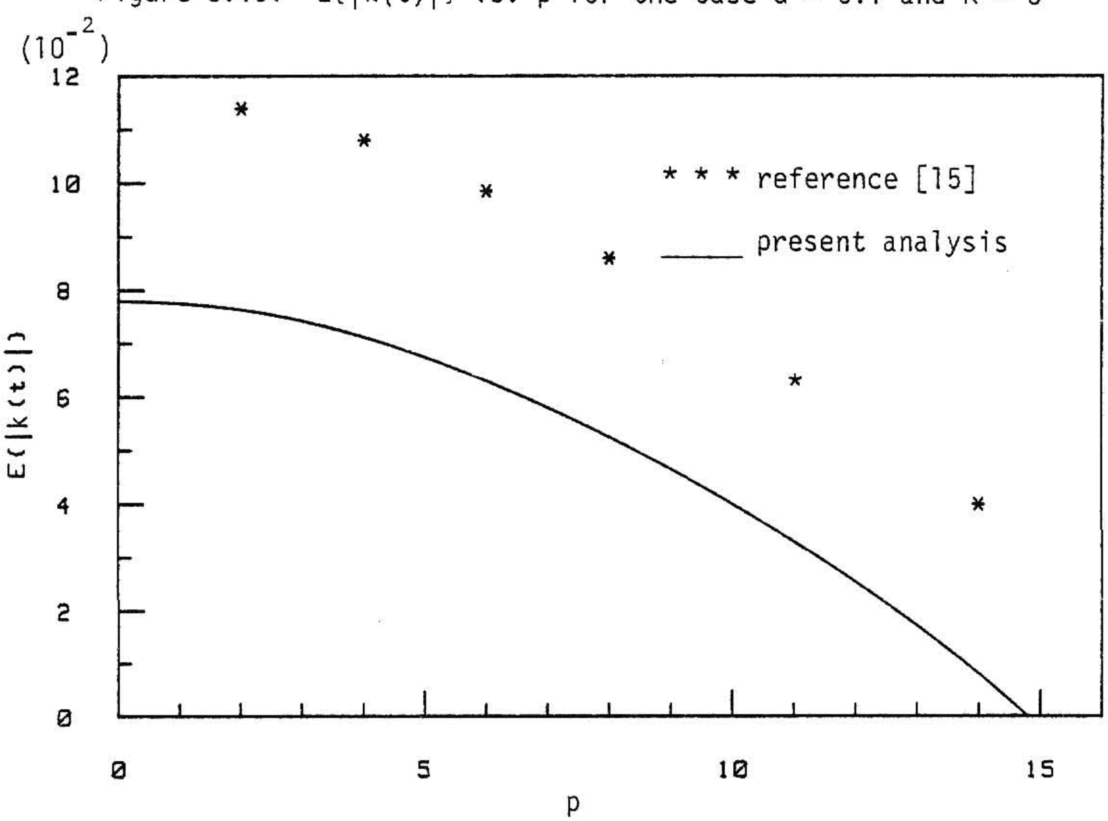

Figure 3.20: E{|k(t)|} vs. p for the Case G = 0.1 and R = 1.5×10^{-3}

74% (or better) of the optimized values reported in reference [15]. In addition, for $R \neq 0$, the results are not as sharp, which may be due to the destabilizing nature of the internal damping coefficient R as mentioned before. The consistent lower values for $E\{|k(t)|\}$ are expected since no optimization scheme was used.

In the following chapter, the present analysis is extended to predict the response bounds for forced motion of discrete linear systems.

IV. RESPONSE BOUNDS FOR FORCED MOTIONS

The determination of conditions for stable operation of engineering systems in the presence of dynamic loads or perturbations is an important problem in analysis and design. The mathematical model for such systems generally possesses an equilibrium state when it is not dynamically perturbed, which is the steady-state point of operation.

The theorem derived in chapter II yields sufficient conditions for the almost sure asymptotic stability of linear systems with stochastic coefficients and systems subjected to dynamic loads or perturbations which do not eliminate the existence of the equilibrium state. In this chapter, the theorem is extended to yield the bounds on the motion of systems in which the perturbations lead to forcing terms in the equation of motion and no equilibrium exists. A column under dynamic transverse load is such an example. Related studies have been undertaken by Caughey and Gray [7], Plaut and Infante [29], Plaut [30], and Holzer [31, 32]. In the following, the response bounds on forced motion for a class of systems, described by a set of second order differential equations with time dependent parameters, are derived. The analysis is based on the development suggested by Plaut and Infante [29]. However, unlike reference [29], the need for optimization is completely eliminated. The perturbations are considered to be of an arbitrary nature.

4.1 Analysis of Response Bounds

Consider

$$M\ddot{x} + [C_o + C(t)]\dot{x} + [K_o + K(t)]x = Q(t); \; x(0) = x_o, \; \dot{x}(0) = \dot{x}_o, \qquad (4\text{-}1)$$

where the right hand side Q(t) is now an n vector which describes the effect of the dynamic loads or perturbations on the system which cannot be represented in the stiffness and damping matrices, K(t) and C(t). The nonzero elements of Q(t), K(t), and C(t) are assumed to be continuous functions of an unspecified nature. All other system matrices are assumed to be the same as given in chapter II.

For equation (4-1), the equilibrium configuration $x = \dot{x} = 0$ of the unperturbed system exists only when $Q(t) \equiv 0$. Therefore, it is desired to determine the appropriate bounds on the vector x(t).

Consider the Liapunov function

$$V(y) = y^T P y \quad , \qquad (2\text{-}8b)$$

where the matrix P is given by equation (2-13) and the 2n vector y is defined in chapter II.

Let

$$U(y) = \{V(y)\}^{1/2} = \{y^T P y\}^{1/2} \; . \qquad (4\text{-}2)$$

The positive root of the scalar function represented by equation (4-2) has all the properties of a norm. This root is used as the measure of the magnitude of the vector y. Taking the time derivative of V(y) and evaluating it along the trajectory of equation (4-1) yields (the calculations are similar to those given in sec. 2.3)

$$\dot{V}(y) = y^T[A_o + C_t + K_t]y + 2y^T Pq(t), \tag{4-3}$$

where

$$q(t) = \begin{Bmatrix} 0 \\ \text{-------} \\ M^{-1}Q(t) \end{Bmatrix} \tag{4-4}$$

and P, A_o, C_t, and K_t are defined by equations (2-13), (2-12), (2-23), (2-24), respectively in chapter II. Next, let $\lambda_{max}(t)$ be the largest eigenvalue of the matrix $\{[A_o + C_t + K_t]P^{-1}\}$ and define $\mu(t)$ by

$$\mu(t) \equiv \{q^T(t)Pq(t)\}^{\frac{1}{2}} \geq 0 \quad . \tag{4-5}$$

Due to Schwarz inequality, the last term of equation (4-3) satisfies

$$y^T Pq \leq \{y^T Py\}^{\frac{1}{2}}\{q^T(t)Pq(t)\}^{\frac{1}{2}} = \mu(t)\{V(y)\}^{\frac{1}{2}} \quad . \tag{4-6}$$

Then, from the results of equation (2-27) and (2-28), obtained in chapter II, and the above result, equation (4-3) yields the following inequality.

$$\dot{V}(t) \leq \lambda_{max}(t)V(t) + 2\mu(t)\{V(y)\}^{\frac{1}{2}} \tag{4-7}$$

Using equation (4-2), inequality (4-7) becomes

$$\dot{U}(t) \leq \tfrac{1}{2}\lambda_{max}(t)U(t) + \mu(t) \quad . \tag{4-8}$$

Integration yields

$$U(t) \leq U(0)\exp\left[\tfrac{1}{2}\int_0^t \lambda_{max}(\tau)d\tau\right] +$$

$$+ \int_0^t \mu(\beta)\exp\left[\tfrac{1}{2}\int_\beta^t \lambda_{max}(\tau)d\tau\right]d\beta \tag{4-9}$$

with $U(0) = \{y^T(0)Py(0)\}^{\frac{1}{2}}$.

The inequality (4-9) gives the upper bound on the norm $U(t)$ of the system represented by equation (4-1). This bound is in terms of the time integrals of $\lambda_{max}(t)$ and $\mu(t)$, which are functions of the dynamic loads and/or perturbations. Using lemma 2 on inequality (4-9), the effects of μ and λ_{max} can be separated as

$$U(t) \leq U(0)\exp\left[\tfrac{1}{2}\int_0^t \lambda_{max}(\tau)d\tau\right] + \left\{\int_0^t \mu^2(\beta)d\beta\right\}^{\frac{1}{2}}\left\{\int_0^t \exp\left[\int_0^t \lambda_{max}(\tau)\,d\tau\right]d\beta\right\}^{\frac{1}{2}} . \tag{4-10}$$

For the special case of $q(t) \equiv 0$, the bound becomes the same as that derived in chapter II. If $C(t) = K(t) \equiv 0$ instead, this leads to

$$U(t) \leq U(0)\exp[\lambda_0 t/2] + \int_0^t \mu(\beta)\exp[\lambda_0(t-\beta)/2]\,d\beta \tag{4-11}$$

from inequality (4-9), or

$$U(t) \leq \exp[\lambda_0 t/2]\left\{U(0) + \left[\int_0^t \mu^2(\beta)d\beta(1-\exp[-\lambda_0 t])/\lambda_0\right]^{\frac{1}{2}}\right\} \tag{4-12}$$

from inequality (4-10), where λ_0 is the maximum eigenvalue of the constant matrix A_0P^{-1}.

The results can also be established in terms of the maximum values of $\lambda_{max}(t)$ and $\mu(t)$. For this purpose, let

$$\lambda_{max}(t) \leq \lambda_M,\ \mu(t) \leq \mu_M; \quad \text{for } 0 \leq t \leq t_1 \tag{4-13}$$

and

$$\lambda_{max}(t) = \lambda_0,\ \mu(t) = 0; \quad \text{for } t > t_1 \ . \tag{4-14}$$

For deterministic $Q(t)$, an estimate of μ_M in inequality (4-13) can be obtained by defining

$$\sup_t |q_{ij}(t)| \leq q_M, \quad \text{for } 0 \leq t \leq t_1 \quad ,$$

(4-15)

$$q(t) = 0, \quad \text{for } t > t_1 \quad .$$

Then it follows from equation (4-5) that

$$\mu_M = \{q_M^T P q_M\}^{\frac{1}{2}} \quad . \tag{4-16}$$

For stochastic $Q(t)$, definition (4-15) should be modified as

$$\text{a.s.} \quad \sup(\sup_t |q_{ij}(t)|) \leq q_M \ , \quad \text{for } 0 \leq t \leq t_1 \quad , \tag{4-17}$$

$$q(t) = 0, \quad \text{for } t > t_1 \quad . \tag{4-18}$$

For stochastic $C(t)$ and $K(t)$, it suffices to require

$$E\{\lambda_{max}(t)\} \leq \lambda_M; \quad \text{for } 0 \leq t \leq t_1 \quad . \tag{4-19}$$

Application of inequality (4-13) and (4-14) to inequality (4-9) yields the following bounds.

$$U(t) \leq [U(0) + 2\mu_M/\lambda_M]\exp[\lambda_M t/2] - 2\mu_M/\lambda_M, \quad \text{for } 0 \leq t \leq t_1 \tag{4-20}$$

and

$$U(t) \leq U(0)\exp[\tfrac{1}{2}(\lambda_M t_1 + \lambda_o(t-t_1))] + 2(\exp[\lambda_M t_1/2] - 1)\mu_M/\lambda_M, \text{for } t > t_1 \ . \tag{4-21}$$

If $\lambda_o < 0$, the bounds given by inequalities (4-20) and (4-21) reduce to

$$U(t) \leq 2(\exp[\lambda_M t_1/2] - 1)\mu_M/\lambda_M \tag{4-22}$$

when $t \to \infty$. If instead, $\lambda_M < 0$ and $t_1 = \infty$, that is the time dependent terms do not vanish, the bounds then reduce to

$$U(t) \leq -2\mu_M/\lambda_M \tag{4-23}$$

when $t \to \infty$.

