

Rational Catalan combinatorial objects and algorithms

by

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Abstract

The Catalan numbers and mathematical objects enumerated by them have been studied since the 1700's. The study of these objects, related objects, and bijections between them is called Catalan combinatorics. In this thesis, we discuss three areas of Catalan combinatorics. First, we discuss rational parking functions and the Pak-Stanley bijection from rational parking functions to alcoves of the Sommers region. Second, we discuss the combinatorics of the polytope created by taking the convex hull over all vector-parking functions. Last, we discuss planar tanglegrams, and provide an algorithm to generate them uniformly at random.

An (m, n) -parking function can be characterized as a function $f : [n] \rightarrow [m]$ such that the partition obtained by reordering the values of f fits inside a right triangle with legs of length m and n . Recent work by McCammond, Thomas, and Williams define an action of words in $[m]^n$ on $\mathbb{R}^n$. They show that rational parking functions are exactly the words that admit fixed points under that action. An (m, n) -invariant set is a set $\Delta \subset \mathbb{Z}$ such that $\Delta + m \subset \Delta$ and $\Delta + n \subset \Delta$. In the first part of this thesis we define an action of words in $[m]^n$ on m -invariant sets by removing the j th m -generator from Δ . We show this action also characterizes (m, n) -parking functions. Further, we show that each (m, n) -invariant set is fixed by a unique monotone parking function. By relating the actions on $\mathbb{R}^m$ and on (m, n) -invariant sets we prove that the set of all the points in $\mathbb{R}^m$ that can be fixed by a parking function is a union of points fixed by monotone parking functions. In the case when $\gcd(m, n) = 1$ we characterize the set of periodic points of the action defined on $\mathbb{R}^m$ and show that the algorithm conjectured by Gorsky, Mazin, and Vazirani becomes affine periodic and provides an inverse to the Pak-Stanley map.

For $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{>0}^n$, a *$\mathbf{b}$ -parking function* is defined to be a sequence $(\beta_1, \dots, \beta_n)$ of positive integers whose nondecreasing rearrangement $\beta'_1 \leq \beta'_2 \leq \dots \leq \beta'_n$ satisfies $\beta'_i \leq b_1 + \dots + b_i$. The $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$ is the convex hull of all $\mathbf{b}$ -parking functions of length n in $\mathbb{R}^n$. Geometric properties of $\mathfrak{X}_n(\mathbf{b})$ were previously explored in the specific case where $\mathbf{b} = (a, b, b, \dots, b)$ and were shown to generalize those of the classical parking-function polytope. In the second part of this thesis, we study $\mathfrak{X}_n(\mathbf{b})$ in full generality. We present a minimal inequality and vertex description for $\mathfrak{X}_n(\mathbf{b})$, prove it is a generalized permutahedron, and study its h -polynomial. Furthermore, we investigate $\mathfrak{X}_n(\mathbf{b})$ through the perspectives of building sets and polymatroids, allowing us to identify its combinatorial types and obtain a bound on its combinatorial diameter.

A tanglegram consists of two rooted binary trees and a perfect matching between their leaves, and a planar tanglegram is one that admits a layout with no crossings. In the third part of this thesis we show that the problem of generating planar tanglegrams uniformly at random reduces to the corresponding problem for irreducible planar tanglegram layouts, which are known to be in bijection with pairs of disjoint triangulations of a convex polygon. We extend the flip operation on a single triangulation to a flip operation on pairs of disjoint triangulations. Interestingly, the resulting flip graph is both connected and regular, and hence a random walk on this graph converges to the uniform distribution. We also show that the restriction of the flip graph to the pairs with a fixed triangulation in either coordinate is connected and give diameter bounds that are near optimal. Our results furthermore yield new insight into the flip graph of triangulations of a convex n -gon with a geometric interpretation on the associahedron.

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Table of Contents

List of Figures	xi
List of Tables	xiv
Acknowledgements	xv
1 Catalan numbers and parking functions	1
1.1 The Catalan Numbers	1
1.1.1 Dyck Paths	2
1.1.2 Regions of Shi hyperplane arrangement in the Weyl Chamber	3
1.1.3 Triangulations of a convex $(n+2)$ -gon	4
1.1.4 Rooted binary trees with $n + 1$ leaves	5
1.2 Extensions of Catalan numbers	5
1.2.1 Rational Catalan numbers	6
1.2.2 b -Catalan numbers	9
1.3 Parking Functions	10
1.3.1 Extensions of Parking Functions	13
1.4 The Affine Symmetric Group and Affine Braid Arrangement	14
1.5 Polytopes	17
2 Periodic points of parking functions	24
2.1 Introduction	24
2.2 Background	26
2.2.1 Acting by a Parking function:	26

2.2.2	Alcoves and the Action of a Parking Function:	26
2.2.3	Invariant sets:	27
2.3	Acting on m-invariant sets	29
2.4	The associated monotone parking function	32
2.5	Periodic Points	37
2.6	Fixed points of parking functions	45
2.7	Inverting the map SP	49
3	The Parking Polytope	56
3.1	Introduction	56
3.2	Polyhedral perspectives on $\mathfrak{X}_n(\mathbf{b})$	58
3.2.1	Vertices, facets, and edges of $\mathfrak{X}_n(\mathbf{b})$	60
3.2.2	$\mathfrak{X}_n(\mathbf{b})$ is equivalent to a generalized permutahedron	71
3.2.3	h -Polynomial of $\mathfrak{X}_n(\mathbf{b})$	75
3.3	A building set perspective	79
3.4	A polymatroid perspective	89
3.5	Conclusion - Further perspectives	92
3.5.1	Relation of $\mathbf{PF}_n$ to Birkhoff/assignment polytopes	93
3.5.2	Relation of $\mathfrak{X}_n(\mathbf{b})$ to partition polytopes	96
4	Tanglegrams	98
4.1	Introduction	98
4.2	Preliminaries	102
4.3	Planar tanglegrams	104
4.4	Pairs of disjoint triangulations	108
4.4.1	A flip graph on pairs of disjoint triangulations	109
4.4.2	Rotations of irreducible planar tanglegram layouts	112
4.5	Open Problems	114

Bibliography	117
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List of Figures

1.1	The Lattice path $LULULUULUULL$	2
1.2	The Dyck path $LULLULUU$, which corresponds to the Dyck word 2110. The heights of the left steps has been listed under each left step.	3
1.3	Shi_3 where the $c_3 = 5$ regions inside the Weyl Chamber are highlighted. . .	4
1.4	A triangulation of a pentagon	4
1.5	The bijection from binary trees to triangulations.	5
1.6	The $(9, 6)$ - Dyck word $LLUULLLLLULUUU$, which has a touch point at $(6, 2)$	6
1.7	Periodic lattice path corresponding to the $(4, 5)$ -invariant set $\Delta = \{0, 3, 4, 5, \dots\}$. The 5-generators of Δ are $\{0, 3, 4, 6, 7\}$ in blue, and the 4-cogenerators are $\{-4, -1, 1, 2\}$ in red.	8
1.8	When $\mathbf{b} = (3, 1, 1, 2)$ we see that 3211 and 6432 are $\mathbf{b}$ -Dyck paths, shown in blue, $p(\mathbf{b})$ is shown in red	9
1.9	This shows that $(LULULLLUUU, \ell)$ corresponds with parking function $\mathbf{p} = 21022$	11
1.10	Shi_3 labeled by parking functions. The Weyl chamber has been highlighted.	12
1.11	Here is $\mathcal{S}_3^4$ with alcoves labeled by affine permutations. The fundamental alcove is labeled by [123].	16
1.12	The 2- and 3-dimensional standard simplices	18
1.13	The face lattice of the 2-dimensional standard simplex Δ_2	18
1.14	The face lattice of the 3-dimensional standard simplex Δ_3	19

1.15	The 2-dimensional simplex Δ_2 with each face labeled by its corresponding nested set.	23
2.1	This is $\mathcal{G}(\Delta) = \mathcal{G}(\Delta')$ from Example 2.2.8.	29
2.2	Acting on the $(5, 3)$ invariant set $\Delta = \{0, 3, 5, 6, 7, 8, 9, 11, 12, \dots\}$ by the parking function $\mathbf{p} = 103$. The 5-generators being replaced in each step is highlighted in red. The 3-generators of Δ are $\{0, 5, 7\}$ which is exactly the set of elements that are removed by $\mathbf{p}$	31
2.3	In this example $\mathbf{w} = 2100$, we consider the four points, the fixed point $x_{\mathbf{w}} = (-5, 0, 5)$ along with $a = (-4, 1, 3)$, $b = (-3, -1, 4)$, and $c = (-5, 0, 5)$. Arrows show where each point is mapped to by the parking function $\mathbf{w} = 2100$. . .	38
2.4	We track the four alcoves that contain $x_{\mathbf{w}}$, a , b and c over the action of $\mathbf{w}$. . .	38
3.1	The polytopes $\mathfrak{X}_3(1, 2, 3)$ (left) and $\mathfrak{X}_3(2, 3, 4)$ (right).	59
3.2	The vertex poset $Q_{\overline{\mathbf{v}}}$ in Lemma 3.2.15(a)	76
3.3	The vertex poset $Q_{\overline{\mathbf{v}}}$ in Lemma 3.2.15(b)	77
3.4	The nestohedron from Theorem 3.3.1.	80
3.5	The combinatorial type of $\mathfrak{X}_n(\mathbf{b})$ when $n = 3$ and $b_1 \geq 2$	88
4.1	Two layouts for the same tanglegram.	98
4.2	The flip graph $\mathcal{D}_5$. Single lines indicate when only one triangulation is changed, and double lines indicate when both are changed.	101
4.3	An example of a meander, with each noncrossing matching color-coded. . . .	101
4.4	A layout for each of the 13 tanglegrams of size four.	103
4.5	The plane dual bijection from Theorem 4.2.2.	104
4.6	A (double) flip at $(2, 4)_1$ with the corresponding irreducible planar tanglegram layouts shown.	109
4.7	A pair of disjoint triangulations where two sequences of flips $(n, 2)_2, (2, 4)_1$ and $(n, 2)_2, (1, 3)_2, (2, 4)_1$ both result in the original pair again.	112

4.8	The rotation operation on rooted binary trees. A portion of the plane duals are shown in blue.	113
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List of Tables

1.1	Computing the h polynomial using Theorem 1.5.14	21
2.1	We use Algorithm 2.7.1 to compute $U(i)$ where $p = 10002$	53
2.2	We use Algorithm 2.7.7 to compute $V(i)$ where $p = 10002$	54
4.1	The number of tanglegrams of size n with irreducible component size k	105
4.2	For each n , the number of iterations needed for the total variation distance from the uniform distribution to be smaller than $1/4$ regardless of the initial vertex chosen, and the second largest eigenvalue of the transition matrix. . .	115

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Chapter 1

Catalan numbers and parking functions

In this thesis, we present three topics based on periodic points of parking functions [Nel25], combinatorics of generalized parking-function polytopes [BBC⁺24], and sampling planar tanglegrams and pairs of disjoint triangulations [BLM⁺23] in Chapters 2, 3, 4 respectively. These topics fall under the general umbrella of Catalan combinatorics. In this chapter, we review many known results and provide background for the rest of the thesis.

1.1 The Catalan Numbers

The Catalan numbers, named after the mathematician Eugène Charles Catalan, have been known and studied since the 1700's [Pak15]. The first several Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862.$$

Many mathematical objects are enumerated by the Catalan numbers. Including Dyck paths (1.1.1), regions of Shi hyperplane arrangement in the Weyl Chamber (1.1.2), triangulations of an $(n+2)$ -gon (1.1.3), rooted binary trees with $n+1$ leaves (1.1.4). Richard Stanley collected a list of these objects that eventually became a book [Sta15] of over 200 mathematical objects

that are enumerated by the Catalan numbers. We can write a closed form for the Catalan numbers as well as a recursive form.

Definition 1.1.1 ([Sta15]). *The n th Catalan number is c_n , is given by*

$$c_n = \frac{1}{n+1} \binom{2n}{n}, \quad \text{or equivalently} \quad c_{n+1} = \sum_{i=0}^n c_i c_{n-i}, \text{ where } c_0 = 1.$$

For the rest of this section, we will highlight several mathematical objects that are enumerated by the Catalan numbers.

1.1.1 Dyck Paths

Definition 1.1.2. *A square-lattice path is a path $\mathbf{P} = P_1 P_2 \dots P_{2n}$ on a $n \times n$ grid starting at $(0, n)$, ending at $(n, 0)$, where the path only takes left steps and up steps. One can record the order of steps as a list of L 's and U 's for left steps and up steps, respectively, or draw the path on the $n \times n$ grid.*

Example 1.1.3. *Figure 1.1 shows the lattice path $LULULULUUULL$.*

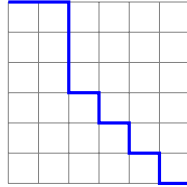


Figure 1.1: The Lattice path $LULULULUUULL$

Definition 1.1.4. *A Dyck path $\mathbf{P} = P_1 P_2 \dots P_{2n}$ is a lattice path where the path stays below or equal to the line $x/n + y/n = 1$*

Definition 1.1.5. *A touch point of a Dyck path is a point of the Dyck path on the line $x/n + y/n = 1$ that is not $(0, n)$ or $(n, 0)$.*

Example 1.1.6. *Figure 1.2 shows the Dyck path $LULLULUU$. Which has one touch point at $(3, 1)$.*

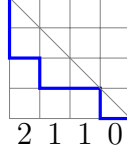


Figure 1.2: The Dyck path $LULLULUU$, which corresponds to the Dyck word 2110. The heights of the left steps has been listed under each left step.

Theorem 1.1.7. *[Sta15] The number of Dyck paths on an $n \times n$ grid is given by c_n*

Definition 1.1.8. *A Dyck word is a word $\mathbf{d} = d_{n-1}d_{n-2}\dots d_0$ in $\{0, 1, \dots, n-1\}$ such that $d_0 \leq d_1 \leq \dots \leq d_{n-1}$ where $d_i \leq i$.*

Dyck words and Dyck paths are in bijection. The bijection between Dyck words and Dyck paths is obtained by taking the heights of the left steps of a Dyck path.

Example 1.1.9. *The Dyck word from the Dyck path in Figure 1.2 is $\mathbf{d} = 2110$.*

1.1.2 Regions of Shi hyperplane arrangement in the Weyl Chamber

The Shi arrangement will be discussed again in Section 1.4. The hyperplane arrangements in this thesis are written in n variables but are of rank $n-1$. For this and other reasons we will consider all our hyperplane arrangements in W where

$$W = \{(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n : x_0 + x_1 + \dots + x_{n-1} = 0\}.$$

Definition 1.1.10 ([Shi87]). *The Shi arrangement is defined by the hyperplanes*

$$Shi_n = \{\{x_i - x_j = k\} : 0 \leq i < j \leq n-1, k = \{0, 1\}\}, \text{ in } W.$$

Definition 1.1.11. *The Weyl Chamber V^n is the subset of W given by*

$$V^n = \{(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n : x_0 + x_1 + \dots + x_{n-1} = 0, x_0 \leq x_1 \leq \dots \leq x_{n-1}\}.$$

Definition 1.1.12. A region of a hyperplane arrangement $\mathcal{H}$ is a connected component of $W \setminus \mathcal{H}$.

Example 1.1.13. Figure 1.3 Shows Shi_3 , where the regions in the Weyl Chamber are highlighted. Notice that there are $c_3 = 5$ regions in the Weyl Chamber.

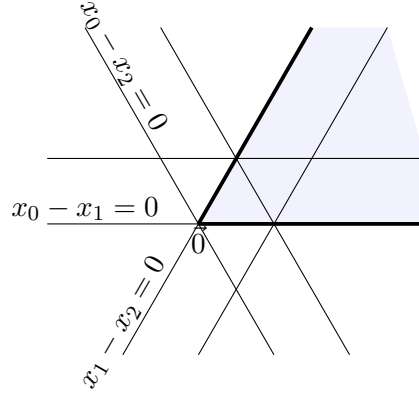


Figure 1.3: Shi_3 where the $c_3 = 5$ regions inside the Weyl Chamber are highlighted.

In Section 1.3 we will Observe that these regions are in bijections in Dyck paths.

1.1.3 Triangulations of a convex $(n+2)$ -gon

Consider a convex polygon whose vertices are labeled with $1, 2, \dots, n$. A diagonal of a convex polygon is the line between vertices i and j such that i and j are not adjacent vertices.

Definition 1.1.14. A triangulation of a convex polygon is a polygon with the maximal number of diagonals such that no diagonals intersect except at vertices.

Example 1.1.15. In Figure 1.4 a triangulation of a pentagon is shown.

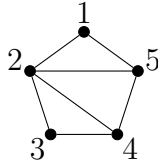


Figure 1.4: A triangulation of a pentagon

Theorem 1.1.16. [Sta15] The number of triangulations of a convex $(n + 2)$ -gon is c_n .

1.1.4 Rooted binary trees with $n + 1$ leaves

A tree is rooted if there is some special vertex called the root, which acts as the start of the tree.

Definition 1.1.17. A rooted binary tree $T = (V, E)$ is a tree where the root has degree 2 and all other nodes have degree 3 or 1.

We sometimes draw an edge from the root that does not connect to any vertex. This will also make the root a node of degree 3 and is helpful in some bijections.

Example 1.1.18. Figure 1.5 shows a binary tree.

Theorem 1.1.19. [Sta15] The set of binary planar trees with $n+1$ leaves and triangulations of a convex $(n+2)$ -gon are in bijection. A tree is sent to its planar dual with each face labeled by $\{1, 2, \dots, n\}$ clockwise.

Example 1.1.20. Figure 1.5 has an example of the bijection between binary trees with $n+1$ leaves and triangulations of a convex $(n+2)$ -gon.

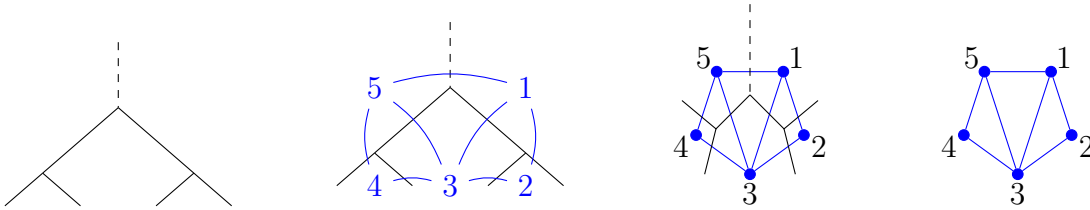


Figure 1.5: The bijection from binary trees to triangulations.

We will be using this bijection extensively in Chapter 4.

1.2 Extensions of Catalan numbers

There are many extensions of the Catalan numbers. In this work, we focus on two of them, rational Catalan numbers and **b**-Catalan numbers. Both can be motivated by constructing a generalized version of Dyck paths and defining the generalized Catalan number as the number of generalized Dyck paths.

1.2.1 Rational Catalan numbers

Catalan numbers count the number of Dyck paths in an $n \times n$ square grid. The rational Catalan numbers are motivated by changing the square grid to a rectangle.

Definition 1.2.1. A (m, n) - Dyck path is a lattice paths in an $m \times n$ grid from $(0, m)$ to $(n, 0)$ that stays weakly below the line $x/n + y/m = 1$.

The definitions for Dyck words and touch points remain analogous to Dyck paths in a square.

Example 1.2.2. Figure 1.6 shows a $(9, 6)$ -Dyck word $LLLUULLLLLULUUU$, which has a touch point at $(6, 2)$ and whose Dyck word is given by $\mathbf{d} = 322222000$.

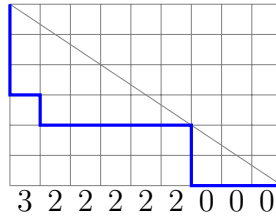


Figure 1.6: The $(9, 6)$ - Dyck word $LLLUULLLLLULUUU$, which has a touch point at $(6, 2)$.

Theorem 1.2.3. [Biz54] The rational Catalan number $c_{m,n}$ is given by the number of (m, n) -Dyck paths. When m and n are coprime, there is a closed formula for rational Catalan numbers given by

$$c_{m,n} = \frac{1}{m+n} \binom{m+n}{n}.$$

When m and n are not coprime, there is a generating function for $c_{m,n}$. Let $m' = m/\gcd(m, n)$ and $n' = n/\gcd(m, n)$ then

$$\sum_{d=0}^{\infty} c_{dm', dn'} z^d = \exp \left(\sum_{k=1}^{\infty} \frac{(km' + kn' - 1)!}{(km')!(kn')!} z^k \right).$$

This is a bit of a miraculous formula, because the terms $\frac{(km' + kn' - 1)!}{(km')!(kn')!}$ are not always integers, but the result is.

Example 1.2.4. Let us demonstrate this formula by finding, $c_{9,6}$ i.e. the number of $(9,6)$ -Dyck paths. We have $(m', n') = (3, 2)$ and we are finding the coefficient of z^3 . Then

$$\begin{aligned}
\sum_{d=0}^{\infty} c_{dm', dn'} z^d &= \exp \left(\sum_{k=1}^{\infty} \frac{(3k+2k-1)!}{(3k)!(2k)!} z^k \right) \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{(3+2-1)!}{(3)!(2)!} z + \frac{(6+4-1)!}{(6)!(4)!} z^2 + \frac{(9+6-1)!}{(9)!(6)!} z^3 + \dots \right) \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{4!}{3!2!} z + \frac{9!}{6!4!} z^2 + \frac{14!}{9!6!} z^3 + \dots \right) \\
&= \exp \left(\sum_{k=1}^{\infty} 2z + 21z^2 + \frac{1001}{3} z^3 + \dots \right) \\
&= 1 + \left(2z + 21z^2 + \frac{1001}{3} z^3 + \dots \right) + \frac{1}{2} (2z + 21z^2 + \dots)^2 + \frac{1}{6} (2z + \dots)^3 + \dots \\
&= 1 + 2z + \left(21 + \frac{1}{2} 4 \right) z^2 + \left(\frac{1001}{3} + \frac{1}{2} 84 + \frac{1}{6} 8 \right) z^3 + \dots \\
&= 1 + 2z + 23z^2 + 377z^3 + \dots
\end{aligned}$$

From this we can see that $c_{9,6} = 377$

The rational Catalan numbers enumerate more than just (m, n) -Dyck paths and (m, n) -Dyck words. They count the number of alcoves in the Sommers Region inside the Weyl Chamber, which we will show in Chapter 2. They also count (m, n) -invariant sets, (m, n) -cores, maximal identity Deograms.

Definition 1.2.5. A set $\Delta \subset \mathbb{Z}$ is an n -invariant set if $\Delta + n \subset \Delta$. A set $\Delta \subset \mathbb{Z}$ is an (m, n) -invariant set Δ if it is both an m -invariant set and n -invariant set. Denote the set of all (m, n) -invariant sets with $M_{m,n}$.

If Δ is an (m, n) -invariant set for any $k \in \mathbb{Z}$ th $\Delta + k$ is still an (m, n) -invariant set. We often consider (m, n) -invariant sets up to these shifts. We have several choices for a representative of each class.

Definition 1.2.6. An (m, n) -invariant set Δ is 0-normalized if $\min(\Delta) = 0$. We say Δ is 0-balanced if $\#\mathbb{Z}_{\geq 0} \setminus \Delta = \#\mathbb{Z}_{< 0} \cap \Delta$.

Theorem 1.2.7. [GM13] *The number of 0-normalized (m, n) -invariant sets is $c_{m,n}$.*

Definition 1.2.8. *If Δ is an n -invariant set, we say that $a \in \Delta$ is an n -generator if $a - n \notin \Delta$. We say $a \notin \Delta$ is an n -cogenerator if $a + n \in \Delta$.*

Definition 1.2.9. *The skeleton of $\Delta \in M_{m,n}$ is the union of m -generators and n -cogenerators.*

When $\gcd(m, n) = 1$ we can visualize (m, n) -invariant sets as periodic lattice paths. Consider a lattice with boxes labeled by $l(x, y) = -nx - my$, where (x, y) are coordinates of the northeast corner of each box. Observe that the labeling is periodic with period $(m, -n)$. Given an (m, n) -invariant set Δ the corresponding periodic path $P(\Delta)$ is the boundary of the union of boxes labeled by integers in Δ . The m -generators are the labels just below $P(\Delta)$ and the n -cogenerators are the labels just right of $P(\Delta)$. The skeleton of Δ are the labels of all boxes with the northeast corner on $P(\Delta)$. (see Figure 1.7).

0	-4	-8								
5	1	-3	-7							
10	6	2	-2	-6						
15	11	7	3	-1	-5	-9				
	16	12	8	4	0	-4	-8			
			13	9	5	1	-3	-7		
				14	10	6	2	-2	-6	-10
					15	11	7	3	-1	-5
							12	8	4	0

Figure 1.7: Periodic lattice path corresponding to the $(4, 5)$ -invariant set $\Delta = \{0, 3, 4, 5, \dots\}$. The 5-generators of Δ are $\{0, 3, 4, 6, 7\}$ in blue, and the 4-cogenerators are $\{-4, -1, 1, 2\}$ in red.

One can recover the Catalan numbers from the rational Catalan numbers:

$$c_n = c_{n+1,n}$$

1.2.2 **b**-Catalan numbers

Catalan numbers count the number of Dyck paths in a $n \times n$ square grid. The **b**-Catalan numbers are motivated by picking a lattice path and counting all the lattice paths under it.

Definition 1.2.10. Let $\mathbf{b} = (b_0, b_1, \dots, b_{n-1}) \in (\mathbb{Z}_{>0})^n$. A word $\mathbf{d} = d_{n-1}d_{n-1} \dots d_0$ in $(\mathbb{Z}_{>0})^n$ is a **b**-Dyck word if $d_i \leq b_0 + b_1 + \dots + b_i$ and $d_0 \leq d_1 \leq \dots \leq d_{n-1}$.

Note the discrepancy with the minimal value being 1 and not 0. Using 1 as the minimal value follows the notation from [Yan01, Yan00, SP02]. These partial sums play an important role. Given $\mathbf{b}$ let $S_i = b_0 + b_1 + \dots + b_i$. If we consider the word $S_{n-1}S_{n-1} \dots S_0$ this corresponds to a lattice path on a $n \times S_{n-1}$ grid.

Definition 1.2.11. Given $\mathbf{b}$ define $p(\mathbf{b})$ as the lattice path on a $n \times S_{n-1}$ grid where the horizontal steps have heights $S_{n-1}, S_{n-1}, \dots, S_0$.

A consequence of having 1 as our minimum value is that the left steps of $p(\mathbf{b})$ all have unique heights.

Definition 1.2.12. A **b**-Dyck path is any lattice path starting at $(n, 1)$ and ending on $(0, S_n)$ that is weakly below $p(\mathbf{b})$.

Example 1.2.13. Let $\mathbf{b} = (3, 1, 1, 2)$, we calculate $\mathbf{S} = (7, 5, 4, 3)$. See that 3211 and 6432 are **b**-Dyck paths. As illustrated on Figure 1.8

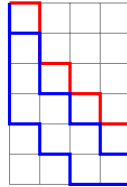


Figure 1.8: When $\mathbf{b} = (3, 1, 1, 2)$ we see that 3211 and 6432 are **b**-Dyck paths, shown in blue, $p(\mathbf{b})$ is shown in red

1.3 Parking Functions

Parking functions arise as a list of parking car preferences, first studied by Konheim and Weiss in [KW66a]. Imagine n cars driving on a one-way street with n parking spots ahead. Each car has a preference $w_i = f(i) \in [n] = \{0, 1, \dots, n-1\}$ for the parking spot in which it wants to park. Each car tries to park in its preferred spot. If the preferred parking spot is available, it does; if not, the car parks at the first available parking spot after. If all cars can park, then the function $f : \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$ is a parking function. We write these functions as words $\mathbf{p} = p_{n-1}p_{n-2} \dots p_1p_0$ where $p_i = f(i)$. The requirement that all cars park can be summarized by

$$\#\{j : p_j < i\} \geq i \text{ for } 0 \leq i \leq n-1.$$

Any permutation of a parking function is also a parking function.

Example 1.3.1. *All parking functions of length 3 are given by*

$$000, 100, 010, 001, 110, 101, 011, 002, 020, 200, 012, 021, 102, 120, 201, 210$$

Theorem 1.3.2. [KW66b] *There are $(n+1)^{n-1}$ parking functions of length n .*

If $\mathbf{p} = p_{n-1}p_{n-2} \dots p_1p_0$ is a parking function and $a_0 \leq a_1 \leq \dots \leq a_{n-1}$ is its nondecreasing rearrangement then $\mathbf{a} = a_{n-1}a_{n-2} \dots a_0$ has the property that $a_i \leq i$. This means that $\mathbf{a} = a_{n-1}a_{n-2} \dots a_0$ is a Dyck word. In fact, parking functions can be seen as labeled Dyck paths.

Definition 1.3.3. *A labeling ℓ of a Dyck path D is a bijection from the left steps of $\mathbf{P} = P_1P_2 \dots P_{2n}$ to $\{0, 1, \dots, n-1\}$ such that if P_i and P_{i+1} are both horizontal steps, then $\ell(P_i) < \ell(P_{i+1})$.*

Theorem 1.3.4. *Labeled Dyck paths are in bijection with parking functions. The bijection is given by setting p_i to be the height of the left step P_j where $\ell(P_j) = i$.*

Example 1.3.5. Consider the Dyck path $LULULLLUUU$ and ℓ given by $\ell(P_0) = 2, \ell(P_2) = 1, \ell(P_4) = 0, \ell(P_5) = 3$ and $\ell(P_6) = 4$. This Dyck path corresponds to the Dyck word $\mathbf{d} = 22210$ and (P, ℓ) corresponds to the parking function 21022 . This is illustrated in Figure 1.9.

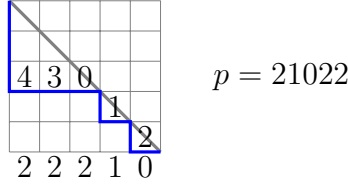


Figure 1.9: This shows that $(LULULLLUUU, \ell)$ corresponds with parking function $\mathbf{p} = 21022$

Next we want to build the bijection from parking functions to regions of the Shi arrangement. This bijection is called the Pak-Stanley labeling. To do this we will develop some notation.

Definition 1.3.6. Let $H_{i,j}^k = x_i - x_j = k$ in $W = \{(x_0, \dots, x_{n-1}) \in \mathbb{R}^n : \sum_{i=0}^{n-1} x_i = 0\}$.

Then the Shi arrangement can be seen as

$$Shi_n = \{H_{i,j}^k : 0 \leq i < j < n, k \in \{0,1\}\}$$

Another important hyperplane arrangement is the affine braid arrangement, which is

$$Br_n = \{H_{i,j}^k : 0 \leq i < j < n, k \in \mathbb{Z}\}$$

The closures of regions of the affine braid arrangement are called alcoves. The fundamental alcove is given by

$$(x_0, \dots, x_{n-1}) \in W : x_0 < x_1 < \dots < x_{n-1} < x_0 + 1$$

Theorem 1.3.7. [Sta98] Parking functions of length n are in bijection with regions of the Shi arrangement.

Following the form of this bijection given in [Maz17]. The Pak-Stanley labeling can be defined inductively using the following rules: Label the fundamental alcove with the zero function $00 \dots 0$. Then follow 2 rules.

- When you cross the hyperplane $H_{i,j}^1$, moving further away from the fundamental alcove, increase $f(i)$ by 1.
- When you cross the hyperplane $H_{i,j}^0$, moving further away from the fundamental alcove, increase $f(j)$ by 1.

If R is some region and $H_R \subset Shi_n$ is the collection of hyperplanes that separate R from the fundamental alcove, then the parking function labeling R is

$$p_R(a) = \#\{H_{i,j}^1 : i = a\} - \#\{H_{i,j}^0 : j = a\}$$

Example 1.3.8. Figure 1.10 Shows Shi_3 , where each region has been labeled by a parking function according to [Maz17]. Notice that the alcoves in the Weyl Chamber are labeled by Dyck words.

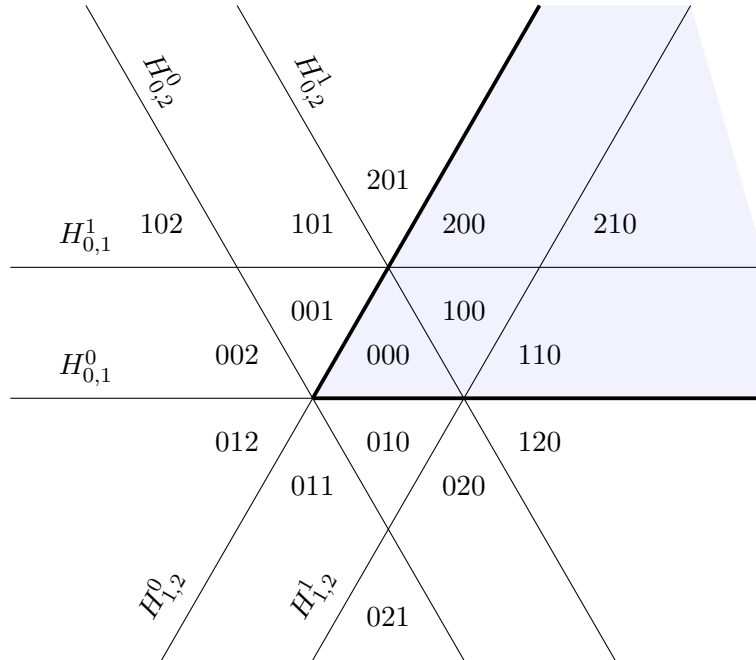


Figure 1.10: Shi_3 labeled by parking functions. The Weyl chamber has been highlighted.

Theorem 1.3.9. *Dyck words are in bijection with regions of the Shi arrangement inside the Weyl chamber.*

1.3.1 Extensions of Parking Functions

The extensions of parking functions in this thesis are both motivated by Theorem 1.3.4.

Definition 1.3.10. *An (m, n) -parking functions is a word $\mathbf{w} = w_{n-1}w_{n-2}\dots w_1w_0 \in [m]^n$ such that*

$$\#\{j : w_j < i\} \geq \frac{in}{m} \text{ for } 1 \leq i \leq m.$$

This is equivalent to a pair (D, ℓ) a rational (m, n) -Dyck path D and row labeling ℓ . And we recover classical parking functions by considering $(n+1, n)$ -parking functions.

Theorem 1.3.11. *[AB15] When m and n are coprime, there are $(m)^{n-1}$ rational parking functions.*

Example 1.3.12. *Observe that*

$$020031, 001230, \text{ and } 321100$$

are all $(4, 6)$ -parking functions.

Definition 1.3.13. *Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{>0}^n$. A $\mathbf{b}$ -parking function is a list $(\beta_1, \dots, \beta_n)$ of positive integers whose nondecreasing rearrangement $\beta'_1 \leq \beta'_2 \leq \dots \leq \beta'_n$ satisfies $\beta'_i \leq b_1 + \dots + b_i$.*

This is equivalent to a pair (D, ℓ) of a $\mathbf{b}$ -Dyck path D and row labeling ℓ . And we recover parking functions by considering $(1, 1, \dots, 1)$ -parking functions.

Example 1.3.14. *See that 3212 is a $(1, 3, 1, 1)$ - parking function, and that 3452 is a $(2, 1, 1, 1)$ - parking function.*

In general, if $b_1 = 1$ then 1 must be the lowest value of a generalized parking function. In Chapter 3 we will observe many dichotomies on if $b_1 = 1$ or $b_1 > 1$.

There is another prominent extension of parking functions, called graphical parking functions. Although there is not much overlap between these types of parking functions and $\mathbf{b}$ -parking functions there is some, which is summarized in [GH16].

1.4 The Affine Symmetric Group and Affine Braid Arrangement

In previous sections, we have had a few results about hyperplane arrangement. Here we unify some of the notation and define the affine symmetric group and affine braid arrangement. There are various bijections from parking functions to regions or alcoves of hyperplane arrangements. Let $H_{i,j}^k = \{x_i - x_j = kn\}$.

Definition 1.4.1. *The affine braid arrangement is the hyperplane arrangement $\mathcal{H} = \{H_{i,j}^k : 0 \leq i < j \leq n-1, k \in \mathbb{Z}\}$ inside of $W = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=0}^{n-1} x_i = 0\}$.*

Note that we scale our hyperplane arrangements by n . The regions of the hyperplane arrangement H are the connected components of $W \setminus H$. The closure of a region of the affine braid arrangement is referred to as an alcove.

Definition 1.4.2. *The fundamental alcove A_0 is the alcove with boundary conditions given by $x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_0 + n$ inside W .*

Note the fundamental alcove has boundaries given by $H_{i-1,i}^0$ for $i \in \{1, \dots, n-1\}$ and $H_{n-1,0}^1$.

Definition 1.4.3. *The height of $H_{i,j}^k$ is $|j-i-nk|$. Note that the height $H_{i,j}^k$ is proportional to the distance between $H_{i,j}^k$ and the center of A_0 .*

The hyperplanes that bound the fundamental alcove are exactly those of height 1.

Definition 1.4.4 ([Shi87]). *The Shi arrangement is $Shi_n = \{H_{i,j}^k : 0 \leq i < j \leq n-1, k = 0 \text{ or } k = 1\}$.*

The Shi arrangement is a sub-hyperplane arrangement of the affine braid arrangement. The regions of Shi_n were shown to be in bijection with classical parking functions in [Sta98] by Stanley with correspondence from Pak.

Definition 1.4.5. *The k -Shi arrangement is given by $k\text{-}Shi_n = \{H_{i,j}^k : 0 \leq i < j \leq n-1, k \in \{-k+1, -k+2, \dots, k\}\}$*

The Pak-Stanley construction extends to the k -Shi case, however, the argument is rather technical. For a more geometric approach, see [Maz17], which took inspiration from the work of Hopkins and Perkinson in [HP15]. For a more general bijection, we need to introduce the Sommers region.

