# ANALYSIS OF SINGLE AND MULTIPLE PENDULUM VIBRATION ABSORBERS APPLIED TO A VERTICAL CANTILEVER BEAM

by

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## NOMENCLATURE

| | |
|---|---|
| $E$ | modulus of elasticity of beam, $lbf/in.^2$ |
| $I$ | moment of inertia of beam for axis perpendicular to the forcing plane, $in.^4$ |
| $A_b$ | cross sectional area of beam, $in.^2$ |
| $L$ | length of beam, in. |
| $\rho$ | density of beam material, $lbm/in.^3$ |
| $F_i$ | amplitude of the $\gamma_i$ component of the beam forcing function, lbf/in. |
| $F$ | beam forcing function, lbf/in. |
| $\gamma$ | circular frequency of vibration, rad/sec. |
| $f$ | frequency of vibration, cycles/sec. |
| $m_i$ | generic pendulum mass, lbm. |
| $\ell_i$ | generic pendulum length, in. |
| $x$ | beam coordinate from fixed end, in. |
| $v(x,t)$ | beam displacement, in. |
| $v_i'(t)$ | displacement of $i^{th}$ generic pendulum mass, in. |
| $t$ | time, sec. |
| $g$ | acceleration of gravity $in./sec.^2$ |
| $V$ | shear in beam lbf. |
| $M$ | bending moment in beam in. lbf. |
| $m_{in}$ | $\sum_{j=1}^{n} m_j$ , lbm . |
| $\beta$ | $\sqrt[4]{\frac{\rho A_b \gamma^2}{EI}}$ , $in.^{-1}$ |

## PART I

## INTRODUCTION

When tall, flexible structures such as light poles, antenna masts; and stacks are subjected to transverse vibratory loadings, large, undesirable deflections sometimes result. Such vibratory loadings are most often caused by winds of high speed which cause a vortex shedding effect. A vortex shedding effect of this nature causes a dynamic loading pattern on the structure in the cross-wind direction. At the present time, little is known about the nature of these forces except that it depends somewhat on wind speed and structural geometry.[1]

Reed and Duncan (ref. 1) have studied methods to suppress wind induced oscillations of the nature just described. Such structures as the Saturn V launch vehicle and Navy antenna systems were considered. In both instances, oscillations were suppressed by attaching energy absorbinb elements to the structure.

In the case of the Saturn V launch vehicle, dampers were placed between the launch vehicle and the tower. Results of a wind tunnel study of a scaled model revealed that the dynamic response of the vehicle was relatively low even at wind speeds of 60 knots.

In the case of the Navy antenna systems, a cluster of plastic coated

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[1]Wilmer H. Reed III and Rodney L. Duncan, Proceedings of the Tenth Midwestern Mechanics Conference. (Fort Collins, Colorado, 1967), pp. 881 - 897.

chains surrounded by a neoprene shroud was attached at the upper end of the antenna. The cluster was placed inside the cylindrical antenna and served as an effective impact type damper. In both of these examples, the vibration absorbers were designed to absorb energy so that large oscillations would not occur.

The purpose of this paper was to study the effects of a pendulum type vibration absorber which, when properly designed, would make the absorbed system insensitive to certain forcing frequencies. As in the case of the previous examples, it was felt that this type of vibration absorber would be well suited for tall flexible structures subjected to dynamic loads in the transverse direction. The purpose of the absorber was not one of energy absorbtion. Instead, the pendulum absorber was to change the resonant frequencies of the absorbed system so that they would not correspond to the forcing frequency or frequencies. Both single and multiple pendulum absorbers were considered.

Since internal damping in structures is usually small, it was neglected in this paper. It was also assumed that the pendulum absorbers were frictionless, and that displacements were sufficiently small so that a linear analysis could be used.

When multiple pendulums with flexible links are excited above the fundamental mode, the displacements tend to become nonplaner. It was assumed in this paper that the pendulum displacements were restricted to the plane of the forcing function. This could be done by suitable link design. It was also assumed that the masses of the pendulums could be considered as point masses. The structure which was analyzed in this paper was a vertical, Hookean, isotropic homogeneous cantilever beam of constant cross section. The absorber was placed at the unfixed end. It was also assumed that the forcing function was uniform along the length of the beam, and that it was composed of a finite number of sinusoidal components. This model is illustrated in figure 1 of chapter 2.

In this paper, displacement equations were developed for various absorbed cantilever systems. From these displacement equations, characteristic equations (equations whose roots yield resonant frequencies) were developed. Computer programs written in Fortran 4, Level G, were developed to evaluate these displacement equations and determine the resonant frequencies of the various cantilever systems.

It was discovered that if none of the forcing function frequencies are resonant frequencies of the fixed-pinned beam, then it is possible to design an absorber such that the absorbed cantilever beam will be effectively pinned at the absorbed end. It can also be guaranteed that resonance will not occur at these frequencies. These statements are proved in Chapter 3.

Experiments were performed in order to verify the validity of the characteristic equations. Chapter 4 contains the description and results of these experiments.

# PART II

## DEVELOPMENT OF DISPLACEMENT EQUATIONS

In this chapter, the displacement equations for various cantilever beam systems are developed. In particular, displacement equations for a cantilever beam with a free end, with a simple pendulum absorber, and with a multiple pendulum absorber are derived. In each case, it is assumed that the beam experiences a forcing function which is uniform along its length and which consists of a finite number of sinusoidal components. This situation is illustrated in figure 1.

A free body diagram of a differential element of the beam at a point x is shown below in figure 2. Equating the sum of the forces in the "v" direction to the product of the mass of the differential element and the acceleration of the differential element in the "v" direction, the following equation is obtained:

$$Fdx + V - \left(V + \frac{\partial V}{\partial x}\,dx\right) = \rho A_b \,\frac{\partial^2 v(x,t)}{\partial t^2}\,dx \,. \tag{2-1}$$

$$V = EI\,\frac{\partial^3 v(x,t)}{\partial x^3} \,. \tag{2-2}$$

Using equation (2-2), the equation from elementary beam theory which relates shear and displacement, equation (2-1) reduces to

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\rho A_b}{EI}\,\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{F}{EI} \,. \tag{2-3a}$$

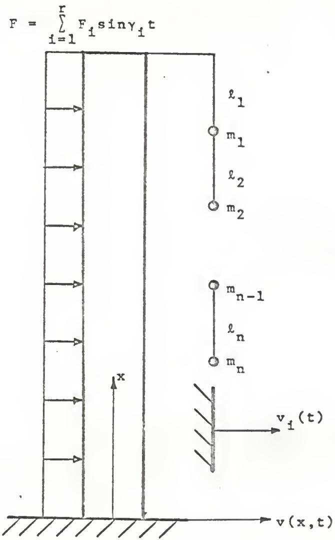


Figure 1

Diagram of Absorbed Beam

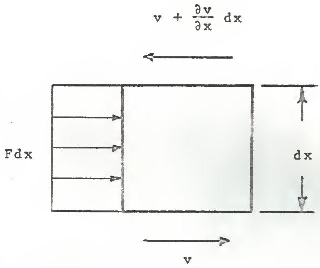


Figure 2

Free-Body Diagram of Differential Beam Element

The boundary conditions for this problem are as follows:

1. $v(0,t) = 0$ . (2-3b)

2. $$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0 \ . \tag{2-3c}$$

3. $$\left.\frac{\partial^2 v(x,t)}{\partial x^2}\right|_{x=L} = 0 \ . \tag{2-3d}$$

4. $$EI \left.\frac{\partial^3 v(x,t)}{\partial x^3}\right|_{x=L} = V\Big|_{x=L} \ . \tag{2-3e}$$

In appendix A, the principle of superposition is proven for this problem. This principle states that if $v_i(x,t)$ is the displacement function for a beam acted upon by a forcing function, $F_i \sin\gamma_i t$, then the resultant displacement function of a beam acted upon by a forcing function

$\sum_{i=1}^{r} F_i \sin\gamma_i t$, is as follows:

$$v(x,t) = \sum_{i=1}^{r} v_i(x,t) \ . \tag{2-4}$$

Due to this principle, the derivations of the displacement equations will be made by assuming a forcing function of a single sinusoidal component

$F(t) = F\sin\gamma t$ .

Equation (2-3a) now takes the following form:

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\rho A_b}{EI}\,\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{F}{EI}\,\sin\gamma t \ . \tag{2-5}$$

By assuming $v(x,t) = r(x)p(t)$, equation (2-5) reduces to:

$$p(t)\,\frac{d^4 r(x)}{dx^4} + \frac{\rho A_b}{EI} r(x)\,\frac{d^2 p(t)}{dt^2} = \frac{F}{EI}\,\sin\gamma t\ . \tag{2-6}$$

The solution of equation (2-6) is the sum of a homogeneous solution and a particular solution. Due to a slight amount of damping, which is present in all physical systems, the steady state solution will contain only the particular solution.

For the steady state solution, it is assumed that $p(t)$ is defined by equation (2-7).

$$p(t) = C_1 \sin\gamma t + C_2 \cos\gamma t\ . \tag{2-7}$$

By combining equations (2-6) and (2-7), the following equation is obtained:

$$(C_1 \sin\gamma t + C_2 \cos\gamma t)\left[\frac{d^4 r(x)}{dx^4} - \frac{\gamma^2 \rho A_b}{EI}\, r(x)\right] = \frac{F\sin\gamma t}{EI}\ . \tag{2-8}$$

Since the right side of equation (2-8) is a function of t only; the term in brackets must be a constant, and $C_2$ must be zero. By letting

$$\left[\frac{d^4 r(x)}{dx^4} - \frac{\gamma^2 \rho A_b}{EI}\, r(x)\right] = Q\ ,$$

$C_1$ must take the form expressed in equation (2-9) .

$$C_1 = \frac{F}{QEI}\ . \tag{2-9}$$

The solution for $r(x)$ now becomes:

$$r(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x - \frac{Q}{\beta^4}\ . \tag{2-10}$$

$$\beta^4 = \frac{\rho A_b \gamma^2}{EI}\ .$$

Since A, B, C, D, and Q are constants which are yet to be determined, no loss of generality will result by expressing $v(x,t)$ as in equation (2-11).

$$v(x,t) = \sin\gamma t \left\{ A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x \frac{-F}{\beta^4 EI} \right\} . \quad (2\text{-}11)$$

By substituting this expression in equations (2-3b), (2-3c), (2-3d), and (2-3e); the four boundary condition equations in terms of A, B, C, and D are shown in equations (2-12a), (2-12b), (2-12c), and (2-12d).

$$B + D - \frac{F}{\beta^4 EI} = 0 . \quad (2\text{-}12a)$$

$$A + C = 0 . \quad (2\text{-}12b)$$

$$-A\sin\beta L - B\cos\beta L + C\sinh\beta L + D\cosh\beta L = 0 . \quad (2\text{-}12c)$$

$$EI\beta^3 \left\{ -A\cos\beta L + B\sin\beta L + C\cosh\beta L + D\sinh\beta L \right\} = V\Big|_{x=L} . \quad (2\text{-}12d)$$

A simultaneous solution of these four equations yields equations (2-13a), (2-13b), (2-13c), and (2-13d).

$$A = \frac{-(V\big|_{x=L}) \dfrac{(\cos\beta L + \cosh\beta L)}{EI\beta^4 \sin\gamma t} + \dfrac{F}{\beta^4 EI}\left\{\cosh\beta L\sin\beta L + \cos\beta L\sinh\beta L\right\}}{2 + 2\cosh\beta L\cos\beta L} . \quad (2\text{-}13a)$$

$$B = \frac{\dfrac{F}{\beta^3 EI}\cos\beta L - A(\sin\beta L + \sinh\beta L)}{\cos\beta L + \cosh\beta L} . \quad (2\text{-}13b)$$

$$C = -A . \quad (2\text{-}13c)$$

$$D = \frac{F}{\beta^4 EI} - B . \quad (2\text{-}13d)$$

A convenient form for the displacement equation can be made by substituting equations (2-13a), (2-13b), (2-13c), and (2-13d) into equation (2-11), and by expressing the displacement equation in terms of A. In this manner, various displacement equations which arise from the consideration of different absorbers may easily be formed.

