UNDAMPED FREE VIBRATION OF A GABLE FRAME

by

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TABLE OF CONTENTS

LIST OF SYMBOLS

T	Kinetic energy of a structural system
q_i	Generalized coordinates
Q_i	Generalized forces
Q_{A_i}	External forces applied to the system
Q_{E_i}	Internal elastic forces due to a change of strain energy
Q_{D_i}	Internal or external damping forces due to energy dissipation in the system
U	Strain energy of the structural system
k_{ij}	Elastic force coefficient
$K' = [k_{ij}]$	A square matrix of elastic force coefficients of order n associated with local coordinates
$K = [k_{ij}]$	A square matrix of elastic force coefficients of order n associated with system coordinates
m_{ij}	Mass inertial coefficient
$M' = [m_{ij}]$	A square matrix of mass inertial coefficients of order n associated with local coordinates
$M = [m_{ij}]$	A square matrix of mass inertial coefficients of order n associated with system coordinates
$\{ \}$	Column matrix
$\lfloor \rfloor$	Row matrix
T_a	Coordinate transformation matrix
T_a^T	Transpose of matrix T_a
$w(x,t)$	Total deflection function
$\Gamma_i(x)$	Deflection function due to unit displacement along the direction of the ith generalized coordinate

m	Mass per unit length
E	Young's modulus of elasticity of the beam element
I	Moment of inertia of beam cross-section
A	Area of beam cross-section
ℓ	Length of beam element
$\{q_j\}$	Structural displacements measured in system coordinates
$\{Q_j\}$	Externally applied forces measured in system coordinates
$\{q'_j\}$	Structural displacements measured in local coordinates
$\{Q'_j\}$	Externally applied forces measured in local coordinates
$\hat{i}$	Unit vector components along x-axis
$\hat{j}$	Unit vector components along y-axis
α	A dimensionless number dependent on the shape of the cross-section
$\beta_i = (q'_i)_s/(q'_i)_b$	Where subscripts s and b denote shear and bending, respectively
G	Modulus of elasticity in sheer
$\overline{V}_P$	The Pth arbitrary trial column vector
ω	Natural frequency of the structural system
α_i	Arbitrary constants
λ	Eigenvalue of the undamped free vibration
$[I]$	Identity matrix
t	Time variable
x'	Local coordinate measured along the beam element
y'	Local coordinate measured normal to the beam element
x	System coordinate measured along the beam element
y	System coordinate measured normal to the beam element

INTRODUCTION

The vibration of structures has played an increasingly important role in structural analysis and design in recent years. Exact solutions of the natural frequencies of beams and of rectangular rigid frames have been obtained in numerous papers and books; however, the exact solutions for the natural frequencies of gable frames are difficult to determine. An approximate method proposed by J. S. Archer[3] will be used in this report. Using the 1410 digital computer, the lowest four natural frequencies and modes of a symmetrical gable frame have been obtained. This was accomplished by using the consistent mass and the lumped mass methods.

The stiffness matrix, the consistent mass matrix and the lumped mass matrix of a simple uniform beam element are derived first. Since a gable frame is a solid continuous medium, it has an infinite number of degrees of freedom. Consequently, this structural system has an infinite number of natural frequencies and natural modes of vibration. But this structural system, as shown in Fig. 11, is considered as four elastic beams joined rigidly. Each individual beam is considered as comprising two beam elements of equal length. By expanding to the structural system with its boundary conditions, the equations of undamped free vibration of a symmetrical gable frame are obtained. These eigenvalue problems were solved with the aid of the 1410 digital computer. The computer programs are shown in the Appendix of this report.

The superiority of the consistent mass matrix over the lumped mass matrix for certain structures has been demonstrated by some authors[3], [4]. For the symmetrical gable frame, however, the percent deviations of these two approaches are insignificant for the first anti-symmetric mode and the first symmetric mode. The percent deviations, however, increase for higher modes.

DIFFERENTIAL EQUATIONS OF UNDAMPED FREE VIBRATION

For a structural system with n degrees of freedom, the generalized form of Lagrange's equations are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i}\right) = Q_i \quad ; \quad (i = 1,2,\cdots,n) \tag{1}$$

These equations express the generalized forces Q_i as a function of the kinetic energy T of the system. Since it is considered that the given system has n finite degrees of freedom, there are n equations corresponding to generalized coordinates $q_i(i = 1,2,\cdots,n)$. The generalized forces Q_i are usually considered to be composed of three distinct forces:

$$Q_i = Q_{A_i} + Q_{E_i} + Q_{D_i} \tag{2}$$

where

Q_{A_i} = external forces applied to the system,

Q_{E_i} = internal elastic forces due to a change of strain energy, and

Q_{D_i} = internal or external damping forces due to the energy dissipation in the system.

According to Castigliano's theorem, when considering the elastic forces Q_{E_i} which have strain energy U, then,

$$Q_{E_i} = -\frac{\partial U}{\partial q_i} \tag{3}$$

The minus sign in Eqs. (3) is introduced because in this theorem, the force is considered as applying to the elastic element as in the spring of the simple spring-mass system shown in Fig. 1. and the generalized force Q_{E_i} acting on the element has the opposite direction of q_i's.

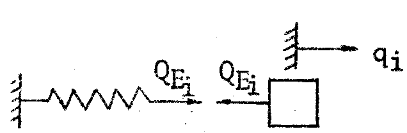


Fig. 1.

Substituting Eqs. (2) and (3) into Eqs. (1), the general form of Lagrange's equations may be reduced to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_{D_i} + Q_{A_i} \quad . \tag{4}$$

In considering the free vibration of an undamped system, the force terms Q_{A_i} and Q_{D_i} are zero, and Eqs. (4) now have the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0 \quad . \tag{5}$$

Since the strain energy of the system is only a function of the coordinates, it can be expanded by Maclaurin's series for n variables about its stable equilibrium position as

$$U(q_1, q_2, \cdots, q_n) = U_o + \sum_{i=1}^{n} \left(\frac{\partial U}{\partial q_i}\right)_o q_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\left(\frac{\partial^2 U}{\partial q_i \partial q_j}\right)_o \cdot q_i q_j \tag{6}$$

+ high order terms,

where subscript o denotes the values at the equilibrium position. In general, the value U_o measured from the equilibrium position may be set at zero. The value of the strain energy in a stable equilibrium position is a relative minimum. This implies its first derivative must vanish; that is

$$\left(\frac{\partial U}{\partial q_i}\right)_o = 0 \quad . \tag{7}$$

Then, assuming small displacements and neglecting all high order terms, the strain energy of the structural system can be expressed as,

$$U = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial q_j \partial q_i}\right)_o q_i q_j \quad , \tag{8}$$

or

$$U = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} k_{ij} q_i q_j \tag{9}$$

where $k_{ij} = k_{ji}$;

thus the stiffness matrix

$$K = [k_{ij}] \tag{10}$$

is a symmetrical matrix. The strain energy of a stable system is always positive. In addition, Eq. (9) is a homogeneous second-degree function of n variables. This implies that U is a positive definite quadratic form of the coordinates q_i. Similarly, since the structural system is assumed to undergo small displacements, kinetic energy T can be considered as a function of velocities $\dot{q}_i$ only,

$$\text{or} \quad T = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \tag{11}$$

where $m_{ij} = m_{ji}$; thus the mass matrix

$$M = [m_{ij}] \tag{12}$$

is a symmetrical matrix. The kinetic energy is always positive; therefore T in Eq. (11) is a positive definite quadratic form of the velocities $\dot{q}_i$. Substituting Eqs. (9) and (11) into Eqs. (5) gives the equations of undamped free vibration as

$$[m_{ij}]\{\ddot{q}_j\} + [k_{ij}]\{q_j\} = 0$$

$$\text{or} \quad M\{\ddot{q}_j\} + K\{q_j\} = 0 \tag{13}$$

Since U and T are quadratic forms of coordinates and velocities, respectively, if high order terms are neglected, then Eqs. (13) are second-order linear ordinary differential equations. The physical meaning of coefficients m_{ij} and k_{ij} is the mass inertial force and the elastic force, respectively, acting at coordinate i due to unit acceleration and displacement, respectively, of coordinate j, with all other coordinates remaining stationary.

CONSISTENT MASS MATRIX AND THE STIFFNESS MATRIX FOR A UNIFORM BEAM ELEMENT IN LOCAL COORDINATES

(1) Consistent Mass Matrix $[M_{ij}]$

A uniform beam element of length ℓ of constant bending stiffness EI, with all generalized coordinates specified as in Fig. 2. is considered. A unit displacement along the direction of the generalized coordinate q_i' of beam element will cause the corresponding deflection function $\Gamma_i(x)$ as shown in Figs. 3. through 8. The results are:

$$\Gamma_1(x) = (1 - \frac{x}{\ell})\ \hat{i} \tag{14}$$

$$\Gamma_2(x) = (1 - \frac{x}{\ell})^2(1 \div 2\ \frac{x}{\ell})\ \hat{j} \tag{15}$$

$$\Gamma_3(x) = x(1 - \frac{x}{\ell})^2\hat{j} \tag{16}$$

$$\Gamma_4(x) = \frac{x}{\ell}\ \hat{i} \tag{17}$$

$$\Gamma_5(x) = (\frac{x}{\ell})^2(3 - 2\ \frac{x}{\ell})\ \hat{j} \tag{18}$$

and

$$\Gamma_6(x) = (\frac{x}{\ell})^2(x - \ell)\ \hat{j} \tag{19}$$

where $\hat{i}$ and $\hat{j}$ denote the unit vectors along x-axis and y-axis, respectively, as shown in Fig. 2.

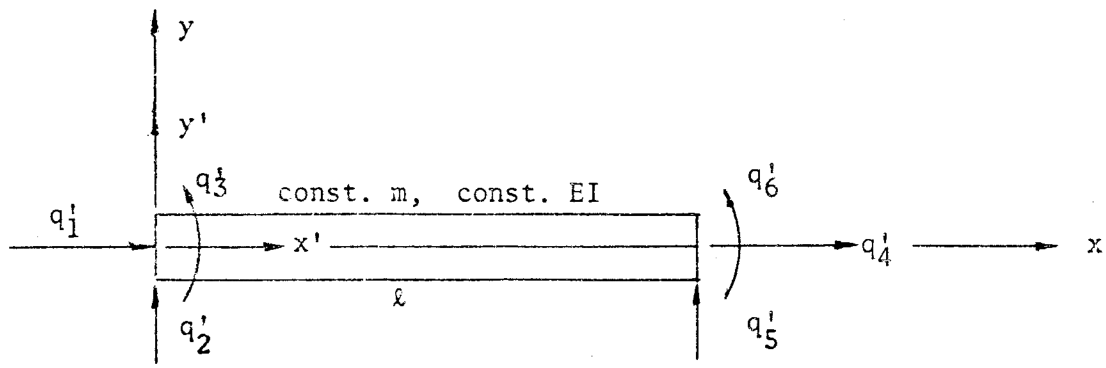


Fig. 2. Local coordinates for a uniform beam element

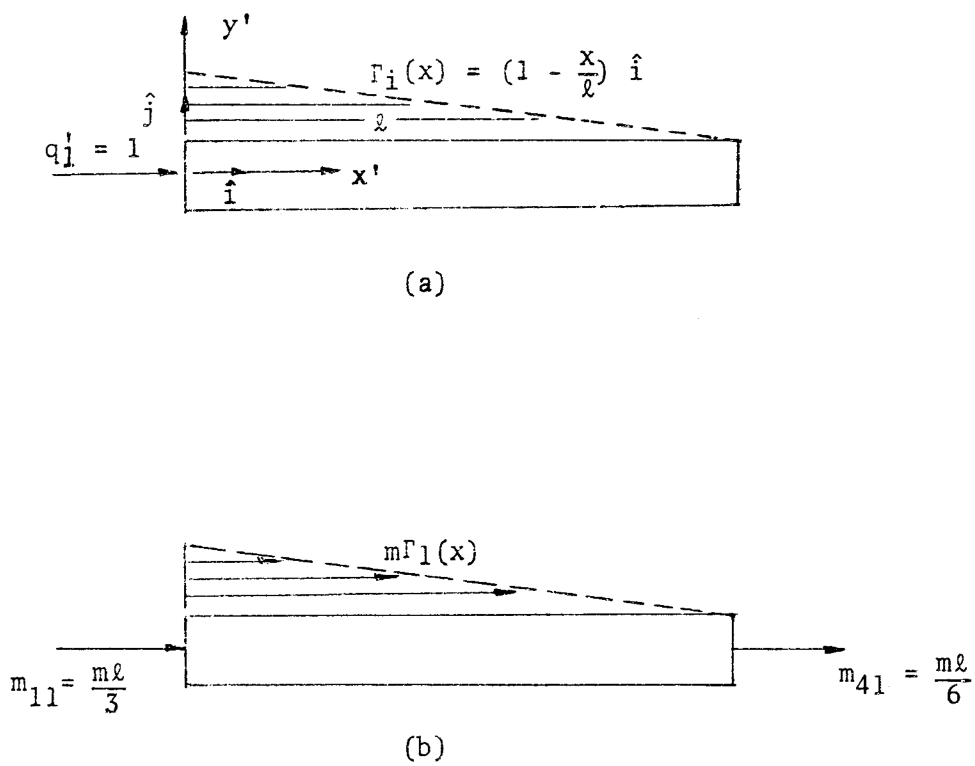
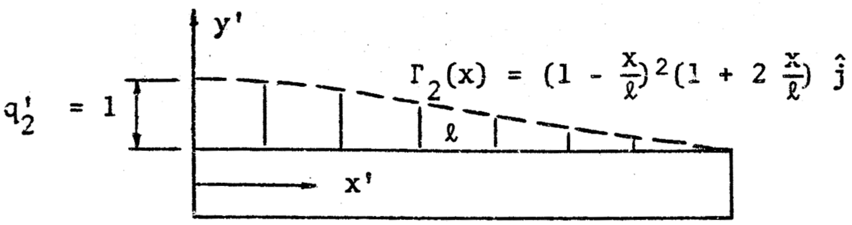
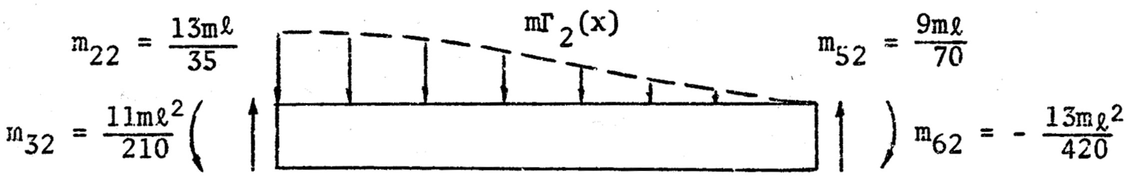