The bounds derived in terms of the norm U(t) depend on the matrix P. This matrix is given by equation (2-13) in chapter II in terms of the constant parameters of the system. An optimization scheme as used in reference [11 or 29] is not needed to minimize the upper bound of the desired form.

4.2 Applications

In the following, some typical examples are presented to illustrate the method. Whenever possible, the results are compared with the bounds available in the literature.

Example 4.2.1. Consider an inverted pendulum, which may be used to model a chimney [34] , shown in figure 4.1. The support has arbitrary movements both in the vertical and in the horizontal directions. The mathematical representation is given as

$$\ddot{x} + 2\zeta\dot{x} + (\omega^2 + f(t))x = h(t), \tag{4-24}$$

where $2\,\zeta = c/(m\ell)$ represents the damping and $\omega^2 = [(k/(m\ell)) - g]$ is the natural frequency of the constant undamped system. The other parameters are defined as in the following.

- g = the gravitational acceleration
- m = the mass of the system
- c = the torsional damping coefficient
- k = the torsional stiffness coefficient
- ℓ = the length of the pendulum
- $f(t)$ = the motion of the support in the vertical direction
- $h(t)$ = the motion of the support in the horizontal direction
- x = the angle of rotation measured from the vertical

Notice that for $h(t) \equiv 0$, equation (4-24) is the same as equation (3-1). Vector $q(t)$, defined by equation (4-4) in the previous section, takes the form

$$q(t) = \begin{Bmatrix} 0 \\ h(t) \end{Bmatrix} . \tag{4-25}$$

Using the calculations done in example 3.1.1, $\lambda_{max}(t)$ is given by equation (3-6) as

$$\lambda_{max}(t) = -2\zeta + \frac{2\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}} \left[1 - \frac{f(t)}{\zeta^2} + \frac{f^2(t)}{4\zeta^4}\right]^{\frac{1}{2}} . \tag{4-26}$$

With P given by equation (2-13), $\mu(t)$ is easily computed as

$$\mu(t) = \{q^T(t)Pq(t)\}^{\frac{1}{2}} = |h(t)| . \tag{4-27}$$

Substitution of equations (4-26) and (4-27) and the initial conditions, $x(0) = x_o$ and $\dot{x}(0) = \dot{x}_o$, into inequality (4-9) yields the bound

$$U(t) \leq \left[(\omega^2 + 2\zeta^2)x_o^2 + 2\zeta\dot{x}_o x_o + \dot{x}_o^2\right]^{\frac{1}{2}} \exp\left[\tfrac{1}{2}\int_o^t \left[-2\zeta + \frac{2\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}}\left[1 - \frac{f(\tau)}{\zeta^2} + \frac{f^2(\tau)}{4\zeta^4}\right]^{\frac{1}{2}}\right] d\tau\right] + \int_o^t |h(\beta)| \exp\left[\tfrac{1}{2}\int_o^t \left[-2\zeta + \frac{2\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}}\left[1 - \frac{f(\tau)}{\zeta^2} + \frac{f^2(\tau)}{4\zeta^4}\right]^{\frac{1}{2}}\right] d\tau\right] d\beta. \tag{4-28}$$

Since

$$V(y) \equiv V(x,\dot{x}) = (\zeta x + \dot{x})^2 + (\omega^2 + \zeta^2)x^2 , \tag{4-29}$$

it follows that

$$(\omega^2 + \zeta^2)x^2 \leq V(t) = \{U(t)\}^2 , \tag{4-30}$$

from which the bound on $x(t)$ is determined as

$$|x(t)| \leq (\omega^2 + \zeta^2)^{-\frac{1}{2}} U(t) \quad . \tag{4-31}$$

To avoid complication, consider the case where $f(t) \equiv 0$ and the initial conditions are zero, $x(0) = \dot{x}(0) = 0$. Then, from inequalities (4-28) and (4-31), the response bound becomes

$$|x(t)| \leq (\omega^2 + \zeta^2)^{-\frac{1}{2}} \int_0^t |h(\beta)| \exp\left[\left(-\zeta + \frac{\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}}\right)(t-\beta)\right] d\beta \ . \tag{4-32}$$

For the special case where there is also no damping ($\zeta = 0$), the bound on $x(t)$ reduces to

$$|x(t)| \leq \frac{1}{\omega} \int_0^t |h(\tau)| d\tau \quad . \tag{4-33}$$

If $h(t)$ is stochastic, then inequality (4-33) can be written in terms of the expected value of $|h(t)|$ as

$$|x(t)| \leq \left[\frac{1}{\omega} E\{|h(t)|\}\right] t \tag{4-34}$$

with probability one.

Instead, if $h(t)$ is deterministic, the inequalities (4-32) and (4-33) can be directly integrated to yield the response bounds. As an example, let $h(t) = A \cos\alpha t$, $A > 0$. Noting that $|h(t)| \leq A$ for all t, inequality (4-32) yields

$$|x(t)| \leq (\omega^2 + \zeta^2)^{-\frac{1}{2}} \int_0^t A \exp\left[\left(-\zeta + \frac{\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}}\right)(t-\beta)\right] d\beta \ . \tag{4-35}$$

Upon integration,

$$|x(t)| \leq \frac{A}{\left[\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - \zeta^2\right]} \left[1 - \exp\left[\left(-\zeta + \frac{\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}}\right) t\right]\right], \ (\zeta > 0), \tag{4-36}$$

which as $t \to \infty$, the bound becomes

$$|x(t)| \le \frac{A}{[\zeta(\omega^2 + \zeta^2)^{\frac{1}{2}} - \zeta^2]} \quad , \quad (\zeta > 0). \tag{4-37}$$

For the undamped system, the bound from inequality (4-33) becomes

$$|x(t)| \le \frac{2A}{\pi\omega} t \quad . \tag{4-38}$$

Next, consider the situation when $h(t)$ is known to be a Gaussian process with zero mean. The probability density function p is given by [17]

$$p(h) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{h^2}{2\sigma^2}\right] \qquad -\infty < h < \infty \ , \tag{4-39}$$

where σ is the standard deviation of h. Assuming the equality of time averages and ensemble averages stated in definition 2 of chapter II, the response bounds for equation (4-24) can be determined through the distributional properties alone. To obtain the response bound in terms of the second moment of the h-process, inequality (4-10) with (4-31) can be used to yield

$$|x(t)| \le (\omega^2 + \zeta^2)^{-\frac{1}{2}} \{\int_0^t h^2(\beta) d\beta\}^{\frac{1}{2}} \{\int_0^t \exp\left[(-2\zeta + \frac{2\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}})(t-\beta)\right] d\beta\}^{\frac{1}{2}} \ , \tag{4-40}$$

which can be rewritten as

$$|x(t)| \le (\omega^2 + \zeta^2)^{-\frac{1}{2}} [E\{h^2(t)\} t]^{\frac{1}{2}} \left|\int_0^t \exp\left[(-2\zeta + \frac{2\zeta^2}{(\omega^2 + \zeta^2)^{\frac{1}{2}}})(t-\beta)\right] d\beta\right|^{\frac{1}{2}} \ . \tag{4-41}$$

$E\{h^2(t)\}$ is easily calculated as

$$E\{h^2(t)\} = E\{h^2(0)\} = \int_{-\infty}^{\infty} h^2 p(h) dh = \sigma^2 \quad . \tag{4-42}$$

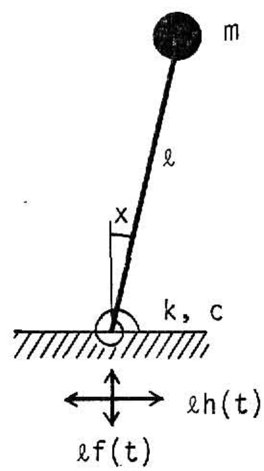

Figure 4.1: Inverted Pendulum with Arbitrary Base Motion

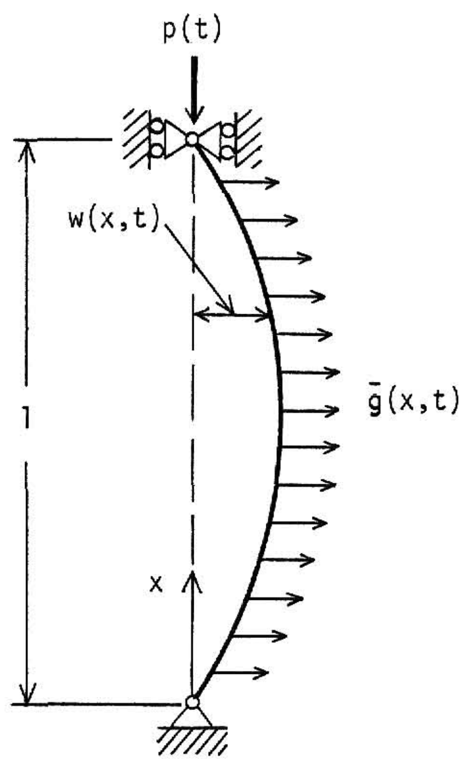

Figure 4.2: Simply Supported Beam-Column Subjected to a Distributed Transverse Load

With this result, upon integration, inequality (4-41) becomes

$$|x(t)| \leq \sigma\left[\frac{t}{\left[(\omega^2+\zeta^2)(2\zeta - \frac{2\zeta^2}{(\omega^2+\zeta^2)^{\frac{1}{2}}})\right]}\left(1 - \exp\left[\left(-2\zeta + \frac{2\zeta^2}{(\omega^2+\zeta^2)^{\frac{1}{2}}}\right)t\right]\right)\right]^{\frac{1}{2}} \tag{4-43}$$

with probability one.