Definition 1.4.6 ([Som05]). *The Sommers region $\mathcal{S}_n^m$ is the region in W bounded by all hyperplanes $H_{i,k}^k$ of height m .*

Theorem 1.4.7 ([MTW19], Theorem 6.20). *The alcoves inside the Sommers region are in bijection with parking functions in PW_n^m .*

The Sommers region and k -Shi arrangements are related through the affine symmetric group.

Definition 1.4.8. *The affine symmetric group $\widetilde{S}_n$ consists of bijections $\omega : \mathbb{Z} \rightarrow \mathbb{Z}$ such that*

$$1) \omega(x+n) = \omega(x) + n \qquad 2) \sum_{k=0}^{n-1} \omega(k) = \frac{(n-1)n}{2}.$$

We write elements of $\widetilde{S}_n$ in window notation $\omega = [\omega(0), \omega(1), \dots, \omega(n-1)]$. The generators $\widetilde{S}_n$ are given by $s_i = [0, 1, \dots, i-1, i+1, i, i+2, \dots, n-1]$ for $i \in \{1, \dots, n-1\}$ and $s_0 = [-1, 1, 2, \dots, n-2, n]$. $\widetilde{S}_n$ acts on W with the generator s_i acting by reflection across the hyperplane $H_{i-1,i}^0$ for $0 < i < n$ and s_0 acting by reflecting across the hyperplane $H_{n-1,0}^1$. This action sends alcoves to alcoves. Further, the action is free and transitive on alcoves. The map $\omega \rightarrow \omega(A_0)$ gives a bijection from $\widetilde{S}_n$ to the alcoves of the affine braid arrangement.

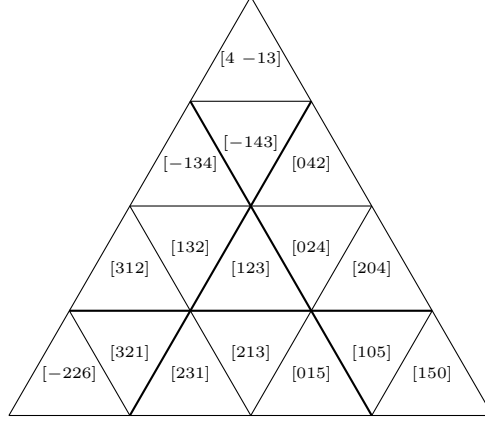


Figure 1.11: Here is $\mathcal{S}_3^4$ with alcoves labeled by affine permutations. The fundamental alcove is labeled by $[123]$.

The affine permutations that correspond to alcoves inside the Sommers region are called m -restricted, the set of these affine permutations is denoted ${}^m\widetilde{S}_n$, it can be seen that

$${}^m\widetilde{S}_n = \{\omega \in \widetilde{S}_n : \omega^{-1}(x) < \omega^{-1}(x+m), \forall x \in \mathbb{Z}\}.$$

If $\omega^{-1} \in {}^m\widetilde{S}_n$ then it is called m -stable. The set of m -stable affine permutations is

$$\widetilde{S}_n^m = \{\omega \in \widetilde{S}_n : \omega(x) < \omega(x+m), \forall x \in \mathbb{Z}\}.$$

Theorem 1.4.9 ([FV10], Theorem 7.1). *The alcoves inside $\mathcal{S}_n^{kn+1}$ region are in bijection with the regions of the k -Shi $_n$. The bijection takes an alcove labeled by $\omega \in \widetilde{S}_n^m$ and maps it to the region containing the alcove labeled by ω^{-1} . The alcove labeled by ω^{-1} is the unique alcove in that region with the least number of hyperplanes $H_{i,j}^k$ between it and A_0 .*

Definition 1.4.10. *The centroid of an alcove is the unique point $(x_0, x_1, \dots, x_{m-1})$ in the interior of the alcove such that all the numbers*

$$\frac{m+1}{2} - mx_0, \quad \frac{m+1}{2} - mx_1, \quad \frac{m+1}{2} - mx_{m-1}$$

are all integers.

Theorem 1.4.11 (Lemma 2.9 [GMV14]). *Centroids of alcoves exist. Moreover, if $(x_0, \dots, x_{m-1}) \in \omega(A_0)$ is the centroid of $\omega(A_0)$ then*

$$\omega^{-1} = \left[\frac{m+1}{2} - mx_0, \quad \frac{m+1}{2} - mx_1, \quad \frac{m+1}{2} - mx_{m-1} \right]$$

1.5 Polytopes

There are 2 equivalent definitions for polytopes.

Definition 1.5.1 (Vertex description). *A polytope is the convex hull of a finite set of points in $\mathbb{R}^n$.*

Definition 1.5.2 (Inequality description). *A polytope is the intersection of finitely many closed half-spaces in $\mathbb{R}^n$.*

Throughout the this section and the thesis, let e_i be the i th standard basis vector in $\mathbb{R}^n$

Example 1.5.3. *Consider the $(n-1)$ -standard simplex Δ_{n-1} given by the convex hull of $\{e_i : 0 \leq i \leq n-1\}$. The standard simplex is in $\mathbb{R}^n$ and has dimension $n-1$. The inequality description if the standard simplex is given by*

$$\{(x_0, \dots, x_{n-1}) \in \mathbb{R}^n : x_0 + \dots + x_{n-1} = 1, x_i \geq 0 \text{ for } 0 \leq i \leq n-1\}$$

We will use the 2- and 3-dimensional standard simplices as examples in this section.

Theorem 1.5.4. (Weyl-Minkowski Theorem) *All polytopes have both a vertex description and an equivalent inequality description.*

Definition 1.5.5. *Given a polytope P and $\lambda \in (\mathbb{R}^n)^*$ the set of points on P where λ is maximized is a face of P .*

When a face is a single point, we call it a vertex, and when a face is 1-dimensional, we call it an edge. A facet is a face whose dimension is one less than the dimension of P . Note

that a face of a polytope is also a polytope. Polytopes being congruent is more rigid than what we will need; instead, we consider when polytopes are combinatorially equivalent.

Definition 1.5.6. *The face lattice of a polytope P is the poset of faces of P with ordered by inclusion. A minimal element given by the empty set is usually added to this poset. Two polytopes are combinatorially equivalent if their face lattices are isomorphic.*

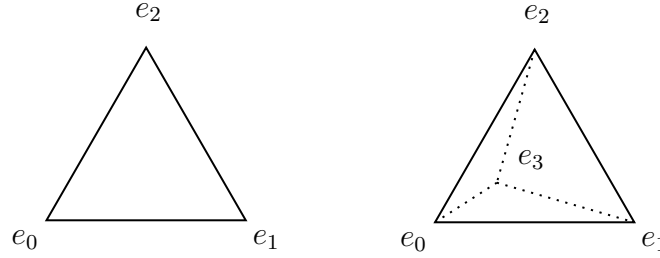


Figure 1.12: The 2- and 3-dimensional standard simplices

Example 1.5.7. *The face lattices of the 2- and 3-dimensional standard simplices are given by Figures 1.13 and 1.14. In the face lattice we have used $e_{i_1} \dots e_{i_k}$ as a shorthand for the face that is the convex hull of $e_{i_1}, \dots, e_{i_k}$.*

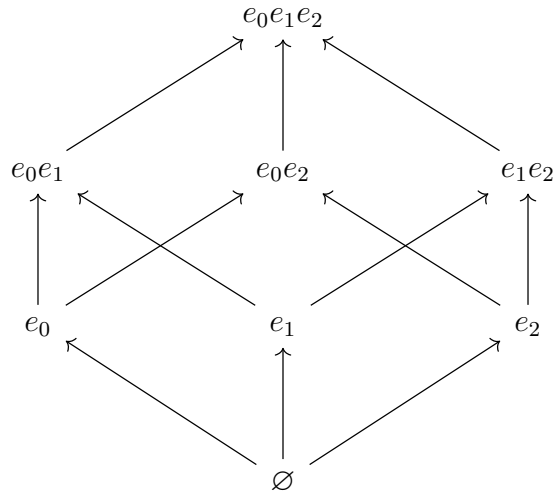


Figure 1.13: The face lattice of the 2-dimensional standard simplex Δ_2

Definition 1.5.8. *Let $f_i(P)$ denote the number of i -dimensional faces of P . The f -polynomial of P is then defined as $f(P; t) := \sum_{i=0}^n f_i(P) t^i$ and the h -polynomial of P is the result of applying a change of parameter given by $h(P; t) := f(P; t - 1)$.*

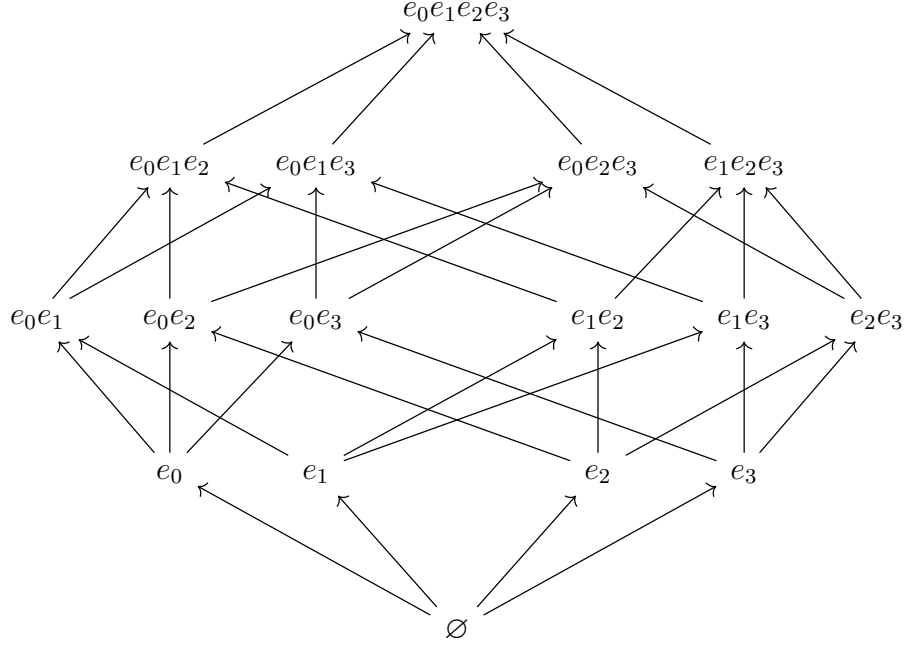


Figure 1.14: The face lattice of the 3-dimensional standard simplex Δ_3

Example 1.5.9. Consider the 2-dimensional standard simplex, $f(\Delta_2) = (3, 3, 1)$. The f -polynomial is $f(\Delta_2, t) = 3 + 3t + t^2$, and the h polynomial is

$$h(\Delta_2; t) = 3 + 3(t - 1) + (t - 1)^2 = 1 + t + t^2.$$

Consider the 3-dimensional, then $f(\Delta_3) = (4, 6, 4, 1)$. The f -polynomial is $f(\Delta_3, t) = 4 + 6t + 4t^2 + t^3$, and the h polynomial is

$$h(\Delta_3; t) = 4 + 6(t - 1) + 4(t - 1)^2 + (t - 1)^3 = 1 + t + t^2 + t^3.$$

Definition 1.5.10. A generalized permutahedron is a polytope P such that every edge is parallel to $e_i - e_j$ for some $0 \leq i, j \leq n$. A simple polytope is a d -dimensional polytope where each vertex is adjacent to exactly d edges.

Observe that for any two vertices v and w that $\sum v_i = \sum w_i = k$ and there is for some

$k > 0$ and unique $i, j \in [n]$ such that

$$w - v = k(e_i - e_j).$$

Definition 1.5.11. *The tangent cone $T_v(P)$ at a vertex v of a polytope is given by*

$$T_v(P) = \left\{ \sum_{w \sim v} \lambda_w (w - v) : \lambda_w \geq 0 \right\}.$$

Where $w \sim v$ mean w and v are adjacent vertices. Intuitively the tangent cone is the set of rays from v that point into P .

Definition 1.5.12. *The polar cone C° of a set C in $\mathbb{R}^n$ is the given by*

$$C^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \text{ for all } x \in C\}.$$

The polar cone is the negative of the dual cone.

Definition 1.5.13. *The normal cone $N_v(P)$ of a vertex v is the polar cone of the tangent cone at v . That is $N_v(P) = T_v(P)^\circ$.*

For a generalized permutahedron P and vertex v observe that

$$\begin{aligned} T_v(P) &= \left\{ \sum_{w \sim v} \lambda_w (w - v) : \lambda_w \geq 0 \right\} \\ &= \left\{ \sum_{w \sim v} \lambda_w (e_{i_w} - e_{j_w}) : \lambda_w \geq 0 \right\}. \end{aligned}$$

Where $w - v = k_w(e_{i_w} - e_{j_w})$ for some $i_w, j_w \in [n]$ and $k_w > 0$. Note that i_w and j_w are unique as $k_w > 0$. The generating vectors $\{e_{i_w} - e_{j_w} : w \sim v\}$ of the tangent cone are the facet-defining outer normals of the normal cone. Which means the normal cone at v is the intersection of closed-half spaces

$$N_v(P) = \bigcap_{w \sim v} \{x_{i_w} \leq x_{j_w}\}.$$

Vertex	Normal Cone	Descents	Term of $h(P, t)$
e_0	$\{x_1 \leq x_0\} \cap \{x_2 \leq x_0\}$	$(1, 0), (2, 0)$	t^2
e_1	$\{x_0 \leq x_1\} \cap \{x_2 \leq x_1\}$	$(2, 1)$	t
e_2	$\{x_0 \leq x_2\} \cap \{x_1 \leq x_2\}$	none	1

Table 1.1: Computing the h polynomial using Theorem 1.5.14

One can associate to each vertex v a *vertex poset* Q_v by setting $i \leq_{Q_v} j$ whenever $x_i \leq x_j$ is in the normal cone of v . We denote the covering relations by $\triangleleft$. The *descent set* $\text{Des}(Q_v)$ of Q_v is the set of ordered pairs (i, j) such that $i \triangleleft_{Q_v} j$ but $i >_{\mathbb{Z}} j$. The *number of descents* of Q_v is $\text{des}(Q_v) := |\text{Des}(Q_v)|$.

Theorem 1.5.14. [PRW08, Theorem 4.2] *When P is a simple generalized permutahedron its h -polynomial is given by*

$$h(P; t) = \sum_{v \in \text{Vert}(P)} t^{\text{des}(Q_v)}.$$

Example 1.5.15. *Consider the 2 dimensional standard simplex. Table 1.1 summarizes the computation of $h(P, t) = 1 + t + t^2$ using Theorem 1.5.14.*

Definition 1.5.16 ([Pos09, Definition 7.1]). *Let E be a finite set. A building set on E is a collection $\mathcal{B}$ of nonempty subsets of E that satisfies the conditions:*

(B1) *If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.*

(B2) *The singleton $\{i\}$ lies in $\mathcal{B}$ for all $i \in E$.*

In this case, we write $\mathcal{B}_{\max} \subset \mathcal{B}$ for the collection of maximal subsets in $\mathcal{B}$ under inclusion.

Definition 1.5.17 ([PRW08, Definition 6.3]). *Let $\mathcal{B}$ be a building set in $[n]$. A nestohedron $P_{\mathcal{B}}$ is a $\mathcal{Y}$ -generalized permutahedron with positive coefficients y_I only for $I \in \mathcal{B}$: $P_n^{\mathcal{Y}}(\{y_I\}) = \sum_{I \in \mathcal{B}} y_I \Delta_I$. This is a Minkowski sum where Δ_I is the convex hull of $\{e_i : i \in I\}$.*

We will relate nestohedra to certain complexes formed from building sets.

Definition 1.5.18 ([PRW08, Definition 6.4]). *Let $\mathcal{B}$ be a building set on a finite set E . A nested set of $\mathcal{B}$ is a subset $N \subseteq \mathcal{B} \setminus \mathcal{B}_{\max}$ that satisfies the following conditions:*

(N1) If $I, J \in N$, then either $I \subseteq J$ or $I \supseteq J$ or $I \cap J = \emptyset$.

(N2) If $J_1, \dots, J_k \in N$ are pairwise disjoint with $k \geq 2$, then $J_1 \cup \dots \cup J_k \notin \mathcal{B}$.

From this definition, it follows that a subset of a nested set is a nested set. Therefore, the collection of all the nested sets of $\mathcal{B}$ forms an abstract simplicial complex with vertex set $\mathcal{B}$, called the *nested set complex* $\Delta_{\mathcal{B}}$ of $\mathcal{B}$. It turns out that the nestohedron $P_{\mathcal{B}}$ is always a simple polytope, and the face lattice of $P_{\mathcal{B}}$ does not depend on the values of the positive parameters y_I , but only on $\mathcal{B}$, and this face lattice is dual to the face lattice of the nested set complex $\Delta_{\mathcal{B}}$; see [Pos09, Theorem 7.4].

Example 1.5.19. The collection $\mathcal{B} = \{\{0\}, \{1\}, \{2\}, \{0, 1, 2\}\}$ is a building set on $[3]$ with $\mathcal{B}_{\max} = \{0, 1, 2\}$. Note that a nested set is a subset of $\mathcal{B}$. Each set $\{i\}$ is abbreviated as i . The corresponding nested sets and their dimensions as faces in $\Delta_{\mathcal{B}}$ are as follows:

$$\left\{ \begin{array}{l} \text{dim} = -1 \quad \emptyset \\ \hline \text{dim} = 0 \quad \{0\}, \{1\}, \{2\}, \\ \hline \text{dim} = 1 \quad \{0, 1\}, \{0, 2\}, \{1, 2\} \end{array} \right\}.$$

Observe that $P_{\mathcal{B}}$ is a shift by $(1, 1, 1)$ the two-dimensional standard simplex. Similarly for the building set $\mathcal{B} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1, 2, 3\}\}$ observe that $P_{\mathcal{B}}$ is a shift by $(1, 1, 1, 1)$ the three-dimensional standard simplex.

Each of the vertices corresponds to a maximal nested set. In fact, each of the faces of the nestohedron corresponds to the nested set, which is the intersection of the maximal nested sets that correspond to the vertices of the face.

Example 1.5.20. In Figure 1.15 is the standard two-dimensional simplex with each face labeled by its corresponding nested set.

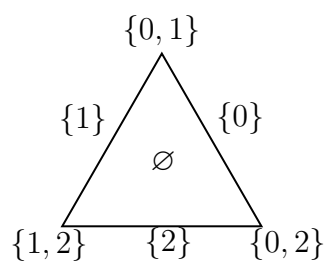


Figure 1.15: The 2-dimensional simplex Δ_2 with each face labeled by its corresponding nested set.

Chapter 2

Periodic points of parking functions

This is a chapter dedicated to periodic points of parking functions. It follows the work in [\[Nel25\]](#).

2.1 Introduction

Parking functions were first defined in [\[KW66b\]](#), and the generalization to rational parking functions was studied in [\[GMV14, ALW15b\]](#). In [\[MTW19\]](#) Jon McCammond, Hugh Tomas, and Nathan Williams give a new characterization of (m, n) -parking functions. They define an action of words in $[m] = \{0, 1, \dots, m-1\}$ on the Weyl Chamber

$$V^m = \{\mathbf{x} = (x_0, x_1, \dots, x_{m-1}) : \sum_i x_i = 0 \text{ and } x_0 \leq x_1 \leq \dots \leq x_{m-1}\},$$

where each letter i acts on $\mathbf{x} \in V^m$ by adding m to x_i , subtracting 1 from each coordinate, then resorting the coordinates. Then a word $w \in [m]^n$ acts on $\mathbf{x} \in V^m$ from right to left. The authors of [\[GMV17\]](#) prove that $\mathbf{w}$ is an (m, n) -parking function if and only if $\mathbf{w}$ has a fixed point in V^m . They further show that the set of fixed points is a convex set of affine dimension $\gcd(m, n) - 1$.

In [\[GMV17\]](#) Eugene Gorsky, Mikhail Mazin, and Monica Vazirani study (m, n) -Dyck paths when m and n are not relatively prime and (m, n) -invariant sets of $\mathbb{Z}$. A set $\Delta \subset \mathbb{Z}$

is (m, n) invariant if $\Delta + m \subset \Delta$ and $\Delta + n \subset \Delta$. When m and n are coprime, there is a bijection from (m, n) -Dyck paths to 0-normalized (m, n) -invariant sets. When m and n are not coprime the authors define an equivalence relation on (m, n) -invariant sets such that the equivalence classes are in bijection with (m, n) -Dyck paths.

In this dissertation, we connect the ideas from [MTW19] and [GMV17]. We define an action of words in $[m]^n$ on bounded and co-bounded m -invariant sets. An (m, n) -invariant set Δ is bounded and co-bounded if there are an $N, M \in \mathbb{N}$ such that $\mathbb{Z}_{\leq -N} \cap \Delta = \emptyset$ and $\mathbb{Z}_{\geq M} \subset \Delta$. The m -generators of Δ are given by $(\Delta + m) \setminus \Delta = \{a_1 < \dots < a_m\}$. The letter $i \in [m]$ will act on Δ by removing the a_i from Δ . A word in $[m]^n$ acts from right to left. We prove:

Theorem 2.1.1. *Let $\mathbf{w}$ be a word in $[m]^n$ and Δ an m -invariant set such that $\mathbf{w} \cdot \Delta = \Delta + n$ then $\mathbf{w}$ is an (m, n) -parking function and Δ is (m, n) invariant. Furthermore for any (m, n) -parking function there exists an (m, n) -invariant set such that $\mathbf{w} \cdot \Delta = \Delta + n$.*

We connect this action to the bijection $\mathcal{G}$ defined in [GMV17] from the equivalence classes of (m, n) -invariant sets to the set of (m, n) -Dyck paths.

Theorem 2.1.2. *The monotone parking function given by the column heights of $\mathcal{G}(\Delta)$ is the unique monotone parking function such that $\mathbf{w} \cdot \Delta = \Delta + n$.*

When $\gcd(m, n) = 1$ we characterize the points in V^m that are periodic under the action of a parking function.

Theorem 2.1.3. *If $\gcd(m, n) = 1$ and $\mathbf{p}$ an (m, n) -parking function with fixed point $\mathbf{x}_p \in V^m$, then $\mathbf{x}_p$ is the centroid of an alcove A_p of the hyperplane arrangement $\mathcal{H} = \{x_i - x_j = km, 1 \leq i < j \leq n, k \in \mathbb{Z}\}$. The set periodic points of $\mathbf{p}$ is exactly $A_p \setminus \mathbf{x}_p$ and all points in $A_p \setminus \mathbf{x}_p$ have period n .*

Structure of this chapter: In Section 2, we provide background on parking functions, the action of words in $[m]^n$ on V^m from [MTW19], the affine braid arrangement, and invariant sets. In Section 3, we define an action on m -invariant sets and then prove Theorem 1.1.

In Section 4, we focus on monotone parking functions and prove Theorem 1.2. In Section 5, we show that the set of fixed points of a parking function is the union of fixed points of monotone parking functions. In Section 6, we consider the case when $\gcd(m, n) = 1$ and classify all the periodic points of an (m, n) -parking function proving Theorem 1.3. In section 7, as a corollary of Theorem 1.3 we are able to resolve conjecture 7.9 from [GMV14] showing an algorithm to invert the map SP does finish in a finite number of steps.

2.2 Background

2.2.1 Acting by a Parking function:

In [MTW19] Jon McCammond, Hugh Tomas, and Nathan Williams build a new characterization of (m, n) -parking functions based on an action. Words in $\{0, 1, \dots, m-1\} = [m]$ act on the Weyl chamber

$$V^m = \{\mathbf{x} = (x_0, x_1, \dots, x_{m-1}) \in \mathbb{R}^m : \sum_i x_i = 0 \text{ and } x_0 \leq x_1 \leq \dots \leq x_{m-1}\},$$

where i acts on $\mathbf{x}$ by adding m to the i th component, subtracting 1 from every component, then resorting. Differing from [MTW19] we use the convention that words act one letter at a time from right to left. If $w \in [m]^n$ we can think of $\mathbf{w}$ as a piecewise-linear function from V^m to V^m .

Theorem 2.2.1 ([MTW19], Theorem 1.1). *Let $w \in [m]^n$, then $\mathbf{w}$ is an (m, n) -parking functions if and only if $\mathbf{w}$ has a fixed point in V^m . The set of fixed points is convex of affine dimension $\gcd(m, n) - 1$ and contained in an affine subset of dimension $\gcd(m, n) - 1$. In particular, when m and n are coprime, each parking function has a unique fixed point.*

2.2.2 Alcoves and the Action of a Parking Function:

We return to a few results about how the action of parking functions relates to alcoves.

Theorem 2.2.2 ([MTW19], Lemma 2.3). *The action of a word $\mathbf{w} \in [m]^n$ on V^m is a weak contraction. If A is an alcove, then $w|_A : A \rightarrow w(A)$ is a linear isometry and $w(A)$ is an alcove.*

Indeed, if $\mathbf{x}, \mathbf{v} \in A$ then acting by letter k translates $\mathbf{x}$ and $\mathbf{v}$ to the same alcove, where they are on the same side of every simple hyperplane $H_{i-1,i}^0$. Thus, these points will be resorted in the same way.

Theorem 2.2.3 ([MTW19], Lemma 2.3). *When m and n are coprime, the unique fixed point of a parking function $\mathbf{p} \in PW_n^m$ is the centroid of an alcove.*

In [GMV14] the authors define a Pak-Stanely map $SP : {}^m\widetilde{S}_n \rightarrow PF_n^m$ given by

$$SP_\omega(\alpha) := \# \{ \beta \mid \beta > \alpha, 0 < \omega(\beta) - \omega(\alpha) < m \}.$$

In conjecture 7.9 of [GMV14] the authors give a possible algorithm to invert SP . In [MTW19] the authors show SP is a bijection. Theorem 2.5.12 shows that the algorithm from [GMV14] converges.

2.2.3 Invariant sets:

We follow the notation from [GMO22] for the equivalence classes defined in [GMV17] for (m, n) -invariant sets. Let $\gcd(m, n) = d \geq 1$ and $\Theta = \{\Delta_0, \Delta_1, \dots, \Delta_{d-1}\}$ be 0-balanced $(m/d, n/d)$ -invariant sets. For every $(x_0, x_1, \dots, x_{d-1}) \in \mathbb{R}^d$ such that $\sum_{i=0}^{d-1} x_i = 0$ let

$$\Delta(x_0, \dots, x_{d-1}) = \bigcup_{i=0}^{d-1} (d\Delta_i + x_i).$$

For each Δ_i let S_i be its skeleton. Consider the space of all possible shifts $x_0, \dots, x_{d-1}$ and consider $\Sigma_\Theta \subset \{\mathbf{x} \in \mathbb{R}^{d-1} : \sum_{i=0}^{d-1} x_i = 0\}$ consisting of all shifts $x_0, \dots, x_{d-1}$ for which there exists i and j such that

$$(dS_i + x_i) \cap (dS_j + x_j) \neq \emptyset.$$

See that Σ_Θ is a hyperplane arrangement. If a shift $(x_0, x_1, \dots, x_{d-1})$ is integer and there is an x_i in each remainder $\pmod{d}$, then $\Delta(x_0, x_1, \dots, x_{d-1})$ is an (m, n) -invariant set.

Definition 2.2.4. *Two (m, n) -invariant sets are equivalent if they correspond to the same connected component of Σ_Θ . See that we get hyperplane arrangements for each tuple Θ of $(m/d, n/d)$ -invariant sets.*

To get all the equivalence classes of bounded and co-bounded (m, n) -invariant sets, one should consider all possible d -tuples of 0-balanced $(m/d, n/d)$ -invariant sets and for each such d -tuple consider the set of connected components of the complement to Σ_Θ in the space of shifts. These should be considered up to symmetry; if two of the sets are equal $\Delta_i = \Delta_j$, then switching the corresponding shift coordinates x_i and x_j interchanges the connected components corresponding to the same equivalence class of (m, n) -invariant sets. Furthermore, if Θ' is a permutation of Θ then the regions correspond to the same permutation of shifts.

In [GMV17] the authors define a bijection $\mathcal{G}$ from equivalence classes of (m, n) -invariant sets $M_{m,n/\sim}$ to (m, n) -Dyck paths $Y_{m,n}$, by recording the order in which n -cogenerators and m -generators are in the skeleton S .

Definition 2.2.5. *Given an (m, n) -invariant set Δ have skeleton S . Let $\mathcal{G} : M_{m,n/\sim} \rightarrow Y_{m,n}$ be the map where an equivalence class of (m, n) -invariant sets is mapped to an (m, n) -Dyck path in the following way. The path of $\mathcal{G}(\Delta)$ can be obtained by rearranging the skeleton of Δ in increasing order, replacing n -generators by up steps and m -cogenerators by left steps, from bottom right to top left.*

Proposition 2.2.6. [ALW15a], proposition 3.2] $\mathcal{G}(\Delta)$ is an (m, n) -Dyck path.

Theorem 2.2.7 ([GMV17], Theorem 1.1). *The map $\mathcal{G}$ is a bijection.*

Example 2.2.8. *Consider the two $(6, 4)$ -invariant sets*

$$\Delta = \mathbb{Z}_{\geq 0} \setminus \{1, 2, 3, 5, 6, 7\}$$

$$\Delta' = \mathbb{Z}_{\geq 0} \setminus \{1, 2, 3, 5, 7, 9, 13\}.$$

Whose respective skeletons are

$$S = \{-4, 0, 2, 4, 5, 8, 9, 11, 13, 17\}$$

$$S' = \{-4, 0, 2, 4, 7, 8, 11, 13, 15, 19\},$$

where we have colored 4-cogenerators in **red** and 6-generators in **blue**. These are in the same region of Σ_θ where $\Theta = \{\mathbb{Z}_{\geq 0} \setminus \{1\}, \mathbb{Z}_{\geq 0} \setminus \{1\}\}$, which means they are in the same equivalence class. This can be seen as the odd generators and cogenerators of Δ can be shifted by 2 without crossing any even generators or cogenerators, making Δ' . Figure 2.1 has the result of $\mathcal{G}(\Delta)$ and $\mathcal{G}(\Delta')$.

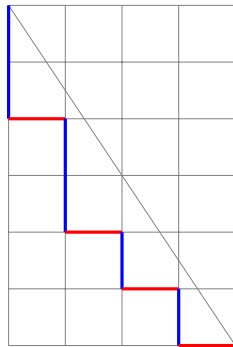


Figure 2.1: This is $\mathcal{G}(\Delta) = \mathcal{G}(\Delta')$ from Example 2.2.8.

2.3 Acting on m -invariant sets

Definition 2.3.1. A subset $\Delta \subset \mathbb{Z}$ is an m -invariant set if $\Delta + m \subset \Delta$. The set of all m -invariant sets is denoted by M_m .

Definition 2.3.2. We define an action of words in $[m] = \{0, 1, \dots, m-1\}$ on the set M_m of m invariant sets. Let $\{a_0 < a_1 < \dots < a_{m-1}\} = \Delta \setminus (\Delta + m)$ be the m -generators of Δ . For $j \in [m]$ set $j \cdot \Delta = \Delta \setminus \{a_j\}$, note that $\Delta \setminus \{a_j\}$ is an m -invariant set. Given a word $w = w_{n-1} \dots w_0$ in $[m]$ we act on Δ one letter at a time from right to left

$$\mathbf{w} \cdot \Delta = w_{n-1}(\dots(w_1(w_0\Delta))\dots).$$

Definition 2.3.3. We define a map $\phi : M_m \rightarrow V^m$ from m -invariant sets into the Weyl Chamber by setting

$$\phi(\Delta) = (a_0, a_1, \dots, a_{m-1}) - \frac{\sum_{j=0}^{m-1} a_j}{m} \vec{1},$$

where $a_0, a_1, \dots, a_{m-1}$ are the m -generators of Δ and $\vec{1} = (1, 1, \dots, 1)$.

Lemma 2.3.4. The map $\phi : M_m \rightarrow V^m$ commutes with actions of words in $[m]$ on V^m and M_m . The image $\phi(\Delta)$ is the centroid of an alcove.

Proof. See that $\sum_{i=1}^{m-1} (a_i - \frac{\sum_{j=1}^{m-1} a_j}{m}) = 0$, thus $\phi(\Delta) \in V^m$ and ϕ is well defined. The m -generators of $i \cdot \Delta$ are

$$\{a_0, \dots, a_{i-1}, a_{i+i}, \dots, a_m, a_i + m\},$$

so

$$\phi(i \cdot \Delta) = \text{sort}((a_0, a_1, \dots, a_i + m, \dots, a_{m-1}) - \vec{1}(\sum_{j=0}^{m-1} (a_j - 1)/m))$$

sorted. This agrees with $i \cdot \phi(\Delta)$. As the set of m -generators of Δ contains exactly one number from each equivalence class modulo m , its clear that $\phi(\Delta)$ is a centroid by Definition 1.4.10. □

Theorem 2.3.5. If $\mathbf{w} \cdot \Delta = \Delta + n$ then $\mathbf{w}$ is an (m, n) -rational parking function.

Proof. The map ϕ sends shifts of Δ to the same point so

$$\mathbf{w} \cdot \phi(\Delta) = \phi(\mathbf{w} \cdot \Delta) = \phi(\Delta + n) = \phi(\Delta).$$

Thus $\mathbf{w}$ has a fixed point in V^m and is an (m, n) -parking function. □

From now on, we denote $\mathbf{p}$ as an (m, n) -parking function.

Lemma 2.3.6. If $\mathbf{p} \cdot \Delta = \Delta + n$ then Δ is an (m, n) -invariant set.

Proof. We know $\mathbf{p} \cdot \Delta \subset \Delta$, so $\Delta + n \subset \Delta$ thus Δ is (m, n) -invariant. □

A slightly stronger statement is true. At every step we must have an (m, n) -invariant set. Let $\Delta^{(i)} = p_{i-1} \dots p_0 \Delta$ where $\Delta^{(0)} = \Delta$.

Corollary 2.3.7. *If $\mathbf{p} \cdot \Delta = \Delta + n$ then $\Delta^{(i)}$ is an (m, n) -invariant set for all $0 \leq i \leq n$.*

Proof. Consider $\Delta^{(i)}$ and set $\mathbf{p}' = p_{i-1} \dots p_0 p_n \dots p_i$ then

$$\mathbf{p}' \cdot \Delta^{(i)} = p_{i-1} \dots p_0 p_n \dots p_i p_{i-1} \dots p_0 \Delta = p_{i-1} \dots p_0 \cdot (\Delta + n) = \Delta^{(i)} + n$$

By Theorem 2.3.6 $\Delta^{(i)}$ is an (m, n) -invariant set. □

Example 2.3.8. *See Figure 2.2 for the action of $p = 103$ on the $(5, 3)$ invariant set $\Delta = \mathbb{Z} \setminus \{1, 2, 4\}$. We show that $p \cdot \Delta = \Delta + 3$, at each step, we record the 5-generators and $\Delta^{(i)}$. You can also notice that at each step $\Delta^{(i)}$ is an (m, n) -invariant set and we remove elements that are also n -generators. Both of these will happen every time you act by a word such that $\mathbf{p} \cdot \Delta = \Delta + n$.*

w	5 - gen	$\Delta^{(i)}$
3	$\{0, 3, 6, \textcolor{red}{7}, 9\}$	$\{0, 3, 5, 6, \textcolor{red}{7}, 8, 9, 10, 11, 12, 13, 14, \dots\}$
0	$\{\textcolor{red}{0}, 3, 6, 9, 12\}$	$\{\textcolor{red}{0}, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, \dots\}$
1	$\{3, \textcolor{red}{5}, 6, 9, 12\}$	$\{3, \textcolor{red}{5}, 6, 8, 9, 10, 11, 12, 13, 14, \dots\}$
	$\{3, 6, 9, 10, 12\}$	$\{3, 6, 8, 9, 10, 11, 12, 13, 14, \dots\}$

Figure 2.2: Acting on the $(5, 3)$ invariant set $\Delta = \{0, 3, 5, 6, 7, 8, 9, 11, 12, \dots\}$ by the parking function $\mathbf{p} = 103$. The 5-generators being replaced in each step is highlighted in red. The 3-generators of Δ are $\{0, 5, 7\}$ which is exactly the set of elements that are removed by $\mathbf{p}$.

Lemma 2.3.9. *Let $\mathbf{p} = p_{n-1}, \dots, p_0$, if $\mathbf{p} \cdot \Delta = \Delta + n$ and a_{p_i} is the p_i th m -generator of $\Delta^{(i)}$ then a_{p_i} is an n -generator of $\Delta^{(i)}$. That is, at each step, we remove elements that are also n -generators of $\Delta^{(i)}$.*

Proof. We know $\Delta^{(i)}$ and $\Delta^{(i+1)}$ are (m, n) -invariant sets from Theorem 2.3.7 and $\Delta^{(i)} \setminus \Delta^{(i+1)} = \{a_{\omega_i}\}$. If a_{p_i} is not an n -generator of $\Delta^{(i)}$ then $a_{p_i} - n \in \Delta^{(i)}$ and $a_{p_i} - n \in \Delta^{(i+1)}$ but $a_{p_i} \notin \Delta^{(i+1)}$ which contradicts $\Delta^{(i+1)}$ being an (m, n) -invariant set. $\square$

Lemma 2.3.10. *If $\mathbf{p} \cdot \Delta = \Delta + n$ and a_{p_i} is the p_i th m -generator of $\Delta^{(i)}$ then a_{p_i} is an n -generator of Δ . That is at each step we remove elements that are also n -generators of Δ the original set.*

Proof. The action of a letter removes one element of Δ . The set of n -generators are given by $\Delta \setminus (\Delta + n)$, which means none of the n -generators of Δ are in $\mathbf{p} \cdot \Delta = \Delta + n$. As $\mathbf{p}$ has length n at each step, we must remove one of the n -generators of Δ . $\square$

Lemma 2.3.11. *If $\mathbf{p} \cdot \Delta = \Delta + n$ then the set $\cup_{i=0}^n \mathbf{mgen}(\Delta^{(i)})$ is the skeleton of $\Delta + n$ as an (n, m) -invariant set, note the switch in n and m . It is the union of m -generators of $\Delta + n$ and n -cogenerators of $\Delta + n$.*

Proof. First we show the skeleton S of $\Delta + n$ is a subset of $\cup_{i=0}^n \mathbf{mgen}(\Delta^{(i)})$. In Lemma 2.3.10 it's shown that of $\mathbf{p}$ only removes the n -generators of Δ . If $x \in \mathbf{mgen}(\Delta^{(i)})$ then either it gets removed at some step of the action or is in $\mathbf{mgen}(\Delta + n)$. If x gets removed, then x must have been an n -generator of Δ . Observe that n -generators of Δ are n -cogenerators of $\Delta + n$.