By performing these steps, the displacement equation becomes equation (2-13e).

$$v(x,t) = \sin\gamma t \left[ A(B_1) + \frac{F}{\beta^4 EI} (B_2) \right] . \tag{2-13e}$$

$$B_1 = \sin\beta x - \sinh\beta x - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta x - \cosh\beta x\right) \tag{2-13f}$$

$$B_2 = \frac{\cos\beta L \cosh\beta x + \cos\beta L \cosh\beta x}{\cos\beta L + \cosh\beta L} - 1 . \tag{2-13g}$$

If the beam is free at the end, $V|_{x=L} = 0$; and, by using equations (2-13), the displacement equation becomes equation (2-14).

$$v(x,t) = \frac{F\sin\gamma t}{\beta^4 EI}\left[ (C_1)(C_2) + C_3) \right] . \tag{2-14a}$$

$$C_1 = \left\{\frac{\cosh\beta L \sin\beta L + \cos\beta L \sinh\beta L}{2 + 2 \cos\beta L \cosh\beta L}\right\} . \tag{2-14b}$$

$$C_2 = \sin\beta x - \sinh\beta x - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta x - \cosh\beta x\right) \tag{2-14c}$$

$$C_3 = \left\{\frac{\cosh\beta L \cos\beta x + \cos\beta L \cosh\beta x}{\cos\beta L + \cosh\beta L} - 1\right\} . \tag{2-14d}$$

By studying equations (2-14c) and 2-14d), it becomes apparent that both terms $C_2$ and $C_3$ are bounded when $\beta$ and x are positive and bounded. It is therefore evident that for resonance, $C_1$ must become infinite. Resonance for a free-ended cantilever will therefore occur when equation (2-15) is satisfied.

$$2 + 2 \cos\beta L \cosh\beta L = 0 . \tag{2-15}$$

Equation 15 is transcendental and hence, values of $\beta$ cannot be solved for directly. A computer program was used to find the first 15 roots of Eq. (2-15).

These results appear in appendix C.

If a simple pendulum is attached to the upper end of the column, $V|_{x=L}$ is no longer zero, and a different displacement equation will result. Consider a pendulum of length $\ell_1$ and mass $m_1$ . Equation (2-16) is obtained from a free body diagram of $m_1$ . (See figure 3)

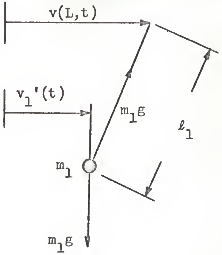


Figure 3.

Free-Body Diagram of Single Mass Pendulum

$$m_1 \frac{\partial^2 v_1(t)}{\partial t^2} = \frac{m_1 g}{\ell_1} \left[ v(L,t) - v_1{}'(t) \right] \quad . \tag{2-16}$$

Equation (2-16) simplifies to equation (2-17) when r(L)sin$\gamma$t is substituted for v(x,t).

$$\frac{\partial^2 v_1(t)}{\partial t^2} + \frac{g}{\ell_1} v_1{}'(t) = \frac{g}{\ell_1} r(L)\sin\gamma t \quad . \tag{2-17}$$

The solution of this equation consists of the sum of a homogeneous solution and a particular solution. Due to a small amount of damping, only the particular solution will remain in the steady state solution. By assuming the form of $v_1'(t)$ expressed by equation (2-18), the steady state solution for $v_1'(t)$ is found to be as expressed in equation (2-19).

$$v_1'(t) = C_1 \sin\gamma t + C_2 \cos\gamma t \quad . \tag{2-18}$$

$$v_1'(t) = \frac{r(L)\sin\gamma t}{1 - \frac{\ell_1 \gamma^2}{g}} \quad . \tag{2-19}$$

The shear force $V|_{x=L}$ at the upper end of the beam must be equal to

$$m_1 \frac{\partial^2 v_1'(t)}{\partial t^2} \quad .$$

Thus,

$$\frac{-m_1 r(L)\sin t}{1 - \frac{\ell_1 \gamma^2}{g}} = V|_{x=L} \quad . \tag{2-20}$$

After substituting equation (2-20) into equations (2-13) and simplifying the results, the displacement equations for the beam, and the pendulum mass reduce to equations (2-21).

$$v(x,t) = \frac{F \sin\gamma t}{\beta^4 EI} \left[ (D_1)(D_2) + D_3 \right] \quad . \tag{2-21a}$$

$$v_1'(t) = \frac{v(L,t)}{1 - \frac{\gamma^2 \ell_1}{g}} \quad . \tag{2-21b}$$

$$D_{11} = \left(1 - \frac{2\cosh\beta L\,\cos\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\frac{-m_1}{\frac{1}{\gamma^2} - \frac{\ell_1}{g}}\right) . \qquad (2\text{-}21c)$$

$$D_{12} = \frac{EI\beta^3(\cosh\beta L\,\sin\beta L + \cos\beta L\,\sinh\beta L)}{\cos\beta L + \cosh\beta L} . \qquad (2\text{-}21d)$$

$$D_{13} = \frac{-m_1}{\frac{1}{\gamma^2} - \frac{\ell_1}{g}} . \qquad (2\text{-}21e)$$

$$D_{14} = \left[\sin\beta L - \sinh\beta L - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta L - \cosh\beta L\right)\right] . \qquad (2\text{-}21f)$$

$$D_{15} = \frac{EI\beta^3(2 + 2\cosh\beta L\,\cos\beta L)}{\cos\beta L + \cosh\beta L} . \qquad (2\text{-}21g)$$

$$D_1 = \frac{D_{11} + D_{12}}{D_{13}(D_{14}) + D_{15}} . \qquad (2\text{-}21h)$$

$$D_2 = \left[\sin\beta x - \sinh\beta x - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta x - \cosh\beta x\right)\right] . \qquad (2\text{-}21i)$$

$$D_3 = \left[\frac{\cosh\beta L\,\cos\beta x + \cos\beta L\,\cosh\beta x}{\cos\beta L + \cosh\beta L} - 1\right] . \qquad (2\text{-}21j)$$

Since $\cos\beta L + \cosh\beta L \neq 0$ for $\beta L \geq 0$, $D_3$ will remain finite for finite values of x. Consequently, if resonance occurs, the product $(D_1)(D_2)$ must grow without bound. Since $D_2$ in equation (2-21) is bounded for positive finite values of $\beta$ and x, resonance must occur when $D_1$ grows without bound. By observing the equations which define $D_1$, it is evident that the characteristic equation for a system consisting of a cantilever beam and a simple pendulum attached at the end, must be equation (2-22).

$$0 = D_{14} - \frac{\left(\frac{1}{\gamma^2} - \frac{\ell_1}{g}\right)}{m_1} D_{15} \quad . \tag{2-22}$$

Equation (2-22) is transcendental and hence the roots cannot be obtained directly. It is interesting to note that not only the beam parameters, but also the pendulum parameters influence the roots of equation (2-22), and hence influence the spectrum of resonant frequencies of the coupled system. A computer program was used to solve for the roots of equation (2-22). A listing of the program is contained in appendix D.

If a multiple pendulum as shown in figure 4 is attached to the free end of a cantilever beam, the shear, V, at x=L must be as expressed by equation (2-23). The value of V at x=L

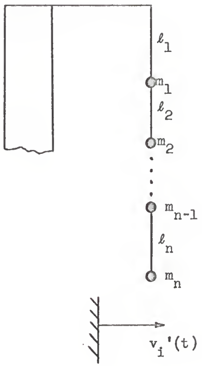


Figure 4.

Diagram of Beam and Multiple Pendulum Absorber

$$V\Big|_{x=L} = \sum_{i=1}^{n} m_i \frac{\partial^2 v_i'(t)}{\partial t^2} \,. \qquad (2\text{-}23)$$

will depend upon the vibration of the multiple pendulum. Since the beam displacement equation is dependent upon $V\big|_{x=L}$, the attachment of such a pendulum will completely alter the vibration characteristics of the beam.

In analyzing the displacement equation, it is convenient to define the following notation:

$$v_i'(t) = \text{displacement of } m_i \,. \qquad (2\text{-}24)$$

$$m_{in} = \sum_{k=i}^{n} m_k \,. \qquad (2\text{-}25)$$

The parameter $m_{in}$ defined in equation (2-25) is convenient for expressing the tensile forces in the lengths of the pendulum. The tensile force $T_i$ in link i now becomes as expressed in equation (2-26).

$$T_i = g\, m_{in} \,. \qquad (2\text{-}26)$$

For mass $m_1$, a free-body diagram yields the following:

$$m_1 \frac{\partial^2 v_1'(t)}{\partial t^2} = T_1 \left[\frac{v(L,t) - v_1'(t)}{\ell_1}\right] + T_2 \left[\frac{v_2'(t) - v_1'(t)}{\ell_2}\right] . \qquad (2\text{-}27)$$

$$m_1 \frac{\partial^2 v_1'(t)}{\partial t^2} = g \left\{ v_1'(t)\left[-\frac{m_{1n}}{\ell_1} - \frac{m_{2n}}{\ell_2}\right] + v_2'(t)\left[\frac{m_{2n}}{\ell_2}\right] + \frac{m_{1n} v(L,t)}{\ell_1} \right\} . \qquad (2\text{-}28)$$

For a generic mass $m_i$, which is an internal mass, a free-body diagram yields:

$$m_i \frac{\partial^2 v_i'(t)}{\partial t^2} = T_i \left[\frac{v_{i-1}'(t) - v_i'(t)}{\ell_i}\right] + T_{i+1} \left[\frac{v_{i+1}'(t) - v_i'(t)}{\ell_{i+1}}\right] . \qquad (2\text{-}29)$$

$$m_i \frac{\partial^2 v_i'(t)}{\partial t^2} = g \left\{ v_{i-1}'(t) \left[\frac{m_{in}}{\ell_i}\right] + v_i'(t)\left[-\frac{m_{in}}{\ell_i} - \frac{m_{i+1\ n}}{\ell_{i+1}}\right] + v_{i+1}'(t)\left[\frac{m_{i+1\ n}}{\ell_{i+1}}\right]\right\}. \tag{2-30}$$

For the end mass, $m_n$, a free-body diagram yields:

$$m_n \frac{\partial^2 v_n'(t)}{\partial t^2} = g \left\{ v_n'(t) \left[\frac{-m_n}{\ell_n}\right] + v_{n-1}(t)\left[\frac{m_n}{\ell_n}\right]\right\} \quad . \tag{2-31}$$

The displacement equations of the "n" mass pendulum become more manageable by using matrix notation. By using matrix notation, equations (2-28), (2-30), and (2-31) reduce to the following matrix equation:

$$(M)\ \{\ddot{v}_i'(t)\} = g(N)\{v_i'(t)\} + \{T\} \sin\gamma t \quad . \tag{2-32}$$

$$\{T\} = \left\{ \begin{matrix} \dfrac{r(L) m_{1n}\ g}{\ell_1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\} \tag{2.32a}$$

$$(M) = \begin{pmatrix} m_1, & 0, & \ldots, & 0 \\ 0\ , & m_2, & 0, \ldots, & 0 \\ & & \vdots & \\ 0\ , & \ldots & , 0, & m_n \end{pmatrix} \tag{2-33}$$

$$(N) = \begin{pmatrix} \frac{-m_{1n}}{\ell_1} \; \frac{-m_{2n}}{\ell_2}, \; \frac{m_{2n}}{\ell_2}, \; 0, \; . \; . \; . \; , \; 0 & & & & & & \\ & & \vdots & & & & \\ 0, \cdots, 0, & \frac{m_{in}}{\ell_i}, & \frac{-m_{in}}{\ell_i} & \frac{-m_{i+1\ n}}{\ell_{i+1}}, & \frac{m_{i+1\ n}}{\ell_{i+1}}, & 0, \cdots, & 0 \\ & & \vdots & & & & \\ 0, \cdots, 0, & & & \frac{m_{nn}}{\ell_n}, & \frac{-m_{nn}}{\ell_n} & & \end{pmatrix}^{*} \quad (2\text{-}34)$$