For $\zeta = 0$, the response bound is determined from inequality (4-34) as

$$|x(t)| \leq (\sigma/\omega)(2/\pi)^{\frac{1}{2}}t \tag{4-44}$$

with probability one. Note that the following relationship has been used.

$$E\{|h(t)|\} = E\{|h(0)|\} = \int_{-\infty}^{\infty}|h|p(h)dh = \sigma(2/\pi)^{\frac{1}{2}} \tag{4-45}$$

Example 4.2.2. As a second example, consider the nondimensional equation for the displacement, $w(x,t)$, of the simply supported beam-column, as discussed in example 3.2.1, subjected to a distributed transverse load. This is shown in figure 4.2. The equation of motion now takes the form

$$\frac{\partial^2 w}{\partial t^2} + 2\zeta\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + p(t)\frac{\partial^2 w}{\partial x^2} = \bar{g}(x,t) \tag{4-46}$$

for $0 \leq x \leq 1$, $t \geq 0$, where $p(t)$ is the axial load, $\bar{g}(x,t)$ is the distributed transverse load, and ζ is the damping coefficient. The boundary conditions are

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(1,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0, \quad t \geq 0. \tag{3-36}$$

As in example 3.2.1, writing the displacement in the form

$$w(x,t) = \sum_{m=1}^{\infty} a_m(t)\sin m\pi x, \tag{3-37}$$

leads to the set of ordinary differential equations

$$\ddot{a}_m + 2\zeta\,\dot{a}_m + \left[m^4\pi^4 - m^2\pi^2 p(t)\right] a_m = h(t), \quad m = 1,2,\ldots, \tag{4-47}$$

where the additional term h(t) is defined via the Fourier integral as

$$h(t) = 2\int_0^1 \bar{g}(x,t)\sin m\pi x\, dx. \tag{4-48}$$

Letting $m^4\pi^4 = \omega^2$, $m^2\pi^2 p(t) = -f(t)$, and $a_m(t) = x(t)$, equation (4-47) reduces exactly to the same form of equation (4-24) for each *m*. Therefore, it follows that the bound on the modal amplitude, $a_m(t)$, can be written directly from inequality (4-31) as

$$|a_m(t)| \leq (m^4\pi^4 + \zeta^2)^{-\frac{1}{2}}\, U(t)\ . \tag{4-49}$$

For the special case of no axial load ($p(t) \equiv 0$), zero damping ($\zeta = 0$), and zero initial conditions ($a_m(0) = \dot{a}_m(0) = 0$), leads to the *bound*

$$|a_m(t)| \leq \frac{2}{m^2\pi^2}\int_0^t \left|\int_0^1 \bar{g}(x,\tau)\sin m\pi x\, dx\right| d\tau\ , \tag{4-50}$$

which is of the form of inequality (4-33). As shown in example 4.2.1, inequalities (4-49) and (4-50) can once again be specialized for particular forms of p(t) and $\bar{g}(x,t)$. Inequality (4-50) is the same as the optimized results obtained in reference [29].

<u>Example 4.2.3</u>. Finally, consider a three-story frame modeled as a three degree-of-freedom system (shear-beam idealization) represented as [31]

$$M\ddot{x} + K_0 x = Q(t) \; , \tag{4-51}$$

where $M = m[I]$, m is the mass coefficient, $[I]$ is the identity matrix,

$$K_0 = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} , \tag{4-52}$$

with k as the stiffness coefficient. Following Holzer [31], the transient lateral load is assumed to be of the form

$$Q(t) = \Phi_1 f(t) \; , \tag{4-53}$$

where

$$f(t) = f_0, \quad 0 \le t \le t_d = \frac{\pi}{\omega_1} \; , \tag{4-54a}$$

$$f(t) = 0, \quad t > t_d \; , \tag{4-54b}$$

$$\Phi_1 = m^{-\frac{1}{2}} \begin{Bmatrix} 0.328 \\ 0.591 \\ 0.737 \end{Bmatrix} \tag{4-55}$$

which is the fundamental modal vector of vibration normalized with respect to M, and $\omega_1 = 0.445(\frac{k}{m})^{\frac{1}{2}}$ is the fundamental circular frequency. The initial conditions are taken as $x(0) = \dot{x}(0) = 0$. Because K_0 is symmetric, P_2 is selected as M. Then, from equation (2-13), matrix P takes the form

$$P = \left[\begin{array}{c|c} K_0 & 0 \\ \hline 0 & M \end{array}\right] . \tag{4-56}$$

Since there is no damping in the system, it is seen, from equation (2-12), that matrix $A_o \equiv 0$. Hence λ_o, the maximum eigenvalue of the A_oP^{-1} matrix, is identically zero. For this example, equation (4-4) takes the form

$$q(t) = m^{\frac{1}{2}}f(t)\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.328 \\ 0.591 \\ 0.737 \end{Bmatrix} . \tag{4-57}$$

Upon simple calculation, one obtains

$$\mu(t) = \{q^T(t)Pq(t)\}^{\frac{1}{2}} = |f(t)|/m . \tag{4-58}$$

Substituting the results from equation (4-58), $\lambda_o = 0$, and $U(0) = 0$ in inequality (4-11), the following bound on $U(t)$ is obtained.

$$U(t) \leq \int_o^t \mu(\beta)d\beta = \int_o^t (|f(\beta)|/m)d\beta \tag{4-59}$$

Or

$$U(t) \leq \frac{f_o}{m}\, t \qquad \text{for} \quad 0 \leq t \leq t_d \tag{4-60a}$$

$$U(t) \leq \frac{f_o\pi}{m\omega_1} \qquad \text{for} \quad t > t_d . \tag{4-60b}$$

But from the definition of V

$$V(y) = V(x,\dot{x}) = x^TK_ox + \dot{x}^TM\dot{x}, \tag{4-61}$$

it then follows that

$$x^TK_ox \leq V(x) = \{U(t)\}^2 . \tag{4-62}$$

Therefore the response bound on x can be represented in terms of the strain energy norm

$$x^T K_o x \leq (\frac{f_o}{m})^2 t^2, \qquad \text{for} \quad 0 \leq t \leq t_d \ , \tag{4-63a}$$

$$x^T K_o x \leq (\frac{f_o \pi}{m\omega_1})^2, \qquad \text{for} \quad t > t_d \ . \tag{4-63b}$$

Inequality (4-63b) is the same as that obtained by Holzer [31] for $t \geq 0$. It is clearly seen that the bound given by the combination of inequalities (4-63a) and (4-63b) is superior to that obtained by inequality (4-63b) alone. Moreover, the result of reference [31] was obtained via an optimization procedure employing the Lagrange multiplier technique which can be time consuming in some cases. As pointed out earlier, no optimization is needed in the present analysis. Observe that the method developed in this study is not restricted to undamped systems while such is the case for the method suggested in reference [31].

In order to get an idea of the sharpness of the bound obtained by the present method, the actual displacement vector x(t) of the system (4-51) is computed by performing the modal analysis. For this purpose, first the eigenvalue problem

$$M \left[\diagdown \omega^2 \diagdown\right] \Phi = K_o \Phi \tag{4-64}$$

is solved. Φ is the modal matrix and $\left[\diagdown \omega^2 \diagdown\right]$ is the diagonal matrix consisting of n eigenvalues, ω^2. Substituting

$$x(t) = \Phi\, \eta(t), \tag{4-65}$$

where η is a vector consisting of a set of time-dependent generalized coordinates, into equation (4-51) and premultiplying by Φ^T yields a set

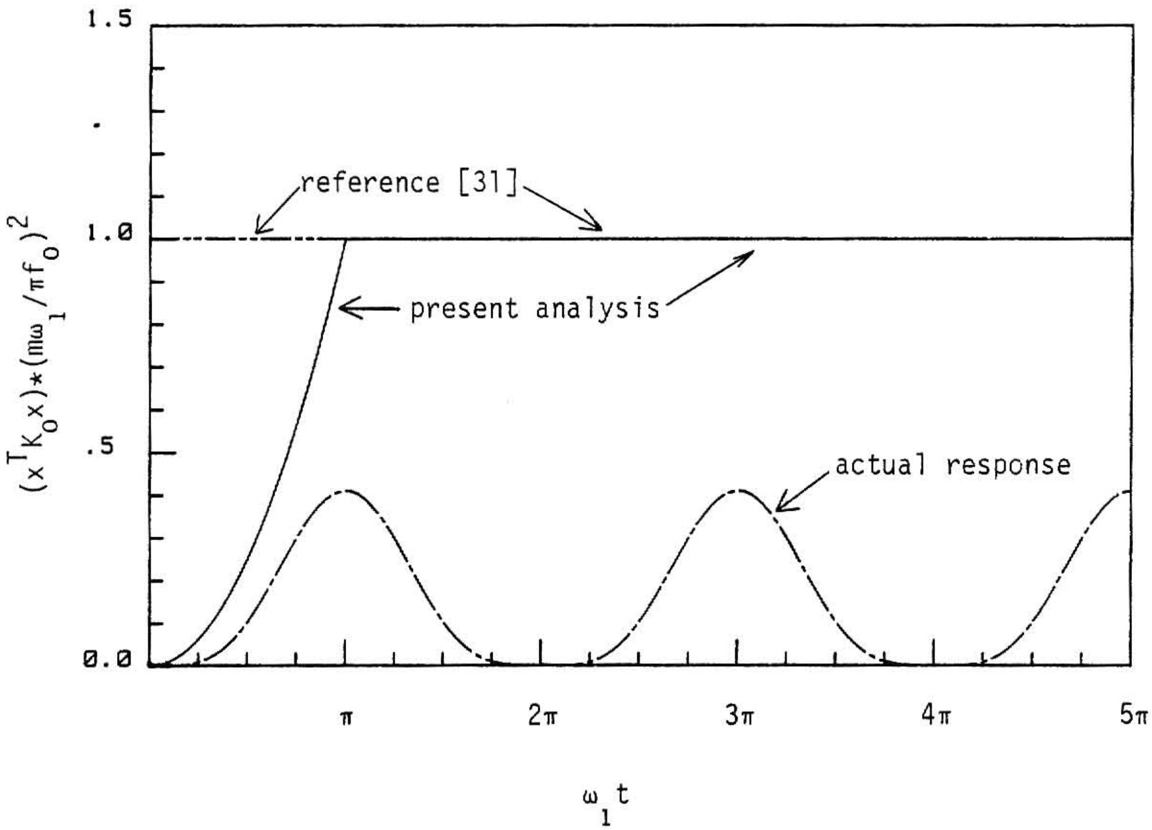

Figure 4.3: Response Bounds for Example 4.2.3

of n uncoupled differential equations where Φ is normalized with respect to M. Solving the uncoupled differential equations, the following displacement vector is obtained.

$$x = \frac{f_o \Phi_1}{m\,\omega_1^2}(1 - \cos\omega_1 t), \tag{4-66}$$

where Φ_1 is the modal vector associated with the fundamental frequency ω_1. From this equation it is seen that the maximum displacement vector is given by

$$x_{max} = \frac{2f_o}{\sqrt{k}\,m\,\omega_1}\begin{Bmatrix} 0.74 \\ 1.33 \\ 1.66 \end{Bmatrix} . \tag{4-67}$$

Further calculation yields

$$x_{max}^T\, K_o\, x_{max} = 0.41\,\left(\frac{\pi f_o}{m\,\omega_1}\right)^2 . \tag{4-68}$$

It is observed that the bound given by inequality (4-63b) is somewhat larger than that of the maximum displacement of the system. Although the bound is not close enough, it may prove useful in engineering analysis for a number of problems, such as, to determine whether a dynamic analysis is required, or to estimate the dynamic load for design purposes. The results are shown graphically in figure 4.3.

V. DISCUSSION AND CONCLUSIONS

An attempt has been made to present a simple computational scheme to guarantee sufficient conditions for the almost sure asymptotic stability of systems described by a set of second order differential equations with stochastic parameters. A theorem and related corollaries, directly applicable to such systems, have been presented through an extension of the approach suggested by Infante [11]. As long as the moments of the time varying parameter ($E\{|f(t)|\}$, $E\{f^2(t)\}$, etc.) exist, the stability bounds can be easily determined from the theorem and/or the corollaries. The technique is not only applicable to stochastic systems but also to systems with deterministic parameters, as pointed out in chapter II.

