The GIT fan for a Mori dream space and the μ-secondary polytope

by

Jamie Peabody

B.A., California State University, Fresno, 2013

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Abstract

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Geometric invariant theory (GIT) was developed by Mumford as a method for constructing quotients by group actions in the context of algebraic geometry. This construction is not canonical and depends on a choice. A way to organize the different choices and their resulting quotients we use a combinatorial object known as the GIT fan. Mori dream spaces were introduced by Hu and Keel as projective varieties whose cone of effective divisors can be decomposed into a finite number of convex polyhedral cones called Mori chambers. As the name suggests they are ideal spaces for Mori's Minimal Model Program, which deals with the classification of varieties up to birational equivalence. It was shown by Hu and Keel that a Mori dream space is naturally a GIT quotient of an affine variety by an algebraic torus. As a consequence, the GIT fan coincides with the Mori chamber decomposition. Thus, the birational geometry of a Mori dream space is an instance of variation of geometric invariant theory quotients (VGIT). In this work we develop a criterion to describe the GIT fan associated to a Mori dream space. This is done by using tropical geometry, toric geometry, and triangulations of marked polytopes. As a result of this approach, we define a polytope called the μ-secondary polytope whose normal fan is the GIT fan, generalizing the result in toric geometry.

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Preface

Geometric invariant theory (GIT) was developed by Mumford [MFK94] and studies quotients by group actions in the context of algebraic geometry. Suppose we have a group G acting on an algebraic variety X. A first consideration for the quotient of X by G is the orbit space X/G, however in general this does not have a nice geometric structure. An issue that can arise is the existence of non-closed orbits of the action. The philosophy of GIT is to remove "bad" points so that what remains has a good categorical quotient that is also a projective variety. The construction through GIT is not canonical and depends on a choice of G-linearized line bundle on X. This choice can sometimes be described in terms of characters of the group, $\widehat{G} = \{\chi : G \longrightarrow \mathbb{G}_m \text{ homomorphism of algebraic groups}\}$. A choice of character defines a subset of points of X called semistable points, the complement of this set is called the unstable locus. The GIT quotient is a good categorical quotient of the set of semistable points by the group G.

Naturally, one might wonder what effect the different choices have on the quotient. When two choices give rise to the same set of unstable points then the resulting quotients are isomorphic, otherwise we obtain something distinct. This raises a further question: what is the relationship between the GIT quotients as we vary the choice of character? The answer to this question leads to the study of variation of geometric invariant theory (VGIT) and has been studied by Dolgachev and Hu in [DH98] and Thaddeus in [Tha96] who found that the GIT quotients are related by birational transformations. One can define an equivalence relation on the space of characters by defining two characters to be equivalent if they define the same set of semistable points. This leads to a combinatorial structure known as the GIT fan. This fan is a collection of convex polyhedral cones, the maximal cones of this fan are called chambers and the codimension one cones are called walls. The characters in the relative interior of a given chamber define the same set of unstable points and thus yield isomorphic GIT quotients.

A class of varieties which can be seen as GIT quotients are normal toric varieties. A toric variety is a variety that contains an algebraic torus as an open dense subset such that the action of the torus on itself extends to an action on the whole variety. Toric varieties are very appealing because they have a nice combinatorial description. The geometry of a toric variety is completely determined by the combinatorics of its associated fan. For certain toric varieties this information is encoded in a polytope.

Suppose we have a closed subgroup $G \subset \mathbb{G}_m^n$ acting on $\mathbb{A}^n$, then the GIT quotient is a toric variety, as seen in [CLS11]. Associated to this group G is a fan $\mathscr{F}_G$, called the secondary fan, which is used to classify all of the toric varieties as GIT quotients of $\mathbb{A}^n$ by G. In Chapter 2 we will discuss GIT quotients of $\mathbb{A}^n$ by G and the secondary fan $\mathscr{F}_G$. In some cases, $\mathscr{F}_G$ coincides with the secondary fan $\mathscr{F}_A$ of a marked polytope (Q, A), where A comes from the action of G on $\mathbb{A}^n$ and $Q = \mathrm{Conv}(A)$. The advantage of being in this case is that the fan $\mathscr{F}_A$ is described by triangulations of the polytope Q. The chambers of this fan correspond to coherent triangulations of Q where each simplex has vertices among the points of the set A, as seen in [GKZ94]. To understand when two chambers are adjacent we look at the triangulations corresponding to each chamber. The triangulations associated to adjacent chambers are related by a circuit modification. A circuit is a minimally affinely dependent set of points which has exactly two coherent triangulations. A circuit modification of a triangulation produces another triangulation by changing only the triangulation of the circuit. These circuit modifications are an essential ingredient to developing the criterion for describing the GIT fan. We will discuss the secondary fan of a marked polytope and circuit modifications in Chapter 1.

A nice fact about the fan $\mathscr{F}_A$, is that it is the normal fan of a polytope $\Sigma(A)$, called the secondary polytope. The vertices of the secondary polytope correspond to coherent triangulations and are defined by

$$\varphi_T := \sum_{a \in A} \Bigg(\sum_{\substack{\sigma \in T \\ a \in \sigma}} \mathrm{vol}(\sigma) \Bigg) \, e_a.$$

Moreover, the face lattice of $\Sigma(A)$ is isomorphic to the poset of coherent subdivisions of (Q, A), ordered by refinement.

A generalization of toric varieties are Mori dream spaces which were introduced by Hu and Keel in [HK00]. We will discuss them in Chapter 4. Mori dream spaces are projective varieties whose effective cone is polyhedral and can be decomposed into a finite number of polyhedral chambers called Mori chambers. As the name suggests, they are ideal for Mori's minimal model program, which deals with the classification of varieties up to birational equivalence. The birational geometry of a Mori dream space can be understood through this chamber decomposition. One of the major results of Hu and Keel is that a Mori dream space can be defined by finite generation of its Cox ring. As a consequence of this a Mori dream space is naturally a GIT quotient of an affine variety by an algebraic torus. This result generalizes Cox's construction of toric varieties as quotients of affine space by an algebraic torus. As a result of this the Mori chamber decomposition coincides with the GIT fan. Thus the birational geometry of a Mori dream space is an instance of VGIT.

Let X be a Mori dream space with $\mathrm{Pic}(X)$ free of rank r and $L_1, \ldots, L_r$ a basis. Then the Cox ring of X is

$$\mathrm{Cox}(X) = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r}).$$

Since X is a Mori dream space we have that $\mathrm{Cox}(X)$ is finitely generated and X is a GIT quotient of $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ by the algebraic torus $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m)$. The fan we want to describe is the GIT fan for quotients of Y by G. Since we have that $\mathrm{Cox}(X)$ is finitely generated it has some presentation as a quotient of a polynomial ring $k[x_1, \ldots, x_n]/I$. This gives Y as a G-equivariant subvariety of $\mathbb{A}^n$ and the GIT fan will be a coarsening of the secondary fan $\mathscr{F}_G$ which describes the quotients of $\mathbb{A}^n$ by G. As stated earlier, under certain conditions we have that this fan coincides with the secondary fan $\mathscr{F}_A$ of a marked polytope (Q, A). If these conditions are not already satisfied, then there is a way to "promote" the data so that we work with the fan $\mathscr{F}_A$.

To describe the GIT fan, we need to determine which walls we lose in the fan $\mathscr{F}_A$. We do this by making use of the correspondence with chambers of $\mathscr{F}_A$ and coherent triangulations of (Q, A). This will be discussed in detail in Section 2.2. The unstable locus for a chamber can be given as a disjoint union of coordinate subspaces indexed by the simplices of the triangulation. The triangulations in adjacent chambers are related by a circuit modification. Using this description, we can determine exactly what part of the unstable locus is changing as we cross a wall, and this is given by the circuit associated to that wall. If we have that Y does not intersect any of what is changing in the unstable locus on either side of a wall, then there will be no change in the GIT quotient and thus that wall is not part of the GIT fan.

To determine if Y intersects the changing unstable loci we use tropical geometry, following the conventions in [MS15]. A process called tropicalization takes an algebraic variety and produces an associated polyhedral complex. The combinatorics of this complex encodes much of the geometry of the original algebraic variety. Using tropical geometry, questions about algebraic varieties can be translated into questions about polyhedral complexes. In Chapter 3 we will discuss tropicalization and how it is used to determine the GIT fan.

The part of the unstable locus that changes as we cross a wall is a disjoint union of certain coordinate subspaces. The tropicalization of these coordinate subspaces correspond to certain polyhedral cones. If the tropicalization of Y intersects the relative interior of these polyhedral cones, then Y intersects the corresponding coordinate subspaces. Since we have that these coordinate subspaces are indexed by the circuit associated to a given wall, we can reduce this criterion to checking if the tropicalization of Y intersects the relative interior of the circuit. This results in the following theorem.

Theorem. 3.2 Suppose $G \subset G_{\text{big}} \simeq \mathbb{G}_m^n$ acts on $\mathbb{A}^n$ and we have a subvariety $X \subset G_{\text{big}}$ with $\overline{X} \subset \mathbb{A}^n$ such that $\overline{X}$ is stable with respect to the action of G. Consider a wall in the secondary fan corresponding to a circuit C. Then $\operatorname{trop}(X)_N$ intersects the relative interior of C (or $\overline{C}$ in the case of a degenerate circuit) if and only if that wall is part of the GIT fan for $\overline{X}$.

The GIT fan was obtained by looking at triangulations of a marked polytope (Q, A) and removing the walls in $\mathscr{F}_A$ where the tropicalization of Y did not intersect the relative interior of the circuit associated to that wall. This process can be quite cumbersome, thus we sought to define a polytope whose normal fan is this coarsened fan, as we have with the fan $\mathscr{F}_A$ and polytope $\Sigma(A)$. This new polytope, which we call the μ-secondary polytope denoted $\Sigma_\mu(A)$ is constructed by defining a measure μ on Q and generalizing the construction of the secondary polytope $\Sigma(A)$ from [GKZ94]. The measure μ is used to define an equivalence relation on coherent triangulations by identifying those that are in adjacent chambers where a wall was lost. This defines a coarsening of the secondary fan which we call the μ-coarsening denoted by $\mathscr{F}_A(\mu)$. Each vertex of $\Sigma_\mu(A)$ is defined by a coherent triangulation T of (Q, A) as

$$\varphi_{T,\mu} := \sum_{a \in A} \left(\sum_{\substack{\sigma \in T \\ a \in \sigma}} \int_{\mathrm{relint}(\sigma)} x_{a,\sigma} \, d\mu \right) e_a$$

where $x_{a,\sigma}$ is the affine function on σ that takes the value 1 on a and 0 at all other vertices of σ.

Theorem. 5.1 The μ-coarsening $\mathscr{F}_A(\mu)$ is the normal fan of the μ-secondary polytope $\Sigma_\mu(A)$.

The polytope associated to the GIT fan of a Mori dream space is defined by a volume measure on the tropical variety.

Theorem. 5.2 Let X be a Mori dream space with $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m) \simeq \mathbb{G}_m^s$ and $Y = \mathrm{Spec}(\mathrm{Cox}(X)) \subset \mathbb{A}^n$. Let (Q, A) be the polytope obtained from the action of G on $\mathbb{A}^n$ (where 0 was added to the set A if the promotion procedure was required). Let μ be a volume measure on the tropicalization of Y. Then the GIT fan is the normal fan to the μ-secondary polytope $\Sigma_\mu(A)$.

This construction will be discussed in Chapter 5 and we will give some examples of coarsened fans and μ-secondary polytopes in Chapter 6.

Chapter 1

Secondary Fan of a Marked Polytope

In this chapter we discuss the secondary fan of a marked polytope, as seen in [GKZ94]. The *secondary fan* is a combinatorial object that describes all subdivisions of a marked polytope. A *marked polytope* is a pair (Q, A) where Q is a polytope and A is a finite set of points containing the vertices, so that $Q = \mathrm{Conv}(A)$.

Suppose we have a full dimensional marked polytope (Q, A) in $\mathbb{R}^n$. A *subdivision* of (Q, A) is a collection $S = \{(Q_i, A_i)\}_{i \in I}$ of marked polytopes such that each A_i is a subset of A, $\dim(Q_i) = n$, any two intersect in a common face, and the union of all Q_i coincides with Q. A *triangulation* of (Q, A) is a particular case of a subdivision where all the Q_i are simplices and A_i is the set of vertices of Q_i.

Figure 1.1: Example of a subdivision and a triangulation

The set of all subdivisions can be made into a poset ordered by refinement. For $S = \{(Q_i, A_i)\}$ and $S' = \{(Q'_j, A'_j)\}$ two subdivisions of (Q, A) we say that S refines S' if for every j there is a collection of (Q_i, A_i) that form a subdivision of (Q'_j, A'_j). The minimal elements of this poset are the triangulations and the maximal element is the subdivision consisting of (Q, A) itself.

Suppose $Q \subset \mathbb{R}^n$ is a full dimensional polytope and $\psi : A \longrightarrow \mathbb{R}$ any function. To construct a subdivision, consider

$$G_\psi = \mathrm{Conv}(\{(a, y) : y \leq \psi(a), a \in A, y \in \mathbb{R}\}) \subset \mathbb{R}^n \times \mathbb{R}.$$

It is an unbounded polyhedron that projects onto Q. The faces that do not contain a vertical half-line form the upper boundary of G_ψ which projects bijectively onto Q. The projection of the bounded faces of G_ψ will form a subdivision of (Q, A). If ψ is generic then the subdivision we obtain is a triangulation.

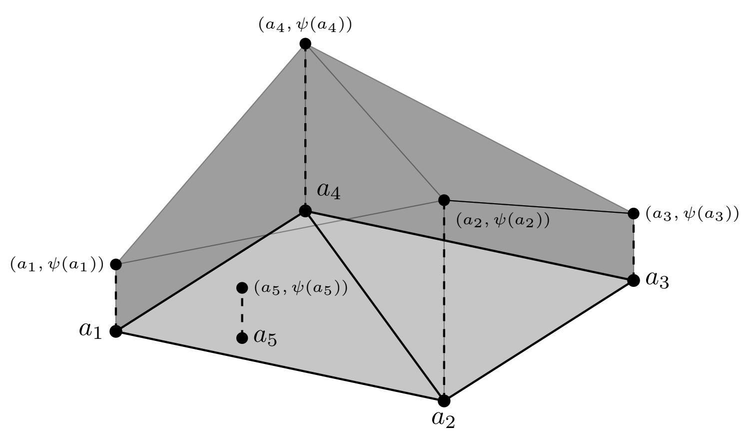


Figure 1.2: Example of constructing a triangulation

1.1 Coherent Triangulations

The triangulations or interest in this work are called *coherent* (or *regular*) triangulations, which turn out to be those triangulations which are constructed as in Figure 1.2. A continuous function $g : Q \longrightarrow \mathbb{R}$ is called *T-piecewise-linear* if it is affine-linear on every simplex of T and *concave* if for any $x, y \in Q$ we have

$$g(tx + (1 - t)y) \;\geq\; tg(x) + (1 - t)g(y)$$

for $0 \leq t \leq 1$. A *domain of linearity* of a concave function g is a subset $U \subset Q$ such that the restriction of g to U is given by some affine-linear function and is maximal with this property.

Definition 1.1. A triangulation T of (Q, A) is called *coherent* if there exists a concave T-piecewise-linear function whose domains of linearity are maximal simplices of T.

For any triangulation T of (Q, A) and any function $\psi : A \longrightarrow \mathbb{R}$ there is a unique T-piecewise-linear function $g_{\psi,T} : Q \longrightarrow \mathbb{R}$. This is defined by setting $g_{\psi,T}(a) = \psi(a)$ for every $a \in A$ that is a vertex of some simplex of T and then affinely interpolating inside of each simplex.

Definition 1.2. Let T be a triangulation of (Q, A). Let $C(T)$ be the cone in $\mathbb{R}^A$ consisting of functions $\psi : A \longrightarrow \mathbb{R}$ with the following properties:

(1) $g_{\psi,T} : Q \longrightarrow \mathbb{R}$ is concave

(2) $g_{\psi,T}(a) \geq \psi(a)$ for all $a \in A$.

A triangulation T is coherent if and only if the interior of $C(T)$ is non-empty.

Proposition 1.1. [GKZ94, Proposition 7.1.5] Let A and $Q = \mathrm{Conv}(A)$ be fixed. The cones $C(T)$ for all coherent triangulations T of (Q, A) together with all the faces of these cones form a complete fan in $\mathbb{R}^A$.

This fan is called the *secondary fan* of A, which we will denote $\mathscr{F}_A$.

1.2 The Secondary Polytope

It turns out that the secondary fan is the normal fan of a polytope called the *secondary polytope*. Let $A \subset \mathbb{R}^n$ be a finite set, $Q = \mathrm{Conv}(A)$ a full dimensional polytope, and T a triangulation of (Q, A). Consider $\varphi_T \in \mathbb{R}^A$ defined by:

$$\varphi_T = \sum_{a \in A} \left(\sum_{\substack{\sigma \in T \\ a \in \sigma}} \mathrm{vol}(\sigma) \right) e_a \tag{1.1}$$

i.e. the value of φ_T on $a \in A$ is the sum of the volumes of the simplices in the triangulation for which a is a vertex.

Definition 1.3. The *secondary polytope*, denoted $\Sigma(A)$, is the convex hull in $\mathbb{R}^A$ of φ_T for all triangulations T of (Q, A).

The normal cone to $\Sigma(A)$ at φ_T , denoted $N_{\varphi_T}\Sigma(A)$, consists of all $\psi \in \mathbb{R}^A$ such that

$$(\varphi_T, \psi) = \max_{\varphi \in \Sigma(A)} (\varphi, \psi) \tag{1.2}$$

Theorem 1.1. [GKZ94, Theorem 7.1.7]

(1) Vertices of $\Sigma(A)$ are precisely the φ_T for all coherent triangulations T of (Q, A). If T is a coherent triangulation of (Q, A) then $\varphi_T \neq \varphi_{T'}$ for any other triangulation T' of (Q, A).

(2) For any triangulation T the normal cone $N_{\varphi_T}\Sigma(A)$ coincides with the cone $C(T)$.

This theorem gives that the secondary fan is the normal fan of the secondary polytope. The vertices of the secondary polytope and the maximal cones of the secondary fan correspond to coherent triangulations. Moreover, the face lattice of $\Sigma(A)$ is isomorphic to the poset of coherent subdivisions of (Q, A), ordered by refinement. Determining when two vertices of the secondary polytope are joined by an edge is equivalent to determining when two maximal cones of the secondary fan are adjacent.