The solution of equation (2-32) consists of the sum of the homogeneous solution and the particular solution. Due to a small amount of damping in the pendulum, the steady state solution will only contain the particular solution. By assuming for the particular solution a form for $\{v_i'(t)\}$ as given in equation (2-35), the matrix differential equation is reduced to equation (2-36).

$$\{v'(t)\} = \{u\} \sin\gamma t \; . \qquad (2\text{-}35)$$

$$(-\gamma^2(M) - g(N))\,\{u\} = \{T\} \qquad (2\text{-}36)$$

By making use of equations (2-37) and (2-38), equation (2-36) becomes equivalent to equation (2-39).

$$(R) = (-\gamma^2(M) - g(N) \; . \qquad (2\text{-}37)$$

$$(S) = (R)^{-1} \; . \; ^{**} \qquad (2\text{-}38)$$

$$\{u\} = (S)\,\{T\} \; . \qquad (2\text{-}39)$$

By denoting the elements of matrix (S) by $S_{ij}$ , equation (2-39) becomes:

$$\{u\} = \frac{r(L)\ m_{1n}\ g}{\ell_1}\,\{S_{i1}\} \qquad (2\text{-}40)$$

* Matrix(N) contains a diagonal band which is three elements wide. The first and last row have only two elements as indicated.

**The case when (R) is singular will be treated later.

In equation (2-40), $\{S_{i1}\}$ is the column vector which has elements identical with the first column of the square matrix (S).

By considering equations (2-23), (2-35), and (2-40), it is evident that the expression for $V|_{x=L}$ becomes equation (2-42).

$$V|_{x=L} = -\gamma^2 \sin\gamma t \sum_{i=1}^{n} m_i u_i \quad . \tag{2-41}$$

$$V|_{x=L} = \frac{-\gamma^2 r(L) m_{1n}\, g^{\sin\gamma t}}{\ell_1} \sum_{i=1}^{n} m_i s_{i1} \quad . \tag{2-42}$$

From equations (2-13), the displacement equation for a cantilever beam with a multiple pendulum (see figure 4) becomes equation (2-43).

$$v(x,t) = \frac{F \sin\gamma t}{\beta^4 EI} \left[ A_1 \; B_1 + B_2 \right] \quad . \tag{2-43a}$$

$$B_1 = \sin\beta x - \sinh\beta x - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta x - \cosh\beta x\right) . \tag{2-43b}$$

$$B_2 = \frac{\cos\beta L \cosh\beta x + \cos\beta L \cosh\beta x}{\cos\beta L + \cosh\beta L} - 1 \quad . \tag{2-43c}$$

$$A_{11} = - \frac{\gamma^2 m_{1n}\, g}{\ell_1} \sum_{i=1}^{n} m_i s_{i1} \left\{ \frac{-2 \cosh\beta L \cos\beta L}{\cos\beta L + \cosh\beta L} + 1 \right\} \quad . \tag{2-43d}$$

$$A_{12} = \frac{EI\, \beta^3}{\cos\beta L + \cosh\beta L} \left\{ \cos\beta L \sin\beta L + \cos\beta L \sinh\beta L \right\} \quad . \tag{2-43e}$$

$$A_{21} = - \frac{\gamma^2 m_{1n}\, g}{\ell_1} \sum_{i=1}^{n} m_i s_{i1} \left\{ \sin\beta L - \sinh\beta L - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\; L}\right)\left(\cos\beta L - \cosh\beta L\right) \right\} . \tag{2-43f}$$

$$A_{22} = \frac{EI\beta^3(2 + 2 \cosh\beta L \cos\beta L)}{\cos\beta L + \cosh\beta L} \quad . \tag{2-43g}$$

$$A_1 = \frac{A_{11} + A_{12}}{A_{21} + A_{22}} \quad . \tag{2-43h}$$

By observing equations (2-43), it is apparent that $B_1$ and $B_2$ are both bounded for values of x, β, and L which are positive and bounded. Therefore, if resonance occurs, then A, must grow without bound. Since the term $\sum_{i=1}^{\infty} m_i s_{i1}$ grows without bound for certain frequencies, it is best to consider as the characteristic equation of the system, equation (2-44a).

$$0 = D_1 + D_2 \ . \tag{2-44a}$$

$$D_1 = \left\{\sin\beta L - \sinh\beta L - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta L - \cosh\beta L\right)\right\} \quad . \tag{2-44b}$$

$$D_2 = \frac{-\ell_1[EI\beta^3(2 + 2 \cosh\beta L \cos\beta L)]}{\left(\gamma^2 m_{1n} \sum_{i=1}^{n} m_i s_{i1}\right)\left(\cos\beta L + \cosh\beta L\right)} \quad . \tag{2-44c}$$

This chapter dealt only with the derivations of the displacement equations and the characteristic equations for the various cantilever systems under study. In chapter 3, these equations will be used to study the effects of pendulum parameters on the vibration characteristics of the cantilever systems.

## PART III

### METHOD OF ABSORBER DESIGN

Suppose that a certain fixed-free beam of the type being considered is subjected to an environment where a uniformly distributed loading of the following form exists in a transverse direction to the beam.

$$F(t) = F_1 \sin \gamma_1 t + \ldots + F_n \sin \gamma_n t \, . \tag{3-1}$$

The displacement equation for this beam would be as given in equation (3-2). In this equation, $v_i(x,t)$ is

$$v(x,t) = \sum_{i=1}^{n} v_i(x,t) \tag{3-2}$$

the displacement function defined by equations (2-14) when

$$\sqrt[4]{\frac{\rho A_b \gamma_i^2}{EI}}$$

is substituted for $\beta$. If any of these $\beta$'s are such that $\beta L$ is equal to or close to values of $\beta L$ in appendix C list 1, then the fixed free beam will experience large amplitudes. The purpose of this chapter is to develop a design procedure for an absorber which, when connected to the upper end of the beam will decrease these amplitudes.

#### Design of Single Mass Pendulum Absorber

Suppose that $F(t)$ in equation (3-1) consists of a single frequency $\gamma_1$. Consider a one mass pendulum which has a natural frequency $\gamma_1$. If $\gamma_1$ is the natural frequency of the pendulum, then it follows that $\ell_1$ must be as defined in equation (3-3).

$$\ell_1 = \frac{g}{\gamma_1^2} \quad . \tag{3-3}$$

From equations (2-21), it is apparent that $D_1$ is in indeterminate form. By use of l' Hospital's Rule, $D_1$ becomes as defined by equation (3-4b) as $\gamma \rightarrow \gamma_1$ . The displacement equation of the beam now becomes equations (3-4).

$$v(x,t) = \frac{F_1 \sin\gamma_1 t}{\beta^4 EI} \left[ D_1 D_2 + D_3 \right] \quad . \tag{3-4a}$$

$$D_1 = \frac{1 - \dfrac{2 \cosh\beta L \cos\beta L}{\cos\beta L + \cosh\beta L}}{\sin\beta L - \sinh\beta L - \left(\dfrac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta L - \cosh\beta L\right)} \quad . \tag{3-4b}$$

$$D_2 = \sin\beta x - \sinh\beta x - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta x - \cosh\beta x\right) \quad . \tag{3-4c}$$

$$D_3 = \left[\frac{\cosh\beta L \cos\beta x + \cos\beta L \cosh\beta x}{\cos\beta L + \cosh\beta L} - 1\right] \quad . \tag{3-4d}$$

Notice that $v(L,t)$ is equal to zero in equation (3-4).

Since $\cos\beta L \neq \cosh\beta L$ for positive values of $\beta L$, $D_2$ and $D_3$ will always remain bounded. Therefore for the absorbed beam, resonance will occur at $\gamma_1$ only when the denominator in equation (3-4b) is equal to zero. By setting the denominator of $D_1$ equal to zero, equation (3-5) is formed.

$$\tan\beta L = \tanh \beta L \, ^{*} \tag{3-5}$$

---

*The first fifteen roots of this equation are contained in appendix C, list 2.

Equation (3-5) is the characteristic equation of the fixed-pinned beam.[1] When equation (3-4) is evaluated for

$$\beta = \sqrt[4]{\frac{\rho A_b \gamma_1^{\,2}}{EI}}$$

it becomes apparent that $v(L,t) = 0$, provided equation (3-5) is not satisfied. Therefore, if equation (3-5) is not satisfied, the absorber under consideration effectively "pins" the upper end of the beam for $\gamma = \gamma_1$. Resonance will occur at $\gamma = \gamma_1$ if and only if $\gamma_1$ is a natural frequency of the fixed-pinned beam.

The pendulum displacement, $v_1'(t)$, as given by equation (2-21b) is in indeterminate form when $\gamma = \gamma_1$. By use of l'Hospital's Rule, equation (2-21b) is reduced to equation (3-6) when $\gamma = \gamma_1$, and $\tan\beta L \neq \tanh\beta L$.

$$v_1'(t) = \frac{-\rho A F_1 \sin\gamma_1 t}{\beta^5 \ EI\ m_1}\left(\cosh\beta L \sin\beta L + \cos\beta L \sinh\beta L\right) \qquad (3\text{-}6)$$

It is interesting to note that the pendulum displacement is inversely proportional to $m_1$. Thus, if design considerations limit the pendulum displacement to small amplitudes, $m_1$ must be larger accordingly.

It should be noted that nothing was stated about resonances or lack of resonances of the absorbed system for frequencies other than $\gamma_1$. The complete spectrum of resonant frequencies for the absorbed system can be found by solving equation (2-22).

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[1]Kin N. Tong, Theory of Mechanical Vibrations. (New York, 1963), p. 258.

## Design of Multiple Mass Pendulum Absorber

If the forcing function acting on the beam consists of more than one sinusoidal component, and if none of these frequencies are such that equation (3-3) is satisfied, then it is possible to design a multiple mass pendulum absorber which will effectively pin the upper end of the beam. Also, the absorber will prevent beam resonance from occurring. The purpose of this section is to show how such an absorber can be designed.

In chapter two, it is shown that the resonant frequencies of a cantilever beam with a multiple pendulum attached are roots of the transcendental equation (3-7).

$$0 = D_1 + D_2 \quad . \tag{3-7a}$$

$$D_1 = \sin\beta L - \sinh\beta L - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)\left(\cos\beta L - \cosh\beta L\right) \quad . \tag{3-7b}$$

$$D_2 = \frac{-\ell_1[EI\beta^3(2 + 2\cosh\beta L\cos\beta L)]}{\left(\gamma^2 m_{1n}\sum_{i=1}^{n} m_i s_{i1}\right)\left(\cos\beta L + \cosh\beta L\right)} \quad . \tag{3-7c}$$

$D_1$ will be zero if and only if equation (3-5) is satisfied. If equation (3-5) is not satisfied, and if $D_2$ is forced to be zero at the forcing function frequencies, then it can be guaranteed that resonance will not occur at these frequencies.