Unlike previous investigations, a Liapunov function, in terms of parameters constituting the constant part of the system, is proposed. This results in a considerable reduction in computational effort since one need not solve the resulting Liapunov matrix equation which can be quite cumbersome for large systems. For systems where the stiffness matrix K_0 is symmetric, the Liapunov function is predetermined in terms of constant matrices M, C_0, and K_0. For systems with non-symmetric K_0, one has to find a satisfactory matrix P_2 which involves only $n(n+1)/2$ unknowns. The suggested form of the Liapunov matrix is especially useful when the damping is proportional to either the mass and/or the stiffness matrix.

In chapter III, illustrative examples show that the method yields stability bounds of practical significance and are comparable to the optimum results. Since the method does not require implementation of any optimization procedure, it is much more economical in terms of computer time. To illustrate this point, following reference [15], it is observed that the complete optimization process involves a total of 10 variables for the example problem 3.2.3 which has only two degrees of freedom. For example 3.2.2, which has six degrees of freedom, a total of 78 variables have to be optimized.

Perhaps, the most important feature of the suggested technique is its ability of handling systems involving follower forces in a rather simple fashion and still produce results which are quite sharp.

The theorem is extended in chapter IV to determine the response bounds for systems with dynamic loads or perturbations which cannot be represented in stiffness and/or damping matrices. It is found that if the maximum eigenvalue $\lambda_{max}(t)$ for the system is negative and in addition if the norm of the forcing term is bounded, then the bounds reach a constant value as t approaches ∞. The bounds are obtained in terms of the forcing function and system parameters. Unlike previous studies, no optimization is required. The results compare very well with those obtained via optimization procedures.

However, careful examination reveals that the method may not be applicable to all time-dependent systems. In order to apply the theorem successfully, it is observed that matrices A_0 and P must remain negative and positive definite, respectively. These conditions may not be satisfied for certain systems although they can be proven asymptotically stable by other techniques. Since a Liapunov function yields only sufficient conditions for stability, this behavior is not unexpected. In all such cases, the suggested form of the Liapunov matrix P cannot be used to determine the stability criteria.

In conclusion, a Liapunov technique is suggested for studying the almost sure asymptotic stability and response bounds of a class of discrete linear systems with stochastic parameters. The approach is simple and very useful from the viewpoint of computation, specially for systems with large number of degrees-of-freedom. The method can be applied to systems involving forces of arbitrary nature and yields results of practical value.

Whereas this investigation has been primarily devoted to the study of discrete or discretized systems, there exist studies which directly deal with continuous systems [33]. While such studies have their merit, any contribution in the study of large discrete systems is of much significance from the practical viewpoint.

It is needless to say that there yet remains much to be accomplished. The stability bounds may be improved significantly by taking into account for more statistical properties of the processes. Some results of this nature are already available for second order scalar equations [14, 35]. However, the problem of finding the necessary as well as the sufficient conditions for stability still remains unsolved in most instances. Except for systems driven by white noise, these conditions have not yet been established. The subject of sample stability continues to pose a difficult challenge.

Some suggested areas for further investigation associated with the present study seem to be the following:

1. Introduce a scaling parameter α in P_3 such that $P_3 = \alpha P_2 M^{-1} C_0$ so as to possibly obtain the optimum Liapunov matrix associated with a par-

ticular system.

