A PROBLEM IN DEPLETED FOURIER SERIES

by

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TABLE OF CONTENTS

INTRODUCTION

If a series $\sum_{n=1}^{\infty} A_n \cos nu$ has all the terms present for n ranging from one to infinity, it is called a complete Fourier Cosine Series. If all terms divisible by p_1 are missing, it is called a depleted series; if all terms divisible by p_1 and p_2 are missing, it is called a doubly depleted series, etc. If p is the product of the k prime numbers $p_1, p_2, \ldots, p_k$, we say the series is depleted by p.

The problem under consideration is to determine the function that can be expressed by $\sum_{n=1}^{\infty} (\cos nx)/n^2$ when this series has been depleted by p.

CONVERGENCE OF $\sum (\cos nx)/n^2$

A standard notation for an infinite series is

$$u_1 + u_2 + u_3 + \dots = \sum_{n=1}^{\infty} u_n = \sum u_n.$$

The nth partial sum of the series is

$$S_n = u_1 + u_2 + u_3 + \dots + u_n.$$

The sum of an infinite series is defined as the limit, as n increases indefinitely, of the sum of the first n terms:

$$S = \lim_{n\to\infty} S_n,$$

provided the limit exists.

If $\sum u_n$ has a sum S, i.e. if S_n approaches a limit when n increases, the series is said to be convergent, or to converge to the value S; if the limit does not exist, the series is divergent.

A series may diverge because S_n increases indefinitely as n increases; or it may diverge because S_n increases and decreases alternately, or oscillates, without approaching any limit. In the latter case the series is called oscillatory.

A necessary condition for convergence is that the general term approach zero as its limit;

$$\lim_{n\to\infty} u_n = 0.$$

If in $\sum u_n$, u_n is a function of n, we have the Integral Test for convergence or divergence of the series:

If the function f(n) is defined not only for positive integral values, but for all positive values of n, and if f(n) never increases with n, then the series $\sum u_n$ converges or diverges according as the integral $\int_1^\infty f(n)dn$ does or does not exist.

If $\sum u_n$ be a series of positive terms to be tested then by the Comparison Test:

(a) If a series $\sum a_n$ of positive terms, known to be convergent, can be found such that $u_n \leqq a_n$, the series to be tested is convergent.

(b) If a series $\sum b_n$ of positive terms, known to be divergent, can be found such that $u_n \geqq b_n$, the series to be tested is diverbent.

The p-series, $\sum 1/n^p$, is convergent for $p > 1$ and divergent for $p \leqq 1$. This can be shown by use of the Integral Test in the following way:

$$1/1^p + 1/2^p + \dots + 1/n^p + \dots = \sum 1/n^p.$$

The general term is $1/n^p$;

$$\int_1^\infty dn/n^p = \int_1^\infty n^{-p}dn = n^{1-p}/(1-p)\Big]_1^\infty$$

Since this is a finite result for $p > 1$, the series is convergent. For $p \leqq 1$, the integral fails to exist; therefore it is divergent.

To test the series

$$(\cos x)/1^2 + (\cos 2x)/2^2 + \dots + (\cos nx)/n^2 + \dots$$

compare with the series

$$1/1^2 + 1/2^2 + 1/3^2 + \dots + 1/n^2 + \dots$$

which is proved above to be convergent. Since

$$(\cos nx)/n^2 \leqq 1/n^2$$

for all values of n, the series $\sum (\cos nx)/n^2$ converges.

If each term of the series is a function of a real variable x for a closed interval $a \leqq x \leqq b$, we can write the series

$$u_1(x) + u_2(x) + u_3(x) + \dots = \sum u_n(x);$$

and its nth partial sum is $S_n(x)$.

The series $\sum u_n(x)$ is uniformly convergent over the interval (a,b) if there is a convergent series of positive constant terms, $\sum a_n$ say, such that $|u_n(x)| \leqq a_n$ for all values of n and x. Therefore the series

$$\sum (\cos nx)/n^2 = (\cos x)/1^2 + (\cos 2x)/2^2 + \dots$$

is uniformly convergent over any interval.

SUMMATION OF COMPLETE SERIES

The summation of the complete series

$$\sum (\cos nx)/n^2 = (\cos x)/1^2 + (\cos 2x)/2^2 + (\cos 3x)/3^2 + \ldots$$

can be attained by expanding each term around $x = \pi$ by means of Taylor's Series and summing the double series thus formed. Taylor's Series gives us:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \ldots,$$

where $f(x)$ is expanded in powers of $(x-a)$ in the vicinity of $x = a$.

