STEP-BY-STEP DETERMINATION OF ICE ACCRETION RATES FOR AIRCRAFT

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INTRODUCTION

Icing of aircraft was not the subject of exhaustive study until the recent war. Although some qualitative analyses were made in the early thirties, e. g., NACA Report No. 403, quantitative examination was delayed until the early forties. About this time the necessity of designing military aircraft for all-weather operation became obvious, and studies which would permit accurate design of aircraft meeting such requirements were initiated and advanced at a feverish pace. The rapidity with which these studies were made and reported resulted in rather inconclusive explanations of the methods employed. In addition, the various authors in this country and in England did not employ any consistent system of symbols, and this inconsistency makes intelligent review of the existing literature difficult.

The author's purpose in preparing this manuscript is to present a coherent outline of the methods employed in a quantitative determination of aircraft icing. It is hoped that this outline will provide the icing analyst with a logically developed derivation of many of the tables, equations, and graphs appearing in such papers as NACA Report No. 4706-5 (10) and AAF Technical Report 5418 (11).

NOMENCLATURE

Symbols

A = coefficient in series solution of a differential equation (Dimensionless)

B = x/K (Dimensionless)

C = diameter of cylinder which is aerodynamically equivalent to a given airfoil over the region of moisture accretion (L)

C_p = specific heat of dry air at constant pressure ($L^2/T^2\theta$)

D = doublet strength (L^3/T)

F = force (ML/T^2)

J = mechanical equivalent of heat (Dimensionless)

K = λ_s/C (Dimensionless)

L = latent heat of vaporization (L^2/T^2)

M = molecular weight (M)

R = Reynolds' number (Dimensionless)

or

R = radius vector in the XY-plane (L)

T = time (T)

U = velocity of fluid stream relative to the equivalent cylinder (L/T). In particular, $-U_s$ is the velocity of the free stream.

V = velocity of a droplet relative to the equivalent cylinder (L/T)

W = velocity of droplet relative to its surrounding fluid stream (L/T)

X = coordinate axis which is a perpendicular intersector of the equivalent cylinder's axis of symmetry and which is parallel to airspeed, U_s (L)

Y = coordinate axis which is the perpendicular intersector of both coordinate axis, X, and the axis of symmetry of the equivalent cylinder (L)

a = radius of droplet (L)

b = v_x/ξ (Dimensionless)

d = differential operator (Dimensionless)

e = 2.71828 ... (Dimensionless)

g = standard acceleration due to gravity (L/T^2)

k = a coefficient of transfer (Dimensionless)

m = mass of droplet (M)

p = pressure (M/LT^2)

q = value of x at which step-by-step trajectory calculations are initiated

r = radius vector in the xy-plane (L)

t = time in units of U_s/C (Dimensionless)

u = stream velocity, U, in units of U_s and with the x-component reversed (Dimensionless)

v = droplet velocity, V, in units of U_s and with the x-component reversed (Dimensionless)

w = droplet velocity, W, in units of U_s and with the x-component reversed (Dimensionless)

x = X-coordinate in units of cylinder radius, C, (Dimensionless)

y = Y-coordinate in units of cylinder radius, C, (Dimensionless)
In particular, $y = y_s$ when $x = \infty$

Δ = finite difference (Dimensionless)

Φ = a function (Dimensionless)

Ψ = stream function (L^2/T)

α = clockwise angle between negative x-axis and u (Dimensionless)

γ = clockwise angle between negative X-axis and W (Dimensionless)

θ = counter-clockwise angle between positive x-axis and radius vector to point of droplet impact on equivalent cylinder. (Dimensionless)

λ = extreme range of droplet having initial velocity, U_s, (L). In particular λ_s is this extreme range, assuming Stokes' law valid.

μ = coefficient of viscosity (M/LT)

ξ = x-1 (Dimensionless)

π = 3.14159 ... (Dimensionless)

ρ = density (M/L^3)

τ = temperature (Θ)

φ = $R_U{}^2/K$ (Dimensionless)

Subscripts

D = aerodynamic drag

I = inertia

M = maximum

X = X-component

Y = Y-component

a = dry air or gaseous constituents of fluid stream

d = droplet

h = heat transfer

n = nth term or nth step

o = stagnation point

q = value when x = q

s = ambient air or undisturbed stream

v = vapor

x = x-component

y = y-component

0, L, 1, 2, ... = power of variable in terms of series whose coefficients are denoted by A

Abbreviations

Ei = Exponential integral

Pr = Prandtl number (Dimensionless)

ANALYSIS

Necessary Meteorological Conditions for Icing

The two types of cloud formations which, under certain conditions, present severe icing hazards are cumulus and stratus, the latter type usually in the formative stage. The presence of water clouds above the freezing level, i. e., the altitude at which atmospheric temperature is 32° F., is a strong indication of supercooling with attendant icing hazards in that region.

Water clouds can form within a wide temperature range and are especially likely to form in air containing condensation nuclei and lacking sublimation nuclei, i. e., in air containing liquid water particles or foreign particles conducive to the formation of water droplets, but devoid of ice crystals or foreign particles conducive to the formation of ice particles. If sublimation nuclei are absent, water clouds may form at temperatures considerably below 32° F.; however, supercooling seldom exceeds 30° F., because the equilibrium of the liquid becomes so unstable at lower temperatures that the most minute disturbance will initiate freezing (1, 2). Once crystallization is initiated, the process continues rapidly until only ice crystals are present or until the temperature of the cloud mass rises to the freezing point.

As sublimation continues from the vapor to the solid state within the cloud, the larger ice crystals precipitate. If the temperature gradient or lapse rate is not inverted, i. e., if atmospheric temperature decreases with increasing altitude, these larger ice crystals or snow flakes form initially in the upper levels of the cloud and fall through the underlying levels of moisture particles. The resultant cooling of the underlying cloud causes rain. At the same time, the upper level of the supercooled droplets within the cloud falls quickly to the freezing level, leaving only ice crystals in that portion of the cloud which is above this altitude.

While the presence of rain from a cloud may, for the reason outlined above, indicate a lowering of the region of water droplets, it does not mean that the region above the freezing level is without icing hazards. The upper level of the ice crystal cloud is frequently capped by water clouds of the alto-cumulus type, because the air at this level has become devoid of sublimation nuclei, and newly initiated upward movements of this air mass results only in the production of supercooled water clouds. It is therefore never safe to assume that icing conditions are totally absent above the freezing level, even though precipitation from the cloud may be heavy (1).

With some experience in observation, ice and rain clouds can be distinguished by characteristic differences in their appearance. It is, with this advantage, even possible to determine the parts of a cloud which contain ice crystals as distinct from those parts containing water vapor, <u>provided the cloud can be clearly observed</u>.

Ice clouds are optically less distinct than water clouds, because the ice crystals of the ice cloud are fewer and larger than the water droplets of the water cloud. Ice clouds are sometimes similar to a heavy mist in appearance; their borders are hazy because the larger particles drop more rapidly and sublimate or melt and evaporate more slowly after falling from the body of the cloud than do the much smaller water drops (1). A parhelion is often observable when the sun's rays fall on an ice cloud and are reflected to the observer. This phenomenon is most frequently observed from aircraft above the cloud level.

The effect of kinetic heating (heating due to impact of the air stream at the leading edges of an immersed body) is to raise the level of aircraft icing above the freezing level of the atmosphere. Any attempt to utilize this advantage in clearing topographical features extending above the freezing level would be disastrous at ordinary speeds, for the temperature rise due to kinetic heating is greatly diminished when the leading edges of exposed surfaces are wet (3). For example, assume an airspeed of 250 miles per hour and an altitude of 10,000 feet above sea level in an atmosphere conforming to the NACA standard atmosphere. From the latter condition, the temperature of the ambient air, τ_s, is 23.3° F, the barometric pressure of the ambient air, p_s, is 698 mb, and the density of the ambient air, ρ_s, is $9.07(10^{-4})$ gm/cm^3. Assuming laminar flow at stagnation, the temperature rise in clear air would be (4):

$$\Delta\tau_o = \frac{U_s^{\;2}}{2gJC_p}\,(Pr)^{\frac{1}{2}} \tag{1}$$

where $\Delta\tau_o$ is the temperature rise at the stagnation point, U_s is the airspeed, g is the gravitational constant, J is the mechanical equivalent of heat, C_p is the specific heat of dry air, and Pr is the Prandtl number. Substitution of the appropriate values in the above equation gives $\Delta T_o = 9.53^{\circ}$ F. Now, applying equation 12 of the NACA Advance Restricted Report No. 5G13 (3) with the assumption that $p_s \gg p_{v_o}$ and, also, $p_s \gg p_{v_s}$:

$$T_s - T_o + \Delta T_o = \frac{k_v}{k_h}\,\frac{M_v}{M_a}\,\frac{p_{v_o} - p_{v_s}}{p_s}\,\frac{L_v}{C_p} \tag{2}$$

where T_o is the temperature at stagnation on the wet leading edge, p_{v_o} is the vapor pressure of water at the stagnation temperature, p_{v_s} is the vapor pressure of water at the ambient air temperature, M_v is the molecular weight of the water, M_a is the molecular weight of the air, L_v is the latent heat of vaporization for water at stagnation temperature, k_v is a dimensionless coefficient of evaporation for water, and k_h is a dimensionless coefficient of heat transfer. Again, according to Hardy (3), at the temperatures under consideration, the ratio k_v/k_h is equal to unity. Solving the preceding equation for stagnation temperature by trial and error: $T_o = 28.4^{\circ}$ F.

It is thus apparent that a flight into an icing cloud would result in a lowering of the stagnation temperature from an apparently safe value of 32.8° F. to a hazardous value of 28.4° F.

Although it was formerly supposed that high liquid-water content was incompatible with the presence of supercooled rain and that the icing hazard incident to flights through such a rain was therefore small, more recent investigations have shown that high liquid-water content and supercooled rain are by no means incompatible in the presence of strong upward air currents (2). Icing due to flight through supercooled rain may always be expected to cover virtually the entire area of the aircraft as projected on a plane normal to the flight path. This result may be expected because of the large inertia of the rain drop as compared to the inertia of the average water droplet present in clouds from which rain is not falling.

Aircraft occasionally encounter regions of supersaturated air at temperatures below 32° F. and in which neither condensation nor sublimation nuclei are present in concentrations sufficient to permit the formation of clouds. Flight through such a region may result in light ice accretion, because the atmospheric disturbance caused by the aircraft is sometimes sufficient to promote condensation of the water vapor upstream from certain of the exposed aircraft surfaces. The droplets resulting from such condensation may then impinge upon these surfaces and freeze, forming a small amount of ice. This particular cause of icing is of little practical interest, as it presents no serious icing hazard (1).