2. The distributional properties of the coefficient processes may be exploited to yield improved sufficient conditions.

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Appendix A

```
10   !                    A COMPUTER PROGRAM
20   !              BASED ON COROLLARIES I AND II
30   !
40   !      This program determines the almost sure asymptotic
50   ! stability of linear discrete systems described by a set
60   ! of second order differential equations with stochastic
70   ! parameters using corollaries I and II.  It finds the
80   ! stability conditions when only one of the matrices, dam-
90   ! ping or stiffness, is multiplied by a "scalar" stochastic
100  ! process.
110  !      After defining all the necessary matrices, the pro-
120  ! gram calculates the eigenvalues of the P matrix to give a
130  ! check for positive definiteness of matrix P.  Then, it
140  ! finds the eigenvalues for the Ao(P_inverse) matrix and the
150  ! Cti(P_inverse) or Ktj(P_inverse) matrix of the particular
160  ! problem.  Using these eigenvalues in the corollaries, the
170  ! stability conditions are found.
180  !     This particular program generates the stability con-
190  ! ditions as a function of the damping coefficient Zeta.  The
200  ! M, Ko, Ci, Kj, and P2 matrices are fed into the computer
210  ! from the keyboard.  The Co matrix elements, as functions of
220  ! Zeta, must be manually inserted into the program between
230  ! lines 1080 and 1100.  The output of the program is the sta-
240  ! bility conditions of corollaries I and II for either
250  ! E(|g(t)|) or E(|f(t)|) at the value Zeta.
260  !
270  !
280  !
290   OPTION BASE 1
300   GCLEAR
310   PRINTER IS 16
320   FIXED 0
330   REAL M(10,10),Co(10,10),Ci(10,10),Ko(10,10),Kj(10,10)
340   REAL P1(10,10),P2(10,10),P3(10,10),Aop(20,20),Ktp(20,20)
350   REAL Ctp(20,20),Minv(10,10),P3t(10,10),P1k(10,10),Z(10,10)
360   REAL D(10,10),O(10,10),A1(10,10),A11(10,10),G1(10,10)
370   REAL P1c(10,10),G11(10,10),Mkj(10,10),Dk(10,10),Fk(10,10)
380   REAL P2mkj(10,10),P2mkjt(10,10),Ktbb(10,10),P2mcit(10,10)
390   REAL Zz(10,10),Mci(10,10),Dc(10,10),Dct(10,10),P2mci(10,10)
400   REAL Aopp(12,12),Ktpp(12,12),Ctpp(12,12),Aevr(12),Aevi(12)
410   REAL Avecr(12,12),Aindic(12),Kevr(12),Kevi(12),Kindic(12)
420   REAL Cevr(12),Cevi(12),Cindic(12),Minvk(10,10),Pinv(12,12)
430   REAL Ao(12,12),Ct(12,12),Kt(12,12),P(12,12),Ktb21(10,10)
440   REAL Ktb12(10,10),Ktb11(10,10),Ctb12(10,10),Ctb21(10,10)
450   REAL Aopabs(12,12),Ktpabs(12,12),Ctpabs(12,12),Ctb22(10,10)
460   REAL Pveci(12,12),Co1(10,10),P21(10,10),P12(10,10),Cb(10,10)
470   REAL Aveci(12,12),Kvecr(12,12),Kveci(12,12),Cvecr(12,12)
480   REAL Cveci(12,12),Pevr(12),Pevi(12),Pindic(12),Pvecr(12,12)
490   !     DIMENSION OF Zeta1 AND F=E(|f(t)|) OR E(|g(t)|)
500                  REAL Zeta1(99),F(99),F1(99)
510   INPUT "ORDER OF MATRICES M(*),C(*),K(*), [# OF ROWS OR # OF
COLUMNS]?",N
```

```
520    PRINT LIN(2),"ORDER OF MATRICES=";N
530    IF N<=0 THEN 510
540    S=2*N
550    REDIM M(N,N),Co(N,N),Ci(N,N),Ko(N,N),Kj(N,N),P1(N,N),P2(N,N)
560    REDIM P3(N,N),Aop(S,S),Ktp(S,S),Ctp(S,S),Minv(N,N),P3t(N,N)
570    REDIM P1k(N,N),Z(N,N),P1c(N,N),D(N,N),O(N,N),Al(N,N),All(N,N)
580    REDIM GI(N,N),GII(N,N),Zz(N,N),Mkj(N,N),Dk(N,N),Fk(N,N)
590    REDIM P2mkj(N,N),P2mkjt(N,N),Ktbb(N,N),Mci(N,N),Dc(N,N)
600    REDIM Dct(N,N),P2mci(N,N),P2mcit(N,N),Cb(N,N),Aopp(S,S)
610    REDIM Ktpp(S,S),Ctpp(S,S),Aevr(S),Aevi(S),Aindic(S),Kevr(S)
620    REDIM Kevi(S),Kindic(S),Cevr(S),Cevi(S),Cindic(S),Minvk(N,N)
630    REDIM Co1(N,N),Pinv(S,S),Ao(S,S),Ct(S,S),Kt(S,S),P(S,S)
640    REDIM Ktb21(N,N),Ktb12(N,N),Ktb11(N,N),Ctb12(N,N),Ctb21(N,N)
650    REDIM Pevr(S),Pevi(S),Pindic(S),Pvecr(S,S),Pveci(S,S)
660    REDIM Ctb22(N,N),P21(N,N),P12(N,N),Avecr(S,S),Aveci(S,S)
670    REDIM Kvecr(S,S),Kveci(S,S),Cvecr(S,S),Cveci(S,S),Aopabs(S,S)
680    REDIM Ktpabs(S,S),Ctpabs(S,S)
690    FLOAT 6
700    ! FEEDING IN OF MATRIX M FROM THE KEYBOARD AND THE GENERATION
710    ! OF ITS INVERSE.
720    CALL Mat_m(N,M(*))
730       MAT Minv=INV(M)
740       IF DET=0 THEN 2740
750    ! FEEDING IN OF MATRIX Ko FROM THE KEYBOARD AND THE GENERATION
760    ! OF THE MATRIX (M_inverse*Ko).
770    CALL Mat_ko(N,Ko(*))
780       MAT Minvk=Minv*Ko
790       PRINT "MAT Minv*Ko",Minvk(*);
800      PRINT "PRESS CONT AFTER YOU HAVE DETERMINED WHAT P2 MATRIX TO
  USE"
810       PAUSE
820    INPUT "DOES P2 EQUAL M, [symmetric Ko?] ? (Y/N)",Y$
830    IF (Y$="N") OR (Y$="n") THEN 880
840    IF (Y$="Y") OR (Y$="y") THEN 860
850    GOTO 820
860    MAT P2=M
870    GOTO 890
880    CALL Mat_p2(N,P2(*))
890       PRINT LIN(2),"MAT P2";P2(*);LIN(2);"IF P2 IS OK, PRESS
CONT.";LIN(2)
900       PAUSE
910    MAT P1k=P2*Minvk
920    PRINT LIN(2),"MAT P2*Minv*Ko",LIN(1),P1k(*);
930    PRINT "CHECK TO SEE IF P2 SYMMETRIZES Minv*Ko. KEY N IF YOU
  WANT TO CHANGE P2. KEY Y TO CONTINUE FOR P2, OK."
940    INPUT "(Y/N)",N$
950    IF (N$="N") OR (N$="n") THEN 820
960    IF (N$="Y") OR (N$="y") THEN 980
970    GOTO 930
980    ! FEEDING IN OF MATRICES Ci and Kj FROM KEYBOARD. !
990    CALL Mat_ci(N,Ci(*))
1000   CALL Mat_kj(N,Kj(*))
1010   ! GENERATION OF Co,Aop,Ktp,Ctp,and P.  ALSO, FINDS Eigenvalues
1020   ! (max. and min.) for Aop,Ktp,Ctp, and P. THIS IS DONE FOR Co
1030   ! CHANGING AS A FUNCTION OF ZETA.
1040   H=0
1050   INPUT "MAXIMUM ZETA?  STEP SIZE? (Zetamax,Step)",Zetamax,Step
1060   FOR Zeta=0 TO Zetamax STEP Step
```

```
1070  ! MATRIX Co as a FUNCTION of Zeta for the system MUST BE
1080  ! INSERTED IN THE PROGRAM HERE!
1090  MAT Co=(Zeta)*Ko
1100  PRINT LIN(3),"ZETA=";Zeta,LIN(2)
1110  MAT Zz=Minv*Co
1120  MAT P3=P2*Zz
1130  MAT P3t=TRN(P3)
1140  MAT Z=P3t*Zz
1150  MAT P1c=(.5)*Z
1160  MAT P1=P1k+P1c
1170  MAT D=P3t*Minvk
1180  MAT O=TRN(D)
1190  MAT A1=D+O
1200  MAT A11=(-.5)*A1
1210  MAT G1=P3t+P3
1220  MAT G11=(-.5)*G1
1230  MAT Mkj=Minv*Kj
1240  MAT Dk=P3t*Mkj
1250  MAT Fk=TRN(Dk)
1260  MAT P2mkj=P2*Mkj
1270  MAT P2mkjt=TRN(P2mkj)
1280  MAT Ktbb=Dk+Fk
1290  MAT Mci=Minv*Ci
1300  MAT Dc=P3t*Mci
1310  MAT Dct=TRN(Dc)
1320  MAT P2mci=P2*Mci
1330  MAT P2mcit=TRN(P2mci)
1340  MAT Cb=P2mcit+P2mci
1350  MAT Ktb21=(-1)*P2mkj
1360  MAT Ktb12=(-1)*P2mkjt
1370  MAT Ktb11=(-.5)*Ktbb
1380  MAT Ctb12=(-.5)*Dc
1390  MAT Ctb21=(-.5)*Dct
1400  MAT Ctb22=(-1)*Cb
1410  MAT P21=(.5)*P3
1420  MAT P12=(.5)*P3t
1430    FOR I=1 TO N
1440      FOR J=1 TO N
1450        Ao(I,J)=A11(I,J)
1460        Kt(I,J)=Ktb11(I,J)
1470        Ct(I,J)=0
1480        P(I,J)=P1(I,J)
1490      NEXT J
1500    NEXT I
1510    R=N+1
1520     FOR K=R TO S
1530      FOR L=1 TO N
1540        Q=K-N
1550        Ao(K,L)=0
1560        Ao(L,K)=0
1570        Kt(K,L)=Ktb21(Q,L)
1580        Kt(L,K)=Ktb12(L,Q)
1590        Ct(K,L)=Ctb21(Q,L)
1600        Ct(L,K)=Ctb12(L,Q)
1610        P(K,L)=P21(Q,L)
1620        P(L,K)=P12(L,Q)
1630      NEXT L
1640     NEXT K
```

```
1650      FOR M=R TO S
1660       FOR T=R TO S
1670        U=M-N
1680        V=T-N
1690        Ao(M,T)=G11(U,V)
1700        Kt(M,T)=0
1710        Ct(M,T)=Ctb22(U,V)
1720        P(M,T)=P2(U,V)
1730       NEXT T
1740      NEXT M
1750      MAT Pinv=INV(P)
1760      IF DET=0 THEN 2720
1770      MAT Aop=Ao*Pinv
1780      MAT Ktp=Kt*Pinv
1790      MAT Ctp=Ct*Pinv
1800      MAT Aopp=Aop
1810      MAT Ktpp=Ktp
1820      MAT Ctpp=Ctp
1830      ! CHECK FOR MATRICES Aop,Ktp,and Ctp EQUAL TO ZERO
1840           MAT Aopabs=ABS(Aop)
1850           MAT Ktpabs=ABS(Ktp)
1860           MAT Ctpabs=ABS(Ctp)
1870           Aopz=SUM(Aopabs)
1880           Ktpz=SUM(Ktpabs)
1890           Ctpz=SUM(Ctpabs)
1900     ! CALCULATION OF Eigenvalues and maximum and minimum
1910     ! eigenvalues for Aop, Ktp, and Ctp.
1920     CALL Eigen((S),P(*),Pevr(*),Pevi(*),Pvecr(*),Pveci(*),
Pindic(*))
1930       PRINT LIN(2),"REAL COMPONENTS OF Eigenvalues for P:",
LIN(1)
1940       MAT PRINT Pevr;
1950       PRINT LIN(2),"IMAGINARY COMPONENTS OF Eigenvalues for
P:",LIN(1)
1960       MAT PRINT Pevi;
1970     BEEP
1980     PRINT "CHECK IF MAT P is positive-definite (all eigenvalues
  are REAL & POSITIVE).";
1990     PRINT "              IF NOT KEY STOP";LIN(2)
2000     WAIT 10000
2010     IF Aopz=Zero THEN 2160
2020      CALL Eigen((S),Aopp(*),Aevr(*),Aevi(*),Avecr(*),Aveci(*),
Aindic(*))
2030        PRINT LIN(2),"REAL COMPONENTS OF Eigenvalues for Aop:",
LIN(1)
2040        MAT PRINT Aevr;
2050        PRINT LIN(2),"IMAGINARY COMPONENTS OF Eigenvalues for
Aop:",LIN(1)
2060        MAT PRINT Aevi;
2070        PRINT "MATRIX Aop",Aop(*);"MATRIX Ao",Ao(*);
2080            Aevr_min=Aevr(1)
2090            Aevr_max=Aevr(1)
2100            FOR I=2 TO S
2110             IF Aevr(I)>Aevr_max THEN Aevr_max=Aevr(I)
2120             IF Aevr(I)<Aevr_min THEN Aevr_min=Aevr(I)
2130            NEXT I
2140            PRINT "Aevr_max=";Aevr_max,"Aevr_min=";Aevr_min;
LIN(2)
```

```
2150       GOTO 2170
2160      PRINT "MAT Aop=ZERO",LIN(2)
2170      IF Ktpz=Zero THEN 2340
2180      CALL Eigen((S),Ktpp(*),Kevr(*),Kevi(*),Kvecr(*),Kveci(*),
Kindic(*))
2190        PRINT LIN(2),"REAL COMPONENTS OF Eigenvalues for Ktp:",
LIN(1)
2200        MAT PRINT Kevr;
2210        PRINT LIN(2),"IMAGINARY COMPONENTS OF Eigenvalues for
Ktp:",LIN(1)
2220        MAT PRINT Kevi;
2230        PRINT "MATRIX Ktp",Ktp(*);
2240            Kevr_min=Kevr(1)
2250            Kevr_max=Kevr(1)
2260            Kabsevr_max=ABS(Kevr(1))
2270            FOR I=2 TO S
2280             IF Kevr(I)>Kevr_max THEN Kevr_max=Kevr(I)
2290             IF Kevr(I)<Kevr_min THEN Kevr_min=Kevr(I)
2300             IF ABS(Kevr(I))>Kabsevr_max THEN Kabsevr_max=
ABS(Kevr(I))
2310            NEXT I
2320            PRINT "Kevr_max=";Kevr_max,"Kevr_min=";Kevr_min,
"Kabsevr_max=";Kabsevr_max;LIN(2)
2330       GOTO 2350
2340      PRINT "MAT Ktp=ZERO",LIN(2)
2350      IF Ctpz=Zero THEN 2520
2360      CALL Eigen((S),Ctpp(*),Cevr(*),Cevi(*),Cvecr(*),Cveci(*),
Cindic(*))
2370        PRINT LIN(2),"REAL COMPONENTS OF Eigenvalues for Ctp:",
LIN(1)
2380        MAT PRINT Cevr;
2390        PRINT LIN(2),"IMAGINARY COMPONENTS OF Eigenvalues for
Ctp:",LIN(1)
2400        MAT PRINT Cevi;
2410        PRINT "MATRIX Ctp",Ctp(*);
2420            Cevr_min=Cevr(1)
2430            Cevr_max=Cevr(1)
2440            Cabsevr_max=ABS(Cevr(1))
2450            FOR I=2 TO S
2460             IF Cevr(I)>Cevr_max THEN Cevr_max=Cevr(I)
2470             IF Cevr(I)<Cevr_min THEN Cevr_min=Cevr(I)
2480             IF ABS(Cevr(I))>Cabsevr_max THEN Cabsevr_max=
ABS(Cevr(I))
2490            NEXT I
2500            PRINT "Cevr_max=";Cevr_max,"Cevr_min=";Cevr_min,
"Cabsevr_max=";Cabsevr_max;LIN(2)
2510      GOTO 2530