Now we show the opposite inclusion. As $\mathbf{p} \cdot \Delta = \Delta + n$ we know $\mathbf{p}$ must remove all n -generators of Δ . For any n -generator x of Δ there is some step where it is removed. That is there is some $i \in \{1, \dots, n-1\}$ such that $\Delta^{(i-1)} \setminus \Delta^{(i)} = \{x\}$ and $x \in \mathbf{mgen}(\Delta^{(i-1)})$ $\square$

The skeleton of Δ as an (m, n) -invariant set S , and as an (n, m) -invariant set S' are related. See that $S' = S + n - m$.

2.4 The associated monotone parking function

In this section, we define the associated monotone parking function to an (m, n) -invariant set.

Definition 2.4.1. A parking function $\mathbf{p} = p_{n-1} \dots p_0$ is monotone if $p_0 \leq p_1 \leq \dots \leq p_{n-1}$.

Theorem 2.4.2. Given an (m, n) -invariant set Δ there is a unique monotone parking function $\mathbf{p}$ such that $\mathbf{p} \cdot \Delta = \Delta + n$. We call this the associated monotone parking function to Δ .

The idea is that we create $\mathbf{p}$ in an iterative process by removing the n -generators of Δ from smallest to largest. In each step, we record the position of the removed generator within the set of current m -generators.

Proof. We show that there exists a monotone parking function that removes the n generators of Δ in order. Let $\{a_0 < a_1 < \dots < a_{n-1}\}$ be the n -generators of Δ . Set $p_0 = 0$, as the minimal element must be an m generator and an n generator. For each $i \in \{1, \dots, n-1\}$ we claim that a_i must be an m generator of $\Delta^{(i)} = \Delta \setminus \{a_0, \dots, a_{i-1}\}$. This claim implies that there exists some $0 \leq p_i < m$ such that $p_i \cdot \Delta^{(i)} = \Delta^{(i+1)}$. See, that $\Delta^{(i)}$ has n generators

$$\{a_0 + n, \dots, a_{i-1} + n, a_i, \dots, a_{n-1}\},$$

not necessarily written in order. For the sake of contradiction, assume that a_i is not an m -generator of $\Delta^{(i)}$ that is $a_i - m \in \Delta^{(i)}$. Then there must be some n generator $\tilde{a}$ of $\Delta^{(i)}$ with $\tilde{a} + kn = a_i - m$ with $k \geq 0$. For some $j < i$, we must have $\tilde{a} = a_j + n$ where a_j is an n generator of Δ . We know that $a_i - n \notin \Delta$, thus $a_i - m - n \notin \Delta$, we arrive at $a_i - m - n = a_j + kn$ the left side is not in Δ , but the right side is, a contradiction. Thus a_i must be an m generator of $\Delta^{(i)}$.

Let $\text{ngen}(\Delta^{(i)}) = \{b_0^{(i)}, \dots, b_{m-1}^{(i)}\}$, there is some j such that $b_j^{(i)} = a_i$, set $p_i = j$. Then we have $p_{i-1} \dots p_0 \Delta = \Delta^{(i)}$ and $p_{n-1} \dots p_0 \Delta = \Delta + n$, so $\mathbf{p} = p_{n-1} \dots p_0$ must be an (m, n) -parking function as it shifts Δ by n .

We show that any monotone parking function $\mathbf{p}$ such that $\mathbf{p} \cdot \Delta = \Delta + n$ must remove the n -generators of Δ from smallest to largest. From Lemma 2.3.10, we know $\mathbf{p}$ must remove all the n -generators of Δ . Say p did not remove the n -generators in order and let a_i be

the first n -generator removed out of order. Consider the m -generators of $\Delta^{(i)}$ which are $\{b_0^{(i)}, b_1^{(i)}, \dots, b_{m-1}^{(i)}\}$, as shown in the first part of this proof, there must be some k with $a_i = b_k^{(i)}$. Then $p_i > k$, as a_i is skipped. Then a_i will remain an m -generator until it is removed. Moreover, a_i will remain the k th smallest m -generator of Δ^t until it is removed, as it is the smallest n -generator of Δ . Say it is removed on step t , then $p_t = k$ and $p_i > p_t$ with $i < t$. Thus, the monotone parking function $\mathbf{p}$ is unique. $\square$

Example 2.4.3. Let $\Delta = \{0, 3, 5, 6, 7, 8, \dots\} \in M_{3,5}$, The 5-generators of Δ are $\{0, 3, 6, 7, 9\}$. At each step we remove the smallest n -generator of Δ that appears as an m -generator of $p_{i-1} \dots p_0 \Delta$.

Stage	p_i	$p_{i-1} \dots p_0 \Delta$	m -generators
0	0	$\{ \textcolor{red}{0}, 3, 5, 6, 7, 8, \dots \}$	$\{ \textcolor{red}{0}, 5, 7 \}$
1	0	$\{ \textcolor{red}{3}, 5, 6, 7, 8, 9, \dots \}$	$\{ \textcolor{red}{3}, 5, 7 \}$
2	1	$\{ 5, \textcolor{red}{6}, 7, 8, 9, 10, \dots \}$	$\{ 5, \textcolor{red}{6}, 7 \}$
3	1	$\{ 5, \textcolor{red}{7}, 8, 9, 10, 11, \dots \}$	$\{ 5, \textcolor{red}{7}, 9 \}$
4	1	$\{ 5, 8, \textcolor{red}{9}, 10, 11, 12, \dots \}$	$\{ 5, \textcolor{red}{9}, 10 \}$

The element removed at each step is in red. See that $11100 \cdot \Delta = \{5, 8, 10, 11, 12, \dots\} = \Delta + 5$, so $\mathbf{p} = 11100$ is the unique monotone $(3, 5)$ -associated monotone parking function to $\Delta = \{0, 3, 5, 6, 7, 8, \dots\}$.

There is a connection between the associated monotone parking function and the map $\mathcal{G}$ from [GMV17].

Lemma 2.4.4. The monotone parking function given by the row lengths of the Dyck path $\mathcal{G}(\Delta)$ is the associated monotone parking function for Δ .

Proof. Let $\{a_0, \dots, a_{n-1}\}$ be the n -generators of Δ and associated monotone parking function $\mathbf{p} = p_{n-1} \dots p_0$. For each $i \in \{0, \dots, n-1\}$ we have that a_i is the p_i th m -generator of $\Delta^{(i)}$. Let S be the skeleton of $\Delta + n$. For each i let $\text{ngen}(\Delta^{(i)}) = \{b_0^{(i)}, \dots, b_{m-1}^{(i)}\}$ be the m generators of $\Delta^{(i)}$.

For any monotone parking function we have $p_0 = 0$ as the minimal element of Δ is an m and n generator. So $a_0 = b_0 = b_{p_0}$ is an m -cogenerator of $\Delta + n$ and is the minimal element of S . Now consider $p_1 \geq p_0$ and $\{b_0^{(1)}, \dots, b_{m-1}^{(1)}\}$ the m -generators of $p_0\Delta$. We know that $\{b_0^{(1)}, \dots, b_{p_1-1}^{(1)}\}$, a possibly empty set, will not be removed by p_1 or by any later letter in $\mathbf{p}$ as $\mathbf{p}$ is monotone. Thus $\{b_0^{(1)}, \dots, b_{p_1-1}^{(1)}\}$ will be m -generators of $\Delta + n$. We also have $b_{p_1}^{(1)} = a_1$ which will be an n cogenerator of $\Delta + n$. So S starts with $\{a_0, b_0^{(1)}, \dots, b_{p_1-1}^{(1)}, a_1\}$.

Consider $p_2 \geq p_1$ and $\{b_0^{(2)}, \dots, b_{m-1}^{(2)}\}$ the m -generators of $p_1 p_0 \Delta$. As before $\{b_0^{(2)}, \dots, b_{p_2-1}^{(2)}\}$ will be m -generators of $\Delta + n$. The m generators $\{b_0^{(2)}, \dots, b_{p_1-1}^{(2)}\} = \{b_0^{(1)}, \dots, b_{p_1-1}^{(1)}\}$ are already accounted for. Again $b_{p_2}^{(2)} = a_2$ is a n -cogenerators of $\Delta + n$. Thus S starts $\{a_0, b_0^{(1)}, \dots, b_{p_1-1}^{(1)}, a_1, b_{p_1}^{(2)}, \dots, b_{p_2-1}^{(2)}, a_2\}$

We can continue this process. At the last step we will need to add $\{b_{p_{n-1}}^{(n)}, \dots, b_{m-1}^{(n)}\}$ as they are m generators of $\Delta + n$ that were not added yet. This process will determine all of S , we have

$$S = \{a_0, b_0^{(1)}, \dots, b_{p_1-1}^{(1)}, a_1, b_{p_1}^{(2)}, \dots, b_{p_2-1}^{(2)}, a_2, \dots, a_i, b_{p_i}^{(i+1)}, \dots, b_{p_{i+1}-1}^{(i+1)}, a_{i+1}, \dots, a_{n-1}, b_{p_{n-1}}^{(n)}, \dots, b_{m-1}^{(n)}\}.$$

See that the number of m -generators between a_{i-1} and a_i is $p_i - p_{i-1}$ which is exactly g_i . The set S is the skeleton of $\Delta + n$.

□

Example 2.4.5. Consider the $(4, 6)$ invariant set

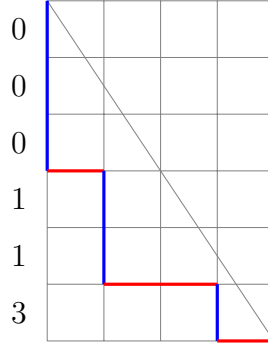
$$\Delta = \{0, 4, 5, 6, 8, 9, 10, 11, 12, \dots\}.$$

For this set the 6-generators are $\{0, 4, 5, 8, 9, 13\}$, The associated monotone parking function is $p = 311000$. At each step, we remove the smallest 6-generator of Δ that appears as an

4-generator of $\Delta^{(i)}$.

i	$\Delta^{(i)}$	4-generators of $(\Delta^{(i)})$	a_i	p_i
0	$\{0, 4, 5, 6, 8, 9, 10, 11, 12, \dots\}$	$\{0, 5, 6, 11\}$	0	0
1	$\{4, 5, 6, 8, 9, 10, 11, 12, 13, \dots\}$	$\{4, 5, 6, 11\}$	4	0
2	$\{5, 6, 8, 9, 10, 11, 12, 13, 14, \dots\}$	$\{5, 6, 8, 11\}$	5	0
3	$\{6, 8, 9, 10, 11, 12, 13, 14, 15, \dots\}$	$\{6, 8, 9, 11\}$	8	1
4	$\{6, 9, 10, 11, 12, 13, 14, 15, 16, \dots\}$	$\{6, 9, 11, 12\}$	9	1
5	$\{6, 10, 11, 12, 13, 14, 15, 16, 17, \dots\}$	$\{6, 11, 12, 13\}$	13	3
6	$\{6, 10, 11, 12, 14, 15, 16, 17, 18, \dots\}$	$\{6, 11, 12, 17\}$		

We have highlighted when an 6-cogenerator of $\Delta + 6$ is removed from Δ in *blue* and the first time an 4-generator of $\Delta + 6$ is skipped by the action *red*. The 4-cogenerators of Δ are $\{-4, 1, 2, 7\}$ and the skeleton of Δ is $\{-4, 0, 1, 2, 4, 5, 7, 8, 9, 13\}$.



Combining Theorem 2.4.4 and Theorem 2.2.7 one obtains.

Lemma 2.4.6. *If Δ and Δ' have the same associated monotone parking function then $\Delta \sim \Delta'$*

Proof. The n cogenerators and m generators in the skeleton of Δ and Δ' are in the same order, thus $\mathcal{G}(\Delta) = \mathcal{G}(\Delta')$ and $\Delta \sim \Delta'$ as $\mathcal{G}$ is a bijection by Theorem 1.1 in [GMV17]. $\square$

2.5 Periodic Points

In this section, we study the points in V^m that are not fixed by an (m, n) -parking functions $\mathbf{p}$. The action of a parking function $\mathbf{p}$ on $\mathbf{x}$ resorts the coordinates of $\mathbf{x}$ several times. In this section, we discuss this sorting.

Definition 2.5.1. *Let $\mathbf{x}$ be in the interior of an alcove. We define the permutation $\sigma_{i,\mathbf{x}}$ of a letter i acting on $\mathbf{x}$ and the permutation $\sigma_{\mathbf{p},\mathbf{x}}$ of a parking function $\mathbf{p}$ acting on $\mathbf{x}$ as follows. Let $\sigma_{i,\mathbf{x}} \in S_m$ be such that*

$$i \cdot \mathbf{x} = \sigma_{i,\mathbf{x}}(x_0 - 1, \dots, x_{i-1} - 1, x_i + m - 1, x_{i+1} - 1, \dots, x_{m-1} - 1)$$

That is, $\sigma_{i,\mathbf{x}}$ is how the coordinates are resorted when i acts on $\mathbf{x}$. Note that since $\mathbf{x}$ is in the interior of an alcove, all the coordinates of $\mathbf{x}$ are distinct, and furthermore, $i \cdot \mathbf{x}$ is also in the interior of an alcove. The permutation of a parking function $\mathbf{p} = p_{m-1} \dots p_0$ on $\mathbf{x}$ is

$$\sigma_{\mathbf{p},\mathbf{x}} = \sigma_{p_{n-1},p_{n-2} \dots p_0 \mathbf{x}} \dots \sigma_{p_1,p_0 \mathbf{x}} \sigma_{p_0,\mathbf{x}}.$$

This works with concatenation. If $\mathbf{p}$ and $\mathbf{q}$ are words then $\sigma_{\mathbf{p},\mathbf{q}\mathbf{x}} \sigma_{\mathbf{q},\mathbf{x}} = \sigma_{\mathbf{pq},\mathbf{x}}$. In particular if $\mathbf{x}$ is fixed by $\mathbf{p}$ then $(\sigma_{\mathbf{p},\mathbf{x}})^\ell = \sigma_{\mathbf{p}^\ell,\mathbf{x}}$ where $\mathbf{p}^\ell$ is $\mathbf{p}$ concatenated with itself ℓ times.

Definition 2.5.2. *Let $\mathbf{x} \in V^m$ for any $i \in \{0, \dots, m-1\}$ let*

$$i \star \mathbf{x} = (x_0 - 1, \dots, x_{i-1} - 1, x_i + m - 1, x_{i+1} - 1, \dots, x_{m-1} - 1).$$

That $i \star \mathbf{x}$ is the shift during the action of i .

This is a motivating example for why we are now starting to track $\sigma_{\mathbf{p}}$.

Example 2.5.3. *Let $\mathbf{w} = 2100 \in PW_4^3$, then the fixed point of $\mathbf{w}$ is $x_{\mathbf{w}} = (-4, 0, 4)$. We consider the centers of the 3 centers of alcoves adjacent to $A_{\mathbf{w}}$ and see where $\mathbf{w}$ maps them. We have labeled the centers a, b, c where $a = (-4, 1, 3)$, $b = (-3, -1, 4)$, and $c = (-5, 0, 5)$. We will find that $\mathbf{w} \cdot a = b$, $\mathbf{w} \cdot b = x_{\mathbf{w}}$, and $\mathbf{w} \cdot c = x_{\mathbf{w}}$.*

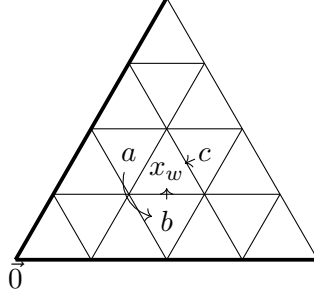


Figure 2.3: In this example $w = 2100$, we consider the four points, the fixed point $x_w = (-5, 0, 5)$ along with $a = (-4, 1, 3)$, $b = (-3, -1, 4)$, and $c = (-5, 0, 5)$. Arrows show where each point is mapped to by the parking function $w = 2100$.

To highlight the geometry of what is happening, we will show each step of the action of w on the four alcoves that contain x_w, a, b and c in Figure 2.4. Recall that the action shifts and then resorts points. Resorting can be viewed as folding back into the Weyl Chamber. At each step we showed where we started in blue, the shift in red and the resort is the blue highlighted area of the next step.

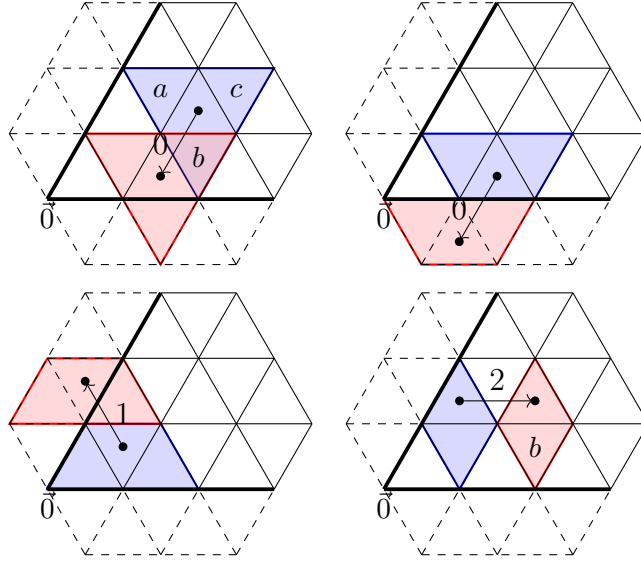


Figure 2.4: We track the four alcoves that contain x_w, a, b and c over the action of w .

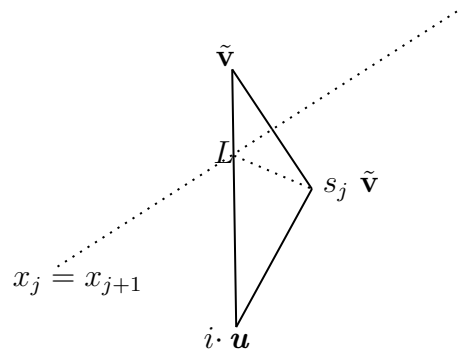
During the first step of the action $0 \star x_w$ and $0 \star c$ are separated by $H_{0,1}^0$ after the shift and after sorting that $0 \cdot x_w = 0 \cdot c$. See that, image $\sigma_{0,x_w} = id$ and $\sigma_{0,b} = s_0$, the image of these points are getting closer to each other. Similarly during the third step $1 \star 00 \cdot x_w$ and

$1 \star 00 \cdot c$ are separated by $H_{1,2}^0$. After the sorting, we have $100 \cdot x_w = 100 \cdot c$. At this we have $\sigma_{100, x_w} = (12)(01)$ and $\sigma_{100, c} = (01)$.

See that after applying all of w we have $w \cdot a = b$. This also means $\|x_w - a\| = \|w \cdot x - w \cdot a\|$. To illustrate Lemma 2.5.4, see that $\sigma_{w, x_w} = \sigma_{w, a}$. It can also be seen that $w^2 \cdot a = x_w$. In this section, we will show that all points outside A_w will eventually be mapped into A_w . This example shows that there are some dynamics about the action that need to be studied.

Lemma 2.5.4. *If $u, v \in V^m$ are in the interior of alcoves A_u and A_v respectively, for $i \in \{0, 1, \dots, m-1\}$ we have $\|u - v\| = \|i \cdot u - i \cdot v\|$ if and only if $\sigma_{i, u} = \sigma_{i, v}$. Furthermore, if $\|u - v\| = \|i \cdot u - i \cdot v\|$ then there is a linear map $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $i|_{A_u} = T|_{A_u}$ and $i|_{A_v} = T|_{A_v}$.*

Proof. Its clear that if $i \cdot u$ and $i \cdot v$ are resorted in the same way then $\|u - v\| = \|i \cdot u - i \cdot v\|$. Now suppose that $\sigma_{i, u} \neq \sigma_{i, v}$. Let $\tilde{v} = \sigma_{i, u}(v_1 - 1, \dots, v_i + m - 1, \dots, v_m - 1)$ be v sorted in the same way as $i \cdot u$. Then $\tilde{v}$ is not in order, so there is some j such that $\tilde{v}_j > \tilde{v}_{j+1}$. We can consider the difference between $\|i \cdot u - \tilde{v}\|$ and $\|i \cdot u - s_j \tilde{v}\|$ where $s_j \tilde{v}$ has v_j swapped with v_{j+1} .



Let L be the point on the intersection of the hyperplane $x_j = x_{j+1}$ and the line segment from $i \cdot u$ to $\tilde{v}$. Then

$$\begin{aligned}
\|i \cdot \mathbf{u} - \tilde{\mathbf{v}}\| &= \|i \cdot \mathbf{u} - L\| + \|L - \tilde{\mathbf{v}}\| \\
&= \|i \cdot \mathbf{u} - L\| + \|L - s_j \tilde{\mathbf{v}}\|.
\end{aligned}$$

By the triangle inequality, which is strict as $\tilde{\mathbf{v}}$ and $s_j \tilde{\mathbf{v}}$ do not coincide

$$\begin{aligned}
\|i \cdot \mathbf{u} - s_j \tilde{\mathbf{v}}\| &< \|i \cdot \mathbf{u} - L\| + \|L - s_j \tilde{\mathbf{v}}\| \\
&= \|i \cdot \mathbf{u} - \tilde{\mathbf{v}}\|.
\end{aligned}$$

As one resorts, the coordinates of $\tilde{\mathbf{v}}$ one goes through several of these reflections. Thus $\|\mathbf{u} - \mathbf{v}\| > \|i \cdot \mathbf{u} - i \cdot \mathbf{v}\|$.

Now, assume that $\|\mathbf{u} - \mathbf{v}\| = \|i \cdot \mathbf{u} - i \cdot \mathbf{v}\|$ we know that the action is piecewise linear on alcoves. Let T and T' be the linear map such that $i|_{A_u} = T|_{A_u}$ and $i|_{A_v} = T'|_{A_v}$. From the definition of the action $T(\mathbf{x}) = \sigma_{\mathbf{u},i}(\mathbf{x} - \mathbf{v})$ and $T'(\mathbf{x}) = \sigma_{\mathbf{v},i}(\mathbf{x} - \mathbf{v})$ where $\mathbf{v} = (1, \dots, 1, 1 - m, 1, \dots, 1)$. As $\sigma_{i,\mathbf{u}} = \sigma_{i,\mathbf{v}}$ we have $T = T'$.

□

This continues to words as well. If $\|\mathbf{w} \cdot \mathbf{x} - \mathbf{w} \cdot \mathbf{v}\| = \|\mathbf{x} - \mathbf{v}\|$ then $\sigma_{\mathbf{w},\mathbf{x}} = \sigma_{\mathbf{w},\mathbf{v}}$, moreover, for each $0 \leq i \leq n$ we have $\sigma_{w_i,\mathbf{x}^{(i)}} = \sigma_{w_i,\mathbf{v}^{(i)}}$. Furthermore, there is a linear map $T : V^m \rightarrow T^m$ such that $\mathbf{w}|_{A_x} = T|_{A_x}$ and $\mathbf{w}|_{A_v} = T|_{A_y}$ where A_x and A_v are alcoves such that $\mathbf{x} \in A_x$ and $\mathbf{v} \in A_v$.

Lemma 2.5.5. *If $\mathbf{x}$ and $\mathbf{v}$ are from the interior of the same alcove, then for all $i \in \{0, \dots, m-1\}$ $\sigma_{i,\mathbf{x}} = \sigma_{i,\mathbf{v}}$.*

Proof. Recall the action of a letter restricted to an alcove is a linear isometry to the alcove it maps to. Thus all points in the alcove must be sorted the same way, thus $\sigma_{i,\mathbf{x}} = \sigma_{i,\mathbf{v}}$ □

This extends to words. If $\mathbf{x}$ and $\mathbf{v}$ are in the same alcove, then $\sigma_{\mathbf{w},\mathbf{x}} = \sigma_{\mathbf{w},\mathbf{v}}$ for all words $\mathbf{w} \in [m]^n$. The following lemma is similar to Lemma 2.5 from [MTW19].

Lemma 2.5.6. *If $\mathbf{x}$ is a fixed point of $\mathbf{p}$ and in the interior of some alcove, then the coordinates of $\mathbf{x}$ can be partitioned into sets $P_0, \dots, P_{d-1}$ of size m/d such that for all $0 \leq i \leq d-1$ and $a \neq b \in P_i$*

$$1) \quad a - b \in d\mathbb{Z} \qquad \qquad 2) \quad a - b \notin m\mathbb{Z}$$

Furthermore, it follows that for $i \neq j$, $a \in P_i$, and $b \in P_j$ that $a - b \notin d\mathbb{Z}$.

Proof. As $\mathbf{x}$ is in the interior of an alcove for any coordinates x_i, x_j we have $x_i - x_j \notin m\mathbb{Z}$. As $\mathbf{x}$ is fixed by $\mathbf{p}$ for each coordinate x_j there is some i and k such that $x_j - n + mk = x_i$. This means $x_j - x_i \in d\mathbb{Z}$ while $x_j - x_i \notin m\mathbb{Z}$. Given a coordinate x_j , consider that m/d cosets

$$x_j + m\mathbb{Z}, \quad x_j + d + m\mathbb{Z}, \quad x_j + 2d + m\mathbb{Z}, \quad \dots, \quad x_j + (m-d) + m\mathbb{Z}.$$

Each set contains exactly one coordinate of $\mathbf{x}$. Then each coordinate belongs to a set of m/d coordinates formed in this fashion. This creates the partition $P_0, P_1, \dots, P_{d-1}$, which have both properties.

Furthermore, if $i \neq j$ with $a \in P_i$ and $b \in P_j$ we cannot have $a - b \in d\mathbb{Z}$, as each coset contains exactly one coordinate of $\mathbf{x}$.

□

Consider $\mathbf{x} = (x_1, x_2, \dots, x_{m-1})$ is a fixed point of $\mathbf{p}$ and in the interior of some alcove. Let $P_0, \dots, P_{d-1}$ be a partition of coordinates of $\mathbf{x}$ from Lemma 2.5.6. For every j and i such that $x_j \in P_i$ we have $x_{\sigma_{\mathbf{p}, \mathbf{x}}(j)}$ is the unique element of P_i such that

$$x_{\sigma_{\mathbf{p}, \mathbf{x}}(j)} - x_j + n \in m\mathbb{Z}$$

Corollary 2.5.7. *If $\mathbf{x}$ is fixed by $\mathbf{p}$ then the permutation $\sigma_{\mathbf{p}, \mathbf{x}}$ is d disjoint cycles of size m/d .*

Proof. From the discussion above, if $x_j \in P_i$ then $x_{\sigma_{p,x}(j)} \in P_i$, Applying the same logic several times we arrive at

$$P_i = \{x_j, x_{\sigma_{p,x}(j)}, x_{\sigma_{p,x}\sigma_{p,x}(j)} = x_{\sigma_{p^2,x}(j)}, \dots, x_{\sigma_{p^{m/d-1},x}(j)}\}.$$

Thus $\sigma_{p,x}$ permutes the coordinates in P_i in a long cycle. There are d parts P_i , thus $\sigma_{p,x}$ is d disjoint cycles of size m/d . $\square$

Lemma 2.5.8. *If $\|\mathbf{p}\mathbf{u} - \mathbf{p}\mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ then for every $\epsilon > 0$ there exists $\mathbf{u}'$ and $\mathbf{v}'$ such that $\mathbf{u}' \in B_\epsilon(\mathbf{u})$, $\mathbf{v}' \in B_\epsilon(\mathbf{v})$ such that $\mathbf{u}'$ and $\mathbf{v}'$ are in the interior of alcoves and $\sigma_{p,\mathbf{u}'} = \sigma_{p,\mathbf{v}'}$. Furthermore, there are alcoves $A_{\mathbf{u}}$ and $A_{\mathbf{v}}$ such that $\mathbf{u} \in A_{\mathbf{u}}$ and $\mathbf{v} \in A_{\mathbf{v}}$ and a linear map T such that $\mathbf{p}|_{A_{\mathbf{u}}} = T|_{A_{\mathbf{u}}}$ and $\mathbf{p}|_{A_{\mathbf{v}}} = T|_{A_{\mathbf{v}}}$.*

Proof. If $\mathbf{u}$ and $\mathbf{v}$ are already in the interior of alcoves, then by Lemma 5.2 we can take $\mathbf{u}' = \mathbf{u}$ and $\mathbf{v}' = \mathbf{v}$ and have $\sigma_{p,\mathbf{u}'} = \sigma_{p,\mathbf{v}'}$. If $\mathbf{u}$ or $\mathbf{v}$ is contained in some hyperplane $H \in \mathcal{H}$, we consider the line segment L connecting $\mathbf{u}$ and $\mathbf{v}$, as $\|\mathbf{p}\mathbf{u} - \mathbf{p}\mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ we know, L is mapped isometrically by $\mathbf{p}$ to its image, which must also be a line segment. For any $\epsilon > 0$ one can pick $\mathbf{u}'' \in B_{\epsilon/2}(\mathbf{u}) \cap L$ and $\mathbf{v}'' \in B_{\epsilon/2}(\mathbf{v}) \cap L$ such that if $\mathbf{u}''$ or $\mathbf{v}''$ is contained in hyperplane $H_{i,j}^k$ then $L \subset H_{i,j}^k$ and each alcove containing $\mathbf{u}''$ also contains $\mathbf{u}$, similarly, each alcove containing $\mathbf{v}''$ also contains $\mathbf{v}$. Let $\mathbf{w} \in B_{\epsilon/2}(0)$ be any point such that $\mathbf{u}' = \mathbf{u}'' + \mathbf{w}$ and $\mathbf{v}' = \mathbf{v}'' + \mathbf{w}$ are in the interiors of alcoves $A_{\mathbf{u}}$ and $A_{\mathbf{v}}$ such that $\mathbf{u}, \mathbf{u}', \mathbf{u}'' \in A_{\mathbf{u}}$ and $\mathbf{v}, \mathbf{v}', \mathbf{v}'' \in A_{\mathbf{v}}$.

Let L' be the line segment from $\mathbf{u}'$ to $\mathbf{v}'$. We claim that $\mathbf{p}$ maps L' to its image isometrically. Indeed, suppose that $\|\mathbf{p}\mathbf{u}' - \mathbf{p}\mathbf{v}'\| < \|\mathbf{u}' - \mathbf{v}'\|$. Let k be such that $\|p_k \dots p_0 \mathbf{u}' - p_k \dots p_0 \mathbf{v}'\| = \|\mathbf{u}' - \mathbf{v}'\|$ and $\|p_{k+1} p_k \dots p_0 \mathbf{u}' - p_{k+1} p_k \dots p_0 \mathbf{v}'\| < \|\mathbf{u}' - \mathbf{v}'\|$. During the action of p_{k+1} before sorting, the images of $\mathbf{u}'$ and $\mathbf{v}'$ must be on opposite sides of a hyperplane $H_{i,i+1}^0 = \{x_i = x_{i+1}\}$ for some i . Thus the alcoves containing the images of $\mathbf{u}'$ and $\mathbf{v}'$ are separated by $H_{i,i+1}^0$. The images of $\mathbf{u}''$ and $\mathbf{v}''$ cannot lie on $H_{i,i+1}^0$, as they are in the interior of alcoves. Thus the images of $\mathbf{u}''$ and $\mathbf{v}''$ are separated by $H_{i,i+1}^0$. Then p_{k+1} necessarily sort the images of $\mathbf{u}''$ and $\mathbf{v}''$ in different ways, thus $\|\mathbf{p} \cdot \mathbf{u}'' - \mathbf{p} \cdot \mathbf{v}''\| < \|\mathbf{u}'' - \mathbf{v}''\|$. But

$\mathbf{u}'', \mathbf{v}'' \in L$ so this contradicts that L is mapped isometrically.

Thus L' is mapped isometrically, which means that $\mathbf{u}' \in B_\epsilon(\mathbf{u})$, $\mathbf{v}' \in B_\epsilon(\mathbf{v})$ such that $\mathbf{u}'$ and $\mathbf{v}'$ are in the interior of alcoves, and $\sigma_{\mathbf{p}, \mathbf{u}'} = \sigma_{\mathbf{p}, \mathbf{v}'}$ by Lemma 5.2.

Furthermore, by lemma 5.2 there is a linear map T such that $\mathbf{p}|_{A_u} = T|_{A_u}$ and $\mathbf{p}|_{A_v} = T|_{A_v}$ as $\mathbf{u}'$ and $\mathbf{v}'$ are in the interior of alcoves with $\|\mathbf{p}\mathbf{u}' - \mathbf{p}\mathbf{v}'\| = \|\mathbf{u}' - \mathbf{v}'\|$. $\square$

From now on assume that m and n are coprime, $\mathbf{p}$ is an (m, n) -parking function, the unique fixed point of $\mathbf{p}$ is $\mathbf{x}_p$, and A_p is the alcove containing $\mathbf{x}_p$. Note that according to Lemma 2.9 from [GMV14] x_p must be the centroid of A_p .

Lemma 2.5.9. *Given an (m, n) -parking function $\mathbf{p}$, for all $\mathbf{x} \in V^m$ there is some $k \in \mathbb{Z}_{\geq 0}$ such that $\mathbf{p}^{k+m}\mathbf{x} = \mathbf{p}^k\mathbf{x}$. That is $\{\mathbf{p}^k\mathbf{x}\}_k$ will eventually become fixed or periodic, with period dividing m .*

Proof. Consider an $\mathbf{x} \in V^m$. Acting by a letter or word is a weak contraction, thus

$$\|\mathbf{p} \cdot \mathbf{x} - \mathbf{x}_p\| \leq \|\mathbf{x} - \mathbf{x}_p\|$$

for all $\mathbf{x} \in V^m$. First consider $\mathbf{x}$ in the interior of some alcove. If we consider the set $\{\mathbf{p}^k \cdot \mathbf{x}\}_k$, because the action is a weak contraction, this set can only intersect finitely many alcoves in V^m as the distance to $\mathbf{x}_p$ is bounded. Thus, for some $k < j$ point $\mathbf{x}^k, \mathbf{x}^j$ are in the same alcove of V^m , say $\mathbf{p}^k\mathbf{x}, \mathbf{p}^j\mathbf{x} \in A$. We know $\mathbf{p}$ isometrically maps alcoves to alcoves, thus $\mathbf{p}^{(j-k)} : A \rightarrow A$. It is well known that alcoves have dihedral symmetry D_{2m} , so $\mathbf{p}^{(j-k)} : A \rightarrow A$ can only be one of $2m$ possible maps. Thus $\mathbf{p}^{(j-k)2m+k}\mathbf{x} = \mathbf{p}^k\mathbf{x}$ and repeated action of $\mathbf{p}$ must become periodic.

When the action becomes periodic, the distance to $\mathbf{x}_p$ will stop decreasing and $\|\mathbf{p}^k\mathbf{x} - \mathbf{x}_p\| = \|\mathbf{p}^{k+j}\mathbf{x} - \mathbf{x}_p\|$ for all $j \geq 0$. By Theorem 2.5.4 $\sigma_{\mathbf{p}, \mathbf{x}_p} = \sigma_{\mathbf{p}, \mathbf{p}^{k+j}\mathbf{x}}$ and by Corollary 2.5.7 $\sigma_{\mathbf{p}, \mathbf{x}_p}$ a cycle of length m . Thus $\sigma_{\mathbf{p}, \mathbf{p}^{k+j}\mathbf{x}} = \sigma_{\mathbf{p}, \mathbf{p}^{k+j}\mathbf{x}_p}$ is the same cycle of length m for all j . Then

$$\sigma_{\mathbf{p}, \mathbf{x}_p}^m = \sigma_{\mathbf{p}^m, \mathbf{x}_p} = id = \sigma_{\mathbf{p}, \mathbf{p}^{k+m}\mathbf{x}_p}$$

The linear map $\mathbf{p}^m|_{A_{\mathbf{x}_p}}$ is the identity map, thus $\mathbf{p}^m|_{\mathbf{p}^j A_x}$ must also be the identity map by Lemma 5.6. Which implies that $\mathbf{p}^{j+m} \cdot \mathbf{x} = \mathbf{p}^j \cdot \mathbf{x}$.

Now consider $\mathbf{x}$ on the boundary of some alcove A' . Let $\mathbf{x}'$ be a point in the interior of A' . Then for any word $\mathbf{w}$ we have $\|\mathbf{w}\mathbf{x} - \mathbf{w}\mathbf{x}'\| = \|\mathbf{x} - \mathbf{x}'\|$, as each letter maps alcoves to alcoves isometrically. By the above, there is some j such that $\mathbf{p}^m|_{\mathbf{p}^j A'} : \mathbf{p}^j A' \rightarrow \mathbf{p}^j A'$ is the identity map and $\mathbf{p}|_{\mathbf{p}^j A'}$ has period m .

□

Now we can classify all the periodic points of a given parking function.