In Appendix B, it is shown that $\gamma^2 m_{in} \sum_{i=1}^{n} m_i s_{i1}$ approaches infinity as $\gamma_i$ approaches $\gamma$ (a natural frequency of the uncoupled pendulum). Thus, if the absorber pendulum has as its uncoupled natural frequencies the frequencies of the forcing function, then $D_2$ will be zero. Thus, if equation (3-5) is not satisfied by any of the forcing function frequencies, then it can be guaranteed that resonance will not occur at these frequencies.

In order to design a multiple pendulum of order n which has n specified uncoupled resonant frequencies, it is necessary to solve equations (3-8) simultaneously for the pendulum parameters.

$$| - \gamma_i^2(M) - g(N)| = 0 \quad ^* \qquad (3\text{-}8)$$

$$i = 1, \dots, n$$

(M)and (N) are defined by equations (2-33) and (2-34) respectively. An n-mass pendulum has the parameters $m_i$ and $\ell_i$, $i = 1, \dots, n$. Thus the system is underdetermined, and there are an infinitude of suitable pendulums to choose from. Some design applications may place restrictions on certain masses or certain lengths. Since there are 2n pendulum parameters and only n restraint equations, considerable freedom is left to the designer in choosing a suitable pendulum.

In most design applications only a few of the frequencies given in equation (3-1) will cause the unabsorbed beam to have large amplitudes. Thus, the designer may feel that it is only necessary to design an absorber to make the coupled system non-resonant at these frequencies. Such a pendulum design would undoubtedly be easier because of the reduced pendulum order. However, in such an instance, the coupled system may be sensitive to frequencies in equation (3-1) for which the free beam was unsensitive. Thus, it is best to use as an absorber a multiple pendulum which has all of the frequencies in

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*It should be noted that these equations are nonlinear in $m_i$ and $\ell_i$. Hence, a numerical solution will probably be necessary when n is large.

equation (3-1) as uncoupled resonant frequencies.

When a beam is acted upon by a forcing function as given in (3-1), and a multiple pendulum absorber (as suggested in this section) is attached at the upper end of the beam, the deflection equation is somewhat simpler than the deflection equation as defined in equations (2-43). The simplified displacement equations become equations (3-9).

$$v(x,t) = \sum_{i=1}^{n} v_i(x,t) \quad . \tag{3-9a}$$

$$\left\{v_i'(t)\right\} = \sum_{i=1}^{n} \{u\}_i \sin\gamma_i t \quad . \tag{3-9b}$$

$$v_i(x,t) = \frac{F_i \sin\gamma_i t}{\beta_i^4 EI}\left(\frac{B_1 B_2}{B_3} + B_4\right) \quad . \tag{3-9c}$$

$$B_1 = 1 - \frac{2\cos\beta_i L \cosh\beta_i L}{\cos\beta_i L + \cosh\beta_i L} \quad . \tag{3-9d}$$

$$B_2 = \sin\beta_i x - \sinh\beta_i x - \left(\frac{\sin\beta_i L + \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}\right)\left(\cos\beta_i x - \cosh\beta_i x\right) \quad . \tag{3-9e}$$

$$B_3 = \sin\beta_i L - \sinh\beta_i L - \left(\frac{\sin\beta_i L + \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}\right)\left(\cos\beta_i L - \cosh\beta_i L\right) \quad . \tag{3-9f}$$

$$B_4 = \frac{\cosh\beta_i L \cos\beta_i x + \cos\beta_i L \cosh\beta_i x}{\cos\beta_i L + \cosh\beta_i L} - 1 \quad . \tag{3-9g}$$

$$\{u\}_i = B_5 \, B_6 \, B_7 \quad . \tag{3-9h}$$

$$B_5 = \frac{-F_i \rho A_b}{g\beta_i^5 EI} \quad . \tag{3-9i}$$

$$B_6 = \frac{\cosh\beta_i L \sin\beta_i L + \cos\beta_i L \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L} \tag{3-9j}$$

$$B_7 = \frac{\{\omega\}_i}{\sum_{j=1}^{n} m_j \, \omega_{ji}} \quad . \tag{3-9k}$$

In equation (3-9k), $\{\omega\}_i$ is the eigenvector of the uncoupled pendulum corresponding to the eigenvalue $\gamma_i^2$. $\omega_{ji}$ is the $j^{th}$ element in $\{\omega\}_i$ .

## PART IV

## EXPERIMENTAL PROCEDURES AND RESULTS

In order to test the derived equations contained in previous portions of this report, various experiments were devised and executed. Since experimental equipment was not available for providing a uniform sinusoidal forcing function as considered previously, it was not possible to test the displacement equations as such. Instead, a point forcing function was used in the experiments to obtain resonant frequencies of various cantilever systems. Since resonance should generally occur at a frequency regardless of the distribution of the forcing function, the characteristic equations (2-15), (2-22), and (2-44) could be tested by using a point forcing function.

A description of the parameters involved in the experiments is listed in table 1. In experiments B and C, absorbers were designed so as to have uncoupled resonant frequencies corresponding to the first resonant frequency of the uncoupled beam (experiment B) and the first two resonant frequencies of the uncoupled beam (experiment C). It was felt that this would be a more rigorous test of absorber performance than tests in which absorbers did not possess these properties.

Three main experiments as described in table 1. were conducted. A schematic of the experimental apparatus used is shown in figure 5. Theoretical resonant frequencies were obtained by using the computer programs found in appendix D.

TABLE 1
EXPERIMENT DESCRIPTIONS

EXPERIMENT A: Unabsorbed cantilever beam

BEAM DIMENSIONS: 60" × 1" × 3/16"

BEAM MATERIAL: Reynolds aluminum No. 2024 T351 QQA 268

BEAM PARAMETERS: $A_b = .188 \text{ in}^2$, $E = 10.3 \times 10^6$ PSI

$\rho = .101 \frac{\text{lbm}}{\text{in}^3}$, $L = 60"$ , $I = 5.53 \times 10^{-4} \text{ in}^4$

ABSORBER DESCRIPTION: No absorber

EXPERIMENT B:

BEAM DIMENSIONS: Same as experiment A

BEAM MATERIAL: Same as experiment A

BEAM PARAMETERS: Same as experiment A

ABSORBER DESCRIPTION: Single pendulum, $\ell_1 = 3.54"$, $m_1 = .104$ lbm

EXPERIMENT C:

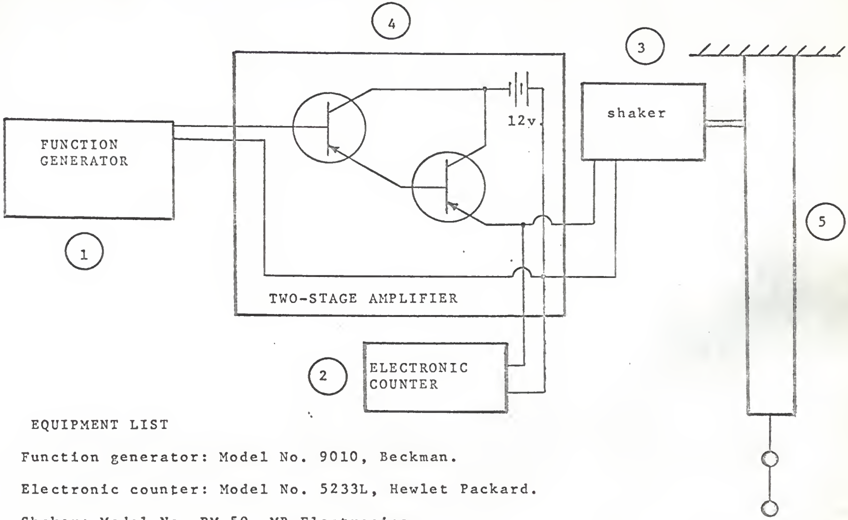
BEAM DIMENSIONS: Same as experiment A

BEAM MATERIAL: Same as experiment A

BEAM PARAMETERS: Same as experiment A

ABSORBER DESCRIPTION: Double mass pendulum, $\ell_1 = 2.00"$ ,

$\ell_2 = 1.63"$, $m_1 = .028$ lbm, $m_2 = .254$ lbm.




EQUIPMENT LIST

1. Function generator: Model No. 9010, Beckman.
2. Electronic counter: Model No. 5233L, Hewlet Packard.
3. Shaker: Model No. PM-50, MB Electronics.
4. Two-Stage amplifier: Fabricated.
5. Absorbed beam specimen.

Figure 5.

Diagram of Experimental Apparatus

TABLE 2

EXPERIMENTAL AND THEORETICAL RESONANT FREQUENCIES FOR EXPERIMENTS A, B, AND C

| | | | | | |
|---|---|---|---|---|---|
| Experiment A (no absorber) | Theoretical Resonant frequencies (cps) | 1.67 | 10.48 | 29.35 | |
| | Experimental resonant frequencies (cps) | 1.90 | 11.40 | 31.0 | |
| Experiment B (single mass absorber: $\ell_1$ = 3.54" $m_1$ = .104 lbm.) | Theoretical resonant frequencies (cps) | 1.23 | 2.23 | 10.6 | |
| | Experimental resonant frequencies (cps) | 1.32 | 2.21 | 11.80 | |
| Experiment C (double mass absorber: $\ell_1$ = 2", $\ell_2$ = 1.63", $m_1$ = .028 lbm. $m_2$ = .254 lbm.) | Theoretical resonant frequencies | 1.03 | 2.59 | 9.78 | 11.34 |
| | Experimental resonant frequencies (cps) | 1.13 | 2.74 | 6.00 | 11.90 |

The theoretical and experimental resonant frequencies for the various experiments are listed in table 2.

The experimental resonant frequencies were found by starting well below the first resonant point, and slowly increasing the frequency until beam displacements increased substantially over a small frequency range. The frequency was then increased until displacements decreased. The frequency range of large displacements was then retraced at a slower rate in order to determine the frequency at which displacements were largest. Since the frequency range contained large displacements, the amplitude of the forcing function was reduced in order to preserve the linearity of the system. An electronic counter was then used to determine this frequency.

The lower resonant frequencies were very easily detected. At higher frequencies, however, resonant detection was more difficult due to smaller displacements. For this reason, only the first several resonant frequencies were found experimentally.

In general, the experimental resonant frequencies were about 10% higher than the theoretical resonant frequencies. It was felt that the main reason for this discrepancy was a slight error in the value of E used in the analytical work. If a larger value of E had been used in the analytical work, then the results in experiment A would have been closer in agreement. The discrepancies in the results of experiments B and C were probably caused by slight errors in the values of E and the pendulum parameters used in the analytical solution. It was felt that the agreement of results in table 2 was sufficient to validate the characteristic equations of chapter two.

The first several resonant mode shapes for the cantilever systems considered were determined by using the computer programs found in appendix D.

Normalized mode shapes for experiments A, B, and C are shown in figures 6, 7, and 8 respectively.