2520      PRINT "MAT Ctp=ZERO",LIN(2)
2530      ! CALCULATION OF E(|g(t)|) or E(|f(t)|) as Zeta varies.
2540            H=H+1
2550            IF Ktpz<>0 THEN 2590
2560            IF Ctpz<>0 THEN 2650
2570            PRINT LIN(1),"NO STOCHASTIC TERMS."
2580            GOTO 2790
2590            F(H)=-2*Aevr_max/(Kevr_max-Kevr_min)
2600            F1(H)=-Aevr_max/Kabsevr_max
2610            Zeta1(H)=Zeta
2620            PRINTER IS 0
```

```
2630              PRINT "Corollary I:  E(|f(t)|)=";F(H),"Corollary II:
E(|f(t)|)=";F1(H),"Zeta=";Zeta,LIN(1)
2640              GOTO 2700
2650              F(H)=-2*Aevr_max/(Cevr_max-Cevr_min)
2660              F1(H)=-Aevr_max/Cabsevr_max
2670              Zeta1(H)=Zeta
2680              PRINTER IS 0
2690              PRINT "Corollary I:  E(|g(t)|)=";F(H),"Corollary II:
E(|g(t)|)=";F1(H),"Zeta=";Zeta,LIN(1)
2700              PRINTER IS 16
2710         GOTO 2780
2720        PRINT "Pinv, DET is zero"
2730        GOTO 2790
2740        PRINT "Minv, DET is zero"
2750        GOTO 2790
2780        NEXT Zeta
2790        STOP
2800 SUB Mat_m(N,M(*))
2810  OPTION BASE 1
2820  PRINT LIN(2),"MATRIX M:",LIN(1)
2830  FIXED 0
2840  FOR I=1 TO N
2850    FOR J=1 TO N
2860        DISP "M(";I;",";J;")";
2870        INPUT M(I,J)
2880    NEXT J
2890  NEXT I
2900  FLOAT 6
2910  MAT PRINT M;
2920  LINPUT "CHANGES (Y/N)?",C$
2930  IF (C$="Y") OR (C$="y") THEN 2960
2940  IF (C$="N") OR (C$="n") THEN 3040
2950  GOTO 2920
2960  INPUT "COORDINATES OF M(*), [ROW,COLUMN]?",I,J
2970  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 2960
2980  FIXED 0
2990  DISP "M(";I;",";J;")";
3000  INPUT M(I,J)
3010  PRINT USING 3020;I,J,M(I,J)
3020  IMAGE "M(",DD,",",DD,")=",4D.6D
3030  GOTO 2920
3040  SUBEND
3050 SUB Mat_co(N,Co(*))
3060  OPTION BASE 1
3070  PRINT LIN(2),"MATRIX Co:",LIN(1)
3080  FIXED 0
3090  FOR I=1 TO N
3100    FOR J=1 TO N
3110        DISP "Co(";I;",";J;")";
3120        INPUT Co(I,J)
3130    NEXT J
3140  NEXT I
3150  FLOAT 6
3160  MAT PRINT Co;
3170  LINPUT "CHANGES (Y/N)?",C$
3180  IF (C$="Y") OR (C$="y") THEN 3210
3190  IF (C$="N") OR (C$="n") THEN 3290
3200  GOTO 3170
```

```
3210  INPUT "COORDINATES OF Co(*), [ROW,COLUMN]?",I,J
3220  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 3210
3230  FIXED 0
3240  DISP "Co(";I;",";J;")";
3250  INPUT Co(I,J)
3260  PRINT USING 3270;I,J,Co(I,J)
3270  IMAGE "Co(",DD,",",DD,")=",4D.6D
3280  GOTO 3170
3290  SUBEND
3300 SUB Mat_ci(N,Ci(*))
3310  OPTION BASE 1
3320  PRINT LIN(2),"MATRIX Ci:",LIN(1)
3330  FIXED 0
3340  FOR I=1 TO N
3350    FOR J=1 TO N
3360        DISP "Ci(";I;",";J;")";
3370        INPUT Ci(I,J)
3380    NEXT J
3390  NEXT I
3400  FLOAT 6
3410  MAT PRINT Ci;
3420  LINPUT "CHANGES (Y/N)?",C$
3430  IF (C$="Y") OR (C$="y") THEN 3460
3440  IF (C$="N") OR (C$="n") THEN 3540
3450  GOTO 3420
3460  INPUT "COORDINATES OF Ci(*), [ROW,COLUMN]?",I,J
3470  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 3460
3480  FIXED 0
3490  DISP "Ci(";I;",";J;")";
3500  INPUT Ci(I,J)
3510  PRINT USING 3520;I,J,Ci(I,J)
3520  IMAGE "Ci(",DD,",",DD,")=",4D.6D
3530  GOTO 3420
3540  SUBEND
3550 SUB Mat_ko(N,Ko(*))
3560  OPTION BASE 1
3570  PRINT LIN(2),"MATRIX Ko:",LIN(1)
3580  FIXED 0
3590  FOR I=1 TO N
3600    FOR J=1 TO N
3610        DISP "Ko(";I;",";J;")";
3620        INPUT Ko(I,J)
3630    NEXT J
3640  NEXT I
3650  FLOAT 6
3660  MAT PRINT Ko;
3670  LINPUT "CHANGES (Y/N)?",C$
3680  IF (C$="Y") OR (C$="y") THEN 3710
3690  IF (C$="N") OR (C$="n") THEN 3790
3700  GOTO 3670
3710  INPUT "COORDINATES OF Ko(*), [ROW,COLUMN]?",I,J
3720  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 3710
3730  FIXED 0
3740  DISP "Ko(";I;",";J;")";
3750  INPUT Ko(I,J)
3760  PRINT USING 3770;I,J,Ko(I,J)
3770  IMAGE "Ko(",DD,",",DD,")=",4D.6D
3780  GOTO 3670
3790  SUBEND
```

```
3800 SUB Mat_kj(N,Kj(*))
3810  OPTION BASE 1
3820  PRINT LIN(2),"MATRIX Kj:",LIN(1)
3830  FIXED 0
3840  FOR I=1 TO N
3850    FOR J=1 TO N
3860        DISP "Kj(";I;",";J;")";
3870        INPUT Kj(I,J)
3880    NEXT J
3890  NEXT I
3900  FLOAT 6
3910  MAT PRINT Kj;
3920  LINPUT "CHANGES (Y/N)?",C$
3930  IF (C$="Y") OR (C$="y") THEN 3960
3940  IF (C$="N") OR (C$="n") THEN 4040
3950  GOTO 3920
3960  INPUT "COORDINATES OF Kj(*), [ROW,COLUMN]?",I,J
3970  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 3960
3980  FIXED 0
3990  DISP "Kj(";I;",";J;")";
4000  INPUT Kj(I,J)
4010  PRINT USING 4020;I,J,Kj(I,J)
4020  IMAGE "Kj(",DD,",",DD,")=",4D.6D
4030  GOTO 3920
4040  SUBEND
4050 SUB Mat_p2(N,P2(*))
4060  OPTION BASE 1
4070  PRINT LIN(2),"MATRIX P2:",LIN(1)
4080  FIXED 0
4090  FOR I=1 TO N
4100    FOR J=1 TO N
4110        DISP "P2(";I;",";J;")";
4120        INPUT P2(I,J)
4130    NEXT J
4140  NEXT I
4150  FLOAT 6
4160  MAT PRINT P2;
4170  LINPUT "CHANGES (Y/N)?",C$
4180  IF (C$="Y") OR (C$="y") THEN 4210
4190  IF (C$="N") OR (C$="n") THEN 4290
4200  GOTO 4170
4210  INPUT "COORDINATES OF P2(*), [ROW,COLUMN]?",I,J
4220  IF (I<=0) OR (I>N) OR (J<=0) OR (J>N) THEN 4210
4230  FIXED 0
4240  DISP "P2(";I;",";J;")";
4250  INPUT P2(I,J)
4260  PRINT USING 4270;I,J,P2(I,J)
4270  IMAGE "P2(",DD,",",DD,")=",4D.6D
4280  GOTO 4170
4290  SUBEND
4300 SUB Eigen(N,A(*),Evr(*),Evi(*),Vecr(*),Veci(*),Indic(*))
4310 Baddta=(N<=0)
4320 IF Baddta=0 THEN 4360
4330 PRINT LIN(2),"ERROR IN SUBPROGRAM Eigen."
4340 PRINT "N=";N,LIN(2)
4350 PAUSE
4360 OPTION BASE 1
4370 INTEGER Local(N)
4380 DIM Prfact(N),Subdia(N),Work(N)
```

```
4390 IF N<>1 THEN 4460
4400 Evr(1)=A(1,1)
4410 Evi(1)=0
4420 Vecr(1,1)=1
4430 Veci(1,1)=0
4440 Indic(1)=2
4450 GOTO 5570
4460 CALL Scale(N,A(*),Veci(*),Prfact(*),Enorm)
4470 Ex=EXP(-39*LOG(2))
4480   CALL Hesqr(N,A(*),Veci(*),Evr(*),Evi(*),Subdia(*),
Indic(*),Eps,Ex)
4490 J=N
4500 I=1
4510 Local(1)=1
4520 IF J=1 THEN 4590
4530 IF ABS(Subdia(J-1))>Eps THEN 4560
4540 I=I+1
4550 Local(I)=0
4560 J=J-1
4570 Local(I)=Local(I)+1
4580 IF J<>1 THEN 4530
4590 K=1
4600 Kon=0
4610 L=Local(1)
4620 M=N
4630 FOR I=1 TO N
4640     Ivec=N-I+1
4650     IF I<=L THEN 4690
4660     K=K+1
4670     M=N-L
4680     L=L+Local(K)
4690     IF Indic(Ivec)=0 THEN 4850
4700     IF Evi(Ivec)<>0 THEN 4800
4710     FOR K1=1 TO M
4720        FOR L1=K1 TO M
4730            A(K1,L1)=Veci(K1,L1)
4740        NEXT L1
4750        IF K1=1 THEN 4770
4760        A(K1,K1-1)=Subdia(K1-1)
4770     NEXT K1
4780     CALL Realve(N,M,Ivec,A(*),Vecr(*),Evr(*),Evi(*),
Work(*),Indic(*),Eps,Ex)
4790     GOTO 4850
4800     IF Kon<>0 THEN 4840
4810     Kon=1
4820     CALL Compve(N,M,Ivec,A(*),Vecr(*),Veci(*),Evr(*),
Evi(*),Indic(*),Subdia(*),Work(*),Eps,Ex)
4830     GOTO 4850
4840     Kon=0
4850 NEXT I
4860 MAT A=IDN
4870 IF N<=2 THEN 5020
4880 M=N-2
4890 FOR K=1 TO M
4900     L=K+1
4910     FOR J=2 TO N
4920        D1=0
4930        FOR I=L TO N
```

```
4940         D2=Veci(I,K)
4950         D1=D1+D2*A(J,I)
4960      NEXT I
4970      FOR I=L TO N
4980         A(J,I)=A(J,I)-Veci(I,K)*D1
4990      NEXT I
5000   NEXT J
5010 NEXT K
5020 Kon=1
5030 FOR I=1 TO N
5040    L=0
5050    IF Evi(I)=0 THEN 5100
5060    L=1
5070    IF Kon=0 THEN 5100
5080    Kon=0
5090    GOTO 5560
5100    FOR J=1 TO N
5110       D1=D2=0
5120       FOR K=1 TO N
5130          D3=A(J,K)
5140          D1=D1+D3*Vecr(K,I)
5150          IF L=0 THEN 5170
5160          D2=D2+D3*Vecr(K,I-1)
5170       NEXT K
5180       Work(J)=D1/Prfact(J)
5190       IF L=0 THEN 5210
5200       Subdia(J)=D2/Prfact(J)
5210    NEXT J
5220    IF L=1 THEN 5340
5230    D1=0
5240    FOR M=1 TO N
5250       D1=D1+Work(M)^2
5260    NEXT M
5270    D1=SQR(D1)
5280    FOR M=1 TO N
5290       Veci(M,I)=0
5300       Vecr(M,I)=Work(M)/D1
5310    NEXT M
5320    Evr(I)=Evr(I)*Enorm
5330    GOTO 5560
5340    Kon=1
5350    Evr(I)=Evr(I)*Enorm
5360    Evr(I-1)=Evr(I)
5370    Evi(I)=Evi(I)*Enorm
5380    Evi(I-1)=-Evi(I)
5390    R=0
5400    FOR J=1 TO N
5410       R1=Work(J)^2+Subdia(J)^2
5420       IF R>=R1 THEN 5450
5430       R=R1
5440       L=J
5450    NEXT J
5460    D3=Work(L)
5470    R1=Subdia(L)
5480    FOR J=1 TO N
5490       D1=Work(J)
5500       D2=Subdia(J)
5510       Vecr(J,I)=(D1*D3+D2*R1)/R
```

```
5520        Veci(J,I)=(D2*D3-D1*R1)/R
5530        Vecr(J,I-1)=Vecr(J,I)
5540        Veci(J,I-1)=-Veci(J,I)
5550     NEXT J
5560 NEXT I
5570 SUBEXIT
5580 SUBEND
5590 SUB Scale(N,A(*),H(*),Prfact(*),Enorm)
5600 OPTION BASE 1
5610 INTEGER I,J,Iter,Ncount
5620 FOR I=1 TO N
5630     FOR J=1 TO N
5640        H(I,J)=A(I,J)
5650     NEXT J
5660 Prfact(I)=1
5670 NEXT I
5680 Bound1=.75
5690 Bound2=1.33
5700 Iter=0
5710 Ncount=0
5720 FOR I=1 TO N
5730     Column=0
5740     Row=0
5750     FOR J=1 TO N
5760        IF I=J THEN 5790
5770        Column=Column+ABS(A(J,I))
5780        Row=Row+ABS(A(I,J))
5790     NEXT J
5800     IF Column=0 THEN 5850
5810     IF Row=0 THEN 5850
5820     Q=Column/Row
5830     IF Q<Bound1 THEN 5870
5840     IF Q>Bound2 THEN 5870
5850     Ncount=Ncount+1
5860     GOTO 5940
5870     Factor=SQR(Q)
5880     FOR J=1 TO N
5890        IF I=J THEN 5920
5900        A(I,J)=A(I,J)*Factor
5910        A(J,I)=A(J,I)/Factor
5920     NEXT J
5930     Prfact(I)=Prfact(I)*Factor
5940 NEXT I
5950 Iter=Iter+1
5960 IF Iter>30 THEN 6130
5970 IF Ncount<N THEN 5710
5980 Fnorm=0
5990 FOR I=1 TO N
6000     FOR J=1 TO N
6010        Q=A(I,J)
6020        Fnorm=Fnorm+Q*Q
6030     NEXT J
6040 NEXT I
6050 Fnorm=SQR(Fnorm)
6060 FOR I=1 TO N
6070     FOR J=1 TO N
6080        A(I,J)=A(I,J)/Fnorm
6090     NEXT J
```

```
6100 NEXT I
6110 Enorm=Fnorm
6120 GOTO 6200
6130 FOR I=1 TO N
6140    Prfact(I)=1
6150    FOR J=1 TO N
6160       A(I,J)=H(I,J)
6170    NEXT J
6180 NEXT I
6190 Enorm=1
6200 SUBEXIT
6210 SUBEND
6220 SUB Hesqr(N,A(*),H(*),Evr(*),Evi(*),Subdia(*),Indic(*),Eps,Ex)
6230 OPTION BASE 1
6240 INTEGER I,J,K,L,M,Maxst,M1,Ns
6250 IF N-2<0 THEN 6830
6260 IF N-2>0 THEN 6290
6270 Subdia(1)=A(2,1)
6280 GOTO 6830
6290 M=N-2
6300 FOR K=1 TO M
6310    L=K+1
6320    S=0
6330    FOR I=L TO N
6340       H(I,K)=A(I,K)
6350       S=S+ABS(A(I,K))
6360    NEXT I
6370    IF S<>ABS(A(K+1,K)) THEN 6410
6380    Subdia(K)=A(K+1,K)
6390    H(K+1,K)=0
6400    GOTO 6780
6410    Sr2=0
6420    FOR I=L TO N
6430       Sr=A(I,K)
6440       Sr=Sr/S
6450       A(I,K)=Sr
6460       Sr2=Sr2+Sr*Sr
6470    NEXT I
6480    Sr=SQR(Sr2)
6490    IF A(L,K)<0 THEN 6510
6500    Sr=-Sr
6510    Sr2=Sr2-Sr*A(L,K)
6520    A(L,K)=A(L,K)-Sr
6530    H(L,K)=H(L,K)-Sr*S
6540    Subdia(K)=Sr*S
6550    X=S*SQR(Sr2)
6560    FOR I=L TO N
6570       H(I,K)=H(I,K)/X
6580       Subdia(I)=A(I,K)/Sr2
6590    NEXT I
6600    FOR J=L TO N
6610       Sr=0
6620       FOR I=L TO N
6630          Sr=Sr+A(I,K)*A(I,J)
6640       NEXT I
6650       FOR I=L TO N
6660          A(I,J)=A(I,J)-Subdia(I)*Sr