Lemma 2.5.10. *If $\mathbf{x}_p$ is fixed by a parking function $\mathbf{p} = p_{n-1} \dots p_0$, then there is some $0 < k < n$ and $0 \leq i < m$ where $p_{k-1} \dots p_0 \mathbf{x}_p = \mathbf{x}_p^{(k)}$ such that $|x_{p,i}^{(k)} - x_{p,i+1}^{(k)}| = 1$. Which means if A_p is the alcove containing $\mathbf{x}_p$ then $p_k \dots p_0 A_p$ has a facet contained in the hyperplane $H_{i,i+1}^0 = \{x_i = x_{i+1}\}$ for some i .*

Proof. By Corollary 2.5.7, we know that $\sigma_{\mathbf{p}, \mathbf{x}_p}$ is a cycle of length m , and all coordinates of $\mathbf{x}_p$ have distinct remainders mod m . Let $x_i = 0 \pmod m$ and $x_j = 1 \pmod m$. Without loss of generality, say that $x_i < x_j$ and thus $i < j$. As $\sigma_{\mathbf{p}, \mathbf{x}_p}$ is a cycle, there is a minimum ℓ such that $(\sigma_{\mathbf{p}, \mathbf{x}_p})^\ell(i) > (\sigma_{\mathbf{p}, \mathbf{x}_p})^\ell(j)$. Then $(\sigma_{\mathbf{p}, \mathbf{x}_p})^{\ell-1}(i) < (\sigma_{\mathbf{p}, \mathbf{x}_p})^{\ell-1}(j)$ and there is a minimal $0 \leq k \leq n-1$ such that

$$\sigma_{p_{k-1} \dots p_0 p^{\ell-1}, \mathbf{x}}(i) > \sigma_{p_{k-1} \dots p_0 p^{\ell-1}, \mathbf{x}}(j).$$

Set $\hat{i} = \sigma_{p_{k-2} \dots p_0 p^\ell}(i)$, $\hat{j} = \sigma_{p_{k-2} \dots p_0 p^\ell}(j)$, and $\hat{x} = p_{k-2} \dots p_0 p^\ell \cdot \mathbf{x}$. See that $\hat{i} < \hat{j}$ and $\hat{x}_{\hat{i}} < \hat{x}_{\hat{j}}$ and $\sigma_{p_{k-1}, \hat{x}}(\hat{i}) > \sigma_{p_{k-1}, \hat{x}}(\hat{j})$. As $\hat{x}_{\hat{i}} - \hat{x}_{\hat{j}}$ changes sign from negative to positive, it must be that $p_{k-1} = \hat{i}$. Acting by a letter does not change the remainders mod m , thus $\hat{x}_{\hat{i}} + 1 \equiv \hat{x}_{\hat{j}} \pmod m$. Acting adds m to a coordinate, thus $\hat{x}_{\hat{i}} + m > \hat{x}_{\hat{j}}$, that means that $\hat{x}_{\hat{j}} = \hat{x}_{\hat{i}} + 1$. Since all the coordinates are integer, distinct, and ordered, we must have $\hat{j} = \hat{i} + 1$. This means that the alcove that contains $\hat{x}$ has a facet contained in $H_{i,i+1}^0$.

□

Lemma 2.5.11. *Given $\mathbf{p}$ a parking function, let $\mathbf{x}_p$ be its fixed point in V^m , and A_p be the alcove containing $\mathbf{x}_p$. The set of periodic points of $\mathbf{p}$ is exactly $A_p \setminus \mathbf{x}_p$ all points of $A_p \setminus \mathbf{x}_p$ have period m .*

Proof. If $\mathbf{x}$ is in the interior of A_p , then $\sigma_{\mathbf{p}, \mathbf{x}_p} = \sigma_{\mathbf{p}, \mathbf{x}}$ which is a cycle of length m . Thus for any point $\mathbf{x} \in A_p$ we have $p^m \cdot \mathbf{x} = \mathbf{x}$, thus all points in the closure are periodic.

For any point $\mathbf{x}$ not in the alcove. By Theorem 2.5.9 there is some k such that $\|\mathbf{p}^k \mathbf{x} - \mathbf{x}_p\| = \|\mathbf{p}^{k+j} \mathbf{x} - \mathbf{x}_p\|$ for all $j \geq 0$. Assume $\mathbf{p}^k \mathbf{x}$ is not in the alcove that contains $\mathbf{x}_p$. Then by Theorem 2.5.10 there is an ℓ such that $|x_i^{(\ell)} - x_{i+1}^{(\ell)}| = 1$, which means $p_{\ell-1} \dots p_1 p_0 \mathbf{x}_p$ is in an alcove with a facet contained in $H_{i,i+1}^0 = \{x_i = x_{i+1}\}$. Let $\mathbf{x}'_p = p_{\ell-1} \dots p_1 p_0 \mathbf{x}_p$ and $\mathbf{x}' = p_{\ell-1} \dots p_1 p_0 \mathbf{x}$, then $\mathbf{p}' = p_{\ell-1} \dots p_0 p_{n-1} \dots p_\ell$ fixes $\mathbf{x}'_p$ and $\{\mathbf{p}'^k \mathbf{x}'\}_k$ becomes periodic. If we consider the points $\{\mathbf{p}'^k \mathbf{x}', \mathbf{p}'^{k+1} \mathbf{x}', \mathbf{p}'^{k+2} \mathbf{x}', \dots, \mathbf{p}'^{k+n-1} \mathbf{x}'\}$ there is t where $\mathbf{p}'^{k+t} \mathbf{x}'$ and $\mathbf{x}'_p$ are separated by $H_{i,i+1}^0$. In this sorting step $\mathbf{p}'^{k+t} \mathbf{x}'$ must have gotten closer to $\mathbf{x}'_p$. Thus $\{\mathbf{p}^k \mathbf{x}\}$ must eventually be in A_p . □

This last lemma shows that if $\mathbf{x}$ is not in the alcove fixed by $\mathbf{p}$ then $|\mathbf{p}^m \mathbf{x} - \mathbf{x}_p| < |\mathbf{x} - \mathbf{x}_p|$. Because alcoves are mapped isometrically to alcoves, we get the following:

Corollary 2.5.12. *The centroid of the fundamental alcove $\mathbf{x}_0 \in A_0$ will converge to the fixed point of $\mathbf{x}_p \in A_p$ under the action of $\mathbf{p}$. i.e. $\mathbf{p}^k \cdot \mathbf{x}_0 \rightarrow \mathbf{x}_p$.*

2.6 Fixed points of parking functions

It was shown in [MTW19] that the set of all fixed points of an (m, n) -parking function is a convex set of dimension $d - 1$ in V^m . Recall the notation from Section 2.2.2. Let $\Theta = (\Delta_0, \Delta_1, \dots, \Delta_{d-1})$ be a tuple of 0-balanced $(m/d, n/d)$ -invariant sets. For each Θ , let V_Θ be a copy of the hyperplane $\{\mathbf{x} \in \mathbb{R}^d : \sum_{i=0}^{d-1} x_i = 0\}$, and consider the hyperplane arrangement $\Sigma_\Theta \subset V_\Theta$ consisting of all shifts $x_0, \dots, x_{d-1}$ for which there exists i and j such that

$$(dS_i + x_i) \cap (dS_j + x_j) \neq \emptyset,$$

where S_i is the skeleton of Δ_i .

A point $(x_0, x_1, \dots, x_{d-1})$ in a region of Σ_Θ corresponds to the shift of $\Theta = (\Delta_0, \Delta_1, \dots, \Delta_{d-1})$, specifically

$$\Delta(x_0, \dots, x_{d-1}) = \bigcup_{i=0}^{d-1} (d\Delta_i + x_i).$$

We consider the configuration space

$$W = \{(\Theta, (x_0, \dots, x_{d-1})) : \Theta = (\Delta_0, \dots, \Delta_{d-1}) \text{ and } (x_0, \dots, x_{d-1}) \notin \Sigma_\Theta\}.$$

One can extend the map $\phi : M_{m,n} \rightarrow V^m$ (Definition 2.3.3 from section 3) to W . Consider a point $(\Theta, (x_0, \dots, x_{d-1})) \in W$. For each invariant set Δ_i of the tuple $\Theta = (\Delta_0, \dots, \Delta_{d-1})$, let A_i be the set of m -generators say $A_i := \{a_{i,0}, a_{i,1}, \dots, a_{i,m/d-1}\}$. Then $\phi : W \rightarrow V^m$ is given by

$$\phi((\Theta, (x_0, \dots, x_{d-1}))) = \text{sort} \left(\bigcup_{i=1}^{d-1} dA_i + x_i \right) - \vec{1}k,$$

where

$$k = \frac{d}{m} \sum_{i=0, j=0}^{d-1, m/d-1} a_{i,j} + \frac{1}{d} \sum_{i=0}^{d-1} x_i.$$

The union is a set of m distinct numbers thought of as a sorted point in V^m . The m -cogenerators are part of the skeleton of each Δ_i , and thus do not intersect; the m -generators are a shift by m/d from m -cogenerators and thus all coordinates of $\phi((\Theta, (x_0, \dots, x_{d-1})))$ are distinct.

We have a result similar to Lemma 2.3.11 in this setting. Let $\text{cord}((x_0, \dots, x_{m/d-1})) = \{x_0, \dots, x_{m/d-1}\}$ be the coordinates of a point as a set. Recall that S_i is the skeleton of Δ_i as an (m, n) -invariant set, now let S'_i the skeleton of Δ_i as an (n, m) -invariant set. Note that this switch of m and n also occurs in Lemma 2.3.11.

Lemma 2.6.1. *Let $\phi((\Theta, \mathbf{x}))$ be a fixed point of $\mathbf{p}$ and for each i let $M_i = p_{i-1} \dots p_0 \phi((\Theta, \mathbf{x}))$. Then*

$$S = \bigcup_{i=0}^n (\text{cord}(M_i) + i)$$

is the skeleton

$$S' = \bigcup_{i=0}^{d-1} dS'_i + x_i + n.$$

Proof. With an unbalanced action, we have $\mathbf{p} \cdot \phi(\Theta, \mathbf{x}) = \phi(\Theta, \mathbf{x}) + n$. If $y \in \text{cord}(\phi(\Theta, \mathbf{x}) + n) \subset S$ then for some j we have $y \in dA_j + x_j + n$, that is y corresponds to an m -generator of Δ_j and $y \in S'$.

The unbalanced action replaces a coordinate $x_j \in \text{cord}(\mathbf{x})$ in V^m by $x_j + m$ and then sorts. If $y \in \text{cord}(S) \setminus \text{cord}((\phi(\Theta, \mathbf{x}) + n))$ then for some $i \in \{0, 1, \dots, n-2\}$ we have $y \in \text{cord}(p_i \dots p_0 \phi(\Theta, \mathbf{x}))$. Then for some j and k we have $y = da + km$ where $a \in A_j$ is an m -generator of Δ_j . Then $y + n = da + km + n \in d\Delta_j + x_j + n$, but $y \notin d\Delta_j + x_j + n$ because $y \notin \text{cord}((\phi(\Theta, \mathbf{x}) + n))$. That is y is an n -cogenerator of $\Delta_j + n$. Thus $y \in S'$. □

Lemma 2.6.2. *If $(\Theta, (x_0, \dots, x_{d-1})) \in W$ and $(\Theta, (y_0, \dots, y_{d-1})) \in W$ are from the same region of Σ_θ and $\mathbf{p} \in PW_m^n$ is such that $\mathbf{p}$ fixes $\phi((\Theta, (x_0, \dots, x_{d-1})))$, then $\mathbf{p}$ fixes $\phi((\Theta, (y_0, \dots, y_{d-1})))$.*

Proof. Let $(\Theta, (x_0, \dots, x_{d-1})) = (\Theta, \mathbf{x})$ and $(\Theta, (y_0, \dots, y_{d-1})) = (\Theta, \mathbf{v})$ be as in the statement of the lemma. For each $i \in \{0, \dots, d-1\}$, let $A_i = \{a_{i,0}, a_{i,1}, \dots, a_{i,n/d-1}\}$ be the m -generators of Δ_i . If $\mathbf{p}$ does not fix $\phi((\Theta, \mathbf{v}))$, then during the action the coordinates must have been sorted differently. Consider the first time they are not sorted the same way. That is some t such that

$$\sigma_{p_{t-1} \dots p_0, \phi((\Theta, \mathbf{x}))} = \sigma_{p_{t-1} \dots p_0, \phi((\Theta, \mathbf{v}))}$$

but

$$\sigma_{p_t p_{t-1} \dots p_0, \phi((\Theta, \mathbf{x}))} \neq \sigma_{p_t p_{t-1} \dots p_0, \phi((\Theta, \mathbf{v}))}.$$

That means there is some (i_1, j_1, k_1) and (i_2, j_2, k_2) such that

$$da_{i_1, j_1} + x_{i_1} + k_1 m + m < da_{i_2, j_2} + x_{i_2} + k_2 m$$

$$da_{i_1, j_1} + y_{i_1} + k_1 m + m > da_{i_2, j_2} + y_{i_2} + k_2 m.$$

Visually changing the shifts from $\mathbf{x}$ to $\mathbf{v}$ changes how elements sort.

$$\begin{array}{ccc}
& m & \\
\text{da}_{i_1 j_1} + x_{i_1} + k_1 m & \xrightarrow{\quad} & \text{da}_{i_2 j_2} + x_{i_1} + k_2 m \\
& m & \\
\text{da}_{i_1 j_1} + y_{i_1} + k_1 m & \xrightarrow{\quad} & \text{da}_{i_2 j_2} + y_{i_2} + k_2 m
\end{array}$$

By lemma 2.6.1, we know $\bigcup_{i=0}^n p_{i-1} \dots p_0 \phi(\Theta, \mathbf{x}) = \bigcup_{i=0}^{d-1} dS'_i + x_i + n = S$ where S'_i is the skeleton of Δ_i as an (n, m) -invariant set. We know that $\text{da}_{i_1 j_1} + x_{i_1} + k_1 m + m \in S$. This inequality is saying that $\bigcup_{i=0}^{d-1} dS'_i + x_i + n$ and $\bigcup_{i=0}^{d-1} dS'_i + y_i + n$ are in a different order. But that means that $\phi(\Theta, \mathbf{x})$ and $\phi(\Theta, \mathbf{v})$ must be in different connected components of the complement to Σ_Θ .

□

Lemma 2.6.3. *If $(\Theta, (x_0, \dots, x_{d-1}))$, $(\Theta, (y_0, \dots, y_{d-1})) \in W$ are from the same region of Σ_Θ and $\mathbf{p} \in PW_m^n$ such that $\mathbf{p}$ fixed $\phi((\Theta, (x_0, \dots, x_{d-1})))$, then for each $k \in \{0, 1, \dots, n-1\}$ the coordinates of $p_k \dots p_0 \phi((\Theta, (y_0, \dots, y_{d-1})))$ are distinct.*

Proof. By Lemma 2.6.2 $\mathbf{p}$ also fixes $\phi(\Theta, (y_0, \dots, y_{d-1}))$. By Lemma 2.6.1 the skeleton S' has unique coordinates. This means for each k the coordinates of $p_k \dots p_0 \phi((\Theta, (y_0, \dots, y_{d-1})))$ are distinct.

□

A region of Σ_Θ can be fixed by multiple parking functions, and some parking functions fix multiple regions. These regions have a unique monotone parking function that fixes every $\Delta(x_0, \dots, x_{d-1})$ in that region.

The image of a region of Σ_Θ is the set of fixed points for the monotone parking function that region corresponds to. The image of all of the Σ_Θ is an arrangement of $(d-1)$ affine subspaces, which is the union of all fixed points of all parking functions.

Example 2.6.4. *Consider $m = 3$ and $n = 4$ and $\Delta = \mathbb{Z}_{\geq 0}$. All of the parking functions $\mathbf{p}$ such that $\mathbf{p} \cdot \Delta = \Delta + n$ are*

$$\{0000, 0001, 0011, 0010, 0100, 0002, 0012\}.$$

Now let $m = 6, n = 4$ and $\Theta = (\Delta_0, \Delta_1)$ where $\Delta_0 = \{-1, 1, 2, 3, 4, 5, \dots\}$ and $\Delta_1 = \mathbb{Z}_{\geq 0}$. Then $S_0 = \{-3, -1, 0, 1, 3\}$ and $S_2 = \{-2, -1, 0, 1, 2\}$ Consider

$$\Delta(-3.5, 3.5) = \{-5.5, -1.5, 0.5, 2.5, 3.5, 4.5, 6.5, \dots\} \text{ and}$$

$$\Delta(-4.5, 4.5) = \{-6.5, -2.5, -0.5, 1.5, 3.5, 4.5, 5.5, 6.5, 7.5, \dots\}.$$

See that $(\Theta, (-3, 5, 3.5))$ and $(\Theta, (-4, 5, 4.5))$ are in different connected components of the complement to Σ_Θ because

$$(2S_0 - 3) \cap (2S_1 + 3) = \{-9, -5, -3, -1, 3\} \cap \{-1, 1, 3, 5, 7\} = \{-1, 3\} \neq \emptyset.$$

These sets correspond to the points

$$\mathbf{x}_0 = \phi(\Theta, (-3, 5, 3.5)) = (-7, -3, 1, 2, 4, 6) \text{ and}$$

$$\mathbf{x}_1 = \phi(\Theta, (-4, 5, 4.5)) = (-8.5, -4.5, -0.5, 2.5, 4.5, 6.5).1$$

Both $\mathbf{x}_0$ and $\mathbf{x}_1$ are fixed by the $(6, 4)$ -parking function $p = 2140$.

Corollary 2.6.5. *An (m, n) -parking function will fix the image of a union of regions from Σ_Θ under ϕ .*

Proof. If $\mathbf{p}$ is a parking function it must fix some Δ . If a parking function fixes any Δ in a region by Theorem 2.6.2 it fixed the whole region. Thus $\mathbf{p}$ fixes a union of regions in Σ_θ . $\square$

2.7 Inverting the map SP

In [GMV14] Gorsky, Mazin, and Vazirani defined a map $SP : {}^m\widetilde{S}_n \rightarrow PF_n^m$, named after the Pak-Stanley map.

$$SP_u(i) = \#\{j > i | 0 < u(i) - u(j) < m\}$$

Gorsky, Mazin, and Vazirani showed that it is bijective for $m = kn \pm 1$, conjectured that it is bijective for all coprime m and n , and provided a conjectured algorithm to invert SP . In [MTW19] McCammond, Thomas, and Williams proved that SP is a bijection for all coprime m and n . Here we present two algorithms to invert SP . The algorithm from [GMV14] is presented in Algorithm 2.7.1 and we translate that algorithm to the notation from [MTW19] in Algorithm 2.7.7. In this section, we show both algorithms create the same intermediate function and that function becomes affine periodic. By affine periodic we mean there is some $N > 0$ such that for all $x > N$ we have $U(x + n) = U(x) + n$. This leads to an inverse to the map SP .

Algorithm 2.7.1. Computing SP^{-1} from [GMV14].

Given $\mathbf{p} \in PF_n^m$. First, we extend $\mathbf{p}$ periodically $p(i + kn) = p(i)$ for $k \in \mathbb{Z}$ and $0 \leq i \leq n - 1$. We construct an injective function $U : \mathbb{N} \rightarrow \mathbb{N}$ as follows: Think about inputting outputs one at a time on the table

i	0	1	2	...
$\mathbf{p}(i)$	$\mathbf{p}(0)$	$\mathbf{p}(1)$	$\mathbf{p}(2)$	...
$U(i)$	$U(0)$	$U(1)$	$U(2)$	...

Initial step: Place 0 under the leftmost 0. That is we set $U(\min\{j : p(j) = 0\}) = 0$.

General Step: We want to place $\alpha \in \mathbb{N}$ on the table. We assume we have already placed $\{0, 1, 2, \dots, \alpha - 1\}$. We place α in the leftmost position i with the following 2 properties:

1. α is to the right of $\alpha - m$. That is $i > U^{-1}(\alpha - m)$
2. If $U(i) = \alpha$ then $\mathbf{p}(i) = \#\{\beta | \alpha - m < \beta < \alpha, U^{-1}(\beta) > i\}$

If we never place a value in the i th position we set $U(i) = \infty$.

The condition that α is placed to the right of $\alpha - m$ insures that $U^{-1}(\alpha - m) < U^{-1}(\alpha)$ which we will call being m -restricted, following the same property from affine permutations. See that we can always place α , as there will be a non-minimal position where $p(i) = 0$ we can place α .

It is useful to track how far positions are away from having property 2 of the general step.

Definition 2.7.2. For all i such that $U(i) > \alpha - 1$ let

$$t_\alpha(i) = p(i) - \#\{\beta | \alpha - m < \beta < \alpha, U^{-1}(\beta) > i\}$$

See that $t_0(i) = p(i)$ and the general step can be rephrased as: Place α in the leftmost position such that $t_\alpha(i) = 0$ and $i > U^{-1}(\alpha - m)$

Lemma 2.7.3. When $t_\alpha(i)$ and $t_{\alpha-1}(i)$ are both defined, we have $t_{\alpha-1}(i) - t_\alpha(i) \in \{0, 1\}$

Proof. See that

$$\begin{aligned} t_{\alpha-1}(i) - t_\alpha(i) &= -\#\{\beta | \alpha - 1 - m < \beta < \alpha - 1, U^{-1}(\beta) > i\} + \#\{\beta | \alpha - m < \beta < \alpha, U^{-1}(\beta) > i\} \\ &= -\#\{\beta | \beta = \alpha - m, U^{-1}(\beta) > i\} + \#\{\beta | \beta = \alpha - 1, U^{-1}(\beta) > i\}. \end{aligned}$$

Each of these sets is either empty or is exactly one element. Thus $|t_{\alpha-1}(i) - t_\alpha(i)| \leq 1$. We have

$$t_\alpha(i) = \begin{cases} t_{\alpha-1}(i) + 1 & i > U^{-1}(\alpha) \text{ and } i < U^{-1}(\alpha - m) \\ t_{\alpha-1}(i) - 1 & i < U^{-1}(\alpha) \text{ and } i > U^{-1}(\alpha - m) \\ t_{\alpha-1}(i) & \text{else.} \end{cases}$$

If $t_{\alpha-1}(i) - t_\alpha(i) = -1$ then $U^{-1}(\alpha - m) > i$ and $i > U^{-1}(\alpha)$ which contradicts that we place α to the right of $\alpha - m$. □

Lemma 2.7.4. While defined $t_\alpha(i)$ is non-negative.

Proof. If $t_{\alpha-1}(i) = 0$ and we were to try to place α such that $t_\alpha(i) = -1$ then $i < U^{-1}(\alpha)$ and $i > U^{-1}(\alpha - m)$ by Lemma 2.7.3, but this means i had both property 1 and 2 and is more left than where we placed α . Thus $t_\alpha(i)$ is non-negative. $\square$

Lemma 2.7.5. *If $U(i) = \infty$ then there is some β with $t_\beta(i) = 0$ and $p(i) = m - 1$*

Proof. For some large α we have placed the last $m - 1$ numbers to the right of i then

$$t_\alpha(i) = p(i) - \#\{\beta | \alpha - m < \beta < \alpha, U^{-1}(\beta) > i\} \leq m - 1 - (m - 1) = 0.$$

As $p(i)$ is bounded by $m - 1$. Thus $p(i) = m - 1$. $\square$

Lemma 2.7.6. *In Algorithm 2.7.1, for every $i \in \mathbb{Z}_{\geq 0}$ we have $U(i) < \infty$. That is there are no gaps left by this algorithm.*

Proof. For the sake of contradiction, let $i \in \mathbb{Z}_{\geq 0}$ such that $U(i) = \infty$. Then $p(i) = m - 1$, by Lemma 2.7.5. Let β be the minimal such that $t_\beta(i) = 0$. When scanning to place β , we must see $\{\beta - 1, \dots, \beta - m + 1\}$ must be placed to the right of i . If we do not place at i it must be that $\beta - m$ is also to the right of i . But then $t_{\beta-1}(i) = 0$, as $\{\beta - 2, \dots, \beta - m\}$ are all to the right of i . $\square$

An alternative to this inverse is the following.

Algorithm 2.7.7. Computing SP^{-1} in the language of [MTW19]

Given $\mathbf{p} \in PF_n^m$. First, we extend $\mathbf{p}$ periodically $p(i + kn) = p(i)$ for $k \in \mathbb{Z}$ and $0 \leq i \leq n - 1$. We will construct a function $V : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ inductively as follows:

0th step: let $i = 0, S_0 = \{0, 1, \dots, m - 1\}$.

For a finite set $S \subset \mathbb{Z}$, let $S[k]$ denote that k th element of S when sorted from smallest to largest.

i th step for $i \geq 0$: Let $\alpha = S_i[p(i)]$, set $V(i) = \alpha$ and $S_{i+1} = (S_i \setminus \{\alpha\}) \cup \{\alpha + m\}$.

In summary, this algorithm is starting at the point $(0, 1, \dots, m-1)$ repeatedly acting by p one letter at a time without balancing and recording the coordinate p_i acted on.

Lemma 2.7.8. *For all $i \in \mathbb{Z}_{\geq 0}$ there is some j with $V(j) = i$. That is there are no gaps in the output of V .*

Proof. See that, as $p \in PF_m^n$ we must have some i with $p(i) = 0$. So in Algorithm 2.7.7, we set $V(i)$ equal to the minimal number of set S_i . This means that there cannot be some number that stays in S_i forever. $\square$

Example 2.7.9. *Consider the parking function $\mathbf{p} = 10002 \in PF_5^3$. First we work out finding $SP^{-1}(p)$ using Algorithm 2.7.1.*

i	0	1	2	3	4	5	6	7	8	9	10	11	12	12	14	15	16	17	18	19
$p(i) = t_0(i)$	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1
$U(i)/t_\alpha(i)$	1	0	0	0	1	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1
	0	0	1	0	1	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	0	1	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	1	2	0	0	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	0	1	4	0	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	5	1	4	0	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	5	1	4	6	0	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	5	0	4	6	7	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	1	2	0	0	0	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	0	1	9	0	0	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	10	1	9	0	0	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	10	1	9	11	0	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	10	0	9	11	12	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	1	2	0	0	0	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	0	1	14	0	0	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	15	1	14	0	0	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	15	1	14	16	0	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	15	0	14	16	17	1
	2	0	1	3	5	8	4	6	7	10	13	9	11	12	15	18	14	16	17	1

Table 2.1: We use Algorithm 2.7.1 to compute $U(i)$ where $p = 10002$

We also go through Algorithm 2.7.7 to find $SP^{-1}(p)$

In both examples, see that the algorithms output the same balanced affine permutation based on the last n entries. In this case, the output is $u = [3, 0, 1, 2, 5]$. Further in these

i	p(i)	S_i	V(i)
0	2	(0, 1, 2)	2
1	0	(0, 1, 5)	0
2	0	(1, 3, 5)	1
3	0	(3, 4, 5)	3
4	1	(4, 5, 6)	5
5	2	(4, 6, 8)	8
6	0	(4, 6, 11)	4
7	0	(6,7, 11)	6
8	0	(7,9, 11)	7
9	1	(9, 10, 11)	10
10	2	(9, 11,13)	13
11	0	(9,11,16)	9
12	0	(11,12,16)	11
13	0	(12,14,16)	12
14	1	(14,15,16)	15
15	2	(14,16,18)	18
16	0	(14,16,21)	14
17	0	(16,17,21)	16
18	0	(17,19,21)	17
19	1	(19,20,21)	20

Table 2.2: We use Algorithm 2.7.7 to compute $V(i)$ where $p = 10002$

examples, we can see that $U(i) = V(i)$ for all i .

Lemma 2.7.10. *Algorithm 2.7.1 and Algorithm 2.7.7 have the same output, that is $U(i) = V(i)$ for all i .*

Proof. Both of these algorithms impose the restriction that if $U(i) = \alpha - m$ and $U(j) = \alpha$ then $i < j$. If we know $U(i) \bmod m$, we can reconstruct $U(i)$ from this. For Algorithm 2.7.1, we start with $U(0) = p_0$ as you need to place $\{0, 1, \dots, p_0 - 1\}$ to have p_0 inversions at $U(0)$. In Algorithm 2, we set $V(0) = \{0, 1, \dots, m - 1\}[p_0] = p_0$. Say $U(j) = V(j)$ for all $j < i$, we consider $U(i)$ and $V(i)$.

On the general step of Algorithm 2.7.1 if we placed α in the i th position, then

1. α is the smallest element in its congruence class $\bmod m$ that was not placed earlier.

Furthermore all smaller numbers in α 's congruence class $\bmod m$ were placed to the left of position i . Therefore $\alpha \in S_i$.

2. Exactly p_i numbers from $\{\alpha - 1, \dots, \alpha - m + 1\}$ were placed to the right of position i .

Let R be the set of these numbers. For every $r \in R$ there is a unique element $r' \in S_i$ such that $m|r - r'$ and $r' \leq r < \alpha$. And vice versa, that is for every element $s' \in S_i$ such that $s' < \alpha$ there is a unique $s \in R$ such that $m|s - s'$ and $s' \leq s < \alpha$.

Therefore, $\alpha = S_i[p_i] = V(i)$ which means $U(i) = V(i)$.

□

Note that for Lemma 2.7.10 we only use that fact that p is a parking function to know that there is some i with $p(i) = 0$. Both these algorithms output the same function as long as there is some i where $p(i) = 0$ even if p is not a parking function. But when p is not a parking function, neither of these algorithms will become affine periodic.

Corollary 2.7.11. *If $\mathbf{p}$ is a parking function, both of these algorithms become affine periodic.*

Proof. We show there is some N such that for $i > N$ we have $U(i + n) = U(i) + n$. The set S_0 can be thought of as the point $\mathbf{x} = (0, 1, 2, \dots, m - 1) \in V^m$, then the general step is acting without balancing by the letter $p(i)$. That means S_{in} is a shift of $\mathbf{p}^i \cdot (0, 1, \dots, m - 1)$. By Theorem 2.5.11 we know $\mathbf{p}^i \cdot (0, 1, \dots, m - 1)$ will converge to the alcove that contains the fixed point of $\mathbf{p}$. Because $(0, 1, 2, \dots, m)$ is a centroid of an alcove, it will converge to the fixed point in a finite number of steps. Once $\mathbf{p}^i \cdot (0, 1, \dots, m - 1)$ converges to the fixed point, the algorithm will become affine periodic in n steps or less.

□

Note that this resolves conjecture 7.9 from [GMV14].

Chapter 3

The Parking Polytope

This chapter is dedicated to the parking polytope. It follows the work in [BBC⁺24].

3.1 Introduction

A *classical parking function* of length n is a list $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of positive integers whose nondecreasing rearrangement $\alpha'_1 \leq \alpha'_2 \leq \dots \leq \alpha'_n$ satisfies $\alpha'_i \leq i$. It is well-known that the number of classical parking functions of length n is $(n+1)^{n-1}$. This number surfaces in a variety of places; for example, it counts the number of labeled rooted forests on n vertices and the number of regions of a Shi arrangement in $\mathbb{R}^n$ (see [Yan15] for further discussion). Let PF_n denote the convex hull in $\mathbb{R}^n$ of all classical parking functions of length n . In 2020, Stanley [Sta20] asked for the volume and the number of vertices, facets, and lattice points of PF_n . These questions were first answered by Amanbayeva and Wang in [AW22] and independently (regarding the volume, number of vertices, and number of facets) by Stong in [Sto22]. Recently, Behrend obtained a general formula for the Ehrhart polynomials of partial permutahedra [BCC⁺23, Theorem 1], the $m = n - 1$, $t = 1$ case of which yields an expression for the number of lattice points in PF_n that is more concise and explicit than the Amanbayeva–Wang formula. In this article, we present results on the combinatorial and geometric properties of the convex hull of a generalization of classical parking functions,

namely $\mathbf{b}$ -parking functions.

Definition 3.1.1. *Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{\geq 0}^n$. A $\mathbf{b}$ -parking function is a list $(\beta_1, \dots, \beta_n)$ of positive integers whose nondecreasing rearrangement $\beta'_1 \leq \beta'_2 \leq \dots \leq \beta'_n$ satisfies $\beta'_i \leq b_1 + \dots + b_i$.*

This generalization of parking functions has been previously explored from an enumerative perspective by Yan [Yan00, Yan01, Yan15] and Pitman and Stanley [SP02]. In [HLVM24], Hanada, Lentfer, and Vindas-Meléndez generalized the classical parking-function polytope to the $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$, i.e., the convex hull of all $\mathbf{b}$ -parking functions of length n in $\mathbb{R}^n$ for $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{\geq 0}^n$. The work of Hanada, Lentfer, and Vindas-Meléndez focused on the $\mathbf{b}$ -parking-function polytope in the special case when $\mathbf{b} = (a, b, b, \dots, b)$. In particular, they described the face structure and provided a closed volume formula for the $\mathbf{b}$ -parking-function polytope in that special case. These parking-function polytopes have surprisingly arisen in different guises and in connection to other polytopes and combinatorial objects, including partial permutahedra [BCC⁺23, BCC⁺22, HS22], Pitman-Stanley polytopes [HLVM24, SP02], polytopes of win vectors [Bac18, BMW97], and stochastic sandpile models [Sel24].

In this dissertation, we explore $\mathbf{b}$ -parking-function polytopes for general $\mathbf{b}$. Our contributions about $\mathbf{b}$ -parking-function polytopes lend themselves to fruitful exploration from different perspectives and propose connections to seemingly unrelated combinatorial objects. We present definitions, terminology, and references in each relevant section. The chapter is structured as follows:

- We begin Section 3.2 by studying the $\mathbf{b}$ -parking-function polytope from a polyhedral perspective. We show that $\mathfrak{X}_n(\mathbf{b})$ is (apart from a single degenerate case) an n -dimensional simple polytope (Theorem 3.2.2 and Theorem 3.2.3(a)), and we give explicit descriptions of the vertices, facets, and edges of $\mathfrak{X}_n(\mathbf{b})$ (Theorems 3.2.3(b) and 3.2.3(c) and Theorem 3.2.7, respectively). We then show that, up to an affine unimodular isomorphism, $\mathfrak{X}_n(\mathbf{b})$ is a generalized permutahedron (Theorem 3.2.12). We conclude the section by presenting the h -polynomial and f -vector of $\mathfrak{X}_n(\mathbf{b})$ (Theorems 3.2.16 and 3.2.17, respectively).

- In Section 3.3 we take a building-set perspective to study $\mathfrak{X}_n(\mathbf{b})$, which allows us to completely characterize the face structure of $\mathfrak{X}_n(\mathbf{b})$ for any $\mathbf{b} \in \mathbb{Z}_{>0}^n$ (Theorem 3.3.9). As a consequence, we show that, up to combinatorial equivalence, every $\mathbf{b}$ -parking-function polytope is either the classical parking-function polytope PF_n or the stellohedron St_n (Theorem 3.3.11).
- We identify $\mathfrak{X}_n(\mathbf{b})$ as a special polymatroid (Theorem 3.4.3) in Section 3.4 and use this perspective to derive linear bounds on its combinatorial diameter (Theorem 3.4.4).
- In Section 3.5, we revisit the classical parking-function polytope PF_n and summarize its known and new connections to other established families of polytopes. We conclude by adding a description of PF_n as a projection of a relaxed Birkhoff polytope (Theorem 3.5.1), and $\mathfrak{X}_n(\mathbf{b})$ as a projection of a relaxed-partition polytope (Theorem 3.5.3).

3.2 Polyhedral perspectives on $\mathfrak{X}_n(\mathbf{b})$

Fix, throughout this work, a positive integer n and a vector $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{>0}^n$. The following definition introduces our main object of study, i.e., a polytope associated to $\mathbf{b}$ -parking functions, which we denote $\mathfrak{X}_n(\mathbf{b})$. Standard references on polytopes include [Gn03] and [Zie95].

Definition 3.2.1. *The $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$ is the convex hull of all $\mathbf{b}$ -parking functions in $\mathbb{R}^n$.*

Recall that the linear action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{R}^n$ is defined by $\pi(\sum_{i=1}^n a_i \mathbf{e}_i) := \sum_{i=1}^n a_i \mathbf{e}_{\pi(i)}$ for all $\pi \in \mathfrak{S}_n$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$. Note that by definition, the polytope $\mathfrak{X}_n(\mathbf{b})$ is invariant under the action of $\mathfrak{S}_n$. For convenience, we use the notation

$$S_i := \sum_{j=1}^i b_j \text{ for } 1 \leq i \leq n \text{ and } \mathbf{v}_k := (1, \dots, 1, S_{k+1}, S_{k+2}, \dots, S_n) \in \mathbb{R}^n \text{ for } 0 \leq k \leq n.$$

Further observe that the $\mathbf{v}_k$ are $\mathbf{b}$ -parking functions, so $\mathbf{v}_k \in \mathfrak{X}_n(\mathbf{b})$ for all k . We will see below that the permutations $\pi(\mathbf{v}_k)$ of the $\mathbf{v}_k$ are precisely the vertices of $\mathfrak{X}_n(\mathbf{b})$ (Theorem 3.2.3(b)).

A theme that will appear early in our study of $\mathfrak{X}_n(\mathbf{b})$ is that the case in which $b_1 = 1$ behaves differently from when $b_1 \geq 2$. Indeed, if $b_1 \geq 2$, then the $\mathbf{b}$ -parking functions $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ are distinct, and the index k in $\mathbf{v}_k$ is precisely the number of entries in $\mathbf{v}_k$ that equal 1 for all $k \geq 0$. However, if $b_1 = 1$, then $\mathbf{v}_0 = \mathbf{v}_1 = (1, S_2, S_3, \dots, S_n)$. This trivial coincidence has important consequences for the structure of the polytope $\mathfrak{X}_n(\mathbf{b})$. Figure 3.1 shows the $\mathbf{b}$ -parking-function polytopes $\mathfrak{X}_3(1, 2, 3)$ (left) and $\mathfrak{X}_3(2, 3, 4)$ (right). Note also that there may be lattice points in $\mathfrak{X}_n(\mathbf{b})$ that are not $\mathbf{b}$ -parking functions. For example, $(2, 2, 2)$ is in $\mathfrak{X}_3(1, 2, 3)$ but is not a $(1, 2, 3)$ -parking function.