The qualitative discussion of icing given in the preceding paragraphs is insufficient as a basis for anti-icing design or analysis. A quantitative discussion of (1) the amount of moisture present in several forms of clouds where icing conditions may exist and (2) of the dimensions of the moisture particles they may contain follows.

The weight of liquid water present in a unit volume of cloud is extremely variable. One German authority reported water contents ranging from 0.1 to 0.6 grams per cubic meter in stratus type clouds, from 0.1 to 1.5 grams per cubic meter in strato-cumulus type clouds, and from 0.3 to 3.0 grams per cubic meter in cumulus type clouds (4).

According to Brock (4):

> The maximum liquid-water contents to be expected in clouds under icing conditions should be less than 3.5 g per cu. m, and present measurements indicate that the maximum (including freezing rain) values are more nearly 2. The effective droplet sizes are expected to vary from 5 to 25 microns radius with larger values expected in drizzles and rain.

Experimental measurements indicate that the actual liquid-water content of cumulus clouds rarely approaches that predicted by calculations based on adiabatic lifting above the cloud base; however, such calculations give values that represent a maximum for clouds of this type (5). Experimental measurements of the liquid-water content of stratus clouds indicate much lower values than that predicted by adiabatic lifting of the air mass above the cloud base.

Mixing of the moist air with the surrounding air takes place at the boundaries of the rising mass. This process tends to reduce the liquid-water content at any given level above the cloud base to some value below that which is predicted when the wet adiabatic lapse rate or temperature gradient is multiplied by the increment of altitude between cloud base and given level. In addition, there is generally a lag in condensation as the cloud mass rises, for sufficient condensation nuclei for immediate condensation are rarely present. It is then apparent that the assumption of adiabatic lifting of the cloud leads to calculated values of the liquid-water content of the cloud which are not likely to be approached in any actual case.

How to estimate the average droplet diameter is a difficult problem, for no theoretical analysis is applicable. Experiment indicates that both mean droplet diameter and maximum droplet diameter depend upon temperature, liquid-water content and type of cloud. But this dependence is not clearly defined. The probable duration of icing conditions, the probable amount of liquid-water content, and the probable mean-effective droplet diameter are functions of the type of cloud. A loose relationship between these icing parameters has been recommended in NACA Report No. 4706-5 (5) as a satisfactory criterion for anti-icing design of all-weather aircraft:

Table 1. Icing conditions which may be encountered by aircraft.

Cloud type	Duration	Liquid-water content (gm/m^3)	Mean droplet diameter (microns)	Free-air temperature (^oF)
		Most probable maximum		
Cumulus	1 minute	2.0	20	0
Stratus or stratocumulus	Continuous	0.8	15	20
		0.5	25	20
		0.5	15	0
		0.25	15	-20
		Typical or normal		
Cumulus	1 minute	0.8	10	0 to 20
			to	
Stratus or stratocumulus	Continuous	0.3	17	10 to 25

Approximate Velocity Distribution of the Air Stream in the Vicinity of an Airfoil

To estimate accretion rates at the leading edge of an airfoil, it is convenient to describe airflow characteristics in this region analytically.

Certain approximations facilitate the determination of such analytical descriptions: First, the air is assumed to behave ideally in the region under investigation. This assumption is justified within an arc of 60 degrees above or below the stagnation point (measured along the leading edge circle) as there is no danger of boundary layer separation in this zone when the airspeed is low enough to make a study of icing hazards necessary. Second, the leading edge of the airfoil is frequently replaced by the equivalent cylinder which best matches the leading edge curvature in the region of accretion. Although the flow pattern of the equivalent cylinder approximates that of the airfoil for this analysis, accretion rates for cylinder and airfoil may differ appreciably. Bergrun (10) has compared accretion rates for a Joukowski airfoil and an equivalent cylinder; the results of this comparison indicate poor agreement when droplet diameters are large. Nevertheless, the equivalent cylinder approximation has been widely used (2, 7). Since a check of step-by-step accuracy is available from an analysis which employed a differential analyzer (11), an equivalent cylinder will be used in the following accretion calculations.

The method of step-by-step calculation may be applied in the case of a Joukowski section. It is only necessary that the stream function* corresponding to that section be applied to the analysis.

Flow of an ideal fluid about a right circular cylinder whose axis of symmetry is at right angles to the direction of fluid flow is represented analytically by the superposition of a uniform stream velocity on a two-dimensional doublet in the same plane (12, 13). For convenience in analysis, the uniform stream velocity is assumed to be parallel to the horizontal coordinate axis and in the negative direction. The axis of the doublet is also assumed to be the horizontal coordinate axis. Then the stream function is;

$$\Psi = -U_s Y + \frac{D}{2\pi}\frac{Y}{X^2 + Y^2} = -U_s Y + \frac{D}{2\pi}\frac{Y}{R^2} .$$

In this equation, Ψ is the stream function, π is the well-known ratio 3.1416, D is the strength of the doublet, U_s is the airspeed or $-U_s$ is the undisturbed stream velocity, X is the distance parallel to U_s, Y is the coordinate distance perpendicular to U_s, and R is the distance from the origin in the XY-plane.

Now, when $\Psi = 0$, $R = C$, where C is the radius of the equivalent cylinder, and $Y \neq 0$, $D = 2\pi U_s C^2$. The point $X = C$, $Y = 0$ corresponds to the stagnation point on the cylinder. Then by substitution,

$$\Psi = -U_s Y \left[1 - \frac{C^2}{X^2 + Y^2} \right].$$

*Along a stream line, Ψ is constant and is therefore called the stream function.

The X and Y components of the stream velocity U can now be found by the conventional method:

$$U_X = \frac{\partial\Psi}{\partial Y} = -U_s\left(1-C^2\frac{X^2 - Y^2}{(X^2+Y^2)^2}\right)$$

$$U_Y = -\frac{\partial\Psi}{\partial X} = U_s C^2 \frac{2XY}{(X^2+Y^2)^2} .$$

In order to reduce these equations to dimensionless form, the velocity components are measured in units of airspeed while the coordinates are measured in units of cylinder radius. If, in addition, the dimensionless horizontal velocity component is assumed positive in the negative direction of the dimensionless horizontal coordinate axis (for later convenience), $u_x = -U_X/U_s$, $u_y = U_Y/U_s$, $x = X/C$, $y = Y/C$, and $r = R/C$.

Then by substitution,

$$u_x = 1 - \frac{x^2-y^2}{(x^2+y^2)^2} = 1 - \frac{x^2-y^2}{r^4} \qquad (3)$$

$$\text{and } u_y = \frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{r^4} \qquad (4)$$

To facilitate rapid step-by-step calculation, u_x and u_y were plotted against upstream distance from the axis of the cylinder for values of y from 0 to 1.0 selected at intervals of 0.1. The resulting graphs constitute Figs. 1 and 2, respectively, while the corresponding tabulated values of these velocity components constitue the Appendix. Although these graphs of u_x and u_y are satisfactory, curves for selected values of these quantities plotted on xy-coordinates would expedite interpolation. Dimensionless stream velocity, u, and the angle it makes with the negative x-axis, α, also were calculated for possible later use. These quantities are defined by the equations,

$$\alpha = \tan^{-1}(u_y/u_x) \text{ and } u = u_x \sec\alpha = u_y \csc\alpha.$$

Although it is relatively easy to determine the characteristics of the stream in the vicinity of the cylinder by plotting the desired quantities, e. g., u_x and u_y, this method is impractical in larger regions of interest. Fortunately, the region of interest beyond that already plotted is included between $4 < x \leqq \infty$ and $0 \leqq y \leqq 1$. And in this region u_x and u_y may be reduced to series form by division. From the equation

$$u_x = 1 - \frac{x^2-y^2}{(x^2+y^2)^2} = 1 - \frac{1}{x^2}\,\frac{1 - y^2/x^2}{(1 + y^2/x^2)^2}$$

$$u_x = 1 - 1/x^2 + 3y^2/x^4 - \dots \quad . \tag{5}$$

And from the equation

$$u_y = \frac{2xy}{(x^2+y^2)^2} = \frac{2}{x^2}\,\frac{y/x}{(1+y^2/x^2)^2}$$

$$u_y = 2y/x^3 - 4y^3/x^5 + 6y^5/x^7 - \dots \quad . \tag{6}$$

When $x > 4$ and $0 \leqq y \leqq 1$ it is apparent that termination of the series for u_x and u_y with the last terms shown above will result in a maximum error in u_x of less than .15 per cent and, in u_y of less than .05 per cent.

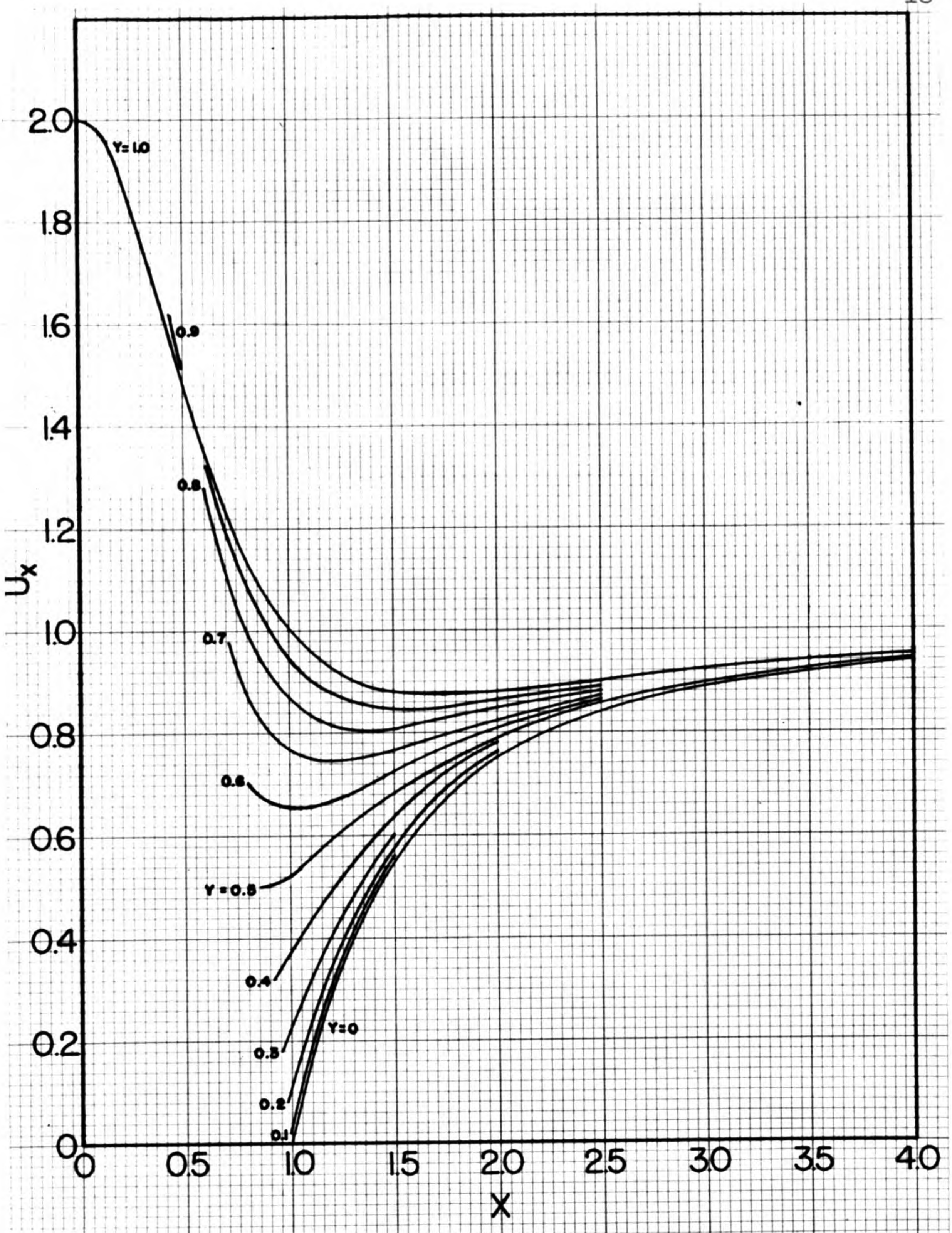

FIGURE I. HORIZONTAL VELOCITY COMPONENT VS UPSTREAM POSITION FROM CYLINDER.