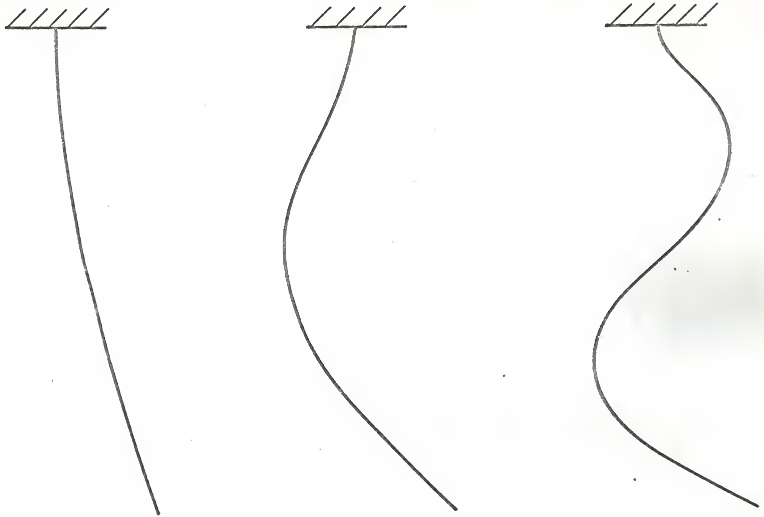
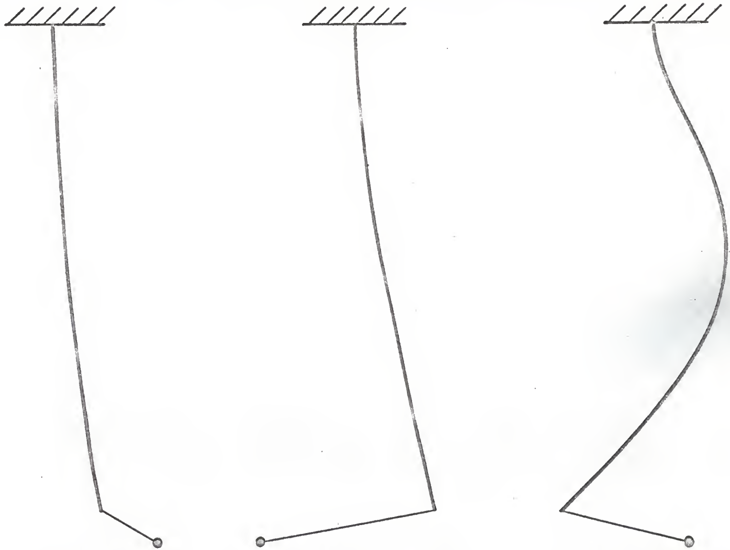
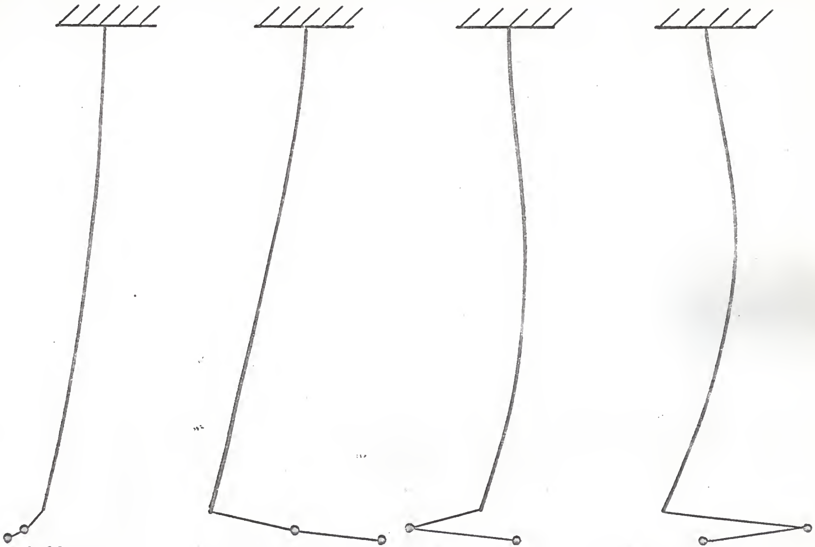



Figure 6. Mode Shapes of Experiment A.



$f_1$ = 1.23 cps. $f_2$ = 2.23 cps. $f_3$ = 10.60 cps.

Figure 7. Mode Shapes of Experiment B.



$f_1$ = 1.03 cps. $f_2$ = 2.59 cps. $f_3$ = 9.78 cps. $f_4$ = 11.34 cps.

Figure 8. Mode Shapes of Experiment C.

## PART V

## SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

It has been shown that for a beam which is forced transversely by a uniform forcing function $F = F_1 \sin\gamma_1 t + \ldots + F_n \sin\gamma_n t$, a properly designed multiple pendulum absorber attached at the unfixed end will prevent resonance from occurring under most conditions. If the multiple pendulum has as its uncoupled resonant frequencies the frequencies of the forcing function, then lack of resonance can be guaranteed provided none of the $\gamma_i$'s correspond to natural frequencies of the fixed-pinned beam. Under these conditions, it can also be guaranteed that the unfixed end of the beam will be effectively pinned.

Since resonant frequencies are, in general, properties of the physical system and not properties of the forcing function distribution, the lack of resonance can, in general, be guaranteed even if the forcing function is not uniform along the length of the beam.

In design instances where the forcing function contains a large number of frequencies, the design of a multiple pendulum of this high order may not be feasible. An alternative approach would be to design a multiple pendulum (of somewhat lower order) which had as its uncoupled natural frequencies, only the forcing function frequencies with large coefficients. Lack of resonance would then be guaranteed at these frequencies. The resonant frequencies of the coupled system could then be obtained by using the computer programs found in appendix D. If none of these frequencies corresponded to frequencies of the forcing function, then lack of resonance could be guaranteed for the coupled system.

SELECTED REFERENCES

1. Reed, Wilmer H. III, and Duncan, Rodney L., "Dampers to Suppress Wind-Induced Oscillations of Tall Flexible Structures." Proceedings of the Tenth Midwestern Mechanics Conference, 1967, 881-897.

2. Thomson, William T. Vibration Theory and Applications. Englewood Cliffs, N. J.: Prentice Hall, Inc., 1965.

3. Tong, Kin N. Theory of Mechanical Vibrations. New York: John Wiley and Sons, Inc., 1960.

# A P P E N D I X

# APPENDIX A

## SUPERPOSITION PROOF

It was shown that for a uniformly distributed forcing function with a single frequency, the following solution exists for the displacement equations.

$$v(x,t) = r(x)\sin\gamma t \quad .$$

$$v'(t) = \{u\}\sin\gamma t \quad .$$

It will now be proven that if the forcing function $F = F_1\sin\gamma_1 t + \ldots + F_r\sin\gamma_r t$ acts on the beam, then the resultant displacement equations will be as shown below.

$$v(x,t) = \sum_{i=1}^{r} r_i\sin\gamma_i t \quad . \tag{A-1}$$

$$v'(t) = \sum_{i=1}^{r} \{u\}_i\sin\gamma_i t \quad . \tag{A-2}$$

In the above equations, $r_i\sin\gamma_i t$ and $\{u\}_i\sin \gamma_i t$ are the resultant solutions when the forcing function is $F = F_i\sin\gamma_i t$.

The governing differential equation and the boundary conditions for a multiple frequency forcing function are given below.

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\rho A_b}{EI}\frac{\partial^2 v(x,t)}{\partial t^2} = \sum_{i=1}^{r}\frac{F_i\sin\gamma_i t}{EI} \quad . \tag{A-3a}$$

$$v(0,t) = 0 \quad . \tag{A-3b}$$

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0 \quad . \tag{A-3c}$$

$$\frac{\partial^2 v(x,t)}{\partial t^2}\Bigg|_{x=0} = 0 \quad . \tag{A-3d}$$

$$EI\,\frac{\partial^3 v(x,t)}{\partial x^3}\Bigg|_{x=L} = \sum_{i=1}^{r}\ \sum_{j=1}^{n} m_j\,\frac{\partial^2\,{}_iv_q'^{*}(t)}{\partial t^2} \quad . \tag{A-3e}$$

It will now be shown that equations (A-3) are satisfied when $v(x,t)$ and $v'(t)$ are as defined in equations (A-1) and (A-2).

By substituting equation (A-1) into equation (A-3a), equation (A-4a) is obtained.

$$\sum_{i=1}^{r}\frac{\partial^4 v_i(x,t)}{\partial x^4} + \frac{\rho A_b}{EI}\sum_{i=1}^{r}\frac{\partial^2 v_i(x,t)}{\partial t^2} = \sum_{i=1}^{r}\frac{F_i \sin\gamma_i t}{EI} \quad . \tag{A-4a}$$

This equation is obviously true because it is true for each $i^{th}$ component. By substituting equation (A-1) into equations (A-3b), (A-3c), and (A-3d), equations (A-4b), (A-4c), and (A-4d) are obtained.

$$\sum_{i=1}^{r} v_i(0,t) = 0 \quad . \tag{A-4b}$$

$$\sum_{i=1}^{r}\frac{\partial v}{\partial x}(x,t)\Bigg|_{x=0} = 0 \quad . \tag{A-4c}$$

$$\sum_{i=1}^{r}\frac{\partial^2 v_i}{\partial x^2}(x,t)\ \Big|_{x=L} = 0 \quad . \tag{A-4d}$$

---

*Note: ${}_iv_q'(t)$ is the displacement of $m_q$ due to the loading $F_i \sin\gamma_i t$ .

As in the case of equation (A-4a), these equations are valid because they are valid for each $i^{th}$ component.

By substituting equations (A-1) and (A-2) into equation (A-3e) the last boundary condition becomes equation (A-4e).

$$\sum_{i=1}^{r} EI \frac{\partial^3 v_i(L,t)}{\partial x^3} = \sum_{i=1}^{r} \sum_{q=1}^{n} m_q \frac{\partial^2 {}_i v_q^{*}{}'(t)}{\partial t^2} \quad . \tag{A-4e}$$

As in the previous cases, this equation is valid because it is valid for each $i^{th}$ component. Thus the resultant displacement equations due to a loading function of the form $F = \sum_{i=1}^{r} F_i \sin \gamma_i t$ are as shown in equations (A-1) and (A-2).

## APPENDIX B
## Singularity Limit Proof

The purpose of this appendix is to prove that as $\gamma$ approaches $\gamma_s$ (where $\gamma_s$ is a natural frequency of the uncoupled pendulum), then $\{s_{i1}\}$ approaches $\infty\{\omega\}$. $\{\omega\}$ is the normalized eigenvector of the uncoupled pendulum corresponding to $\gamma_s$. This proof is divided into 3 parts.

### Part 1

The purpose of this part is to show that the two multiple pendulums in the following figure have no common resonant frequencies.

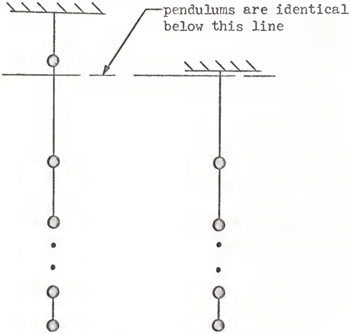



Figure 9.

Similar Multiple Pendulums

These two pendulums are identical except for one added mass and link as shown.

When pendulums 1 and 2 are two mass and one mass pendulums respectively, the resultant characteristic equations are as given in equations (A-5) and

(A-6).

ONE MASS PENDULUM: $0 = -\gamma^2 m_2 + \frac{gm_2}{\ell_2}$ . (A-5)

TWO MASS PENDULUM:

$$0 = \begin{vmatrix} -\gamma^2 m_1 + g\,\frac{m_{1\,2}}{\ell_1} + \frac{m_{2\,2g}}{\ell_2} & -g\,\frac{m_{2\,2}}{\ell_2} \\ \\ \frac{-g\,m_{22}}{\ell_n} & -\gamma^2 m_2 + \frac{g\,m_{22}}{\ell_2} \end{vmatrix} . \qquad \text{(A-6)}$$

Notice that the lower right element in equation (A-6) is the right side of equation (A-5). If these two pendulums have a common natural frequency, then the common frequency must be as defined by equation (A-5). If this is so, then by equation (A-6), $\left(\frac{gm_2}{\ell_2}\right)^2$ must be zero. Since this is impossible, the two pendulums do not have a common natural frequency.