Fig. 3. (a) Generalized deformation curve due to a unit axial displacement along q_1'

(b) Effective distributed mass



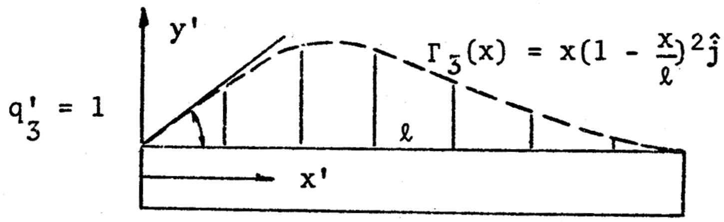
(a)



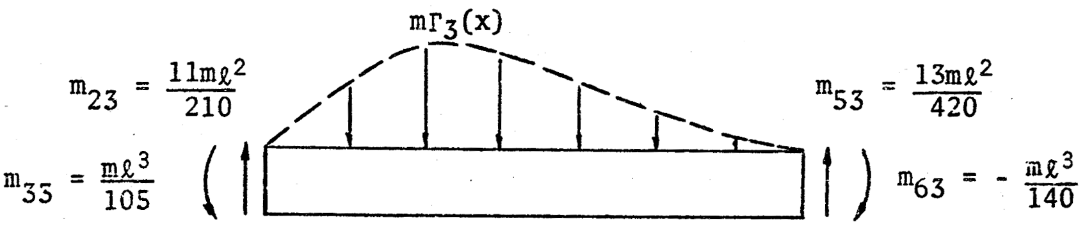
(b)

Fig. 4. (a) Generalized deformation curve due to a unit displacement along q'_2

(b) Effective distributed mass



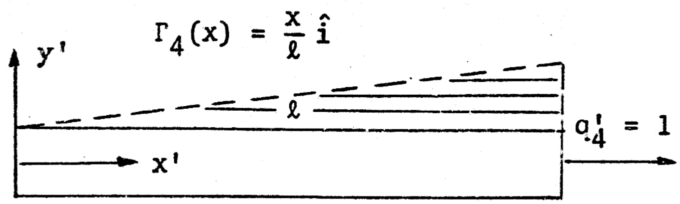
(a)



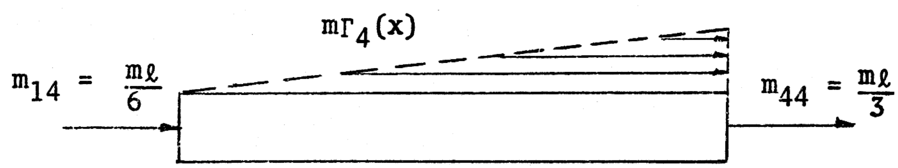
(b)

Fig. 5. (a) Generalized deformation curve due to a unit displacement along q'_3

(b) Effective distributed mass



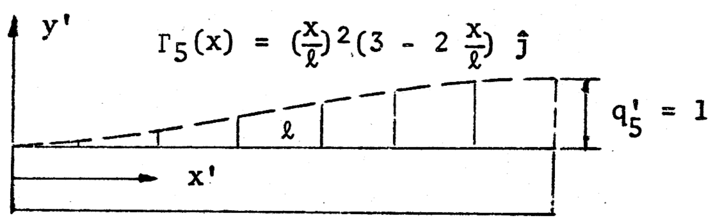
(a)



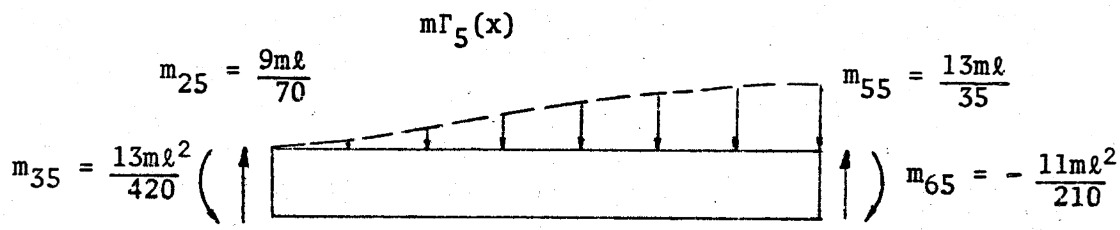
(b)

Fig. 6. (a) Generalized deformation curve due to a unit axial displacement along q_4'

(b) Effective distributed mass



(a)



(b)

Fig. 7. (a) Generalized deformation curve due to a unit displacement along q_5'

(b) Effective distributed mass

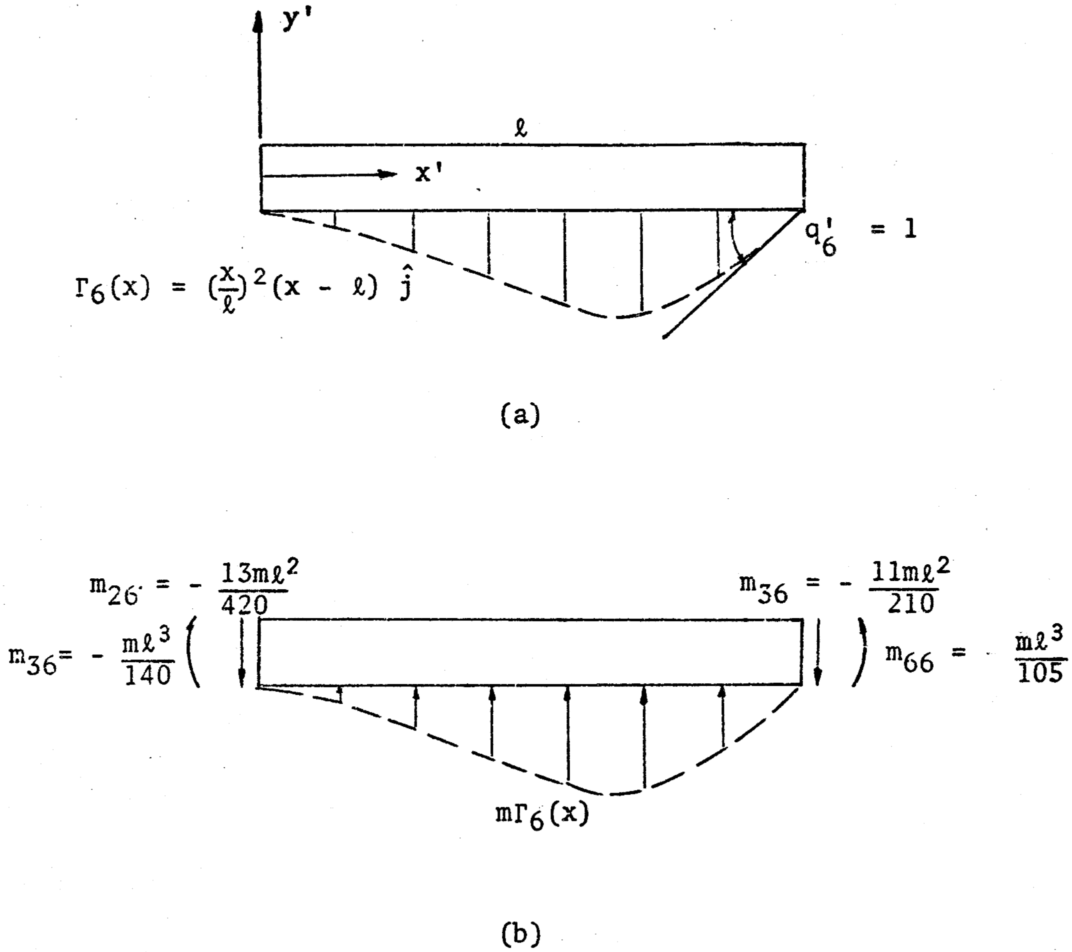


Fig. 8. (a) Generalized deformation curve due to a unit displacement along q_6'

(b) Effective distributed mass

By using superposition, the actual deflection at any point x of the beam element can be expressed as

$$w(x,t) = \sum_{i=1}^{n} \Gamma_i(x) q_i'(t). \qquad \text{at time } t \tag{20}$$

where the coordinates $q_i'(t)(i = 1,2,\cdots,6)$ determine the amplitudes of the respective functions $\Gamma_i(x)$ which contribute to the total deflection $w(x,t)$. Differentiating with respect to time t, Eq. (20) becomes

$$\dot{w}(x,t) = \sum_{i=1}^{n} \Gamma_i(x) \dot{q}_i'(t), \tag{21}$$

Or, in matrix form, it is

$$\dot{w}(x,t) = \lfloor \Gamma \rfloor \{\dot{q}'\} = \lfloor \dot{q}' \rfloor \{\Gamma\} \quad . \tag{22}$$

The kinetic energy of the beam element with distributed mass can be written as

$$T = \frac{1}{2} \int_0^{\ell} m\dot{w}^2(x,t)dx \tag{23}$$

Substituting Eq. (22) into Eq. (23), the kinetic energy yields

$$\begin{aligned} T &= \frac{1}{2} \int_0^{\ell} m \lfloor \dot{q}' \rfloor \{\Gamma\} \cdot \lfloor \Gamma \rfloor \{\dot{q}'\} dx \\ &= \frac{1}{2} \lfloor \dot{q}' \rfloor \int_0^{\ell} m \lfloor \Gamma \rfloor \cdot \{\Gamma\} dx \cdot \{\dot{q}'\} \\ &= \frac{1}{2} \lfloor \dot{q}' \rfloor [m_{ij}] \{\dot{q}'\} \\ &= \frac{1}{2} \lfloor \dot{q}' \rfloor M' \{\dot{q}'\} \quad , \end{aligned} \tag{24}$$

where

$$M' = [m_{ij}] = \int_0^{\ell} m \lfloor \Gamma \rfloor \cdot \{\Gamma\} dt \tag{25}$$

and

$$m_{ij} = \int_0^{\ell} m \cdot \Gamma_i(x)\Gamma_j(x)dt \tag{26}$$

The component of the mass inertial coefficient m_{ij} is defined as a line integral over the beam element of the product of mass per unit length m and the deflection functions $\Gamma_i(x)$ and $\Gamma_j(x)$.

Thus the mass inertial coefficients can be evaluated by substituting Eqs.(14) through (19) into Eq. (26); thus they yield,

$$m_{11} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot (1 - \frac{x}{\ell})\ \hat{i}dx = \frac{m\ell}{3}$$

$$m_{21} = m_{12} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot (1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\hat{j}dx = 0$$

$$m_{31} = m_{13} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot x(1 - \frac{x}{\ell})^2\hat{j}dx = 0$$

$$m_{41} = m_{14} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot \frac{x}{\ell}\hat{i}dx = \frac{m\ell}{6}$$

$$m_{51} = m_{15} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot (\frac{x}{\ell})^2(3 - 2\frac{x}{\ell})\ \hat{j}dx = 0$$

$$m_{61} = m_{16} = \int_0^{\ell} m(1 - \frac{x}{\ell})\ \hat{i} \cdot (\frac{x}{\ell})^2(x - \ell)\ \hat{j}dx = 0$$

$$m_{22} = \int_0^{\ell} m(1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j} \cdot (1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j}dx = \frac{13}{35}m\ell$$

$$m_{32} = m_{23} = \int_0^{\ell} m(1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j} \cdot x(1 - \frac{x}{\ell})^2\hat{j}dx = \frac{11}{210}m\ell^2$$

$$m_{42} = m_{24} = \int_0^{\ell} m(1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j} \cdot \frac{x}{\ell}\ \hat{i}dx = 0$$

$$m_{52} = m_{25} = \int_0^{\ell} m(1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j} \cdot (\frac{x}{\ell})^2(3 - 2\frac{x}{\ell})\ \hat{j}dx = \frac{9}{70}\ m\ell$$

$$m_{62} = m_{26} = \int_0^{\ell} m(1 - \frac{x}{\ell})^2(1 + 2\frac{x}{\ell})\ \hat{j} \cdot (\frac{x}{\ell})^2(x - \ell)\ \hat{j}dx = -\frac{13}{420}\ m\ell^2$$

$$m_{33} \equiv \int_0^{\ell} mx(1 - \frac{x}{\ell})^2\ \hat{j} \cdot x(1 - \frac{x}{\ell})^2\hat{j}dx = \frac{m\ell^3}{105}$$

$$m_{43} = m_{34} = \int_0^\ell mx(1 - \frac{x}{\ell})^2 \hat{j} \cdot \frac{x}{\ell}\,\hat{i}dx = 0$$

$$m_{53} = m_{35} = \int_0^\ell mx(1 - \frac{x}{\ell})^2 \hat{j} \cdot (\frac{x}{\ell})^2 (3 - 2\frac{x}{\ell})\ \hat{j}dx = \frac{13}{420}\, m\ell^2$$

$$m_{63} = m_{36} = \int_0^\ell mx(1 - \frac{x}{\ell})^2 \hat{j} \cdot (\frac{x}{\ell})^2 (x - \ell)\ \hat{j}dx = -\ \frac{m\ell^3}{140}$$

$$m_{44} = \int_0^\ell m\, \frac{x}{\ell}\, \hat{i} \cdot \frac{x}{\ell}\, \hat{i}dx = \frac{m\ell}{3}$$

$$m_{54} = m_{45} = \int_0^\ell m\, \frac{x}{\ell}\, \hat{i} \cdot (\frac{x}{\ell})^2 (3 - 2\,\frac{x}{\ell})\ \hat{j}dx = 0$$

$$m_{64} = m_{46} = \int_0^\ell m\, \frac{x}{\ell}\, \hat{i} \cdot (\frac{x}{\ell})^2 (x - \ell)\ \hat{j}dx = 0$$

$$m_{55} = \int_0^\ell m(\frac{x}{\ell})^2 (3 - 2\,\frac{x}{\ell})\ \hat{j} \cdot (\frac{x}{\ell})^2 (3 - 2\frac{x}{\ell})\ \hat{j}dx = \frac{13}{35}\, m\ell$$

$$m_{65} = m_{56} = \int_0^\ell m(\frac{x}{\ell})^2 (3 - 2\,\frac{x}{\ell})\ \hat{j} \cdot (\frac{x}{\ell})^2\ (x - \ell)\ \hat{j}dx = -\ \frac{11}{210}\, m\ell^2$$

$$m_{66} = \int_0^\ell m(\frac{x}{\ell})^2 (x - \ell)\ \hat{j} \cdot (\frac{x}{\ell})^2 (x - \ell)\ \hat{j}dx = \frac{m\ell^3}{105}$$

Finally, the results thus obtained can be written as:

$$M' = [m_{ij}] = m\ell \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ & \frac{13}{35} & \frac{11}{210}\ell & 0 & \frac{9}{70} & -\frac{13}{420}\ell \\ & & \frac{\ell^2}{105} & 0 & \frac{13}{420}\ell & -\frac{\ell^2}{140} \\ & & & \frac{1}{3} & 0 & 0 \\ & & & & \frac{13}{35} & -\frac{11}{210}\ell \\ & \text{Symmetric} & & & & \frac{\ell^2}{105} \end{bmatrix} \tag{27}$$

(2) Stiffness Matrix $[k_{ij}]$

Axial stiffness and bending stiffness including shear deformation are considered for a uniform beam element as shown in Fig. 2. in the following paragraph.