1.3 Circuit Modifications

We can determine when two maximal cones in the secondary fan are adjacent through the relationship between the triangulations that correspond to each cone. This description requires the notion of a circuit.

Definition 1.4. A finite set C of points in $\mathbb{R}^n$ is called a *circuit* if any proper subset $C' \subsetneq C$ is affinely independent while C itself is affinely dependent.

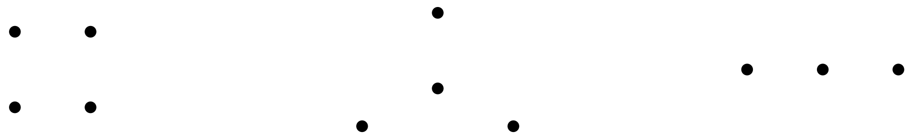


Figure 1.3: Examples of circuits

A circuit can always be obtained by adding one point to the set of vertices of a simplex. There is a unique (up to scaling) affine relation among the elements of C:

$$\sum_{c \in C} a_c \cdot c = 0, \quad \sum a_c = 0.$$

This gives a decomposition of C into two subsets.

$$C_+ = \{c \in C : a_c > 0\} \text{ and } C_- = \{c \in C : a_c < 0\}$$

This decomposition $C = C_+ \coprod C_-$ is unique up to interchanging C_+ and C_-.

Proposition 1.2. [GKZ94, Proposition 7.1.2] Let $C \subset \mathbb{R}^n$ be any circuit and $Q = \mathrm{Conv}(C)$. Then Q has exactly two triangulations with vertices in C: the triangulation T_+ with simplices $\mathrm{Conv}(C - \{c\})$, $c \in C_+$, and the triangulation T_- with simplices $\mathrm{Conv}(C - \{c\})$, $c \in C_-$.

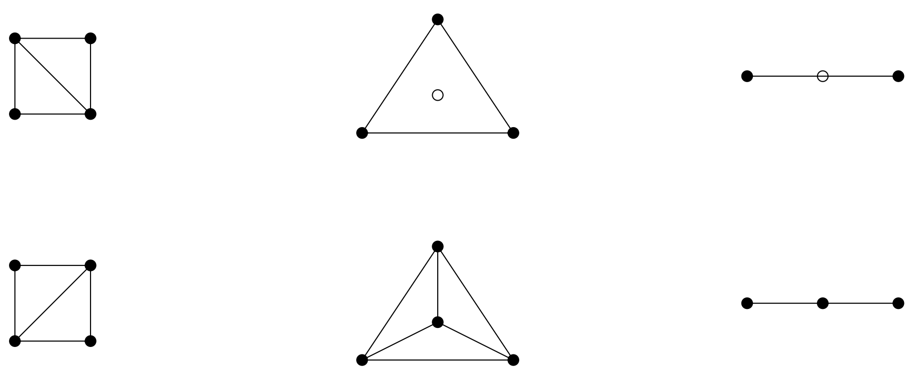


Figure 1.4: Examples of triangulations of circuits

We need to distinguish between two types of circuits which we will call non-degenerate and degenerate. A circuit in $\mathbb{R}^n$ is *non-degenerate* if it affinely spans $\mathbb{R}^n$ otherwise it is *degenerate*. We will now describe how to modify a triangulation along a circuit, starting with the case of a non-degenerate circuit.

Let $Q = \mathrm{Conv}(A) \subset \mathbb{R}^n$ be a full dimensional polytope and T a triangulation of (Q, A). Suppose we have a non-degenerate circuit C where $\mathrm{Conv}(C)$ is a union of simplices of T with vertices in C. The triangulation of $\mathrm{Conv}(C)$ induced by T must be one of the two triangulations of $(\mathrm{Conv}(C), C)$, say T_+. We obtain a new triangulation T' of (Q, A) by replacing T_+ with T_- in $\mathrm{Conv}(C)$ and doing nothing to the other simplices of T. The triangulation T' is called the *modification of T along C*. An example of this can be seen in the following Figure 1.5.

T $\qquad\qquad$ T'

Figure 1.5: A modification along a non-degenerate circuit

Proposition 1.3. [GKZ94, Proposition 7.2.8] Let a triangulation T and circuit C be as above and T' the modification of T along C. If both T and T' are coherent triangulations then the corresponding cones in the secondary fan are adjacent.

Now let us consider the case when $C \subset \mathbb{R}^n$ is a degenerate circuit. First we need to extend the circuit to a spanning set $C^e = C \cup B$. There is a unique (up to scaling) affine relation between the elements of C^e where the elements of C are the only points with a non-zero coefficient.

Figure 1.6: Example of a degenerate circuit in $\mathbb{R}^2$

As with circuits, $(\mathrm{Conv}(C^e), C^e)$ has exactly two triangulations. They are obtained from the two triangulations of the circuit C by taking cones over their simplices.

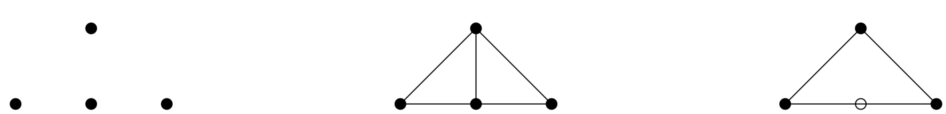


Figure 1.7: An extended circuit in $\mathbb{R}^2$ and its two triangulations

Definition 1.5. Let T be a triangulation and C a circuit. We say that T is supported on C if the following hold:

(1) There are no vertices of T inside $\mathrm{Conv}(C)$ besides elements of C itself

(2) The polytope $\mathrm{Conv}(C)$ is a union of faces of simplices of T

(3) If $\mathrm{Conv}(I)$ and $\mathrm{Conv}(I')$ are maximal simplices of one of the two triangulations of $\mathrm{Conv}(C)$ then for every $F \subset A - C$ we have $\mathrm{Conv}(I \cup F)$ is a simplex of T if and only if $\mathrm{Conv}(I' \cup F)$ is a simplex of T

Let $Q = \mathrm{Conv}(A) \subset \mathbb{R}^n$ be a full dimensional polytope, C a degenerate circuit, and T a triangulation supported on C. Then T induces one of the two possible triangulations on $\mathrm{Conv}(C)$, say T_+. Each maximal simplex of that triangulation of $\mathrm{Conv}(C)$ is a face of a simplex of an extended circuit whose convex hull is a union of simplices of T. This allows us to form

$$\overline{C} = \bigcup_{k \in K} C_k^e \tag{1.3}$$

where K is some finite indexing set. We obtain a new triangulation T' by replacing T_+ with T_- for each C_k^e and leaving all other simplices untouched.

An example of this is shown in the following figure. Here were are modifying along a degenerate circuit similar to the one shown in Figure 1.6. The $\overline{C}$ in this case consists of two extended circuits like the one seen Figure 1.7.

Figure 1.8: A modification along a degenerate circuit

Theorem 1.2. [GKZ94, Theorem 7.2.10] Let T and T' be two coherent triangulations of (Q, A). The associated cones in the secondary fan are adjacent if and only if there is a circuit

$C \subset A$ such that T and T' are both supported on C and obtained from each other by a modification along C.

Chapter 2

Geometric Invariant Theory

Geometric Invariant Theory (GIT) provides a method for constructing quotients for group actions on algebraic varieties in a way that preserves some geometric structure. Suppose we have an algebraic variety X being acted on by a reductive algebraic group G. In general the orbit space X/G is not an algebraic variety, a reason for this is the existence of non-closed orbits. We would want the projection $X \longrightarrow X/G$ to be a regular map of algebraic varieties and thus have closed fibers, which requires all orbits to be closed in the Zariski topology of X. A way to deal with this problem, which was proposed by Mumford [MFK94], is to form a GIT quotient. Such a quotient is constructed by finding an open G-invariant subset U of X so that $U \longrightarrow U/G$ exists and U is as large as possible. One problem with this construction is that the set U is not canonical and depends on a choice of a G-linearized line bundle on X. In general we can see different quotients resulting from different choices of G-linearized line bundles. This variation of GIT quotients can be described by a combinatorial structure known as the GIT fan.

Let us fix some notation, let k be an algebraically closed field, $\mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \ldots, x_n])$, and $\mathbb{G}_m$ the multiplicative group of units. We consider the case when G a closed subgroup of $\mathbb{G}_m^n$ acting on $\mathbb{A}^n$. To construct a GIT quotient of $\mathbb{A}^n$ by G we need to first make a choice of G-linearized line bundle on $\mathbb{A}^n$. To obtain such a line bundle we lift the G-action on $\mathbb{A}^n$ to the trivial line bundle $\mathbb{A}^n \times \mathbb{A}^1 \longrightarrow \mathbb{A}^n$. This lifting of the G action can be described by the characters of the group G, and in this case all G-linearized line bundles are obtained this way. The *character group* of G is

$$\widehat{G} := \{\chi : G \to \mathbb{G}_m : \chi \text{ is a homomorphism of algebraic groups}\}.$$

For a character $\chi \in \widehat{G}$ the action of G on $\mathbb{A}^n \times \mathbb{A}^1$ is given by

$$g \cdot (\mathbf{x}, t) = (g \cdot \mathbf{x}, \chi(g)t),\ g \in G, (\mathbf{x}, t) \in \mathbb{A}^n \times \mathbb{A}^1.$$

This lifts the action of G on $\mathbb{A}^n$ to the trivial line bundle $\mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$. Let $\mathscr{L}_\chi$ denote the sheaf of sections of $\mathbb{A}^n \times \mathbb{A}^1$ with this G-action, which we call the *G-linearized line bundle* for the character χ. Note that for $d \in \mathbb{Z}$ we have $\mathscr{L}_\chi^{\otimes d} = \mathscr{L}_{\chi^d}$.

Now that we have a choice of G-linearized line bundle we need to construct the open set U. A global section $s \in \Gamma(\mathbb{A}^n, \mathscr{L}_\chi)$ can be written as

$$s(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x})) \in \mathbb{A}^n \times \mathbb{A}^1,\ \mathbf{x} \in \mathbb{A}^n,$$

for a unique $f \in k[x_1, \ldots, x_n]$. Then we have G acts on global sections by

$$(g \cdot s)(\mathbf{x}) = (\mathbf{x}, \chi(g) f(g^{-1} \cdot \mathbf{x})),\ \mathbf{x} \in \mathbb{A}^n.$$

The (G, χ)-*invariant global sections* are

$$\Gamma(\mathbb{A}^n, \mathscr{L}_\chi)^G := \{f \in k[x_1, \ldots, x_n] : f(g \cdot \mathbf{x}) = \chi(g) f(\mathbf{x}) \text{ for } g \in G,\ \mathbf{x} \in \mathbb{A}^n\}.$$

Example 2.1. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ by $t \cdot \mathbf{x} = (t \cdot x_1, \ldots, t \cdot x_n)$. We have $\widehat{\mathbb{C}^*} \simeq \mathbb{Z}$ where $a \in \mathbb{Z}$ gives the character defined by $\chi^a(t) = t^a$. The $\mathbb{C}^*$-invariant global sections $\Gamma(\mathbb{A}^n, \mathscr{L}_\chi)^{\mathbb{C}^*}$ is isomorphic to the ring of homogeneous degree a polynomials.

Definition 2.1. Fix $G \subset \mathbb{G}_m^n$ and $\chi \in \widehat{G}$, with G-linearized line bundle $\mathscr{L}_\chi$.

(1) $\mathbf{x} \in \mathbb{A}^n$ is *semistable* if there exists $d > 0$ and $f \in \Gamma(\mathbb{A}^n, \mathscr{L}_{\chi^d})^G$ such that $f(\mathbf{x}) \neq 0$.

(2) $\mathbf{x} \in \mathbb{A}^n$ is *stable* if there exists $d > 0$ and $f \in \Gamma(\mathbb{A}^n, \mathscr{L}_{\chi^d})^G$ such that $f(\mathbf{x}) \neq 0$, the stabilizer $G_\mathbf{x}$ is finite, and all G-orbits are closed in the set $\{\mathbf{x} \in \mathbb{A}^n : f(\mathbf{x}) \neq 0\}$.

(3) $\mathbf{x} \in \mathbb{A}^n$ is *unstable* if it is not semistable.

(4) The set of semistable, stable, and unstable points will be denoted $(\mathbb{A}^n)_\chi^{ss}$, $(\mathbb{A}^n)_\chi^s$, and $(\mathbb{A}^n)_\chi^{us}$ respectively.

Example 2.2. Consider $\mathbb{C}^*$ acting on $\mathbb{C}^n$ as in Example 2.1. The sets of semistable and stable points depends on the choice of character $a \in \mathbb{Z}$.

$$\begin{array}{ll} a > 0 & (\mathbb{C}^n)_{\chi^a}^{ss} = (\mathbb{C}^n)_{\chi^a}^{s} = \mathbb{C}^n \setminus \{0\} \\ a = 0 & (\mathbb{C}^n)_{\chi^a}^{ss} = \mathbb{C}^n \text{ and } (\mathbb{C}^n)_{\chi^a}^{s} = \emptyset \\ a < 0 & (\mathbb{C}^n)_{\chi^a}^{ss} = (\mathbb{C}^n)_{\chi^a}^{s} = \emptyset \end{array}$$

Consider the graded ring

$$R_\chi = \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{A}^n, \mathscr{L}_{\chi^d})^G.$$

Definition 2.2. The *GIT quotient* of $\mathbb{A}^n$ by G with respect to χ is

$$\mathbb{A}^n /\!\!/_\chi\, G := \mathrm{Proj}\ (R_\chi)$$

Proposition 2.1. [CLS11, Proposition 14.1.12] For $G \subset \mathbb{G}_m^n$ and $\chi \in \widehat{G}$, we have:

(1) There is a projective morphism $\mathbb{A}^n /\!\!/_\chi\, G \to \mathbb{A}^n /\!\!/ G = \mathrm{Spec}(k[x_1, \ldots, x_n]^G)$.

(2) $\mathbb{A}^n /\!\!/_\chi\, G \neq \emptyset$ if and only if $(\mathbb{A}^n)_\chi^{ss} \neq \emptyset$.

(3) The GIT quotient $\mathbb{A}^n /\!\!/_\chi\, G$ is a good categorical quotient of $(\mathbb{A}^n)_\chi^{ss}$ under the action of G.

(4) If $(\mathbb{A}^n)^s_\chi \neq \emptyset$ then the action of G on $(\mathbb{A}^n)^s_\chi$ has a geometric quotient $(\mathbb{A}^n)^s_\chi/G$ isomorphic to a nonempty open subset of $\mathbb{A}^n /\!\!/_\chi G$. Thus $\mathbb{A}^n /\!\!/_\chi G$ is an almost geometric quotient and $\dim \mathbb{A}^n /\!\!/_\chi G = n - \dim G$.

Example 2.3. We continue with the case from Example 2.2. Let $a \in \mathbb{Z}$ be a character, we have three cases for the GIT quotient.

$$\begin{array}{lll} a > 0 & \mathbb{C}^n /\!\!/_{\chi^a} \mathbb{C}^* \simeq & \mathbb{P}^{n-1} \\ a = 0 & \mathbb{C}^n /\!\!/_{\chi^a} \mathbb{C}^* = & \{\mathrm{pt}\} \\ a < 0 & \mathbb{C}^n /\!\!/_{\chi^a} \mathbb{C}^* = & \emptyset \end{array}$$

This construction depends on a choice. In some instances two different choices will yield isomorphic GIT quotients, but we could also obtain quotients that are non-isomorphic. To keep track of this we use a combinatorial structure known as the GIT fan.

Definition 2.3. For $G \subset \mathbb{G}_m^n$ acting on $\mathbb{A}^n$, let $\widehat{G}_{\mathbb{Q}} := \widehat{G} \otimes_{\mathbb{Z}} \mathbb{Q}$ be its rational character space.

(1) The *weight cone* is the convex cone in $\widehat{G}_{\mathbb{Q}}$ generated by all $\chi \in \widehat{G}$ for which a semistable point with respect to χ exists.

(2) The *orbit cone* of $\mathbf{x} \in \mathbb{A}^n$ is the convex cone in $\widehat{G}_{\mathbb{Q}}$ generated by all $\chi \in \widehat{G}$ for which x is semistable with respect to χ.

(3) The *GIT cone* of a character $\chi \in \widehat{G}$ is the intersection of all orbit cones containing χ.

(4) The *GIT fan* is the collection of all GIT cones.

Theorem 2.1. [BH05, Theorem 2.11] The GIT fan is in fact a fan in $\widehat{G}_{\mathbb{Q}}$ having the weight cone as its support.

In each GIT cone the set of semistable points is constant and thus any two choices of character within a given GIT cone will yield isomorphic GIT quotients. If we choose characters in different GIT cones we obtain non-isomorphic quotients.

Example 2.4. The GIT fan for Example 2.3 is:

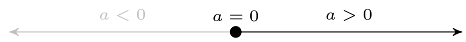


Figure 2.1: Example of GIT fan

Let us now explore more of the combinatorial aspects of this construction. The character group of $\mathbb{G}_m^n$ is isomorphic to $\mathbb{Z}^n$ thus the inclusion $G \subset \mathbb{G}_m^n$ induces a homomorphism $\gamma : \mathbb{Z}^n \to \widehat{G}$ with kernel M. This gives the following exact sequence:

$$0 \longrightarrow M \xrightarrow{\delta} \mathbb{Z}^n \xrightarrow{\gamma} \widehat{G}$$

where we denote the image of $\mathbf{a} \in \mathbb{Z}^n$ in $\widehat{G}$ by $\chi^{\mathbf{a}}$. Let N be the dual to M, then the dual of δ is a map $\mathbb{Z}^n \to N$ that sends the standard basis $e_1, \dots, e_n \in \mathbb{Z}^n$ to elements $\nu_1, \dots, \nu_n \in N$. We then have a formula for the map δ by

$$\delta(m) = (\langle m, \nu_1 \rangle, \dots, \langle m, \nu_n \rangle)\,,\ m \in M.$$

Lemma 2.1. [CLS11, Lemma 14.2.1]

(1) The map γ is surjective, so we have a short exact sequence:

$$0 \longrightarrow M \xrightarrow{\delta} \mathbb{Z}^n \xrightarrow{\gamma} \widehat{G} \longrightarrow 0 \tag{2.1}$$

(2) $G = \{(t_1, \dots, t_n) \in \mathbb{G}_m^n : \prod_{i=1}^n t_i^{\langle m, \nu_i \rangle} = 1,$ for all $m \in M\}$.