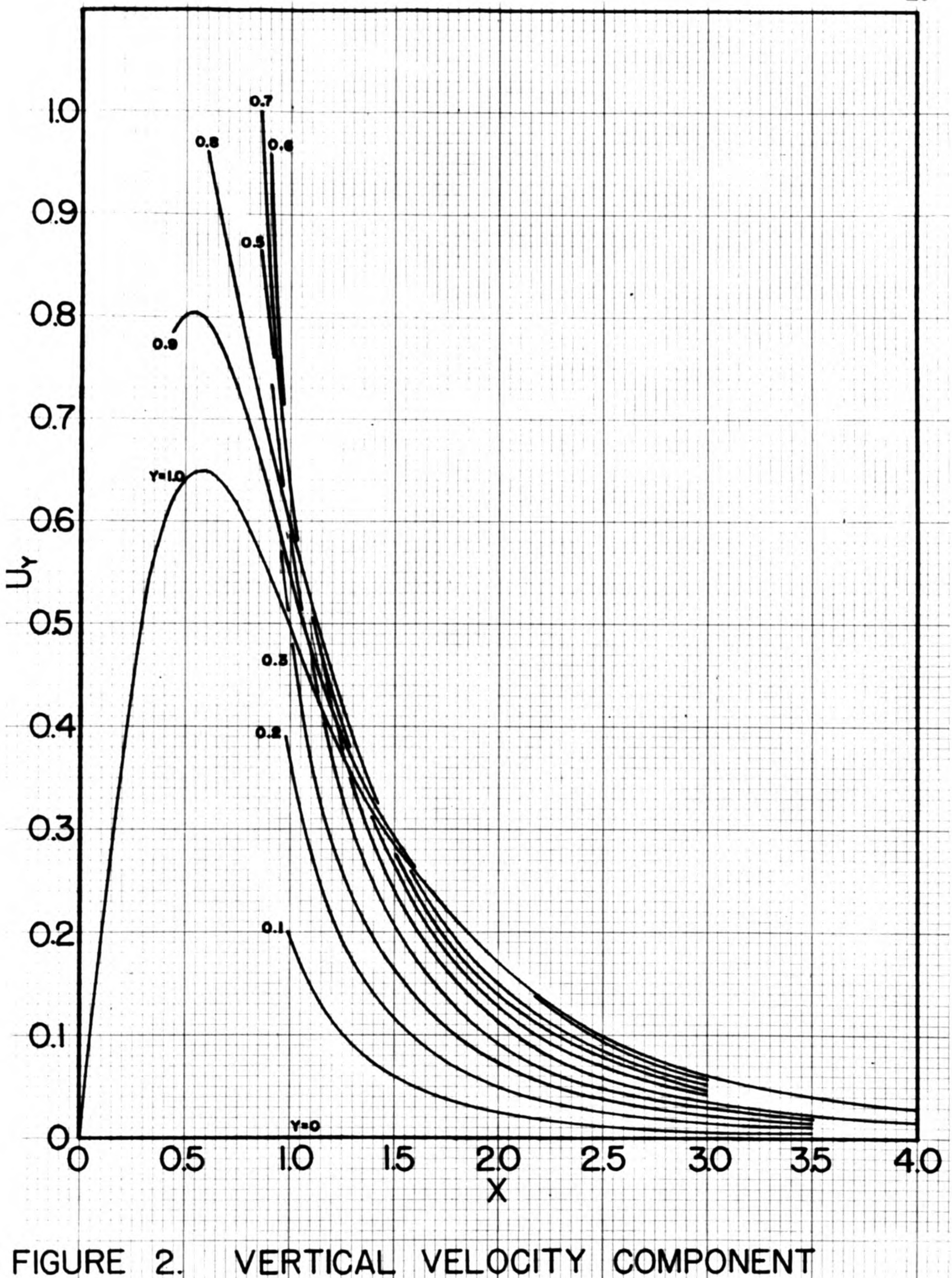

FIGURE 2. VERTICAL VELOCITY COMPONENT VS UPSTREAM POSITION FROM CYLINDER.

Behaviour of a Sphere Immersed in a Fluid

Before the trajectories of droplets suspended in a fluid stream can be determined, it is necessary to describe the behaviour of such a droplet as it moves through a motionless body of the immersing fluid. First attempts (14) to calculate accretion rates assumed the droplet Reynolds' number to be so low as to allow application of Stokes' law without empirical correction; however, more recent calculations have recognized the fact that Stokes' law cannot be directly applied to the determination of droplet trajectories when the velocity of the droplet relative to the immersing fluid is more than infinitesimal.

If Stokes' law is invalid, the force acting on the droplet can only be evaluated when the droplet drag coefficient is a known fuction of the droplet Reynolds' number. By definition,

$$R = 2a\rho_a W/\mu_a \tag{7}$$

and in particular, when $W = U_s$,

$$R_U = 2a\rho_a U_s/\mu_a \tag{8}$$

where a is the radius of the droplet, W is the velocity of the droplet relative to the stream, μ_a is the coefficient of viscosity for the immersing fluid, ρ_a is the density of the immersing fluid, and R is the droplet Reynolds' number.

Mathematically, Stokes' law may be expressed as

$$F_D = -6\pi\mu_a aW. \tag{9}$$

In the above equation F_D is the force imposed on the droplet by aerodynamic drag. When this force is expressed in terms of the coefficient of drag,

$$F_D = -\tfrac{1}{2} C_D \rho_a \pi a^2 W^2 \qquad (10)$$

where C_D is the coefficient of drag. Then when Stokes' law is valid,

$$6\pi\mu_a = \tfrac{1}{2} C_D \rho_a a W \pi \quad \text{or} \quad (C_D/12)(a\rho_a W/\mu_a) = 1.$$

By substitution, $C_D R/24 = 1$ under these conditions. At higher Reynolds' numbers the actual aerodynamic drag on the droplet exceeds that predicted by Stokes' law and, consequently, $C_D R/24 > 1$.

When Stokes' law applies it is relatively easy to calculate the extreme range or maximum travel of a droplet possessing a given initial velocity relative to a stagnant immersing fluid. Applying d'Alembert's principle,

$$F_D + F_I = 0.$$

In this equation F_D is, as previously defined, the force imposed on the droplet by aerodynamic drag while F_I is the kinetic reaction imposed on the droplet when it is accelerated. Since $F_I = -m\, dW/dT$ and $F_D = -6\pi\mu_a aW$, by substitution,

$$m\, dW/dT + 6\pi\mu_a aW = 0$$

or

$$mW\, dW/dS = -6\pi\mu_a aW.$$

The mass of the droplet, m, may be replaced by the product of droplet volume by the density of the liquid of which the droplet is composed, ρ_d. In the previous equations, T is time while S is distance measured along the droplet trajectory. Solving the preceding equation of kinetic equilibrium for dS and substituting the equivalent expression $(4/3)\rho_d \pi a^3$ for m,

$$dS = -\frac{2}{9} \rho_d a^2 \, dW/\mu_a .$$

If λ_s is defined as the extreme range or maximum travel of a droplet whose initial velocity relative to the stagnant immersing fluid is U_s,

$$\int_0^{\lambda_s} dS = -\frac{2}{9} \rho_d a^2 /\mu_a \int_{U_s}^{0} dW .$$

Performing the indicated integrations,

$$\lambda_s = (2/9)\rho_d a^2 U_s/\mu_a \tag{11}$$

When Stokes' law is invalid, a calculation of the extreme range of a droplet having an initial velocity U_s is more difficult. In this case the aerodynamic drag, F_D, may be expressed as the product of Stokes' drag by the correction factor $C_D R/24$. Then the equation of kinetic equilibrium becomes

$$m \, dW/dT + 6\pi\mu_a a W \, (C_D R/24) = 0.$$

But $W = \mu_a R/2a\rho_a = dS/dT$. Again, by substitution,

$$\frac{4}{3}\rho_d \pi a^3 \; \frac{\mu_a}{2a\rho_a} \frac{dR}{dT} = -6\pi\mu_a a \; dS/dT \; (C_D R/24) \quad .$$

If λ is the extreme range under these conditions,

$$\lambda = -(2/9)\,\frac{\rho_d a^2}{\mu_a}\,\frac{\mu_a}{2a\rho_a}\int_{R_U}^{0}\frac{dR}{(C_D R/24)} \tag{12}$$

Then the ratio of actual extreme range to that predicted by Stokes' law is

$$\lambda/\lambda_s = \frac{\frac{2}{9}\,\frac{\rho_d a^2}{\mu_a}\,\frac{\mu_a}{2a\rho_a}\int_{0}^{R_U}\frac{dR}{(C_D R/24)}}{(2/9)\rho_d a^2 U_s/\mu_a}$$

and this reduces to

$$\lambda/\lambda_s = \frac{1}{R_U}\int_{0}^{R_U}\frac{dR}{(C_D R/24)}\quad . \tag{13}$$

Since C_D is an empirical function of R, equation 13 cannot be integrated directly. The calculation of the ratio, λ/λ_s, is therefore tedious. A graph of $C_D R/24$ and λ/λ_s as functions of R constitutes Fig. 3 of this thesis; data for its construction were obtained from AAF Technical Report 5418 (11).