When pendulums 1 and 2 are 3 mass and 2 mass pendulums respectively, then the resultant characteristic equations are as shown in equations (A-7) and (A-8).

$$\begin{vmatrix} -\gamma^2 m_2 + g\frac{m_{2\,3}}{\ell_2} + g\frac{m_{3\,3}}{\ell_3} & , & -\frac{g\,m_{3\,3}}{\ell_3} \\ \\ -\frac{g\,m_{3\,3}}{\ell_3} & , & -\gamma^2 m_3 + \frac{g\,m_{3\,3}}{\ell_3} \end{vmatrix} \qquad \text{(A-7)}$$

$$0 = \begin{vmatrix} -\gamma^2 m_1 + \frac{g m_{13}}{\ell_1} + \frac{g m_{23}}{\ell_2} & , & -\frac{g m_{23}}{\ell_2} & , & 0 \\ -\frac{g m_{23}}{\ell_2} & , & -\gamma^2 m_2 + \frac{g m_{23}}{\ell_2} + \frac{g m_{33}}{\ell_3} & , & -\frac{g m_{33}}{\ell_3} \\ 0 & , & -\frac{g m_{33}}{\ell_3} & , & -\gamma^2 m_3 + \frac{g m_{33}}{\ell_3} \end{vmatrix} \tag{A-8}$$

Notice that the elements in equation (A-7) are identically equal to the 2X2 array of elements in the lower right corner of equation (A-8). By denoting the characteristic determinants by $D_3$, $D_2$, and $D_1$ for a 3 mass, 2 mass, and 1 mass pendulum, respectively; equations (A-7) and (A-8) become $D_2 = D_3 = 0$ when equations (A-7) and (A-8) are satisfied simultaneously. By expanding equation (A-8) by minors of the first column, equation (A-9) is obtained.

$$D_3 = 0 = \left(-\gamma^2 m_1 + g\frac{m_{13}}{\ell_1} + g\frac{m_{23}}{\ell_2}\right) D_2 - \left(g\frac{m_{23}}{\ell_2}\right)^2 D_1 \quad . \tag{A-9}$$

If equations (A-7) and (A-8) are satisfied by the same value of $\gamma$, then $D_3 = D_2 = 0$. From equation (A-9), $D_1 = 0$ for this value of $\gamma$. I previously proved that $D_1$ and $D_2$ are not both zero for any common value of $\gamma$. Hence, the 3-mass and the 2-mass pendulums under consideration have no common natural frequencies.

It will now be proven that if a K-1 mass pendulum and a K mass pendulum have no common natural frequencies, then a K + 1 mass pendulum and a K mass pendulum will have no common natural frequencies. It must be kept in mind that the pendulums under consideration are identical except for the addition of masses and lengths at the upper ends of the pendulums. The preceeding statement will be proven by contradiction.

Assume that the K-1 mass pendulum and the K mass pendulum have no common natural frequencies. Assume also that the K+1 mass pendulum and the K mass pendulum do have at least one common natural frequency. The characteristic equation for the K+1 mass pendulum is given in equation (A-10).

$$0 = \begin{vmatrix} -\gamma^2 m_1 + \frac{gm_{1\ k+1}}{\ell_1} + \frac{gm_{2\ k+1}}{\ell_2} \ , & -\frac{gm_{2\ k+1}}{\ell_2} & 0, \ldots, 0 \\ -\frac{gm_{2\ k+1}}{\ell_2} & \multicolumn{2}{c}{k^{th} \text{ order characteristic determinant}} \\ 0 & & k\text{-}1^{th} \text{ order characteristic determinant} \\ \vdots & & \\ 0 & & \end{vmatrix} \qquad \text{(A-10)}$$

By expanding equation (A-10) by minors of the first row, equation (A-11) is obtained.

$$0 = D_{k+1} = \left(-\gamma^2 + \frac{gm_{1\ k+1}}{\ell_1} + \frac{gm_{2\ k+1}}{\ell_2}\right) D_k - \left(\frac{gm_{2\ k+1}}{\ell_2}\right)^2 D_{k-1} \ . \qquad \text{(A-11)}$$

If $D_{k+1}$ and $D_k$ are both zero for some value of $\gamma$, then, by equation (A-10), $D_{k-1}$ must be zero for the same value of $\gamma$. Since it was assumed that $D_{k-1}$ and $D_k$ are not both zero for any value of $\gamma$, $D_{k+1}$ and $D_k$ are not both zero for any value of $\gamma$.

It was shown previously that a 3-mass pendulum and a 2 mass pendulum of the types being considered have no common natural frequencies. Hence, by using the previous argument and mathematical induction, the n mass pendulum and the n-1 mass pendulum being considered have no common natural frequencies.

Part 2

The purpose of this part is to show that as $\gamma$ approaches one of the values of $\gamma_s$ then $s_{11}$ approaches $\infty$ . $s_{11}$ is the first element in the column vector $\{s_{i1}\}$, from equation (2-38),

$$(S) = (R)^{-1} = \frac{(\text{Adj } r_{ij})}{|R|} \quad . \tag{A-12}$$

The element $s_{11}$ now becomes:

$$s_{11} = \frac{(-1)^2 \, D_{n-1}}{D_n} \quad . \tag{A-13}$$

From part 1, $D_{n-1}$ does not approach zero as $D_n$ approaches zero. Since $D_n$ approaches zero as $\gamma$ approaches $\gamma_s$, $s_{11}$ approaches infinity as $\gamma$ approaches $\gamma_s$.

Part 3

The purpose of this part is to show that $\{s_{i1}\}$ approaches $\infty\{\omega\}_s$ as $\gamma$ approaches $\gamma_s$ . From equation (A-12),

$$\{s_{i1}\} = \frac{\{\text{Adj } r_{i1}\}}{|R|} \quad . \tag{A-14}$$

I shall now solve for $\{\omega\}_s$, assuming that the first element in $\{\omega\}_s$ is unity.

$$\omega_1 = 1 \; . \tag{A-15a}$$

$$(R)\{\omega\}_s = \{0\} \; . \tag{A-15b}$$

Expanding equations (A-15) yields equations (A-16) .

$$r_{12}\,\omega_2 + \dots + r_{1n}\omega_n = -r_{11}\ .$$

$$r_{22}\,\omega_2 + \dots + r_{2n}\omega_n = -r_{21}\ . \qquad (A\text{-}16)$$

$$r_{n2}\,\omega_2 + \dots + r_{nn}\omega_n = -r_{n1}\ .$$

Solving the lower n-1 equations of (A-16) by Cramer's rule results in (A-17).

$$\frac{\begin{vmatrix} r_{22} & , & -r_{11} & , \cdots , & r_{1n} \\ r_{23} & , & -r_{21} & , \cdots , & r_{2n} \\ & & \vdots & & \\ r_{2n} & , & -r_{n1} & , \cdots , & r_{nn} \end{vmatrix}}{\text{adj } r_{11}}\ . \qquad (A\text{-}17)$$

The i-1 column in the numerator in equation (A-17) is the right side of (A-16).

Noting that (R) is symmetric, and that the interchange of any two columns in a determinant results in a sign change; it becomes apparent that equation (A-17) is equivalent to equation (A-18).

$$\omega_i = \frac{(-1)^{1+i}\ M_{i1}}{\text{Adj } r_{11}} = \frac{\text{Adj } r_{i1}}{\text{Adj } r_{11}}\ . \qquad (A\text{-}18)$$

By comparing equations (A-14) and equation (A-18); it becomes apparent that as $\gamma$ approaches $\gamma_s$, and (R) approaches singularity, then $\{s_{i1}\}$ becomes as defined in equation (A-19).

$$\lim_{\gamma \to \gamma_s} \{s_{i1}\} = \infty\,\{\omega\}_s\ . \qquad (A\text{-}19)$$

It is interesting to note that in some instances $\sum_{i=1}^{n} \omega_i = 0$, but in no instance is equation (A-20) satisfied.

$$\sum_{i=1}^{n} \omega_i m_i = 0 \ . \tag{A-20}$$

If equation (A-20) were true, the momentum of the pendulum in the horizontal direction would necessarily be a constant. This, in turn would mean that $\ell_1$ would have to be vertical at all times. If $\ell_1$ were vertical at all times, $\ell_2$ would have to be vertical at all times, etc. Thus, the trivial case $\{\omega\}_i = \{0\}$, would exist.

Since $\sum_{i=1}^{n} m_i \omega_i$ is never zero, $\lim_{\gamma \to \gamma_s} \gamma m_{1n} \sum_{i=1}^{n} m_i s_{i1} = \infty$ . (A-21)

## APPENDIX C

### ROOTS OF TRANSCENDENTAL EQUATIONS*

| Root No. | LIST 1 | LIST 2 |
|---|---|---|
| | $\cosh\beta L \cos\beta L + 1 = 0$ | $\tan\beta L - \tanh\beta L = 0$ |
| 1 | 1.87 | 3.93 |
| 2 | 4.69 | 7.07 |
| 3 | 7.85 | 10.21 |
| 4 | 10.99 | 13.35 |
| 5 | 14.14 | 16.49 |
| 6 | 17.27 | 19.64 |
| 7 | 20.42 | 22.78 |
| 8 | 23.56 | 25.92 |
| 9 | 26.70 | 29.06 |
| 10 | 29.84 | 32.21 |
| 11 | 32.98 | 35.35 |
| 12 | 36.12 | 38.49 |
| 13 | 39.26 | 41.63 |
| 14 | 42.41 | 44.77 |
| 15 | 45.55 | 47.91 |

*This is only a partial list. For $n > 15$, Root list 1 $\simeq n\pi - \frac{\pi}{2}$

Root list 2 $\simeq n\pi + \frac{\pi}{4}$ .