From the short exact sequence in eq. (2.1) we obtain the $\widehat{G}$-grading on the polynomial ring $k[x_1, \dots, x_n]$ by $\deg(\mathbf{x}^{\mathbf{a}}) = \gamma(\mathbf{a}) \in \widehat{G}$.

To each character we can associate a polyhedron in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. For $\chi = \chi^{\mathbf{a}} \in \widehat{G}$ with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we define the polyhedron

$$P_{\mathbf{a}} = \{m \in M_{\mathbb{R}} : \langle m, \nu_i \rangle \geq -a_i, 1 \leq i \leq n\} \subset M_{\mathbb{R}}.$$

For any $\mathbf{b}$ giving the same character χ we have $\mathbf{a}-\mathbf{b}=\delta(m)$ for some $m\in M$ thus $P_{\mathbf{a}}$ and $P_{\mathbf{b}}$ are translations of each other by some element of M. We can also define the polyhedron associated to χ independent of the choice of $\mathbf{a}$. First tensor eq. (2.1) with $\mathbb{R}$ and get the following short exact sequence:

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\delta_{\mathbb{R}}} \mathbb{R}^n \xrightarrow{\gamma_{\mathbb{R}}} \widehat{G}_{\mathbb{R}} \longrightarrow 0$$

Let $\chi\otimes 1\in \widehat{G}_{\mathbb{R}}$ be the image of χ under $\widehat{G}\to\widehat{G}_{\mathbb{R}}=\widehat{G}\otimes_{\mathbb{Z}}\mathbb{R}$. Then define the polyhedron $P_\chi\subset\mathbb{R}^n$ by

$$P_\chi=\{\mathbf{b}\in\mathbb{R}^n_{\geq 0}:\gamma_{\mathbb{R}}(\mathbf{b})=\chi\otimes 1\}=\gamma_{\mathbb{R}}^{-1}(\chi\otimes 1)\cap\mathbb{R}^n_{\geq 0}$$

To relate these two descriptions consider $\chi=\chi^{\mathbf{a}}$ for $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n$ then

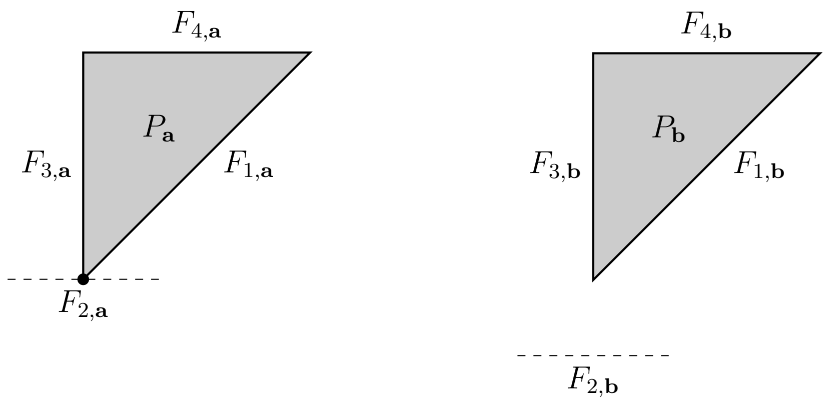
$$P_\chi=\{(\langle m,\nu_1\rangle+a_1,\dots,\langle m,\nu_n\rangle+a_n):m\in P_{\mathbf{a}}\}.$$

The facets of a polyhedron are important for understanding its geometry and combinatorics. For a polyhedron P_χ we want to consider not only its facets but also what are called virtual facets. First consider $\chi=\chi^{\mathbf{a}}$ with polyhedron $P_{\mathbf{a}}$. For $1\leq i\leq n$ the set

$$F_{i,\mathbf{a}}=\{m\in P_{\mathbf{a}}:\langle m,\nu_i\rangle=-a_i\}$$

is called a virtual facet of $P_{\mathbf{a}}$. All of the facets of $P_{\mathbf{a}}$ are among the $F_{i,\mathbf{a}}$ but some of the $F_{i,\mathbf{a}}$ might be faces of smaller dimension or empty. From the relationship between P_χ and $P_{\mathbf{a}}$ we see that a virtual facet $F_{i,\mathbf{a}}\subset P_{\mathbf{a}}$ corresponds to the subset of P_χ where the ith coordinate is 0. Denote this set by $F_{i,\chi}=P_\chi\bigcap\{x_i=0\}$.

Example 2.5. Consider the group $G=\{(t_1,t_1^{-1}t_2,t_1,t_2):t_1,t_2\in\mathbb{C}^*\}\simeq(\mathbb{C}^*)^2\subset(\mathbb{C}^*)^4$. We have $M\simeq\mathbb{Z}^2$ and $\nu_1=-e_1+e_2,\nu_2=e_2,\nu_3=e_1,\nu_4=-e_2$. For $\mathbf{a}=(0,0,0,1)$ we get a polytope with three true facets and one virtual facet that is a vertex. For $\mathbf{b}=(0,1,0,1)$ we get a polytope with three true facets and one empty virtual facet. These can be seen in Figure 2.2.



$P_{\mathbf{a}} \subset \mathbb{R}^2$ for $\mathbf{a} = (0,0,0,1)$ $\qquad$ $P_{\mathbf{b}} \subset \mathbb{R}^2$ for $\mathbf{b} = (0,1,0,1)$

Figure 2.2: Example of polyhedra associated to a character

The virtual facets can be used to determine the set of semistable points.

Proposition 2.2. [CLS11, Proposition 14.2.21] Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{A}^n$ and define $J(\mathbf{x}) = \{j : x_j = 0\}$. Then $\mathbf{x} \in (\mathbb{A}^n)^{ss}_{\chi}$ if and only if $\bigcap_{j \in J(\mathbf{x})} F_{j,\chi} \neq \emptyset$.

Consider the ideal

$$B(\chi) = \left\langle \prod_{j \notin J} x_j : J \subset \{1, \ldots, n\}, \bigcap_{j \in J} F_{j,\chi} \neq \emptyset \right\rangle \subset k[x_1, \ldots, x_n] \tag{2.2}$$

which we call the irrelevant ideal for the character χ by Proposition 2.2 . Its vanishing locus is the unstable locus $(\mathbb{A}^n)^{us}_{\chi}$. We can also define the irrelevant ideal in terms of the vertices of the polyhedron P_{χ},

$$B(\chi) = \left\langle \prod_{\mathbf{b} \notin F_{i,\chi}} x_i : \mathbf{b} \text{ is a vertex of } P_{\chi} \right\rangle \tag{2.3}$$

Example 2.6. Consider Example 2.5 from above. We compute the irrelevant ideals using the description in terms of the vertices of the polyhedron, as in eq. (2.3). The irrelevant ideal for $\chi^{\mathbf{a}}$ is

$$B(\chi^{\mathbf{a}}) = \langle x_4, x_2x_3, x_1x_2 \rangle$$

and the irrelevant ideal for $\chi^{\mathbf{b}}$ is

$$B(\chi^{\mathbf{b}}) = \langle x_2x_4, x_2x_3, x_2x_1\rangle.$$

Now we want to consider certain cones that are useful in understanding the different GIT quotients. Consider $\chi_i = \gamma(e_i) \in \widehat{G}$ which generate $\widehat{G}$, then $\widehat{G}_{\mathbb{R}}$ is spanned by the vectors $\beta_i = \chi_i \otimes 1 = \gamma_{\mathbb{R}}(e_i)$. We also have the vectors $\nu_i = \delta^{\vee}(e_i) \in N \subset N_{\mathbb{R}}$. Then we define the cones

$$C_\beta = \mathrm{Cone}(\beta_1, \ldots, \beta_n) \subset \widehat{G}_{\mathbb{R}}$$

$$C_\nu = \mathrm{Cone}(\nu_1, \ldots, \nu_n) \subset N_{\mathbb{R}}$$

with lattices $\widehat{G} \otimes 1$ and N respectively. The cone C_β will allow us to determine when semistable points exist and thus when the GIT quotient is non-empty. This is the content of the following proposition.

Proposition 2.3. [CLS11, Proposition 14.3.5] If $\chi \in \widehat{G}$ is a character, then the following are equivalent:

(1) $\mathbb{A}^n /\!\!/_\chi G \neq \emptyset$

(2) $(\mathbb{A}^n)^{ss}_\chi \neq \emptyset$

(3) $\chi \otimes 1 \in C_\beta$

If we just consider the interior of the cone C_β then we can determine whether there are any stable points.

Proposition 2.4. [CLS11, Proposition 14.3.6] If $\chi \otimes 1 \in C_\beta$, then the following are equivalent:

(1) $\mathbb{G}_m^n \subset (\mathbb{A}^n)^s_\chi$

(2) $(\mathbb{A}^n)^s_\chi \neq \emptyset$

(3) $\chi \otimes 1$ is in the interior of C_β

(4) $F_{i,\chi}$ is a proper face of P_χ for all i

Moreover, the above imply that

(5) $\dim \mathbb{A}^n /\!\!/_\chi G = n - \dim G$,

when the ν_i are nonzero vectors (5) is equivalent to (1) − (4).

From this proposition we see that when the ν_i are nonzero vectors then stable points exist if and only if the GIT quotient has the expected dimension. Also the GIT quotient is nonempty and of smaller than expected dimension if and only if the character is on the boundary of C_β.

2.1 The Secondary Fan

We will now aim to define the secondary fan for G acting on $\mathbb{A}^n$, which under certain conditions will coincide with the fan $\mathscr{F}_A$ mentioned in Chapter 1. In some cases the secondary fan will be the same as the GIT fan. First notice that there are only finitely many distinct GIT quotients up to isomorphism, since $\mathbb{A}^n /\!\!/_\chi G \simeq (\mathbb{A}^n \backslash Z(\chi)) /\!\!/ G$ were $Z(\chi)$ is the vanishing locus of the irrelevant ideal

$$B(\chi) = \left\langle \prod_{j \notin J} x_j : J \subset \{1, \dots, n\}, \bigcap_{j \in J} F_{j,\chi} \neq \emptyset \right\rangle \subset k[x_1, \dots, x_n].$$

There are only finitely many such ideals and thus only finitely many GIT quotients. The lattice points of C_β can be partitioned into finitely many subsets where the GIT quotient is constant on each subset. Each subset then lies in the relative interior of a cone of the secondary fan, whose support is C_β. We will give the combinatorial description of this fan as seen in [CLS11].

For $\chi \in \widehat{G}$ let Σ_χ be the normal fan of P_χ in $N_\mathbb{R}$ and $I_\emptyset = \{i : F_{i,\chi} = \emptyset\}$ the set of empty virtual facets of P_χ. A subset $J \subset \{1, \dots, n\}$ is called a β-basis if $|J| = \dim G$ and $\{\beta_i : i \in J\}$ is a basis of $\widehat{G}_\mathbb{R}$. Then the cone $\mathrm{Cone}(\beta_J) = \mathrm{Cone}(\beta_i : i \in J) \subset \widehat{G}_\mathbb{R}$ is a full dimensional simplicial cone. For σ a maximal cone in Σ_χ define the set $J_\sigma = \{i : \nu_i \notin \sigma(1) \text{ or } i \in I_\emptyset\}$.

Then J_σ is a β-basis and we define a chamber in the secondary fan by

$$C(\chi) := \bigcap_{\substack{\sigma \in \Sigma_\chi \\ \text{maximal}}} \operatorname{Cone}(\beta_{J_\sigma}) \tag{2.4}$$

The secondary fan is the collection of all of the chambers $C(\chi)$ for $\chi \in \widehat{G}$. We will denote the secondary fan for quotients of $\mathbb{A}^n$ by G by $\mathscr{F}_G$.

Suppose $\nu_1, \ldots, \nu_n \in N$ are distinct and lie in an integral affine hyperplane $H \subset N_\mathbb{R}$ not containing 0. Then $Q_\nu = \operatorname{Conv}(\nu_1, \ldots, \nu_n)$ is a lattice polytope in H. Assume Q_ν is of full dimension in H, so that it has codimension 1 in $N_\mathbb{R}$, and C_ν has full dimension in $N_\mathbb{R}$. A triangulation T of $\{\nu_1, \ldots, \nu_n\}$ is a collection of simplices such that each simplex has codimension 1 in $N_\mathbb{R}$ with vertices in $\{\nu_1, \ldots, \nu_n\}$, the intersection of any two simplices is a face of each, and the union of all the simplices is Q_ν.

There is a bijective correspondence between triangulations of $\{\nu_1, \ldots, \nu_n\}$ and simplicial fans Σ such that $|\Sigma| = C_\nu$ with $\Sigma(1) \subset \{\operatorname{Cone}(\nu_i) : 1 \leq i \leq n\}$. Given a triangulation T, the cones of a fan Σ are the cones over the simplices of T and their faces. Conversely a fan Σ, defines simplices of a triangulation T as $\sigma \cap Q_\nu$ where $\sigma \in \Sigma$ is a maximal cone.

Proposition 2.5. [CLS11, Proposition 15.2.9] For $\nu_1, \ldots, \nu_n \in N$ distinct and in an integral affine hyperplane $H \subset N_\mathbb{R}$ not containing 0 there is a bijection between chambers of the secondary fan and coherent triangulations of $\{\nu_1, \ldots, \nu_n\}$.

This result allows us to use what we know about the secondary fan of a marked polytope from Chapter 1. We saw that two maximal cones in the secondary fan are adjacent if the corresponding triangulations can be obtained from each other by a modification along a circuit. By using this description we want to develop a way to determine how the unstable locus changes as we move between adjacent chambers in the secondary fan. A first step is to describe the unstable locus in terms of triangulations of a marked polytope.

For a fixed $\chi \in \widehat{G}$ we have a polyhedron P_χ and its normal fan Σ_χ. Each maximal cone σ in Σ_χ corresponds to a vertex $\mathbf{b}$ of P_χ thus ($\nu_i \notin \sigma(1)$ or $i \in I_\emptyset$) $\iff$ $\mathbf{b} \notin F_{i,\chi}$. This allows us to write the irrelevant ideal as

$$B(\chi) = \left\langle \prod_{i \in J_\sigma} x_i : \sigma \text{ maximal cone in } \Sigma_\chi \right\rangle \tag{2.5}$$

recall that $J_\sigma = \{i : \nu_i \notin \sigma(1) \text{ or } i \in I_\emptyset\}$. Also, each maximal cone σ in Σ_χ corresponds to a simplex in a triangulation of Q_ν where the vertices of the simplex $\sigma \cap Q_\nu$ are the complement of J_σ. This allows us to describe the irrelevant ideal in terms of a triangulation of a marked polytope. For a triangulation $T = \{(Q_i, A_i)\}_{i \in I}$ of the marked polytope (Q, A), the irrelevant ideal is

$$B(T) = \left\langle \prod_{j \notin A_i} x_j : i \in I \right\rangle \tag{2.6}$$

Definition 2.4. Let $T = \{(Q_i, A_i)\}_{i \in I}$ be a triangulation of (Q, A). We say $J \subset A$ is a primitive collection if $J \not\subset A_i$ for any $i \in I$ and for every $J' \subsetneq J$ there exists $i \in I$ such that $J' \subset A_i$.

Proposition 2.6. For $T = \{(Q_i, A_i)\}_{i \in I}$ a triangulation of (Q, A) we have

$$B(T) = \bigcap_J \langle x_j : j \in J \rangle$$

where the intersection is over all primitive collections $J \subset A$.

Example 2.7. Consider $G = \{(t_1, t_2, t_1^{-1}t_2^{-2}, t_1 t_2, t_1^{-1}) : t_1, t_2 \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^5$ acting on $\mathbb{C}^5$. We have $M \simeq \mathbb{Z}^3$, $\widehat{G} \simeq \mathbb{Z}^2$, and the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^5 \xrightarrow{B} \mathbb{Z}^2 \longrightarrow 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

This gives

$$
\begin{aligned}
\nu_1 &= e_1 + e_3 & \qquad \beta_1 &= e_1 \\
\nu_2 &= e_1 + e_2 + e_3 & \qquad \beta_2 &= e_2 \\
\nu_3 &= e_2 + e_3 & \qquad \beta_3 &= -e_1 - 2e_2 \\
\nu_4 &= -e_1 + e_2 + e_3 & \qquad \beta_4 &= e_1 + e_2 \\
\nu_5 &= e_3 & \qquad \beta_5 &= -e_1
\end{aligned}
$$

We see that the ν_i are all distinct and lie in the hyperplane $e_3^\vee = 1$. The convex hull of the ν_i in the hyperplane gives the polytope Q_ν which has five triangulations with vertices among the ν_i.

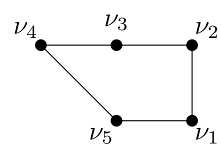


Figure 2.3: The polytope Q_ν in the hyperplane $e_3^\vee = 1$

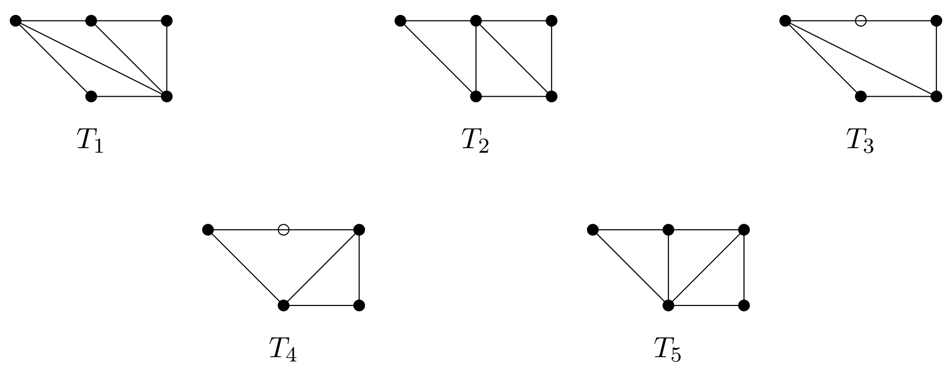


Figure 2.4: The five triangulations of Q_ν

The secondary fan has support $\mathrm{Cone}(\beta_1, \ldots, \beta_5) = \mathbb{R}^2$ and each chamber corresponds to one of the triangulations of Q_ν.