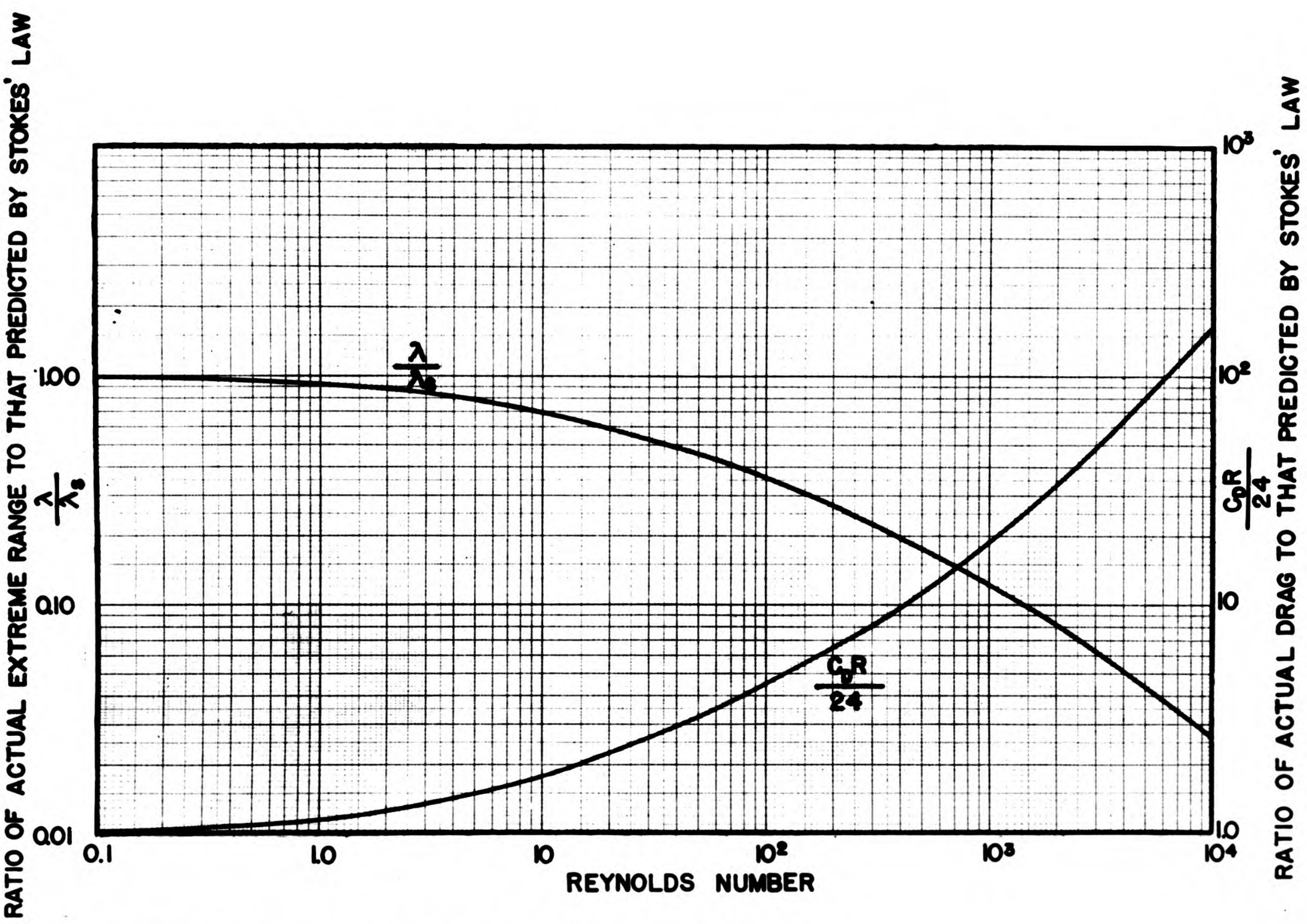

FIGURE 3. $C_D R/24$ AND λ/λ_s VS R.

Kinetics of a Droplet Immersed in an Accelerating Fluid

If d'Alembert's principle is applied to a droplet which is immersed in an accelerating fluid,

$$F_{I_X} + F_{D_X} = 0 .$$

The X-component of the droplet's kinetic reaction is, as universally defined,

$$F_{I_X} = -m\, dV_X/dT$$

and, on substitution for m,

$$F_{I_X} = -\frac{4}{3}\rho_d \pi a^3\, dV_X/dT.$$

The X-component of the drag exerted on the droplet as it moves relative to the immersing fluid may be expressed by the equation,

$$F_{D_X} = -\tfrac{1}{2}\, C_D \rho_a (\pi a^2)\, W^2 \cos\gamma$$

where γ is the angle between the relative velocity, W, and the X-axis.

Now by substitution for F_{I_X} and F_{D_X} in the equation of kinetic equilibrium,

$$-\frac{4}{3}\rho_d \pi a^2\, dV_X/dT - \tfrac{1}{2}\, C_D \rho_a\, (\pi a^2)\, W^2 \cos\gamma = 0.$$

Then by transposition,

$$dV_X/dT = -\frac{3}{8}\,\frac{1}{a}\, C_D\, \frac{\rho_a}{\rho_d}\, W W_X \qquad (14)$$

and, similarly,

$$dV_Y/dT = -\frac{3}{8}\,\frac{1}{a}\, C_D\, \frac{\rho_a}{\rho_d}\, W W_Y \qquad (15)$$

The product of the droplet velocity component by the derivative of that velocity component with respect to distance along the corresponding axis may be substituted for the derivative of the droplet velocity component with respect to time. Thus, for a given droplet, d,

$$\frac{dV_X}{dT} = V_X \frac{dV_X}{dX} \text{ and } \frac{dV_Y}{dT} = V_Y \frac{dV_Y}{dY}$$

or, in dimensionless terms,

$$\frac{dv_x}{dt} = -v_x \frac{dv_x}{dx} \quad \text{and} \quad \frac{dv_y}{dt} = v_y \frac{dv_y}{dy} \ .$$

In these equations V is the droplet velocity while v is the droplet velocity in units of U_s with the sign of the component in the x-direction changed.

The previously expressed extreme range of the droplet, as calculated by Stokes' law, is now employed in the reduction of the equations of kinetic equilibrium to dimensionless form. Multiplication of equation 14 by equation 11 gives

$$\lambda_s \ dV_X/dT = - \frac{3}{8}\frac{1}{a} C_D \left[\frac{2}{9} \frac{\rho_d \ a^2 U_s}{\mu_a} \right] \frac{\rho_a}{\rho_d} WW_X \ .$$

If the parameter, K, is defined as the quotient of λ_s by C, and if the appropriate substitutions are made for V and W, the above equation reduces to

$$CK \ U_s \frac{dv_x}{dT} = - \frac{C_D}{24} \left[\frac{2a\rho_a \ W}{\mu_a} \right] U_s^{\ 2} \ (v_x - u_x)$$

or

$$\frac{dv_x}{dt} = - \frac{C_D R}{24K} (v_x - u_x)$$

where $t = U_s T/C$ is a dimensionless measure of time. By substitution for the term on the left side of this equation,

$$v_x \frac{dv_x}{dx} = \frac{C_D R}{24K}(v_x - u_x).$$

Similarly, the kinetic equilibrium of the droplet in the y-direction is

$$\frac{dv_y}{dt} = v_y \frac{dv_y}{dy} = -\frac{C_D R}{24K}(v_y - u_y).$$

Reduced to its simplest form, the quotient of R by R_U is

$$\frac{R}{R_U} = \frac{W}{U_s}.$$

When the above equation is squared and $W_X^2 + W_Y^2$ is substituted for W^2,

$$\left[\frac{R}{R_U}\right]^2 = \left[\frac{W_X}{U_s}\right]^2 + \left[\frac{W_Y}{U_s}\right]^2$$

or

$$\left[\frac{R}{R_U}\right]^2 = (v_x - u_x)^2 + (v_y - u_y)^2$$

Recapitulating:

$$\frac{dv_x}{dt} = -\frac{C_D R}{24K}(v_x - u_x) \text{ or } v_x \frac{dv_x}{dx} = \frac{C_D R}{24K}(v_x - u_x) \tag{16}$$

$$\frac{dv_y}{dt} = -\frac{C_D R}{24K}(v_y - u_y) \text{ or } v_y \frac{dv_y}{dy} = -\frac{C_D R}{24K}(v_y - u_y) \tag{17}$$

$$\left[\frac{R}{R_U}\right]^2 = (v_x - u_x)^2 + (v_y - u_y)^2 \tag{18}$$

The coefficient of drag for the droplet is analytically defined only when creeping motion exists. Thus, at greater relative velocities step-by-step calculation of droplet velocity and displacement components is necessary. This method of analysis is relatively simple if the components of droplet acceleration may be assumed linear during some small increment of time. By this assumption the components of droplet velocity and droplet displacement are

$$v_{x_{n+1}} = v_{x_n} - \frac{C_D R}{24K} (v_{x_n} - u_{x_n}) \Delta t \qquad (19)$$

$$v_{y_{n+1}} = v_{y_n} - \frac{C_D R}{24K} (v_{y_n} - u_{y_n}) \Delta t \qquad (20)$$

$$x_{n+1} = x_n - v_{x_n} \Delta t + \frac{1}{2} \frac{C_D R}{24K} (v_{x_n} - u_{x_n}) (\Delta t)^2 \qquad (21)$$

and

$$y_{n+1} = y_n + v_{y_n} \Delta t - \frac{1}{2} \frac{C_D R}{24K} (v_{y_n} - u_{y_n})(\Delta t)^2 \qquad (22)$$

when time is chosen as the independent variable.

Again, if the components of droplet acceleration may be assumed linear during some small increment of distance, the components of droplet velocity and droplet displacement are

$$v_{x_{n+1}} = v_{x_n} + \frac{C_D R}{24K} (v_{x_n} - u_{x_n}) \frac{\Delta x}{v_{x_n}} \qquad (23)$$

$$v_{y_{n+1}} = v_{y_n} + \frac{C_D R}{24K} (v_{y_n} - u_{y_n}) \frac{\Delta x}{v_{x_n}} \qquad (24)$$

$$x_{n+1} = x_n + \Delta x \qquad (25)$$

and

$$y_{n+1} = y_n - \frac{v_{y_n}}{v_{x_n}} \Delta x - \frac{1}{2} \frac{C_D R}{24K} (v_{y_n} - u_{y_n}) \left[\frac{\Delta x}{v_{x_n}}\right]^2 \qquad (26)$$

when x is chosen as the independent variable.

An Explanation of the Inverse Problem

The present discussion is devoted to the solution of the accretion problem when free stream velocity, equivalent cylinder radius and equivalent droplet radius are known. The free stream Reynolds' number, R_U, and the dimensionless constant, K, may thus be calculated directly, and the equations of motion may then be solved.