# APPENDIX D

## COMPUTER PROGRAMS

```
C/////////////////////////////////////////////////////////////////////
C  THIS IS THE MAIN PROGRAM FOR CALCULATING THE ROOTS OF
C  (COSH(BL))*(COS(BL))+1.=0.0   (OPTION 1)
C**********************************************************************
C**********************************************************************
C  THIS IS ALSO THE MAIN PROGRAM FOR CALCULATING THE ROOTS OF
C  TAN(BL)-TANH(BL)=0.  (OPTION 2)
C**********************************************************************
C**********************************************************************
C  SUBPROGRAMS NEEDED ARE...
C  SUBROUTINE ROOT(F,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
C  FUNCTION FUN(X)
C  NOTE...  USE THE FUNCTION FUN(X) CORRESPONDING TO THE OPTION DESIRED
C**********************************************************************
C**********************************************************************
C**********************************************************************
  101 FORMAT(4E14.5)
      DELI=.01
      X=.5
      DDEL=.1
      DELX=.01
      XMAX=100.
      ILAST=1
      IPRINT=0
      DO100I=1,50
      CALL ROOT(X,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
      X=XL+3.05
      WRITE(3,101)XS,YS,XL,YL
  100 CONTINUE
      STOP
      END
C/////////////////////////////////////////////////////////////////////
```

```
C/////////////////////////////////////////////////////////////////////
C  THIS IS THE MAIN PROGRAM FOR CALCULATING THE RESONANT FREQUENCIES
C  FOR A CANTILEVER BEAM WITH A SINGLE MASS PENDULUM ATTACHED AT THE
C  UPPER END.
C  SUBPROGRAMS NEEDED ARE.......
C  FUNCTION(F)   OPTION 3
C  SUBROUTINE ROOT(F,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
C**********************************************************************
C**********************************************************************
C**********************************************************************
  101 FORMAT(4E14.5)
```

```
      DELI=.1
      F=.2
      DDEL=.1
      DELX=.01
      XMAX=10000.
      ILAST=1
      IPRINT=0
      DO100I=1,20
      CALL ROOT(F,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
      F=XL
      WRITE(3,101)XS,YS,XL,YL
  100 CONTINUE
      STOP
      END
C//////////////////////////////////////////////////////////////////////


C//////////////////////////////////////////////////////////////////////
C  THIS IS THE MAIN PROGRAM FOR CALCULATING THE DISPLACEMENT EQUATION
C  FOR A VERTICAL CANTILEVER BEAM WITH A SINGLE MASS PENDULUM ABSORBER.
C**********************************************************************
C**********************************************************************
C  BEAM DISPLACEMENTS FOR THE UNABSORBED BEAM ARE ALSO CALCULATED FOR
C  THE SAME FREQUENCIES AND VALUES OF X.
C**********************************************************************
C**********************************************************************
C  THE BEAM FOR WHICH THE PARAMETER ARE ENTERED IS THE BEAM IN
C  EXPERIMENTS A,B,AND C.
C**********************************************************************
C**********************************************************************
C  THE BEAM LOADING IS 1.0 POUND PER INCH.
C**********************************************************************
C**********************************************************************
C  BEAM DISPLACEMENTS ARE CALCULATED FOR X=0,6,12,18,...,60 INCHES.
C**********************************************************************
C**********************************************************************
C  THE FREQUENCY RANGE IS 1.0 RAD./SEC. TO 300.0 RAD./SEC.  AS THE
C  PROGRAM STANDS .
C**********************************************************************
C**********************************************************************
C  SUBPROGRAMS NEEDED ARE......
C  FUNCTION SUBPROGRAM VFB(X,B,AL)
C  FUNCTION SUBPROGRAM VSM(X,B,AL,E,AI,AM,D,G)
C**********************************************************************
C**********************************************************************
C**********************************************************************
C    AM=PENDULUM MASS IN LBM.
C    D=PENDULUM LENGTH IN INCHES.
C    AL=BEAM LENGTH IN INCHES.
C    E=MODULUS OF ELASTICITY IN PSI.
C    AB=CROSS SECTIONAL AREA OF BEAM.
```

```
C  AG=ACCELERATION OF GRAVITY (386 IPS).
C   RC=BEAM MATERIAL DENSITY IN LBM/IN**3.
C   B=BETA.
C   FR=FREQUENCY (CPS).
C   G=GAMMA  (RAD/SEC).
C   VF(J)=FREE BEAM DISPLACEMENT.
C   VP(J)=ABSORBED BEAM DISPLACEMENT.
C   VPP=PENDULUM MASS DISPLACEMENT.
C***********************************************************************
C***********************************************************************
C***********************************************************************
      DIMENSION VF(20),VP(20)
  101 FORMAT(1H0,F7.3,F7.3,F6.5,11E10.3)
  102 FORMAT(1H ,F7.3,F7.3,F6.5,11E10.3)
  103 FORMAT(1H ,E10.3)
      AM=.104
      D=3.54
      AL=60.
      E=10.3E6
      AI=5.53E-4
      AB=.188
      RO=.101
      AG=386.
      DO1I=1,3000
      G=FLOAT(I)
      FR=G/2./3.14159
      B=((RO*AB*G*G)/(E*AI*386.))**.25
      X=-6.
      GS=G*G
      DO2J=1,11
      X=X+6.
      VF(J)=VFB(X,B,AL)
      VP(J)=VSM(X,B,AL,E,AI,AM,D,G)
    2 CONTINUE
      WRITE(3,101)G,FR,B,(VF(J),J=1,11)
      WRITE(3,102)G,FR,B,(VP(J),J=1,11)
      VPP=VP(11)/(1.-GS*D/AG)
    1 WRITE(3,103)VPP
      STOP
      END
C///////////////////////////////////////////////////////////////////////


C///////////////////////////////////////////////////////////////////////
C  THIS IS THE MAIN PROGRAM FOR CALCULATING THE DISPLACEMENTS OF
C  A CANTILEVER BEAM WITH A MULTIPLE PENDULUM ATTACHED AT THE UNFIXED
C  END.  LOADING IS 1 POUND PER INCH.
C***********************************************************************
C***********************************************************************
C  THIS PROGRAM ALSO DETERMINES THE RESONANT FREQUENCIES OF A CANTILEVER
C  BEAM WITH A MULTIPLE PENDULUM ATTACHED AT THE UNFIXED END.
```

```
C*****************************************************************
C*****************************************************************
C  DEFLECTIONS OF UNABSORBED BEAM ARE ALSO CALCULATED.
C*****************************************************************
C*****************************************************************
C  FREQUENCY RANGE IS .25 RAD/SEC TO 75 RAD/SEC AS THE PROGRAM STANDS.
C*****************************************************************
C*****************************************************************
C  RESONANT FREQUENCIES OCCUR WHEN C2 AND C8 CHANGE SIGNS SIMULTANEOUSL
C*****************************************************************
C*****************************************************************
C  DISPLACEMENTS ARE CALCULATED FOR X=0,6,12,18,...,60 INCHES.
C*****************************************************************
C*****************************************************************
C  IF PENDULUM ORDER IS GREATER THAN 10, DIMENSION STATEMENTS MUST BE C
C*****************************************************************
C*****************************************************************
C  BEAM PARAMETERS ARE THOSE USED IN EXPERIMENTS A,B,AND C.
C*****************************************************************
C*****************************************************************
C  PENDULUM PARAMETERS ARE THOSE USED IN EXPERIMENT C.
C*****************************************************************
C*****************************************************************
C  SUBPROGRAMS NEEDED ARE........
C  FUNCTION VFB(X,B,AL)
C  SUBROUTINE VMP(X,B,AL,E,AI,G,D1,VNP)
C  SUBROUTINE MINV(A,N,D,L,M)
C  FUNCTION AAM(I)
C*****************************************************************
C*****************************************************************
C  AM(I)=MASS OF THE ITH. PENDULUM MASS  (LBM.)
C  CL(I)=LENGTH OF LINK(I)
C  K=ORDER OF THE PENDULUM
C  VF(J)=FREE BEAM DISPLACEMENT AT X=6*J (IN)
C  VAB(J)=ABSORBED BEAM SISPLACEMENT AT X=6*J  (IN)
C  VPP(L)=DISPLACEMENT OF LTH. PENDULUM MASS IN INCHES
C   AL=BEAM LENGTH IN INCHES.
C   E=MODULUS OF ELASTICITY IN PSI.
C   AB=CROSS SECTIONAL AREA OF BEAM.
C  AG=ACCELERATION OF GRAVITY (386 IPS).
C   RC=BEAM MATERIAL DENSITY IN LBM/IN**3.
C   B=BETA.
C   FR=FREQUENCY (CPS).
C   G=GAMMA  (RAD/SEC).
C*****************************************************************
C*****************************************************************
C*****************************************************************
      DIMENSION VF(20),VPP(10),VAB(20)
      COMMON AM(10),CL(10),AR(100),K
  102 FORMAT(1H ,F7.3,F7.3,F6.5,11E10.3)
  103 FORMAT(1H ,E10.3)
  104 FORMAT(10E12.3)
```

```
  105 FORMAT(I5)
  109 FORMAT(17H-START OF NEW RUN)
      K=2
      AM(1)=.0276
      AM(2)=.2535
      CL(1)=2.0
      CL(2)=1.635
      AL=60.
      E=10.3E6
      AI=5.53E-4
      AB=.188
      RO=.101
      AG=386.
      DO1I=1,300
      G=FLOAT(I)/4.
      FR=G/2./3.14159
      B=((RO*AB*G*G)/(E*AI*386.))**.25
      X=-6.
      GS=G*G
      DO2J=1,11
      X=X+6.
      VF(J)=VFB(X,B,AL)
      CALL VMP(X,B,AL,E,AI,G,D1,VNP)
      VAB(J)=VNP
    2 CONTINUE
      WRITE(3,102)G,FR,B,(VF(J),J=1,11)
      WRITE(3,102)G,FR,B,(VAB(J),J=1,11)
      DO3L=1,K
    3 VPP(L)=(AR(L))*VAB(11)*(AAM(1))*AG/(CL(1))
      WRITE(3,104)(VPP(L),L=1,K)
    1 CONTINUE
      STOP
      END
C///////////////////////////////////////////////////////////////////////
```