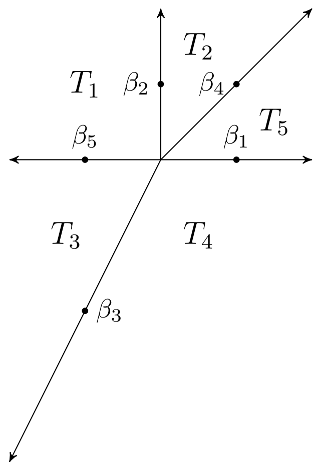


Figure 2.5: The secondary fan with chambers labeled by the corresponding triangulation

Using eq. (2.6), the irrelevant ideals for each chamber can be found by looking at the corresponding triangulation.

$$B(T_1) = \langle x_2x_3, x_2x_5, x_4x_5\rangle$$

$$B(T_2) = \langle x_1x_2, x_2x_4, x_4x_5\rangle$$

$$B(T_3) = \langle x_2x_3, x_3x_5\rangle$$

$$B(T_4) = \langle x_1x_3, x_3x_4\rangle$$

$$B(T_5) = \langle x_1x_2, x_1x_4, x_3x_4\rangle$$

2.2 Changes in the Unstable Locus

Using what we have done thus far we can describe how the unstable locus changes when we cross a wall in the secondary fan in terms of the triangulations that correspond to each chamber. We first fix some notation. For $J \subset \{1, \dots, n\}$ define

$$X_J := \{(x_1, \dots, x_n) : x_j = 0 \iff j \in J\} \tag{2.7}$$

Let

$$Y_{I_1,I_2} := \coprod_{J\in[I_1,I_2)} X_J$$

where the union is taken over all J that contain I_1 and are properly contained in I_2. Let

$$Y_{I_1,I_2,I_3} := \coprod_{J\in[I_1,I_3)\setminus[I_2,I_3)} X_J$$

where the union is taken over all J that contain I_1, are properly contained in I_3, and do not contain all of I_2.

For a triangulation $T = \{(Q_i, A_i)\}_{i\in I}$ the unstable locus is given by the vanishing of the irrelevant ideal $B = \bigcap_J \langle x_j : j \in J\rangle$ which is $\bigcup_J \overline{X_J}$, where J are primitive collections. A stratification of the unstable locus is

$$(\mathbb{A}^n)^{us}_T = \coprod_{\substack{J\not\subseteq A_i\\ \text{any } i\in I}} X_J.$$

Let T and T' be two coherent triangulations of (Q, A) on either side of a wall in the secondary fan and C the circuit such that T and T' are both supported on C and obtained from each other by a modification along C. The triangulation T induces one of the two possible triangulations on $\mathrm{Conv}(C)$, say T_+. The changes in the unstable locus are described in the following proposition.

Proposition 2.7. Let T, T' and C be as above. Then the unstable locus for T' is obtained from the unstable locus of T by replacing $Y_{C_+,C}$ with $Y_{C_-,C}$ if C is a non-degenerate circuit or by replacing $\bigcup_{k\in K} Y_{C_+,C,C^e_k}$ with $\bigcup_{k\in K} Y_{C_-,C,C^e_k}$ if C is a degenerate circuit with $\overline{C} = \bigcup_{k\in K} C^e_k$ from eq. (1.3).

Proof. First suppose C is a non-degenerate circuit. Since T' is the modification of T along C then any simplex of T with a vertex not in C is still a simplex of T'. Thus the X_J where J contains an element corresponding to a vertex not in C will be in the unstable locus for both T and T'. Thus the only changes in the unstable locus will occur for $J \subsetneq C$.

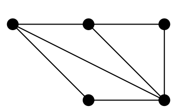
We have $C = C_+ \coprod C_-$ and T_+ has simplices $\mathrm{Conv}(C - \{c\})$ for $c \in C_+$ and T_- has simplices $\mathrm{Conv}(C - \{c\})$ for $c \in C_-$. Any J not containing all of C_+ will be contained in the set of vertices of some simplex of T_+. Thus we need only consider those J that contain all of C_+. Now every X_J for $J \in [C_+, C)$ is part of the unstable locus for T since those J are not contained in the set of vertices of any simplex of T_+. Also every $J \in [C_+, C)$ is contained in some $C - \{c\}$ for $c \in C_-$ and so those X_J are not part of the unstable locus for T'. Similarly X_J for $J \in [C_-, C)$ is part of the unstable locus for T' but not for T. Thus to get the unstable locus for T' from T we are replacing $Y_{C_+,C}$ with $Y_{C_-,C}$.

Now consider the case when C is a degenerate circuit, with $\overline{C} = \bigcup_{k \in K} C^e_k$, as discussed in Section 1.3. Any simplex of T with a vertex not in $\overline{C}$ is also a simplex of T', so we need only consider the case when we are completely in $\overline{C}$. Any set containing vertices from $C^e_k \setminus C$ and $C^e_{k'} \setminus C$ for $k \neq k'$ will either be contained in a simplex that is in both T and T' or not contained in any simplex. Thus we will only see changes in the unstable locus for $J \subsetneq C^e_k$.

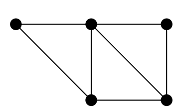
For each k, T_+ has simplices $\mathrm{Conv}(C^e_k - \{c\})$ for $c \in C_+$ and T_- has simplices $\mathrm{Conv}(C^e_k - \{c\})$ for $c \in C_-$. Then for each $k \in K$ and $J \in [C_+, C^e_k)$, we have X_J is in the unstable locus for T. Similarly, we have for $J \in [C_-, C^e_k)$ that X_J is in the unstable locus for T'. But, there is overlap when $J \in [C, C^e_k)$. Thus the X_J that are in the unstable locus for T but not for T' are when $J \in [C_+, C^e_k) \setminus [C, C^e_k)$. Similarly, X_J for $J \in [C_-, C^e_k) \setminus [C, C^e_k)$ is in the unstable locus for T' but not for T. Thus we are replacing $\bigcup_{k \in K} Y_{C_+,C,C^e_k}$ in the unstable locus for T with $\bigcup_{k \in K} Y_{C_-,C,C^e_k}$ in the unstable locus for T'.

$\square$

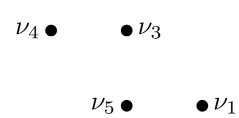
Example 2.8. Consider Example 2.7 and let us determine the changes in the unstable locus. As a first case we consider going between the chambers corresponding to T_1 and T_2. These two triangulations are related by a modification along the non-degenerate circuit $\{\nu_1, \nu_3, \nu_4, \nu_5\}$.



T_1



T_2



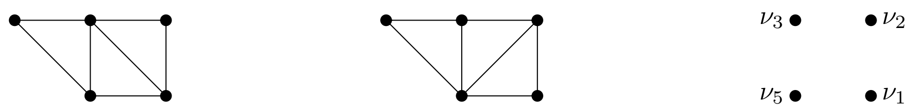
circuit

Going from the triangulation T_1 to the triangulation T_2 we are replacing

$$X_{\{3,5\}} \coprod X_{\{3,5,1\}} \coprod X_{\{3,5,4\}} \quad \text{with} \quad X_{\{1,4\}} \coprod X_{\{1,4,3\}} \coprod X_{\{1,4,5\}}$$

in the unstable locus.

Next we consider the change when going from T_2 to T_5. These triangulations are related by a modification along the circuit $\{\nu_1, \nu_2, \nu_3, \nu_5\}$.



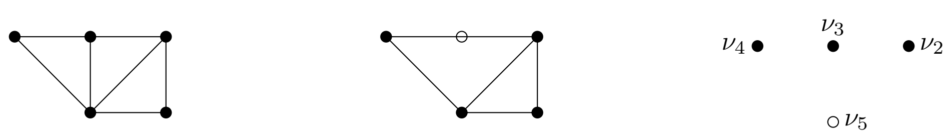
T_2 T_5 circuit

The change when going from T_2 to T_5 is we are replacing

$$X_{\{2,5\}} \coprod X_{\{2,5,1\}} \coprod X_{\{2,5,3\}} \quad \text{with} \quad X_{\{1,3\}} \coprod X_{\{1,3,2\}} \coprod X_{\{1,3,5\}}$$

in the unstable locus.

Now we consider what happens between the triangulations T_5 and T_4. These two triangulations are related by a modification along the degenerate circuit $\{\nu_2, \nu_3, \nu_4\}$ which we need to extend by adding ν_5.



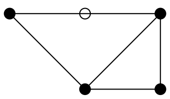
T_5 T_4 circuit

When going from the triangulation T_5 to the triangulation T_4 we are replacing

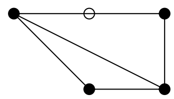
$$X_{\{2,4\}} \coprod X_{\{2,4,5\}} \;\; \text{with} \;\; X_{\{3\}} \coprod X_{\{3,2\}} \coprod X_{\{3,4\}} \coprod X_{\{3,5\}} \coprod X_{\{3,2,5\}} \coprod X_{\{3,4,5\}}$$

in the unstable locus.

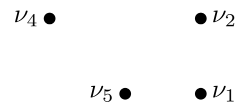
Now for the change when going from the chamber corresponding to T_4 to that for T_3 we are modifying along the circuit $\{\nu_1, \nu_2, \nu_4, \nu_5\}$.



T_4



T_3

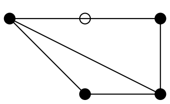


circuit

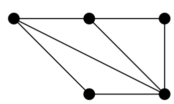
In the unstable locus we are replacing

$$X_{\{1,4\}} \coprod X_{\{1,4,2\}} \coprod X_{\{1,4,5\}} \text{ with } X_{\{2,5\}} \coprod X_{\{2,5,1\}} \coprod X_{\{2,5,4\}}.$$

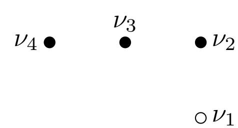
Lastly we have the change when moving from the chamber for T_3 to the chamber for T_1. In this case we are modifying along the degenerate circuit $\{\nu_2, \nu_3, \nu_4\}$ that we extend by adding ν_1.



T_3



T_1



circuit

When moving from T_3 to T_1 in the unstable locus we are replacing

$$X_{\{3\}} \coprod X_{\{3,2\}} \coprod X_{\{3,4\}} \coprod X_{\{3,1\}} \coprod X_{\{3,1,2\}} \coprod X_{\{3,1,4\}} \text{ with } X_{\{2,4\}} \coprod X_{\{2,4,1\}}.$$

The following table summarizes the changes in the unstable locus as we move between chambers labeled by triangulations.

$T_1 \rightleftarrows T_2$	$X_{\{3,5\}} \coprod X_{\{3,5,1\}} \coprod X_{\{3,5,4\}} \rightleftarrows X_{\{1,4\}} \coprod X_{\{1,4,3\}} \coprod X_{\{1,4,5\}}$
$T_2 \rightleftarrows T_5$	$X_{\{2,5\}} \coprod X_{\{2,5,1\}} \coprod X_{\{2,5,3\}} \rightleftarrows X_{\{1,3\}} \coprod X_{\{1,3,2\}} \coprod X_{\{1,3,5\}}$
$T_5 \rightleftarrows T_4$	$X_{\{2,4\}} \coprod X_{\{2,4,5\}} \rightleftarrows X_{\{3\}} \coprod X_{\{3,2\}} \coprod X_{\{3,4\}} \coprod X_{\{3,5\}} \coprod X_{\{3,2,5\}} \coprod X_{\{3,4,5\}}$
$T_4 \rightleftarrows T_3$	$X_{\{1,4\}} \coprod X_{\{1,4,2\}} \coprod X_{\{1,4,5\}} \rightleftarrows X_{\{2,5\}} \coprod X_{\{2,5,1\}} \coprod X_{\{2,5,4\}}$
$T_3 \rightleftarrows T_1$	$X_{\{3\}} \coprod X_{\{3,2\}} \coprod X_{\{3,4\}} \coprod X_{\{3,1\}} \coprod X_{\{3,1,2\}} \coprod X_{\{3,1,4\}} \rightleftarrows X_{\{2,4\}} \coprod X_{\{2,4,1\}}$

Consider a subvariety $X \subset \mathbb{A}^n$ that is stable under the action by G. We are interested in determining the GIT fan for quotients of X by G. We can use what we know about quotients of $\mathbb{A}^n$ by G by the relationship between the semistable and unstable points of $\mathbb{A}^n$ and X. Observe for $\chi \in \widehat{G}$ we have $X^{ss}_\chi = X \cap (\mathbb{A}^n)^{ss}_\chi$ and $X^{us}_\chi = X \cap (\mathbb{A}^n)^{us}_\chi$. We have constructed the secondary fan for the action of G on $\mathbb{A}^n$ and we know how the unstable locus changes as we cross walls in this fan. Consider a wall in the secondary fan, we want to check if X intersects any of the strata X_J in the part of the unstable locus that is changing as we cross the wall. If X does not intersect any of the X_J then the GIT quotient of X will not change when crossing that wall and so it will not be a part of the GIT fan for X. To determine whether or not X intersects any of these strata we will use tropical geometry.

Chapter 3

Tropical Geometry and the GIT Fan

3.1 Tropical Geometry

First we will introduce some basic tropical geometry following conventions in [MS15]. Let k be a field with a valuation. We consider the ring $S = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials over k in n variables. For $f = \sum_{\mathbf{a} \in \mathbb{Z}^n} c_{\mathbf{a}} \, \mathbf{x}^{\mathbf{a}} \in S$ the *tropicalization of f* is a piecewise linear function defined by

$$\operatorname{trop}(f)(\mathbf{w}) := \min_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ c_{\mathbf{a}} \neq 0}} (\operatorname{val}(c_{\mathbf{a}}) + \mathbf{a} \cdot \mathbf{w}) \tag{3.1}$$

The *classical variety* of f is a hypersurface in the algebraic torus T^n given by the zero locus of f

$$V(f) = \{\mathbf{x} \in T^n : f(\mathbf{x}) = 0\}.$$

Definition 3.1. The *tropical hypersurface* $\operatorname{trop}(V(f))$ is

$$\{\mathbf{w} \in \mathbb{R}^n : \exists\ \mathbf{a}_1 \neq \mathbf{a}_2 \text{ with } \operatorname{val}(c_{\mathbf{a}_1}) + \mathbf{a}_1 \cdot \mathbf{w} = \operatorname{trop}(f)(\mathbf{w}) = \operatorname{val}(c_{\mathbf{a}_2}) + \mathbf{a}_2 \cdot \mathbf{w}\} \tag{3.2}$$

Example 3.1. Let $k = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series.

Let $f(x, y) = x + y + 1$, using eq. (3.1) we have

$$\operatorname{trop}(f)(\mathbf{w}) = \min(w_1, w_2, 0).$$

For $g(x, y) = tx^2 + xy + ty^2 + x + y + t^2$ we have

$$\operatorname{trop}(g)(\mathbf{w}) = \min(1 + 2w_1, w_1 + w_2, 1 + 2w_2, w_1, w_2, 2).$$

These tropical hypersurfaces can be seen in Figure 3.1.

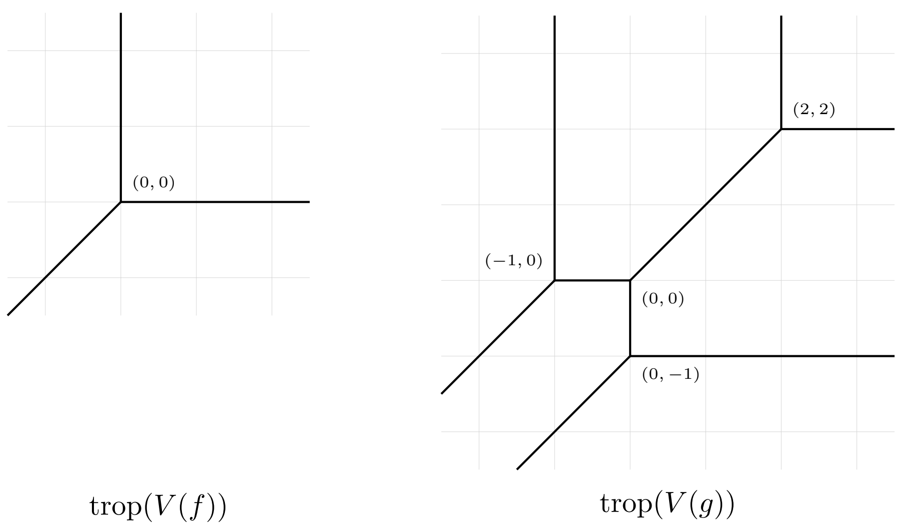


Figure 3.1: Examples of tropical hypersurfaces

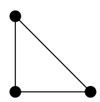
For $f = \sum_{\mathbf{a}\in\mathbb{Z}^n} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in S$ the *Newton polytope* of f is the polytope

$$\operatorname{Newt}(f) = \operatorname{Conv}(\mathbf{a} : c_{\mathbf{a}} \neq 0) \subset \mathbb{R}^n \tag{3.3}$$

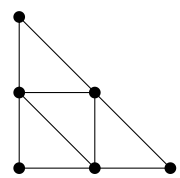
Proposition 3.1. [MS15, Proposition 3.1.6] Let $f \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface $\operatorname{trop}(V(f))$ is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of $f = \sum_{\mathbf{a}\in\mathbb{Z}^n} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ given by the weights $\operatorname{val}(c_{\mathbf{a}})$ on the lattice points in $\operatorname{Newt}(f)$.

In the case when the valuations of the coefficients of f are all zero the tropical hypersurface is a fan in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the normal fan to the Newton polytope of f. This can be seen in Example 3.1. For the polynomial f the valuations of the coefficients were all 0 and the tropical hypersurface is a fan in $\mathbb{R}^2$. Whereas the polynomial g has some coefficients with non-zero valuation and the tropical hypersurface is not a fan in $\mathbb{R}^2$.

Example 3.2. For the polynomials in Example 3.1 we have the following Newton polytopes and regular subdivisions with weights coming from the valuation on the coefficients.



Newt(f)



Newt(g)

Figure 3.2: Examples of Newton polytopes with subdivision

Definition 3.2. Let I be an ideal in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $X = V(I)$ its variety in the torus T^n. The *tropicalization of* X is

$$\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}(V(f)) \subset \mathbb{R}^n \tag{3.4}$$

Let $\operatorname{trop}(X_{\text{triv}})$ denote the tropicalization of X with respect to the trivial valuation.

Theorem 3.1. [MS15, Theorem 6.3.4] Consider a toric variety X_Σ with torus T^n. Consider $X \subset T^n$ and $\overline{X}$ its closure in X_Σ. For $\sigma \in \Sigma$ we have $\overline{X} \cap \mathcal{O}_\sigma \neq \emptyset$ if and only if $\operatorname{trop}(X_{\text{triv}})$ intersects the relative interior of the cone σ.