A second problem presents itself when the free stream velocity, the equivalent cylinder radius, and the rate of accretion are known, and the droplet effective diameter is required. Since the droplet radius is not known it is necessary to eliminate this quantity from the equations of motion; this may be done if the square of free stream Reynolds' number is divided by the dimenless constant, K. The resulting dimensionless constant, ϕ, is then given by the formula

$$\phi = R_u^{\,2}/K = 18\rho_a^{\,2} \quad CU_s/\mu_a\rho_d \ .$$

This second type of problem has been solved indirectly (11); however, the scope of this thesis does not include a step-by-step analysis for this case.

The Effect of K on Accretion

Consider the trajectory of a droplet in the immediate vicinity of the stagnation point. A translation of the coordinates to the stagnation point may be effected by substituting ξ+1 for x. Then from equations 5 and 6, setting x = ξ+ 1 and ignoring those terms which are negligible when ξ and y are small, the horizontal and vertical velocity components of the stream are, very nearly, $u_x = 2\xi$ and $u_y = 2y$.

If the droplets are not too large nor the free stream velocity too high, Stokes' law holds and the equation of kinetic equilibrium in the horizontal direction is

$$v_x \frac{dv_x}{d\xi} = \frac{1}{K}(v_x - 2\xi) \quad .$$

This is an ordinary first order homogeneous equation, integrable by the substitution, $v_x = b\xi$. By this transformation, $d\xi/\xi = -b\,db/(b^2 - b/K + 2/K)$.

At some small distance, ξ_o, ahead of the stagnation point, a droplet approaching the airfoil along the streamline $\Psi = 0$ will have a velocity v_{x_o} for which the corresponding value of b is b_o. For the droplet to strike the cylinder, ξ must have no positive real value when $b = o$ for if $\xi > o$ when $v_x = o$ the droplet can never reach the cylinder. Now

$$\xi = \xi_o e^{-\int_{b_o}^{0} b\,db/(b^2 - b/K + 2/K)}.$$

When $K < 1/8$ the denominator of the integrand has real roots and $\xi > o$; when $K = 1/8$ the denominator of the integrand is zero and $\xi = o$, indicating that the droplet reaches the stagnation point but with zero velocity; when $K > 1/8$ the denominator of the integrand has complex roots and ξ has no real positive value for which $b = o$. This latter case indicates that the droplets have a finite velocity at the stagnation point, and under this condition some moisture accretion must result.

Initiation of Step-by-Step Calculations

Through the use of previously presented equations and Figs. 1, 2, and 3, the path may be determined for any droplet in the region outside the cylinder and bounded by $(0 \leqq x \leqq 4,\ 0 \leqq y \leqq 1)$. But it is usually required that the path be determined for some particle whose ordinate at $x = \infty$ is $0 \leqq y_s \leqq 1$.

It is impractical to plot u_x and u_y in the region $(4 < x \leqq \infty,\ 0 \leqq y \leqq 1)$ as was done in Figs. 1 and 2 for the region $(0 \leqq x \leqq 4,\ 0 \leqq y \leqq 1)$; however, these components of stream velocity have been expressed in series form by equations 4 and 5. Moreover, the accuracy of these expressions was seen to be high for the region presently considered.

Further investigation shows u_x is monotonic decreasing while u_y is monotonic increasing in this region (15). Since droplet acceleration depends upon relative velocity it is apparent that v_x is also monotonic decreasing while v_y is monotonic increasing; moreover, $u_x < v_x < 1$ and $u_y > v_y > 0$ for any given droplet in the region. Again, referring to the series solutions for components of the stream velocity, the minimum value of u_x is, to a close approximation, 0.9375 and the maximum value of u_y is 0.0277. Thus $(v_x - u_x) < 0.0625$ and $|(v_y - u_y)| < 0.0277$ in the region $(4 < x \leqq \infty,\ 0 \leqq y \leqq 1)$.

If the Reynolds' number for the droplet is sufficiently small, $C_D R/24 \approx 1$ and the equations of kinetic equilibrium are, for a given droplet,

$$v_x \frac{dv_x}{dx} = \frac{1}{K}(v_x - u_x) \text{ and } v_x \frac{dv_y}{dx} = \frac{1}{K}(v_y - u_y) \quad .$$

Note that x has been chosen as the independent variable in the equations just stated.

Substituting for u_x, the equation for kinetic equilibrium in the x-direction is

$$v_x \frac{dv_x}{dx} = \frac{1}{K}\left[v_x - \left(1 - \frac{1}{x^2} + 3\frac{y^2}{x^4} - \dots\right)\right] .$$

Assuming a series solution for this x-component of droplet velocity:

$$v_x = A_o + A_L \log x + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \dots \quad .$$

By the usual method of equating coefficients,

$A_o = 1$, $A_L = 0$, $A_1 = 0$, $A_2 = -1$, $A_3 = 2K$, $A_4 = 3y^2 - 6K^2$,

$A_5 = 24K^3 - (12y^2 + 2)K$, $A_6 = 10K^2 - 5y^4$, ...

and substituting,

$$v_x = 1 - \frac{1}{x^2} + \frac{2K}{x^3} + \frac{(3y^2 - 6K^2)}{x^4} + \frac{24K^3 - (12y^2 + 2)K}{x^5} +$$

$$\frac{10K^2 - 5y^4}{x^6} + \dots \quad .$$

Expanding,

$$v_x = 1 - \frac{1}{x^2} + \frac{2}{Bx^2} - \frac{6}{B^2x^2} + \frac{24}{B^3x^2} - \dots\dots + \frac{3y^2}{x^4} - \frac{12y^2}{Bx^4}$$

$$- \frac{2}{Bx^4} + \frac{10}{B^2x^4} - \frac{5y^4}{x^6} + \dots$$

where $B = x/K$.

Neglecting the second set of terms,

$$v_x = 1 - \frac{1}{x^2}\left(1 - \frac{2!}{B} + \frac{3!}{B^2} - \frac{4!}{B^3} + \dots\right),$$

and if

$$\Phi(B) = 1 - \frac{2!}{B} + \frac{3!}{B^2} - \frac{4!}{B^3} + \ldots, \tag{27}$$

then by substitution,

$$v_x = 1 - \frac{1}{x^2}\Phi(B).$$

According to Langmuir and Blodgett (11), $\Phi(B)$ can be expressed in terms of the exponential integral as

$$\Phi(B) = B + B^2 e^B \int_{-B}^{\infty} \frac{dB}{B\ e^B} = B + B^2 e^B \ \mathrm{Ei}\ (-B). \tag{28}$$

By a method similar to that outlined above for v_x when $x = q$, where q is the starting abscissa for step-by-step trajectory calculations,

$$\frac{v_y}{y_q} = \frac{1}{Kq^2}\left[1 - \Phi(B_q)\right] - 4\,\frac{y_q^{\,2}}{q^5}\ .$$

In this equation $y = y_q$ when $x = q$.

To initiate step-by-step calculations, y_q as well as v_x and v_y must be determined for droplets whose ordinate at $x = \infty$ was y_s. Now

$$\frac{\Delta y_q}{y_s} = \int_{y_s}^{y_q} \frac{dy}{y_s} = \frac{1}{y_s}\int_{\infty}^{q} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\ dx,$$

but $\frac{dy}{dt} = v_y$ and $\frac{dx}{dt} = -v_x$.

Hence, by substitution,

$$\frac{\Delta y_q}{y_s} = -\frac{1}{y_s}\int_{\infty}^{q} \frac{v_y}{v_x}\ dx.$$

Since $y_q = y_s + \Delta y_q$,

$$\frac{1}{y_s}\frac{v_y}{v_x}dx = \left(1 + \frac{\Delta y_q}{y_s}\right)\frac{v_y}{y_q}\frac{dx}{v_x} .$$

Again, by substitution,

$$\frac{\Delta y_q}{y_s} = - \left[1+\frac{\Delta y_q}{y_s}\right]\left[\int_\infty^q \frac{1}{Kx^2}\;\frac{1-\Phi(B)}{1-\Phi(B)/x^2}\,dx - 4y_s^2\int_\infty^q \frac{1}{x^5}\;\frac{\left(1+\frac{\Delta y_q}{y_s}\right)^2}{1-\Phi(B)/x^2}\,dx\right].$$

Performing the indicated divisions and resubstituting x/K for B in the integrands gives

$$\int_\infty^q \frac{1}{Kx^2}\;\frac{1-\Phi(B)}{(1-\Phi(B)/x^2)}\,dx \approx \int_\infty^q \left[\frac{2}{x^3} - \frac{6K}{x^4} - \frac{24K^2}{x^5} - \frac{120K^3}{x^6} - \dots\right]dx$$

and

$$\int_\infty^q \frac{1}{x^5}\left[\frac{\left(1+\frac{\Delta y_q}{y_s}\right)^2}{1-\Phi(B)/x^2}\right]dx \approx \int_\infty^q \frac{1}{x^5}\left[1 + \Phi(B)/x^2\right]dx$$

for the first and second integrals, respectively.

When the integrations are performed and the results applied to the equation for $\Delta y_q/y_s$,

$$\frac{\Delta y_q}{y_s} = -\left[1+\frac{\Delta y_q}{y_s}\right]\left[-\frac{1}{q^2}\;\Phi\,(B_q)+\frac{{y_s}^2}{q^4}\right].$$

Then by transposition, $\Delta y_q/y_s$ is, to a close approximation, given by the equation

$$\frac{\Delta y_q}{y_s} = \frac{\Phi\,(B_q)}{q^2} + \frac{1}{q^4}\,(1-{y_s}^2).$$

Finally y_q of the equation for v_y/y_q is replaced by $y_s + \Delta y_q$ and the equation then reduces to the nearly exact form,

$$\frac{v_{y_q}}{y_s} = \left[1 + \Phi(B_q)/q^2\right]\left[\frac{1-\Phi(B_q)}{Kq^2} - \frac{4y_s^{\,2}}{q^5}\right]$$

or

$$\frac{v_{y_q}}{y_s} = \frac{1}{Kq^2}\left[1-\Phi(B_q)\right] + \frac{2}{q^5}\,(1-2y_s^{\,2}).$$

Recapitulating:

$$v_q = 1 - \frac{1}{q^2}\,\Phi(B_q) \tag{29}$$

$$\frac{v_{y_q}}{y_s} = \frac{1}{Kq^2}\left[1 - \Phi(B_q)\right] + \frac{2}{q^5}\,(1-2y_s^{\,2}) \tag{30}$$

and

$$\frac{\Delta y_q}{y_s} = \frac{1}{q^2}\,\Phi(B_q) + \frac{1}{q^4}\,(1-y_s^{\,2}) \ . \tag{31}$$

Evaluation of Physical Constants

Application of the previously developed formulas to the case of an airfoil moving through the earth's atmosphere allows certain physical quantities to be assumed constant. In the usual analysis the airfoil is assumed to move with constant velocity through a meteorologically constant atmosphere.

