SOLVABILITY OF THE DIRECT LYAPUNOV FIRST MATCHING CONDITION IN TERMS OF THE GENERALIZED COORDINATES

by

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B.S., Technological University of Panama, 1988
M.S., Technological University of Panama, 2001

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mechanical and Nuclear Engineering
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Abstract

There are a number of different types of mechanical systems which can be termed as underactuated. The degrees of freedom (DOF) of a system are defined by the system's number of independent movements. Underactuated mechanical systems have fewer actuators than DOF. Some examples such as satellites, air craft, overhead crane loads, and missiles have at least one unactuated DOF.

The work presented here develops a nonlinear control law for the asymptotic stabilization of underactuated systems. This is accomplished by finding the solution of matching conditions that arise from Lyapunov's second method, analogous to the dissipation of energy. The direct Lyapunov approach (DLA) offers a wide range of applications for underactuated systems due to the fact that the algebraic equations, ordinary differential equations, and partial differential equations stemming from the matching conditions are more tractable than those appearing in other approaches.

Two lemmas of White et al. (2007) are applied for the positive definiteness and symmetry condition of the KD matrix which is used to define an analogous kinetic energy for the system. The defined KD matrix and the Lyapunov candidate function are developed to ensure stability. The KD matrix is analogous to the mass matrix of the dynamic system. The candidate Lyapunov function, involving the analogous kinetic energy and an undefined potential of the generalized position coordinates, is presented. By computing the time derivative of the Lyapunov candidate function, three equations called matching conditions emerge and parts of their solution provide the nonlinear control law that stabilizes the system.

This dissertation presents the derivation of the DLA, provides a new method to solve the first matching condition (FMC), and shows the tools for the control law design. The stability is achieved from the proper shape of the potential, the positive definiteness of the KD matrix, and the non-positive rate of change of the Lyapunov function. The ball and beam, the inverted pendulum cart, and, a more complicated system, the ball and arc are presented to demonstrate the importance of the results because the methods to solve the matching equations, emerging from the system examples, are simple and easier. The presented controller design formulation satisfies the FMC exactly without introducing control law terms that are quadratic in the velocities or approximations. This methodology allows the development of the first nonlinear stabilizing

control law for the ball and arc system, a simple and effective formulation to find a control law for the inverted pendulum cart, and a stabilizing control of the ball and beam apparatus without the necessity of approximations to solve the FMC. To illustrate the formulation, the derivation is performed using the symbolic manipulation program Maple and it is simulated in the Matlab/Simulink environment.

The dissertation on the solvability of the first matching condition for stabilization is organized into six different chapters. The introduction of the problem and the previous approaches are presented in Chapter 1. Techniques for solving of the first matching condition, as well as the limitations, are provided in Chapter 2. The application of this general strategy to the ball and beam system appears in Chapter 3. Chapter 4 and 5 present the application of the method to the ball and arc apparatus and to the inverted pendulum cart, respectively. The difficulties for each application are also presented. Particularly, Chapter 5 shows the application of the produced material to obtain an easier formulation for the inverted pendulum cart compared to previous published controller examples. Finally, some conclusions and recommendations for future work are presented.

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Sometimes is only your faith that keeps you strong enough, thank you Lord God.

Deyka

April, 2012

Dedication

To my best friend and husband
Marcelo

To my kids
Andres, Reggeany, Alan, and Alec

To my parents
Roberto and Francisca

To my brother and his family
Roberto Jr.

To my sister and her family
Bleysin Damaris

To my country
Panamá
and the Technological University of Panama

Chapter 1 - Introduction Problem and Previous Approaches

1.1 Introduction

For mechanical systems, there is an abundance of methodological designs for controllers of linear systems; however, this usually requires the system to operate within a specific range. The situation is radically different for nonlinear systems due to the fact that there are not many tools that could be applied to the controller design of nonlinear systems having large ranges of operation. In high performance applications, where a wide range of operating conditions are encountered, linear methods fail. The situation gets even more complicated when the nonlinear system to be controlled is underactuated. Underactuated means having fewer actuators than degrees of freedom. Many control techniques developed for fully actuated mechanical systems cannot be directly used to stabilize underactuated systems, such as feedback linearization, linear parameterization, or passivity techniques developed in optimal, robust, and adaptive control, Slotine and Li (2002). Furthermore, undesirable higher relative degree is not well defined at some singular points, Hauser et al. (1992) and non-minimum phase behavior may be present, Khalil (2002). The large diversity of nonlinear phenomena suggests that a simple design approach to deal with complex nonlinear phenomena is a need. Thus, this is an important motivation for the design of new tools and procedures.