```

```
6670        NEXT I
6680     NEXT J
6690     FOR J=1 TO N
6700        Sr=0
6710        FOR I=L TO N
6720           Sr=Sr+A(J,I)*A(I,K)
6730        NEXT I
6740        FOR I=L TO N
6750           A(J,I)=A(J,I)-Subdia(I)*Sr
6760        NEXT I
6770     NEXT J
6780 NEXT K
6790 FOR K=1 TO M
6800     A(K+1,K)=Subdia(K)
6810 NEXT K
6820 Subdia(N-1)=A(N,N-1)
6830 Eps=0
6840 FOR K=1 TO N
6850     Indic(K)=0
6860     IF K<>N THEN Eps=Eps+Subdia(K)^2
6870     FOR I=K TO N
6880        H(K,I)=A(K,I)
6890        Eps=Eps+A(K,I)^2
6900     NEXT I
6910 NEXT K
6920 Eps=Ex*SQR(Eps)
6930 Shift=A(N,N-1)
6940 IF N<=2 THEN Shift=0
6950 IF A(N,N)<>0 THEN Shift=0
6960 IF A(N-1,N)<>0 THEN Shift=0
6970 IF A(N-1,N-1)<>0 THEN Shift=0
6980 M=N
6990 Ns=0
7000 Maxst=N*10
7010 FOR I=2 TO N
7020     FOR K=I TO N
7030        IF A(I-1,K)<>0 THEN 7120
7040     NEXT K
7050 NEXT I
7060 FOR I=1 TO N
7070     Indic(I)=1
7080     Evr(I)=A(I,I)
7090     Evi(I)=0
7100 NEXT I
7110 GOTO 8150
7120 K=M-1
7130 M1=K
7140 I=K
7150 IF K<0 THEN 8150
7160 IF K=0 THEN 7900
7170 IF ABS(A(M,K))<=Eps THEN 7900
7180 IF M-2=0 THEN 7950
7190 I=I-1
7200 IF ABS(A(K,I))<=Eps THEN 7230
7210 K=I
7220 IF K>1 THEN 7190
7230 IF K=M1 THEN 7950
7240 S=A(M,M)+A(M1,M1)+Shift
7250 Sr=A(M,M)*A(M1,M1)-A(M,M1)*A(M1,M)+.25*Shift^2
```

```
7260 A(K+2,K)=0
7270 X=A(K,K)*(A(K,K)-S)+A(K,K+1)*A(K+1,K)+Sr
7280 Y=A(K+1,K)*(A(K,K)+A(K+1,K+1)-S)
7290 R=ABS(X)+ABS(Y)
7300 IF R=0 THEN Shift=A(M,M-1)
7310 IF R=0 THEN 7230
7320 Z=A(K+2,K+1)*A(K+1,K)
7330 Shift=0
7340 Ns=Ns+1
7350 FOR I=K TO M1
7360    IF I=K THEN 7420
7370    X=A(I,I-1)
7380    Y=A(I+1,I-1)
7390    Z=0
7400    IF I+2>M THEN 7420
7410    Z=A(I+2,I-1)
7420    Sr2=ABS(X)+ABS(Y)+ABS(Z)
7430    IF Sr2=0 THEN 7470
7440    X=X/Sr2
7450    Y=Y/Sr2
7460    Z=Z/Sr2
7470    S=SQR(X*X+Y*Y+Z*Z)
7480    IF X<0 THEN 7500
7490    S=-S
7500    IF I=K THEN 7520
7510    A(I,I-1)=S*Sr2
7520    IF Sr2<>0 THEN 7550
7530    IF I+3>M THEN 7870
7540    GOTO 7840
7550    Sr=1-X/S
7560    S=X-S
7570    X=Y/S
7580    Y=Z/S
7590    FOR J=I TO M
7600       S=A(I,J)+A(I+1,J)*X
7610       IF I+2>M THEN 7630
7620       S=S+A(I+2,J)*Y
7630       S=S*Sr
7640       A(I,J)=A(I,J)-S
7650       A(I+1,J)=A(I+1,J)-S*X
7660       IF I+2>M THEN 7680
7670       A(I+2,J)=A(I+2,J)-S*Y
7680    NEXT J
7690    L=I+2
7700    IF I<M1 THEN 7720
7710    L=M
7720    FOR J=K TO L
7730       S=A(J,I)+A(J,I+1)*X
7740       IF I+2>M THEN 7760
7750       S=S+A(J,I+2)*Y
7760       S=S*Sr
7770       A(J,I)=A(J,I)-S
7780       A(J,I+1)=A(J,I+1)-S*X
7790       IF I+2>M THEN 7810
7800       A(J,I+2)=A(J,I+2)-S*Y
7810    NEXT J
7820    IF I+3>M THEN 7870
7830    S=-A(I+3,I+2)*Y*Sr
7840    A(I+3,I)=S
```

```
7850      A(I+3,I+1)=S*X
7860      A(I+3,I+2)=S*Y+A(I+3,I+2)
7870    NEXT I
7880    IF Ns>Maxst THEN 8150
7890    GOTO 7120
7900    Evr(M)=A(M,M)
7910    Evi(M)=0
7920    Indic(M)=1
7930    M=K
7940    GOTO 7120
7950    R=.5*(A(K,K)+A(M,M))
7960    S=.5*(A(M,M)-A(K,K))
7970    S=S*S+A(K,M)*A(M,K)
7980    Indic(K)=1
7990    Indic(M)=1
8000    IF S<0 THEN 8080
8010    T=SQR(S)
8020    Evr(K)=R-T
8030    Evr(M)=R+T
8040    Evi(K)=0
8050    Evi(M)=0
8060    M=M-2
8070    GOTO 7120
8080    T=SQR(-S)
8090    Evr(K)=R
8100    Evi(K)=T
8110    Evr(M)=R
8120    Evi(M)=-T
8130    M=M-2
8140    GOTO 7120
8150    SUBEXIT
8160    SUBEND
8170    SUB Realve(N,M,Ivec,A(*),Vecr(*),Evr(*),Evi(*),Work(*),
         Indic(*),Eps,Ex)
8180    Baddta=(N<=0) OR (M<=0) OR (Ivec<=0)
8190    IF Baddta=0 THEN 8230
8200    PRINT LIN(2),"ERROR IN SUBPROGRAM Realve."
8210    PRINT "N=";N,"M=";M,"Ivec=";Ivec,LIN(2)
8220    PAUSE
8230    OPTION BASE 1
8240    INTEGER Iwork(N)
8250    INTEGER I,Iter,J,K,L,Ns
8260    Vecr(1,Ivec)=1
8270    IF M=1 THEN 9220
8280    Evalue=Evr(Ivec)
8290    IF Ivec=M THEN 8380
8300    K=Ivec+1
8310    R=0
8320    FOR I=K TO M
8330      IF Evalue<>Evr(I) THEN 8360
8340      IF Evi(I)<>0 THEN 8360
8350      R=R+3
8360    NEXT I
8370    Evalue=Evalue+R*Ex
8380    FOR K=1 TO M
8390      A(K,K)=A(K,K)-Evalue
8400    NEXT K
```

```
8410  K=M-1
8420  FOR I=1 TO K
8430    L=I+1
8440    Iwork(I)=0
8450    IF A(I+1,I)<>0 THEN 8490
8460    IF A(I,I)<>0 THEN 8610
8470    A(I,I)=Eps
8480    GOTO 8610
8490    IF ABS(A(I,I))>=ABS(A(I+1,I)) THEN 8560
8500    Iwork(I)=1
8510    FOR J=I TO M
8520       R=A(I,J)
8530       A(I,J)=A(I+1,J)
8540       A(I+1,J)=R
8550    NEXT J
8560    R=-A(I+1,I)/A(I,I)
8570    A(I+1,I)=R
8580    FOR J=L TO M
8590       A(I+1,J)=A(I+1,J)+R*A(I,J)
8600    NEXT J
8610  NEXT I
8620  IF A(M,M)<>0 THEN 8640
8630  A(M,M)=Eps
8640  FOR I=1 TO N
8650    IF I>M THEN 8680
8660    Work(I)=1
8670    GOTO 8690
8680    Work(I)=0
8690  NEXT I
8700  Bound=.01/(Ex*N)
8710  Ns=0
8720  Iter=1
8730  R=0
8740  FOR I=1 TO M
8750    J=M-I+1
8760    S=Work(J)
8770    IF J=M THEN 8830
8780    L=J+1
8790    FOR K=L TO M
8800       Sr=Work(K)
8810       S=S-Sr*A(J,K)
8820    NEXT K
8830    Work(J)=S/A(J,J)
8840    T=ABS(Work(J))
8850    IF R>=T THEN 8870
8860    R=T
8870  NEXT I
8880  FOR I=1 TO M
8890    Work(I)=Work(I)/R
8900  NEXT I
8910  R1=0
8920  FOR I=1 TO M
8930    T=0
8940    FOR J=I TO M
8950       T=T+A(I,J)*Work(J)
8960    NEXT J
8970    T=ABS(T)
8980    IF R1>=T THEN 9000
```

```
8990    R1=T
9000  NEXT I
9010  IF Iter=1 THEN 9030
9020  IF Previs<=R1 THEN 9220
9030  FOR I=1 TO M
9040    Vecr(I,Ivec)=Work(I)
9050  NEXT I
9060  Previs=R1
9070  IF Ns=1 THEN 9220
9080  IF Iter>6 THEN 9230
9090  Iter=Iter+1
9100  IF R<Bound THEN 9120
9110  Ns=1
9120  K=M-1
9130  FOR I=1 TO K
9140    R=Work(I+1)
9150    IF Iwork(I)=0 THEN 9190
9160    Work(I+1)=Work(I)+Work(I+1)*A(I+1,I)
9170    Work(I)=R
9180    GOTO 9200
9190    Work(I+1)=Work(I)*A(I+1,I)+Work(I+1)
9200  NEXT I
9210  GOTO 8730
9220  Indic(Ivec)=2
9230  IF M=N THEN 9280
9240  J=M+1
9250  FOR I=J TO N
9260    Vecr(I,Ivec)=0
9270  NEXT I
9280  SUBEXIT
9290  SUBEND
9300  SUB Compve(N,M,Ivec,A(*),Vecr(*),H(*),Evr(*),Evi(*),Indic(*),
       Subdia(*),Work(*),Eps,Ex)
9310  Baddta=(N<=0) OR (M<=0) OR (Ivec<=0)
9320  IF Baddta=0 THEN 9360
9330  PRINT LIN(2),"ERROR IN SUBPROGRAM Compve."
9340  PRINT "N=";N,"M=";M,"Ivec=";Ivec,LIN(2)
9350  PAUSE
9360  OPTION BASE 1
9370  INTEGER Iwork(N)
9380  DIM Work1(N),Work2(N)
9390  INTEGER I,I1,I2,Iter,J,K,L,Ns
9400  Fksi=Evr(Ivec)
9410  Eta=Evi(Ivec)
9420  IF Ivec=M THEN 9530
9430  K=Ivec+1
9440  R=0
9450  FOR I=K TO M
9460    IF Fksi<>Evr(I) THEN 9490
9470    IF ABS(Eta)<>ABS(Evi(I)) THEN 9490
9480    R=R+3
9490  NEXT I
9500  R=R*Ex
9510  Fksi=Fksi+R
9520  Eta=Eta+R
9530  R=Fksi*Fksi+Eta*Eta
9540  S=2*Fksi
```

```
9550  L=M-1
9560  FOR I=1 TO M
9570    FOR J=I TO M
9580       D=0
9590       A(J,I)=0
9600       FOR K=I TO J
9610          D=D+H(I,K)*H(K,J)
9620       NEXT K
9630       A(I,J)=D-S*H(I,J)
9640    NEXT J
9650    A(I,I)=A(I,I)+R
9660  NEXT I
9670  FOR I=1 TO L
9680    R=Subdia(I)
9690    A(I+1,I)=-S*R
9700    I1=I+1
9710    FOR J=1 TO I1
9720       A(J,I)=A(J,I)+R*H(J,I+1)
9730    NEXT J
9740    IF I=1 THEN 9760
9750    A(I+1,I-1)=R*Subdia(I-1)
9760    FOR J=I TO M
9770       A(I+1,J)=A(I+1,J)+R*H(I,J)
9780    NEXT J
9790  NEXT I
9800  K=M-1
9810  FOR I=1 TO K
9820    I1=I+1
9830    I2=I+2
9840    Iwork(I)=0
9850    IF I=K THEN 9870
9860    IF A(I+2,I)<>0 THEN 9910
9870    IF A(I+1,I)<>0 THEN 9910
9880    IF A(I,I)<>0 THEN 10140
9890    A(I,I)=Eps
9900    GOTO 10140
9910    IF I=K THEN 9970
9920    IF ABS(A(I+1,I))>=ABS(A(I+2,I)) THEN 9970
9930    IF ABS(A(I,I))>=ABS(A(I+2,I)) THEN 10070
9940    L=I+2
9950    Iwork(I)=2
9960    GOTO 10000
9970    IF ABS(A(I,I))>=ABS(A(I+1,I)) THEN 10050
9980    L=I+1
9990    Iwork(I)=1
10000   FOR J=I TO M
10010      R=A(I,J)
10020      A(I,J)=A(L,J)
10030      A(L,J)=R
10040   NEXT J
10050   IF I<>K THEN 10070
10060   I2=I1
10070   FOR L=I1 TO I2
10080      R=-A(L,I)/A(I,I)
10090      A(L,I)=R
10100      FOR J=I1 TO M
10110         A(L,J)=A(L,J)+R*A(I,J)
10120      NEXT J
```

```
10130   NEXT L
10140 NEXT I
10150 IF A(M,M)<>0 THEN 10170
10160 A(M,M)=Eps
10170 FOR I=1 TO N
10180   IF I>M THEN 10220
10190   Vecr(I,Ivec)=1
10200   Vecr(I,Ivec-1)=1
10210   GOTO 10240
10220   Vecr(I,Ivec)=0
10230   Vecr(I,Ivec-1)=0
10240 NEXT I
10250 Bound=.01/(Ex*N)
10260 Ns=0
10270 Iter=1
10280 FOR I=1 TO M
10290   Work(I)=H(I,I)-Fksi
10300 NEXT I
10310 FOR I=1 TO M
10320   D=Work(I)*Vecr(I,Ivec)
10330   IF I=1 THEN 10350
10340   D=D+Subdia(I-1)*Vecr(I-1,Ivec)
10350   L=I+1
10360   IF L>M THEN 10400
10370   FOR K=L TO M
10380      D=D+H(I,K)*Vecr(K,Ivec)
10390   NEXT K
10400   Vecr(I,Ivec-1)=D-Eta*Vecr(I,Ivec-1)
10410 NEXT I
10420 K=M-1
10430 FOR I=1 TO K
10440   L=I+Iwork(I)
10450   R=Vecr(L,Ivec-1)
10460   Vecr(L,Ivec-1)=Vecr(I,Ivec-1)
10470   Vecr(I,Ivec-1)=R
10480   Vecr(I+1,Ivec-1)=Vecr(I+1,Ivec-1)+A(I+1,I)*R
10490   IF I=K THEN 10510
10500   Vecr(I+2,Ivec-1)=Vecr(I+2,Ivec-1)+A(I+2,I)*R
10510 NEXT I
10520 FOR I=1 TO M
10530   J=M-I+1
10540   D=Vecr(J,Ivec-1)
10550   IF J=M THEN 10610
10560   L=J+1
10570   FOR K=L TO M
10580      D1=A(J,K)
10590      D=D-D1*Vecr(K,Ivec-1)
10600   NEXT K
10610   Vecr(J,Ivec-1)=D/A(J,J)
10620 NEXT I
10630 FOR I=1 TO M
10640   D=Work(I)*Vecr(I,Ivec-1)
10650   IF I=1 THEN 10670
10660   D=D+Subdia(I-1)*Vecr(I-1,Ivec-1)
10670   L=I+1
10680   IF L>M THEN 10720
10690   FOR K=L TO M
10700      D=D+H(I,K)*Vecr(K,Ivec-1)
```

```
10710   NEXT K
10720   Vecr(I,Ivec)=(Vecr(I,Ivec)-D)/Eta
10730 NEXT I
10740 L=1
10750 S=0
10760 FOR I=1 TO M
10770   R=Vecr(I,Ivec)^2+Vecr(I,Ivec-1)^2
10780   IF R<=S THEN 10810
10790   S=R
10800   L=I
10810 NEXT I
10820 U=Vecr(L,Ivec-1)
10830 V=Vecr(L,Ivec)
10840 FOR I=1 TO M
10850   B=Vecr(I,Ivec)
10860   R=Vecr(I,Ivec-1)
10870   Vecr(I,Ivec)=(R*U+B*V)/S
10880   Vecr(I,Ivec-1)=(B*U-R*V)/S
10890 NEXT I
10900 B=0
10910 FOR I=1 TO M
10920   R=Work(I)*Vecr(I,Ivec-1)-Eta*Vecr(I,Ivec)
10930   U=Work(I)*Vecr(I,Ivec)+Eta*Vecr(I,Ivec-1)
10940   IF I=1 THEN 10970
10950   R=R+Subdia(I-1)*Vecr(I-1,Ivec-1)
10960   U=U+Subdia(I-1)*Vecr(I-1,Ivec)
10970   L=I+1
10980   IF L>M THEN 11030
10990   FOR J=L TO M
11000      R=R+H(I,J)*Vecr(J,Ivec-1)
11010      U=U+H(I,J)*Vecr(J,Ivec)
11020   NEXT J
11030   U=R*R+U*U
11040   IF B>=U THEN 11060
11050   B=U
11060 NEXT I
11070 IF Iter=1 THEN 11090
11080 IF Previs<=B THEN 11200
11090 FOR I=1 TO N
11100   Work1(I)=Vecr(I,Ivec)
11110   Work2(I)=Vecr(I,Ivec-1)
11120 NEXT I
11130 Previs=B
11140 IF Ns=1 THEN 11240
11150 IF Iter>6 THEN 11260
11160 Iter=Iter+1
11170 IF Bound>SQR(S) THEN 10310
11180 Ns=1
11190 GOTO 10310
11200 FOR I=1 TO N
11210   Vecr(I,Ivec)=Work1(I)
11220   Vecr(I,Ivec-1)=Work2(I)
11230 NEXT I
11240 Indic(Ivec-1)=2
11250 Indic(Ivec)=2
11260 SUBEND
11270 END
```