By this assumption the undisturbed stream velocity and such atmospheric parameters as temperature, pressure, density, humidity, air viscosity, liquid-water content, and water droplet effective diameter are fixed.

When the atmospheric temperature is low enough to create an icing hazard the vapor pressure exerted by the atmospheric moisture is a negligible fraction of atmospheric pressure. Thus, atmospheric pressure may be assumed to be due solely to the gaseous constituents of the atmosphere.

One method of prescribing a meteorologically constant atmosphere assumes the airfoil to move at constant altitude through an atmosphere whose gaseous constituents are described by some standard atmosphere (6). The liquid-water content per unit volume of the atmosphere and the mean droplet diameter are then chosen to provide realistic design conditions.

According to von Mises (6), the viscosity of air is closely approximated by the equation

$$\mu_a = 234.1 \frac{(T_a + 459.4)^{3/2}}{T_a + 682.6} \, 10^{-10} \tag{32}$$

where T_a is the air temperature in degrees Fahrenheit and μ_a is the viscosity of the air in slugs per foot-second.

Step-by-Step Solution of an Example

To demonstrate by a specific example, let it be assumed that an airfoil moves with a constant velocity of 250 miles per hour through NACA standard air (6) at a constant altitude of 10,000 feet above sea level. Furthermore, let it be assumed that the liquid-water content of the atmosphere is 0.8 grams per cubic meter and that the effective droplet diameter of this moisture is 15 microns; these meteorological conditions are typical of stratus clouds. Finally, let it be assumed that the equivalent cylinder has a radius of 3.28 inches.

Summarizing:

U_s = 367.5 ft./sec. μ_a = 3.512(10^{-7}) slugs/ft.-sec.

τ_a = 23.3° F. a = 2.46(10^{-5}) ft.

p_a = 10.11 lb./in.^{2} abs. ρ_d = 1.94 slugs/ft.^{3}

ρ_a = 1.756(10^{-3}) slugs/ft.^{3} C = .2733 ft.

From these data,

$$R_U = 2a\ {}_aU_s/\ {}_a = \frac{4.92(10^{-5})1.756(10^{-3})367.5}{3.512(10^{-7})}$$

$$= 90.4$$

$$\lambda_s = \frac{2}{9}\,\frac{\rho_d a^2 U_s}{\mu_a} = .222\,\frac{(1.94)6.05(10^{-10})367.5}{3.512(10^{-7})}$$

$$= .2733 \text{ ft.},$$

$$K = \lambda_s/C$$

$$= 1.00,$$

and

$$\phi = R_U{}^2/K$$

$$= 8.17(10^3).$$

Initial values of v_x, v_y, and y can be determined by means of equations 29, 30, and 31, respectively. If, in the present example, step-by-step calculations are initiated at x = 4, $\Phi(B_4) = \Phi(4.0)$ by substitution. From "Tables of Functions..." by Jahnke and Emde (16), the value of the exponential integral, Ei(-4.0), is -.003779. Then by equation 28, Φ(B) = .700. Substituting for x and Φ(B) in the previously mentioned equations;

$$v_{x_4} = 1 - .0625(.700) = 1 - .0438 \approx .956,$$

$$v_{y_4}/y_s = .0625(.300) + .00195(1 - 2y_s{}^2) \approx .0207 - .0039y_s{}^2\ ,$$

and

$$\Delta y_4/y_s = .0438 + .0039(1 - y_s^2) \approx .0479 - .0039y_s^2.$$

Values of v_{x_4}, v_{y_4}, and y_4 corresponding to previously stated conditions of the present example are presented in Table 2. The selected initial ordinates are shown in the left column of this table.

Table 2. Droplet velocity components and ordinate at x = 4.

y_s	v_{x_4}	v_{y_4}/y_s	v_{y_4}	$\Delta y_4/y_s$	Δy_4	y_4
0.00	0.956	0.00000	0.000000	0.00000	0.000000	0.0000
0.05	0.956	0.02070	0.001035	0.04770	0.002385	0.0524
0.10	0.956	0.02066	0.002066	0.04786	0.004786	0.1048
0.15	0.956	0.02061	0.003093	0.04781	0.007171	0.1572
0.20	0.956	0.02054	0.004108	0.04774	0.009548	0.2095

Equations 18, 23, 24, 25, and 26 provide a convenient means for step-by-step extension of the droplet trajectories from x = 4 to the immediate vicinity of the cylinder. As a first example of this method, the trajectory of a droplet whose ordinate, y_s, was zero is tabulated in Table 3. From symmetry it is apparent that such a droplet can have no vertical velocity component at any point along its path. For this reason v_y and y are zero throughout the table. At x = 1.00 the value of v_x is seen to be 0.254. By comparison, Fig. 8 of AAF Technical Report 5418 (11) indicates that the droplet of the example problem would strike the stagnation point of the cylinder with a velocity, v_x, of 0.265. Thus, in the present example, finite summation appears to have introduced an error of 4.15 per cent into the velocity with which

Table 3. Tabulated trajectory calculation for a droplet whose ordinate, y_s, = 0.000 at $x = \infty$.

(1) x_n	(2) Δx	(3) x_{n+1}	(4) v_{x_n}	(5) u_{x_n}	(6) (4)-(5)	(7) v_{y_n}	(8) u_{y_n}	(9) (7)-(8)	(10) $(6)^2$	(11) $(9)^2$	(12) $\sqrt{(10)+(11)}$	(13) R	(14) $\frac{C_D R}{24K}$	(15) (2)/(4)	(16) (14)(6)(15)	(17) $v_{x_{n+1}}$	(18) (14)(9)(15)	(19) $v_{y_{n+1}}$	(20) y_n	(21) (7)/(4)	(22) -(21)(2)	(23) $\frac{-(18)(15)}{2}$	(24) y_{n+1}
4.00	-0.50	3.50	.956	.937	.019	0	0	0	-	0	.019	1.718	1.225	-.522	-.0122	.944	0	0	0	0	0	0	0
3.50	-0.50	3.00	.944	.917	.027	0	0	0	-	0	.027	2.44	1.30	-.530	-.0186	.925	0	0	0	0	0	0	0
3.00	-0.50	2.50	.925	.888	.037	0	0	0	-	0	.037	3.34	1.38	-.540	-.0276	.897	0	0	0	0	0	0	0
2.50	-0.25	2.25	.897	.840	.057	0	0	0	-	0	.057	5.15	1.50	-.279	-.0238	.873	0	0	0	0	0	0	0
2.25	-0.25	2.00	.873	.802	.071	0	0	0	-	0	.071	6.41	1.58	-.287	-.0321	.851	0	0	0	0	0	0	0
2.00	-0.20	1.80	.851	.752	.099	0	0	0	-	0	.099	8.95	1.72	-.235	-.0400	.811	0	0	0	0	0	0	0
1.80	-0.20	1.60	.811	.688	.123	0	0	0	-	0	.123	11.11	1.82	-.247	-.0552	.756	0	0	0	0	0	0	0
1.60	-0.10	1.50	.756	.604	.152	0	0	0	-	0	.152	13.73	1.97	-.1322	-.0396	.716	0	0	0	0	0	0	0
1.50	-0.10	1.40	.716	.547	.169	0	0	0	-	0	.169	15.28	2.03	-.1397	-.0479	.668	0	0	0	0	0	0	0
1.40	-0.10	1.30	.668	.480	.188	0	0	0	-	0	.188	16.99	2.12	-.1497	-.0596	.608	0	0	0	0	0	0	0
1.30	-0.05	1.25	.608	.398	.210	0	0	0	-	0	.210	18.98	2.20	-.0823	-.0380	.570	0	0	0	0	0	0	0
1.25	-0.05	1.20	.570	.350	.220	0	0	0	-	0	.220	19.89	2.25	-.0877	-.0434	.527	0	0	0	0	0	0	0
1.20	-0.05	1.15	.527	.295	.232	0	0	0	-	0	.232	20.97	2.30	-.0949	-.0506	.476	0	0	0	0	0	0	0
1.15	-0.05	1.10	.476	.235	.241	0	0	0	-	0	.241	21.80	2.33	-.1051	-.0590	.417	0	0	0	0	0	0	0
1.10	-0.05	1.05	.417	.160	.257	0	0	0	-	0	.257	23.22	2.37	-.1199	-.0730	.344	0	0	0	0	0	0	0
1.05	-0.05	1.00	.344	.085	.259	0	0	0	-	0	.259	23.41	2.39	-.1454	-.0900	.254	0	0	0	0	0	0	0
1.00	-	-	.254	-	-	0	-	-	-	-	-	-	-	-	-	-	-	-	0	-	-	-	-

droplets will strike the stagnation point of the cylinder.

The general method of constructing a table such as Table 3 is as follows: Values of v_x, v_y, and y corresponding to a chosen value of y_s and an arbitrarily selected value of $x \geqq 4$ are calculated by means of equations 29, 30, and 31; x, v_x, v_y, and y are then substituted in columns (1), (4), (7), and (20), respectively. An increment Δx must then be assumed. Care should be taken to choose this increment of such magnitude as to avoid large changes of v_x, v_y, or y in any one step. The algebraic sum of columns (1) and (2) is column (3) by equation 25. Values of u_x and u_y are determined from Figs. 1 and 2, respectively. This is possible since x and y are known. The value determined for u_x is entered in column (5), while the value determined for u_y is entered in column (8). The difference $(v_x - u_x)$ is w_x and appears in column (6); similarly, the difference $(v_y - u_y)$ is w_y and appears in column (9). The square of the value in column (6), which is generally* entered in column (10), and the square of the value in column (9), which is entered in column (11), are added; the square root of the resulting sum is w and appears in column (12). Knowing R_U and w, equation 18 can be used to determine R. The value of R, thus determined, is entered in column (13). Since K is known in a problem of this type and since $C_D R/24$ can be determined from Fig. 3, entering this graph with the value of R from column (13), column (14) may now be determined. Column (15) is the

*When $y_s = 0$ as in Table 3, $u = \sqrt{u_x^2 + 0^2}$ and it is thus pointless to square u_x only to extract the square root at some later stage of the calculations.

quotient of column (2) by column (4). Column (16) is the product of columns (14), (6), and (15). By equation 23, column (4) plus column (16) gives the value of v_x at the end of the step. This sum appears in column (17). In like manner, the product of columns (14), (9), and (15) constitutes column (18). By equation 24, column (7) plus column (18) gives the value of v_y at the end of the step. This sum appears in column (19). The quotient of column (7) by column (4) is entered in column (21). Column (22) is formed as the product of column (21) by column (2) with the sign changed. Column (23) is one-half the product of column (18) by column (5) with the sign changed. By equation 26, the value of y at the end of the step, column (24), is the sum of columns (20), (22), and (23).