To find the sum of $\sum (\cos nx)/n^2$ over the interval $0 \leqq x \leqq 2\pi$, we shall choose the mid-point $x = \pi$ around which to expand the function.

$f(x) = (\cos nx)/n^2$	$f(\pi) = (-1)/n^2$
$f'(x) = -(\sin nx)/n$	$f'(\pi) = 0$
$f''(x) = -\cos nx$	$f''(\pi) = (-1)^{n-1}$
$f'''(x) = n \sin nx$	$f'''(\pi) = 0$
$f^{\text{iv}}(x) = n^2 \cos nx$	$f^{\text{iv}}(\pi) = (-1)^n/n^2$
$f^{\text{v}}(x) = -n^3 \sin nx$	$f^{\text{v}}(\pi) = 0$
$\vdots$	$\vdots$

From the above we find

$$(\cos nx)/n^2 = (-1)^n/n^2 + (-1)^{n-1}(x-\pi)^2/2! + (-1)^n n^2 (x-\pi)^4/4! + \ldots;$$

hence

$$\sum(\cos nx)/n^2 = \sum(-1)^n/n^2 + (x-\pi)^2/2!\sum(-1)^{n-1}$$
$$+ (x-\pi)^4/4!\sum(-1)^n n^2 + (x-\pi)^6/6!\sum(-1)^{n-1}n^4 + \dots.$$

Therefore $\sum(\cos nx)/n^2$ expands into an infinite set of infinite series commonly called a double series.

$$\begin{aligned}\sum(\cos nx)/n^2 = {} & \left[-1/1^2 + 1/2^2 - 1/3^2 + 1/4^2 - 1/5^2 + \dots\right] \\ & + (x-\pi)^2/2!\left[1 - 1 + 1 - 1 + 1 - 1 + \dots\right] \\ & + (x-\pi)^4/4!\left[-1 + 4 - 9 + 16 - 25 + \dots\right] \\ & + (x-\pi)^6/6!\left[1 - 16 + 81 - 256 + \dots\right] \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + (-1)^s(x-\pi)^{2(s+1)}/(2s+2)!\left[1-2^{2s}+3^{2s}- \dots\right] \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

where s is equal to 0, 1, 2, 3, ...

*By the formula $B_r = (2r)!/(2^{2r-1}\pi^{2r})\sum 1/n^{2r}$, we can find the $\sum 1/n^2$ by letting $r = 1$ and knowing that $B_1 = 1/6$, where B_1 is the first Bernoulli number.

$$1/6 = 1/\pi^2\sum 1/n^2, \text{ or } \sum 1/n^2 = \pi^2/6.$$

Since the series

$$-1 + 1/4 - 1/9 + 1/16 - 1/25 + \dots = -1/n^2 + 2/2^2n^2,$$

then

$$\sum(-1)\ /n^2 = -\sum 1/n^2 + 1/2\sum 1/n^2 = -\pi^2/6 + \pi^2/12 = -\pi^2/12.$$

Some divergent series are summable. By letting $e^{-x}(u_0 + u_1x + u_2x^2/2! + u_3x^3/3! + \dots) = e^{-x}u(x)$ and assum-

*See Bromwich, Theory of Infinite Series, Art. 93.

ing that the coefficients u_n are such that the series $u(x)$ converges for all values of x, we may give the following definition for a summable divergent series:

Provided that the integral $\int_0^\infty e^{-x}u(x)dx$ is convergent, we may agree to associate its value with the series $\sum u_n$, if this series is not convergent; this integral may then be called the "sum" of the series; and the series may be called summable. The sum may be denoted by the symbol

$$\mathscr{S}_0^\infty u_n.$$

This definition is due to Borel and is regarded as the fundamental definition.