First, the components of the axial stiffness are derived by calculating the external forces required in the direction of the coordinates q_i' $(i = 1,2,\cdots,6)$ to sustain a unit axial displacement of the coordinates q_1' and q_4', respectively, with all other coordinates constrained rigidly. The results are:

$$\left.\begin{array}{ll} k_{11} = k_{44} = \frac{AE}{\ell} & \\ k_{41} = k_{14} = -\frac{AE}{\ell} & \\ k_{i1} = k_{1i} = 0 & \text{for } i = 2, 3, 5, 6 \\ k_{i4} = k_{4i} = 0 & \text{for } i = 2, 3, 5, 6 \end{array}\right] \tag{28}$$

Next, the strain energy in pure bending is

$$U = \frac{1}{2} EI \int_o^\ell \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx \tag{29}$$

From Eq. (20)

$$w(x,t) = \sum_{i=1}^{n} \Gamma_i(x) q_i'(t)$$

Taking second derivatives with respect to x

$$\frac{\partial^2 w}{\partial x^2} = \sum_{i=1}^{n} \frac{d^2\Gamma_i(x)}{dx^2} q_i'(t) \tag{30}$$

or, in matrix form,

$$\frac{\partial^2 w}{\partial x^2} = \left\lfloor \frac{d^2\Gamma}{dx^2} \right\rfloor \{q'\} = \lfloor q' \rfloor \left\{\frac{d^2\Gamma}{dx^2}\right\} \tag{31}$$

Substituting Eq. (31) into Eq. (29) yields

$$U = \frac{1}{2} EI \int_{0}^{\ell} \lfloor \frac{d^2\Gamma}{dx^2} \rfloor \{q'\} \cdot \lfloor q' \rfloor \{\frac{d^2\Gamma}{dx^2}\} \, dx$$

$$= \frac{1}{2} \lfloor q' \rfloor \int_{0}^{\ell} EI \lfloor \frac{d^2\Gamma}{dx^2} \rfloor \cdot \{\frac{d^2\Gamma}{dx^2}\} \, dx \cdot \{q'\} \tag{32}$$

The matrix form of Eq. (9) is

$$U = \frac{1}{2} \lfloor q' \rfloor [k_{ij}] \{q'\} \tag{33}$$

where

$$K' = [k_{ij}] = EI \int_{0}^{\ell} \lfloor \frac{d^2\Gamma}{dx^2} \rfloor \cdot \{\frac{d^2\Gamma}{dx^2}\} \, dx \tag{34}$$

and

$$k_{ij} = EI \int_{0}^{\ell} \frac{d^2\Gamma_i}{dx^2} \cdot \frac{d^2\Gamma_j}{dx^2} \, dx \tag{35}$$

Substituting the deflection functions $\Gamma_2(x)$, $\Gamma_3(x)$, $\Gamma_5(x)$ and $\Gamma_6(x)$ into Eqs. (35), the coefficients of bending stiffness are:

$$k_{22} = EI \int_{0}^{\ell} (-\frac{6}{\ell^2} + \frac{12x}{\ell^3}) \hat{j} \cdot (-\frac{6}{\ell^2} + \frac{12x}{\ell^3}) \hat{j} dx = \frac{12}{\ell^3} EI$$

$$k_{32} = k_{23} = EI \int_{0}^{\ell} (-\frac{4}{\ell} + \frac{6x}{\ell^2}) \hat{j} \cdot (-\frac{6}{\ell^2} + \frac{6x}{\ell^3}) \hat{j} dx = \frac{6}{\ell^2} EI$$

$$k_{52} = k_{25} = EI \int_{0}^{\ell} (-\frac{6}{\ell^2} + \frac{12x}{\ell^3}) \hat{j} \cdot (\frac{6}{\ell^2} - \frac{12x}{\ell^3}) \hat{j} dx = -\frac{12}{\ell^3} EI$$

$$k_{62} = k_{26} = EI \int_{0}^{\ell} (-\frac{6}{\ell^2} + \frac{12x}{\ell^3}) \hat{j} \cdot (-\frac{2}{\ell} + \frac{6x}{\ell^2}) \hat{j} dx = \frac{6EI}{\ell^2}$$

$$k_{33} = EI \int_{0}^{\ell} (-\frac{4}{\ell} + \frac{6x}{\ell^2}) \hat{j} \cdot (-\frac{4}{\ell} + \frac{6x}{\ell^2}) \hat{j} dx = \frac{4EI}{\ell}$$

$$k_{53} = k_{35} = EI \int_{0}^{\ell} (-\frac{4}{\ell} + \frac{6x}{\ell^2}) \hat{j} \cdot (\frac{6}{\ell^2} - \frac{12x}{\ell^3}) \hat{j} dx = -\frac{6EI}{\ell^2}$$

$$k_{63} = k_{36} = EI \int_{0}^{\ell} (-\frac{2}{\ell} + \frac{6x}{\ell^2}) \hat{j} \cdot (-\frac{4}{\ell} + \frac{6x}{\ell^2}) \hat{j} dx = \frac{2EI}{\ell}$$

$$k_{55} = EI \int_{o}^{\ell} \left(\frac{6}{\ell^2} - \frac{12x}{\ell^3} \right) \hat{j} \cdot \left(\frac{6}{\ell^2} - \frac{12x}{\ell^3} \right) \hat{j} dx = \frac{12EI}{\ell^3}$$

$$k_{65} = k_{56} = EI \int_{o}^{\ell} \left(\frac{6}{\ell^2} - \frac{12x}{\ell^3} \right) \hat{j} \cdot \left(- \frac{2}{\ell} + \frac{6x}{\ell^2} \right) \hat{j} dx = - \frac{6EI}{\ell^2}$$

$$k_{66} = EI \int_{o}^{\ell} \left(- \frac{2}{\ell} + \frac{6x}{\ell^2} \right) \hat{j} \cdot \left(- \frac{2}{\ell} + \frac{6x}{\ell^2} \right) \hat{j} dx = \frac{4EI}{\ell}$$

From the above results and Eqs.(28), the stiffness matrix can be expressed as:

$$K' = [k_{ij}] = \begin{bmatrix} \frac{AE}{\ell} & 0 & 0 & -\frac{AE}{\ell} & 0 & 0 \\ & \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & 0 & -\frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ & & \frac{4EI}{\ell} & 0 & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} \\ & & & \frac{AE}{\ell} & 0 & 0 \\ & & & & \frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} \\ & \text{Symmetric} & & & & \frac{4EI}{\ell} \end{bmatrix} \qquad (36)$$

For the bending stiffness with the shear deformation included, Eqs.(35) becomes

$$k_{ij} = \frac{EI}{(1 + \beta_i)(1 + \beta_j)} \left\{ \int_{o}^{\ell} \frac{d^2\Gamma_i}{dx^2} \frac{d^2\Gamma_j}{dx^2} dx + \frac{\alpha GA}{EI} \beta_i \beta_j \int_{o}^{\ell} \frac{d\Gamma_i}{dx} \frac{d\Gamma_j}{dx} dx \right\} \qquad (37)$$

where

α = a dimensionless number dependent on the shape of the cross section,

$\beta_i = (q_i')s/(q_i')b$ where the subscripts s and b denote shear and bending, respectively, and

G = modulus of elasticity in shear.

The correction factor $1/(1 + \beta_i)$ is less than 1, because the stiffness decreases when the shear deformation is included. Timoshenko[5] has shown that the correction factor to the fundamental frequency is about 99% for a simply supported uniform I-beam with a thin web, but the shear effect is relatively significant in higher modes. Here it is neglected.

TRANSFORMATION OF MATRICES FOR A UNIFORM BEAM ELEMENT FROM LOCAL COORDINATES INTO SYSTEM COORDINATES

Let (x', y') and (x, y) be local and system coordinates, respectively, with a common origin as shown in Fig. 9. The table of direction cosines of the axes becomes

	x'	y'	z'
x	$\cos\theta$	$-\sin\theta$	0
y	$\sin\theta$	$\cos\theta$	0
z	0	0	1

Table 1. Direction cosines

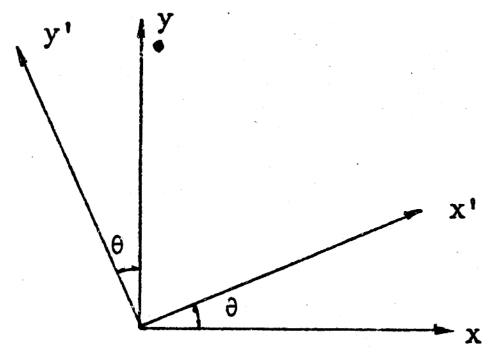


Fig. 9. Transformation of rectangular coordinates

In matrix form, the transformation is

$$\{q\} = T_a\{q'\} \tag{38}$$

where the transform matrix is

$$T_a = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

for order 6 x 6 of a uniform beam element in system coordinates as shown in Fig. 10.

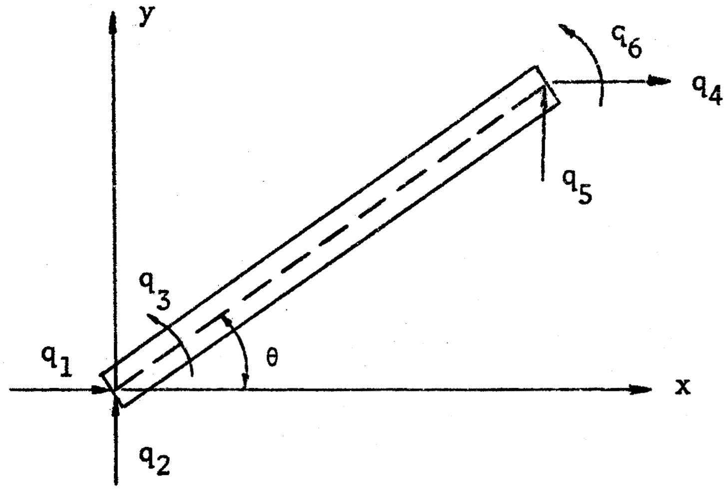


Fig. 10. System coordinates for a uniform beam element

Equation (39) is an orthogonal matrix; thus inversion of Eq. (38) yields

$$\{q'\} = T_a^{-1}\{q\} = T_a^T\{q\} \quad \text{, or} \tag{40}$$

$$\lfloor q' \rfloor = \lfloor q \rfloor T_a \tag{41}$$

The derivatives with respect to time become

$$\{\dot{q}'\} = T_a^T \{\dot{q}\} \qquad \text{or} \tag{42}$$

$$\lfloor \dot{q}' \rfloor = \lfloor \dot{q} \rfloor T_a \tag{43}$$

Substituting Eqs. (42) and (43) into Eq. (24) gives

$$T = \frac{1}{2} \lfloor \dot{q} \rfloor T_a M' T_a^T \{\dot{q}\} \tag{44}$$

Comparing Eqs. (24) and (44), the consistent mass matrix M for system coordinates can be written

$$M = T_a M' T_a^T \tag{45}$$

Similarly, from Eqs. (33), (38) and (39), the stiffness matrix K for system coordinates can be written

$$K = T_a K' T_a^T \tag{46}$$

Expansions of Eqs. (45) and (46) yield:

$M = [m_{ij}] = m\ell\Phi$

Where

$$\Phi = \begin{bmatrix} \frac{1}{3}\mathrm{Cos}^2\theta + \frac{13}{35}\mathrm{Sin}^2\theta, & -\frac{4}{105}\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & -\frac{11}{210}\ell\,\mathrm{Sin}\theta, & \frac{1}{6}\mathrm{Cos}^2\theta + \frac{9}{70}\mathrm{Sin}^2\theta, & \frac{4}{105}\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & \frac{13}{420}\ell\,\mathrm{Sin}\theta \\ & \frac{1}{3}\mathrm{Sin}^2\theta + \frac{13}{35}\mathrm{Cos}^2\theta, & \frac{11}{210}\ell\,\mathrm{Cos}\theta, & \frac{4}{105}\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & \frac{1}{6}\mathrm{Sin}^2\theta + \frac{9}{70}\mathrm{Cos}^2\theta, & -\frac{13}{420}\ell\,\mathrm{Cos}\theta \\ & & \frac{1}{105}\ell^2, & -\frac{13}{420}\ell\,\mathrm{Sin}\theta, & \frac{13}{420}\ell\,\mathrm{Cos}\theta, & -\frac{1}{140}\ell^2 \\ & & & \frac{1}{3}\mathrm{Cos}^2\theta + \frac{13}{35}\mathrm{Sin}^2\theta, & -\frac{4}{105}\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & \frac{11}{210}\ell\,\mathrm{Sin}\theta \\ & & & & \frac{1}{3}\mathrm{Sin}^2\theta + \frac{13}{35}\mathrm{Cos}^2\theta, & -\frac{11}{210}\ell\,\mathrm{Cos}\theta \\ & \text{Symmetric} & & & & \frac{1}{105}\ell^2 \end{bmatrix} \quad (47)$$

$$K = [k_{ij}] = \frac{EI}{\ell}\Psi$$

Where

$$\Psi = \begin{bmatrix} \frac{12}{\ell^2}\mathrm{Sin}^2\theta + \frac{A}{I}\mathrm{Cos}^2\theta, & (\frac{A}{I} - \frac{12}{\ell^2})\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & -\frac{6}{\ell}\mathrm{Sin}\theta, & -\frac{12}{\ell^2}\mathrm{Sin}^2\theta - \frac{A}{I}\mathrm{Cos}^2\theta, & (\frac{12}{\ell^2} - \frac{A}{I})\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & -\frac{6}{\ell}\mathrm{Sin}\theta \\ & \frac{12}{\ell^2}\mathrm{Cos}^2\theta + \frac{A}{I}\mathrm{Sin}^2\theta, & \frac{6}{\ell}\mathrm{Cos}\theta, & (\frac{12}{\ell^2} - \frac{A}{I})\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & -\frac{12}{\ell^2}\mathrm{Cos}^2\theta - \frac{A}{I}\mathrm{Sin}^2\theta, & \frac{6}{\ell}\mathrm{Cos}\theta \\ & & 4, & \frac{6}{\ell}\mathrm{Sin}\theta, & -\frac{6}{\ell}\mathrm{Cos}\theta, & 2 \\ & & & \frac{12}{\ell^2}\mathrm{Sin}^2\theta + \frac{A}{I}\mathrm{Cos}^2\theta, & (\frac{A}{I} - \frac{12}{\ell^2})\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta, & \frac{6}{\ell}\mathrm{Sin}\theta \\ & & & & \frac{12}{\ell^2}\mathrm{Cos}^2\theta + \frac{A}{I}\mathrm{Sin}^2\theta, & -\frac{6}{\ell}\mathrm{Cos}\theta \\ & \text{Symmetric} & & & & 4 \end{bmatrix} \qquad (48)$$

LUMPED MASS MATRIX FOR A UNIFORM BEAM ELEMENT

In Fig. 2. let one-half of the mass of the beam element be concentrated at each of the two ends of the element. From Eq. (24), the kinetic energy in matrix form can be written as:

$$T = \frac{1}{2} \lfloor \dot{q}' \rfloor M' \{\dot{q}'\}$$

Then the lumped mass matrix for the uniform beam element in local coordinates is

$$M' = \frac{1}{2} m\ell \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 0 & & & \\ & & & 1 & & \\ & \text{Symmetric} & & & 1 & \\ & & & & & 0 \end{bmatrix} \tag{49}$$

Form Eq. (45),

$$M = T_a M' T_a^T$$

This is the lumped mass matrix M for system coordinates. Substituting Eqs. (39) and (49) into Eq. (45) results in

$$M = \frac{1}{2} m\ell \begin{bmatrix} \mathrm{Cos}\theta & -\mathrm{Sin}\theta & 0 & 0 & 0 & 0 \\ \mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Cos}\theta & -\mathrm{Sin}\theta & 0 \\ 0 & 0 & 0 & \mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 0 & & & \\ & & & 1 & & \\ & \text{Symmetric} & & & 1 & \\ & & & & & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} \mathrm{Cos}\theta & \mathrm{Sin}\theta & 0 & 0 & 0 & 0 \\ -\mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Cos}\theta & \mathrm{Sin}\theta & 0 \\ 0 & 0 & 0 & -\mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} m\ell \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ \text{Symmetric} & & & & & 0 \end{bmatrix} \tag{50}$$

The result shows that the lumped mass matrix, which is a diagonal matrix, is independent of its coordinates.

ASSEMBLY OF THE EIGENVALUE PROBLEM FOR A FRAME WITH RIGID CONNECTIONS

A symmetrical gable frame as shown in Fig. 11 can be considered as four elastic beams joined rigidly. Each individual beam of the frame is considered to be composed of two beam elements of equal length. Let A, B, C, D, E, F, G and H denote each beam element.

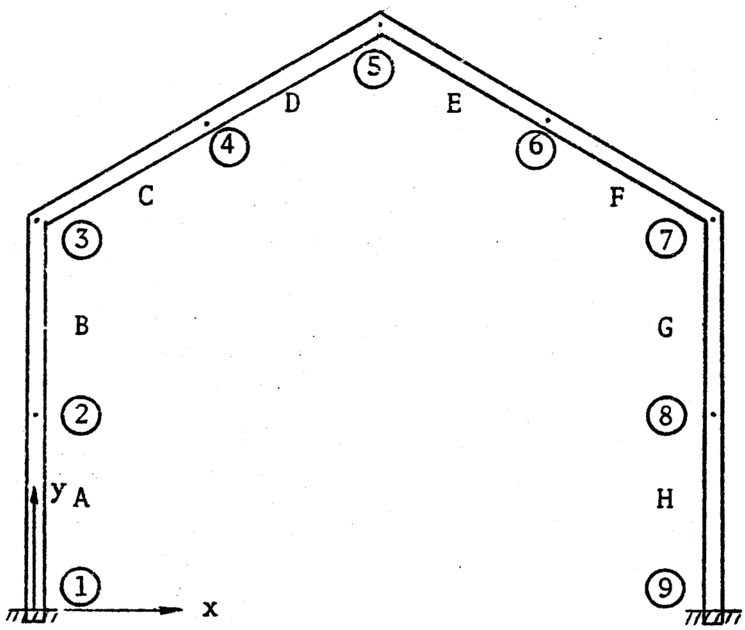


Fig. 11. A symmetrical gable frame with clamped ends and rigid connected joints

From elementary mechanics for each beam element, it follows that

$$\{Q\} = K\ \{q\} \tag{51}$$

where

K denotes the stiffness matrix (6 x 6).

$$\{Q\}^T = \lfloor Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \rfloor$$

is a row matrix of externally applied forces measured in system coordinates.

$$\{q\}^T = \lfloor q_1, q_2, q_3, q_4, q_5, q_6 \rfloor$$

is a row matrix of structural displacements measured in system coordinates.

Equation (51) can be expressed as

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} K_{11}^A & K_{12}^A \\ K_{21}^A & K_{22}^A \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad \text{for beam element A,} \tag{52}$$

Similarly, the following equations are developed:

$$\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} K_{11}^B & K_{12}^B \\ K_{21}^B & K_{22}^B \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} \quad \text{for beam element B,} \tag{53}$$

$$\begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} K_{11}^C & K_{12}^C \\ K_{21}^C & K_{22}^C \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} \quad \text{for beam element C,} \tag{54}$$

$$\begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} = \begin{bmatrix} K_{11}^D & K_{12}^D \\ K_{21}^D & K_{22}^D \end{bmatrix} \begin{Bmatrix} q_4 \\ q_5 \end{Bmatrix} \quad \text{for beam element D,} \tag{55}$$

$$\begin{Bmatrix} Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} K_{11}^E & K_{12}^E \\ K_{21}^E & K_{22}^E \end{bmatrix} \begin{Bmatrix} q_5 \\ q_6 \end{Bmatrix} \quad \text{for beam element E,} \tag{56}$$

$$\begin{Bmatrix} Q_6 \\ Q_7 \end{Bmatrix} = \begin{bmatrix} K_{11}^F & K_{12}^F \\ K_{21}^F & K_{22}^F \end{bmatrix} \begin{Bmatrix} q_6 \\ q_7 \end{Bmatrix} \quad \text{for beam element F,} \tag{57}$$

$$\begin{Bmatrix} Q_7 \\ Q_8 \end{Bmatrix} = \begin{bmatrix} K_{11}^G & K_{12}^G \\ K_{21}^G & K_{22}^G \end{bmatrix} \begin{Bmatrix} q_7 \\ q_8 \end{Bmatrix} \quad \text{for beam element G,} \tag{58}$$

and

$$\begin{Bmatrix} Q_8 \\ Q_9 \end{Bmatrix} = \begin{bmatrix} K_{11}^H & K_{12}^H \\ K_{21}^H & K_{22}^H \end{bmatrix} \begin{Bmatrix} q_8 \\ q_9 \end{Bmatrix} \quad \text{for beam element H,} \tag{59}$$

where

Q_i $(i = 1, 2, \cdots, 9)$ is a three-element vector of the generalized external forces applied at node i;

q_i $(i = 1, 2, \cdots, 9)$ is a three-element vector of the generalized displacements at node i; and

K_{ij}^P $(P = A, B, \cdots, H)$ is a matrix (3 x 3) obtained by partitioning the stiffness matrix (6 x 6) for the Pth beam element in system coordinates.

Since the beam elements are joined rigidly, Eqs. (53) through (59) may be expanded thus:

$$Q_1 = K_{11}^{A}q_1 + K_{12}^{A}q_2$$

$$Q_2 = K_{21}^{A}q_1 + (K_{22}^{A} + K_{11}^{B})q_2 + K_{12}^{B}q_3$$

$$Q_3 = K_{21}^{B}q_2 + (K_{22}^{B} + K_{11}^{C})q_3 + K_{12}^{C}q_4$$

$$Q_4 = K_{21}^{C}q_3 + (K_{22}^{C} + K_{11}^{D})q_4 + K_{12}^{D}q_5$$

$$Q_5 = K_{21}^{D}q_4 + (K_{22}^{D} + K_{11}^{E})q_5 + K_{12}^{E}q_6$$

$$Q_6 = K_{21}^{E}q_5 + (K_{22}^{E} + K_{11}^{F})q_6 + K_{12}^{F}q_7$$

$$Q_7 = K_{21}^{F}q_6 + (K_{22}^{F} + K_{11}^{G})q_7 + K_{12}^{G}q_8$$

$$Q_8 = K_{21}^{G}q_7 + (K_{22}^{G} + K_{11}^{H})q_8 + K_{12}^{H}q_9$$

$$Q_9 = K_{21}^{H}q_8 + K_{22}^{H}q_9$$

In matrix form

$$
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{Bmatrix}
=
\begin{bmatrix}
K_{11}^{A} & K_{12}^{A} & & & & & & & \\
K_{21}^{A} & K_{22}^{A}+K_{11}^{B} & K_{12}^{B} & & & & & & 0 \\
 & K_{21}^{B} & K_{22}^{B}+K_{11}^{C} & K_{12}^{C} & & & & & \\
 & & K_{21}^{C} & K_{22}^{C}+K_{11}^{D} & K_{12}^{D} & & & & \\
 & & & K_{21}^{D} & K_{22}^{D}+K_{11}^{E} & K_{12}^{E} & & & \\
 & & & & K_{21}^{E} & K_{22}^{E}+K_{11}^{F} & K_{12}^{F} & & \\
 & & & & & K_{21}^{F} & K_{22}^{F}+K_{11}^{G} & K_{12}^{G} & \\
\text{Symmetric} & & & & & & K_{21}^{G} & K_{22}^{G}+K_{11}^{H} & K_{12}^{H} \\
 & & & & & & & K_{21}^{H} & K_{22}^{H}
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{Bmatrix}
\tag{60}
$$

Since

$$\{Q\} = M\{\ddot{q}\} \tag{61}$$

where M denotes the mass matrix (6 x 6).

$\{Q\}^T = \lfloor Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \rfloor$ is a row matrix of externally applied forces measured in system coordinates, and

$\{\ddot{q}\}^T = \lfloor \ddot{q}_1, \ddot{q}_2, \ddot{q}_3, \ddot{q}_4, \ddot{q}_5, \ddot{q}_6 \rfloor$ is a row matrix of structural accelerations measured in system coordinates.

Equation (61) can be rewritten as:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} M_{11}^{A} & M_{12}^{A} \\ M_{21}^{A} & M_{22}^{A} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} \quad \text{for beam element A,} \tag{62}$$

Similarly, the following equations are developed:

$$\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} M_{11}^{B} & M_{12}^{B} \\ M_{21}^{B} & M_{22}^{B} \end{bmatrix} \begin{Bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} \quad \text{for beam element B,} \tag{63}$$

$$\begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} M_{11}^{C} & M_{12}^{C} \\ M_{21}^{C} & M_{22}^{C} \end{bmatrix} \begin{Bmatrix} \ddot{q}_3 \\ \ddot{q}_4 \end{Bmatrix} \quad \text{for beam element C,} \tag{64}$$

$$\begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} = \begin{bmatrix} M_{11}^{D} & M_{12}^{D} \\ M_{21}^{D} & M_{22}^{D} \end{bmatrix} \begin{Bmatrix} \ddot{q}_4 \\ \ddot{q}_5 \end{Bmatrix} \quad \text{for beam element D,} \tag{65}$$

$$\begin{Bmatrix} Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} M_{11}^{E} & M_{12}^{E} \\ M_{21}^{E} & M_{22}^{E} \end{bmatrix} \begin{Bmatrix} \ddot{q}_5 \\ \ddot{q}_6 \end{Bmatrix} \quad \text{for beam element E,} \tag{66}$$

$$\begin{Bmatrix} Q_6 \\ Q_7 \end{Bmatrix} = \begin{bmatrix} M_{11}^{F} & M_{12}^{F} \\ M_{21}^{F} & M_{22}^{F} \end{bmatrix} \begin{Bmatrix} \ddot{q}_6 \\ \ddot{q}_7 \end{Bmatrix} \quad \text{for beam element F,} \tag{67}$$

$$\begin{Bmatrix} Q_7 \\ Q_8 \end{Bmatrix} = \begin{bmatrix} M_{11}^{G} & M_{12}^{G} \\ M_{21}^{G} & M_{22}^{G} \end{bmatrix} \begin{Bmatrix} \ddot{q}_7 \\ \ddot{q}_8 \end{Bmatrix} \quad \text{for beam element G,} \tag{68}$$

and

$$\begin{Bmatrix} Q_8 \\ Q_9 \end{Bmatrix} = \begin{bmatrix} M_{11}^H & M_{12}^H \\ M_{21}^H & M_{22}^H \end{bmatrix} \begin{Bmatrix} \ddot{q}_8 \\ \ddot{q}_9 \end{Bmatrix} \quad \text{for beam element H,} \qquad (69)$$

where

$Q_i (i = 1, 2, \cdots, 9)$ is a three-element vector of the generalized external forces applied at node i;

$\ddot{q}_i (i = 1, 2, \cdots, 9)$ is a three-element vector of the generalized accelerations at node i; and

$M_{ij}^P (P = A, B, \cdots, H)$ is a matrix (3 x 3) obtained by partitioning the mass matrix (6 x 6) for the Pth beam element in system coordinates.