To initiate the next step, the values in columns (3), (17), (19), and (24) of the completed line are entered in columns (1), (4), (7), and (20), respectively, on the next line below. It is only necessary to select an appropriate value for Δx before proceeding with this next step.

As a second example of the method, the trajectory of a droplet whose initial ordinate, y_s, was 0.15 is tabulated in Table 4. In this case, the droplet velocity possesses a vertical component, and step-by-step determination of the trajectory is more difficult. Graphical construction of this trajectory in the vicinity of the cylinder locates the coordinates of droplet impact on the cylinder: x = .933, y = .357. The tangent of this trajectory at the point of impact intersects the x-axis at x = 1.265. Now, by Theorem I,

Table 4. Tabulated trajectory calculations for a droplet whose ordinate, y_s, = 0.15 at $x = \infty$.

(1) x_n	(2) Δx	(3) x_{n+1}	(4) v_{x_n}	(5) u_{x_n}	(6) (4)-(5)	(7) v_{y_n}	(8) u_{y_n}	(9) (7)-(8)	(10) $(6)^2$	(11) $(9)^2$	(12) $\sqrt{(10)+(11)}$	(13) R	(14) $\frac{C_D R}{24K}$	(15) (2)/(4)	(16) (14)(6)(15)	(17) $v_{x_{n+1}}$	(18) (14)(9)(15)	(19) $v_{y_{n+1}}$	(20) y_n	(21) (7)/(4)	(22) -(21)(2)	(23) $-\frac{(18)(15)}{2}$	(24) y_{n+1}
4.00	-0.50	3.50	.956	.940	.016	.0031	.006	-.003	.000256	.000009	.0163	1.471	1.20	-.522	-.0104	.946	.00188	.0050	.1572	.00324	.00162	.000490	.1593
3.50	-0.50	3.00	.946	.919	.027	.0050	.008	-.003	.000729	.000009	.0272	2.45	1.30	-.528	-.0185	.927	.00206	.0071	.1593	.00529	.00264	.000544	.1625
3.00	-0.50	2.50	.927	.892	.035	.0071	.011	-.004	.001225	.000016	.0352	3.18	1.37	-.539	-.0258	.901	.00295	.0101	.1625	.00766	.00383	.000696	.1670
2.50	-0.25	2.25	.901	.846	.055	.0101	.019	-.009	.00303	.000081	.0558	5.04	1.50	-.2774	-.0229	.878	.00374	.0138	.1670	.01121	.00280	.000519	.1703
2.25	-0.25	2.00	.878	.809	.069	.0138	.029	-.015	.00476	.000225	.0706	6.39	1.58	-.285	-.0311	.847	.00675	.0206	.1703	.01571	.00393	.000962	.1752
2.00	-0.20	1.80	.847	.760	.087	.0206	.042	-.021	.00757	.000441	.0895	8.09	1.68	-.236	-.0344	.813	.00833	.0289	.1752	.0243	.00487	.000984	.1811
1.80	-0.20	1.60	.813	.704	.109	.0289	.060	-.031	.01188	.000961	.1129	10.20	1.79	-.246	-.0480	.765	.01366	.0426	.1811	.0356	.00711	.001680	.1899
1.60	-0.10	1.50	.765	.628	.137	.0426	.091	-.048	.01877	.00230	.1419	12.81	1.92	-.1307	-.0344	.731	.01204	.0546	.1899	.0557	.00557	.000786	.1963
1.50	-0.10	1.40	.731	.575	.156	.0546	.116	-.061	.02435	.00372	.1676	15.14	2.02	-.1368	-.0431	.688	.01687	.0715	.1963	.0747	.00747	.001151	.2049
1.40	-0.10	1.30	.688	.513	.175	.0715	.142	-.070	.0306	.00490	.1884	17.02	2.13	-.1454	-.0542	.634	.02168	.0932	.2049	.1040	.01040	.001578	.2169
1.30	-0.05	1.25	.634	.446	.188	.0932	.177	-.084	.0353	.00706	.2058	18.60	2.19	-.0778	-.0308	.603	.01379	.1070	.2169	.1450	.00725	.000536	.2247
1.25	-0.05	1.20	.603	.410	.193	.1070	.209	-.102	.0372	.01040	.2181	19.71	2.23	-.0830	-.0357	.567	.01887	.1259	.2247	.1774	.00887	.000783	.2343
1.20	-0.05	1.15	.567	.370	.197	.1259	.239	-.113	.0388	.01277	.2270	20.52	2.27	-.0881	-.0394	.528	.02260	.1485	.2343	.2218	.01109	.000995	.2464
1.15	-0.05	1.10	.528	.333	.195	.1485	.283	-.135	.0380	.0182	.2370	21.42	2.30	-.0947	-.0425	.485	.0294	.1779	.2464	.2812	.01406	.001392	.2619
1.10	-0.05	1.05	.485	.286	.199	.1779	.327	-.149	.0396	.0222	.2483	22.45	2.35	-.1031	-.0482	.437	.0361	.2140	.2619	.3665	.01832	.001859	.2820
1.05	-0.05	1.00	.437	.260	.177	.2140	.402	-.188	.0313	.0353	.2575	23.26	2.38	-.1144	-.0482	.389	.0512	.2652	.2820	.490	.0245	.00293	.3094
1.00	-0.025	0.975	.389	.240	.149	.2652	.509	-.244	.0222	.0596	.286	25.9	2.48	-.0643	-.0237	.365	.0388	.3040	.3094	.681	.01702	.001249	.3177
0.975	-0.025	0.950	.365	.229	.136	.3040	.550	-.246	.01848	.0605	.281	25.4	2.47	-.0685	-.0230	.342	.0416	.3456	.3177	.833	.02082	.001425	.3399
0.950	-0.025	0.925	.342	.229	.113	.3456	.620	-.274	.01267	.0751	.279	25.2	2.47	-.0731	-.0204	.322	.0495	.3672	.3399	1.011	.02528	.001808	.3670
0.925	-0.025	0.900	.322	.260	.062	.3672	.670	-.303	.00384	.0919	.309	28.0	2.55	-.0776	-.0123	.310	.0600	.4272	.3670	1.140	.02350	.002327	.3928
0.900	-0.05	0.850	.310	.303	.007	.4272	.750	-.323	.000049	.1042	.323	28.2	2.56	-.1613	-.0029	.307	.1333	.5605	.3928	1.380	.0690	.01077	.4726
0.85	-	-	.307	-	-	.5605	-	-	-	-	-	-	-	-	-	-	-	-	.4381	1.827	-	-	-

Appendix D, of AAF Technical Report 5418 (11):

> If tangents be drawn to all the trajectories (for a given set of values for K and φ) at the point where these meet the cylinder (or sphere), these tangents have a common point of intersection which lies on the x-axis at a distance x_a from the origin.

The maximum ordinate of the impact point can thus be determined by constructing the tangent to the cylinder which passes through the point (1.265, 0). The corresponding angle between the positive x-axis and a cylinder radius through this point, θ_M, is found to be 37.9 degrees in this example. From Fig. 5 of AAF Technical Report 5418 (11), θ_M is determined as 35 degrees, using K and φ as determined for the present example. Thus, the method of finite summation appears to have introduced an error of 8.29 per cent in the determination of θ_M of the example. Nevertheless, it should be noted that this error would result in a conservative estimate of icing hazard in the present instance.

CONCLUSIONS

It has been demonstrated in the preceding analysis that step-by-step determination of droplet trajectories in the vicinity of a right circular cylinder is entirely practical. Furthermore, a satisfactory accuracy has been shown for the examples chosen. Accuracy would undoubtedly be improved by increasing the number of steps used in calculating the trajectories and by constructing Figs. 1, 2, and 3 on a larger scale. Additional curves on Figs. 1 and 2 for small values of the parameter, y, would also be advantageous.

Complete determination of the rate of ice accretion for the example airfoil would require the calculation of an additional trajectory, tangent to or barely missing the equivalent cylinder. No interpolation would be necessary for the determination of y_{s_M}, the maximum initial ordinate for a droplet striking the cylinder, if the tangent trajectory were fortuitously chosen; however, a trajectory which barely misses the equivalent cylinder is more likely to be selected and, in this case it would then be necessary to interpolate to determine y_{s_M}.

In either case, determination of this maximum initial ordinate would allow calculation of the rate of ice accretion per foot of span of the example airfoil.

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APPENDIX

The data appearing in Table 5 were calculated as a necessary preliminary to the construction of graphs of u_x and u_y. Each of these components was then plotted as a function of x with y serving as a parameter. Due to symmetry of the stream about the x-axis it was only necessary to tabulate data in the first quadrant for complete exploration in the region of accretion.

Values of α and of u were evaluated for possible later use as an aid in the step-by-step analysis.

Equations employed in the tabulation of these data were:

$$u_x = 1 - \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$u_y = \frac{2xy}{(x^2+y^2)^2}$$

$$\alpha = \tan^{-1}(u_y/u_x)$$

$$u = u_x \sec \alpha = u_y \csc \alpha$$

Table 5. Tabulated stream velocity components in vicinity of the cylinder.

x	x^2	$x^2 - y^2$	$x^2 + y^2$	$(x^2 + y^2)^2$	$\frac{x^2 - y^2}{(x^2 + y^2)^2}$	u_x	$2xy$	u_y	Tan α	α	u
					$y = 0;\ y^2 = 0$						
4.0	16.00	16.00	16.00	256.00	.0625	.9375	0	0	0	0	.9375
3.5	12.25	12.25	12.25	150.06	.0816	.9184	0	0	0	0	.9184
3.0	9.00	9.00	9.00	81.00	.1111	.8889	0	0	0	0	.8889
2.5	6.25	6.25	6.25	39.06	.1600	.8400	0	0	0	0	.8400
2.0	4.00	4.00	4.00	16.00	.2500	.7500	0	0	0	0	.7500
1.5	2.25	2.25	2.25	5.0925	.4444	.5556	0	0	0	0	.5556
1.0	1.00	1.00	1.00	1.0000	1.000	.0000	0	0	0	0	.0000
0.5	0.25	0.25	0.25	0.0625	4.000	-3.000	0	0	0	0	-3.0000
0.0	0.00	0.00	0.00	0.0000	-	-	-	-	-	-	-
					$y = 0.1;\ y^2 = 0.01$						
4.0	16.00	15.99	16.01	256.32	.0622	.9378	.80	.00312	.00333	.19	.938
3.5	12.25	12.24	12.26	150.32	.0814	.9186	.70	.00465	.00507	.29	.919
3.0	9.00	8.99	9.01	81.18	.1107	.8893	.60	.00739	.00831	.45	.889
2.5	6.25	6.24	6.26	39.19	.1592	.8408	.50	.01276	.01517	.87	.841
2.0	4.00	3.99	4.01	16.08	.2483	.7517	.40	.02488	.0331	1.90	.752
1.5	2.25	2.24	2.26	5.1375	.4362	.5638	.30	.0584	.1036	5.91	.567
1.0	1.00	.99	1.01	1.0200	.971	.029	.20	.1960	6.76	81.58	.198
0.5	0.25	.24	.26	.0676	3.55	-2.55	.10	1.480	-.580	144.5	3.13
0.0	0.00	-.01	.01	.0001	-100.0	101.0	0	0	0	0	101.0