*Further, if C is any factor independent of n,

$$\mathscr{S}_0^\infty Cu_n = C\,\mathscr{S}_0^\infty u_n.$$

Now, in the series $1 - 1 + 1 - 1 + 1 - \ldots$,

$$u(x) = 1 - x + x^2/2! - x^3/3! + \ldots = e^{-x},$$

and so

$$\int_0^\infty e^{-x}u(x)dx = \int_0^\infty e^{-2x}dx = 1/2.$$

By letting

$$C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots$$

and $$S = 0 + \sin\theta + \sin 2\theta + \sin 3\theta + \ldots,$$

we obtain $C + iS = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots$

*See Bromwich, Theory of Infinite Series, Art. 102.

from which we see that the associated function is

$$u(x) = 1 + xe^{i\theta} + x^2e^{2i\theta}/2! + \dots = e^{xe^{i\theta}},$$

or $$u(x) = e^{x(\cos\theta + i\sin\theta)}.$$

Hence, provided that θ is not zero or a multiple of 2π, we find the sum

$$\int_0^\infty e^{-x}u(x)dx = \int_0^\infty e^{-x(1-\cos\theta-i\sin\theta)}dx = 1/(1-\cos\theta-i\sin\theta)$$
$$= 1/2(1 + i\cot\theta/2).$$

Therefore, the real part $C = 1 + \cos\theta + \cos 2\theta + \dots = 1/2$.

According to Bromwich*, we may differentiate the series found for $\mathscr{S}_1^\infty \cos n\theta$ and $\mathscr{S}_1^\infty \sin n\theta$ as often as we please, provided that θ is not a multiple of 2π. Hence we find

$$\mathscr{S}_1^\infty n^{2s}\cos n\theta = 0, \text{ and } \mathscr{S} n^{2s-1}\sin n\theta = 0.$$

Taking $\theta = \pi$ in the first equation, we find the result:

$$1^{2s} - 2^{2s} + 3^{2s} - 4^{2s} + \dots = 0.$$

Since the series $\sum(\cos nx)/n^2$ is uniformly convergent for all values of x, we shall expect, as is the case, that each series in the double series is summable. Therefore

$$\sum(\cos nx)/n^2 = -\pi^2/12 + (x-\pi)^2/(2!\cdot 2) + (x-\pi)^4/4!(0) + 0.$$
$$\sum(\cos nx)/n^2 = (x-\pi)^2/4 - \pi^2/12 = 1/12\left[3(x-\pi)^2 - \pi^2\right]$$

where $0 \leqq x \leqq 2\pi$; or by taking any interval of length 2π, we may write

$$\sum(\cos nx)/n^2 = 1/12\left\{3\left[x - (2k+1)\pi\right]^2 - \pi^2\right\}$$

where $2k\pi \leqq x \leqq 2(k+1)\pi$, and $k = 0, 1, 2, 3, \dots$.

*See Bromwich, Theory of Infinite Series, Art. 109 and 110.

$y_\phi = \sum(\cos n\phi x)/(n\phi)^2$ is a series composed of all the terms of the complete series $\sum(\cos nx)/n^2$ divisible by ϕ. This series may be summed in the following way:

$$\sum(\cos n\phi x)/(n\phi)^2 = 1/\phi^2 \sum(\cos nu)/n^2, \text{ where } u = \phi x.$$

$$\sum(\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{3\left[u - (2k+1)\pi\right]^2 - \pi^2\right\}$$

or $\sum(\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{3\left[\phi x - (2k+1)\pi\right]^2 - \pi^2\right\}$

where $2k\pi/\phi \leqq x \leqq 2(k+1)\pi/\phi$, and $k = 1, 2, 3, \ldots$.

SUMMATION OF $\sum (\cos nx)/n^2$, DEPLETED BY SIX

By depleting a series by some number p, in this case 6, it is meant to omit all terms of the complete series divisible by any of the prime factors of p. In the series

$y = \sum (\cos nx)/n^2$, depleted by 6 $= (\cos x)/1^2 + (\cos 5x)/5^2$
$\quad + (\cos 7x)/7^2 + (\cos 11x)/11^2 + (\cos 13x)/13^2 + \ldots,$

we have $$y = y_1 - y_2 - y_3 + y_6$$

where $y_1 = (\cos x)/1^2 + (\cos 2x)/2^2 + (\cos 3x)/3^2 + \ldots,$

$$y_2 = (\cos 2x)/2^2 + (\cos 4x)/4^2 + (\cos 6x)/6^2 + \ldots$$
$$= 1/2^2 \sum (\cos nu)/n^2, \text{ where } u = 2x,$$
$$y_3 = (\cos 3x)/3^2 + (\cos 6x)/6^2 + (\cos 9x)9^2 + \ldots$$
$$= 1/3^2 \sum (\cos nu)/n^2, \text{ where } u = 3x, \text{ and}$$
$$y_6 = (\cos 6x)/6^2 + (\cos 12x)/12^2 + (\cos 18x)/18^2 + \ldots$$
$$= 1/6^2 \sum (\cos nu)/n^2, \text{ where } u = 6x.$$