Since the beam elements are joined rigidly, Eqs. (62) through (69) can be expanded and written in matrix form

$$
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{Bmatrix}
=
\begin{bmatrix}
M_{11}^A & M_{12}^A & & & & & & & \\
M_{21}^A & M_{22}^A + M_{11}^B & M_{12}^B & & & & & 0 & \\
 & M_{21}^B & M_{22}^B + M_{11}^C & M_{12}^C & & & & & \\
 & & M_{21}^C & M_{22}^C + M_{11}^D & M_{12}^D & & & & \\
 & & & M_{21}^D & M_{22}^D + M_{11}^E & M_{12}^E & & & \\
 & & & & M_{21}^E & M_{22}^E + M_{11}^F & M_{12}^F & & \\
 & & & & & M_{21}^F & M_{22}^F + M_{11}^G & M_{12}^G & \\
 & & & & & & M_{21}^G & M_{22}^G + M_{11}^H & M_{12}^H \\
\text{Symmetric} & & & & & & & M_{21}^H & M_{22}^H
\end{bmatrix}
\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \\ \ddot{q}_8 \\ \ddot{q}_9 \end{Bmatrix}
\tag{70}
$$

The equations of motion of the gable frame may be written as

$$\begin{bmatrix} M_{11}^{A} & M_{12}^{A} & & & & & & & \\ M_{21}^{A} & M_{22}^{A}+M_{11}^{B} & M_{12}^{B} & & & & & 0 & \\ & M_{21}^{B} & M_{22}^{B}+M_{11}^{C} & M_{12}^{C} & & & & & \\ & & M_{21}^{C} & M_{22}^{C}+M_{11}^{D} & M_{12}^{D} & & & & \\ & & & M_{21}^{D} & M_{22}^{D}+M_{11}^{E} & M_{12}^{E} & & & \\ & & & & M_{21}^{E} & M_{22}^{E}+M_{11}^{F} & M_{12}^{F} & & \\ & & & & & M_{21}^{F} & M_{22}^{F}+M_{11}^{G} & M_{12}^{G} & \\ & & & & & & M_{21}^{G} & M_{22}^{G}+M_{11}^{H} & M_{12}^{H} \\ \text{Symmetric} & & & & & & & M_{21}^{H} & M_{22}^{H} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \\ \ddot{q}_8 \\ \ddot{q}_9 \end{Bmatrix} +$$

$$
+\begin{bmatrix}
K_{11}^{A} & K_{12}^{A} & & & & & & & \\
K_{21}^{A} & K_{22}^{A}+K_{11}^{B} & K_{12}^{B} & & & & & 0 & \\
 & K_{21}^{B} & K_{22}^{B}+K_{11}^{C} & K_{12}^{C} & & & & & \\
 & & K_{21}^{C} & K_{22}^{C}+K_{11}^{D} & K_{12}^{D} & & & & \\
 & & & K_{21}^{D} & K_{22}^{D}+K_{11}^{E} & K_{12}^{E} & & & \\
 & & & & K_{21}^{E} & K_{22}^{E}+K_{11}^{F} & K_{12}^{F} & & \\
 & & & & & K_{21}^{F} & K_{22}^{F}+K_{11}^{G} & K_{12}^{G} & \\
 & & & & & & K_{21}^{G} & K_{22}^{G}+K_{11}^{H} & K_{12}^{H} \\
 & \text{Symmetric} & & & & & & K_{21}^{H} & K_{22}^{H}
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{Bmatrix}
=
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{Bmatrix}
\tag{71}
$$

In Fig. 11 the gable frame is clamped at the feet, thus $q_1 = q_9 = 0$. Also, there are no external forces applied at nodes 2, 3, 4, 5, 6, 7 and 8. These conditions may be acknowledged in Eqs. (71) and the equation partitioned as

$$
\left[\begin{array}{c|ccccccc|c}
M_{11}^{A} & M_{12}^{A} & & & & & & 0 & 0 \\
\hline
M_{21}^{A} & M_{22}^{A} + M_{11}^{B} & M_{12}^{B} & & & & & 0 & 0 \\
 & M_{21}^{B} & M_{22}^{B} + M_{11}^{C} & M_{12}^{C} & & & & & \\
 & & M_{21}^{C} & M_{22}^{C} + M_{11}^{D} & M_{12}^{D} & & & & \\
 & & & M_{21}^{D} & M_{22}^{D} + M_{11}^{E} & M_{12}^{E} & & & \\
 & & & & M_{21}^{E} & M_{22}^{E} + M_{11}^{F} & M_{12}^{F} & & \\
 & & & & & M_{21}^{F} & M_{22}^{F} + M_{11}^{G} & M_{12}^{G} & \\
0 & 0 & & & & & M_{21}^{G} & M_{22}^{G} + M_{11}^{H} & M_{12}^{H} \\
\hline
0 & 0 & & & & & & M_{21}^{H} & M_{22}^{H}
\end{array}\right]
\left\{\begin{array}{c}
0 \\
\hline
\ddot{q}_2 \\
\ddot{q}_3 \\
\ddot{q}_4 \\
\ddot{q}_5 \\
\ddot{q}_6 \\
\ddot{q}_7 \\
\ddot{q}_8 \\
\hline
0
\end{array}\right\} +
$$

$$
+\left[\begin{array}{c|ccccccc|c}
K_{11}^{A} & K_{12}^{A} & & & & & & 0 & 0\\ \hline
K_{21}^{A} & K_{22}^{A}+K_{11}^{B} & K_{12}^{B} & & & & & 0 & 0\\
 & K_{21}^{B} & K_{22}^{B}+K_{11}^{C} & K_{12}^{C} & & & & & \\
 & & K_{21}^{C} & K_{22}^{C}+K_{11}^{D} & K_{12}^{D} & & & & \\
 & & & K_{21}^{D} & K_{22}^{D}+K_{11}^{E} & K_{12}^{E} & & & \\
 & & & & K_{21}^{E} & K_{22}^{E}+K_{11}^{F} & K_{12}^{F} & & \\
 & & & & & K_{21}^{F} & K_{22}^{F}+K_{11}^{G} & K_{12}^{G} & \\
0 & 0 & & & & & K_{21}^{G} & K_{22}^{G}+K_{11}^{H} & K_{12}^{H}\\ \hline
0 & 0 & & & & & & K_{21}^{H} & K_{22}^{H}
\end{array}\right]
\left\{\begin{array}{c}
0\\ \hline q_2\\ q_3\\ q_4\\ q_5\\ q_6\\ q_7\\ q_8\\ \hline 0
\end{array}\right\}
=
\left\{\begin{array}{c}
Q_1\\ \hline 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ \hline Q_9
\end{array}\right\}
\tag{72}
$$

Equation (72) may be rewritten into three separate equations, as:

$$
[M_{12}^{A} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]
\left\{\begin{array}{c}
\ddot{q}_2\\ \ddot{q}_3\\ \ddot{q}_4\\ \ddot{q}_5\\ \ddot{q}_6\\ \ddot{q}_7\\ \ddot{q}_8
\end{array}\right\}
+ [K_{12}^{A} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]
\left\{\begin{array}{c}
q_2\\ q_3\\ q_4\\ q_5\\ q_6\\ q_7\\ q_8
\end{array}\right\}
= Q_1
\tag{73}
$$

$$\begin{bmatrix} M_{22}^{A}+M_{11}^{B} & M_{12}^{B} & & & & & 0 \\ M_{21}^{B} & M_{22}^{B}+M_{11}^{C} & M_{12}^{C} & & & & \\ & M_{21}^{C} & M_{22}^{C}+M_{11}^{D} & M_{12}^{D} & & & \\ & & M_{21}^{D} & M_{22}^{D}+M_{11}^{E} & M_{12}^{E} & & \\ & & & M_{21}^{E} & M_{22}^{E}+M_{11}^{F} & M_{12}^{F} & \\ & & & & M_{21}^{F} & M_{22}^{F}+M_{11}^{G} & M_{12}^{G} \\ \text{Symmetric} & & & & & M_{21}^{G} & M_{22}^{G}+M_{11}^{H} \end{bmatrix} \begin{Bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \\ \ddot{q}_8 \end{Bmatrix} +$$

$$+\begin{bmatrix} K_{22}^{A}+K_{11}^{B} & K_{12}^{B} & & & & & 0 \\ K_{21}^{B} & K_{22}^{B}+K_{11}^{C} & K_{12}^{C} & & & & \\ & K_{21}^{C} & K_{22}^{C}+K_{11}^{D} & K_{12}^{D} & & & \\ & & K_{21}^{D} & K_{22}^{D}+K_{11}^{E} & K_{12}^{E} & & \\ & & & K_{21}^{E} & K_{22}^{E}+K_{11}^{F} & K_{12}^{F} & \\ & & & & K_{21}^{F} & K_{22}^{F}+K_{11}^{G} & K_{12}^{G} \\ \text{Symmetric} & & & & & K_{21}^{G} & K_{22}^{G}+K_{11}^{H} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (74)$$

and

$$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad M_{21}^{H}] \begin{Bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \\ \ddot{q}_8 \end{Bmatrix} + [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad K_{21}^{H}] \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = Q_9 \tag{75}$$

Equations (73) and (75) yield the reaction at the clamped ends of the frame due to motions of the structure and may be ignored for the purpose of this paper. Equations (74) are again simplified as

$$M\,\{\ddot{q}_j\} + K\,\{q_j\} = 0 \quad , \qquad (j = 2, 3, 4, \cdots, 8) \tag{76}$$

where

$$M = \begin{bmatrix} M_{22}^{A} + M_{11}^{B} & M_{12}^{B} & & & & & 0 \\ M_{21}^{B} & M_{22}^{B} + M_{11}^{C} & M_{12}^{C} & & & & \\ & M_{21}^{C} & M_{22}^{C} + M_{11}^{D} & M_{12}^{D} & & & \\ & & M_{21}^{D} & M_{22}^{D} + M_{11}^{E} & M_{12}^{E} & & \\ & & & M_{21}^{E} & M_{22}^{E} + M_{11}^{F} & M_{12}^{F} & \\ & & & & M_{21}^{F} & M_{22}^{F} + M_{11}^{G} & M_{12}^{G} \\ \text{Symmetric} & & & & & M_{21}^{G} & M_{22}^{G} + M_{11}^{H} \end{bmatrix} \tag{77}$$

and

$$K = \begin{bmatrix} K_{22}^{A} + K_{11}^{B} & K_{12}^{B} & & & & & 0 \\ K_{21}^{B} & K_{22}^{B} + K_{11}^{C} & K_{12}^{C} & & & & \\ & K_{21}^{C} & K_{22}^{C} + K_{11}^{D} & K_{12}^{D} & & & \\ & & K_{21}^{D} & K_{22}^{D} + K_{11}^{E} & K_{12}^{E} & & \\ & & & K_{21}^{E} & K_{22}^{E} + K_{11}^{F} & K_{12}^{F} & \\ & \text{Symmetric} & & & K_{21}^{F} & K_{22}^{F} + K_{11}^{G} & K_{12}^{G} \\ & & & & & K_{21}^{G} & K_{22}^{G} + K_{11}^{H} \end{bmatrix} \tag{78}$$

The displacements q_j ($j = 2, 3, \cdots, 8$) in Eqs. (76) are assumed to be sinusoidal functions of time t with constant frequency ω as shown in Fig. 12. That is:

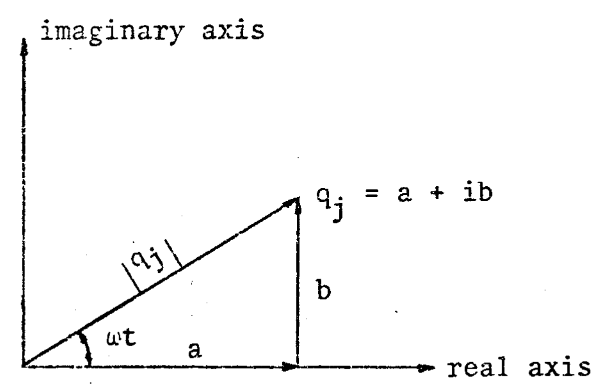


Fig. 12.

$$q_j = |q_j| e^{i\omega t} \tag{79}$$

where $i = \sqrt{-1}$ and the $|q_j|$ are the moduli of displacements q_j. Then Eqs. (79) form a set of linear second-order differential equations, which can be simplified as follows:

$\ddot{q}_j = -\omega^2 q_j \quad , \; (j = 2, 3, \cdots, 8)$ or

$$\{\ddot{q}_j\} = -\omega^2\{q_j\} \tag{80}$$

Substituting Eqs. (80) into Eqs. (76) yields

$$K\,\{q_j\} = \omega^2 M\,\{q_j\} \tag{81}$$

Since Eqs. (76) are linearly independent, this implies that the nonsingular square matrix M or K has an inverse, M^{-1} or K^{-1}, respectively. Thus,

$$K^{-1} M\,\{q_j\} = \frac{1}{\omega^2}\,\{q_j\} \tag{82}$$

Letting the dynamical matrix $[c_{ij}] = K^{-1} M$, Eqs. (82) are simplified as

$$\left([c_{ij}] - \frac{1}{\omega^2}\,[I]\right)\{q_j\} = \{0\} \tag{83}$$

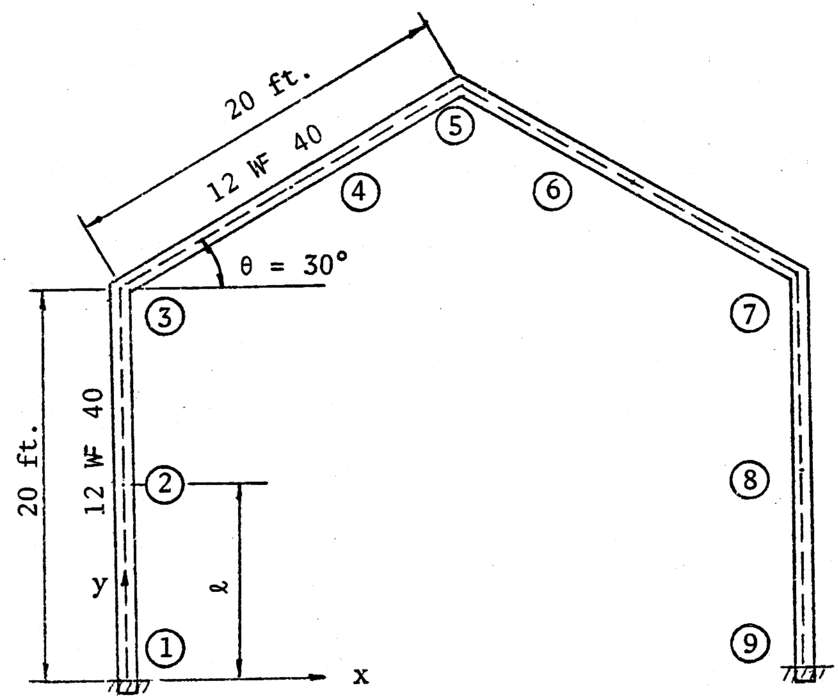
where [I] is an identity matrix. For convenience, the matrix $[c_{ij}]$ is designated as C and $\frac{1}{\omega^2}$ as λ . That is

$$(C - \lambda I)\,\{q_j\} = \{0\} \tag{84}$$

This yields an eigenvalue problem.

NUMERICAL EXAMPLE

A symmetrical gable frame with clamped ends and rigid joints is now considered and is illustrated in Fig. 13.



$\ell = 10$ ft.

$E = 29.5 \times 10^6 \#/\text{in.}^2$

$I = 310.1 \text{ in.}^4$

$A = 11.77 \text{ in.}^2$

$\frac{A\ell^2}{I} = 546.55917$

Fig. 13. A symmetrical gable frame with clamped ends and rigid connected joints.

If Eqs. (47) and (48) are expressed as the matrices (2 x 2) shown in Eqs. (52) and (62), then by substituting into Eqs. (77) and (78), the consistent mass matrix and the stiffness matrix are obtained as follows:

The consistent mass matrix is,

$$
M = m\ell \left[\begin{array}{ccccccccccccccccccccc}
\frac{26}{35} & 0 & 0 & \frac{9}{70} & 0 & \frac{13\ell}{420} \\
 & \frac{2}{3} & 0 & 0 & \frac{1}{6} & 0 \\
 & & \frac{2\ell^2}{105} & -\frac{13}{420}\ell & 0 & -\frac{1}{140}\ell^2 & & & & & & & & & & & & & & 0 \\
 & & & \frac{5}{7} & -\frac{\sqrt{3}}{105} & \frac{11}{420}\ell & \frac{11}{70} & \frac{\sqrt{3}}{105} & \frac{13\ell}{840} \\
 & & & & \frac{73}{105} & \frac{11\sqrt{3}\ell}{420} & \frac{\sqrt{3}}{105} & \frac{37}{280} & -\frac{13\sqrt{3}\ell}{840} \\
 & & & & & \frac{2\ell^2}{105} & -\frac{13\ell}{840} & \frac{13\sqrt{3}\ell}{840} & -\frac{\ell^2}{140} \\
 & & & & & & \frac{24}{35} & -\frac{2\sqrt{3}}{105} & 0 & \frac{11}{70} & \frac{\sqrt{3}}{105} & \frac{13\ell}{840} \\
 & & & & & & & \frac{76}{105} & 0 & \frac{\sqrt{3}}{105} & \frac{37}{280} & -\frac{13\sqrt{3}\ell}{840} \\
 & & & & & & & & \frac{2\ell^2}{105} & -\frac{13\ell}{840} & \frac{13\sqrt{3}\ell}{840} & -\frac{\ell^2}{140} \\
 & & & & & & & & & \frac{24}{35} & 0 & \frac{11\ell}{210} & \frac{11}{70} & -\frac{\sqrt{3}}{105} & -\frac{13\ell}{840} \\
 & & & & & & & & & & \frac{76}{105} & 0 & -\frac{\sqrt{3}}{105} & \frac{37}{280} & -\frac{13\sqrt{3}\ell}{840} \\
 & & & & & & & & & & & \frac{2\ell^2}{105} & \frac{13\ell}{840} & \frac{13\sqrt{3}\ell}{840} & -\frac{\ell^2}{140} \\
 & & & & & & & & & & & & \frac{24}{35} & \frac{2\sqrt{3}}{105} & 0 & \frac{11}{70} & -\frac{\sqrt{3}}{105} & -\frac{13\ell}{840} \\
 & & & & & & & & & & & & & \frac{76}{105} & 0 & -\frac{\sqrt{3}}{105} & \frac{37}{280} & -\frac{13\sqrt{3}\ell}{840} \\
 & & & & & & & & & & & & & & \frac{2\ell^2}{105} & \frac{13\ell}{840} & \frac{13\sqrt{3}\ell}{840} & -\frac{\ell^2}{140} \\
 & & & & & & & & & & & & & & & \frac{5}{7} & \frac{\sqrt{3}}{105} & \frac{11\ell}{420} & \frac{9}{70} & 0 & -\frac{13\ell}{420} \\
 & & & & & & & & & & & & & & & & \frac{73}{105} & -\frac{11\sqrt{3}\ell}{420} & 0 & \frac{1}{6} & 0 \\
 & & & & & & & & & & & & & & & & & \frac{2\ell^2}{105} & \frac{13\ell}{420} & 0 & -\frac{\ell^2}{140} \\
 & & \text{Symmetric} & & & & & & & & & & & & & & & & \frac{26}{35} & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & \frac{2}{3} & 0 \\
 & & & & & & & & & & & & & & & & & & & & \frac{2\ell^2}{105}
\end{array} \right] \quad (85)
$$

The stiffness matrix is,

$$
K = \frac{EI}{\ell^3}\left[\begin{array}{*{21}{c}}
24 & 0 & 0 & -12 & 0 & -6\ell \\
& \frac{2A\ell^2}{I} & 0 & 0 & -\frac{A\ell^2}{I} & 0 & & & & & & & & & & & & & & 0 \\
& & 8\ell^2 & 6\ell & 0 & 2\ell^2 \\
& & & 15+\frac{3}{4}\frac{A\ell^2}{I} & \frac{\sqrt{3}}{4}\left(\frac{A\ell^2}{I}-12\right) & 3\ell & -3-\frac{3}{4}\frac{A\ell^2}{I} & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -3\ell \\
& & & & 9+\frac{5}{4}\frac{A\ell^2}{I} & 3\sqrt{3}\ell & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -9-\frac{1}{4}\frac{A\ell^2}{I} & 3\sqrt{3}\ell \\
& & & & & 8\ell^2 & 3\ell & -3\sqrt{3}\ell & 2\ell^2 \\
& & & & & & 6+\frac{3}{2}\frac{A\ell^2}{I} & \frac{\sqrt{3}}{2}\left(\frac{A\ell^2}{I}-12\right) & 0 & -3-\frac{3}{4}\frac{A\ell^2}{I} & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -3\ell \\
& & & & & & & 18+\frac{1}{2}\frac{A\ell^2}{I} & 0 & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -9-\frac{1}{4}\frac{A\ell^2}{I} & 3\sqrt{3}\ell \\
& & & & & & & & 8\ell^2 & 3\ell & -3\sqrt{3}\ell & 2\ell^2 \\
& & & & & & & & & 6+\frac{3}{2}\frac{A\ell^2}{I} & 0 & 6\ell & -3-\frac{3}{4}\frac{A\ell^2}{I} & -\frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & 3\ell \\
& & & & & & & & & & 18+\frac{1}{2}\frac{A\ell^2}{I} & 0 & -\frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -9-\frac{1}{4}\frac{A\ell^2}{I} & 3\sqrt{3}\ell \\
& & & & & & & & & & & 8\ell^2 & -3\ell & -3\sqrt{3}\ell & 2\ell^2 \\
& & & & & & & & & & & & 6+\frac{3}{2}\frac{A\ell^2}{I} & -\frac{\sqrt{3}}{2}\left(12+\frac{A\ell^2}{I}\right) & 0 & -3-\frac{3}{4}\frac{A\ell^2}{I} & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & 3\ell \\
& & & & & & & & & & & & & 18+\frac{1}{2}\frac{A\ell^2}{I} & 0 & \frac{\sqrt{3}}{4}\left(12-\frac{A\ell^2}{I}\right) & -9-\frac{1}{4}\frac{A\ell^2}{I} & 3\sqrt{3}\ell \\
& & & & & & & & & & & & & & 8\ell^2 & -3\ell & -3\sqrt{3}\ell & 2\ell^2 \\
& & & & & & & & & & & & & & & 15+\frac{3}{4}\frac{A\ell^2}{I} & -\frac{\sqrt{3}}{4}\left(\frac{A\ell^2}{I}-12\right) & 3\ell & -12 & 0 & 6\ell \\
& & & & & & & & & & & & & & & & 9+\frac{5}{4}\frac{A\ell^2}{I} & -3\sqrt{3}\ell & 0 & -\frac{A\ell^2}{I} & 0 \\
& & & & & & & & & & & & & & & & & 8\ell^2 & -6\ell & 0 & 2\ell^2 \\
& & & & & & & & & & & & & & & & & & 24 & 0 & 0 \\
& & \text{Symmetric} & & & & & & & & & & & & & & & & & \frac{2A\ell^2}{I} & 0 \\
& & & & & & & & & & & & & & & & & & & & 8\ell^2
\end{array}\right] \qquad (86)
$$

The lumped mass matrix is,

$$
M = m\ell \begin{bmatrix}
1 &&&&&&&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&&&&&&& \\
&& 0 &&&&&&&&&&&&&&& 0 &&& \\
&&& 1 &&&&&&&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&&&&&&& \\
&&&&& 0 &&&&&&&&&&&&&&& \\
&&&&&& 1 &&&&&&&&&&&&&& \\
&&&&&&& 1 &&&&&&&&&&&&& \\
&&&&&&&& 0 &&&&&&&&&&&& \\
&&&&&&&&& 1 &&&&&&&&&&& \\
&&&&&&&&&& 1 &&&&&&&&&& \\
&&&&&&&&&&& 0 &&&&&&&&& \\
&&&&&&&&&&&& 1 &&&&&&&& \\
&&&&&&&&&&&&& 1 &&&&&&& \\
&&&&&&&&&&&&&& 0 &&&&&& \\
&&&&&&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&&&&&&& 1 &&&& \\
\text{Symmetric} &&&&&&&&&&&&&&&&& 0 &&& \\
&&&&&&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&&&&&&&&& 0
\end{bmatrix} \tag{87}
$$

MATRIX ITERATION

Matrix iteration is a convenient and useful method for solving the matrix equation as derived in Eqs. (84). Let the eigenvalues of the matrix C with order n x n be

$\lambda_1, \lambda_2, \cdots\cdots, \lambda_n$ $\qquad$ (λ_i · real positive value)

with

$$\lambda_1 > \lambda_2 > \lambda_3 > \cdots\cdots > \lambda_n \tag{88}$$

and the corresponding eigenvectors

$\{q_j\}_1, \{q_j\}_2, \cdots\cdots\cdots, \{q_j\}_n$

Then let the first arbitrary trial column vector $\overline{V}_1$ be a linear superposition of the n eigenvectors. This is possible since the eigenvectors are linearly independent in n space.

$$\overline{V}_1 = \alpha_1 \{q_j\}_1 + \alpha_2 \{q_j\}_2 + \cdots\cdots\cdots + \alpha_n \{q_j\}_n \tag{89}$$

Multiplying by the dynamical matrix gives

$$C\overline{V}_1 = \alpha_1 C \{q_j\}_1 + \alpha_2 C \{q_j\}_2 + \cdots\cdots + \alpha_n C \{q_j\}_n \tag{90}$$

From Eqs. (84), the following holds

$$\left.\begin{array}{l} C \{q_j\}_1 = \lambda_1 \{q_j\}_1 \\ C \{q_j\}_2 = \lambda_2 \{q_j\}_2 \\ -------- \\ C \{q_j\}_n = \lambda_n \{q_j\}_n \end{array}\right] \tag{91}$$

Hence Eq. (90) yields

$$C\overline{V}_1 = \overline{V}_2$$

$$= \alpha_1 \lambda_1 \{q_j\}_1 + \alpha_2 \lambda_2 \{q_j\}_2 + \cdots\cdots + \alpha_n \lambda_n \{q_j\}_n \quad (92)$$

By repeating the process, the following results:

$$C\overline{V}_2 = C^2 \overline{V}_1$$

$$= \alpha_1 \lambda_1^2 \{q_j\}_1 + \alpha_2 \lambda_2^2 \{q_j\}_2 + \cdots\cdots + \alpha_n \lambda_n^2 \{q_j\}_n \quad . \quad (93)$$

If the process is repeated, the Pth iteration will yield

$$C\overline{V}_P = C^P \overline{V}_1$$

$$= \alpha_1 \lambda_1^P \{q_j\}_1 + \alpha_2 \lambda_2^P \{q_j\}_2 + \cdots\cdots + \alpha_n \lambda_n^P \{q_j\}_n \quad , \quad (94)$$

where $\overline{V}_P$ is the Pth trial column vector. The constants α_1, α_2, α_3, $\cdots\cdots$, α_n are arbitrary.