SUBPROGRAMS

```
C///////////////////////////////////////////////////////////////////////
      FUNCTION FUN(X)
C**********************************************************************
C  OPTION 1
C**********************************************************************
      FUN=(COSH(X))*(COS(X))+1.
      RETURN
      END
C///////////////////////////////////////////////////////////////////////
```

```
C///////////////////////////////////////////////////////////////////////
      FUNCTION FUN(X)
C***********************************************************************
C  OPTION 2
C***********************************************************************
      FUN=TAN(X)-TANH(X)
      RETURN
      END
C///////////////////////////////////////////////////////////////////////


C///////////////////////////////////////////////////////////////////////
      FUNCTION FUN(F)
C***********************************************************************
C  OPTION 3
C***********************************************************************
      G=F*2.*3.14159
      GS=G*G
      RO=.101
      AL=60.
      E=10.3E6
      AI=5.53E-4
      AB=.188
      AG=386.
      B4=RO*AB*G*G/(E*AI*386.)
      B=B4**.25
      BL=B*AL
      AM=.104
      D=3.54
      D11=1./GS-D/AG
      D1=-AM/D11
      D2=E*AI*B*B/(COS(BL)+COSH(BL))
      D2=D2*386.
      C21=2.+2.*COSH(BL)*COS(BL)
      C22=D2*C21
      C23=(SIN(BL)+SINH(BL))*(COS(BL)-COSH(BL))
      C24=C23/(COS(BL)+COSH(BL))
      C25=SIN(BL)-SINH(BL)-C24
      C2=D1*C25+C22
      FUN=C2
      RETURN
      END
C///////////////////////////////////////////////////////////////////////


C///////////////////////////////////////////////////////////////////////
      SUBROUTINE ROOT(X,DELI,DDEL,DELX,XMAX,ILAST,XS,XL,YS,YL,IPRINT)
C  SUBROUTINE ROOT WAS WRITTEN BY DR. H. S. WALKER, MECHANICAL ENGINEER
C  DEPARTMENT, KANSAS STATE UNIVERSITY
C     THIS SUBROUTINE CALLS FUN(X) *************************************
```

```
C       IPRINT=1 PRINTS XS,XL,YS,YL.  IPRINT=0 DOES NOT PRINT. ********
C       ROOTS OF THE TRANSCENDENTAL EQUATION FUN(X) ********************
C       DELI IS THE INITIAL INCREMENT **********************************
C       DDEL IS THE FACTOR FOR DIVIDING THE INTERVAL *******************
C       DELX IS THE ACCURACY ON THE SOLUTION *MMMMMMMMMM---------  -
C       XMAX IS THE MAXIMUM THAT THE ARGUMENT X MAY BECOME ************
C       ILAST IS THE NUMBER OF ROOTS DESIRED **************************
      4 FORMAT(4E20.8)
        IUI=0
        X=X-DELI
     10 DEL = DELI
        X = X + DEL
        Y=FUN(X)
        IF(Y) 98,500,99
     98 ZAZ=-1.0
        GO TO 100
     99 ZAZ=+1.0
    100 XS = X
        YS = Y
        IF(X.GT.XMAX)  GO TO 510
        X = X + DEL
        Y=FUN(X)
        IF(ZAZ*Y) 110,500,100
    110 XL = X
        YL = Y
    120 X = XS - YS*((XL-XS)/(YL-YS))
        IF(XL-XS-DELX) 500,130,130
    130 Y=FUN(X)
        DEL = DEL*DDEL
        IF(ZAZ*Y) 140,500,100
    140 XL = X
        YL = Y
        X = X - DEL
        Y=FUN(X)
        IF(ZAZ*Y) 140,500,150
    150 XS = X
        YS = Y
        GO TO 120
    500 IUI=IUI+1
        IF(IPRINT.EQ.0) GO TO 900
        WRITE(3,4)  XS,XL,YS,YL
    900 CONTINUE
    502 IF(ILAST-IUI) 510,510,10
    510 CONTINUE
C       * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
        RETURN
        END
C///////////////////////////////////////////////////////////////////////
```

```
C///////////////////////////////////////////////////////////////////
      FUNCTION VFB(X,B,AL)
C     DEFLECTION FOR BEAM WITH NO PENDULUUM
C*******************************************************************
C*******************************************************************
      AI=5.5E-4
      E=10.3E6
      BL=B*AL
      BX=B*X
      C1=(COSH(BL))*(SIN(BL))+(COS(BL))*(SINH(BL))
      C2=C1/(2.+2.*(COS(BL))*(COSH(BL)))
      C3=COS(BX)-COSH(BX)
      C4=C3*(SIN(BL)+SINH(BL))/(COS(BL)+COSH(BL))
      C5=SIN(BX)-SINH(BX)-C4
      C6=(COSH(BL))*(COS(BX))+(COS(BL))*(COSH(BX))
      C7=C6/(COS(BL)+COSH(BL))
      C8=C7-1.
      VFB=(C2*C5+C8)/(B**4.)
      VFB=VFB/E/AI
      RETURN
      END
C///////////////////////////////////////////////////////////////////


C///////////////////////////////////////////////////////////////////
      FUNCTION VSM(X,B,AL,E,AI,AM,D,G)
C     DEFLECTION FOR BEAM WITH A SIMPLE PENDULLM
C*******************************************************************
C*******************************************************************
      BX=B*X
      AG=386.
      BL=B*AL
      C11=COSH(BL)*COS(BX)+COS(BL)*COSH(BX)
      C12=C11/(COS(BL)+COSH(BL))
      C1=C12-1.
      GS=G*G
      D11=1./GS-D/AG
      D1=-AM/D11
      D2=E*AI*B*B*B/(COS(BL)+COSH(BL))
      D2=D2*386.
      C21=2.+2.*COSH(BL)*COS(BL)
      C22=D2*C21
      C23=(SIN(BL)+SINH(BL))*(COS(BL)-COSH(BL))
      C24=C23/(COS(BL)+COSH(BL))
      C25=SIN(BL)-SINH(BL)-C24
      C2=D1*C25+C22
      C31=(SIN(BL)+SINH(BL))*(COS(BX)-COSH(BX))
      C32=C31/(COS(BL)+COSH(BL))
      C33=SIN(BX)-SINH(BX)-C32
      C34=COSH(BL)*SIN(BL)+COS(BL)*SINH(BL)
      C35=D2*C34
```

```
      C36=1.-2.*COSH(BL)*COS(BL)/(COS(BL)+COSH(BL))
      C37=C36*D1
      C3=(C37+C35)*C33
      VSM=(C3/C2+C1)/(B**4.)
      VSM=VSM/E/AI
      RETURN
      END
C///////////////////////////////////////////////////////////////////////


C///////////////////////////////////////////////////////////////////////
      SUBROUTINE VMP(X,B,AL,E,AI,G,D1,VNP)
C***********************************************************************
C  THIS SUBROUTINE CALCULATES THE DEFLECTION OF A CANTILEVER BEAM
C  WITH A MULTIPLE PENDULUM ABSORBER.
C***********************************************************************
      DIMENSION VF(20),VPP(10),VAB(20)
      DIMENSION AN(10,10),SAM(10,10),R(10,10),N(10),L(10)
      COMMON AM(10),CL(10),AR(100),K
  101 FORMAT(1H0,5E14.5)
C     ALSO NEEDS FUNCTION SUBPROGRAM AAM(I)
C     COMPUTES DEFLECTION FOR BEAM WITH MULTIPLE PENDULUUM
      DO1I=1,K
      DO1J=1,K
      AN(I,J)=0.0
    1 CONTINUE
      DO2I=1,K
      DO2J=1,K
      SAM(I,J)=0.0
    2 CONTINUE
C     K IS THE ORDER OF THE PENDULUM
      AN(1,1)=-AAM(1)/CL(1)-AAM(2)/CL(2)
      AN(1,2)=AAM(2)/CL(2)
      AN(K,K-1)=AM(K)/CL(K)
      AN(K,K)=-AM(K)/CL(K)
      IF(K.EQ.2)GO TO 25
      M=K-1
      DO3I=2,M
      AN(I,I-1)=AAM(I)/CL(I)
      AN(I,I)=-AAM(I)/CL(I)-AAM(I+1)/CL(I+1)
      AN(I,I+1)=AAM(I+1)/CL(I+1)
    3 CONTINUE
   25 CONTINUE
      DO4I=1,K
      SAM(I,I)=AM(I)
    4 CONTINUE
      DO9I=1,K
      DO5J=1,K
      R(I,J)=-G*G*SAM(I,J)-386.*AN(I,J)
    5 CONTINUE
    9 CONTINUE
      DO10I=1,K
```

```
      DO6J=1,K
    6 AR((J-1)*K+I)=R(I,J)
   10 CONTINUE
      CALL MINV(AR,K,D,L,N)
      H=0.
      DO7I=1,K
      H=H+(AM(I))*AR(I)
    7 CONTINUE
      D1=-G*G*386.*AAM(1)*H/CL(1)
      BX=B*X
      AG=386.
      BL=B*AL
      C11=COSH(BL)*COS(BX)+COS(BL)*COSH(BX)
      C12=C11/(COS(BL)+COSH(BL))
      C1=C12-1.
      GS=G*G
      D2=E*AI*B*B*B/(COS(BL)+COSH(BL))
      D2=D2*386.
      C21=2.+2.*COSH(BL)*COS(BL)
      C22=D2*C21
      C23=(SIN(BL)+SINH(BL))*(COS(BL)-COSH(BL))
      C24=C23/(COS(BL)+COSH(BL))
      C25=SIN(BL)-SINH(BL)-C24
      C2=D1*C25+C22
      C31=(SIN(BL)+SINH(BL))*(COS(BX)-COSH(BX))
      C32=C31/(COS(BL)+COSH(BL))
      C33=SIN(BX)-SINH(BX)-C32
      C34=COSH(BL)*SIN(BL)+COS(BL)*SINH(BL)
      C35=D2*C34
      C36=1.-2.*COSH(BL)*COS(BL)/(COS(BL)+COSH(BL))
      C37=C36*D1
      C8=C2/(C37+C35)
      IF(X.LT.55.)GO TO 8
      WRITE(3,101)C8,D,C2,C37,C35
    8 CONTINUE
      C3=(C37+C35)*C33
      VNP=((C3/C2)+C1)/B**4./E/AI
      RETURN
      END
C/////////////////////////////////////////////////////////////////////


C/////////////////////////////////////////////////////////////////////
      FUNCTION AAM(I)
C*********************************************************************
C  THIS SUBROUTINE CALCULATES.....
C  AM(I)+AM(I+1)+...+AM(K)
C*********************************************************************
      COMMON AM(10),CL(10),AR(100),K
      AAM=0.
```

```
      DO1I1=I,K
      AAM=AAM+AM(I1)
    1 CONTINUE
      RETURN
      END
C//////////////////////////////////////////////////////////////////////


C//////////////////////////////////////////////////////////////////////
C
C  THIS SUBROUTINE WAS TAKEN FROM IBM SYSTEM/360 SCIENTIFIC SUBROUTINE
C  PACKAGE (360A-CM-03X) VERSION II PROGRAMER,S MANUAL.
C     ..............................................................
C
C        SUBROUTINE MINV
C
C        PURPOSE
C           INVERT A MATRIX
C
C        USAGE
C           CALL MINV(A,N,D,L,M)
C
C        DESCRIPTION OF PARAMETERS
C           A - INPUT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY
C               RESULTANT INVERSE.
C           N - ORDER OF MATRIX A
C           D - RESULTANT DETERMINANT
C           L - WORK VECTOR OF LENGTH N
C           M - WORK VECTOR OF LENGTH N
C
C        REMARKS
C           MATRIX A MUST BE A GENERAL MATRIX
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT
C           IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT
C           THE MATRIX IS SINGULAR.
C
C     ..............................................................
C
      SUBROUTINE MINV(A,N,D,L,M)
      DIMENSIONA(36),L(6),M(6)
C
C        ...........................................................
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION
C        STATEMENT WHICH FOLLOWS.
```

```
C
C     DOUBLE PRECISION A,D,BIGA,HOLD
C
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS
C        ROUTINE.
C
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  ABS IN STATEMENT
C        10 MUST BE CHANGED TO DABS.
C
C        ...............................................................
C
C        SEARCH FOR LARGEST ELEMENT
C
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF( ABS(BIGA)- ABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
C
C        INTERCHANGE ROWS
C
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI) =HOLD
C
C        INTERCHANGE COLUMNS
C
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
```

```
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI) =HOLD
C
C        DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C        CONTAINED IN BIGA)
C
   45 IF(BIGA) 48,46,48
   46 D=0.0
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
C
C        REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
      A(IJ)=A(IK)*A(KJ)+A(IJ)
   65 CONTINUE
C
C        DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C        PRODUCT OF PIVOTS
C
      D=D*BIGA
C
C        REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
   80 CONTINUE
C
C        FINAL ROW AND COLUMN INTERCHANGE
C
```

```
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI) =HOLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI) =HOLD
      GO TO 100
  150 RETURN
      END
C//////////////////////////////////////////////////////////////////////

PROCESSING ENDED AT EOD
```

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ANALYSIS OF SINGLE AND MULTIPLE PENDULUM VIBRATION ABSORBERS APPLIED TO A VERTICAL CANTILEVER BEAM

by

JOHN RICHARD FRILEY

B. S., Kansas State University, 1967

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AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of

the requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1969

Name: John R. Friley Date of Degree: January, 1969

Institution: Kansas State University Location: Manhattan, Kansas

Title of Study: ANALYSIS OF SINGLE AND MULTIPLE PENDULUM VIBRATION ABSORBERS APPLIED TO A VERTICAL CANTILEVER BEAM

Pages in Study: 61 Candidate for Degree of Master of Science

Major Field: Mechanical Engineering

Scope and Method of Study: An analytical analysis was made of the displacements of a cantilever beam with an undamped pendulum attached at the unfixed end. Both single and multiple pendulums were considered. The beam was considered to be uniformly loaded in the transverse direction. The forcing function was considered to have been composed of a finite number of sinusoidal components. From the displacement equations, characteristic equations were derived for both the multiple pendulum case and the single pendulum case. Computer programs written in Fortran 4, Level G, were written to evaluate displacement equations and to solve the characteristic equations. Experiments were devised and executed in order to determine the first several natural frequencies of two different absorbed systems. The results of these experiments indicated that the characteristic equations were correct.

Findings and Conclusions: It was found that for a uniform transverse forcing function $F = F_1 t + \ldots + F_n \sin \gamma t$, the unfixed end of the beam could be effectively pinned and lack of resonance could be guaranteed provided none of the frequencies $\gamma_n$ corresponded to resonant frequencies of the fixed-pinned beam. It was found that a pendulum which would produce these results was a pendulum whose uncoupled resonant frequencies were those of the forcing function. Such a pendulum would be an $n^{th}$ order multiple pendulum.

An attempt was not made to devise a scheme for designing a pendulum which possesses these properties. Such a design would require the simultaneous solution of n $n^{th}$ order algebraic equations. It was assumed that such a solution did indeed exist. Since the pendulum parameters consist of 2n quantities (n masses and n links), the set of equations is underdetermined. Thus, a family of suitable pendulums exist, and various design constraints could be imposed.

Since, in general, resonant-frequencies are properties of the physical system and independent of the forcing function, it is felt that in most instances a pendulum absorber as described above would prevent resonance from occurring at natural frequencies of the uncoupled pendulum when the forcing function was nonuniform.

MAJOR PROFESSOR'S APPROVAL Hugh S. Walker

VITA

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