There has been extensive research during recent years on the control of the underactuated mechanical systems due to the broad range of real life applications. Stabilization of these systems has been reached and widely studied until now through benchmark examples involving a lack of symmetry. Symmetry of the system refers to the fact that the mass matrix does not depend on the unactuated coordinate, Olfati-Saber (2000). Stabilization is about driving the state vector $\boldsymbol{x}$ from a given initial condition back to the equilibrium point. The control designs to stabilize these systems based on symmetry have been developed by methods that usually modify the Lagrangian of the system.

Using the Euler-Lagrange method, Goldstein (1981), the equations of motion of the mechanical system are determined as

$$\frac{d}{dt}\left(\frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}}=\mathbf{Q}' \tag{1.1}$$

where $\mathbf{q}, \dot{\mathbf{q}}$, and $\ddot{\mathbf{q}} \in \Re^n$ represent the vector of the generalized coordinates, velocities, and accelerations, respectively. $L(\mathbf{q}, \dot{\mathbf{q}}): \Re^{2n} \rightarrow \Re$ is the Lagrangian defined as the kinetic energy minus the potential energy of the system. The vector $\mathbf{Q}'$ contains the generalized constraints and the applied generalized non-conservative forces/moments as the control input (s).

The Lyapunov second method, Ogata (1998), is applied for the development of the control law. The candidate Lyapunov function is made of intrinsic positive quantities and part is described as a quadratic matrix product, White et al. (2008 and 2009). The candidate Lyapunov function used in the Lyapunov direct method is

$$\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\dot{\mathbf{q}} + \mathbf{\Phi}(\mathbf{q}) \tag{1.2}$$

where $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}): \Re^{2n} \rightarrow \Re$ is the candidate Lyapunov function, $\mathbf{\Phi}(\mathbf{q})$ is a real scalar potential function of the generalized coordinates, and

$$\mathbf{K_D} = \mathbf{P}(\mathbf{q})\mathbf{M}(\mathbf{q}) \tag{1.3}$$

where $\mathbf{P}(\mathbf{q}) \in \Re^{n \times n}$ is a matrix defined so that $\mathbf{K_D} \in \Re^{n \times n}$ is a symmetric and positive-definite matrix. The matrix $\mathbf{M}(\mathbf{q}) \in \Re^{n \times n}$ is the positive definite symmetric mass matrix of the dynamic system.

The time derivative of the candidate function is made non-positive and this concept is the basis for the Lyapunov application to nonlinear control problems. The control challenge arises from the nonlinear nature of the governing equations and the underactuation. The time derivative of the candidate Lyapunov function, together with the equations of motion results in an equation that is solved by a matching method. When this method is applied, the terms quadratic in the velocities are grouped together obtaining a set of linear ordinary differential equations (ODEs). These equations are called the first matching condition. Grouping terms which are linear in the velocities results in linear algebraic equations (LAEs) and these equations are called the second matching condition. The third matching condition involves only position coordinates resulting in linear partial differential equations (PDEs). This methodology is called the direct Lyapunov approach (DLA).

1.2 Previous approaches

The main challenge when designing controls for underactuated mechanical systems is the non-linearity of the equations of motion that govern the dynamics together with the manipulation of those equations so that a controller can be found. The application of any method is, in general, a rather difficult task because getting the needed controller involves solving ordinary and partial differential equations. Generally, control strategies for the stabilization of underactuated systems can be found in the literature that introduce some non-linear approximations, modify the symmetry conditions, or switch the controller through singularities. The early references and inspirational developments pertaining to the current work are categorized as non-matching based and matching based. Matching based approaches are developed methods to design a controller by satisfying a set of matching conditions and non-matching based approaches are those methods applied to underactuated systems that follow other well-known non-linear control methods.

1.2.1 Non-matching based

Backstepping has become a very popular control design method for nonlinear control systems because it can guarantee global stability, tracking, and transient performance for strict-feedback systems, Khalil (1996). Backstepping as a design procedure for the feedback stabilization of underactuated systems and several controller designs based on the cascade and passivity paradigms development are presented by Kokotovic et al. (1996). In Olfati-Saber (2000 - 2001), cascade normal forms for underactuated mechanical systems are introduced for control design that reduces the system order. The cascade normal forms are three classes of nonlinear systems, namely, systems in strict feedback form, feedforward form, and non-triangular linear quadratic form due to the particular lower-triangular, upper-triangular, and non-triangular structure, respectively, in which the state variables appear in the dynamics of the corresponding nonlinear systems. Backstepping also was extended to stabilize nonholonomic systems.