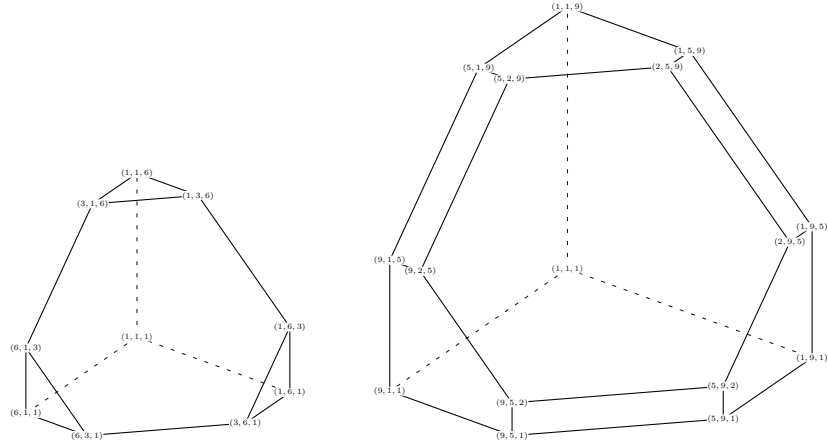


Figure 3.1: The polytopes $\mathfrak{X}_3(1, 2, 3)$ (left) and $\mathfrak{X}_3(2, 3, 4)$ (right).

We begin by proving that, with the exception of a single “degenerate” case, $\mathfrak{X}_n(\mathbf{b})$ is an n -dimensional polytope.

Proposition 3.2.2. *If $n = 1 = b_1$, then $\dim(\mathfrak{X}_n(\mathbf{b})) = 0$. However, if $n \geq 2$, or if $n = 1$ and $b_1 \geq 2$, then $\dim(\mathfrak{X}_n(\mathbf{b})) = n$.*

Proof. In the degenerate case $n = 1 = b_1$, the polytope $\mathfrak{X}_n(\mathbf{b})$ contains only the point (1) in $\mathbb{R}^1$. In all other cases, $\mathfrak{X}_n(\mathbf{b})$ is n -dimensional because, if $b_1 \geq 2$, then $\{\mathbf{v}_k : 0 \leq k \leq n\}$ is a set of $n + 1$ affinely independent points in $\mathfrak{X}_n(\mathbf{b})$, and, if $b_1 = 1$, but $n \geq 2$, then

$\{(S_2, 1, S_3, \dots, S_n)\} \cup \{\mathbf{v}_k : 1 \leq k \leq n\}$ is a set of $n+1$ affinely independent points in $\mathfrak{X}_n(\mathbf{b})$.

□

In view of Theorem 3.2.2, we exclude the degenerate case $n = 1 = b_1$ from this point forward. That is, we proceed throughout the rest of this chapter with the additional assumption that either $n \geq 2$ or, if $n = 1$, then $b_1 \geq 2$. Thus, $\dim \mathfrak{X}_n(\mathbf{b}) = n$.

3.2.1 Vertices, facets, and edges of $\mathfrak{X}_n(\mathbf{b})$

In this subsection, we describe the vertices, facets, and edges of the $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$. In particular, we find that $\mathfrak{X}_n(\mathbf{b})$ is a simple polytope, meaning that the number of edges of $\mathfrak{X}_n(\mathbf{b})$ that are incident to each vertex of $\mathfrak{X}_n(\mathbf{b})$ is n . Here and below, we use the standard notation $[n] := \{1, 2, \dots, n\}$.

Theorem 3.2.3.

- (a) *The polytope $\mathfrak{X}_n(\mathbf{b})$ is simple.*
- (b) *The set of vertices of $\mathfrak{X}_n(\mathbf{b})$ is $\{\pi(\mathbf{v}_k) : \pi \in \mathfrak{S}_n \text{ and } 0 \leq k \leq n\}$.*
- (c) *The minimal inequality description of $\mathfrak{X}_n(\mathbf{b})$ is as the set of points $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that $x_i \geq 1$ for $1 \leq i \leq n$ and*

$$\sum_{i \in I} x_i \leq \sum_{i=n-|I|+1}^n S_i, \quad (\mathbf{b}\text{-parking})$$

or, equivalently,

$$\sum_{i \in I} x_i \leq \sum_{j=1}^n \min\{j, |I|\} b_{n-j+1},$$

for all nonempty $I \subseteq [n]$ if $b_1 \geq 2$, and for all nonempty $I \subseteq [n]$ with $|I| \neq n-1$ if $b_1 = 1$. In other words, these inequalities are precisely the facet-defining inequalities for $\mathfrak{X}_n(\mathbf{b})$.

Before proceeding to the proof of Theorem 3.2.3, we note that, even when $b_1 = 1$, every point in $\mathfrak{X}_n(\mathbf{b})$ satisfies the (b-parking) inequalities for *all* subsets $I \subseteq [n]$, including those where $|I| = n - 1$. However, when $b_1 = 1$, these “ $|I| = n - 1$ ” (b-parking) inequalities are redundant because they follow from the (b-parking) inequality for $[n]$ together with the inequalities $x_i \geq 1$. For, let $m \in [n]$ and $I := [n] \setminus \{m\}$. Then the (b-parking) inequality for $[n]$ (with $b_1 = 1$) is

$$\sum_{i \in [n]} x_i \leq \sum_{j=1}^{n-1} j b_{n-j+1} + n.$$

Thus, subtracting the inequality $x_m \geq 1$ yields

$$\sum_{i \in I} x_i \leq \sum_{j=1}^{n-1} j b_{n-j+1} + (n - 1) = \sum_{j=1}^n \min \{j, |I|\} b_{n-j+1},$$

which is the (b-parking) inequality for I . Nonetheless, none of the other inequalities can be eliminated, as the following proof demonstrates *inter alia*.

We also remark here that Theorem 3.2.3 can be proved using the established theory of polymatroids. See Section 3.4 for further discussion of this connection. However, for completeness, we provide a proof here of Theorem 3.2.3 that does not rely on that theory. In fact, it is the details of the proof of Theorem 3.2.3 that motivate the forthcoming connections with generalized permutahedra and polymatroids.

Proof of Theorem 3.2.3. We temporarily write $P(\mathbf{b})$ for the polytope of points in $\mathbb{R}^n$ satisfying the system of inequalities given in Theorem 3.2.3(c). The outline of the proof is as follows. We first prove that $\mathfrak{X}_n(\mathbf{b}) \subseteq P(\mathbf{b})$. We then prove that the vertices of $P(\mathbf{b})$ are all of the form $\pi(\mathbf{v}_k)$ as in Theorem 3.2.3(b). Thus, the vertices of $P(\mathbf{b})$ are **b**-parking functions, which proves the converse containment $P(\mathbf{b}) \subseteq \mathfrak{X}_n(\mathbf{b})$, giving us $\mathfrak{X}_n(\mathbf{b}) = P(\mathbf{b})$ and the “ $\subseteq$ ” direction of Theorem 3.2.3(b). In the course of proving that every vertex of $P(\mathbf{b})$ is of the form $\pi(\mathbf{v}_k)$, we will have shown that each such vertex satisfies precisely n of the inequalities given in Theorem 3.2.3(c), which demonstrates that $\mathfrak{X}_n(\mathbf{b})$ is simple (Theorem 3.2.3(a)). We will then show that every point of the form $\pi(\mathbf{v}_k)$ is in fact a vertex

of $\mathfrak{X}_n(\mathbf{b})$, completing the proof of Theorem 3.2.3(b). Finally, we will see that each of the inequalities in Theorem 3.2.3(c) is satisfied with equality by some point in $\mathfrak{X}_n(\mathbf{b})$. This fact, together with the fact that every vertex of $\mathfrak{X}_n(\mathbf{b})$ satisfies exactly n such inequalities, implies that the system of inequalities in Theorem 3.2.3(c) is minimal and thus will complete the proof of Theorem 3.2.3.

To prove that $\mathfrak{X}_n(\mathbf{b}) \subseteq P(\mathbf{b})$, it suffices to show that every vertex of $\mathfrak{X}_n(\mathbf{b})$ is in $P(\mathbf{b})$. To this end, let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ be a vertex of $\mathfrak{X}_n(\mathbf{b})$, and let $I \subseteq [n]$. Write $\mathbf{v}' = (v'_1, \dots, v'_n)$ for the weakly increasing reordering of $\mathbf{v}$. Thus, the final $|I|$ coordinates of $\mathbf{v}'$, namely $v'_{n-|I|+1}, v'_{n-|I|+2}, \dots, v'_n$, are the $|I|$ weakly largest coordinates of $\mathbf{v}$. Moreover, $\mathbf{v}$ is a $\mathbf{b}$ -parking function (because the vertices of the convex hull of a finite set are elements of that set), so $1 \leq v'_i \leq \sum_{j=1}^i b_j$ for $1 \leq i \leq n$. Hence, $v_i \geq 1$ for $1 \leq i \leq n$, and

$$\sum_{i \in I} v_i \leq \sum_{i=n-|I|+1}^n v'_i \leq \sum_{i=n-|I|+1}^n \sum_{j=1}^i b_j = \sum_{j=1}^n \sum_{i=\max\{j, n-|I|+1\}}^n b_j = \sum_{j=1}^n \min\{n-j+1, |I|\} b_j.$$

Reversing the order of summation yields that $\mathbf{v}$ satisfies Inequality (b-parking). Thus, $\mathfrak{X}_n(\mathbf{b}) \subseteq P(\mathbf{b})$.

We now prove the converse containment. Let $\mathbf{w} = (w_1, \dots, w_n)$ be a vertex of $P(\mathbf{b})$. It suffices to show that $\mathbf{w}$ is a $\mathbf{b}$ -parking function. Since $P(\mathbf{b})$ and the set of $\mathbf{b}$ -parking functions are both invariant under the action of $\mathfrak{S}_n$ on $\mathbb{R}^n$, we may suppose that $w_1 \leq \dots \leq w_n$. We will show that $\mathbf{w} = \mathbf{v}_k$ for some k , making $\mathbf{w}$ evidently a $\mathbf{b}$ -parking function.

Let $\mathcal{I}$ be the family of all nonempty subsets $I \subseteq [n]$ such that

$$\sum_{i \in I} w_i = \sum_{i=n-|I|+1}^n S_i, \tag{3.1}$$

and furthermore such that $|I| \neq n-1$ if $b_1 = 1$. Thus, $\mathcal{I}$ indexes the set of inequalities of the form (b-parking) that the vertex $\mathbf{w}$ satisfies with equality. Let k be the number of coordinates of $\mathbf{w}$ that equal 1. Since $P(\mathbf{b})$ is a polytope in $\mathbb{R}^n$, and $\mathbf{w}$ is a vertex of this polytope, $\mathbf{w}$ satisfies with equality at least n of the inequalities that define $P(\mathbf{b})$. Hence,

$$|\mathcal{I}| \geq n - k.$$

For each $I \subseteq [n]$, let $I_+ := \{n - |I| + 1, n - |I| + 2, \dots, n\}$, so that I_+ comprises the last $|I|$ elements of $[n]$. (We will see below that in fact $I = I_+$ for all $I \in \mathcal{I}$.) Since

$$\sum_{i=n-|I|+1}^n S_i = \sum_{i \in I} w_i \leq \sum_{i=n-|I|+1}^n w_i \leq \sum_{i=n-|I|+1}^n S_i \quad \text{for all } I \in \mathcal{I}$$

(where the middle inequality holds because $\mathbf{w}$ is weakly increasing, and the last inequality holds because $\mathbf{w} \in P(\mathbf{b})$), we have that

$$\sum_{i \in I} w_i = \sum_{i=n-|I|+1}^n w_i \quad \text{for all } I \in \mathcal{I} \quad (3.2)$$

and, in particular, by Equation (3.1), that $I_+ \in \mathcal{I}$ for all $I \in \mathcal{I}$. Moreover, note that Equation (3.2) is an equation of two sums, each of which is a sum of the same number $|I|$ of terms, and that the ℓ^{th} term on the left-hand side of Equation (3.2) is bounded above by the ℓ^{th} term on the right-hand side for $1 \leq \ell \leq |I|$. It follows that the ℓ^{th} term on the left-hand side equals the ℓ^{th} term on the right-hand side for $1 \leq \ell \leq |I|$. In particular, the first terms on each side are equal:

$$w_{\min(I)} = w_{n-|I|+1} \quad \text{for all } I \in \mathcal{I}. \quad (3.3)$$

Since $\mathbf{w} \in P(\mathbf{b})$, we have in particular that

$$\sum_{i=n-|I|+2}^n w_i \leq \sum_{i=n-|I|+2}^n S_i \quad \text{for all } I \subseteq [n], \quad (3.4)$$

while, from Equations (3.1) and (3.2),

$$\sum_{i=n-|I|+1}^n w_i = \sum_{i=n-|I|+1}^n S_i \quad \text{for all } I \in \mathcal{I}. \quad (3.5)$$

Subtracting Inequality (3.4) from Equation (3.5) yields that

$$w_{n-|I|+1} \geq S_{n-|I|+1} \quad \text{for all } I \in \mathcal{I}. \quad (3.6)$$

Moreover, since $\mathbf{w} \in P(\mathbf{b})$,

$$\sum_{i=n-|I|}^n w_i \leq \sum_{i=n-|I|}^n S_i \quad \text{for all } I \subseteq [n] \text{ such that } |I| \leq n-1. \quad (3.7)$$

Subtracting Equation (3.5) from Inequality (3.7) yields

$$w_{n-|I|} \leq S_{n-|I|} \quad \text{for all } I \in \mathcal{I} \text{ such that } |I| \leq n-1. \quad (3.8)$$

Since $S_{n-|I|} < S_{n-|I|+1}$, it follows from Inequalities (3.6) and (3.8) that

$$w_{n-|I|} < w_{n-|I|+1} \quad \text{for all } I \in \mathcal{I} \text{ such that } |I| \leq n-1. \quad (3.9)$$

We now get that, in fact, $I = I_+$ for all $I \in \mathcal{I}$, since otherwise we would have an $I \in \mathcal{I}$ with $|I| \leq n-1$ and $\min(I) < \min(I_+) = n-|I|+1$, and so, from Equation (3.3) and $w_{\min(I)} \leq w_{n-|I|} \leq w_{n-|I|+1}$, we would get that $w_{n-|I|} = w_{n-|I|+1}$, contradicting Inequality (3.9). Thus, every element of $\mathcal{I}$ is of the form $I = I_j := \{j, j+1, \dots, n\}$ for some $1 \leq j \leq n$.

We will now show that $\mathbf{w} = \mathbf{v}_k = (1, \dots, 1, S_{k+1}, S_{k+2}, \dots, S_n)$. However, we must separately consider two cases. For the first case, suppose that $w_j > 1$ for all j such that $I_j \in \mathcal{I}$. Since $w_1 = \dots = w_k = 1$, it follows that $j \geq k+1$ for all j such that $I_j \in \mathcal{I}$. In other words, every $I \in \mathcal{I}$ is of the form I_j with $k+1 \leq j \leq n$. But there are only $n-k$ such subsets of $[n]$, and $|\mathcal{I}| \geq n-k$, so we must have that $\mathcal{I}$ is *precisely* the set $\{I_{k+1}, I_{k+2}, \dots, I_n\}$. Subtracting the (**b-parking**) equation for I_{j+1} from the equation for I_j , we get that $w_j = S_j$ for $k+1 \leq j \leq n-1$, and the I_n equation itself states that $w_n = S_n$. Thus, $\mathbf{w} = \mathbf{v}_k \in \mathfrak{X}_n(\mathbf{b})$, and moreover $\mathbf{w}$ satisfies precisely $k + |\mathcal{I}| = n$ of the inequalities in Theorem 3.2.3(c) with equality.

This leaves the case in which $w_j = 1$ for some j such that $I_j \in \mathcal{I}$. From Inequality (3.6), it follows that $S_j = 1$. Since $1 \leq S_1 < \cdots < S_n$, we get that $j = 1$ and hence $b_1 = S_1 = 1 = w_1$ and $[n] = I_1 \in \mathcal{I}$. (Recall that we are also now assuming that $n \geq 2$ because we have excluded the degenerate case $n = 1 = b_1$ in the remarks following Theorem 3.2.2.) Furthermore, $k = 1$ because, since $[n] = I_1 \in \mathcal{I}$, we may subtract the (b-parking) inequality for I_k from the (b-parking) equation for $[n]$ to get that

$$k = \sum_{i=1}^k w_i \geq \sum_{i=1}^k S_i \geq \sum_{i=1}^k i,$$

which, together with $k \geq 1$ (because $w_1 = 1$), implies that $k = 1$. Thus, $|\mathcal{I}| \geq n - 1$. Now, since $b_1 = 1$, the set $\mathcal{I}$ does not contain the subset I_2 because $|I_2| = n - 1$, so $\mathcal{I}$ must contain all $n - 2$ subsets $I_3, \dots, I_n$.¹ Hence, $\mathcal{I} = \{[n], I_3, \dots, I_n\}$. Subtracting the (b-parking) equation for I_3 from the (b-parking) equation for $[n]$ and applying $w_1 = 1 = b_1$ yields $w_2 = S_2$. For $3 \leq j \leq n - 1$, subtract the (b-parking) equation for I_{j+1} from the equation for I_j to get that $w_j = S_j$. Finally, the I_n equation itself states that $w_n = S_n$. Thus, we have again found that $\mathbf{w} = \mathbf{v}_k \in \mathfrak{X}_n(\mathbf{b})$ and that $\mathbf{w}$ satisfies precisely $k + |\mathcal{I}| = 1 + 1 + (n - 2) = n$ of the inequalities in Theorem 3.2.3(c) with equality.

Since, in both cases, $\mathbf{w} \in \mathfrak{X}_n(\mathbf{b})$, we now have that $P(\mathbf{b}) = \mathfrak{X}_n(\mathbf{b})$. We have also found that every vertex of $P(\mathbf{b})$ equals $\pi(\mathbf{v}_k)$ for some $\pi \in \mathfrak{S}_n$ and $0 \leq k \leq n$, so we have now proved that every vertex of $\mathfrak{X}_n(\mathbf{b})$ has this same form. In addition, we have found that the number of inequalities in Theorem 3.2.3(c) that the vertex $\mathbf{w}$ satisfies with equality is precisely n , so $\mathfrak{X}_n(\mathbf{b})$ is simple (proving Theorem 3.2.3(a)) and moreover no inequality given in Theorem 3.2.3(c) that supports $\mathfrak{X}_n(\mathbf{b})$ is redundant. In particular, since every point of the form $\mathbf{v}_k$ with $0 \leq k \leq n$ satisfies precisely n of the inequalities in Theorem 3.2.3(c) with equality, it follows that every such point is a vertex of $\mathfrak{X}_n(\mathbf{b})$. This completes the proof of Theorem 3.2.3(b).

The final claim is that this is a minimal system of inequalities. Since $\mathfrak{X}_n(\mathbf{b})$ is simple, it

¹The claim that there are exactly $n - 2$ such subsets is the point at which this argument fails in the degenerate case $n = 1 = b_1$.

remains only to show that every inequality in this system supports $\mathfrak{X}_n(\mathbf{b})$. Of course, the inequalities of the form $x_i \geq 1$ are all satisfied with equality by the point $(1, \dots, 1) \in \mathfrak{X}_n(\mathbf{b})$. For the **(b-parking)** inequalities, let $I \subseteq [n]$ be given, and let $k := n - |I|$. Then $\mathbf{v}_k$ satisfies the **(b-parking)** inequality for I_+ with equality. Let $\pi \in \mathfrak{S}_n$ be any permutation under which the image of I_+ is I . Then $\pi(\mathbf{v}_k)$ is a point in $\mathfrak{X}_n(\mathbf{b})$ that satisfies the **(b-parking)** inequality for I itself with equality. $\square$

As a corollary of Theorem 3.2.3(b), we enumerate the vertices of $\mathfrak{X}_n(\mathbf{b})$. When $b_1 = 1$ (and so $\mathbf{v}_0 = \mathbf{v}_1$), the number of vertices is the same as in the case of the classical parking-function polytope PF_n . However, when $b_1 \geq 2$, an additional $n!$ vertices come into play.

Corollary 3.2.4. *If $b_1 = 1$, then the number of vertices of $\mathfrak{X}_n(\mathbf{b})$ is $n! \left(\frac{1}{1!} + \dots + \frac{1}{n!} \right)$. Otherwise, if $b_1 \geq 2$, then the number of vertices of $\mathfrak{X}_n(\mathbf{b})$ is $n! \left(\frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{n!} \right)$.*

Proof. When $b_1 \geq 2$ and $k \geq 0$, or when $b_1 = 1$ and $k \geq 2$, each vertex $\mathbf{v}$ of $\mathfrak{X}_n(\mathbf{b})$ of the form $\pi(\mathbf{v}_k)$ ($\pi \in \mathfrak{S}_n$) corresponds to the unique pair (I, σ) of the k -subset $I \subseteq [n]$ of entries of $\mathbf{v}$ that equal 1 and the bijection $\sigma: [n - k] \rightarrow [n] \setminus I$ such that $\sigma(j)$ is the position of S_j in $\mathbf{v}$ for $k + 1 \leq j \leq n$. On the other hand, when $b_1 = 1$ and $k \in \{0, 1\}$, so that $\mathbf{v}_0 = \mathbf{v}_1$, each vertex of the form $\pi(\mathbf{v}_k)$ corresponds to the unique permutation in $\mathfrak{S}_n$ (namely, π) such that $\pi(j)$ is the position of S_j in $\mathbf{v}$ for $1 \leq j \leq n$. Thus, $\mathfrak{X}_n(\mathbf{b})$ has $\sum_{k=0}^n \binom{n}{k} (n - k)!$ vertices if $b_1 \geq 2$, or $n! + \sum_{k=2}^n \binom{n}{k} (n - k)!$ vertices if $b_1 = 1$. $\square$

As an immediate corollary of Theorem 3.2.3(c), we enumerate the facets of $\mathfrak{X}_n(\mathbf{b})$.

Corollary 3.2.5. *If $b_1 = 1$, then the number of facets of $\mathfrak{X}_n(\mathbf{b})$ is $2^n - 1$. Otherwise, if $b_1 \geq 2$, then the number of facets of $\mathfrak{X}_n(\mathbf{b})$ is $2^n + n - 1$.*

See Theorem 3.2.17 below for a formula for the number of k -dimensional faces of $\mathfrak{X}_n(\mathbf{b})$ for all $k = 0, 1, \dots, n$.

From the characterization of the vertices and facets of $\mathfrak{X}_n(\mathbf{b})$ in Theorem 3.2.3, it is straightforward to describe the facets that contain a given nondecreasing vertex $\mathbf{v}_k$. To this

end, define the following notation:

$$L_j := \{(x_1, \dots, x_n) \in \mathfrak{X}_n(\mathbf{b}) : x_j = 1\} \quad \text{for } 1 \leq j \leq n,$$

$$U_j := \{(x_1, \dots, x_n) \in \mathfrak{X}_n(\mathbf{b}) : \sum_{i=j}^n x_i = \sum_{i=j}^n S_i\} \quad \text{for } 1 \leq j \leq n, \text{ with } j \neq 2 \text{ if } b_1 = 1.$$

Thus, L_j is the facet of $\mathfrak{X}_n(\mathbf{b})$ defined by the inequality $x_j \geq 1$, and U_j is the facet of $\mathfrak{X}_n(\mathbf{b})$ defined by the (**b-parking**) inequality for the subset $I = I_j := \{j, j+1, \dots, n\}$ (where $j \neq 2$ if $b_1 = 1$). In particular, it is routine to check that the facets that contain the vertex $\mathbf{v}_k$ are as follows.

Lemma 3.2.6. *Fix $k \in \mathbb{Z}$ with $0 \leq k \leq n$.*

(a) *If $b_1 \geq 2$, or if $b_1 = 1$ and $k \geq 2$, then the n distinct facets that contain $\mathbf{v}_k$ are*

$$L_1, L_2, \dots, L_k, U_{k+1}, U_{k+2}, \dots, U_n.$$

(b) *If $b_1 = 1$ and $k \in \{0, 1\}$ (i.e., $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$), then the n distinct facets that contain $\mathbf{v}_k$ are*

$$L_1, U_1, U_3, U_4, \dots, U_n.$$

Proof. Since $\mathfrak{X}_n(\mathbf{b})$ is simple and n -dimensional, it suffices to confirm that the n given facets contain $\mathbf{v}_k$. □

We will now use Theorem 3.2.6 to describe the precise conditions under which two vertices of $\mathfrak{X}_n(\mathbf{b})$ share an edge. An informal summary of the conditions is as follows: Two vertices $\mathbf{v}$ and $\mathbf{w}$ share an edge if and only if either

1. the vertices are the same, except that the minimum entry > 1 in one vertex is a 1 in the other vertex; or
2. the vertices are the same, except that two entries of the form S_{j-1} and S_j swap positions.

In addition, we find that the edge vectors are all parallel to roots in the root system B_n , with lengths easily expressed in terms of the entries of the vector $\mathbf{b}$ defining $\mathfrak{X}_n(\mathbf{b})$ or by their partial sums.

Proposition 3.2.7. *Let $\mathbf{v}$ and $\mathbf{w}$ be vertices of $\mathfrak{X}_n(\mathbf{b})$. Fix $\pi \in \mathfrak{S}_n$ and $k \in \mathbb{Z}$ with $0 \leq k \leq n$ such that $\mathbf{v} = \pi(\mathbf{v}_k)$.*

(a) *If $b_1 \geq 2$, or if $b_1 = 1$ and $k \geq 2$, then $\mathbf{v}$ and $\mathbf{w}$ share an edge of $\mathfrak{X}_n(\mathbf{b})$ if and only if one of the following conditions holds:*

(1) *$\mathbf{w} = (\pi \circ (j, k))(\mathbf{v}_{k-1})$ for some $1 \leq j \leq k$, in which case*

$$\mathbf{w} - \mathbf{v} = (S_k - 1)\mathbf{e}_{\pi(j)}$$

(informally, the vertices are the same, except that one of the 1 entries in $\mathbf{v}$ is S_k in $\mathbf{w}$); or

(2) *$\mathbf{w} = (\pi \circ (j - 1, j))(\mathbf{v}_k)$ for some $k + 2 \leq j \leq n$, in which case*

$$\mathbf{w} - \mathbf{v} = b_j(\mathbf{e}_{\pi(j-1)} - \mathbf{e}_{\pi(j)})$$

(informally, the vertices are the same, except that the S_{j-1} and S_j entries swap positions);² or

(3) *$k \leq n - 1$ and $\mathbf{w} = \pi(\mathbf{v}_{k+1})$, in which case*

$$\mathbf{w} - \mathbf{v} = -(S_{k+1} - 1)\mathbf{e}_{\pi(k+1)}$$

(informally, the vertices are the same, except that the S_{k+1} entry in $\mathbf{v}$ is a 1 in $\mathbf{w}$).³

(b) *If $b_1 = 1$ and $k \in \{0, 1\}$ (i.e., $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$), then $\mathbf{v}$ and $\mathbf{w}$ share an edge of $\mathfrak{X}_n(\mathbf{b})$ if and only if one of the following conditions holds:*

²When $b_1 = 1$ and $k \in \{0, 1\}$ (so that $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$), we must use $k = 0$ for this equation to hold.

³When $b_1 = 1$ and $k \in \{0, 1\}$ (so that $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$), we must use $k = 1$ for this equation to hold.

(1) $\mathbf{w} = (\pi \circ (j-1, j))(\mathbf{v}_1)$ for some $2 \leq j \leq n$, in which case

$$\mathbf{w} - \mathbf{v} = b_j(\mathbf{e}_{\pi(j-1)} - \mathbf{e}_{\pi(j)})$$

(informally, the vertices are the same, except that the S_{j-1} and S_j entries swap positions); or

(2) $\mathbf{w} = \pi(\mathbf{v}_2)$, in which case

$$\mathbf{w} - \mathbf{v} = -b_2\mathbf{e}_{\pi(2)}$$

(informally, the vertices are the same, except that the S_2 entry in $\mathbf{v}$ is a 1 in $\mathbf{w}$).

Proof. For both parts of the theorem to be proved, note that we will be done once we have proved the “if” direction, since we will then have found n vertices that are adjacent to $\mathbf{v}$, which must be all of the vertices that are adjacent to $\mathbf{v}$, because $\mathfrak{X}_n(\mathbf{b})$ is simple and of dimension n by Theorem 3.2.3(a) and Theorem 3.2.2. Thus, the “only if” direction will follow immediately. Note also that the calculation of the difference vector $\mathbf{w} - \mathbf{v}$ under each condition is straightforward.

To prove the “if” direction of Proposition 3.2.7(a), suppose that $b_1 \geq 2$, or $b_1 = 1$ and $k \geq 2$, as in Lemma 3.2.6(a). Thus, the facets that contain $\mathbf{v}_k$ are

$$L_1, L_2, \dots, L_k, U_{k+1}, U_{k+2}, \dots, U_n. \quad (3.10)$$

By the $\mathfrak{S}_n$ invariance of $\mathfrak{X}_n(\mathbf{b})$, the vertices $\mathbf{v}$ and $\mathbf{w}$ share an edge if and only if $\pi^{-1}(\mathbf{w})$ lies on all but one of these facets. Now, if $\mathbf{w} = (\pi \circ (j, k))(\mathbf{v}_{k-1})$ for some $1 \leq j \leq k$, then $\pi^{-1}(\mathbf{w}) = (j, k)(\mathbf{v}_{k-1})$ lies on all facets in (3.10) except L_j . If $\mathbf{w} = (\pi \circ (j-1, j))(\mathbf{v}_k)$ for some $k+2 \leq j \leq n$, then $\pi^{-1}(\mathbf{w}) = (j-1, j)(\mathbf{v}_k)$ lies on all facets in (3.10) except U_j . Finally, if $k \leq n-1$ and $\mathbf{w} = \pi(\mathbf{v}_{k+1})$, then $\pi^{-1}(\mathbf{w}) = \mathbf{v}_{k+1}$ lies on all facets in (3.10) except U_{k+1} .

To prove the “if” direction of Proposition 3.2.7(b), suppose that $b_1 = 1$ and $k \in \{0, 1\}$,

as in Lemma 3.2.6(b). Thus, the facets that contain $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$ are

$$L_1, U_1, U_3, U_4, \dots, U_n. \quad (3.11)$$

If $\mathbf{w} = (\pi \circ (1, 2))(\mathbf{v}_0)$, then $\pi^{-1}(\mathbf{w}) = (1, 2)(\mathbf{v}_0)$ lies on all facets in (3.11) except L_1 . If $\mathbf{w} = (\pi \circ (j - 1, j))(\mathbf{v}_0)$ for some $3 \leq j \leq n$, then $\pi^{-1}(\mathbf{w}) = (j - 1, j)(\mathbf{v}_0)$ lies on all facets in (3.11) except U_j . Finally, if $\mathbf{w} = \pi(\mathbf{v}_2)$, then $\pi^{-1}(\mathbf{w}) = \mathbf{v}_2$ lies on all facets in (3.11) except U_1 . $\square$

Recall that, given a vertex $\mathbf{v}$ of a polytope P , the *tangent cone* $\mathcal{T}_{\mathbf{v}}(P)$ at $\mathbf{v}$ is the cone generated by the edge vectors pointing from $\mathbf{v}$ to its neighbors in the edge graph of P . That is,

$$\mathcal{T}_{\mathbf{v}}(P) := \left\{ \sum_{\substack{\mathbf{w} \in \text{Vert}(P) : \\ \mathbf{w} \sim \mathbf{v}}} \lambda_{\mathbf{w}}(\mathbf{w} - \mathbf{v}) : \lambda_{\mathbf{w}} \geq 0 \text{ for all vertices } \mathbf{w} \sim \mathbf{v} \right\},$$

where we write $\mathbf{w} \sim \mathbf{v}$ when $\mathbf{w}$ and $\mathbf{v}$ are adjacent vertices of P . As a corollary of Theorem 3.2.7, we get an explicit description of the tangent cone at each vertex of $\mathfrak{X}_n(\mathbf{b})$.

Corollary 3.2.8. *Let $\mathbf{v}$ be a vertex of $\mathfrak{X}_n(\mathbf{b})$. Fix $\pi \in \mathfrak{S}_n$ and $k \in \mathbb{Z}$ with $0 \leq k \leq n$ such that $\mathbf{v} = \pi(\mathbf{v}_k)$. Let $\mathcal{T}_{\mathbf{v}} := \mathcal{T}_{\mathbf{v}}(\mathfrak{X}_n(\mathbf{b}))$ be the tangent cone at $\mathbf{v}$.*

(a) *Suppose that $b_1 \geq 2$, or that $b_1 = 1$ and $k \geq 2$. Then $\mathcal{T}_{\mathbf{v}}$ is generated by*

$$\{\mathbf{e}_{\pi(j)} : 1 \leq j \leq k\} \quad \cup \quad \{-\mathbf{e}_{\pi(k+1)}\} \quad \cup \quad \{\mathbf{e}_{\pi(j-1)} - \mathbf{e}_{\pi(j)} : k+2 \leq j \leq n\}$$

if $k \leq n-1$, and by $\{\mathbf{e}_j : 1 \leq j \leq n\}$ if $k = n$.

(b) *Suppose that $b_1 = 1$ and $k \in \{0, 1\}$. Then $\mathcal{T}_{\mathbf{v}}$ is generated by*

$$\{\mathbf{e}_{\pi(j-1)} - \mathbf{e}_{\pi(j)} : 2 \leq j \leq n\} \quad \cup \quad \{-\mathbf{e}_{\pi(2)}\}.$$

3.2.2 $\mathfrak{X}_n(\mathbf{b})$ is equivalent to a generalized permutahedron

Generalized permutahedra (sometimes spelled *permutohedra*) form a widely studied class of polytopes with a rich theory and many interesting subclasses. One definition of a generalized permutahedron is that it is a polyhedron whose normal fan is a coarsening of the braid arrangement. For sources on generalized permutahedra, we recommend [Dok11, JR22, Pos09, PRW08]. We use an equivalent characterization. (See [JR22, Theorem 2.3].) Here and below, we write $\mathbf{e}_1, \dots, \mathbf{e}_n$ for the standard basis vectors in $\mathbb{R}^n$, and $\widehat{\mathbf{e}}_1, \dots, \widehat{\mathbf{e}}_n, \widehat{\mathbf{e}}_{n+1}$ for the standard basis vectors in $\mathbb{R}^{n+1}$.

Definition 3.2.9. *A polytope in $\mathbb{R}^n$ is a generalized permutahedron if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in [n]$ with $i \neq j$.*

We show that the lifting of $\mathfrak{X}_n(\mathbf{b})$ is a generalized permutahedron, from which it, and $\mathfrak{X}_n(\mathbf{b})$ itself, immediately inherit a variety of combinatorial and geometric results and properties. Let $B := \sum_{k=1}^n S_k = \sum_{k=1}^n (n-k+1)b_k$, and let $\ell: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be the affine map that “vertically projects” $\mathbb{R}^n$ onto the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^{n+1} : x_1 + \dots + x_n + x_{n+1} = B\}$ via $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, B - \sum_{i=1}^n x_i)$. Using this mapping, Proposition 3.2.11 below follows.

Definition 3.2.10. *Let $\overline{\mathfrak{X}}_n(\mathbf{b}) := \{\ell(\mathbf{x}) : \mathbf{x} \in \mathfrak{X}_n(\mathbf{b})\}$ be the lifting of $\mathfrak{X}_n(\mathbf{b})$ to the hyperplane H . We call $\overline{\mathfrak{X}}_n(\mathbf{b})$ the lifted $\mathbf{b}$ -parking-function polytope.*

Proposition 3.2.11. *The vertex set of $\overline{\mathfrak{X}}_n(\mathbf{b})$ is $\{\ell(\mathbf{v}) : \mathbf{v} \text{ is a vertex of } \mathfrak{X}_n(\mathbf{b})\}$. Furthermore, the minimal inequality description of $\overline{\mathfrak{X}}_n(\mathbf{b})$ is the minimal inequality description of $\mathfrak{X}_n(\mathbf{b})$ together with the additional equality $x_1 + \dots + x_n + x_{n+1} = B$.*

Building off this combinatorial structure, we have the main result of this subsection.

Theorem 3.2.12. *The lifted $\mathbf{b}$ -parking-function polytope $\overline{\mathfrak{X}}_n(\mathbf{b})$ is a generalized permutahedron.*

Proof. Proposition 3.2.11 implies that the edges of $\mathfrak{X}_n(\mathbf{b})$ lift to edges of $\overline{\mathfrak{X}}_n(\mathbf{b})$ in the natural way: $[\mathbf{v}, \mathbf{w}]$ is an edge of $\mathfrak{X}_n(\mathbf{b})$ if and only if $[\ell(\mathbf{v}), \ell(\mathbf{w})]$ is an edge of $\overline{\mathfrak{X}}_n(\mathbf{b})$. If $[\mathbf{v}, \mathbf{w}]$ is an

edge of $\mathfrak{X}_n(\mathbf{b})$ with $\mathbf{v} - \mathbf{w} = c(\mathbf{e}_i - \mathbf{e}_j)$, then $\sum_{i=1}^n v_i = \sum_{i=1}^n w_i$. Thus $\ell(\mathbf{v})_{n+1} = B - \sum_{i=1}^n v_i = B - \sum_{i=1}^n w_i$, so $\ell(\mathbf{v}) - \ell(\mathbf{w}) = c(\widehat{\mathbf{e}}_i - \widehat{\mathbf{e}}_j)$. If $[\mathbf{v}, \mathbf{w}]$ is an edge of $\mathfrak{X}_n(\mathbf{b})$ with $\mathbf{v} - \mathbf{w} = c\mathbf{e}_j$, then $\sum_{i=1}^n v_i - \sum_{i=1}^n w_i = c$. Thus, $\ell(\mathbf{v})_{n+1} = B - \sum_{i=1}^n v_i = B - \sum_{i=1}^n w_i - c = \ell(\mathbf{w})_{n+1} - c$, so $\ell(\mathbf{v}) - \ell(\mathbf{w}) = c(\widehat{\mathbf{e}}_j - \widehat{\mathbf{e}}_{n+1})$. By Proposition 3.2.7, all edges of $\mathfrak{X}_n(\mathbf{b})$ are of this form. Hence, $\overline{\mathfrak{X}}_n(\mathbf{b})$ is a generalized permutahedron. $\square$

We may now apply established results about generalized permutahedra to $\overline{\mathfrak{X}}_n(\mathbf{b})$. One such result is [JR22, Corollary 4.8], which provides a formula for the number of lattice points of a generalized permutahedron. Since lifting a polytope does not change the number of lattice points of a polytope, by applying the formula we obtain the number of lattice points of $\mathfrak{X}_n(\mathbf{b})$. Furthermore, we note that the enumeration of $\mathbf{b}$ -parking functions is not known in the form of a completely closed formula [Yan00, Yan01], but by applying the formula of [JR22, Corollary 4.8], we obtain an upper-bound for the number of $\mathbf{b}$ -parking functions since all $\mathbf{b}$ -parking functions will be lattice points in $\mathfrak{X}_n(\mathbf{b})$.