Since

$y_\phi = \sum (\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{3\left[\phi x - (2k+1)\pi\right]^2 - \pi^2\right\}$

where $2k\pi/\phi \leqq x \leqq 2(k+1)\pi/\phi$, the summation of the above series can be written.

$$y_2 = 1/(4\cdot 12) \left\{3\left[2x - (2k+1)\pi\right]^2 - \pi^2\right\},$$
$$\text{when } k\pi \leqq x \leqq (k+1)\pi.$$
$$y_3 = 1/(9\cdot 12) \left\{3\left[3x - (2k+1)\pi\right]^2 - \pi^2\right\},$$
$$\text{when } 2k\pi/3 \leqq x \leqq 2(k+1)\pi/3.$$

And $$y_6 = 1/(36\cdot 12)\left\{3\left[6x - (2k+1)\pi\right]^2 - \pi^2\right\},$$
$$\text{when } k\pi/3 \leqq x \leqq (k+1)\pi/3.$$

Let v_6 be the integral part of $v/6$, and r_6 the remainder; then $v = 6v_6 + r_6$.

By making these substitutions for k, we arrive at the summation of the depleted series.

$$\begin{aligned} y &= y_1 - y_2 - y_3 + y_6 \\ &= 1/12 \left\{3\left[x - (2v_6+1)\pi\right]^2 - \pi^2\right\} \\ &\quad - 1/48 \left\{3\left[2x - (2v_3+1)\pi\right]^2 - \pi^2\right\} \\ &\quad - 1/108 \left\{3\left[3x - (2v_2+1)\pi\right]^2 - \pi^2\right\} \\ &\quad + 1/432 \left\{3\left[6x - (2v+1)\pi\right]^2 - \pi^2\right\} \end{aligned}$$
$$\text{where } v\pi/3 \leqq x \leqq (v+1)\pi/3.$$

After expanding and collecting, we get

$y = A\pi x + A_1\pi^2$, where

$A = -v_6 + v_3/2 + v_2/3 - v/6 - 1/6$, and

$A_1 = v_6(v_6+1) - v_3(v_3+1)/4 - v_2(v_2+1)/9 - v(v+1)/36 + 1/9$

$$\text{when } v\pi/3 \leqq x \leqq (v+1)\pi/3.$$

But $v_6 = (v-r_6)/6$, $v_3 = (v-r_3)/3$, and $v_2 = (v-r_2)/2$; therefore, by substitution, we get

$$A = (r_6 - r_3 - r_2 - 1)/6,$$

and $$A_1 = -Av/3 + B$$

where $$B = \left[r_6(r_6-6) - r_3(r_3-3) - r_2(r_2-2) + 4\right]/36.$$

A has at most six different values, and B has at most six different values.

$y = A\pi x - A_1\pi^2$ is a series of straight lines which can be represented by a Fourier series depleted by six.

By using different values of v we can find the remainders r_6, r_3, and r_2; and, therefore, values of A, B, and A_1, from which we can write the equations for the six regions in the interval from $x = 0$ to $x = 2\pi$.

v	0	1	2	3	4	5
r_6	0	1	2	3	4	5
r_3	0	1	2	0	1	2
r_2	0	1	0	1	0	1
A	-1/6	-1/3	-1/6	1/6	1/3	1/6
B	1/9	1/18	-1/18	-1/9	-1/18	1/18
A_1	1/9	1/6	1/18	-5/18	-1/2	-2/9

For $v = 0$, $y = (-\pi/6)x + \pi^2/9$, $0 \leqq x \leqq \pi/3$.

$v = 1$, $y = (-\pi/3)x + \pi^2/6$, $\pi/3 \leqq x \leqq 2\pi/3$.

$v = 2$, $y = (-\pi/6)x + \pi^2/18$, $2\pi/3 \leqq x \leqq \pi$.

$v = 3$, $y = (\pi/6)x - 5\pi^2/18$, $\pi \leqq x \leqq 4\pi/3$.

$v = 4$, $y = (\pi/3)x - \pi^2/2$, $4\pi/3 \leqq x \leqq 5\pi/3$.

$v = 5$, $y = (\pi/6)x - 2\pi^2/9$, $5\pi/3 \leqq x \leqq 2\pi$.