x	x^2	$x^2 - y^2$	$x^2 + y^2$	$(x^2 + y^2)^2$	$\frac{x^2 - y^2}{(x^2 + y^2)^2}$	u_x	$2xy$	u_y	Tan α	α	u
$y = 0.2;\ y^2 = 0.04$											
4.0	16.00	15.96	16.04	257.28	.0620	.9380	1.60	.00622	.00664	.38	.938
3.5	12.25	12.21	12.29	151.08	.0808	.9192	1.40	.00926	.01008	.58	.919
3.0	9.00	8.96	9.04	81.72	.1097	.8903	1.20	.01469	.01649	.95	.890
2.5	6.25	6.21	6.29	39.56	.1570	.8430	1.00	.0253	.0300	1.72	.844
2.0	4.00	3.96	4.04	16.32	.2425	.7575	.80	.0490	.0647	3.70	.759
1.5	2.25	2.21	2.29	5.2441	.422	.578	.60	.1145	.1981	11.20	.589
1.0	1.00	.96	1.04	1.0816	.887	.113	.40	.3702	3.34	73.32	.387
0.5	0.25	.21	.29	0.0841	2.496	-1.496	.20	2.378	-1.590	122.15	2.81
0.0	0.00	-0.04	.04	0.0016	-25.0	26.0	0	0	0	0	26.0
$y = 0.3;\ y^2 = 0.09$											
4.0	16.00	15.91	16.09	259.00	.0615	.9385	2.40	.00926	.00988	.57	.939
3.5	12.25	12.16	12.34	152.00	.0800	.9200	2.10	.01381	.01501	.86	.920
3.0	9.00	8.91	9.09	82.6	.1090	.8910	1.80	.02179	.02445	1.40	.891
2.5	6.25	6.16	6.34	40.2	.1531	.8469	1.50	.0373	.0440	2.52	.848
2.0	4.00	3.91	4.09	16.75	.2334	.7666	1.20	.0716	.0935	5.34	.771
1.5	2.25	2.16	2.34	5.47	.3945	.6055	.90	.1645	.2715	15.77	.630
1.0	1.00	.91	1.09	1.188	.7665	.2335	.60	.504	2.146	65.03	.555
0.5	0.25	.16	.34	.1153	1.387	-.387	.30	2.598	-6.71	98.49	2.63
0.0	0.00	-.09	.09	.0081	-11.11	12.11	0	0	0	0	12.11
$y = 0.4;\ y^2 = 0.16$											
4.0	16.00	15.84	16.16	261.00	.0607	.9393	3.20	.01227	.01307	.75	.939
3.5	12.25	12.09	12.41	153.9	.0786	.9214	2.80	.01811	.01976	1.13	.921
3.0	9.00	8.84	9.16	84.0	.1051	.8949	2.40	.02859	.0319	1.84	.895
2.5	6.25	6.09	6.41	41.0	.1487	.8513	2.00	.0488	.0573	3.28	.854
2.0	4.00	3.84	4.16	17.35	.2213	.7787	1.60	.0922	.1185	6.76	.784
1.5	2.25	2.09	2.41	5.80	.3602	.6398	1.20	.207	.3234	17.91	.673
1.0	1.00	.84	1.16	1.346	.6245	.3755	.80	.592	1.567	57.44	.703
0.5	0.25	.09	.41	.1681	.535	.465	.40	2.379	5.12	78.93	2.425
0.0	0.00	-.16	.16	.0256	-6.25	7.25	0	0	0	0	7.25

x	x^2	$x^2 - y^2$	$x^2 + y^2$	$(x^2 + y^2)^2$	$\frac{x^2 - y^2}{(x^2 + y^2)^2}$	u_x	$2xy$	u_y	Tan α	α	u
					$y = 0.5$; $y^2 = 0.25$						
4.0	16.00	15.75	16.25	264.00	.0596	.9404	4.00	.01516	.01610	.92	.940
3.5	12.25	12.00	12.50	156.0	.0769	.9231	3.50	.02245	.02431	1.39	.923
3.0	9.00	8.75	9.25	85.6	.1022	.8978	3.00	.03503	.0390	2.23	.898
2.5	6.25	6.00	6.50	42.2	.1422	.8578	2.50	.0592	.0690	3.96	.860
2.0	4.00	3.75	4.25	17.70	.2118	.7882	2.00	.1130	.1434	8.16	.798
1.5	2.25	2.00	2.50	6.25	.3200	.6800	1.50	.2400	.353	19.46	.721
1.0	1.00	.75	1.25	1.563	.4795	.5205	1.00	.641	1.236	51.0	.825
0.5	0.25	.00	.50	.250	.000	1.000	.50	2.00	2.000	63.43	2.238
0.0	0.00	-.25	.25	.0625	-4.00	5.00	0	0	0	0	5.00
					$y = 0.6$; $y^2 = 0.36$						
4.0	16.00	15.64	16.36	268.00	.0584	.9416	4.80	.01791	.01900	1.08	.941
3.5	12.25	11.89	12.61	158.7	.0750	.9250	4.20	.02648	.02862	1.64	.925
3.0	9.00	8.64	9.36	87.7	.0984	.9016	3.60	.0410	.0455	2.61	.902
2.5	6.25	5.89	6.61	43.6	.1383	.8617	3.00	.0704	.0818	4.69	.864
2.0	4.00	3.64	4.36	19.01	.1916	.8084	2.40	.1262	.1561	8.89	.819
1.5	2.25	1.89	2.61	6.81	.2772	.7228	1.80	.2641	.3655	20.08	.770
1.0	1.00	.64	1.36	1.85	.346	.654	1.20	.649	.992	44.75	.922
0.5	0.25	-.11	.61	.372	-.2956	1.2956	.60	1.613	1.245	51.24	2.070
0.0	0.00	-.36	.36	.1296	-2.78	3.78	0	0	0	0	3.78
					$y = 0.7$; $y^2 = 0.49$						
4.0	16.00	15.51	16.49	272.00	.0571	.9429	5.60	.02060	.02184	1.25	.943
3.5	12.25	11.76	12.74	162.1	.0725	.9275	4.90	.0302	.0326	1.54	.927
3.0	9.00	8.51	9.49	90.0	.0946	.9054	4.20	.0467	.0516	2.93	.906
2.5	6.25	5.76	6.74	45.5	.1267	.8733	3.50	.0769	.0881	5.05	.877
2.0	4.00	3.51	4.49	20.15	.1742	.8258	2.80	.1390	.1681	9.55	.839
1.5	2.25	1.76	2.74	7.50	.2347	.7653	2.10	.280	.366	20.10	.816
1.0	1.00	.51	1.49	2.22	.230	.770	1.40	.631	.820	39.37	.996
0.5	0.25	-.24	.74	.547	-.438	1.438	.70	1.279	.890	41.64	1.923
0.0	0.00	-.51	.49	.2401	-2.04	3.04	0	0	0	0	3.04

x	x^2	$x^2 - y^2$	$x^2 + y^2$	$(x^2 + y^2)^2$	$\frac{x^2 - y^2}{(x^2 + y^2)^2}$	u_x	$2xy$	u_y	Tan α	α	u
$y = 0.8;\ y^2 = 0.64$											
4.0	16.00	15.36	16.64	277.00	.0555	.9445	6.40	.02310	.02445	1.40	.945
3.5	12.25	11.61	12.89	166.0	.0700	.9300	5.60	.03373	.03666	2.10	.931
3.0	9.00	8.36	9.64	92.9	.0900	.9100	4.80	.0517	.0569	3.26	.910
2.5	6.25	5.61	6.89	47.5	.1180	.8820	4.00	.0842	.0954	5.48	.887
2.0	4.00	3.36	4.64	21.5	.1562	.8438	3.20	.1488	.1762	10.00	.856
1.5	2.25	1.61	2.89	8.35	.1927	.8073	2.40	.2872	.3557	19.58	.857
1.0	1.00	.36	1.64	2.69	.134	.866	1.60	.595	.687	34.50	1.051
0.5	0.25	-.39	.89	.790	-.494	1.494	.80	1.013	.6775	34.13	1.751
0.0	0.00	-.64	.64	.4096	-1.56	2.56	0	0	0	0	2.56
$y = 0.9;\ y^2 = 0.81$											
4.0	16.00	15.19	16.81	283.00	.0537	.9463	7.20	.02543	.02688	1.53	.946
3.5	12.25	11.44	13.06	170.2	.0846	.9154	6.30	.03698	.04035	2.31	.916
3.0	9.00	8.19	9.81	96.2	.0851	.9149	5.40	.0561	.0614	3.55	.915
2.5	6.25	5.44	7.06	49.9	.1090	.8910	4.50	.0902	.1013	5.78	.897
2.0	4.00	3.19	4.81	23.15	.1378	.8622	3.60	.1556	.1802	10.22	.876
1.5	2.25	1.44	3.06	9.37	.1538	.8462	2.70	.288	.3402	18.80	.894
1.0	1.00	.19	1.81	3.28	.058	.942	1.80	.549	.582	30.21	1.092
0.5	0.25	-.56	1.06	1.122	-.498	1.498	.90	.801	.535	28.13	1.648
0.0	0.00	-.81	.81	.657	-1.23	2.23	0	0	0	0	2.23
$y = 1.0;\ y^2 = 1.0$											
4.0	16.00	15.00	17.00	289.00	.0519	.9481	8.00	.02768	.02920	1.67	.948
3.5	12.25	11.25	13.25	175.5	.0601	.9399	7.00	.0399	.0425	2.43	.941
3.0	9.00	8.00	10.00	100.0	.0800	.9200	6.00	.0600	.0652	3.64	.920
2.5	6.25	5.25	7.25	52.6	.0999	.9001	5.00	.0951	.1045	6.03	.907
2.0	4.00	3.00	5.00	25.0	.1200	.8800	4.00	.1600	.1802	10.22	.895
1.5	2.25	1.25	3.25	10.55	.1185	.8815	3.00	.2843	.3225	17.88	.926
1.0	1.00	0.00	2.00	4.00	0	1.00	2.00	.500	.500	26.57	1.120
0.5	0.25	-0.75	1.25	1.56	-.481	1.481	1.00	.641	.433	23.40	1.616
0.0	0.00	-1.00	1.00	1.00	-1.00	2.00	0	0	0	0	2.00