We are specifically interested in the case when the toric variety is affine space. Consider $\mathbb{A}^n$ with torus T^n. Let $e_1, \ldots, e_n$ be a basis of $\widehat{T^n} \simeq \mathbb{Z}^n$, then the fan for $\mathbb{A}^n$ consists of the cones:

$$\operatorname{cone}(J) = \operatorname{cone}(e_i : i \in J) \text{ for all } J \subset \{1, \ldots, n\} \tag{3.5}$$

Recall from eq. (2.7), $X_J = \{(x_1, \ldots, x_n) : x_j = 0 \iff j \in J\}$.

Corollary 3.1. Let $X \subset T^n$ and $\overline{X} \subset \mathbb{A}^n$. For $J \subset \{1, \ldots, n\}$ we have $\overline{X} \cap X_J \neq \emptyset$ if and only if $\operatorname{trop}(X_{\text{triv}})$ intersects the relative interior of $\operatorname{cone}(J)$.

3.2 The GIT Fan

Recall that we are interested in describing the GIT fan for quotients of a subvariety $X \subset \mathbb{A}^n$ that is stable under the action by a group $G \subset G_{\text{big}} \simeq \mathbb{G}_m^n$. We have an understanding of the secondary fan $\mathscr{F}_G$ from Section 2.1 which describes the GIT quotients of $\mathbb{A}^n$ by G. We have also seen how, under certain conditions, the secondary fan $\mathscr{F}_G$ is equivalent to the secondary fan $\mathscr{F}_A$ as discussed in Chapter 1 (where A comes from the action of G on $\mathbb{A}^n$ and the chambers correspond to coherent triangulations). We also know, by Proposition 2.7, how the unstable locus changes as we cross walls in the secondary fan, which is described by circuit modifications of triangulations (Section 1.3). The GIT fan for X will be a coarsening of the fan $\mathscr{F}_G$, where we lose a wall if X does not intersect any of the strata that are changing in the unstable locus. Now we want to connect this with the tropical geometry we discussed in the previous section. Specifically, we want to use Corollary 3.1 to determine how X intersects what is changing in the unstable locus as we cross a wall. Understanding this will allow us to say when a given wall of the fan $\mathscr{F}_G$ is part of the GIT fan for X.

From the inclusion $G \subset G_{\text{big}}$ we obtain the following short exact sequences:

$$0 \longrightarrow M \xrightarrow{\ \delta\ } \widehat{G}_{\text{big}} \xrightarrow{\ \gamma\ } \widehat{G} \longrightarrow 0 \tag{3.6}$$

$$0 \longleftarrow N \xleftarrow{\ \delta^\vee\ } (\widehat{G}_{\text{big}})^\vee \xleftarrow{\ \gamma^\vee\ } \widehat{G}^\vee \longleftarrow 0 \tag{3.7}$$

In order to apply Corollary 3.1, we need to understand the tropicalization of X with respect to the trivial valuation. In what follows, for ease of notation, all tropicalizations will be with respect to the trivial valuation. For $f = \sum\limits_{\mathbf{a} \in \widehat{G}_{\text{big}}} c_{\mathbf{a}}\, \mathbf{x}^{\mathbf{a}} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ its tropicalization is the function

$$\mathrm{trop}(f)(\mathbf{w}) = \min_{\substack{\mathbf{a}\in\widehat{G}_{\mathrm{big}} \\ c_{\mathbf{a}}\neq 0}} (\langle \mathbf{a}, \mathbf{w}\rangle) \quad \mathbf{w} \in (\widehat{G}_{\mathrm{big}})^{\vee}_{\mathbb{R}} \simeq \mathbb{R}^n \tag{3.8}$$

and the tropical hypersurface is

$$\mathrm{trop}(V(f)) = \{\mathbf{w} \in (\widehat{G}_{\mathrm{big}})^{\vee}_{\mathbb{R}} : \exists\, \mathbf{a}_1 \neq \mathbf{a}_2 \text{ with } \langle \mathbf{a}_1, \mathbf{w}\rangle = \mathrm{trop}(f)(\mathbf{w}) = \langle \mathbf{a}_2, \mathbf{w}\rangle\}.$$

If $X = V(I)$, then the tropicalization of X is

$$\mathrm{trop}(X) = \bigcap_{f\in I} \mathrm{trop}(V(f)) \subset (\widehat{G}_{\mathrm{big}})^{\vee}_{\mathbb{R}} \simeq \mathbb{R}^n.$$

Recall that $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is $\widehat{G}$ graded by $\deg(\mathbf{x}^{\mathbf{a}}) = \gamma(\mathbf{a})$, where γ is the map in eq. (3.6). When we have $X = V(I)$ we want to consider the ideal I^h generated by homogeneous elements of I with respect to this grading. Then we will be able to determine $\mathrm{trop}(X)$ by using the ideal I^h. First we need the following lemma, which will be useful for relating the tropical hypersurface defined by a polynomial to the tropical hypersurfaces defined by its homogeneous pieces.

Lemma 3.1. *Suppose f and g are Laurent polynomials such that if $\mathbf{x}^{\mathbf{a}}$ appears in either f or g then $\mathbf{x}^{\mathbf{a}}$ appears in $f+g$. Then*

$$\mathrm{trop}(V(f)) \cap \mathrm{trop}(V(g)) \subset \mathrm{trop}(V(f+g)).$$

Proof. Suppose $\mathbf{w} \in \mathrm{trop}(V(f)) \cap \mathrm{trop}(V(g))$. Since $\mathbf{w} \in \mathrm{trop}(V(f))$ the minimum in $\mathrm{trop}(f)(\mathbf{w})$ is achieved at least twice, thus there exists $\mathbf{a}_1 \neq \mathbf{a}_2$ such that $\langle \mathbf{a}_1, \mathbf{w}\rangle = \langle \mathbf{a}_2, \mathbf{w}\rangle$ and is minimal among all other exponents of f. Similarly, since $\mathbf{w} \in \mathrm{trop}(V(g))$ we have $\mathbf{b}_1 \neq \mathbf{b}_2$ such that $\langle \mathbf{b}_1, \mathbf{w}\rangle = \langle \mathbf{b}_2, \mathbf{w}\rangle$ and is minimal among all other exponents of g. Without loss of generality, suppose $\langle \mathbf{a}_1, \mathbf{w}\rangle \leq \langle \mathbf{b}_1, \mathbf{w}\rangle$. Since, by assumption, any exponent appearing in f and g also appears as an exponent in $f+g$ we have that the minimum in $\mathrm{trop}(f+g)(\mathbf{w})$

is achieved at least twice, thus $\mathbf{w} \in \operatorname{trop}(V(f+g))$. $\square$

Proposition 3.2. Let $X = V(I) \subset T^n$. Let I^h be the ideal generated by homogeneous elements of I. Then

$$\operatorname{trop}(X) = \bigcap_{f \in I^h} \operatorname{trop}(V(f)).$$

Proof. Since $I^h \subset I$ we have

$$\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}(V(f)) \subset \bigcap_{g \in I^h} \operatorname{trop}(V(g)).$$

Let $f \in I$ then $f = \sum_{i=1}^m f_i$ where f_i are homogeneous. By Lemma 3.1 we have

$$\bigcap_{i=1}^m \operatorname{trop}(V(f_i)) \subset \operatorname{trop}(V(f)).$$

Thus

$$\bigcap_{g \in I^h} \operatorname{trop}(V(g)) \subset \bigcap_{f \in I} \operatorname{trop}(V(f)).$$

$\square$

The tropical variety is contained in $(\widehat{G}_{\text{big}})^{\vee}_{\mathbb{R}} \simeq \mathbb{R}^n$, but we want to consider its image in $N_{\mathbb{R}}$ under the map $\delta^{\vee}_{\mathbb{R}}$ as in eq. (3.7). The following proposition is giving that this is a projection, since in $(\widehat{G}_{\text{big}})^{\vee}_{\mathbb{R}}$ the tropical variety is invariant under translations by the image of $\gamma^{\vee}$.

Proposition 3.3. If f is homogeneous then $\operatorname{trop}(V(f))$ is parallel to the image of $\gamma^{\vee}$.

Proof. Suppose $f = \sum\limits_{\mathbf{a} \in \widehat{G}_{\text{big}}} c_{\mathbf{a}}\, x^{\mathbf{a}}$ is homogeneous then $\gamma(\mathbf{a}) = \chi \in \widehat{G}$ for all $\mathbf{a}$ such that $c_{\mathbf{a}} \neq 0$. Let $\varphi \in (\widehat{G})^{\vee}$ then

$$\begin{aligned}
\operatorname{trop}(f)(\mathbf{w}+\gamma^{\vee}(\varphi)) &= \min_{c_{\mathbf{a}}\neq 0}(\langle \mathbf{a}, \mathbf{w}+\gamma^{\vee}(\varphi)\rangle) \\
&= \min_{c_{\mathbf{a}}\neq 0}(\langle \mathbf{a}, \mathbf{w}\rangle + \langle \mathbf{a}, \gamma^{\vee}(\varphi)\rangle) \\
&= \min_{c_{\mathbf{a}}\neq 0}(\langle \mathbf{a}, \mathbf{w}\rangle + \langle \gamma(\mathbf{a}), \varphi\rangle) \\
&= \min_{c_{\mathbf{a}}\neq 0}(\langle \mathbf{a}, \mathbf{w}\rangle + \langle \chi, \varphi\rangle) \\
&= \min_{c_{\mathbf{a}}\neq 0}(\langle \mathbf{a}, \mathbf{w}\rangle) + \langle \chi, \varphi\rangle \\
&= \operatorname{trop}(f)(\mathbf{w}) + \gamma^{\vee}(\varphi)
\end{aligned}$$

Thus $\operatorname{trop}(f)$ applied to a translate of $\mathbf{w}$ by an element in the image of $\gamma^{\vee}$ is the same as translating $\operatorname{trop}(f)(\mathbf{w})$ by that same element in the image of $\gamma^{\vee}$. Thus we have $\operatorname{trop}(V(f))$ is parallel to the image of $\gamma^{\vee}$. □

Let $\operatorname{trop}(X)_N$ denote the image of $\operatorname{trop}(X)$ in $N_{\mathbb{R}}$ under the map $\delta_{\mathbb{R}}^{\vee}$, which by proposition 3.3 is a projection. Now we want to put everything together to give the main result describing the GIT fan. Suppose $G \subset G_{\text{big}} \simeq \mathbb{G}_m^n$ acts on $\mathbb{A}^n$ and $X \subset G_{\text{big}}$ is a subvariety with $\overline{X} \subset \mathbb{A}^n$ such that $\overline{X}$ is stable with respect to the action of G. Let $A = \{\nu_i := \gamma(e_i) : i = 1, \dots, n\} \subset N$, in order to use the correspondence between the fans $\mathscr{F}_A$ and $\mathscr{F}_G$, we need the points in A to be distinct and lie in an integral affine hyperplane $H \subset N_{\mathbb{R}}$ not containing 0, as in Proposition 2.5. With this condition we have that the chambers of the secondary fan $\mathscr{F}_G$ correspond to coherent triangulations of the marked polytope $(\operatorname{Conv}(A), A)$. The walls of $\mathscr{F}_G$ can be described by the circuit C for which the triangulations in the chambers separated by the wall are supported and are modifications of each other along the circuit.

Theorem 3.2. *Let G, X, and $\overline{X}$ be as above. Consider a wall in the secondary fan corresponding to the circuit C. Then $\operatorname{trop}(X)_N$ intersects the relative interior of C (or $\overline{C}$ in the case of a degenerate circuit) if and only if that wall is part of the GIT fan for $\overline{X}$.*

Proof. First we consider the case when C is a non-degenerate circuit. By Proposition 2.7, the unstable locus of the chambers on either side of the wall differ by interchanging

$$\coprod_{J\in[C_+,C)} X_J \qquad \text{and} \qquad \coprod_{J\in[C_-,C)} X_J \tag{3.9}$$

Then $\overline{X}$ intersects any of these strata if and only if the GIT quotient changes when crossing the wall if and only if that will is part of the GIT fan for $\overline{X}$.

By Corollary 3.1, $\overline{X}$ intersects X_J if and only if $\mathrm{trop}(X)$ intersects the relative interior of $\mathrm{cone}(e_i : i \in J)$. After moving to $N_{\mathbb{R}}$ via the map $\delta_{\mathbb{R}}^{\vee}$ and applying Proposition 3.2, then $\overline{X}$ intersects X_J if and only if $\mathrm{trop}(X)_N$ intersects the relative interior of $\mathrm{cone}(\nu_i : i \in J)$. This gives that $\overline{X}$ will intersect any of the strata in eq. (3.9) if and only if $\mathrm{trop}(X)_N$ intersects the relative interior of any cone in

$$\bigcup_{J\in[C_+,C)} \mathrm{cone}(\nu_i : i \in J) \qquad \text{or} \qquad \bigcup_{J\in[C_-,C)} \mathrm{cone}(\nu_i : i \in J) \tag{3.10}$$

We have $\mathrm{trop}(X)_N$ intersects the relative interior of any cone in eq. (3.10) if and only if $\mathrm{trop}(X)_N$ intersects the relative interior of C, since the union of the relative interiors of all the cones in eq. (3.10) is precisely the relative interior of C. Thus we have the desired result in the case when C is a non-degenerate circuit.

Now consider the case when C is a degenerate circuit. First determine $\overline{C} = \bigcup\limits_k C_k^e$ as in eq. (1.3), then by Proposition 2.7, the unstable locus on either side interchanges

$$\bigcup_k \left(\coprod_{J\in[C_+,C)\setminus[C_k^e,C)} X_J \right) \quad \text{and} \quad \bigcup_k \left(\coprod_{J\in[C_-,C)\setminus[C_k^e,C)} X_J \right) \tag{3.11}$$

Similarly to the previous case we have that $\overline{X}$ intersects any of the strata that is changing in the unstable locus if and only if $\mathrm{trop}(X)_N$ intersects the relative interior any cone in

$$\bigcup_k \left(\bigcup_{J\in[C_+,C)\setminus[C_k^e,C)} \mathrm{cone}(\nu_i : i \in J) \right) \quad \text{or} \quad \bigcup_k \left(\bigcup_{J\in[C_-,C)\setminus[C_k^e,C)} \mathrm{cone}(\nu_i : i \in J) \right) \tag{3.12}$$

As in the previous case we also have $\operatorname{trop}(X)_N$ intersects the relative interior of any cone in eq. (3.12) if and only if $\operatorname{trop}(X)_N$ intersects the relative interior of $\overline{C}$. Thus we also obtain the desired result in the case C is a degenerate circuit. $\square$

Chapter 4

Mori Dream Spaces

Mori dream spaces were first introduced by Hu and Keel in [HK00], as a class of varieties for which the minimal model program can be run in a general way. Another defining characteristic is finite generation of the Cox ring. They also are varieties that admit a good decomposition of the cone of effective divisors into chambers called Mori chambers. Using a presentation of the Cox ring, Hu and Keel showed that a Mori dream space is naturally a GIT quotient of an affine variety by an algebraic torus and the Mori chambers coincide with the GIT fan.

First we fix some notation. For a variety X, the *Picard group*, $\mathrm{Pic}(X)$, is the group of line bundles on X. We will often make the usual mix of notation between divisors D and their corresponding line bundles $\mathcal{O}(D)$. Let $N^1(X)$ denote the *Neron-Severi group* of numerically equivalent divisors.

Definition 4.1. A normal projective variety X is a *Mori dream space* if

(1) the group of line bundles $\mathrm{Pic}(X)$ is a finitely generated abelian group,

(2) $\mathrm{nef}(X)$ is the affine hull of finitely many semi-ample line bundles

(3) there is a finite collection of birational maps $f_i : X \dashrightarrow X_i$ which are isomorphisms in codimension one, such that each X_i satisfies (2) and $\mathrm{Mov}(X)$ is the union of $f_i^*(\mathrm{nef}(X_i))$

For a line bundle L on X, the *section ring* is

$$R_L := \bigoplus_{n\in\mathbb{N}} H^0(X, L^n).$$

When D is an effective divisor and R_D is finitely generated, there is an induced rational map $f_D : X \dashrightarrow \mathrm{Proj}(R_D)$. Two divisors D_1 and D_2 with finitely generated section rings are called *Mori equivalent* if the induced rational maps f_{D_1} and f_{D_2} have isomorphic images making the natural triangular diagram commute. The equivalence classes are called *Mori chambers* and impose a fan structure on the cone of effective divisors. This fan will correspond to a GIT fan which we will discuss later in Section 4.1. These fan structures are related to the following ring.

Definition 4.2. Suppose the classes of the line bundles $L_1, \ldots, L_s$ are a basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. For $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$ let $L^{\mathbf{n}} := L_1^{\otimes n_1} \otimes \cdots \otimes L_s^{\otimes n_s}$. A *Cox ring* of X is

$$\mathrm{Cox}(X) = \bigoplus_{\mathbf{n}\in\mathbb{Z}^s} H^0(X, L^{\mathbf{n}})$$

There is a natural grading of $\mathrm{Cox}(X)$ by $\mathrm{Pic}(X)$ which determines an action of a torus of dimension s equal to the rank of $\mathrm{Pic}(X)$. This torus is $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m) \simeq \mathbb{G}_m^s$. For a section x in $H^0(X, L^{\mathbf{a}})$ with $\mathbf{a} = (a_1, \ldots, a_s)$ the action is given by

$$(t_1, \ldots, t_s) \cdot x = t_1^{a_1} \cdots t_s^{a_s} x.$$

Example 4.1. As an example we will consider X a degree 5 del Pezzo surface [BP04]. Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blow-up of four points in general position. Then $\mathrm{Pic}(X)$ is a free abelian group of rank 5 generated by H, the pullback of a line in $\mathbb{P}^2$, and the exceptional divisors E_1, E_2, E_3, E_4. $\mathrm{Cox}(X)$ has 10 generators corresponding to the (-1)-curves on X. We will denote the generators by e_i and f_{ij} with $\deg(e_i) = E_i$ and $\deg(f_{ij}) = H - E_i - E_j$. We have $\mathrm{Cox}(X) \simeq \mathbb{C}[e_1, \ldots, e_4, f_{12}, \ldots, f_{34}]/I$, where the ideal I is generated by the following five quadrics:

$$f_{14}f_{23} - f_{12}f_{34} - f_{13}f_{24}$$
$$e_2f_{12} - e_3f_{13} - e_4f_{14}$$
$$e_1f_{12} - e_3f_{23} - e_4f_{24}$$
$$e_1f_{13} - e_2f_{23} - e_4f_{34}$$
$$e_1f_{14} - e_2f_{24} - e_3f_{34}$$

The characterization of Mori dream spaces that is most useful in this work is in terms of finite generation of its Cox ring. This is a result of Hu and Keel and is stated in the following proposition.