From the inequality $v\pi/3 \leqq x \leqq (v+1)\pi/3$, we can determine v. For instance, let $x = 200^\circ$:

$$v\ /3 < 200^\circ < (v+1)\pi/3$$

$$v < 600^\circ/180^\circ < v+1$$

$$v < 3\ 1/3 < v+1$$

Therefore, $v = 3.$

Since the cosine function has a period of 2π, these equations will suffice for all values of x if properly selected.

SUMMATION OF $\sum(\cos nx)/n^2$, DEPLETED BY P

First we shall solve for the summation of the series when depleted by p where p is a prime number or is taken alone. With this condition in mind

$$y = \sum(\cos nx)/n^2, \text{ depleted by } p = y_1 - y_p.$$

From equations derived in part three, it is obvious that

$$y_1 = 1/12 \left\{3\left[x - (2v_p+1)\pi\right]^2 - \pi^2\right\},$$

and $$y_p = 1/(12p^2) \left\{3\left[px - (2v+1)\pi\right]^2 - \pi^2\right\}.$$

Then $$y = 1/12 \left\{3\left[x - (2v_p+1)\pi\right]^2 - \pi^2\right\} - 1/12p^2 \left\{3\left[px - (2v+1)\pi\right]^2 - \pi^2\right\}$$

$$= \left[(2v+1)/2p - (2v_p+1)/2\right]\pi x - \left[(2v_p+1)^2/4 - (2v+1)^2/4p^2 + (1-p^2)/12p^2\right]\pi^2$$

$$= A\pi x + A_1\pi^2$$

where $A = (2v+1)/2p - v_p - 1/2$,

and $A_1 = (2v_p+1)^2/4 - (2v+1)^2/4p^2 - (1-p^2)/12p^2$.

Since $v_p = (v-r_p)/p$, where v_p is the integral part of v/p and r_p the remainder, by substituting this value for v_p we get

$$y = A\pi x + A_1\pi^2$$

when $$A = r_p/p - 1/2,$$

and $$A = -2Av/p + B$$

where $$B = r_p(r_p-p)/p^2 + (p^2-1)/6p^2,$$

$$\text{when } 2v\pi/p \leqq x \leqq 2(v+1)\pi/p.$$

Now let us consider the case $y = \sum(\cos nx)n^2$, depleted by p, where p_1 and p_2 are the prime factors of p, and $p \equiv p_1p_2$.

Here $y = y_1 - y_{p_1} - y_{p_2} + y_p$

$$= 1/12\left\{3\left[x-(2v_{p_{11}}+1)\pi\right]^2-\pi^2\right\} - 1/12p_1^2\left\{3\left[p_1x-(2v_{p_2}+1)\pi\right]^2-\pi^2\right\} - 1/12p_2^2\left\{3\left[p_2x-(2v_{p_1}+1)\pi\right]^2-\pi^2\right\} + 1/12p^2\left\{3\left[px-(2v+1)\pi\right]^2-\pi^2\right\}$$

when $2v\pi/p \leqq x \leqq 2(v+1)\pi/p$.

By simplifying and substituting $(v-r_p)/p$ for v_p, $(v-r_{p_1})/p_1$ for v_{p_1}, and $(v-r_{p_2})/p_2$ for v_{p_2}, we get

$$y = A\pi x + A_1\pi^2$$

where $\quad A = (r_p-r_{p_1}-r_{p_2})/p + (p_1+p_2-p-1)/2p$,

and $\quad A_1 = -2Av/p + B$

when $\quad B = 1/p^2\left[r_p(r_p-p) - r_{p_1}(r_{p_1}-p_1) - r_{p_2}(r_{p_2}-p_2)\right] + 1/6p^2\left[p^2 - p_1^2 - p_2^2 + 1\right]$

in the interval $2v\pi/p \leqq x \leqq 2(v+1)\pi/p$.

We shall take one more case before generalizing in the number of prime factors of p. That is $y = \sum(\cos nx)/n^2$ where p_1, p_2, and p_3 are the three prime factors of p, and $p = p_1p_2p_3$. With an increasing number of prime factors of p, the problem rapidly increases in complexity.