Additionally, by a theorem of Backman and Liu [BL24], Theorem 3.2.12 implies that $\overline{\mathfrak{X}}_n(\mathbf{b})$, and hence $\mathfrak{X}_n(\mathbf{b})$ itself, admits a regular unimodular triangulation. For a definition of and further details about regular unimodular triangulations, see [HPPS21]. We can also express $\overline{\mathfrak{X}}_n(\mathbf{b})$ as a Minkowski sum of simplices. In order to do that, we begin by expressing $\overline{\mathfrak{X}}_n(\mathbf{b})$ in the notation introduced by Postnikov [Pos09] for generalized permutahedra. Following from [Pos09, Section 6], every generalized permutahedron can be expressed as

$$P_n^{\mathbb{Z}}(\{z_I\}) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = z_{[n]}, \sum_{i \in I} x_i \geq z_I \text{ for any subset } I \subset [n] \right\},$$

for some collection of real parameters $\{z_I\}$ over subsets I of $[n]$ with $z_{\emptyset} = 0$.

Proposition 3.2.13. *The lifted $\mathbf{b}$ -parking-function polytope $\overline{\mathfrak{X}}_n(\mathbf{b})$ is the generalized permutahedron $P_{n+1}^{\mathbb{Z}}(\{z_I\})$ in which the values of the parameters z_I for $\emptyset \neq I \subseteq [n+1]$ are given by*

$$z_I := \begin{cases} |I| & \text{if } n+1 \notin I, \\ S_1 + S_2 + \dots + S_{|I|-1} & \text{if } n+1 \in I. \end{cases}$$

Proof. From Theorem 3.2.11, consider the defining inequalities,

$$\sum_{i \in I} x_i \leq S_n + S_{n-1} + \cdots + S_{n-|I|+1},$$

for any I with $|I| = k$, $n+1 \notin I$, and $x_1 + \cdots + x_{n+1} = \sum_{i=1}^n S_i$. Subtracting these two yields the inequality

$$\sum_{i \in [n+1] \setminus I} x_i \geq S_1 + S_2 + \cdots + S_{|I|-1}.$$

This shows that $\bar{\mathfrak{X}}_n(\mathbf{b})$ can be described in terms of the following inequalities:

$$\begin{cases} x_i \geq 1, & \text{for } 1 \leq i \leq n, \\ \sum_{i \in I} x_i \geq \sum_{i=1}^{|I|-1} S_i, & \text{for all } I \subset [n+1] \text{ with } n+1 \in I, \\ \sum_{i=1}^{n+1} x_i = \sum_{i=1}^n S_i, \end{cases}$$

since they recover the defining inequalities in Theorem 3.2.3. To match with Postnikov's definition of $P_{n+1}^{\mathcal{Z}}(\{z_I\})$, we can add the following redundant inequalities obtained from $x_i \geq 1$ which have no effect on the solutions of the above system of inequalities:

$$\sum_{i \in I} x_i \geq |I| \quad \text{for all } I \subset [n+1] \text{ such that } n+1 \notin I.$$

These inequalities give all the values of z_I for all nonempty $I \subset [n+1]$. □

In [ABD10], Ardila, Benedetti, and Doker prove that any generalized permutahedron can be written as a signed Minkowski sum of scaled standard simplices. Specifically,

$$P_n^{\mathcal{Z}}(\{z_I\}) = P_n^{\mathcal{Y}}(\{y_I\}) := \sum_{I \subseteq [n]} y_I \triangle_I,$$

where $y_I = \sum_{J \subseteq I} (-1)^{|I|-|J|} z_J$ and $\triangle_I := \text{conv}\{\mathbf{e}_i : i \in I\}$, for each $I \subseteq [n]$. In the case that $y_I \geq 0$ for all I , we call this a $\mathcal{Y}$ -generalized permutahedron, which enjoys a variety of special

properties including, if it is an integral polytope, Ehrhart positivity and h^* -real-rootedness [Pos09]; for more on these properties, see [BS21, Bra16, Liu19].

Proposition 3.2.14. *The lifted $\mathbf{b}$ -parking-function polytope $\overline{\mathfrak{X}}_n(\mathbf{b})$ is the signed Minkowski sum $\sum_{I \subseteq [n+1]} y_I \Delta_I$, where*

$$y_I := \begin{cases} 1 & \text{if } n+1 \notin I \text{ and } |I| = 1, \\ 0 & \text{if } I = \{n+1\}, \text{ or if } n+1 \notin I \text{ and } |I| \geq 2, \\ b_1 - 1 & \text{if } n+1 \in I \text{ and } |I| = 2, \\ \sum_{j=0}^{|I|-3} (-1)^{|I|+j-1} \binom{|I|-3}{j} b_{j+2} & \text{if } n+1 \in I, |I| \geq 3, \end{cases}$$

and $\Delta_I := \text{conv} \{\widehat{\mathbf{e}}_i : i \in I\}$ for all $I \subseteq [n+1]$.

Proof. We write $\overline{\mathfrak{X}}_n(\mathbf{b}) = P_{n+1}^{\mathcal{Z}}(\{z_I\})$, with z_I as given in Theorem 3.2.13. By [Dok11, Prop. 2.2.4], this becomes $P_{n+1}^{\mathcal{Z}}(\{z_I\}) = \sum_{I \subseteq [n+1]} y_I \Delta_I$, where the y_I s are determined by $z_I = \sum_{J \subseteq I} y_J$ for all nonempty $I \subseteq [n+1]$.

$$\text{First note that } y_{\{i\}} = z_{\{i\}} = \begin{cases} 1 & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i = n+1. \end{cases}$$

For $I \subseteq [n+1]$ not containing $n+1$ with $|I| \geq 2$, we apply the principle of inclusion–exclusion to get

$$y_I = \sum_{J \subseteq I} (-1)^{|I|-|J|} z_J = \sum_{k=0}^{|I|} (-1)^{|I|-k} \binom{|I|}{k} k = 0.$$

For $I \subseteq [n+1]$ with $n+1 \in I$, $|I| \geq 2$, then again by the principle of inclusion–exclusion,

$$\begin{aligned} y_I &= \sum_{J \subseteq I} (-1)^{|I|-|J|} z_J = \sum_{J \subseteq I \setminus \{n+1\}} (-1)^{|I|-|J|} z_J + \sum_{\substack{J \subseteq I \\ n+1 \in J}} (-1)^{|I|-|J|} z_J \\ &= -y_{I \setminus \{n+1\}} + \sum_{k=1}^{|I|-1} (-1)^{|I|-k-1} \binom{|I|-1}{k} \sum_{\ell=1}^k S_\ell. \end{aligned} \tag{3.12}$$

When $|I| = 2$, $y_I = -1 + b_1$. For $|I| \geq 3$, the second term on the right-hand side of

Equation (3.12) can be rewritten as

$$\begin{aligned}
\sum_{\ell=1}^{|I|-1} S_{\ell} \left(\sum_{k=\ell}^{|I|-1} (-1)^{|I|-1-k} \binom{|I|-1}{k} \right) &= \sum_{\ell=1}^{|I|-1} S_{\ell} \left(\sum_{k=0}^{|I|-1-\ell} (-1)^k \binom{|I|-1}{k} \right) \\
&= \sum_{\ell=1}^{|I|-1} \left(\sum_{i=1}^{\ell} b_i \right) (-1)^{|I|-1-\ell} \binom{|I|-2}{|I|-1-\ell} = \sum_{i=1}^{|I|-1} b_i \sum_{\ell=i}^{|I|-1} (-1)^{|I|-1-\ell} \binom{|I|-2}{|I|-1-\ell} \\
&= \sum_{i=1}^{|I|-1} b_i \sum_{\ell=0}^{|I|-1-i} (-1)^{\ell} \binom{|I|-2}{\ell} = \sum_{i=2}^{|I|-1} (-1)^{|I|-1-i} \binom{|I|-3}{|I|-1-i} b_i.
\end{aligned}$$

Hence, for $|I| \geq 3$,

$$y_I = \sum_{i=2}^{|I|-1} (-1)^{|I|-1-i} \binom{|I|-3}{|I|-1-i} b_i = (-1)^{|I|-1} \sum_{i=0}^{|I|-3} (-1)^i \binom{|I|-3}{i} b_{i+2}. \quad \square$$

Theorem 3.2.14 allows us to determine from $\mathbf{b} = (b_1, \dots, b_n)$ whether $\overline{\mathfrak{X}}_n(\mathbf{b})$ is a $\mathcal{Y}$ -generalized permutahedron. In particular, the liftings of $\mathbf{PF}_n$ and of the specific $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(a, b)$ studied in [HLVM24] (where $(a, b) := \mathbf{b} = (a, b, b, \dots, b)$) are both $\mathcal{Y}$ -generalized permutahedra.

3.2.3 h -Polynomial of $\mathfrak{X}_n(\mathbf{b})$

We now consider the h -polynomial of the $\mathbf{b}$ -parking-function polytope. Recall the background on h -polynomials and its computation via vertex posets in Section 1.5. The stellohedron $\mathbf{St}_n$ is a simple generalized permutahedron first defined by Postnikov, Reiner, and Williams [PRW08] as the nestohedron for a certain building set (see Section 3.3 for more details), which is intimately connected to $\mathfrak{X}_n(\mathbf{b})$. The h -polynomial of $\mathbf{St}_n$ is the *binomial Eulerian polynomial*

$$\tilde{A}_n(z) = 1 + z \sum_{k=1}^n \binom{n}{k} A_k(z),$$

where $A_k(z) = \sum_{\sigma \in \mathfrak{S}_k} z^{\text{des}(\sigma)}$ is the Eulerian polynomial [PRW08].

From Theorem 3.2.3(a) and Theorem 3.2.12, we have that $\overline{\mathfrak{X}}_n(\mathbf{b})$ is a simple general-

ized permutahedron. By [PRW08, Theorem 4.2], the h -polynomial of a simple generalized permutahedron can be computed using its vertex posets.

Since the face numbers of $\mathfrak{X}_n(\mathbf{b})$ do not change after the lifting ℓ , we may use this equation to compute the h -polynomial of $\mathfrak{X}_n(\mathbf{b})$. We characterize the vertex poset for every vertex of $\overline{\mathfrak{X}}_n(\mathbf{b})$.

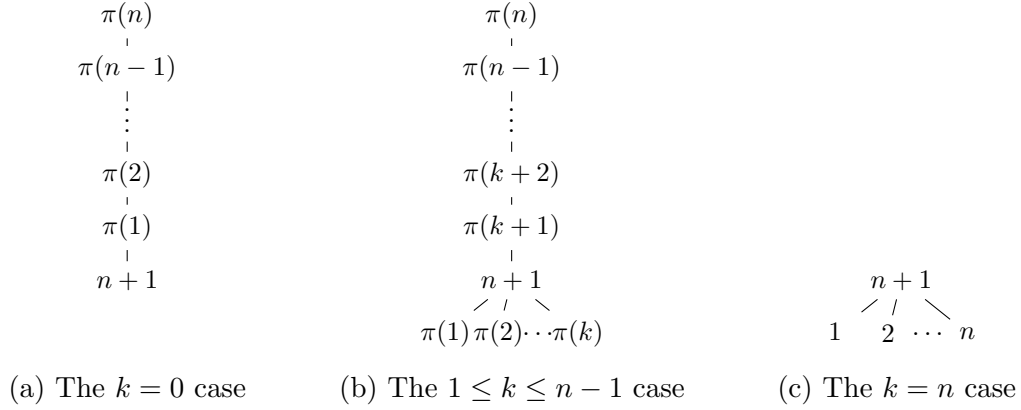


Figure 3.2: The vertex poset $Q_{\bar{\mathbf{v}}}$ in Lemma 3.2.15(a)

Lemma 3.2.15. *Let $\bar{\mathbf{v}}$ be a vertex of $\overline{\mathfrak{X}}_n(\mathbf{b})$. Fix $\pi \in \mathfrak{S}_n$ and $k \in \mathbb{Z}$ with $0 \leq k \leq n$ such that $\bar{\mathbf{v}}$ is the image under the lifting ℓ of the vertex $\pi(\mathbf{v}_k)$ of $\mathfrak{X}_n(\mathbf{b})$.*

(a) *If $b_1 \geq 2$, or if $b_1 = 1$ and $k \geq 2$, then the vertex poset $Q_{\bar{\mathbf{v}}}$ has the covering relations*

$$\begin{aligned} \pi(j) &\lessdot n+1 && \text{for } 1 \leq j \leq k, \\ n+1 &\lessdot \pi(k+1), \\ \pi(j-1) &\lessdot \pi(j) && \text{for } k+2 \leq j \leq n. \end{aligned}$$

(See the Hasse diagrams in Figure 3.2.)

(b) *If $b_1 = 1$ and $k \in \{0, 1\}$ (in which case $\mathbf{v}_k = \mathbf{v}_0 = \mathbf{v}_1$), then the vertex poset $Q_{\bar{\mathbf{v}}}$ has*

the covering relations

$$\pi(1) \triangleleft \pi(2) \triangleleft \cdots \triangleleft \pi(n),$$

$$n+1 \triangleleft \pi(2).$$

(See the Hasse diagram in Figure 3.3.)

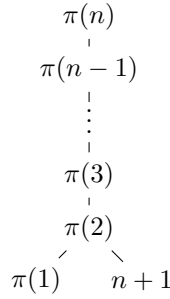


Figure 3.3: The vertex poset $Q_{\bar{\mathbf{v}}}$ in Lemma 3.2.15(b)

Proof. Let $\mathbf{v} := \pi(\mathbf{v}_k)$, so that $\bar{\mathbf{v}} = \ell(\mathbf{v})$. The normal cone at $\bar{\mathbf{v}}$ is the polar of the tangent cone at $\bar{\mathbf{v}}$. From the formula for the affine projection ℓ , we have that $\ell(\mathbf{w}) - \ell(\mathbf{v}) = L(\mathbf{w} - \mathbf{v})$ for all $\mathbf{w} \in \mathbb{R}^n$, where $L: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ is the linear map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, -\sum_{i=1}^n x_i)$. To compute the tangent cone of $\bar{\mathfrak{X}}_n(\mathbf{b})$ at $\bar{\mathbf{v}}$, we apply L to the generators of $\mathcal{T}_{\mathbf{v}}(\mathfrak{X}_n(\mathbf{b}))$ given in Theorem 3.2.8. Now,

$$L(\mathbf{e}_j) = \widehat{\mathbf{e}}_j - \widehat{\mathbf{e}}_{n+1}$$

for $1 \leq j \leq n$. Thus, the tangent cone $\mathcal{T}_{\bar{\mathbf{v}}}$ of $\bar{\mathfrak{X}}_n(\mathbf{b})$ at $\bar{\mathbf{v}}$ is generated as follows:

(a) If $b_1 \geq 2$, or if $b_1 = 1$ and $k \geq 2$, then $\mathcal{T}_{\bar{\mathbf{v}}}$ is generated by

$$\{\widehat{\mathbf{e}}_{\pi(j)} - \widehat{\mathbf{e}}_{n+1} : 1 \leq j \leq k\} \cup \{-\widehat{\mathbf{e}}_{\pi(k+1)} + \widehat{\mathbf{e}}_{n+1}\} \cup \{\widehat{\mathbf{e}}_{\pi(j-1)} - \widehat{\mathbf{e}}_{\pi(j)} : k+2 \leq j \leq n\}$$

if $k \leq n-1$, and by $\{\widehat{\mathbf{e}}_j - \widehat{\mathbf{e}}_{n+1} : 1 \leq j \leq n\}$ if $k = n$.

(b) If $b_1 = 1$ and $k \in \{0, 1\}$, then $\mathcal{T}_{\bar{\mathbf{v}}}$ is generated by

$$\{\widehat{\mathbf{e}}_{\pi(j-1)} - \widehat{\mathbf{e}}_{\pi(j)} : 2 \leq j \leq n\} \cup \{-\widehat{\mathbf{e}}_{\pi(2)} + \widehat{\mathbf{e}}_{n+1}\}.$$

These generating vectors of the tangent cone are the facet-defining outer normals of the normal cone. Thus, for example, in the case where $b_1 \geq 2$ and $k \leq n - 1$, the normal cone at $\bar{\mathbf{v}}$ is defined by the inequalities

$$\begin{aligned} x_{\pi(j)} &\leq x_{n+1} && \text{for } 1 \leq j \leq k, \\ x_{n+1} &\leq x_{\pi(k+1)}, \\ x_{\pi(j-1)} &\leq x_{\pi(j)} && \text{for } k+2 \leq j \leq n; \end{aligned}$$

and the other cases are handled similarly. These inequalities determine the covering relations given in the statement of Theorem 3.2.15. $\square$

Theorem 3.2.16. *The h -polynomial of $\mathfrak{X}_n(\mathbf{b})$ is*

$$h(\mathfrak{X}_n(\mathbf{b}); z) = \begin{cases} \tilde{A}_n(z) - nzA_{n-1}(z) & \text{if } b_1 = 1, \\ \tilde{A}_n(z) & \text{if } b_1 \geq 2. \end{cases}$$

When $b_1 = 1$ one can write $h(\mathfrak{X}_n(\mathbf{b}); z) = 1 + z \sum_{k=1, k \neq n-1}^n \binom{n}{k} A_k(z)$.

Proof. Consider first the case in which $b_1 \geq 2$. From Figure 3.2 (corresponding to Lemma 3.2.15(a)), we see that $Q_{\overline{\pi(\mathbf{v}_k)}}$ contains $1 + \text{des}(\pi(k+1), \dots, \pi(n))$ descents if $0 \leq k \leq n-1$, and 0 descents if $k = n$. Thus, enumerating vertices as in the proof of Theorem 3.2.4,

$$\sum_{\mathbf{v} \in \text{Vert}(\mathfrak{X}_n(\mathbf{b}))} z^{\text{des}(Q_{\bar{\mathbf{v}}})} = \sum_{k=0}^{n-1} \sum_{\substack{I \subseteq [n]: \\ |I|=n-k}} \sum_{\sigma \in \mathfrak{S}_{n-k}} z^{1+\text{des}(\sigma)} + 1 = 1 + z \sum_{k=1}^n \binom{n}{k} \sum_{\sigma \in \mathfrak{S}_k} z^{\text{des}(\sigma)} = \tilde{A}_n(z).$$

Now consider the case in which $b_1 = 1$. From Figures 3.2 and 3.3, we see that $Q_{\overline{\pi(\mathbf{v}_k)}}$ contains $1 + \text{des}(\pi)$ descents if $k = 1$, $1 + \text{des}(\pi(k+1), \dots, \pi(n))$ descents if $2 \leq k \leq n-1$,

and 0 descents if $k = n$. Again enumerating vertices as in the proof of Theorem 3.2.4,

$$\begin{aligned} \sum_{\mathbf{v} \in \text{Vert}(\mathfrak{X}_n(\mathbf{b}))} z^{\text{des}(Q_{\overline{\mathbf{v}}})} &= \sum_{\pi \in \mathfrak{S}_n} z^{1+\text{des}(\pi)} + \sum_{k=2}^{n-1} \sum_{\substack{I \subseteq [n]: \\ |I|=n-k}} \sum_{\sigma \in \mathfrak{S}_{n-k}} z^{1+\text{des}(\sigma)} + 1 \\ &= 1 + z \sum_{k=1}^{n-2} \binom{n}{k} \sum_{\sigma \in \mathfrak{S}_k} z^{\text{des}(\sigma)} + z \sum_{\pi \in \mathfrak{S}_n} z^{\text{des}(\pi)} = \tilde{A}_n(z) - n z A_{n-1}(z). \quad \square \end{aligned}$$

Since the h -polynomial of $\mathfrak{X}_n(\mathbf{b})$ depends only on whether $b_1 = 1$ or $b_1 \geq 2$, comparing with results on faces numbers of $\mathfrak{X}_n(a, b, b, \dots, b)$ in [HLVM24, Prop. 3.14], we immediately have the following result giving all face numbers of $\mathfrak{X}_n(\mathbf{b})$ for all $\mathbf{b}$.

Corollary 3.2.17. *Let f_k be the number of k -dimensional faces of $\mathfrak{X}_n(\mathbf{b})$ for $k = 0, 1, \dots, n$. Then,*

$$f_k = \begin{cases} \sum_{j=0, j \neq 1}^{n-k} \binom{n}{j} (n-k-j)! S(n-j+1, n-k-j+1) & \text{if } b_1 = 1, \\ \sum_{j=0}^{n-k} \binom{n}{j} (n-k-j)! S(n-j+1, n-k-j+1) & \text{if } b_1 \geq 2, \end{cases}$$

where $S(n, k)$ is the Stirling number of the second kind.

3.3 A building set perspective

In this section, we completely characterize the face structure and identify the combinatorial types of all $\mathbf{b}$ -parking-function polytopes. In particular, we show that even though not every $\mathbf{b}$ -parking-function polytope is a nestohedron (see Theorem 1.5.17), up to combinatorial equivalence, every $\mathbf{b}$ -parking-function polytope is either the classical parking-function polytope PF_n (if $b_1 = 1$) or the stellohedron St_n (if $b_1 \geq 2$).

We consider the $\mathcal{Y}$ -generalized permutahedron associated with a certain class of collections $\mathcal{B}$. Recall some background on building sets, nestohedra, and nested set complexes from Section 1.5.

Example 3.3.1. Consider the collection

$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

is a building set on $[4]$. Note that a nested set is a subset of $\mathcal{B}$. To simplify the notation, in this example and in Figure 3.4 each set $\{i\}$ is abbreviated as i , and each set $\{i, j, k\}$ is abbreviated as ijk . The corresponding nested sets and their dimensions as faces in $\Delta_{\mathcal{B}}$ are as follows:

$$\left\{ \begin{array}{l} \text{dim} = -1 \qquad \qquad \qquad \emptyset \\ \hline \text{dim} = 0 \qquad \qquad \{1\}, \{2\}, \{3\}, \{4\}, \{124\}, \{134\}, \{234\} \\ \hline \text{dim} = 1 \qquad \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 124\}, \{1, 134\}, \\ \qquad \qquad \qquad \{2, 124\}, \{2, 234\}, \{3, 134\}, \{3, 234\}, \{4, 124\}, \{4, 134\}, \{4, 234\} \\ \hline \text{dim} = 2 \qquad \{1, 2, 3\}, \{1, 2, 124\}, \{1, 3, 134\}, \{2, 3, 234\}, \{1, 4, 124\}, \\ \qquad \qquad \qquad \{1, 4, 134\}, \{2, 4, 124\}, \{2, 4, 234\}, \{3, 4, 134\}, \{3, 4, 234\} \end{array} \right\}.$$

This building set's nestohedron is the simple polytope in Figure 3.4, which is combinatorially equivalent to the classical parking function. Each of the vertices corresponds to a maximal nested set. In fact, each of the faces of the nestohedron corresponds to the nested set that is the intersection of the maximal nested sets that correspond to the vertices of the face.

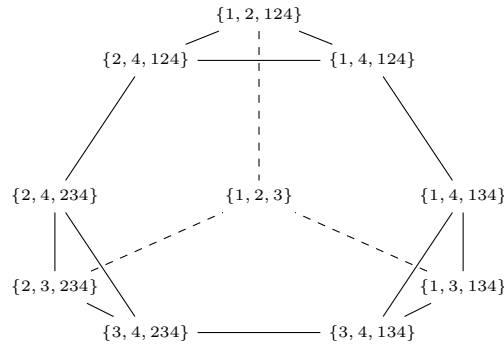


Figure 3.4: The nestohedron from Theorem 3.3.1.

There are two special families of vectors $\mathbf{b}$ for which $\overline{\mathfrak{X}}_n(\mathbf{b})$ is a translation by $\widehat{\mathbf{e}}_{n+1}$ of a

nestohedron whose face structure can be determined without much more work.

Definition 3.3.2 ([PRW08, Section 10.4]). A stellohedron $\mathbf{St}_n$ is a nestohedron for the building set

$$\{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : S \subseteq [n]\}.$$

Proposition 3.3.3. Let $\mathbf{b} \in \mathbb{Z}_{>0}^n$. Let y_I ($I \subseteq [n+1]$) be the associated coefficients given in Theorem 3.2.14. Assume that $y_I = \sum_{j=0}^{|I|-3} (-1)^{|I|+j-1} \binom{|I|-3}{j} b_{j+2} > 0$ for all sets I for which $n+1 \in I$ and $|I| \geq 3$,

- (a) If $b_1 \geq 2$, then the $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$ is combinatorially equivalent to the stellohedron $\mathbf{St}_n$.
- (b) If $b_1 = 1$, then $\mathfrak{X}_n(\mathbf{b})$ is combinatorially equivalent to the nestohedron for the building set

$$\{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : S \subseteq [n], |S| \neq 1\}.$$

Proof. From the assumption of Theorem 3.3.3, observe that $\mathfrak{X}_n(\mathbf{b})$ is combinatorially equivalent to the following Minkowski sum:

$$\mathfrak{X}_n(\mathbf{b}) \cong \bar{\mathfrak{X}}_n(\mathbf{b}) \cong \sum_{i=1}^n \Delta_{\{i\}} + \chi(b_1) \sum_{i=1}^n \Delta_{\{i, n+1\}} + \sum_{S \subseteq [n], |S| \geq 2} \Delta_{S \cup \{n+1\}},$$

where $\chi : \mathbb{Z}_{>0} \rightarrow \{0, 1\}$ is the indicator function getting value 1 if $b_1 \geq 2$ and getting value 0 if $b_1 = 1$. Note that if $b_1 \geq 2$ the indices of the standard simplices in the above Minkowski sum range over the collection $\{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : \emptyset \neq S \subseteq [n]\}$. If we add $\{n+1\}$ to this collection, the building set that results corresponds to the stellohedron $\mathbf{St}_n$. Adding $\Delta_{\{n+1\}}$ to the Minkowski sum only translates the polytope by $\hat{\mathbf{e}}_{n+1}$ and does not change the face structure. Hence, we have

$$\mathfrak{X}_n(\mathbf{b}) \cong \sum_{i=1}^n \Delta_{\{i\}} + \sum_{S \subseteq [n], S \neq \emptyset} \Delta_{S \cup \{n+1\}} + \Delta_{\{n+1\}} \cong \mathbf{St}_n.$$

If $b_1 = 1$, after adding $\Delta_{\{n+1\}}$ to the Minkowski sum, we have

$$\mathfrak{X}_n(\mathbf{b}) \cong \sum_{i=1}^n \Delta_{\{i\}} + \sum_{S \subseteq [n], |S| \geq 2} \Delta_{S \cup \{n+1\}} + \Delta_{\{n+1\}}.$$

The indices of the standard simplices in the above Minkowski sum range over the collection

$$\{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : S \subseteq [n], |S| \neq 1\},$$

which is also a building set. Therefore, $\mathfrak{X}_n(\mathbf{b})$ is combinatorially equivalent to the nestohedron corresponding to this building set. $\square$

We will see later in Theorem 3.3.11 that the two face structures in Theorem 3.3.3 are the only combinatorial types that can occur on $\mathfrak{X}_n(\mathbf{b})$ for *any* $\mathbf{b} \in \mathbb{Z}_{>0}^n$, even if the corresponding y_I do not satisfy the positivity conditions of Theorem 3.3.3. We introduce the notation

$$\mathcal{B}_{\text{St}_n} := \{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : S \subseteq [n]\},$$

$$\mathcal{B}_{\text{PF}_n} := \{\{1\}, \{2\}, \dots, \{n\}\} \cup \{S \cup \{n+1\} : S \subseteq [n], |S| \neq 1\},$$

for the two building sets in Theorem 3.3.3. We also introduce some terminology from [BHM⁺22] to characterize the nested sets. We say a subset $I \subseteq [n]$ and a (nonempty) chain $\mathcal{F} = (A_1 \subsetneq \dots \subsetneq A_k)$ in the Boolean lattice $(2^{[n]}, \subseteq)$ are *compatible*, denoted by $I \leq \mathcal{F}$, if $I \subseteq \min(\mathcal{F}) = A_1$. If $\mathcal{F}$ is the empty chain $\emptyset$, then we consider every subset I to be compatible with $\emptyset$. The nested sets for $\mathcal{B}_{\text{St}_n}$ and $\mathcal{B}_{\text{PF}_n}$ are characterized in the next propositions.

Proposition 3.3.4 ([Lia24, Prop. 5.3]). *The nested sets for $\mathcal{B}_{\text{St}_n}$ have form $\{\{i\}\}_{i \in I} \cup \{A \cup \{n+1\}\}_{A \in \mathcal{F}}$, where $I \subseteq [n]$ and $\mathcal{F} \subset 2^{[n]} \setminus \{[n]\}$ is a chain such that $I \leq \mathcal{F}$ is a compatible pair.*

The building set and the collection of nested sets in Theorem 3.3.1 is the case of $n = 3$ of the proposition below.

Proposition 3.3.5. *The nested set for $\mathcal{B}_{\text{PF}_n}$ is of the form $\{\{i\}\}_{i \in I} \cup \{A\}_{A \in \mathcal{F}}$, where $I \subseteq [n+1]$ and $|I| \leq 2$ if $n+1 \in I$, and $\mathcal{F} \subset 2^{[n+1]} \setminus \{[n+1]\}$ is a chain such that $|\min(\mathcal{F})| \geq 3$ and $n+1 \in \min(\mathcal{F})$, and $I \leq \mathcal{F}$ is a compatible pair.*

Proof. Let N be a nested set for $\mathcal{B}_{\text{PF}_n}$. Let I be the subset consisting of $i \in [n+1]$ for any singleton $\{i\} \in N$ and $\mathcal{F}$ be the collection of all subsets in N that are not a singleton. By the definition of nested set for $\mathcal{B}_{\text{PF}_n}$, we must have $n+1 \in A$ and $|A| \geq 3$ for any $A \in \mathcal{F}$. Pick any $A_1, A_2 \in \mathcal{F}$. Then we always have $n+1 \in A_1 \cap A_2 \neq \emptyset$. Hence one of A_1 or A_2 must be a subset of the other and therefore $\mathcal{F}$ is a chain. For any $i \in I$, we must have $i \in A$ for all $A \in \mathcal{F}$; otherwise, there exists $A' \in \mathcal{F}$ such that $\{i\}$ and A' are disjoint subsets in N with $A' \cup \{i\} \in \mathcal{B}_{\text{PF}_n}$, which violates the condition (N2) in the definition of nested sets (Theorem 1.5.18). This implies $I \subseteq \min(\mathcal{F})$, so $I \leq \mathcal{F}$. If $n+1 \in I$ and $|I| \geq 3$, then the union of all singletons in I , which is I itself, is in $\mathcal{B}_{\text{PF}_n}$, which again violates condition (N2). Thus, we have $|I| \leq 2$ if $n+1 \in I$. Conversely, it is routine to check that such a collection $\{\{i\}\}_{i \in I} \cup \{A\}_{A \in \mathcal{F}}$ satisfies conditions (N1) and (N2) of Theorem 1.5.18. $\square$

In order to understand the face lattice of $\mathfrak{X}_n(\mathbf{b})$ in terms of building sets, we now generalize the notation U_i for certain facets of $\mathfrak{X}_n(\mathbf{b})$ introduced immediately before Theorem 3.2.6 by setting

$$U_I := \left\{ (x_1, \dots, x_n) \in \mathfrak{X}_n(\mathbf{b}) : \sum_{i \in I} x_i = \sum_{i=n-|I|+1}^n S_i \right\},$$

for all nonempty $I \subseteq [n]$, except in the case $|I| = n-1$ if $b_1 = 1$. Thus, U_I is the facet of $\mathfrak{X}_n(\mathbf{b})$ defined by the (**b-parking**) inequality for the subset I . In terms of this notation, the facets U_j defined prior to Theorem 3.2.6 are given by $U_j = U_{I_j}$, where $I_j := \{j, j+1, \dots, n\}$. Comparing the supporting hyperplanes of the facets of $\mathfrak{X}_n(\mathbf{b})$ in Theorem 3.2.3(c) and the two building sets in Theorem 3.3.3, we immediately arrive at the following lemma.

Lemma 3.3.6. *Let $\mathcal{B} := \mathcal{B}_{\text{St}_n}$ if $b_1 \geq 2$, and let $\mathcal{B} := \mathcal{B}_{\text{PF}_n}$ if $b_1 = 1$. The facets of $\mathfrak{X}_n(\mathbf{b})$ are in one-to-one correspondence with the elements of $\mathcal{B} \setminus \mathcal{B}_{\max} = \mathcal{B} \setminus \{[n+1]\}$ via the following*

bijection:

$$L_i \longmapsto \{i\}, \quad U_{[n] \setminus S} \longmapsto S \cup \{n+1\}$$

for all $i \in [n]$, all proper subsets $S \subsetneq [n]$ if $b_1 \geq 2$, and all proper subsets $S \subsetneq [n]$ with $|S| \neq 1$ if $b_1 = 1$.

For any $B \in \mathcal{B} \setminus \{[n+1]\}$, we denote by F_B the corresponding facet of $\mathfrak{X}_n(\mathbf{b})$ under the bijection in Theorem 3.3.6.

Lemma 3.3.7. *Let $\mathcal{B} := \mathcal{B}_{\text{PF}_n}$ if $b_1 = 1$, and let $\mathcal{B} := \mathcal{B}_{\text{St}_n}$ if $b_1 \geq 2$. Then the map*

$$N \longmapsto \bigcap_{B \in N} F_B$$

is a bijection between the maximal nested sets N (i.e., the facets of $\Delta_{\mathcal{B}}$) and the vertices of $\mathfrak{X}_n(\mathbf{b})$.

Proof. Consider $b_1 = 1$. The maximal nested sets for $\mathcal{B}_{\text{PF}_n}$ are of the form $\{\{i\}\}_{i \in I} \cup \{A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_{n-|I|}\}$ satisfying the condition of Theorem 3.3.5 and $|A_{i+1}| - |A_i| = 1$ for all i . If $n+1 \notin I$, then the corresponding vertex can be obtained as a vector in $\mathbb{R}^n$ by placing 1s in positions I , then placing S_n in position $[n+1] \setminus A_{n-|I|}$, placing S_{n-1} in position $A_{n-|I|} \setminus A_{n-|I|-1}$, and so on, and in the final step placing $S_{|I|+1}$ in position $A_2 \setminus A_1$. In this way, we fill out all n entries of the vector in $\mathbb{R}^n$, and the resulting vector is the permutation of $\mathbf{v}_{|I|}$ which is a vertex by Theorem 3.2.3(b). If $n+1 \in I$, then $I = \{\{i\}, \{n+1\}\}$ for some $i \in [n]$. Then we obtain the vertex as a vector in $\mathbb{R}^n$ by placing $1 = b_1 = S_1$ in position i , placing S_n in position $[n+1] \setminus A_{n-2}$, placing S_{n-1} in position $A_{n-2} \setminus A_{n-3}$, etc., placing S_3 in position $A_2 \setminus A_1$, and in the final step placing S_2 in position $A_1 \setminus \{i, n+1\}$. In this case, the resulting vector is the permutation of $\mathbf{v}_1 = \mathbf{v}_0$ which is also a vertex by Theorem 3.2.3(b). As a result, this gives a bijection between the maximal nested set for $\mathcal{B}_{\text{PF}_n}$ and the vertices of $\mathfrak{X}_n(\mathbf{b})$ for $b_1 = 1$ via the intersections of facets corresponding to the maximal nested sets.

For the case of $b_1 \geq 2$, the proof is similar. $\square$

Remark 3.3.8. In both cases $b_1 = 1$ and $b_1 \geq 2$, one can further consider the vertex posets Q_v for every v in the set of vertices $\text{Vert}(\mathfrak{X}_n(\mathbf{b}))$ (see Subsection 3.2.3) cooperating with the bijection in Theorem 3.3.7. It gives the following commutative diagram

$$\begin{array}{ccc} \text{Vert}(\mathfrak{X}_n(\mathbf{b})) & \xrightarrow{\quad\quad\quad} & \{\text{maximal nested sets for } \mathcal{B}\} \\ & \searrow \quad \quad \nearrow & \\ & \{Q_v : v \in \text{Vert}(\mathfrak{X}_n(\mathbf{b}))\} & \end{array}$$

For example, the following are the correspondences between the weakly increasing vertices of $\mathfrak{X}_n(\mathbf{b})$ for $b_1 = 1$ and their vertex posets and the maximal nested sets:

$$\begin{array}{ccccc} \left(\underbrace{1, \dots, 1}_k, S_{k+1}, \dots, S_n \right) & \longleftrightarrow & \begin{array}{c} n \\ | \\ n-1 \\ \vdots \\ k+2 \\ | \\ k+1 \\ | \\ n+1 \\ / \quad \backslash \\ 1 \quad 2 \quad \dots \quad k \end{array} & \longleftrightarrow & \left\{ \begin{array}{l} \{1\}, \{2\}, \dots, \{k\}, \\ [k] \cup \{n+1\}, \\ [k+1] \cup \{n+1\}, \\ \dots, \\ [n-1] \cup \{n+1\} \end{array} \right\} \\ & & & & \\ (1, S_2, \dots, S_n) & \longleftrightarrow & \begin{array}{c} n \\ | \\ n-1 \\ \vdots \\ 3 \\ | \\ 2 \\ / \quad \backslash \\ 1 \quad n+1 \end{array} & \longleftrightarrow & \left\{ \begin{array}{l} \{1\}, \{n+1\}, \\ [2] \cup \{n+1\}, \\ [3] \cup \{n+1\}, \\ \dots, \\ [n-1] \cup \{n+1\} \end{array} \right\} \end{array}$$

With the aid of Lemma 3.3.6 and Lemma 3.3.7, we can completely characterize the face structure of the $\mathbf{b}$ -parking-function polytope for any $\mathbf{b}$ as follows.