Proposition 4.1. [HK00, Proposition 2.9] Let X be a smooth projective variety with $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Then X is a Mori dream space if and only if $\mathrm{Cox}(X)$ is finitely generated.

4.1 Mori Dream Spaces and GIT

Another result of Hu and Keel is that a Mori Dream Space is naturally a GIT quotient of an affine variety by an algebraic torus. This generalizes Cox's construction of toric varieties as quotients of affine space as we discussed in Chapter 2.

Proposition 4.2. [HK00, Proposition 2.9] If X is a Mori dream space then X is a GIT quotient of $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ by the algebraic torus $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m) \simeq \mathbb{G}_m^s$.

We want to use what we developed in Section 3.2 to describe the GIT fan for quotients of Y by G. We want to see, as in Theorem 3.2, that this fan is a coarsening of a fan coming from toric geometry. When $\mathrm{Cox}(X)$ is finitely generated then we can write $\mathrm{Cox}(X)$ as a quotient of a polynomial ring

$$\mathrm{Cox}(X) \simeq k[x_1, \ldots, x_n]/I.$$

This presentation can be chosen so that the action of G on $\mathrm{Cox}(X)$ lifts to an action of G on $k[x_1, \ldots, x_n]$. Then the action of G on Y is induced by an action of G on $\mathbb{A}^n$ and Y is a G-equivariant subvariety of $\mathbb{A}^n$.

Now that we have an action of an algebraic torus on affine space we can consider the secondary fan $\mathscr{F}_G$ as discussed in Section 2.1. We have $G \subset G_{\text{big}} \simeq \mathbb{G}_m^n$ and the following short exact sequences:

$$0 \longrightarrow M \xrightarrow{\delta} \widehat{G}_{\text{big}} \xrightarrow{\gamma} \widehat{G} \longrightarrow 0$$

$$0 \longleftarrow N \xleftarrow{\delta^\vee} (\widehat{G}_{\text{big}})^\vee \xleftarrow{\gamma^\vee} \widehat{G}^\vee \longleftarrow 0$$

In order for Theorem 3.2 to apply, we need $\nu_i := \delta^\vee(e_i)$ for $i = 1, \ldots, n$ to be distinct and lie in an integral affine hyperplane not containing 0. This will often not be satisfied but we can proceed by first applying the following procedure to promote the data so that the hyperplane condition is satisfied.

4.2 The Promotion

From the exact sequences above we have $\beta_i := \gamma(e_i)$ and $\nu_i := \delta^\vee(e_i)$ for $i = 1, \ldots, n$. We use this to construct the following short exact sequences:

$$0 \longrightarrow M \oplus \mathbb{Z} \xrightarrow{\widetilde{\delta}} \widehat{G}_{\text{big}} \oplus \mathbb{Z} \xrightarrow{\widetilde{\gamma}} \widehat{G} \longrightarrow 0$$

$$0 \longleftarrow N \oplus \mathbb{Z} \xleftarrow{\widetilde{\delta}^\vee} (\widehat{G}_{\text{big}})^\vee \oplus \mathbb{Z} \xleftarrow{\widetilde{\gamma}^\vee} \widehat{G}^\vee \longleftarrow 0$$

by setting $\widetilde{\gamma}(e_i) = \beta_i$ for $i = 1, \ldots, n$ and $\widetilde{\gamma}(e_{n+1}) = -\sum_{i=1}^n \beta_i$. And $\widetilde{\delta}^\vee(e_i) = (\nu_i, 1)$ for $i = 1, \ldots, n$ and $\widetilde{\delta}^\vee(e_{n+1}) = e_{n+1}$. This puts the ν_i in an integral affine hyperplane not containing 0 and then we obtain the polytope $Q = \text{Conv}(\nu_1, \ldots, \nu_n, 0)$ in this hyperplane.

Let D be the divisor corresponding to the character $-\sum_{i=1}^n \beta_i$ and $\mathscr{L}_D$ the G-linearized line bundle on Y corresponding to D. Let $\widetilde{Y} = \text{Tot}(\mathscr{L}_D)$. The GIT fan obtained by Theorem 3.2 is for quotients of $\widetilde{Y}$ by G.

4.2.1 The Promotion for del Pezzo Surfaces

For an example of this promotion procedure we will look at the cases of del Pezzo surfaces. A description of the Cox ring of a del Pezzo surface can be found in [BP04] and [Der06].

Let X_r be a smooth Del Pezzo surface obtained from $\mathbb{P}^2$ by blowing up $r \leq 8$ points $p_1, \ldots, p_r$ in general position. Let $\pi : X_r \longrightarrow \mathbb{P}^2$ be the corresponding projective morphism. Then we have $\mathrm{Pic}(X_r) \simeq \mathbb{Z}^{r+1}$ has basis $H := [\pi^*\mathcal{O}(1)]$ and $E_i := [\pi^{-1}(p_i)]$ for $i = 1, \ldots, r$. The anticanonical divisor of X_r is $-K_{X_r} = 3H - E_1 - \cdots - E_r$ and the degree of X_r is $d_r = 9 - r$.

The generators of $\mathrm{Cox}(X_r)$ are given in [BP04, Theorem 3.2]. For $3 \leq r \leq 7$ the Cox ring of X_r is generated by non-zero sections of the (-1)-curves on X_r. For $r = 8$ we need to add two linearly independent global sections of $-K_{X_8}$. The number N_r of (-1)-curves on X_r is given in the following table:

r	3	4	5	6	7	8
N_r	6	10	16	27	56	240

The (-1)-curves on X_r are described in [BP04, Theorem 2.1]. They are the following:

(1) blown-up points

(2) lines through pairs of points

(3) conics through 5 of the points ($r \geq 5$)

(4) cubics containing 7 of the points, 1 of them double ($r \geq 7$)

(5) quartics containing 8 of the points, 3 of them double ($r = 8$)

(6) quintics containing 8 of the points, 6 of them double ($r = 8$)

(7) sextics containing 8 of the points, 7 of them double, 1 of them triple ($r = 8$)

For $r = 3$ there are 6 generators of $\mathrm{Cox}(X_3)$ which correspond to the three exceptional divisors $E_1, \ldots, E_3$ and the three divisors $H - E_i - E_j$. The β_i are the columns of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

When applying the promotion procedure we consider the character $-\sum_{i=1}^{6}\beta_i = (-3,1,1,1)$. The divisor corresponding to this character is $-3H + E_1 + E_2 + E_3$.

For $r = 4$ we have 10 generators of $\mathrm{Cox}(X_4)$ which correspond to the 4 exceptional divisors $E_1, \ldots, E_4$ and the 6 divisors $H - E_i - E_j$. The β_i are the columns of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

In this case we have the character $-\sum_{i=1}^{10}\beta_i = (-6,2,2,2,2)$ and the corresponding divisor is $-6H + 2E_1 + 2E_2 + 2E_3 + 2E_4$.

For $r = 5$ there are 16 generators of $\mathrm{Cox}(X_5)$ which correspond to the 5 exceptional divisors $E_1, \ldots, E_5$, the 10 divisors $H - E_i - E_j$ and the divisor $2H - E_1 \cdots - E_5$. The divisor from the promotion is $-12H + 4E_1 + 4E_2 + 4E_3 + 4E_4 + 4E_5$.

Here are all the divisors we get from the promotion for $r = 3, \ldots, 8$

$r = 3$	$-3H + E_1 + \cdots + E_3 \quad = K_{X_3}$
$r = 4$	$-6H + 2(E_1 + \cdots + E_4) \quad = 2K_{X_4}$
$r = 5$	$-12H + 4(E_1 + \cdots + E_5) \quad = 4K_{X_5}$
$r = 6$	$-27H + 9(E_1 + \cdots + E_6) \quad = 9K_{X_6}$
$r = 7$	$-84H + 28(E_1 + \cdots + E_7) \quad = 28K_{X_7}$
$r = 8$	$-726H + 242(E_1 + \cdots + E_8) \quad = 242K_{X_8}$

Let g_r denote the number of generators of $\mathrm{Cox}(X_r)$, which only differs from N_r when

$r = 8$, in that case we have 242 generators. The divisor we obtain when doing the promotion is $\frac{g_r}{d_r} K_{X_r}$.

Chapter 5

The Coarsening

In the following we use Theorem 3.2 and adapt constructions in [GKZ94] to describe a polytope associated to the coarsened fan. Recall we have an algebraic torus G acting on affine space $\mathbb{A}^n$ and a G-equivariant subvariety X. The action of G on $\mathbb{A}^n$ defines a marked polytope (Q, A). We require that this polytope satisfy a hyperplane condition (Proposition 2.5) so as to use the correspondence between the secondary fan $\mathscr{F}_G$ (which describes quotients of $\mathbb{A}^n$ by G Section 2.1) and the secondary fan $\mathscr{F}_A$ (which describes coherent subdivisions of (Q, A) Section 1.1). In Section 1.2, we discussed the secondary polytope $\Sigma(A)$ which has as its normal fan the secondary fan $\mathscr{F}_A$. The GIT fan for quotients of X by G was obtained as a coarsening of the fan $\mathscr{F}_A$ by removing walls based on intersections with the tropicalization of X. From here a natural result to seek is if this coarsened fan is the normal fan to a polytope in a similar manner as seen with the fan $\mathscr{F}_A$ and polytope $\Sigma(A)$. This is indeed the case and we will now present the construction.

Let A be a finite set of points in N that lie in a hyperplane $H \subset N_{\mathbb{R}}$ not containing 0. Let $Q = \mathrm{Conv}(A) \subset H$ be a full dimensional polytope. We have $\mathscr{F}_A$, the secondary fan in $\mathbb{R}^A$ with maximal cones

$$C(T) = \{\psi : g_{\psi,T} \text{ concave and } g_{\psi,T}(a) \geq \psi(a)\}$$

for all coherent triangulations T. Recall that $g_{\psi,T}$ was defined by setting $g_{\psi,T}(a) = \psi(a)$ for each $a \in A$ that is a vertex of some simplex of T and then affinely interpolating inside each simplex. Also recall (eq. (1.1)) the secondary polytope $\Sigma(A)$ was defined as the convex hull of

$$\varphi_T \;=\; \sum_{a \in A} \left(\sum_{\substack{\sigma \in T \\ a \in \sigma}} \mathrm{vol}(\sigma) \right) e_a$$

for all coherent triangulations T. The polytope we construct is defined similarly but we will consider a different measure.

Suppose we have a measure μ on Q. We define an equivalence relation on the set of triangulations of Q with respect to μ. Let T_1 and T_2 be two triangulations of Q and define

$$T_1 \sim_\mu T_2 \text{ if } \sigma \in T_1 \text{ with } \int\limits_{\mathrm{relint}(\sigma)} d\mu \neq 0 \Rightarrow \sigma \in T_2.$$

This relation is essentially saying two triangulations T_1 and T_2 are equivalent with respect to μ if and only if the simplices that intersect $\mathrm{supp}(\mu)$ are in both triangulations.

Definition 5.1. The *μ-coarsening* $\mathscr{F}_A(\mu)$ is a stratification of $\mathbb{R}^A$ given by cones

$$C([T]) \;=\; \bigcup_{T' \in [T]} C(T')$$

for each equivalence class and their intersections.

Lemma 5.1. Let $Q \subset N_{\mathbb{R}}$, μ a measure on Q, and T a triangulation of Q. Then for each $\sigma \in T$ there exists $\{p_{a,\sigma}\}_{a \in \sigma}$ such that

$$\int\limits_{\mathrm{relint}(\sigma)} g \; d\mu \;=\; \sum_{a \in \sigma} p_{a,\sigma} \; g(a)$$

for any affine function $g : N_{\mathbb{R}} \longrightarrow \mathbb{R}$.

Proof. Let $x_{a,\sigma}$ be the affine function on σ that takes the value 1 at a and 0 at all other vertices of σ.

Then define

$$p_{a,\sigma} = \int\limits_{\mathrm{relint}(\sigma)} x_{a,\sigma}\ d\mu$$

Let $g : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be any affine function and note $g = \sum_{a\in\sigma} g(a)\ x_{a,\sigma}$ on σ. Then we have

$$\begin{aligned}
\int\limits_{\mathrm{relint}(\sigma)} g\ d\mu &= \int\limits_{\mathrm{relint}(\sigma)} \left(\sum_{a\in\sigma} g(a)\ x_{a,\sigma}\right)\ d\mu \\
&= \sum_{a\in\sigma}\left(\int\limits_{\mathrm{relint}(\sigma)} g(a)\ x_{a,\sigma}\ d\mu\right) \\
&= \sum_{a\in\sigma}\left(g(a)\int\limits_{\mathrm{relint}(\sigma)} x_{a,\sigma}\ d\mu\right) \\
&= \sum_{a\in\sigma} g(a)\ p_{a,\sigma}
\end{aligned}$$

□

For a triangulation T define

$$\varphi_{T,\mu} := \sum_{a\in A}\left(\sum_{\substack{\sigma\in T\\ a\in\sigma}}\ \int\limits_{\mathrm{relint}(\sigma)} x_{a,\sigma}\ d\mu\right)\ e_a$$

where the sum is over all simplices not just maximal ones. Note that if we have $T_1 \sim_\mu T_2$ then $\varphi_{T_1,\mu} = \varphi_{T_2,\mu}$. This follows since the only non-zero $p_{a,\sigma}$ will come from those simplices σ such that $\sigma \cap \mathrm{supp}(\mu) \neq \emptyset$. In the case when $T_1 \sim_\mu T_2$ we have that the simplices that intersect $\mathrm{supp}(\mu)$ are the same in both triangulations and thus the $p_{a,\sigma}$ are the same for T_1 and T_2.

For a given triangulation T and $a \in A$ we define a function $x_{a,T}$ on Q where on each simplex of T we assign the value 1 to a and 0 at all other vertices. Then we can write

$$\varphi_{T,\mu} = \sum_{a \in A} \left(\int x_{a,T} \; d\mu \right) e_a.$$

Lemma 5.2. For $\psi \in \mathbb{R}^A$ $(\varphi_{T,\mu}, \psi) = \int\limits_Q g_{\psi,T} \; d\mu$.

Proof. Let $\psi \in \mathbb{R}^A$ and note $\psi = \sum\limits_{a \in A} \psi(a) \; e_a$. Then we have

$$\begin{aligned}
(\varphi_{T,\mu}, \psi) &= \sum_{a \in A} \varphi_{T,\mu}(a) \; \psi(a) \\
&= \sum_{a \in A} \left(\sum_{\substack{\sigma \in T \\ a \in \sigma}} \int\limits_{\mathrm{relint}(\sigma)} x_{a,\sigma} \; d\mu \right) \psi(a) \\
&= \sum_{a \in A} \left(\sum_{\substack{\sigma \in T \\ a \in \sigma}} p_{a,\sigma} \; \psi(a) \right) \\
&= \sum_{\sigma \in T} \left(\sum_{a \in \sigma} p_{a,\sigma} \; \psi(a) \right) \\
&= \sum_{\sigma \in T} \left(\int\limits_{\mathrm{relint}(\sigma)} g_{\psi,T} \; d\mu \right) \\
&= \int\limits_Q g_{\psi,T} \; d\mu
\end{aligned}$$

□

Definition 5.2. The *μ-secondary polytope* $\Sigma_\mu(A)$ is the convex hull of $\varphi_{T,\mu}$.

For any $\psi \in \mathbb{R}^A$ define

$$G_\psi = \mathrm{Conv}\big((a, y) : \; y \leq \psi(a), \; a \in A, \; y \in \mathbb{R}\big).$$

The upper boundary of G_ψ can be given as the graph of the function $g_\psi : Q \longrightarrow \mathbb{R}$ defined by $g_\psi(x) = \max\{y : \; (a, y) \in G_\psi\}$. Note that by definition the function g_ψ is concave. We also have that G_ψ defines a subdivision of Q which we will denote $S(\psi) = \{(Q_i, A_i)\}_{i\in I}$.

Lemma 5.3. For a fixed $\psi \in \mathbb{R}^A$ we have

(1) For any triangulation T and any $x \in Q$ $g_\psi(x) \geq g_{\psi,T}(x)$ with equality if and only if the minimal simplex $\sigma \in T$ containing x is contained in some Q_i in the subdivision $S(\psi)$.

(2) $\max\limits_{\varphi\in\Sigma_\mu(A)} (\varphi, \psi) = \int\limits_Q g_\psi \; d\mu.$

Proof. For (1) the inequality holds since $g_{\psi,T}$ is affine on each simplex σ of the triangulation T and $g_\psi(a) \geq \psi(a) = g_{\psi,T}(a)$ for any vertex $a \in \sigma$.

Suppose $g_\psi(x) = g_{\psi,T}(x)$ for some $x \in Q$. Then x is in some Q_i in the subdivision $S(\psi)$. Let σ be the minimal simplex of the triangulation T containing x. Then we have $x \in \mathrm{relint}(\sigma)$ and we may assume $g_\psi(a) = g_{\psi,T}(a)$ for all vertices a of σ since otherwise by concavity of g_ψ and $g_{\psi,T}$ an affine function on σ we would have $g_\psi(x) > g_{\psi,T}(x)$ for all $x \in \mathrm{relint}(\sigma)$. Now we have a concave function $f := g_\psi - g_{\psi,T}$ on σ such that $f(a) = 0$ for all $a \in \mathrm{vert}(\sigma)$ and $f(x) = 0$ for $x \in \mathrm{relint}(\sigma)$, thus $f(y) = 0$ for all $y \in \sigma$. Since g_ψ coincides with $g_{\psi,T}$ on σ we get that σ is contained in Q_i.

Conversely, assume the minimal simplex $\sigma \in T$ containing x is contained in some Q_i and we have $g_\psi(a) = g_{\psi,T}(a)$ for all vertices $a \in \sigma$. Then the graph of $g_{\psi,T}$ on that simplex will be contained in the upper boundary of G_ψ thus it lies on the graph of g_ψ which implies $g_{\psi,T}(x) = g_\psi(x)$ for $x \in \sigma$.