$$y = y_1 - y_{p_1} - y_{p_2} - y_{p_2} + y_{p_1p_2} + y_{p_1p_3} + y_{p_2p_3} - y_p$$

$$y = 1/12\left\{3\left[x - (2v_p+1)\pi\right]^2 - \pi^2\right\} - 1/12p_1^2\left\{3\left[p_1x - (2v_{p_2p_3}+1)\pi\right]^2 - \pi^2\right\} - 1/12p_2^2\left\{3\left[p_2x - (2v_{p_1p_3}+1)\pi\right]^2 - \pi^2\right\}$$
$$-1/12p_3^2\left\{3\left[p_3x - (2v_{p_1p_2}+1)\pi\right]^2 - \pi^2\right\} + 1/(12p_1^2p_2^2)\left\{3\left[p_1p_2x - (2v_{p_3}+1)\pi\right]^2 - \pi^2\right\} + 1/(12p_1^2p_3^2)\left\{3\left[p_1p_3x - (2v_{p_2}+1)\pi\right]^2 - \pi^2\right\}$$
$$+1/(12p_2^2p_3^2)\left\{3\left[p_2p_3x - (2v_{p_1}+1)\pi\right]^2 - \pi^2\right\} - 1/12p^2\left\{3\left[px - (2v+1)\pi\right]^2 - \pi^2\right\}$$

Since $v = (v-r_p)/p$, $v_{p_1} = (v-r_{p_1})/p_1$, $v_{p_2} = (v-r_{p_2})/p_2$, $v_{p_1p_2} = (v-r_{p_1p_2})/p_1p_2$, $v_{p_1p_3} = (v-r_{p_1p_3})/p_1p_3$, and $v_{p_2p_3} = (v-r_{p_2p_3})/p_2p_3$, we may substitute these values and simplify to attain

$$y = A\pi x + A_1\pi^2$$

where $A = 1/p(r_p - r_{p_1p_2} - r_{p_1p_3} - r_{p_2p_3} + r_{p_1} + r_{p_2} + r_{p_3})$
$\quad + 1/2p(p_1p_2 + p_1p_3 + p_2p_3 - p_1 - p_2 - p_3 - p + 1)$,

and $A_1 = -2Av/p + B$

where $B = 1/p^2\left[r_p(r_p-p) - r_{p_1p_2}(r_{p_1p_2}-p_1p_2)\right.$
$\quad - r_{p_1p_3}(r_{p_1p_3}-p_1p_3) - r_{p_2p_3}(r_{p_2p_3}-p_2p_3)$
$\quad \left. + r_{p_1}(r_{p_1}-p_1) + r_{p_2}(r_{p_2}-p_2) + r_{p_3}(r_{p_3}-p_3)\right]$
$\quad + 1/6p^2\left[p^2 - p_1^2p_2^2 - p_1^2p_3^2 - p_2^2p_3^2 + p_1^2 + p_2^2 + p_3^2 - 1\right]$

when $2v\pi/p \leq x \leq 2(v+1)\pi/p$.

With the results of these three particular cases of depleted series we can write from mathematical induction a general solution for the summation of the series $\sum (\cos nx)/n^2$ depleted by any number p which was a finite number k of prime factors $p_1, p_2, p_3, \ldots, p_i, \ldots, p_k$, and $p = p_1p_2p_3\cdots p_k$.

$\sum (\cos nx)/n^2$, depleted by $p = A\pi x + A_1\pi^2$ where

$$A = 1/p\left[r_{q_k} - \sum r_{q_{k-1}} + \sum r_{q_{k-2}} - \ldots + (-1)^{k-1}\sum r_{q_1}\right]$$
$$+ 1/2p\left[\sum q_{k-1} - \sum q_{k-2} + \ldots + (-1)^k \sum q_1 - p + (-1)^{k-1}\right],$$

and $A_1 = -2Av/p + B$ where

$$B = 1/p^2\left[r_{q_k}(r_{q_k} - q_k) - \sum r_{q_{k-1}}(r_{q_{k-1}} - q_{k-1}) + \sum r_{q_{k-2}}(r_{q_{k-2}}\right.$$
$$\left. - q_{k-2}) - \ldots + (-1)^{k-1}\sum r_{q_1}(r_{q_1} - q_1)\right] + 1/6p^2\left[q_k^2 - \sum q_{k-1}^2\right.$$
$$\left. + \sum q_{k-2}^2 - \ldots + (-1)^{k-1}\sum q_1^2 + (-1)^k\right],$$

$$\text{when } 2v\pi/p \leqq x \leqq 2(v+1)\pi/p.$$

Here q_m is the products of the numbers in the combinations of the k prime factors of p taken m of them at a time, and in $\sum r_{q_m}(r_{q_m} - q_m)$ the same value of q_m is used in all three positions simultaneously.

CONCLUSION

$f(x) = A\pi x + A_1\pi^2$, a series of straight lines, is a function which can be expressed by $\sum(\cos nx)/n^2$ when depleted by p.

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