According to Eq. (88), as P becomes sufficiently large and if the arbitrary constant α_1 is larger than the rest of the α's, the first term on the right-hand side of Eq. (94) is the only significant one; that is:

$$\overline{V}_{P+1} = C^P \overline{V}_1 = \alpha_1 \lambda_1^P \{q_j\}_1 \quad (95)$$

This means the (P + 1)th trial column vector is same as the first eigen-vector multiplied by a constant. One more iteration yields

$$\overline{V}_{P+2} = C^{P+1} \overline{V}_1 = \alpha_1 \lambda_1^{P+1} \{q_j\}_1 \quad \text{or}$$

$$C\ (C^P \overline{V}_1) = \lambda_1\ (\alpha_1 \lambda_1^P \{q_j\}_1)$$

$$= \lambda_1\ (C^P \overline{V}_1) \quad (96)$$

This implies that the first eigenvector with the largest eigenvalue λ_1 corresponding to the lowest natural frequency ω_1 is obtained.

To determine the intermediate modes and their corresponding frequencies, let a new matrix D have the same set of eigenvectors as the matrix C, and let its corresponding eigenvalues be

$$U_1, U_2, \cdots\cdots, U_n$$

The product matrix P becomes

$$P = CD \tag{97}$$

Since

$$P \{q_j\}_i = C (D \{q_j\}_i) = CU_i \{q_j\}_i = \lambda_i U_i \{q_j\}_i$$

This means the matrix P has the same set of eigenvectors as the matrix C with eigenvalues

$$\lambda_1 U_1, \lambda_2 U_2, \cdots\cdots, \lambda_n U_n$$

An alternate form of Eq. (97) is

$$P = C (\lambda_1 I - C) \tag{98}$$

which has the same set of eigenvectors as the matrix C. That is

$$\{q_j\}_1, \{q_j\}_2, \{q_j\}_3, \cdots\cdots\cdots, \{q_j\}_n$$

Its corresponding eigenvalues become

$$0 \quad \lambda_2(\lambda_1 - \lambda_2), \ \lambda_3(\lambda_1 - \lambda_3), \ \ldots\ldots, \ \lambda_n(\lambda_1 - \lambda_n)$$

This result gives us a way of evaluating intermediate modes and their corresponding frequencies.

Determining λ_1 and $\{q_j\}_1$ from the matrix C, then iterating the matrix P yields the eigenvector with its largest eigenvalue λ_p. This eigenvector cannot be $\{q_j\}_1$ again, since the eigenvalue of the matrix P in Eq. (98) corresponding to $\{q_j\}_1$ is zero and thus cannot be the largest one. This implies the iteration of the matrix P will yield its largest eigenvalue-eigenvector pair

$$\lambda_P \quad \text{and} \quad \{q_j\}_2$$

Comparing with Eq. (98) which has the eigenvalue-eigenvector pair

$$\lambda_2 \; (\lambda_1 - \lambda_2) \quad \text{and} \quad \{q_j\}_2$$

Thus

$$\lambda_2 \, (\lambda_1 - \lambda_2) = \lambda_P$$

Two roots can be evaluated from the quadratic equation, but only one root λ_2 has meaning for the matrix C. Consequently, the second natural frequency ω_2 and its corresponding mode $\{q_j\}_2$ of the matrix C are obtained.

By the same iteration process, the mth eigenvalue-eigenvector pair is

$$\lambda_Q \quad \text{and} \quad \{q_j\}_m \quad , \quad (1 < m < n)$$

where λ_Q is similar to λ_P except the mth iteration.

The mth eigenvalue of the matrix C can be solved from the polynomial equation

$$\prod_{i=1}^{m-1} \lambda_m \; (\lambda_i - \lambda_m) = \lambda_Q \tag{99}$$

which has only one meaningful root λ_m. Finally, the mth natural frequency ω_m and its corresponding mode $\{q_j\}_m$ of the matrix C are obtained.

NUMERICAL RESULTS

Using the 1410 digital computer, the results are tabulated as follows:

Modes	(1) CM	(2) LM	$\frac{CM-LM}{CM}$ %
1 Anti-Symmetric	11.3382	11.2994	0.3423
2 Symmetric	26.9528	26.9266	0.0972
3 Anti-Symmetric	61.9981	60.5726	2.2992
4 Symmetric	91.4203	92.3770	-1.0464

Table 2. Natural frequencies of a symmetric gable frame

Note:

(1) Solutions obtained using the consistent mass matrix.

(2) Solutions obtained using the lumped mass matrix.

	First mode	Second mode	Third mode	Fourth mode
q_{12}	.10000000E 01	.10000000E 01	.10000000E 01	.10000000E 01
q_{22}	.22232225E-02	.89382952E-02	.15584609E-01	.25317256E-01
q_{32}	-.48727065E 00	-.39433232E 00	-.24121173E 00	-.79708314E-01
q_{13}	.25336057E 01	.14646409E 01	.31083212E 00	-.53979782E 00
q_{23}	.44444047E-02	.17845461E-01	.30907082E-01	.49722162E-01
q_{33}	-.33575858E 00	.23974195E 00	.63110308E 00	.78407162E 00
q_{14}	.27679103E 01	.59999050E 00	-.54143045E 00	-.99286493E 00
q_{24}	-.39524185E 00	.15241593E 01	.15052583E 01	.85552963E 00
q_{34}	.34596705E-01	.60889297E 00	.50662781E-01	-.48418942E 00
q_{15}	.25418625E 01	-.11074083E-08	.32118814E 00	.15576803E-06
q_{25}	-.26361131E-08	.25641219E 01	.14422763E-06	-.81242459E 00
q_{35}	.19042079E 00	.19854212E-08	-.85238418E 00	.11804428E-06
q_{16}	.27679103E 01	-.59999051E 00	-.54143065E 00	.99286455E 00
q_{26}	.39524184E 00	.15241593E 01	-.15052584E 01	.85553018E 00
q_{36}	.34596708E-01	-.60889298E 00	.50662736E-01	.48418923E 00
q_{17}	.25336057E 01	-.14646409E 01	.31083219E 00	.53979768E 00
q_{27}	-.44444047E-02	.17845460E-01	-.30907081E-01	.49722202E-01
q_{37}	-.33575858E 00	-.23974196E 00	.63110309E 00	-.78407142E 00
q_{18}	.10000000E 01	-.10000000E 01	.10000000E 01	-.99999914E 00
q_{28}	-.22232225E-02	.89382962E-02	-.15584614E-01	.25317280E-01
q_{38}	-.48727065E 00	.39433233E 00	-.24121168E 00	.79708366E-01

Table 3. Natural modes of vibration of a symmetrical gable frame by using the consistent mass method

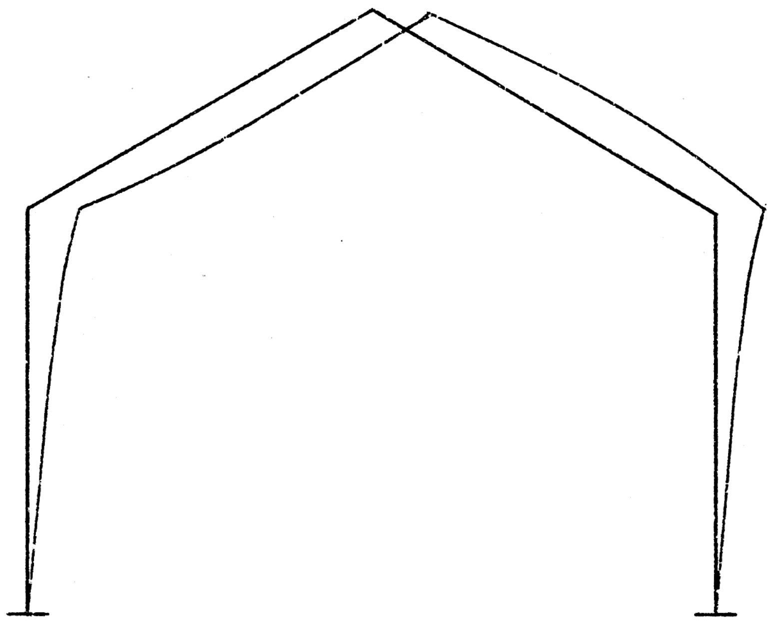
Note: q_{ij} The component of displacement along the direction i at node j.

	First mode	Second mode	Third mode	Fourth mode
q_{12}	.10000000E 01	.10000000E 01	.10000000E 01	.10000000E 01
q_{22}	.23135929E-02	.88866956E-02	.13977959E-01	.15684829E-01
q_{32}	-.48648042E 00	-.38920274E 00	-.21499177E-00	-.51066741E-01
q_{13}	.25363075E 01	.14730662E 01	.30170499E-00	-.49424806E-00
q_{23}	.46250711E-02	.17742479E-01	.27730637E-01	.30876185E-01
q_{33}	-.33676435E 00	.23109395E 00	.58850205E-00	.64915742E-00
q_{14}	.27694991E 01	.60581610E 00	-.56491566E-00	-.94563788E-00
q_{24}	-.39313532E 00	.15286707E 01	.15309259E 01	.83455128E-00
q_{34}	.35882201E-01	.60695759E 00	.50972961E-01	-.37130487E-00
q_{15}	.25446744E 01	.30423797E-09	.31518175E-00	-.15718941E-05
q_{25}	.30475190E-09	.25787531E 01	-.25847694E-08	-.75656698E-00
q_{35}	.18584876E 00	-.79162357E-09	-.82009746E-00	.16016191E-05
q_{16}	.27694991E 01	-.60581610E 00	-.56491565E-00	.94563775E-00
q_{26}	.39313532E 00	.15286706E 01	-.15309259E 01	.83455639E-00
q_{36}	.35882201E-01	-.60695759E 00	.50972963E-01	.37130380E-00
q_{17}	.25363075E 01	-.14730662E 01	.30170499E-00	.49424542E-00
q_{27}	-.46250711E-02	.17742479E-01	-.27730637E-01	.30876640E-01
q_{37}	-.33676435E 00	-.23109395E 00	.58850204E-00	-.64915575E-00
q_{18}	.99999999E 00	-.99999999E 00	.99999999E-00	-.99999094E-00
q_{28}	-.23135929E-02	.88866955E-02	-.13977959E-01	.15685062E-01
q_{38}	-.48648042E 00	.38920274E 00	-.21499176E-00	.51066918E-01

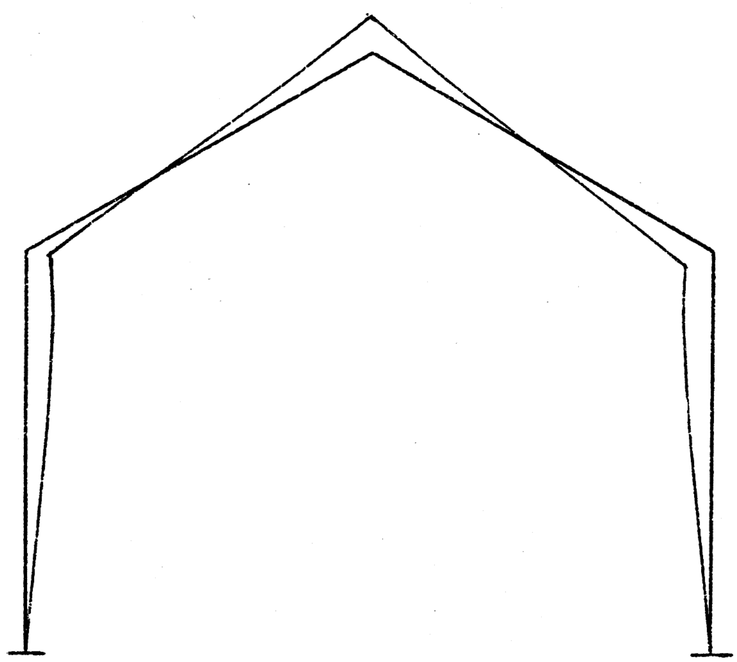
Table 4. Natural modes of vibration of a symmetrical gable frame by using the lumped mass method

Note: q_{ij} The component of displacement along the direction i at node j

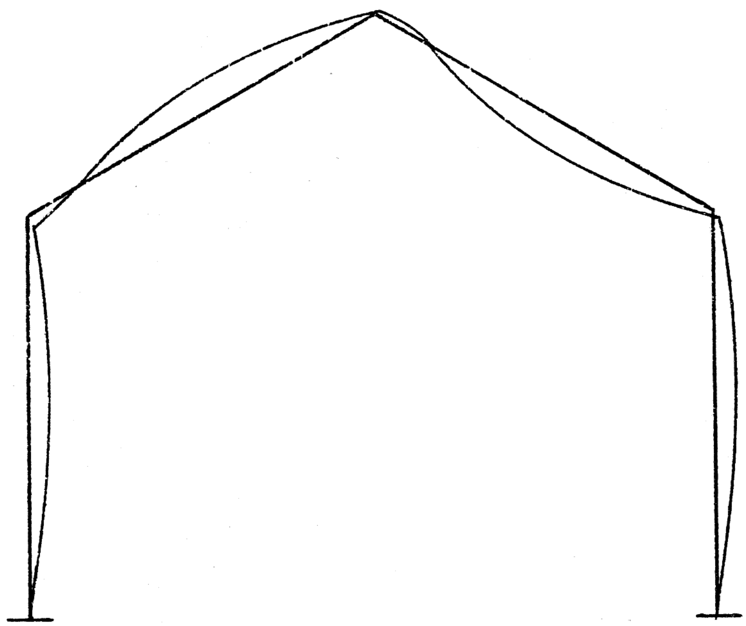
Using the data in Tables 3. and 4., the natural modes of vibration are drawn as follows:



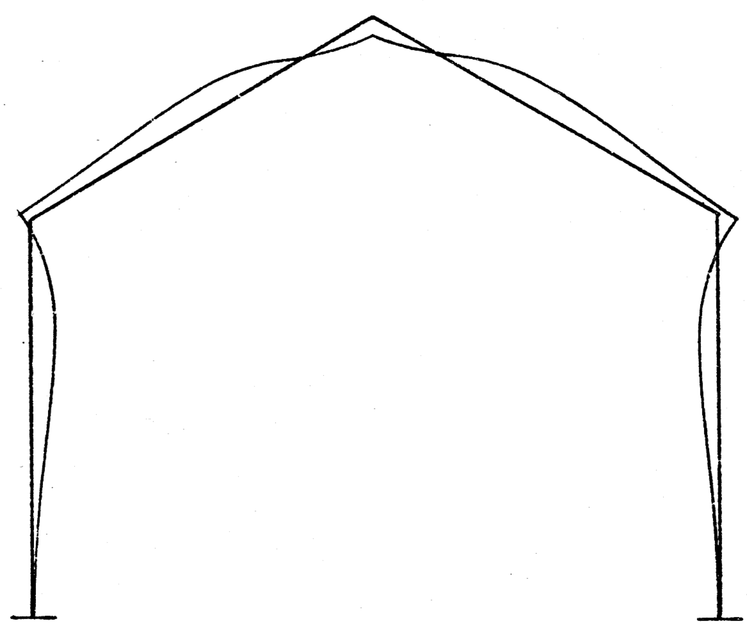
(a) First mode



(b) Second mode



(c) Third mode



(d) Fourth mode

Fig. 14. Natural modes of vibration of a symmetrical gable frame

APPENDIX

```
      EIGENVALUE PROBLEM OF FIRST MODE   CHEN-I WANG
      COMMONA(21,42)
    1 FORMAT(4E16.8)
    2 FORMAT(/)
    3 FORMAT(1H1)
      N=21
      READ(1,1) ((A(I,J),J=1,N),I=1,N)
      CALL INVRS(N)
      WRITE(3,1)((A(I,J),J=1,N),I=1,N)
      WRITE(2,1)((A(I,J),J=1,N),I=1,N)
      STOP
      END
     MON$$      EXEQ FORTRAN,,,18
      SUBROUTINE INVRS (N)
      COMMONA(21,42)
    2 FORMAT(40X,10HNO INVERSE)
      NN=N+N
      DO 130 I=1,N
      IN1=I+N
      DO 120 J=1,N
      JN1=J+N
  120 A(I,JN1)=0.
  130 A(I,IN1)=1.
      DO 100 M=1,N
    5 DIV=A(M,M)
      IF(DIV.EQ.0.0) GO TO 40
      DO 10 J=1,NN
   10 A(M,J)=A(M,J)/DIV
      DO 30 I=1,N
      IF(I.EQ.M) GO TO 30
      AMUL=A(I,M)
      DO 20 J=1,NN
   20 A(I,J)=A(I,J)-AMUL*A(M,J)
   30 CONTINUE
      GO TO 100
   40 DO 60 I=M,N
      IF(A(I,M).EQ.0.0) GO TO60
      DO 50 J=1,NN
      DUMY=A(I,J)
      A(I,J)=A(M,J)
   50 A(M,J)=DUMY
      GO TO 5
   60 CONTINUE
      WRITE (3,2)
      GO TO 110
  100 CONTINUE
      DO 140 I=1,N
      DO 140 J=1,N
      JN=J+N
  140 A(I,J)=A(I,JN)
  110 RETURN
      END
```

```
      DIMENSIONB(21,21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((B(I,J),J=1,N),I=1,N)
      DO20J=1,N
      WRITE(3,1)(B(I,J),I=1,N)
   20 WRITE(2,1)(B(I,J),I=1,N)
      STOP
      END
```

```
      DIMENSION A(21,21),B(21),C(21,21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((A(I,J),J=1,N),I=1,N)
      DO 20 M=1,N
      READ(1,1)(B(K),K=1,N)
      DO20I=1,N
      C(I,M)=0
      DO20J=1,N
   20 C(I,M)=C(I,M)+A(I,J)*B(J)
      ESUM=0
      DO 30 M=1,N
   30 ESUM=ESUM+C(M,M)
      WRITE(3,1)ESUM
      WRITE(2,1)ESUM
      WRITE(3,1)((C(I,J),J=1,N),I=1,N)
      WRITE(2,1)((C(I,J),J=1,N),I=1,N)
      STOP
      END
```

```
      COMMON C(21,21),R(21),S(21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((C(I,J),J=1,N),I=1,N)
      CALL ITER(N,WSQ)
      EI=WSQ
      WSQ=1./WSQ
      W=SQRT(WSQ)
      WRITE(3,1)WSQ,W
      WRITE(2,1)WSQ,W
      WRITE(3,1)EI
      WRITE(2,1)EI
      WRITE(3,1)(R(I),I=1,N)
      WRITE(2,1)(R(I),I=1,N)
      STOP
      END
     MON$S      EXEQ FORTRAN,,,18
      SUBROUTINE ITER (N,WSQ)
      COMMON C(21,21),R(21),S(21)
    1 FORMAT(4E16.8)
      EPSI=0.00000001
      DO 20 I=1,N
   20 R(I)=1.
   30 DO 40 I=1,N
      S(I)=0.
      DO40J=1,N
   40 S(I)=S(I)+C(I,J)*R(J)
      DO 45 I=1,N
      IF (S(I).NE.0.0)GO TO48
   45 CONTINUE
      GO TO 120
   48 WSQ=S(I)
      ERROR=0.
      DO 50 I=1,N
      RI=F(I)
      R(I)=S(I)/WSQ
      IF (ABS(R(I)-RI).GT.ERROR)  ERROR=ABS(R(I)-RI)
   50 CONTINUE
      WRITE(3,1) WSQ
      WRITE(3,1)(R(I),I=1,N)
      IF(ERROR.GT.EPSI)  GO TO 30
      GO TO 100
  120 DO 130 I=1,N
      V=I
      V=V+1./V
  130 R(I)=R(I)+V
      GO TO 30
  100 RETURN
      END
```

```
      EIGENVALUE PROBLEM OF INTERMEDIATE MODES
      DIMENSIONC(21,21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((C(I,J),J=1,N),I=1,N)
      READ(1,1) EI
      DO40I=1,N
      DO20J=1,N
   20 C(I,J)=-C(I,J)
   40 C(I,I)=EI+C(I,I)
      DO60J=1,N
      WRITE(3,1)(C(I,J),I=1,N)
   60 WRITE(2,1)(C(I,J),I=1,N)
      STOP
      END
```

```
      DIMENSIONC(21,21),D(21),E(21,21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((C(I,J),J=1,N),I=1,N)
      DO20M=1,N
      READ(1,1) (D(K),K=1,N)
      DO20I=1,N
      E(I,M)=0
      DO20J=1,N
   20 E(I,M)=E(I,M)+C(I,J)*D(J)
      WRITE(3,1)((E(I,J),J=1,N),I=1,N)
      WRITE(2,1)((E(I,J),J=1,N),I=1,N)
      STOP
      END
```

```
      COMMON C(21,21),R(21),S(21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((C(I,J),J=1,N),I=1,N)
      CALL ITER(N,WSQ)
      EI=WSQ
      WSQ=1./WSQ
      W=SQRT(WSQ)
      WRITE(3,1)WSQ,W
      WRITE(2,1)WSQ,W
      WRITE(3,1)EI
      WRITE(2,1)EI
      WRITE(3,1)(R(I),I=1,N)
      WRITE(2,1)(R(I),I=1,N)
      STOP
      END
     MON$$        EXEQ FORTRAN,,,18
      SUBROUTINE ITER (N,WSQ)
      COMMON C(21,21),R(21),S(21)
    1 FORMAT(4E16.8)
      EPSI=0.00000001
      DO 20 I=1,N
   20 R(I)=1.
   30 DO 40 I=1,N
      S(I)=0.
      DO40J=1,N
   40 S(I)=S(I)+C(I,J)*R(J)
      DO 45 I=1,N
      IF (S(I).NE.0.0)GO TO48
   45 CONTINUE
      GO TO 120
   48 WSQ=S(I)
      ERROR=0.
      DO 50 I=1,N
      RI=R(I)
      R(I)=S(I)/WSQ
      IF (ABS(R(I)-RI).GT.ERROR)  ERROR=ABS(R(I)-RI)
   50 CONTINUE
      WRITE(3,1) WSQ
      WRITE(3,1)(R(I),I=1,N)
      IF(ERROR.GT.EPSI)  GO TO 30
      GO TO 100
  120 DO 130 I=1,N
      V=I
      V=V+1./V
  130 R(I)=R(I)+V
      GO TO 30
  100 RETURN
      END
```

```
C     REAL ROOTS-NTH DEGREE EQUATION
C     C(J) IS THE COEFFICIENT OF XE**(N+1-J)
      DIMENSION C(20),D(20),Y(20)
    1 FORMAT(4E16.8)
    2 FORMAT(I5,E16.8)
    3 FORMAT(////)
      DO 60 MOON=1,1
      WRITE(3,3)
      READ(1,2)N,EPSI
      N1=N+1
      READ(1,1)(C(I),I=1,N1)
      NN=N
      IK=1
      XE=0.
      IT=0
    4 Z=C(N1)
      DO 10 J=1,N
      Y(J)=C(J)
      N2=N-J+1
      DO 5K=1,N2
    5 Y(J)=Y(J)*XE
   10 Z=Z+Y(J)
C     TEST FOR ACCURACY
      IF(ABS(Z)-EPSI)45,45,20
   20 DZ=C(N)
      IF(N-1)70,70,71
   71 N3=N-1
      DO30J=1,N3
      N2=N-J
      P=N2+1
      Y(J)=P*C(J)
      DO 25K=1,N2
   25 Y(J)=Y(J)*XE
   30 DZ=DZ+Y(J)
      GO TO 36
   70 DZ=C(1)
   36 IF(DZ)35,40,35
   35 XE=XE-Z/DZ
      GO TO 4
C     TEST TO SKIP THE EXTREMUM VALUE OF Z
   40 XE=XE+Z*5.
      IT=IT+1
      IF(IT.GT.5)GOTO 60
      GOTO 4
   45 WRITE(3,2) IK,XE
      WRITE(2,2)IK,XE
      IF(IK-NN)49,60,60
   49 IK=IK+1
      IT=0
C     REDUCING THE DEGREE OF EQUATION BY 1
      D(1)=C(1)
      DO50 I=2,N
      J=I-1
```

```
50 D(I)=C(I)+XE*D(J)
   DO 55 I=1,N
55 C(I)=D(I)
   N=N-1
   N1=N+1
   GO TO 4
60 CONTINUE
   STOP
   END
```

```
      COMMON DET, N, A(21,21), B(21,21)
    1 FORMAT(4E16.8)
      N=21
      READ(1,1) ((B(I,J),J=1,N),I=1,N)
      DO 100 IT=1,2
      READ(1,1) EI
      DO 20 I=1,N
      DO 10 J=1,N
   10 A(I,J)=B(I,J)
   20 A(I,I)=B(I,I)-EI
      CALL DETER
  100 WRITE(3,1) DET
      STOP
      END
     MON$$      EXEQ FORTRAN,,,18
      SUBROUTINE DETER
      COMMON DET, N, A(21,21), B(21,21)
    1 FORMAT(4E16.8)
      DET=1.
      DO200 M=1,N
    5 DIV=A(M,M)
      IF(DIV.EQ.0.0) GO TO 150
      DET=DET*DIV
      IF(M.EQ.N) GO TO 250
      DO 2- J=M,N
   20 A(M,J)=A(M,J)/DIV
      M1=M+1
      DO 6- I=M1,N
      AIJ=-A(I,M)
      DO 50 J=M,N
   50 A(I,J)=A(I,J)+AIJ*A(M,J)
   60 CONTINUE
      GO TO 200
  150 DO 18- I=M1,N
      IF(A(I,M).EQ.0.0) GO TO 180
      DO 160 J=M,N
      DUMY=A(M,J)
      A(M,J)=A(I,J)
  160 A(I,J)=DUMY
      GO TO 5
  180 CONTINUE
      WRITE(3)M, A(M,M), DET
      DET=*=
      GO TO 250
  200 CONTINUE
      WRITE(3) DET
  250 RETURN
      END
```

```
      COMMON N, A(21,21),B(21),C(21)
    1 FORMAT(4E16.8)
      N=21
      DO 100 IT=1,2
      READ(1,1) ((A(I,J),J=1,N),I=1,N)
      READ(1,1)  (B(I), I=1,N)
      READ(1,1) EI
      DO 20 I=1,N
   20 A(I,I)=A(I,I)-EI
  100 CALL MATMUL
      STOP
      END
     MON$$      EXEQ FORTRAN,,,18
      SUBROUTINE MATMUL
      COMMON N, A(21,21),B(21),C(21)
    1 FORMAT(4E16.8)
      DO 100 I=1,N
      C(I)=0
      DO 100 J=1,N
  100 C(I)=C(I)+A(I,J)*B(J)
      WRITE(3,1) (C(I),I=1,N)
      RETURN
      END
```

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UNDAMPED FREE VIBRATION OF A GABLE FRAME

by

CHEN I WANG

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AN ABSTRACT OF MASTER'S REPORT

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ABSTRACT

The exact solutions of the natural frequencies and modes for certain structures have been obtained in numerous papers and books. However, for gable frames, the exact solutions of the natural frequencies and modes are difficult to determine. Using the 1410 digital computer, attempts have been made to solve such difficult problems by using the consistent mass method.

In this report, undamped free vibration of a symmetrical gable frame clamped at ends and joined rigidly is considered. Since the frame is considered as four elastic beams joined rigidly and each individual beam is composed of two beam elements of equal length, this structural system has twenty-one degrees of freedom. Only the lowest four natural frequencies and natural modes are presented. Another approximate method, lumped mass, is employed to check the results obtained from previous method. The percent deviations of these two approaches for the first anti-symmetric mode and first symmetric mode are insignificant, but they tend to increase for higher modes.