Theorem 3.3.9. Let $\mathcal{B} := \mathcal{B}_{\text{PF}_n}$ if $b_1 = 1$, and let $\mathcal{B} := \mathcal{B}_{\text{St}_n}$ if $b_2 \geq 1$. Then the map

$$\Delta_{\mathcal{B}} \longrightarrow \{\text{nonempty faces of } \mathfrak{X}_n(\mathbf{b})\}, \quad \text{where} \quad N \longmapsto \mathcal{F}_N := \bigcap_{B \in N} F_B$$

is an isomorphism between the dual of the face lattice of the nested set complex $\Delta_{\mathcal{B}}$ and the face lattice of $\mathfrak{X}_n(\mathbf{b})$.

Proof. We will give the proof in detail only for the case in which $b_1 = 1$. A similar approach works for the case in which $b_1 \geq 2$.

Consider $\mathbf{b} = (b_1, \dots, b_n)$ with $b_1 = 1$. We first prove that the map from the nested sets in $\Delta_{\mathcal{B}_{\text{PF}_n}}$ to the set of nonempty faces of $\mathfrak{X}_n(\mathbf{b})$ is well-defined. From Theorem 3.3.6, the singleton nested set $\{B\} \in \Delta_{\mathcal{B}_{\text{PF}_n}}$ corresponds to the facet F_B of $\mathfrak{X}_n(\mathbf{b})$. Since any

intersection of facets is a face of the polytope, we know $\mathcal{F}_N$ is a face of $\mathfrak{X}_n(\mathbf{b})$ for any nested set $N \in \Delta_{\mathcal{B}_{\text{PF}_n}}$. For any nested set N , the face $\mathcal{F}_N$ is nonempty by Theorem 3.3.7 because $\mathcal{F}_N$ contains those vertices that correspond to the maximal nested sets containing N . Hence, the map is well-defined.

Now we check the injectivity of the map. By Theorem 3.2.3(a), $\mathfrak{X}_n(\mathbf{b})$ is a simple polytope. In a simple polytope, every nonempty intersection of facets determines a unique face. Hence, the injectivity of the map follows.

We now show that the map is surjective. Let N' be a subset of $\mathcal{B}_{\text{PF}_n}$ such that the face $\bigcap_{B \in N'} F_B$ is nonempty. Say $N' = \{\{i\}\}_{i \in I'} \cup \{A\}_{A \in \mathcal{F}'}$ for some $I' \subset [n+1]$ and $\mathcal{F}'$ consisting of non-singleton subsets in $\mathcal{B}_{\text{PF}_n}$, then

$$\bigcap_{B \in N'} F_B = \left(\bigcap_{i \in I'} L_i \right) \cap \left(\bigcap_{A \in \mathcal{F}'} U_{[n] \setminus A} \right).$$

We now show that N' must be a nested set for $\mathcal{B}_{\text{PF}_n}$ (Recall Proposition 3.3.5).

- By definition of $\mathcal{B}_{\text{PF}_n}$, we have $n+1 \in A$ and $|A| \geq 3$ for any $A \in \mathcal{F}'$ right away.
- First, for any $A \in \mathcal{F}'$ we must have $I' \subset A$; otherwise, there is $j \in I'$ but $j \notin A$ for some $A \in \mathcal{F}'$. Then this face $\bigcap_{B \in N'} F_B$ is contained in the intersection of the facets $F_j \cap U_{[n] \setminus A}$, which is an empty set since $j \in [n] \setminus A$. One can check the defining equations of the hyperplanes that there are no vertices of $\mathfrak{X}_n(\mathbf{b})$ in this intersection. This contradicts the assumption that the face is nonempty.
- Second, we claim if $n+1 \in I'$, then $|I'| \leq 2$. This is because the face $\bigcap_{B \in N'} F_B$ is contained in the facet $U_{[n]}$ whose supporting hyperplane is $\sum_{i=1}^n x_i = S_n + S_{n-1} + \dots + S_1$. Any vertex of $\mathfrak{X}_n(\mathbf{b})$ on $U_{[n]}$ is the permutation of $(S_1, S_2, \dots, S_n) = (1, S_2, \dots, S_n)$ which has exactly one coordinate being 1. Thus, I' does not contain more than one element that is not $n+1$.
- In the final step, we show that $\mathcal{F}'$ must be a chain. Let $A_1, A_2 \in \mathcal{F}'$ with $|A_1| \leq |A_2|$ then we must have $A_1 \subseteq A_2$. If not, there exists $j \in A_1$, but $j \notin A_2$. Notice that any vertex

$v = (v_1, \dots, v_n)$ of $\mathfrak{X}_n(\mathbf{b})$ on the facet $U_{[n] \setminus A_2}$ has coordinates satisfying

$$\{v_i : i \in [n] \setminus A_2\} = \{S_n, S_{n-1}, \dots, S_{|A_2|+1}\};$$

while the vertex v on $U_{[n] \setminus A_1}$ has coordinates satisfying

$$\{v_i : i \in [n] \setminus A_1\} = \{S_n, \dots, S_{|A_2|+1}, \dots, S_{|A_1|+1}\}.$$

Then there is no vertex of $\mathfrak{X}_n(\mathbf{b})$ that lies on $U_{[n] \setminus A_2}$ and $U_{[n] \setminus A_1}$ simultaneously since all the vertices have at most one coordinate equal to any one of $S_n, S_{n-1}, \dots, S_{|A_2|+1}$.

Therefore, by Theorem 3.3.5 the subset N' is a nested set for $\mathcal{B}_{\mathbf{PF}_n}$.

Now the map is a bijection. By the definition of the map, it is clear that any two nested sets satisfy $N_1 \subseteq N_2$ if and only if $\mathcal{F}_{N_1} \supseteq \mathcal{F}_{N_2}$. Therefore, the map is a poset isomorphism. $\square$

Recall, as discussed immediately following Theorem 1.5.18, that given a building set $\mathcal{B} \subset 2^{[n]}$, the face lattice of the nestohedron $P_{\mathcal{B}} = \sum_{I \in \mathcal{B}} y_I \Delta_I$ (where $y_I > 0$) is completely determined by the building set $\mathcal{B}$ and is dual to the face lattice of the nested set complex $\Delta_{\mathcal{B}}$. It is somewhat surprising that the face lattice of $\mathfrak{X}_n(\mathbf{b})$ is determined by the two building sets $\mathcal{B}_{PF_n}$ and $\mathcal{B}_{St_n}$ because from Proposition 3.2.14 we have seen that for most cases of $\mathbf{b}$ the polytope $\mathfrak{X}_n(\mathbf{b})$ is not a nestohedron (or a translation of one). It will be interesting to understand how the value of y_I affects the face structure of $\sum_{\emptyset \neq I \subseteq [n]} y_I \Delta_I$. More concretely, can we characterize the parameters $\{y_I\}$ (with no positivity constraint on $y_I \in \mathbb{R}$) such that $\sum_{\emptyset \neq I \subseteq [n]} y_I \Delta_I$ is combinatorially equivalent to $P_{\mathcal{B}}$ for a given building set $\mathcal{B}$?

From Theorem 3.3.9, we can explicitly tell which facets intersect at a given face and all the vertices on this face by observing the nested set corresponding to the face, as illustrated in the following example.

Example 3.3.10. For $\mathbf{b} = (1, b_2, \dots, b_6)$, consider the nested set $N = \{\{1\}, \{3\}\} \cup \{\{1, 3, 4, 7\}, \{1, 2, 3, 4, 7\}\}$

Then the face $\mathcal{F}_N$ is the intersection of the following facets of $\mathfrak{X}_6(\mathbf{b})$:

$$L_1 = \{\mathbf{x} \in \mathfrak{X}_6(\mathbf{b}) : x_1 = 1\}, L_3 = \{\mathbf{x} \in \mathfrak{X}_6(\mathbf{b}) : x_3 = 1\},$$

$$U_{\{2,5,6\}} = \{\mathbf{x} \in \mathfrak{X}_6(\mathbf{b}) : x_2 + x_5 + x_6 = S_6 + S_5 + S_4\}, U_{\{5,6\}} = \{\mathbf{x} \in \mathfrak{X}_6(\mathbf{b}) : x_5 + x_6 = S_6 + S_5\}.$$

The face $\mathcal{F}_N$ is 2-dimensional and has 4 vertices:

$$(1, S_4, 1, 1, S_5, S_6), (1, S_4, 1, S_3, S_5, S_6), (1, S_4, 1, 1, S_6, S_5), (1, S_4, 1, S_3, S_6, S_5).$$

Corollary 3.3.11. For any $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{Z}_{>0}^n$, the $\mathbf{b}$ -parking-function polytope $\mathfrak{X}_n(\mathbf{b})$ is combinatorially equivalent to the classical parking-function polytope PF_n if $b_1 = 1$ or is combinatorially equivalent to the stellohedron St_n if $b_1 \geq 2$.

Next, we provide a visualization of the two 3-dimensional cases of Theorem 3.3.11.

Example 3.3.12. The combinatorial type of $\mathfrak{X}_n(\mathbf{b})$ is as in Figure 3.4 when $n = 3$ and $b_1 = 1$ and is as in Figure 3.5 when $n = 3$ and $b_1 \geq 2$. The coordinate of each vertex can be obtained by using the bijection described in the proof of Theorem 3.3.7.

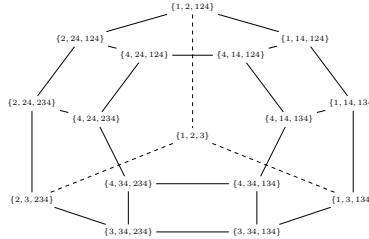


Figure 3.5: The combinatorial type of $\mathfrak{X}_n(\mathbf{b})$ when $n = 3$ and $b_1 \geq 2$.

Up to this point we have observed $\mathfrak{X}_n(\mathbf{b})$ as a generalized permutahedron and have characterized its combinatorial types through the use of building sets. In the next section, we exhibit $\mathfrak{X}_n(\mathbf{b})$ as a special polymatroid, which allows us to obtain further results.

3.4 A polymatroid perspective

Here we study the $\mathbf{b}$ -parking-function polytope through a polymatroid lens. Polymatroids are a class of polyhedra that were introduced by Edmonds [Edm03] in 1970 as polyhedral generalizations of matroids. We note that generalized permutahedra are equivalent (up to translation) to polymatroids. By exhibiting $\mathfrak{X}_n(\mathbf{b})$ as a polymatroid, we are able to draw conclusions about properties such as its combinatorial diameters. For background on combinatorial diameters see [YKK84, KS10]. We begin by recalling a few helpful terms and definitions from matroid theory [Oxl11].

Definition 3.4.1. *Let $f : 2^{[n]} \rightarrow \mathbb{R}$ be a set function. The function f is said to be submodular if for every $X, Y \subseteq [n]$ with $X \subseteq Y$ and every $z \in [n] \setminus Y$, the following inequality holds: $f(X \cup \{z\}) - f(X) \geq f(Y \cup \{z\}) - f(Y)$. The polymatroid associated with such a function f is the polytope*

$$P_f := \left\{ \mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in [n], \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq [n] \right\}.$$

A set function $f : 2^{[n]} \rightarrow \mathbb{R}$ is said to be *nondecreasing* if, for every $X, Y \subseteq [n]$ with $X \subseteq Y$, we have $f(X) \leq f(Y)$. Henceforth, let $f_{\mathbf{b}} : 2^{[n]} \rightarrow \mathbb{Z}_{\geq 0}$ be the set function defined by setting $f_{\mathbf{b}}(\emptyset) := 0$ and

$$f_{\mathbf{b}}(I) := -|I| + \sum_{j=1}^n \min\{j, |I|\} b_{n-j+1}, \quad \text{for } \emptyset \neq I \subseteq [n].$$

Proposition 3.4.2. *The function $f_{\mathbf{b}}$ is a nondecreasing submodular function.*

Proof. By construction, $f_{\mathbf{b}}$ is integer and nondecreasing. It remains to be shown that $f_{\mathbf{b}}$ is submodular. Observe that $f_{\mathbf{b}}(I) = -|I| + \sum_{j=1}^n \min\{j, |I|\} b_{n-j+1}$ only depends on the cardinality $|I|$ of I , so we write $f_{\mathbf{b}}(|I|) = f_{\mathbf{b}}(I)$. It suffices to prove that $I \subset J \subsetneq [n] \Rightarrow f_{\mathbf{b}}(|I| + 1) - f_{\mathbf{b}}(|I|) \geq f_{\mathbf{b}}(|J| + 1) - f_{\mathbf{b}}(|J|)$, or equivalently

$$i \leq j < n \quad \Rightarrow \quad f_{\mathbf{b}}(i + 1) - f_{\mathbf{b}}(i) \geq f_{\mathbf{b}}(j + 1) - f_{\mathbf{b}}(j).$$

The difference $f_{\mathbf{b}}(i+1) - f_{\mathbf{b}}(i)$ is

$$f_{\mathbf{b}}(i+1) - f_{\mathbf{b}}(i) = -(i+1) - (-i) + \sum_{k=1}^n (\min\{i+1, k\} - \min\{i, k\}) b_{n-k+1} = -1 + \sum_{k=i+1}^n b_{n-k+1}.$$

As $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_{>0}^n$,

$$\sum_{k=i+1}^n b_{n-k+1} \geq \sum_{k=j+1}^n b_{n-k+1} \quad \text{for } i \leq j.$$

Thus, $f_{\mathbf{b}}$ is submodular. □

We translate the inequality description of $\mathfrak{X}_n(\mathbf{b})$ (Theorem 3.2.3(c)) to a description of the form P_f . Setting $y_i := x_i - 1$, one obtains

$$x_i \geq 1 \quad \Leftrightarrow \quad y_i \geq 0 \quad \text{for all } i \in [n],$$

and the remaining constraints in Theorem 3.2.3(c) take the form

$$\sum_{i \in I} x_i \leq \sum_{j=1}^n \min\{j, |I|\} b_{n-j+1} \quad \Leftrightarrow \quad \sum_{i \in I} y_i \leq -|I| + \sum_{j=1}^n \min\{j, |I|\} b_{n-j+1}.$$

Note that this transformation corresponds to the translation of $\mathfrak{X}_n(\mathbf{b})$ by $-(1, \dots, 1) \in \mathbb{R}^n$.

The resulting polymatroid is now described as

$$\mathfrak{X}'_n(\mathbf{b}) := \left\{ \mathbf{y} \in \mathbb{R}^n : y_i \geq 0 \text{ for all } i \in [n], \sum_{i \in I} y_i \leq f_{\mathbf{b}}(I) \text{ for all } I \subseteq [n] \right\},$$

which proves the following.

Theorem 3.4.3. *The polytope $\mathfrak{X}'_n(\mathbf{b})$ is a polymatroid. In fact, $\mathfrak{X}'_n(\mathbf{b})$ is a special polymatroid for which upper bounds on the sum of entries in any set I depend only on the cardinality $|I|$.*

The polytopes $\mathfrak{X}_n(\mathbf{b})$ and $\mathfrak{X}'_n(\mathbf{b})$ are combinatorially equivalent. Thus, we can leverage

the theory of polymatroids to obtain results about $\mathfrak{X}_n(\mathbf{b})$. In particular, we note that known results about polymatroids allow for alternate directions to recover Theorem 3.2.3. Topkis [Top84] studied extreme point adjacency in polymatroids [Top84, Corollary 5.4], and his work can be used to deduce Theorem 3.2.3(a). Shapley’s observation on the structure of the vertices of a polymatroid [Sha71] can be used to obtain Theorem 3.2.3(b). Edmonds [Edm03] showed that we can optimize linear functions over polymatroids in strongly polynomial time by certifying *total dual integrality* through the greedy algorithm. In turn, this yields another direction for proving the inequality description in Theorem 3.2.3(c).

Next, using Theorem 3.4.3 and Theorem 3.2.7, we can carry over known upper bounds on the combinatorial diameters of polymatroids [Top92] to $\mathfrak{X}_n(\mathbf{b})$, of the form $\min\{2n, \frac{1}{2}n(n - 1) + 1\}$. Recall that the *combinatorial diameter* of a polytope is the diameter of its edge graph, that is, the maximum length of a shortest edge-path between two vertices. We now provide a direct proof of these upper bounds, which allows us to improve them for the case $b_1 = 1$ and see tightness in all cases.

Theorem 3.4.4. *If $b_1 > 1$, the combinatorial diameter of $\mathfrak{X}_n(\mathbf{b})$ is exactly $\min\{2n, \frac{1}{2}n(n - 1) + 1\}$. If $b_1 = 1$, the combinatorial diameter is exactly $\min\{2(n - 1), \frac{1}{2}n(n - 1)\}$.*

Proof. Note that $2n \leq \frac{1}{2}n(n - 1) + 1$ and $2(n - 1) < \frac{1}{2}n(n - 1)$ if and only if $n \geq 5$. For $n \leq 4$, tightness of the bounds can be verified exhaustively. For example, for $n = 3$, it is clear from Figure 3.1 that the combinatorial diameters of $\mathfrak{X}_3(1, 2, 3)$ and $\mathfrak{X}_3(2, 3, 4)$ are 3 and 4, respectively.

We provide an argument for the bounds $2n$ and $2(n - 1)$. Let $\mathbf{w}_k = \pi(\mathbf{v}_k)$ be a vertex of $\mathfrak{X}_n(\mathbf{b})$ where $k < n$. By Theorem 3.2.7, $\mathbf{w}_k$ is adjacent to $\mathbf{w}_{k+1} = \pi(\mathbf{v}_{k+1})$, which corresponds to reducing the $(k + 1)^{\text{th}}$ entry of $\mathbf{v}_k$ to 1. Consider the path $P_k = (\mathbf{w}_k, \mathbf{w}_{k+1}, \dots, \mathbf{w}_{n-1}, \mathbf{1})$ on $\mathfrak{X}_n(\mathbf{b})$, where P_k ends at $\mathbf{v}_n = \mathbf{w}_n = \mathbf{1}$. Because P_k begins at $\mathbf{w}_k$, the length of P_k is at most n when $b_1 > 1$. Concatenating paths when necessary, we have a path of length at most $2n$ between any two vertices of $\mathfrak{X}_n(\mathbf{b})$ with $b_1 > 1$. A path of this length is realized when traveling between the vertices $\mathbf{w}_0 = (b_1, S_2, \dots, S_n)$ and $\pi'(\mathbf{w}_0) = (S_n, S_{n-1}, \dots, S_2, b_1)$. When $b_1 = 1$, $\mathbf{w}_0 = \mathbf{w}_1$, so P_0 has length at most $n - 1$. A path of length $2(n - 1)$ is realized

when travelling between vertices $\mathbf{w}_1 = (1, S_2, \dots, S_n)$ and $\pi'(\mathbf{w}_1) = (S_n, S_{n-1}, \dots, S_2, 1)$.

It remains to show tightness. We exhibit that the paths constructed above are shortest paths between the given pairs of vertices. For $b_1 > 1$, the bound $2n$ for a path between $\mathbf{w}_0$ and $\pi'(\mathbf{w}_0)$ is achieved through a combination of n reductions and n increases of entries S_i , and no swaps. A path that exclusively uses swaps requires at least $\frac{1}{2}n(n-1)$ steps, and so is not better. Let S_j be the largest value that is reduced during an edge walk. It can be seen that any walk where $j = 1$ has to perform the same number of swaps as one exclusively using swaps; any walk where $j = n-1$ has to perform at least two further steps (in addition to $n-1$ reductions and $n-1$ increases). Let now $2 \leq j \leq n-2$. Then the number of swaps required to arrive at a vertex where the j largest values $S_n, \dots, S_{n-j+1}$ are in the target positions is bounded below by $j(n-j)$. (For every $1 \leq i_1 < i_2 \leq n$, it takes $i_2 - i_1$ swaps to move S_{i_2} to the position in which S_{i_1} lies; in a best case, the target positions for $S_n, \dots, S_{n-j+1}$ start these swaps with values $S_j, \dots, S_1$.) Combined with the at least j reductions and j increases, the path has length at least $2j + j(n-j) \geq 2j + 2(n-2) \geq 2n$.

An analogous argument with $n-1$ in place of n shows tightness of $2(n-1)$ for $b_1 = 1$. $\square$

3.5 Conclusion - Further perspectives

Recall that the classical parking-function polytope PF_n is the convex hull of all parking functions of length n in $\mathbb{R}^n$. As noted in [HLVM24, Remark 2.10] and [BCC⁺22, Remarks 4.8 and 4.9], there are other alternative perspectives on PF_n . It can be viewed as:

- a special case of the main object of study for this chapter, namely the $\mathbf{b}$ -parking-function polytope when $\mathbf{b} = (1, 1, 1, \dots, 1)$,
- a particular *partial permutahedron* $\mathcal{P}(n, n-1)$ as defined in [HS22] and further studied in [BCC⁺23, BCC⁺22],
- a specific *polytope of win vectors* as defined by Bartels et al. in [BMW97] for the case of the complete graph on n vertices (the work of Backman [Bac18, Theorem 4.5 and Corollary

4.6] provides alternate routes for computing the volume and lattice-point enumerator of PF_n), and

- essentially as the stochastic sandpile model for the complete graph [Sel24].

In addition to these interpretations, as direct corollaries to Theorem 3.2.12, Theorem 3.3.9, and Theorem 3.4.3, we can also add that (a lifted version of) PF_n :

- is a special generalized permutahedron,
- can be associated with a building set, and
- is a special type of polymatroid, respectively.

We conclude by adding an interpretation of the classical parking-function polytope as a projection of a relaxed Birkhoff or assignment polytope, and an interpretation of $\mathbf{b}$ -parking-function polytopes as projections of relaxed partition polytopes.

3.5.1 Relation of PF_n to Birkhoff/assignment polytopes

Consider an assignment problem for n cars and n parking spots, represented with decision variables $x_{ij} \in \{0, 1\}$ indicating car i being assigned to spot j . The linear relaxation of this problem yields the *Birkhoff polytope*, also called the *assignment polytope* or *the polytope of doubly stochastic matrices*:

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= 1 \quad \text{for all } 1 \leq i \leq n \\ \sum_{i=1}^n x_{ij} &= 1 \quad \text{for all } 1 \leq j \leq n \\ x_{ij} &\geq 0 \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq n. \end{aligned}$$

See [Bir46, BR74] and [Zie95, Example 0.12] and [Bru06, Chapter 9], as well as references within, for the extensive literature on these classical polytopes. Due to the total unimodu-

larity of the underlying constraint matrix and integrality of the right-hand sides, the vertices satisfy $x_{ij} \in \{0, 1\}$ [Bir46, BR74, VN53]. In fact, they are in one-to-one correspondence with all possible assignments of cars to parking spots. We will prove that a linear projection of a mild relaxation of this polytope gives PF_n .

To this end, note that either of the two types of equality constraints in the description of the assignment polytope, combined with the nonnegativity constraints, implies an upper bound of $x_{ij} \leq 1$. This allows an equivalent description of the form

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= 1 \quad \text{for all } 1 \leq i \leq n \\ \sum_{i=1}^n x_{ij} &\geq 1 \quad \text{for all } 1 \leq j \leq n \\ x_{ij} &\geq 0 \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq n. \end{aligned}$$

Informally, each car is assigned to precisely one spot, and each parking spot requires at least one car assigned to it, which is only possible if one retains a one-to-one assignment.

We now perform a relaxation where we allow for the assignment of multiple cars to the same (preferred) parking spot, following the design of PF_n : the i^{th} car to arrive picks from the first i spots. To represent the corresponding conditions $\alpha'_i \leq i$ in a nondecreasing rearrangement of the entries, one can replace the set of constraints $\sum_{i=1}^n x_{ij} \geq 1$ by sums of the first k constraints for $1 \leq k \leq n$:

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= 1 \quad \text{for all } 1 \leq i \leq n \\ \sum_{j=1}^k \sum_{i=1}^n x_{ij} &\geq k \quad \text{for all } 1 \leq k \leq n \\ x_{ij} &\geq 0 \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq n. \end{aligned}$$

As $\sum_{i=1}^n x_{ij} \geq 1$ for all $1 \leq j \leq n$ implies $\sum_{j=1}^k \sum_{i=1}^n x_{ij} \geq k$ for all $1 \leq k \leq n$, this feasible set, which we call BF_n , is a relaxation of the assignment polytope. Note that the associated

constraint matrix lost total unimodularity. As part of our arguments below, we will prove that $\mathbf{BF}_n$ is integral, i.e., it has integral vertices.

We consider the linear projection $\psi_n : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^n$, $(x_{ij}) \rightarrow (\sum_{j=1}^n jx_{ij})_{i=1,\dots,n}$, i.e., where the i^{th} entry is $\sum_{j=1}^n jx_{ij}$. Note that for integral (x_{ij}) in $\mathbf{BF}_n$ or $\mathbf{PF}_n$, $x_{ij} = 1$ holds for exactly one j , the parking spot j for car i . We show that the projection of $\mathbf{BF}_n$ results in exactly $\mathbf{PF}_n$, i.e., $\psi_n(\mathbf{BF}_n) = \mathbf{PF}_n$.

Theorem 3.5.1. *The polytope $\mathbf{PF}_n$ is the linear projection of the relaxed Birkhoff/assignment polytope $\mathbf{BF}_n$, under ψ_n .*

Proof. Recall that $\mathbf{PF}_n$ is an integral polytope in $\mathbb{R}^n$. Thus, for $\mathbf{a} \in \mathbf{PF}_n$, which may have fractional components, there exists a convex combination $\mathbf{a} = \sum_{\ell=1}^n \lambda_{\ell} \mathbf{a}^{\ell}$ of integral parking functions $\mathbf{a}^{\ell}$. Let $\mathbf{x}^{\ell} = (x_{ij}^{\ell}) \in \{0,1\}^{n^2}$ be defined by $x_{ij}^{\ell} = 1$ if and only if $a_i^{\ell} = j$. Then $\mathbf{x} = \sum_{\ell=1}^n \lambda_{\ell} \mathbf{x}^{\ell}$ lies in $\mathbf{BF}_n$ and projects onto $\mathbf{a}$ under ψ_n .

Conversely, let now $\mathbf{x} \in \mathbf{BF}_n$. We have to prove that its projection $\psi_n(\mathbf{x}) = \mathbf{a}$ lies in $\mathbf{PF}_n$. To this end, we first prove that $\mathbf{BF}_n$ is integral. Let $\mathbf{x}$ be fractional and consider a bipartite graph G with $V(G) = C \cup P$, $|C| = n = |P|$, and edges (i,j) from C to P if and only if $0 < x_{ij} < 1$; informally, it is a representation of only the fractional assignments of parking spots to cars. If there exists a cycle in the undirected graph underlying G , one can send flow along either orientation of that cycle and obtain a new feasible solution. In this case, $\mathbf{x}$ is not a vertex of $\mathbf{BF}_n$.

If there does not exist a cycle in G , then there exist at least two nodes with degree 1 in each connected component, as any tree has at least two leaves. By $\sum_{j=1}^n x_{ij} = 1$ for all cars i , the degree of $i \in C$ in any component is at least two. Thus, there are at least two parking spots $j', j'' \in P$ with degree 1, as well as a path between j' and j'' . Let $j' < j''$ and note that the constraints $\sum_{j=1}^{j'} \sum_{i=1}^n x_{ij} \geq j'$ and $\sum_{j=1}^{j''} \sum_{i=1}^n x_{ij} \geq j''$ must be satisfied strictly. By sending (maximal) flow along the oriented path from j' to j'' , one obtains a new feasible solution with a strictly larger set of active constraints. This again implies that $\mathbf{x}$ is not a vertex of $\mathbf{BF}_n$.

We conclude that $\mathbf{BF}_n$ is an integral polytope with all vertices in $\{0,1\}^{n^2}$. By the same

arguments as above, $\mathbf{x} = \sum_{\ell=1}^t \lambda_{\ell} \mathbf{x}^{\ell}$ is a convex combination with $\mathbf{x}^{\ell} \in \{0, 1\}^{n^2}$. Let $\mathbf{a}^{\ell} = \psi_n(\mathbf{x}^{\ell})$ be the projection of $\mathbf{x}^{\ell}$ under ψ_n , and observe it is an integral parking function. Thus, $\mathbf{x}$ projects to a convex combination $\mathbf{a} = \sum_{\ell=1}^n \lambda_{\ell} \mathbf{a}^{\ell}$ of integral parking functions $\mathbf{a}^{\ell}$. This proves the claim. $\square$

Remark 3.5.2. *As an anonymous reviewer noted, it is shown in [HS22, Theorems 5.27 & 5.28] that similar projections of other relaxations of the Birkhoff polytope give partial permutohedra, of which certain cases are (up to translation) $\mathbf{b}$ -parking-function polytopes. As an example, consider [HS22, Theorem 5.27] with $n \geq m - 1$ and $z = (1, 2, \dots, n)$. Then the projection of the polytope of $m \times n$ doubly substochastic matrices $\text{PPerm}(m, n)$ under $\phi_z : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m$, such that*

$$(x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \mapsto \left(\sum_{j=1}^n j x_{ij} \right)_{1 \leq i \leq m},$$

is the partial permutahedron $\mathcal{P}_z(m, n)$, which coincides (up to translation) with $\mathfrak{X}_m(\mathbf{b})$ where $\mathbf{b} = (n - m + 2, 1, \dots, 1)$.

3.5.2 Relation of $\mathfrak{X}_n(\mathbf{b})$ to partition polytopes

Most of the proof for Theorem 3.5.1 transfers to $\mathbf{b}$ -parking-function polytopes $\mathfrak{X}_n(\mathbf{b})$. By using $x_{ij} \in \{0, 1\}$ to indicate car i being assigned a number $1 \leq j \leq S_n$, one can formulate the set of $\mathbf{b}$ -parking functions in the form

$$\begin{aligned} \sum_{j=1}^{S_n} x_{ij} &= 1 \quad \text{for all } 1 \leq i \leq n \\ \sum_{j=1}^{S_k} \sum_{i=1}^n x_{ij} &\geq k \quad \text{for all } 1 \leq k \leq n \\ x_{ij} &\geq 0 \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq S_n, \end{aligned}$$

and the projection $(x_{ij}) \rightarrow (\sum_{j=1}^{S_n} j x_{ij})_{i=1, \dots, n}$ gives $\mathfrak{X}_n(\mathbf{b})$ by the same arguments.

A relation of this system to well-known polytopes is more involved than for PF_n . The

system can be seen as a relaxation of so-called *(bounded-size) partition polytopes* [BV21, BV22]. Partition polytopes are special transportation polytopes and generalized Birkhoff or assignment polytopes. They are named for representing the (possibly fractional) assignment of n items to ℓ clusters, where one specifies the total weight κ_j assigned to each cluster j . A bounded-size partition polytope specifies lower and/or upper bounds on this weight instead of an exact weight. For our purposes, we set $\ell = S_n$ and only use lower bounds. Formally,

$$\begin{aligned} \sum_{j=1}^{S_n} x_{ij} &= 1 \quad \text{for all } 1 \leq i \leq n \\ \sum_{i=1}^n x_{ij} &\geq \kappa_j \quad \text{for all } 1 \leq j \leq S_n \\ x_{ij} &\geq 0 \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq S_n. \end{aligned}$$

The first type of constraint guarantees that each item is assigned to a cluster (or partially to multiple clusters). The second type of constraint guarantees that each cluster receives a certain minimum weight. Note that $S_n > n$ for any $\mathfrak{X}_n(\mathbf{b})$ that is not also a classical parking function $\mathbf{PF}_n$; informally, there are more clusters than items. For the existence of a feasible solution, it is necessary that at least some of the κ_j satisfy $\kappa_j < 1$.

Let $S_0 = 0$ and $t \in [n]$, and set $\kappa_j = \frac{1}{S_t - S_{t-1}}$ for $S_{t-1} < j \leq S_t$. One can represent the conditions $\beta'_i \leq S_i$ in a nondecreasing rearrangement of the entries by replacing the set of constraints $\sum_{i=1}^n x_{ij} \geq \kappa_j$ by sums of the first S_k constraints for $1 \leq k \leq n$. This gives the above relaxed system.

Corollary 3.5.3. *The polytope $\mathfrak{X}_n(\mathbf{b})$ is a linear projection of a relaxed bounded-size partition polytope.*

Chapter 4

Tanglegrams

This chapter is dedicated to tanglegrams. It follows the work in [BLM⁺23].

4.1 Introduction

A *tanglegram* consists of two rooted binary trees and a perfect matching between their leaves. They initially arose in computer science and biology [BBB⁺12, Cha02, MBKK15]. Tanglegrams are drawn in the plane using *layouts* such as the ones shown in Figure 4.1. Layouts with the fewest number of crossings possible are of interest in applications such as estimating the number of horizontal gene transfers between species [VASJG09].

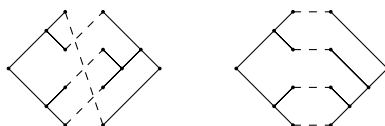


Figure 4.1: Two layouts for the same tanglegram.

Combinatorial interest has grown recently, and enumerating several variations of tanglegrams has been studied [BKM17, Ges21, RRW18]. Algorithms for uniform sampling of tanglegrams were established in [BKM17] and [Fus16], and properties of random tanglegrams were studied in [KW16].

A tanglegram is *planar* if it has a layout with no crossings, and we refer the reader to [CSW19, Liu22, LPR⁺07, RRW18] for many interesting results about them. The second

author established a characterization of all planar layouts of a planar tanglegram in [Liu22] and then proposed the problem of efficiently sampling planar tanglegrams uniformly at random. In this dissertation, we consider this problem. Note that one can use the algorithms in [BKM17] and [Fus16] to sample tanglegrams uniformly at random until a planar tanglegram is generated, but results in [BKM17] and [RRW18] imply that planar tanglegrams are rare.

In their work enumerating planar tanglegrams, Ralaivaosaona, Ravelomanana, and Wagner [RRW18] introduced *irreducible* planar tanglegrams, which are planar tanglegrams that cannot be constructed from smaller tanglegrams. We make this definition precise in Section 4.2. Let $\mathcal{P}$ and $\mathcal{I}$ respectively denote the set of planar and irreducible planar tanglegrams. [RRW18, Theorem 1] gives a relation between the generating functions

$$\begin{aligned} T(x) &= \sum_{\mathcal{T} \in \mathcal{P}} x^{|\mathcal{T}|}, \\ H(x) &= \frac{1}{2}x^2 + \sum_{\mathcal{T} \in \mathcal{I}: |\mathcal{T}| > 2} x^{|\mathcal{T}|}, \end{aligned} \tag{4.1}$$

where $|\mathcal{T}|$ denotes the number of leaves in the component trees of $\mathcal{T}$. Their sequences of coefficients can be found at [OEI, A257887, A349408]. Letting $\text{Irr}(\mathcal{T})$ denote the irreducible tanglegram formed by contracting each of the smaller tanglegrams in $\mathcal{T}$ to a pair of matched leaves, we generalize these generating functions to also account for $\text{Irr}(\mathcal{T})$ by defining

$$\begin{aligned} T(x, y) &= \sum_{\mathcal{T} \in \mathcal{P}} x^{|\mathcal{T}|} y^{|\text{Irr}(\mathcal{T})|}, \\ H(x, y) &= H(xy) = \frac{1}{2}x^2 y^2 + \sum_{\mathcal{T} \in \mathcal{I}: |\mathcal{T}| > 2} x^{|\mathcal{T}|} y^{|\mathcal{T}|}. \end{aligned} \tag{4.2}$$

We establish a generalization of the relation on $T(x)$ and $H(x)$ from [RRW18, Theorem 1]. Note that substituting $y = 1$ recovers the original result.

Theorem 4.1.1. *The following holds:*

$$T(x, y) = H(T(x), y) + \frac{T(x^2)y^2}{2} + xy. \tag{4.3}$$

Using this result, we establish Theorem 4.3.2, which reduces the problem of generating planar tanglegrams uniformly at random to computing coefficients of $T(x, y)$ and generating irreducible planar tanglegram layouts uniformly at random. In their work enumerating planar tanglegrams, Ralaivaosaona, Ravelomanana, and Wagner [RRW18] also established a natural bijection between irreducible planar tanglegram layouts and pairs of triangulations of a convex polygon that do not share a diagonal. We call these *pairs of disjoint triangulations*. Hence, one can instead consider the problem of generating pairs of disjoint triangulations uniformly at random.

Triangulations of a convex n -gon are one of many objects enumerated by the Catalan numbers. Sampling Catalan objects uniformly or approximately uniformly at random has been an active area of research over the last several decades. Approaches include direct methods using properties of Catalan numbers [AS92, BBJ17, R85], Boltzmann sampling with the generating function relation [DFLS03], and Markov chains on flip graphs of Catalan objects [EF22, MT97, MRS01]. Enumerative results suggest that pairs of disjoint triangulations are significantly more complicated than individual triangulations. In particular, there is no known simple formula in n for the number of pairs of disjoint triangulations of an n -gon, even after fixing one of the triangulations [AR16, RRW18].

We extend the flip operation on single triangulations to a new flip operation on the pairs of disjoint triangulations of an n -gon. We write $\mathcal{D}_n$ for the graph whose vertices are pairs of disjoint triangulations and whose edges are given by flips, and we call this the *flip graph* on pairs of disjoint triangulations. The flip graph $\mathcal{D}_5$ is shown in Figure 4.2. Our flip operation implies many surprising properties for $\mathcal{D}_n$. In particular, we can define a *Markov chain* (X_t) from a random walk on $\mathcal{D}_n$. We set X_0 to be an arbitrary vertex and, for $t > 0$, choose X_t uniformly at random among the vertices adjacent to X_{t-1} .

Theorem 4.1.2. *For a fixed $n \geq 5$, let (X_t) be the Markov chain defined above. Then as $t \rightarrow \infty$, (X_t) converges to the uniform distribution on the vertices of $\mathcal{D}_n$ in total variance distance.*

Theorem 4.1.2 gives a method for near uniform sampling of pairs of disjoint triangulations.