To prove (2) first note that the max can be taken over $\varphi_{T',\mu}$ since $\Sigma_\mu(A)$ is the convex hull of $\varphi_{T',\mu}$. By (1) and Lemma 5.2 it suffices to find a triangulation T such that for any $x \in Q$ the minimal simplex containing x is contained in some Q_i. To find such a triangulation consider each (Q_i, A_i) in the subdivision $S(\psi)$. By perturbing ψ slightly we can obtain a triangulation of (Q_i, A_i) for each $i \in I$. Then we obtain a triangulation T of (Q, A) such that any simplex of T is contained in some Q_i. □

We want to compute the normal fan to the polytope $\Sigma_\mu(A)$. To do so we need to determine the normal cones at each $\varphi_{T,\mu}$. Since any two triangulations in the same equivalence class

$[T]$ have the same $\varphi_{T,\mu}$ we define a normal cone for each equivalence class. The normal cone to $\Sigma_\mu(A)$ at $[T]$ is

$$N_{[T]}\Sigma_\mu(A) := \left\{\psi \in \mathbb{R}^A : (\varphi_{\widetilde{T},\mu}, \psi) = \max_{\varphi\in\Sigma_\mu(A)} (\varphi, \psi),\ \widetilde{T} \in [T]\right\}.$$

Theorem 5.1. $\mathscr{F}_A(\mu)$ is the normal fan of the polytope $\Sigma_\mu(A)$.

Proof. Suppose $\psi \in C(T)$ then $g_{\psi,T}$ is concave and $g_{\psi,T}(a) \geq \psi(a)$ for all $a \in A$. Then each simplex of T will be contained in some Q_i in the subdivision $S(\psi)$ thus we have $g_\psi = g_{\psi,T}$ on all of Q by (1) of Lemma 5.3. Applying Lemma 5.2 and (2) of Lemma 5.3 we have

$$\begin{aligned}(\varphi_{T,\mu}, \psi) &= \int_Q g_{\psi,T}\ d\mu \\ &= \int_Q g_\psi\ d\mu \\ &= \max_{\varphi\in\Sigma_\mu(A)} (\varphi, \psi)\end{aligned}$$

thus $C(T) \subset N_{[T]}\Sigma_\mu(A)$.

If $T_1 \sim_\mu T_2$ then $\varphi_{T_1,\mu} = \varphi_{T_2,\mu}$. Thus $C(T_1)$ and $C(T_2)$ are both contained in $N_{[T]}\Sigma_\mu(A)$ and we have $C([T]) \subset N_{[T]}\Sigma_\mu(A)$.

For $T_1 \not\sim_\mu T_2$ we want to show that $\varphi_{T_1,\mu} \neq \varphi_{T_2,\mu}$. By definition we have there exists a simplex $\sigma \in T_1$ with $\int_{\mathrm{relint}(\sigma)} \mathrm{d}\mu \neq 0$ and $\sigma \notin T_2$. Suppose by contradiction that $\varphi_{T_1,\mu} = \varphi_{T_2,\mu}$. There are two cases to consider, either $\sigma \not\subset \tau$ for all $\tau \in T_2$ or there exists $\tau \in T_2$ with $\sigma \subsetneq \tau$.

First consider the case $\sigma \not\subset \tau$ for all $\tau \in T_2$. Choose $\psi \in$ relint $C(T_2)$ then $g_\psi = g_{\psi,T_2}$. By Lemma 5.3 (2) and Lemma 5.2 we have

$$\max_{\varphi\in\Sigma_\mu(A)} (\varphi, \psi) = \int_Q g_\psi\ d\mu = \int_Q g_{\psi,T_2}\ d\mu = (\varphi_{T_2,\mu}, \psi) = (\varphi_{T_1,\mu}, \psi) = \int_Q g_{\psi,T_1}\ d\mu$$

Thus for any $x \in \mathrm{supp}(\mu) \cap \sigma$ we have $g_\psi(x) = g_{\psi,T_1}(x)$. By Lemma 5.3 (1) we have that the minimal simplex of T_1 containing x is contained in some simplex of the triangulation T_2, since the subdivision $S(\psi)$ for $\psi \in \operatorname{relint} C(T_2)$ is T_2. But this implies that σ is contained in some simplex of T_2, which contradicts our assumption that σ is not contained in any simplex of T_2.

Now consider the case that there exists $\tau \in T_2$ with $\sigma \subsetneq \tau$. Choose $\psi \in \operatorname{relint} C(T_1)$ then $g_\psi = g_{\psi,T_1}$. By a similar argument as in the previous case, we obtain that $g_\psi = g_{\psi,T_2}$. Now for any $x \in \operatorname{supp}(\mu) \cap \tau$ we have $g_\psi(x) = g_{\psi,T_2}(x)$. Using Lemma 5.3 (2) we obtain that the minimal simplex of T_2 containing x is contained in a simplex of T_1. But this would imply $\tau \subset \sigma$ and further that $\tau = \sigma$ which contradicts the assumption that $\sigma \notin T_2$.

$\square$

Now we want to relate this polytope back to the situation of Mori dream spaces and GIT fans. If X is a Mori dream space then we have X is a GIT quotient of $Y = \operatorname{Spec}(\operatorname{Cox}(X))$ by the algebraic torus $G = \operatorname{Hom}(\operatorname{Pic}(X), \mathbb{G}_m) \simeq \mathbb{G}_m^s$. By a presentation of the Cox ring we have Y as a G-equivariant subvariety of $\mathbb{A}^n$. Let (Q, A) be the marked polytope obtained from the action of G on $\mathbb{A}^n$. In order for Theorem 3.2 to apply we needed to satisfy a hyperplane condition. We saw in Section 4.2, if this condition was not already satisfied we could promote the data. This promotion produced a divisor D with G-linearized line bundle $\mathscr{L}_D$ and the space $\widetilde{Y} = \operatorname{Tot}(\mathscr{L}_D)$. The resulting fan describes GIT quotients of $\widetilde{Y}$ by G and contains as a subfan the GIT fan for quotients of Y by G. The result of the promotion on (Q, A) is to add 0 as a marked point of Q.

Theorem 5.2. Let X be a Mori dream space with G, Y and (Q, A) as above. Let μ be a volume measure on the tropicalization of Y. Then the GIT fan is the normal fan to the μ-secondary polytope $\Sigma_\mu(A)$.

Proof. If μ is a volume measure on $\operatorname{trop}(Y)_N$ then the equivalence relation gives that two triangulations are equivalent if and only if the simplices that intersect $\operatorname{trop}(Y)_N$ are in both triangulations. If T_1 and T_2 are two triangulations from the same equivalence class then $\operatorname{trop}(Y)_N$ does not intersect the relative interior of the circuit along which the triangulations T_1 and T_2 are modifications of each other. Thus by Theorem 3.2 that wall is not part of the GIT fan. This measure μ gives the GIT fan as the μ-coarsening $\mathscr{F}_A(\mu)$. Lastly, by Theorem 5.1 we have that $\mathscr{F}_A(\mu)$ is the normal fan of the μ-secondary polytope $\Sigma_\mu(A)$. $\square$

Chapter 6

Examples

6.1 A Toric Example

Consider the polytope Q given as the convex hull of the points

$$a_1 = (1,0),\ a_2 = (1,1),\ a_3 = (1,2),\ \text{and } a_4 = (1,3)\ \in N.$$

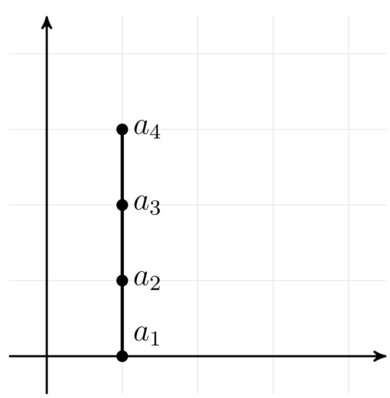


Figure 6.1: The polytope $Q \subset N$

We have the following short exact sequences:

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{B} \mathbb{Z}^2 \longrightarrow 0$$

$$0 \longleftarrow N \xleftarrow{A^T} \mathbb{Z}^4 \xleftarrow{B^T} \mathbb{Z}^2 \longleftarrow 0$$

$$\text{with } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

The action of G on $\mathbb{C}^4$ is given by

$$(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_1^{-2} t_2 x_2, t_1 t_2^{-2} x_3, t_2 x_4).$$

There are four coherent triangulations of Q, as shown in the figure below.

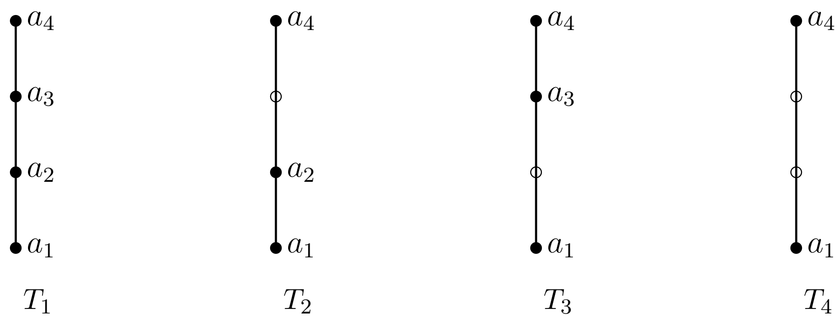


Figure 6.2: The four triangulations of Q

The secondary fan has a maximal chamber corresponding to each coherent triangulation and is shown in the figure below.

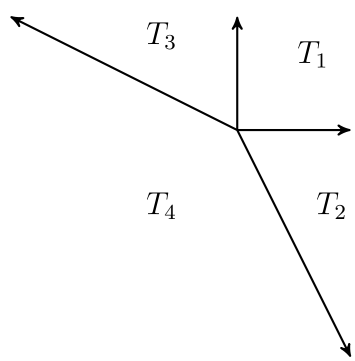


Figure 6.3: The secondary fan with chambers labeled by the corresponding triangulation

Now consider X defined by the polynomial $f = x_1 - x_3 x_4^2$. We need to determine how $\mathrm{trop}(V(f))_N$ intersects Q.

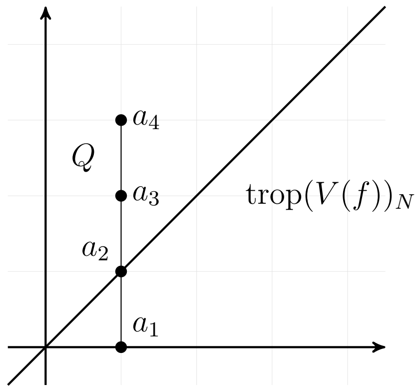


Figure 6.4: Q and $\mathrm{trop}(V(f))_N$

We see that $\mathrm{trop}(V(f))_N$ intersects Q in the point a_2, so we take μ to be the measure supported at a_2. Let us see which triangulations are equivalent with respect to μ. For the triangulation T_1 the only simplex for which $\int\limits_{\mathrm{relint}(\sigma)} d\mu \neq 0$ is the simplex which is just the point a_2. There is only one other triangulation that has a_2 as a simplex and that is the triangulation T_2. Thus we have $T_1 \sim_\mu T_2$. For the triangulations T_3 and T_4 there is only one simplex in each for which $\int\limits_{\mathrm{relint}(\sigma)} d\mu \neq 0$ but those simplices are not in any other triangulation, thus no other triangulations are equivalent with respect to μ. Then we have $\mathscr{F}_A(\mu)$ consists of three maximal cones, one is the union of the cones corresponding to T_1 and T_2 and the others are the cones for T_3 and T_4.

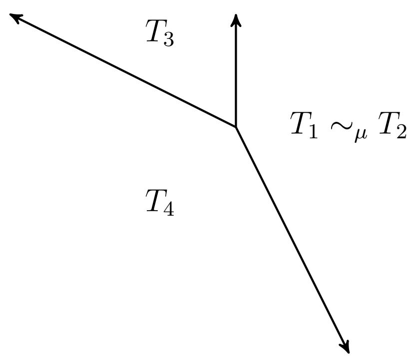
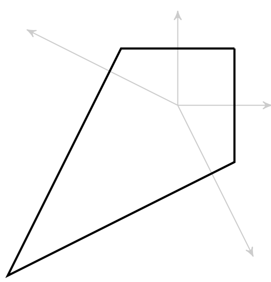


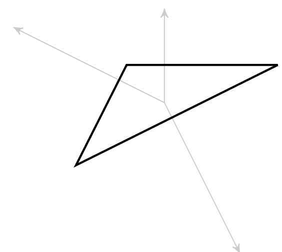
Figure 6.5: The μ-coarsening $\mathscr{F}_A(\mu)$

Now let us determine the $\varphi_{T,\mu}$ for each triangulation. First consider the triangulation T_1. The only non-zero integral will be over the simplex $\{a_2\}$ and in this case the value is 1. Thus $\varphi_{T_1,\mu} = (0, 1, 0, 0)$. Since $T_1 \sim_\mu T_2$ we have $\varphi_{T_2,\mu} = (0, 1, 0, 0)$. For the triangulation

T_3 the only non-zero integral will come from the simplex $\text{Conv}(a_1, a_3)$ and in this case we have $\varphi_{T_3,\mu} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$. Lastly for the triangulation T_4 the only non-zero integral is over the simplex $\text{Conv}(a_1, a_4)$ and we obtain $\varphi_{T_4,\mu} = \left(\frac{2}{3}, 0, 0, \frac{1}{3}\right)$.



secondary polytope



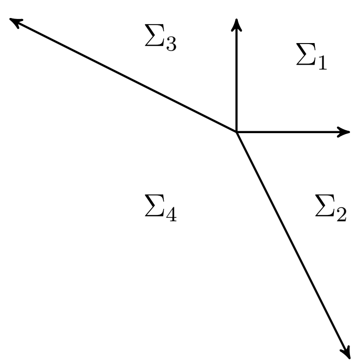
μ-secondary polytope

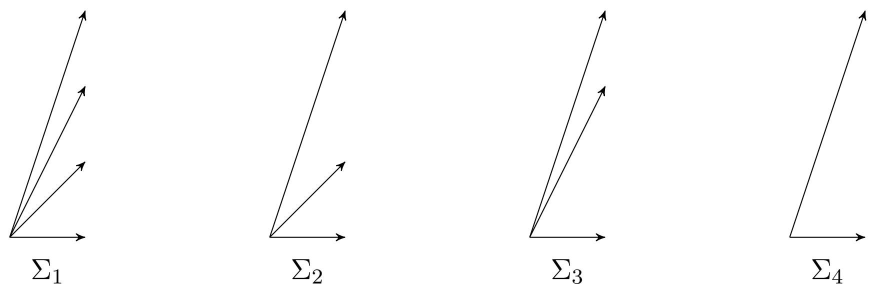
Figure 6.6: Example of secondary polytope and μ-secondary polytope

Now let us discuss the GIT quotients appearing in this example. First consider the GIT quotients of $\mathbb{C}^4$ with action of $G \simeq \mathbb{C}^{*2}$ given by

$$(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_1^{-2} t_2 x_2, t_1 t_2^{-2} x_3, t_2 x_4).$$

The secondary fan has four chambers and the GIT quotient in a given chamber is the toric variety of a fan coming from the triangulation of Q labeling that chamber. We can label the secondary fan by the fan defining the toric variety that is the GIT quotient obtained in a given chamber.



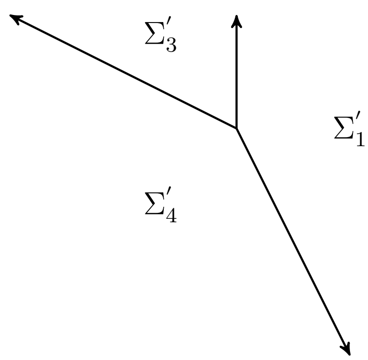


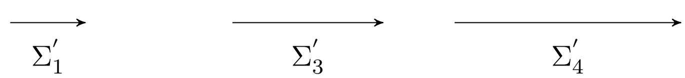
The fan Σ_4 defines a singular toric surface with an A_2 singularity. In terms of stacks this is $[\mathbb{C}^2/\mathbb{Z}_3]$. The fans Σ_2 and Σ_3 define toric varieties that are partial resolutions of this singularity. The fan Σ_1 defines a smooth toric variety.

For the GIT quotients of X by G, notice that $X \simeq \mathbb{C}^3$ in variables (x_2, x_3, x_4), giving the action by G as

$$(t_1, t_2) \cdot (x_2, x_3, x_4) = (t_1^{-2} t_2 x_2, t_1 t_2^{-2} x_3, t_2 x_4).$$

We found the GIT fan using Theorem 3.2, and we can label it by the fans defining the GIT quotients.





For Σ_1' we get $\mathbb{C}$, for Σ_3' we get $[\mathbb{C}/\mathbb{Z}_2]$, and for Σ_4' we get $[\mathbb{C}/\mathbb{Z}_3]$.

6.2 The Return of Example 2.7

Now let us use the setup from Example 2.7 to find another μ-secondary polytope. We have the polytope $Q \subset H$ as the convex hull of

$$a_1 = (1,0),\ a_2 = (1,1),\ a_3 = (0,1),\ a_4 = (-1,1),\ a_5 = (0,0).$$

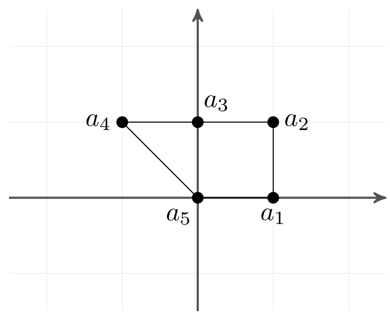
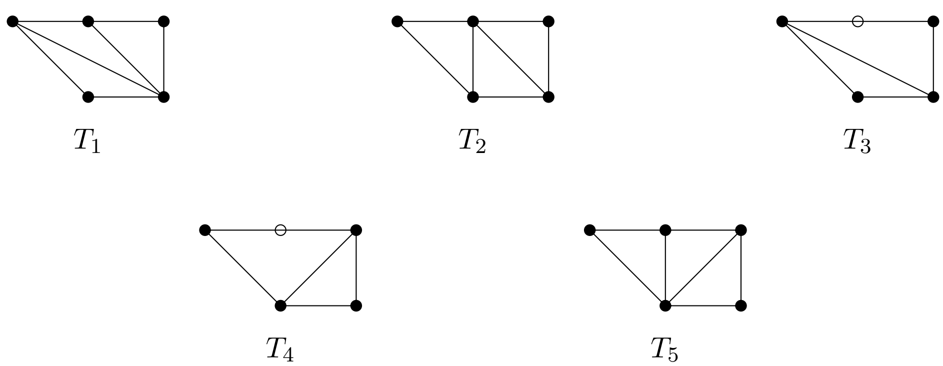
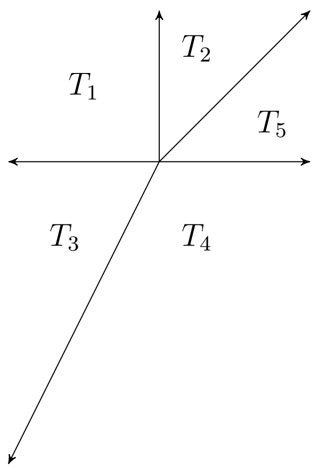


Figure 6.9: The polytope $Q \subset H$

Recall that there were five coherent triangulations



and secondary fan



Let us take for the measure μ on Q a volume measure supported on the line segment from $(0,0)$ to $(1,1)$.