APPENDIX B

Element Properties [26]

element mass matrix

$$m = \frac{m\ell}{420}\begin{bmatrix} 156 & & \text{Sym.} & \\ 22 & 4 & & \\ 54 & 13 & 156 & \\ -13 & -3 & -22 & 4 \end{bmatrix}$$

element geometric stiffness matrix

$$k_G(t) = \frac{p_o + p(t)}{\ell}[k_G] \equiv \frac{p_o + p(t)}{\ell}\begin{bmatrix} -\frac{6}{5} & & \text{Sym.} & \\ -\frac{1}{10} & -\frac{2}{15} & & \\ \frac{6}{5} & \frac{1}{10} & -\frac{6}{5} & \\ -\frac{1}{10} & \frac{1}{30} & \frac{1}{10} & -\frac{2}{15} \end{bmatrix}$$

element elastic stiffness matrix

$$k = \frac{EI}{\ell^3}\begin{bmatrix} 12 & & \text{Sym.} & \\ 6 & 4 & & \\ -12 & -6 & 12 & \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

element k_o matrix

$$k_o = k + \frac{p_o}{\ell}[k_G]$$

element vector

$$q = \text{col}\{v_i, \theta_i\ell, v_j, \theta_j\ell\}$$

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by

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ABSTRACT

A Liapunov function specially suitable for the study of the almost sure asymptotic stability of a class of linear discrete systems, described by a set of second order differential equations with stochastic parameters, is presented. A theorem and related corollaries, applicable to systems involving general types of forces, are obtained. The proposed technique is shown to be useful in minimizing the computational efforts associated with relatively large dynamical systems. Several examples, including systems involving follower forces, are included to demonstrate the effectiveness of the method.

In addition, the theorem is extended to study the response bounds of systems for which the dynamic loads or perturbations lead to forcing terms in the equations of motion eliminating the existence of an equilibrium state. Illustrative examples are also included. The results, in general, are found to be of significant practical value.