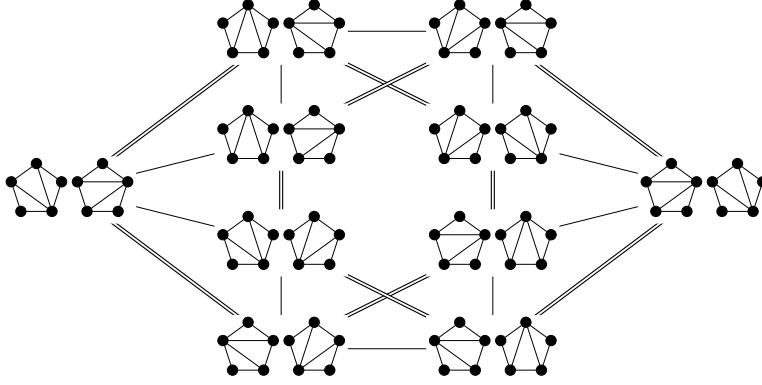


Figure 4.2: The flip graph $\mathcal{D}_5$. Single lines indicate when only one triangulation is changed, and double lines indicate when both are changed.

The mixing time of X_t is a topic of continued study. Our general method is similar to one used by Heitsch and Tetali to study *meanders*, which are also pairs of Catalan objects subject to some property [HT11]. More specifically, a meander is a pair of noncrossing matchings of $2n$ points that form a cycle, such as the one shown in Figure 4.3. Heitsch and Tetali also constructed a Markov chain that converges to the uniform distribution, and the mixing time of their Markov chain also remains open.

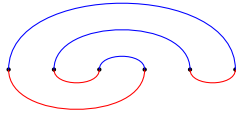


Figure 4.3: An example of a meander, with each noncrossing matching color-coded.

In order to prove Theorem 4.1.2, we show that $\mathcal{D}_n$ is regular and connected. In the process, we bound the diameter of $\mathcal{D}_n$. In particular, we prove the following.

Theorem 4.1.3. *For any positive integer $n \geq 5$, the graph $\mathcal{D}_n$ is simple, connected, and $2(n - 3)$ -regular, with diameter at most $4n - 16$.*

The proof of Theorem 4.1.3 provides novel insight into the classical flip graph of triangulations of an n -gon. Namely, we show in Theorem 4.4.5 that the induced subgraph of triangulations of an n -gon disjoint from some fixed triangulation is always connected. The flip graph in this case is the set of vertices and edges of the associahedron, a polytope that plays a fundamental role in algebraic and geometric combinatorics, see [CSZ15] and

references therein. Our results may be rephrased within that context. The facets of the associahedron are in natural bijection with the set of diagonals of the polygon. A triangulation corresponds to a vertex of a given facet if and only if that triangulation contains the diagonal corresponding to that facet. Therefore, we have shown that for any fixed vertex v of the associahedron, the induced subgraph consisting of all vertices that do not share a facet with v is still connected with small diameter, a property that one can study for any polytope as discussed in Problem 4.

This chapter is organized as follows. In Section 4.2, we outline background on tanglegrams and summarize relevant results in [RRW18] on planar tanglegrams, including their connection with pairs of disjoint triangulations. In Section 4.3, we prove Theorem 4.1.1 and apply it to construct our algorithm for sampling planar tanglegrams. In Section 4.4, we define our flip graphs on pairs of disjoint triangulations and establish Theorems 4.1.2 and 4.1.3. We conclude in Section 4.5 with open problems.

4.2 Preliminaries

A *rooted binary tree* is a tree with a distinguished vertex called the *root* where each vertex has zero or two children. We consider children to be unordered, so these are different from the plane binary trees enumerated by the Catalan numbers.

A *tanglegram* $\mathcal{T} = (L, R, \sigma)$ is formed from an ordered pair of rooted binary trees (L, R) with the same number of leaves and a perfect matching σ between the leaves of L and R . The *size* of a tanglegram, denoted $|\mathcal{T}|$, is the common number of leaves in the two trees forming the tanglegram. An isomorphism between two tanglegrams $\mathcal{T} = (L, R, \sigma)$ and $\mathcal{T}' = (L', R', \sigma')$ is an isomorphism of the underlying graphs that maps L to L' and R to R' . See [BKM17, MBKK15] for more details. All tanglegrams in this chapter are considered up to isomorphism.

Tanglegrams are drawn in the plane using *layouts*, where L is embedded in the plane left of the line $x = 0$ with leaves on $x = 0$, R is embedded in the plane right of $x = 1$ with leaves on $x = 1$, and the matching σ is drawn using straight lines. A *crossing* in a layout is any

intersecting pair of lines induced by σ . In general, a tanglegram has multiple layouts, and a tanglegram is *planar* if it has a layout in which none of the straight lines induced by σ cross. Examples are shown in Figure 4.4.

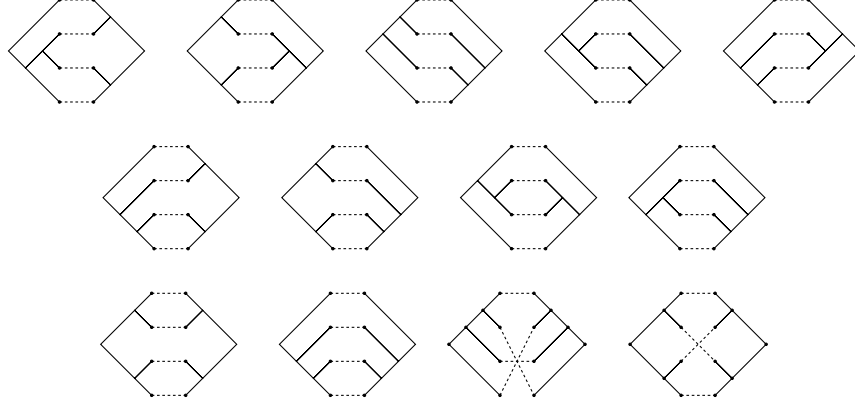


Figure 4.4: A layout for each of the 13 tanglegrams of size four.

For a tanglegram $\mathcal{T} = (L, R, \sigma)$, suppose that for the internal vertices $u \in L$ and $v \in R$, the descendants of u and v are matched by σ . When u and v are not the roots of L and R , the subtrees rooted at v_1 and v_2 with the matching induced by σ is a *proper subtanglegram* of $\mathcal{T}$.

A tanglegram $\mathcal{T}$ is *irreducible* if it contains no proper subtanglegrams. For any tanglegram $\mathcal{T}$, its irreducible component, denoted $\text{Irr}(\mathcal{T})$, is the tanglegram obtained by contracting each maximal proper subtanglegram to a pair of matched leaves. If $\mathcal{T}'$ is an irreducible tanglegram, then $\mathcal{T}'$ *extends to* $\mathcal{T}$ if $\text{Irr}(\mathcal{T}) = \mathcal{T}'$. Observe that in Figure 4.4, the last two tanglegrams are irreducible and not planar. Of the eleven planar tanglegrams, five are irreducible, three have an irreducible component of size two, and three have an irreducible component of size three.

We now state two results of [RRW18]. Note that the first result can also be derived as a consequence of [Lin22, Theorem 1.1].

Proposition 4.2.1. [RRW18, Proposition 5] *Every irreducible planar tanglegram $\mathcal{T}$ with $|\mathcal{T}| \geq 3$ has exactly two planar layouts. Moreover, the two planar layouts are mirror images of one another.*

Theorem 4.2.2. [RRW18, Theorem 4] *For any $n \geq 2$, there is a bijection between the following sets:*

- *the set of planar layouts of irreducible planar tanglegrams of size n , and*
- *the set of ordered pairs of disjoint triangulations of an $(n + 1)$ -gon.*

We describe this bijection. Starting with the two plane binary trees in a planar layout of $\mathcal{T} = (L, R, \sigma)$, draw lines from the root and all leaves to infinity. Then take the plane dual. Label the region above the root in each tree as 1. In L , proceed clockwise, and in R , proceed counterclockwise. An example is shown in Figure 4.5.

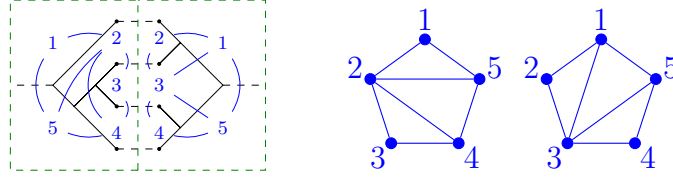


Figure 4.5: The plane dual bijection from Theorem 4.2.2.

4.3 Planar tanglegrams

In this section, we consider the problem of uniformly sampling planar tanglegrams. We first establish Theorem 4.1.1 and its consequences. We then apply these results to reduce the problem of uniformly sampling planar tanglegrams to the corresponding problem for irreducible planar tanglegrams or their layouts in Theorem 4.3.2.

Recall the generating functions defined in Equations (4.1) and (4.2). We now prove Theorem 4.1.1, which generalizes part of [RRW18, Theorem 1].

Proof of Theorem 4.1.1. We rewrite the right hand side of Equation (4.3) as

$$\left(H(T(x), y) - \frac{T(x)^2 y^2}{2} \right) + \left(\frac{T(x)^2 y^2 + T(x^2) y^2}{2} \right) + xy \quad (4.4)$$

The third summand xy accounts for the unique tanglegram where both trees consist of a single vertex. For the remaining summands, we consider those with irreducible component

size two and greater than two separately.

To interpret the first summand, observe that $H(x, y) - \frac{x^2 y^2}{2}$ counts irreducible tanglegrams of size at least 3, where each term $x^k y^k$ corresponds to an irreducible planar tanglegram of size k . Theorem 4.2.1 implies that irreducible tanglegrams of size at least 3 have no symmetry. Consequently, given an irreducible tanglegram $\mathcal{T}$ of size $k > 2$, fixing a layout of $\mathcal{T}$ and replacing matched leaves from top-to-bottom with planar tanglegrams $(\mathcal{T}_i)_{i=1}^k$ produces a distinct tanglegram for each selection of $(\mathcal{T}_i)_{i=1}^k$. The generating function $T(x)^k$ counts ordered k -tuples of planar tanglegrams, and replacing pairs of matched leaves with planar tanglegrams corresponds to replacing x with $T(x)$. Hence, $H(T(x), y) - \frac{T(x)^2 y^2}{2}$ is the generating function for planar tanglegrams with $|\text{Irr}(\mathcal{T})| \geq 3$.

To interpret the second summand, tanglegrams with $|\text{Irr}(\mathcal{T})| = 2$, we must start with the unique layout for the unique tanglegram of size two and replace matched leaves with $\mathcal{T}_1$ and $\mathcal{T}_2$. However, interchanging the order of $\mathcal{T}_1$ and $\mathcal{T}_2$ produces an isomorphic tanglegram. Observe that $T(x)^2 y^2$ double counts the cases when $\mathcal{T}_1 \neq \mathcal{T}_2$, and $T(x^2) y^2$ counts the cases when $\mathcal{T}_1 = \mathcal{T}_2$. Hence, $\frac{T(x)^2 y^2 + T(x^2) y^2}{2}$ enumerates planar tanglegrams with $|\text{Irr}(\mathcal{T})| = 2$. $\square$

Note that once we can efficiently compute the coefficients of $T(x)$ and $H(x)$, we can efficiently perform the composition to generate the coefficients of $T(x, y)$. Using known values of $T(x)$ and $H(x)$ from [OEI, A257887, A349408], we give some coefficients of $T(x, y)$ in Table 4.1.

n, k	2	3	4	5	6	7	8	total
2	1							1
3	1	1						2
4	3	3	5					11
5	13	9	20	34				76
6	90	46	70	170	273			649
7	747	312	360	680	1638	2436		6173
8	7040	2580	2435	3570	7371	17052	23391	63429

Table 4.1: The number of tanglegrams of size n with irreducible component size k .

We respectively use the notation t_n and $t_{n,k}$ for the coefficient of x^n in $T(x)$ and $x^n y^k$ in $T(x, y)$. We also use the notation h_n for $t_{n,n}$, which is the number of irreducible planar

tanglegrams of size n . Recall that a *composition* of n is a decomposition of n into an ordered sum of positive integers, and we use the notation $(a_i)_{i=1}^k \models n$ to denote this. The preceding theorem and proof imply the following corollary.

Corollary 4.3.1. *Let $2 \leq k \leq n$. The number of ways to extend a size k irreducible planar tanglegram into a size n planar tanglegram, denoted $c_{n,k}$, is independent of the irreducible planar tanglegram. Moreover,*

- (a) *if n is even, then $t_{n,2} = \frac{1}{2} (\sum_{i=1}^{n-1} t_i t_{n-i} + t_{n/2})$,*
- (b) *if n is odd, then $t_{n,2} = \frac{1}{2} \sum_{i=1}^{n-1} t_i t_{n-i}$, and*
- (c) *if $k \neq 2$, then $t_{n,k} = h_k \cdot \sum_{(a_i)_{i=1}^k \models n} t_{a_1} t_{a_2} \dots t_{a_k}$.*

Note that $c_{n,k} = t_{n,k}/h_k$.

Using these $c_{n,k}$ constants, we define a procedure for uniformly sampling planar tanglegrams that encodes the techniques from Theorem 4.1.1, assuming an algorithm for uniformly sampling irreducible planar tanglegram layouts. Hence, this reduces the problem of generating planar tanglegrams to generating irreducible planar tanglegrams or their layouts.

Theorem 4.3.2. *The following procedure generates a planar tanglegram of size $n \geq 3$ uniformly at random.*

1. *Choose an integer $2 \leq k \leq n$ with probability $\frac{h_k c_{n,k}}{t_n}$ and generate an irreducible planar tanglegram layout $\mathcal{L}$ of size k uniformly at random.*
2. (a) *If $k \neq 2$, select $(a_i)_{i=1}^k \models n$ with probability $\frac{t_{a_1} t_{a_2} \dots t_{a_k}}{c_{n,k}}$ and independently generate planar tanglegrams $(\mathcal{T}_i)_{i=1}^k$ of sizes $(a_i)_{i=1}^k$ uniformly at random.*
 (b) *If n is odd and $k = 2$, select $(a_1, a_2) \models n$ with probability $\frac{t_{a_1} t_{a_2}}{2c_{n,2}}$ and independently generate planar tanglegrams $(\mathcal{T}_1, \mathcal{T}_2)$ of sizes (a_1, a_2) uniformly at random.*
 (c) *If n is even and $k = 2$,*
 - *with probability $\frac{t_{n/2}}{2c_{n,2}}$, generate a single tanglegram $\mathcal{T}_1 = \mathcal{T}_2$ of size $n/2$ uniformly at random, and*

- otherwise, select $(a_1, a_2) \models n$ with probability $\frac{t_{a_1} t_{a_2}}{\sum_{i=1}^{n-1} t_i t_{n-i}}$ and independently generate planar tanglegrams $(\mathcal{T}_1, \mathcal{T}_2)$ of sizes (a_1, a_2) uniformly at random.

3. In all cases, output the tanglegram corresponding to $\mathcal{L}$ with matched leaves replaced from top to bottom by $\{\mathcal{T}_i\}_{i=1}^k$.

Proof. The definition of $c_{n,k}$ and the results of Theorem 4.3.1 imply that all of the necessary quantities in steps (1) and (2) sum to 1. We show that each tanglegram of size n can be generated in two ways, and each of these possibilities has probability $\frac{1}{2t_n}$.

Consider a planar tanglegram $\mathcal{T}$ with $|\text{Irr}(\mathcal{T})| \geq 3$. To generate $\mathcal{T}$ in the algorithm, we must first generate one of the layouts of $\text{Irr}(\mathcal{T})$ in step (1). Theorem 4.2.1 implies that there are two possibilities $\mathcal{L}_1$ and $\mathcal{L}_2$, and observe that each of them has probability $\frac{h_k c_{n,k}}{t_n} \cdot \frac{1}{2h_k} = \frac{c_{n,k}}{2t_n}$ of being generated. For each $\mathcal{L}_i$, a unique list of tanglegrams $(\mathcal{T}_{i,j})_{j=1}^k$ must replace the matched leaves in $\mathcal{L}_i$ from top-to-bottom to construct $\mathcal{T}$. Letting $a_{i,j} = |\mathcal{T}_{i,j}|$, we see in that the probability $(\mathcal{T}_{i,j})_{j=1}^k$ is generated in step (2) is given by $\frac{t_{a_1} t_{a_2} \dots t_{a_k}}{c_{n,k}} \cdot \frac{1}{t_{a_1} t_{a_2} \dots t_{a_k}} = \frac{1}{c_{n,k}}$. Hence, each of the two ways of generating $\mathcal{T}$ has probability $\frac{c_{n,k}}{2t_n} \cdot \frac{1}{c_{n,k}} = \frac{1}{2t_n}$ of being generated, so the probability $\mathcal{T}$ is generated is $\frac{1}{t_n}$.

Next, consider a planar tanglegram $\mathcal{T}$ with $|\text{Irr}(\mathcal{T})| = 2$. This requires first generating the unique layout $\mathcal{L}$ for the unique planar tanglegram of size 2. If $n = |\mathcal{T}|$ is odd, then two possibilities (T_1, T_2) and (T_2, T_1) extend $\mathcal{L}$ to $\mathcal{T}$. Letting $a_1 = |\mathcal{T}_1|$ and $a_2 = |\mathcal{T}_2|$, the probability of obtaining $\mathcal{T}$ is given by

$$\frac{h_2 c_{n,2}}{t_n} \cdot \left(\frac{t_{a_1} t_{a_2}}{2c_{n,2}} \cdot \frac{1}{t_{a_1} t_{a_2}} + \frac{t_{a_2} t_{a_1}}{2c_{n,2}} \cdot \frac{1}{t_{a_2} t_{a_1}} \right) = \frac{1}{t_n}.$$

Note that $h_2 = 1$, so this term disappears in the product above.

Now consider when $n = |\mathcal{T}|$ is even. Suppose $\mathcal{T}$ requires replacing the matched leaves in $\mathcal{L}$ with the same tanglegram $\mathcal{T}'$. With probability $\frac{t_{n/2}}{2c_{n,2}} \cdot \frac{1}{t_{n/2}}$, we generate $\mathcal{T}'$ twice in the first case of (2c), so $\mathcal{T}$ has a

$$\frac{h_2 c_{n,2}}{t_n} \cdot \frac{t_{n/2}}{2c_{n,2}} \cdot \frac{1}{t_{n/2}} = \frac{1}{2t_n}$$

probability of being generated this way. The probability of generating $\mathcal{T}$ by generating $\mathcal{T}'$

twice in the second case of (2c) is

$$\frac{h_2 c_{n,2}}{t_n} \cdot \left(1 - \frac{t_{n/2}}{2c_{n,2}}\right) \cdot \frac{t_{n/2} t_{n/2}}{\sum_{i=1}^{n-1} t_i t_{n-i}} \cdot \frac{1}{t_{n/2} t_{n/2}}.$$

Using Theorem 4.3.1, $h_2 = 1$, and $c_{n,2} = t_{n,2}$, this simplifies to

$$\frac{c_{n,2}}{t_n} \cdot \frac{\sum_{i=1} h_i h_{n-i}}{2c_{n,2}} \cdot \frac{1}{\sum_{i=1}^{n-1} t_i t_{n-i}} = \frac{1}{2t_n}.$$

The case when $\mathcal{T}$ requires replacing matched leaves in $\mathcal{L}$ with two distinct tanglegrams is done using the same properties, where we note that there is no way to generate $\mathcal{T}$ in the first case of (2c) but two ways to generate it in the second case. $\square$

Note that the procedure in Theorem 4.3.2 can be applied recursively when generating $\{\mathcal{T}_i\}_{i=1}^k$, except that the tanglegrams of size 1 and 2 should be generated directly since they are unique. For efficiency reasons, one can also directly generate all planar tanglegrams below a certain size. From this, we see that uniform sampling of planar tanglegrams can be reduced to computation of the coefficients in $T(x, y)$ and uniform sampling of irreducible planar tanglegram layouts.

4.4 Pairs of disjoint triangulations

In this section, we consider the problem of uniformly sampling pairs of disjoint triangulations, which is equivalent to uniformly sampling irreducible planar tanglegram layouts by Theorem 4.2.2. In Subsection 4.4.1, we construct our flip graph $\mathcal{D}_n$ on the pairs of disjoint triangulations of an n -gon. We then establish Theorems 4.1.2 and 4.1.3 using Theorem 4.4.4 and Theorem 4.4.6. In Subsection 4.4.2, we describe the operation on irreducible planar tanglegram layouts that corresponds to flips in $\mathcal{D}_n$.

4.4.1 A flip graph on pairs of disjoint triangulations

Throughout this subsection, we fix a labeling of the convex n -gon using $[n] = \{1, 2, \dots, n\}$, and use pairs (a, b) with $a, b \in [n]$ to denote diagonals. For a triangulation T , we use the notation $(a, b) \in T$ to denote that the diagonal (a, b) is in the triangulation T .

If (a, b) is a diagonal of a triangulation T , then deleting (a, b) creates a unique quadrilateral (a, a', b, b') for some $a', b' \in [n]$. A *flip* at $(a, b) \in T$ replaces (a, b) with (a', b') , resulting in another triangulation of the n -gon. We extend the flip operation to pairs of disjoint triangulations. An example is shown in Figure 4.6.

Definition 4.4.1. Let (T_1, T_2) be an ordered pair of disjoint triangulations of an n -gon, and suppose $(a, b) \in T_i$ for some $i \in [2]$. A *flip* at $(a, b)_i \in (T_1, T_2)$ is defined as

- (a) flip $(a, b) \in T_i$, and
- (b) if the resulting diagonal (a', b') is in T_j for $j \neq i$, then flip $(a', b') \in T_j$.

When only (a) is performed, we refer to this as a *single flip*, and when both (a) and (b) are performed, we refer to this as a *double flip*.

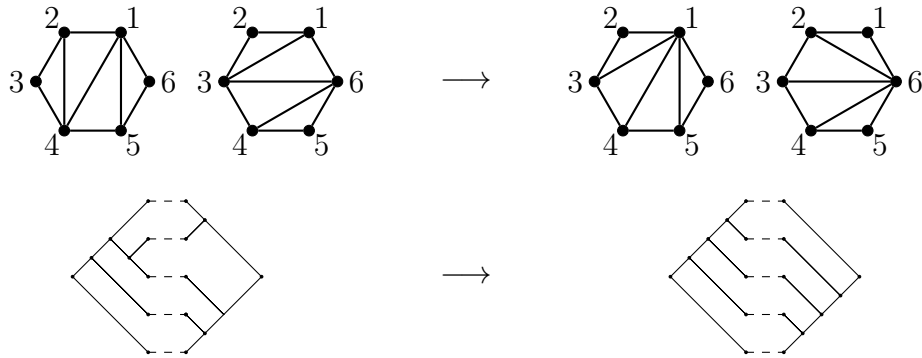


Figure 4.6: A (double) flip at $(2, 4)_1$ with the corresponding irreducible planar tanglegram layouts shown.

Lemma 4.4.2. Let (T_1, T_2) be an ordered pair of disjoint triangulations of an n -gon. If (T'_1, T'_2) is obtained from (T_1, T_2) by a flip at $(a, b)_i$, then (T'_1, T'_2) is also a pair of disjoint triangulations. Furthermore, (T_1, T_2) can also be obtained from (T'_1, T'_2) by some flip.

Proof. Without loss of generality, assume we flip the edge $(a, b) \in T_1$ to obtain T'_1 . If after the flip, the new diagonal (a', b') does not coincide with any diagonals of T_2 , then we are done. Note that in this case, we have that $T_2 = T'_2$, and the flip $(a', b')_2$ allows us to obtain (T_1, T_2) from (T'_1, T'_2) .

Otherwise, we flip (a', b') in the second polygon to obtain T'_2 . The resulting diagonal (a'', b'') crosses (a', b') , and hence cannot appear in T'_1 . Hence, (T'_1, T'_2) is a pair of disjoint triangulations. In this case, observe that the flip $(a'', b'')_2$ allows us to obtain (T_1, T_2) from (T'_1, T'_2) . $\square$

The mutual reachability between two pairs of disjoint triangulations allows us to now formally define the *flip graph on pairs of disjoint triangulations*.

Definition 4.4.3. Let $\mathcal{D}_n$ denote the (undirected) graph with vertices corresponding to ordered pairs of disjoint triangulations of an n -gon and adjacency given by flips.

Observe that $\mathcal{D}_3$ is a single vertex graph and $\mathcal{D}_4$ is a path graph on two vertices. Hence, we are primarily interested in the cases $n \geq 5$.

For a given $(T_1, T_2) \in \mathcal{D}_n$, there are $2(n-3)$ diagonals on which a flip can be performed. It is not difficult to see that if $n \geq 5$, then flipping at different diagonals results in different pairs of disjoint triangulations.

Lemma 4.4.4. *For any integer $n \geq 5$, the flip graph $\mathcal{D}_n$ is simple and $2(n-3)$ -regular.*

Proof. It suffices to show that for any pair (T_1, T_2) of disjoint triangulations, a flip at $(a, b) \in T_1$ and a flip at $(c, d) \in T_2$ cannot result in the same pair (T'_1, T'_2) . Let $(a', b') \in T'_1$ and $(c', d') \in T'_2$ be the diagonals obtained by flipping $(a, b) \in T_1$ and $(c, d) \in T_2$ respectively. If (T'_1, T'_2) is obtained from (T_1, T_2) by flipping $(a, b) \in T_1$, then $(a', b') = (c, d)$, and if (T'_1, T'_2) is obtained from (T_1, T_2) by flipping $(c, d) \in T_2$, then $(c', d') = (a, b)$. This implies that (a, c, b, d) is a quadrilateral in both T_1 and T_2 . Since $n \geq 5$, this implies that T_1 and T_2 share a diagonal, which is a contradiction. $\square$

Let $\mathcal{T}_n$ denote the flip graph for triangulations of an n -gon. For any triangulation S of an n -gon, the subgraph of $\mathcal{T}_n$ induced by the set of triangulations disjoint from S is denoted

$\mathcal{T}_n(S)$. Pournin showed in [Pou14] that the diameter of $\mathcal{T}_n$ is $2n - 10$ for $n > 12$. We show connectedness of $\mathcal{D}_n$ and a linear diameter bound for $\mathcal{D}_n$ by first showing corresponding statements for $\mathcal{T}_n(S)$.

Theorem 4.4.5. *Let $n \geq 5$, and let S be a triangulation of the n -gon. Then $\mathcal{T}_n(S)$ is connected, and its diameter is at most $2n - 8$.*

Proof. Every triangulation contains a diagonal of the form $(i, i+2)$, so we assume without loss of generality that S contains the diagonal $(2, n)$. Let Δ denote the triangulation consisting of $\{(1, i) : 3 \leq i \leq n-1\}$, which is called a *standard triangulation* in Chapter 1 of [DLRS10]. We show that every triangulation T disjoint from S is connected to Δ by a path in $\mathcal{T}_n(S)$ of length at most $n - 4$. In fact, we claim that if T contains d edges of the form $\{(1, i) : 3 \leq i \leq n-1\}$, then it is connected to Δ by a path in $\mathcal{T}_n(S)$ of length at most $n - 3 - d$. Note that $d \geq 1$, as $(2, n) \notin T$ implies the existence of some diagonal of the form $(1, i)$.

We prove the claim by induction on $n - 3 - d$. If $d = n - 3$, then $T = \Delta$, and these triangulations are connected by a path of length $n - 3 - (n - 3) = 0$ as needed. Now suppose $d < n - 3$, and let $3 \leq i_1 < i_2 < \dots < i_d \leq n - 1$ denote the indices such that $(1, i_j) \in T$ for all i_j . Since $d < n - 3$, it must be that $i_{j+1} - i_j > 1$ for some $j \in \{1, \dots, d\}$. Consider the polygon with vertices $\{1, i_j, i_j + 1, i_j + 2, \dots, i_{j+1}\}$ with triangulation T_1 induced by T . Observe that T_1 cannot contain any edges of the form $(1, i)$, and since $i_j, 1, i_{j+1}$ are consecutive vertices, this can only occur if $(i_j, i_{j+1}) \in T_1$. Flipping this diagonal results in $(1, i')$ for some $i_j < i' < i_{j+1}$. Note that this diagonal $(1, i')$ cannot appear in S since S contains $(2, n)$. Hence, flipping (i_j, i_{j+1}) in T results in some triangulation T' disjoint from S that contains $d + 1$ diagonals of the form $\{(1, i) : 3 \leq i \leq n - 1\}$. By the inductive hypothesis, T' and Δ are connected by a path in $\mathcal{T}_n(S)$ of length at most $n - 3 - d - 1$, and hence T is connected to Δ by a path in $\mathcal{T}_n(S)$ of length at most $n - 3 - d$.

Now let T_1 and T_2 be any two triangulations disjoint from S . Choosing Δ as above, we see that each T_i is connected to Δ by a path of length at most $n - 4$ in $\mathcal{T}_n(S)$, implying T_1 and T_2 are connected by a path of length at most $2n - 8$. $\square$

Pournin's result for $\text{diam}(\mathcal{T}_n)$ when $n > 12$ combined with known values for $5 \leq n \leq 12$

implies that for all $n \geq 5$, we have $\text{diam}(\mathcal{T}_n) \leq 2n - 8$. Hence, the above result implies the following statements for $\mathcal{D}_n$.

Corollary 4.4.6. *For $n \geq 5$, the flip graph $\mathcal{D}_n$ is connected and*

$$\text{diam}(\mathcal{D}_n) \leq \text{diam}(\mathcal{T}_n) + 2n - 8 \leq 4n - 16.$$

Proof. Let $(T_1, T_2), (T_3, T_4) \in \mathcal{D}_n$. Then there is a path of length at most $\text{diam}(\mathcal{T}_n)$ in $\mathcal{D}_n$ from (T_1, T_2) to (T_3, T) for some T disjoint from T_3 . By Theorem 4.4.5, there is a path from (T_3, T) to (T_3, T_4) of length at most $2n - 8$ in $\mathcal{D}_n$. $\square$

It is well known that a random walk on a regular connected graph converges to the uniform distribution in total variation distance if it is aperiodic. Aperiodicity follows from the fact that $\mathcal{D}_n$ contains cycles of size 3 for all $n \geq 5$, which is shown using the standard triangulations in Figure 4.7. This implies that Theorem 4.1.2 follows from Theorem 4.1.3, which is immediate from Theorem 4.4.4 and Theorem 4.4.6.

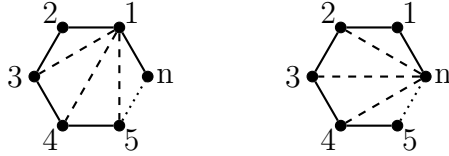


Figure 4.7: A pair of disjoint triangulations where two sequences of flips $(n, 2)_2, (2, 4)_1$ and $(n, 2)_2, (1, 3)_2, (2, 4)_1$ both result in the original pair again.

4.4.2 Rotations of irreducible planar tanglegram layouts

In this subsection, we establish the operation on planar layouts of irreducible planar tanglegrams that corresponds to flips in pairs of disjoint triangulations. Our operation uses the well-known rotation on plane binary trees, which is given in Figure 4.8. Note that our resulting correspondence is not the first of its kind, and we refer the reader to [STT88, Lemma 1] for one such example.

Lemma 4.4.7. *Let P be a rooted plane tree and T be its plane dual triangulation. The following operations correspond:*

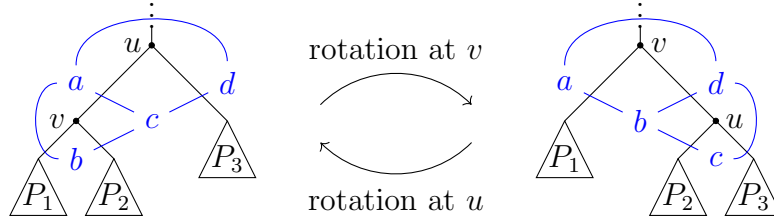


Figure 4.8: The rotation operation on rooted binary trees. A portion of the plane duals are shown in blue.

- a rotation at vertex $v \in V(P)$, and
- a flip at the diagonal in T corresponding to the two regions separated by the edge between v and its parent.

Proof. Suppose v is the left child of its parent u . If the regions of the plane dual surrounding v and u are $\{a, b, c, d\}$ as in Figure 4.8, then observe that a rotation at v replaces the diagonal (a, c) in the dual with (b, d) while preserving all remaining diagonals. This is precisely a flip at the diagonal (a, c) . When v is the right child of its parent, then a similar argument applies. $\square$

We now generalize the rotation operation to planar layouts of irreducible planar tanglegrams. Note that any such planar layout is determined by the two component plane binary trees P_1 and P_2 , as all edges between matched leaves must be horizontal. Hence, we denote a planar layout simply as (P_1, P_2) . Furthermore, in the plane dual bijection of Theorem 4.2.2, if two vertices u and v form the roots of a proper subtanglegram, then the regions adjacent to their parent edges form the shared diagonal in the dual of (P_1, P_2) .

Definition 4.4.8. Let (P_1, P_2) be a planar layout of an irreducible planar tanglegram. A rotation at $u \in V(P_1) \cup V(P_2)$ is defined as

- rotate $u \in V(P_i)$, and
- if a proper subtanglegram is formed, then rotate at the vertex $v \in V(P_j)$ whose subtree forms part of a proper subtanglegram, where $j \neq i$.

Theorem 4.4.9. *Let $\mathcal{L}_n$ be the graph on planar layouts of irreducible planar tanglegrams of size n with edges given by rotations. Then $\mathcal{L}_n$ is isomorphic to $\mathcal{D}_{n+1}$.*

Proof. A bijection between the vertices of $\mathcal{L}_n$ and $\mathcal{D}_{n+1}$ is given by Theorem 4.2.2, so it suffices to show that the rotation and flip operations correspond appropriately. Let $(P_1, P_2), (P'_1, P'_2) \in V(\mathcal{L}_n)$ with respective plane dual pairs of triangulations $(T_1, T_2), (T'_1, T'_2) \in V(\mathcal{D}_{n+1})$. We claim that (P'_1, P'_2) can be obtained by some rotation in (P_1, P_2) if and only if (T'_1, T'_2) can be obtained by some flip in (T_1, T_2) .

Suppose (P'_1, P'_2) can be obtained from (P_1, P_2) by some rotation, which we assume without loss of generality is at some $v \in V(P_1)$. By Theorem 4.4.7, we have that T'_1 is obtained from T_1 by some flip. If (P'_1, P_2) does not contain any proper subtanglegrams, then the result follows. Otherwise, a proper subtanglegram in (P'_1, P_2) corresponds to a shared diagonal in (T'_1, T_2) , and we know that there is only one shared diagonal in (T'_1, T_2) . Observe that there is a unique flip in T_2 that removes the shared diagonal in (T'_1, T_2) . The rotation in P_2 described in Theorem 4.4.8(b) removes the proper subtanglegram in (P'_1, P_2) , and hence this rotation in P_2 and the necessary flip in T'_2 must coincide. We conclude that (T'_1, T'_2) is obtained from (T_1, T_2) by some flip. A similar argument implies the converse. $\square$

4.5 Open Problems

Running the Markov chain on $\mathcal{D}_n$ from Theorem 4.1.2 sufficiently many iterations allows for approximately uniform sampling of pairs of disjoint triangulations. Determining the number of iterations needed remains open, similar to the Markov chain in [HT11].

Problem 1. *Determine the mixing time of the Markov chain from Theorem 4.1.2 and the meanders Markov chain from [HT11].*

Note that a trivial upper bound for the mixing time of the Markov chain on $\mathcal{D}_n$ is $O(|V(\mathcal{D}_n)|^2)$ [LP17, Proposition 10.28]. We suspect that this bound can be improved significantly because the mixing time for a random walk on $\mathcal{T}_n$ is polynomial in n [MT97, MRS01],

while $|V(\mathcal{D}_n)|$ grows rapidly with respect to n [RRW18]. The computer data given in Table 4.2 supports our suspicion that $O(|V(\mathcal{D}_n)|^2)$ is not a useful upper bound for the mixing time of $\mathcal{D}_n$.

n	5	6	7	8	9
$ V(\mathcal{D}_n) $	10	68	546	4872	46782
iterations	3	7	14	25	39
σ_2	0.5590...	0.7287...	0.8478...	0.9512...	0.9677...

Table 4.2: For each n , the number of iterations needed for the total variation distance from the uniform distribution to be smaller than $1/4$ regardless of the initial vertex chosen, and the second largest eigenvalue of the transition matrix.

The Markov chains discussed in this chapter allow for near-uniform generation of the corresponding elements. Another natural question is what properties a “typical” element has. One surprising result of [KW16] is that as $n \rightarrow \infty$, generating a tanglegram of size n uniformly at random becomes similar to generating two rooted plane binary trees on n leaves independently and uniformly at random. Konvalinka and Wagner used this result to establish asymptotic properties of tanglegrams.

Problem 2. *Study properties of pairs of disjoint triangulations, meanders, and planar tanglegrams generated uniformly at random. In the case of planar tanglegrams, establish analogues of results in [KW16].*

Known enumerative results for meanders [OEI, A005315] and pairs of disjoint triangulations [RRW18] imply that these objects are not in bijection with each other. However, it is possible that other connections exist by applying certain bijections between Catalan objects.

Problem 3. *Explore connections between pairs of disjoint triangulations, meanders, and other pairs of Catalan objects satisfying some property.*

Finally, we pose a problem involving our construction of $\mathcal{D}_n$. Our techniques on $\mathcal{D}_n$ can be generalized much further in the broader language of polyhedra.

Problem 4. *For a given simple polytope P , explore the graph on pairs*

$$\{(u, v) : u, v \in V(P), u \text{ and } v \text{ are not contained in the same facet}\}$$

with edges constructed in a manner similar to in $\mathcal{D}_n$. Determine what analogues of Theorem 4.1.3 hold and when this is the graph of an abstract polytope.

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