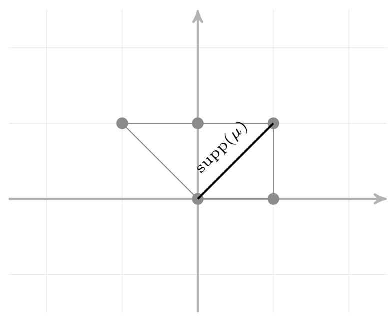


Figure 6.11: The support of the measure μ

When we compute $\varphi_{T,\mu}$ for each coherent triangulation we obtain:

$$\varphi_{T_1,\mu} = \left(\frac{1}{3},\ \frac{1}{4},\ \frac{1}{6},\ \frac{1}{12},\ \frac{1}{6}\right) \qquad \varphi_{T_4,\mu} = \left(0,\ \frac{1}{2},\ 0,\ 0,\ \frac{1}{2}\right)$$
$$\varphi_{T_2,\mu} = \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 0,\ \frac{1}{4}\right) \qquad \varphi_{T_5,\mu} = \left(0,\ \frac{1}{2},\ 0,\ 0,\ \frac{1}{2}\right)$$
$$\varphi_{T_3,\mu} = \left(\frac{1}{3},\ \frac{1}{3},\ 0,\ \frac{1}{6},\ \frac{1}{6}\right)$$

We see that for T_4 and T_5 the $\varphi_{T,\mu}$ are equal which means these two triangulations are identified under the equivalence relation. In the secondary fan we lose the wall between the chambers for T_4 and T_5.

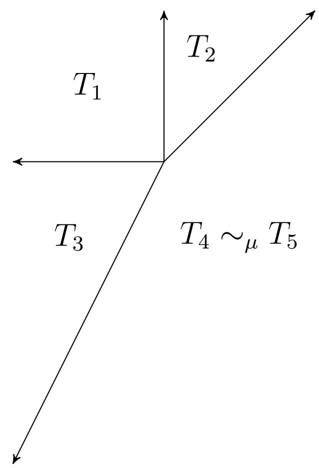


Figure 6.12: The μ-coarsening $\mathscr{F}_A(\mu)$

We can compare the secondary polytope and the μ-secondary polytope in the following figures.

secondary polytope $\qquad$ μ-secondary polytope

Figure 6.13: The secondary polytope and μ-secondary polytope

6.3 A non-toric example

Consider X a degree 5 del Pezzo surface. As we saw in example 4.1, $\mathrm{Pic}(X)$ is rank 5 and generated by H, the pullback of a line in $\mathbb{P}^2$, and the exceptional divisors E_1, E_2, E_3, and E_4. The Cox ring of X has 10 generators corresponding to the (-1)-curves on X which we will denote e_i and f_{ij} with $\deg(e_i) = E_i$ and $\deg(f_{ij}) = H - E_i - E_j$. We have $Y = \mathrm{Spec}(\mathrm{Cox}(X)) \subset \mathbb{C}^{10}$ and $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}^*) \simeq (\mathbb{C}^*)^5$. This gives the following short exact sequence:

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^{10} \xrightarrow{B} \mathbb{Z}^5 \longrightarrow 0$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

A simple calculation shows that the ν_i, which correspond to the rows of the matrix A, do not lie in an integral affine hyperplane not containing 0. Thus we need to apply the promotion procedure described in Section 4.2. After the promotion we have the exact sequence:

$$0 \longrightarrow M \oplus \mathbb{Z} \xrightarrow{\widetilde{A}} \mathbb{Z}^{11} \xrightarrow{\widetilde{B}} \mathbb{Z}^5 \longrightarrow 0$$

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \widetilde{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -6 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 2 \end{bmatrix}$$

We have $Q = \mathrm{Conv}(\mathcal{A})$ where

$$\mathcal{A} = \left\{ \begin{array}{l} (0,0,1,1,1),\ (1,1,0,0,1),\ (-1,0,-1,0,-1),\ (0,-1,0,-1,-1), \\ (1,1,1,1,1),\ (-1,0,0,0,0),\ (0,-1,0,0,0),\ (0,0,-1,0,0), \\ (0,0,0,-1,0),\ (0,0,0,0,-1),\ (0,0,0,0,0) \end{array} \right\}.$$

By a computer calculation using SageMath, we find that the marked polytope (Q, A) has 1080 coherent triangulations, which correspond to the chambers of the secondary fan $\mathscr{F}_A$ and the vertices of the secondary polytope $\Sigma(A)$.

To determine the tropicalization of Y recall that $\mathrm{Cox}(X) \simeq \mathbb{C}[e_1, \ldots, e_4, f_{12}, \ldots, f_{34}]/I$ where I is the ideal generated by the five quadrics

$$f_{14}f_{23} - f_{12}f_{34} - f_{13}f_{24} \tag{6.1}$$

$$e_2f_{12} - e_3f_{13} - e_4f_{14} \tag{6.2}$$

$$e_1f_{12} - e_3f_{23} - e_4f_{24} \tag{6.3}$$

$$e_1f_{13} - e_2f_{23} - e_4f_{34} \tag{6.4}$$

$$e_1f_{14} - e_2f_{24} - e_3f_{34} \tag{6.5}$$

For Theorem 3.2, we are interested in $\mathrm{trop}(Y)_N$, so consider a change of variables using the map A. We want to change from variables $\mathbf{y} = (e_1, e_2, e_3, e_4, f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})$ to $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$. First consider

$$f_{14}f_{23} - f_{12}f_{34} - f_{13}f_{24}$$

and divide through by $f_{14}f_{23}$ which gives

$$1 - \frac{f_{12}f_{34}}{f_{14}f_{23}} - \frac{f_{13}f_{24}}{f_{14}f_{23}}.$$

In terms of $\mathbf{y}$ this polynomial is

$$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(0,0,0,0,1,0,-1,-1,0,1)} - \mathbf{y}^{(0,0,0,0,0,1,-1,-1,1,0)}.$$

The preimage of each exponent under the map A gives the polynomial in terms of $\mathbf{x}$

$$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(0,1,1,0,-1)} - \mathbf{x}^{(-1,1,1,-1,0)}.$$

Writing this in terms of the x_i we obtain

$$x_1x_4x_5 - x_1x_2x_3x_4 - x_2x_3x_5.$$

eq. (6.1)	$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(0,0,0,0,1,0,-1,-1,0,1)} - \mathbf{y}^{(0,0,0,0,0,1,-1,-1,1,0)}$	$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(0,1,1,0,-1)} - \mathbf{x}^{(-1,1,1,-1,0)}$
eq. (6.2)	$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(0,-1,1,0,-1,1,0,0,0,0)} - \mathbf{y}^{(0,-1,0,1,-1,0,1,0,0,0)}$	$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(-1,0,0,0,0)} - \mathbf{x}^{(0,-1,0,0,0)}$
eq. (6.3)	$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(-1,0,1,0,-1,0,0,1,0,0)} - \mathbf{y}^{(-1,0,0,1,-1,0,0,0,1,0)}$	$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(0,0,-1,0,0)} - \mathbf{x}^{(0,0,0,-1,0)}$
eq. (6.4)	$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(-1,1,0,0,0,-1,0,1,0,0)} - \mathbf{y}^{(-1,0,0,1,0,-1,0,0,0,1)}$	$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(1,0,-1,0,0)} - \mathbf{x}^{(1,0,0,0,-1)}$
eq. (6.5)	$\mathbf{y}^{\mathbf{0}} - \mathbf{y}^{(-1,1,0,0,0,0,-1,0,1,0)} - \mathbf{y}^{(-1,0,1,0,0,0,-1,0,0,1)}$	$\mathbf{x}^{\mathbf{0}} - \mathbf{x}^{(0,1,0,-1,0)} - \mathbf{x}^{(0,1,0,0,-1)}$

In terms of the x_i we get the following:

$$x_1x_4x_5 - x_1x_2x_3x_4 - x_2x_3x_5$$

$$x_1x_2 - x_2 - x_1$$

$$x_3x_4 - x_4 - x_3$$

$$x_3x_5 - x_1x_5 - x_1x_3$$

$$x_4x_5 - x_2x_5 - x_2x_4$$

From a computer calculation in SageMath, we find that $\mathrm{trop}(Y)_N$ is dimension 2 with 10 rays and 15 cones. Combinatorially, $\mathrm{trop}(Y)_N$ is the Petersen graph. The vertices of the Petersen graph correspond to the rays of $\mathrm{trop}(Y)_N$ and coincide with the vertices of Q. This tropicalization has also been described in [RSS16].

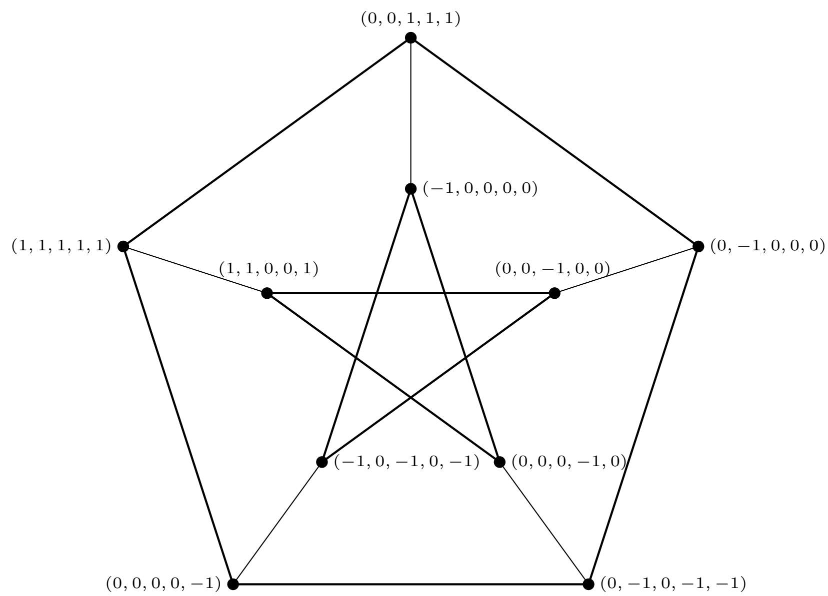


Figure 6.14: Petersen graph for $\text{trop}(Y)_N$

Using Theorem 5.2 let μ be a volume measure on $\text{trop}(Y)_N$ and we will compute the μ-secondary polytope $\Sigma_\mu(A)$. Doing this computation using SageMath and the functions defined in Appendix A, we see that $\Sigma_\mu(A)$ is a 5 dimensional polytope in $\mathbb{R}^{11}$ with 121 vertices.

Interestingly, the Petersen graph also shows up as the graph of exceptional curves on X.

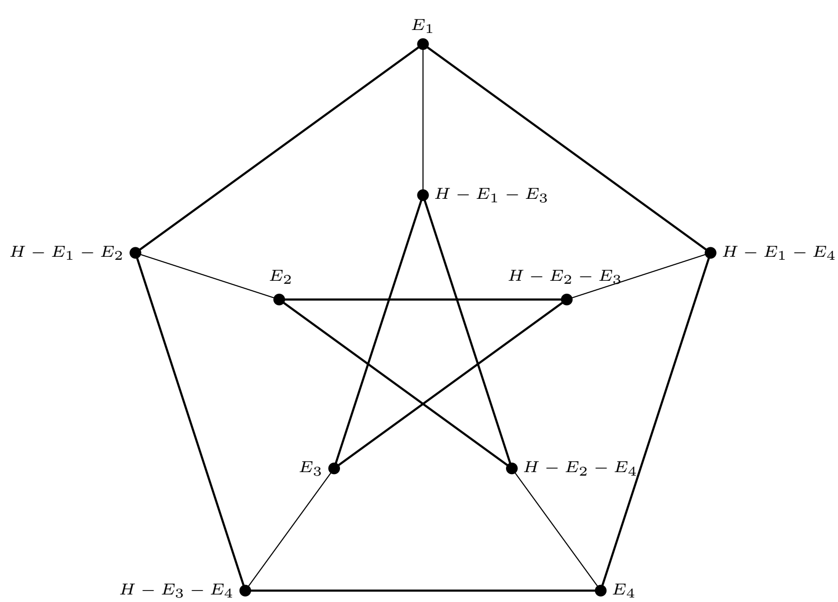


Figure 6.15: Graph of exceptional curves on X

Lemma 6.1. Let P be the Petersen graph for $\text{trop}(Y)_N$. If $e \in E(P)$ then $0 \notin \text{relint}(\text{Conv}\{v \in V(P) \mid v \notin e\})$.

Corollary 6.1. Let $T = \{(Q_i, A_i)\}_{i \in I}$ be a triangulation of (Q, A). If 0 is a vertex of Q_i for some $i \in I$ then $v_T = \{v \in A \mid v \notin Q_j \text{ any } j \in I \text{ with } 0 \in A_j\}$ contains non-adjacent vertices of P.

Doing computations using SageMath we see that there are 76 equivalence classes of triangulations for which 0 is a vertex of some simplex in the triangulation. For one of the equivalence classes we get that every simplex has 0 as a vertex thus the set v_T is empty. For all other equivalence classes they each correspond to a set of non-adjacent vertices of the graph P. In a chamber corresponding to one of these triangulations T what we will obtain is the space obtained by blowing down the exceptional curves corresponding to the vertices in the set v_T.

Bibliography

[BH05] Florian Berchtold and Jürgen Hausen. Git-equivalence beyond the ample cone. *arXiv preprint math/0503107*, 2005.

[BP04] Victor V. Batyrev and Oleg N. Popov. *The Cox Ring of a Del Pezzo Surface*, pages 85–103. Birkhäuser Boston, Boston, MA, 2004.

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Society, 2011.

[Der06] Ulrich Derenthal. On the cox ring of del pezzo surfaces. *arXiv preprint math/0603111*, 2006.

[DH98] Igor V Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 87(1):5–51, 1998.

[Dol03] Igor Dolgachev. *Lectures on invariant theory*. Number 296. Cambridge University Press, 2003.

[GKZ94] I.M. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. 1994.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and git. *Michigan Math. J.*, 48(1):331–348, 2000.

[McK10] James McKernan. Mori dream spaces. *Japanese Journal of Mathematics*, 5(1):127–151, 2010.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[MS15] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry:*. Graduate Studies in Mathematics. American Mathematical Society, 2015.

[RSS16] Qingchun Ren, Kristin Shaw, and Bernd Sturmfels. Tropicalization of del pezzo surfaces. *Advances in Mathematics*, 300:156–189, 2016.

[Tha96] Michael Thaddeus. Geometric invariant theory and flips. *Journal of the American Mathematical Society*, 9(3):691–723, 1996.

Appendix A

SageMath code to compute μ-secondary polytope

The following are functions written in SageMath to compute the secondary polytope. Let $A \subset \mathbb{R}^n$ be a finite set of points and $Q = \mathrm{Conv}(A)$. Let μ be a measure on Q. These functions rely on the situation when the intersection of a simplex with the support of μ is a simplex. To run the functions we need to use $PC(A) := \mathrm{PointConfiguration}(\mathrm{vertices} = A, \mathrm{regular} = \mathrm{true})$. Also we need to define supp_mu, a list of Polyhedra that are the pieces of the support of the measure μ.

The input for this function is T a triangulation of $PC(A)$ and an integer min_dim. It returns a list of simplices of the triangulation that have dimension between min_dim and the dimension of Q.

```
def dim_simps(T, min_dim):
    simps = []
    v = T.point_configuration()
    for d in range(min_dim, v.dim()+1):
        for s in T.simplicial_complex().faces()[d]:
            simps.append(Polyhedron([v[i] for i in s]))
    return simps
```

This function returns a dictionary where the keys are simplices of the triangulation and the value associated to a key (simplex) is a list of the intersections of the simplex with each piece of the support of μ.

```
def simps_int_supp_mu(T, supp_mu):
    D = dict([])
    d = min(p.dim() for p in supp_mu)
    S = dim_simps(T, d)
    for s in S:
        D[s] = []
    for p in supp_mu:
        if s.intersection(p).dim() >= d and
            s.relative_interior_contains(s.intersection(p).center()):
            D[s].append(s.intersection(p))
    for s in D.keys():
        if len(D[s]) == 0:
            D.pop(s)
    return D
```

This function is used to find the $p_{a,\sigma}$ from Chapter 5.

```
def matrix_for_coeffs(simp, supp_mu_part, dim):
    M = Matrix(QQ, dim+1, simp.dim()+2, 0)
    for i in range(M.ncols()):
        M[0,i] = 1
    L = simp.vertices_list()
    for j in range(len(L)):
        for i in range(1, M.nrows()):
            M[i,j] = L[j][i-1]
    C = simp.intersection(supp_mu_part).center()
    for i in range(1, M.nrows()):
        M[i, M.ncols()-1] = C[i-1]
    return M.rref()
```

This function computes $\varphi_{T,\mu}$.

```
def phiT_mu(T, A, supp_mu):
```

```
    Q = Polyhedron(A)
    phi = [0 for a in A]
    D = dict([])
    for a in A:
        D[a] = 0
    S = simps_int_supp_mu(T, supp_mu)
    for s in S.keys():
        for p in S[s]:
            M = matrix_for_coeffs(s,p,Q.dim())
                for i in range(len(s.vertices_list())):
                    D[tuple(s.vertices_list()[i])] +=
                        M[i,M.ncols()-1]*p.volume(measure = 'induced_rational')
    for i in range(len(A)):
        phi[i] = D[A[i]]
    return phi
```

This function returns the μ-secondary polytope $\Sigma_\mu(A)$.

```
def mu_secondary_polytope(A, supp_mu):
    triangs = PointConfiguration(A, regular = true).triangulations_list()
    phi = []
    for T in triangs:
        phi.append(phiT_mu(T, A, supp_mu))
    return Polyhedron(phi)
```