Explicit formulas of smooth time-varying state feedbacks, which make the origin of an underactuated surface vessel globally, uniformly, asymptotically stable, are proposed by Mazenc, Pettersen, and Nijmeijer (2002). In this case, the construction of the feedback extensively relies on the backstepping approach. The feedbacks constructed are time-periodic functions. Underactuated surface vessels are an example of a non-holonomic system.

Methods for feedback linearization are discussed in Krener, Isidori and Respondek (1983). It was discussed how a system can be partially and totally linearized by state feedback and coordinate change. The robustness of this linearization is dependent on the ability to stabilize the infinitesimal approximation of the transformed system; however situations can arise where to obtain stability, linearization must be sacrificed. Further work is presented for designing asymptotic observers for a class of nonlinear systems. Partial feedback linearization and the study and analysis of the resulting internal or zero dynamic is crucial in understanding the behavior of the overall system as discussed by Spong et al. (1995). The collocated partial feedback linearization result and the application and discussion about the effect of the system approximation on the size of the region of stability for the Acrobot example are given by Murray and Hauser (1991).

Stabilization techniques for nonlinear underactuated systems in the Riemannian geometry context are presented by Bullo (2000). The nonlinear system theory based on differential geometry methods is presented in Grillo et al. (2009-2011) for stabilizing underactuated mechanical systems by imposing kinematic constraints that ensures the existence of a Lyapunov function. The relationship between closed-loop mechanical systems and mechanical systems with constrains is applied to stabilize a general underactuated system.

1.2.2 Matching based

Andreev et al. (2000 and 2002) discuss matching control laws for underactuated systems characterized by a linear system of first order partial differential equations. The λ-method transforms the first matching condition of that method into a set of linear PDEs that are sequentially solved.

The controlled Lagrangian method of Bloch et al. (1997-2000) uses energy shaping to ensure stabilization of the underactuated mechanical systems with symmetry. The new terms that appear in the equations of motion define the control inputs. Here the closed-loop dynamics has an energy conservation law associated with the Lagrangian. The generalized matching conditions are represented by an over determined set of PDEs. Related and significant works involving energy methods are in Gomez et al. (2004). In similar works, simplified conditions for generalized matching of controlled Lagrangians for stabilization of underactuated mechanical systems are used in Hamberg (2000). Bloch and co-workers developed a method for the stabilization of mechanical systems with symmetry based on the technique of controlled Lagrangians. This method systematically provides a control law that stabilizes an unstable equilibrium by modifying the Lagrangian and also by applying energy shaping. The controlled

Lagrangian for control of underactuated mechanical systems using both kinetic and potential shaping are in Woolsey et al. (2004).

The interconnection and damping assignment-passivity based control (IDA-PBC) is a method based on the Hamiltonian to stabilize an equilibrium point with a feedback law that changes the internal interconnection structure of the system. This change provides an extra degree of freedom to the IDA-PBC method with respect to the controlled Lagrangian. The concept of stabilization is passivity. Here a set of nonlinear PDEs need to be solved for the closed loop Hamiltonian and interconnection structure. In Ortega et al. (2002) and in Blankenstein, Ortega, and Van der Schaft (2005), it is shown that the controlled Lagrangian is a special case of the interconnection and damping assignment-passivity based control method by choosing an appropriate close-loop interconnection structure. In Gomez-Estern, Ortega, Rubio, Aracil (2001), the stabilization of underactuated mechanical systems simplifies the PDEs imposed by IDA-PBC technique to a nonlinear ODE system using parametrization of the closed-loop inertia matrix. An energy-based balance control approach to the ball and beam system differing from Ortega et al. (2001), is presented in Muralidharan, Anantharaman and Mahindrakar (2010). Here the model of the system is modified to change the equilibrium points for an operating point and the matching equations are obtained through the Hamiltonian formulation. More recently Sandoval, Kelly and Santibáñez (2011) consider the IDA-PBC to compensate friction by means of a nonlinear observer. Based on the Lyapunov direct method it was shown that the closed- loop system is asymptotically stable.

The DLA is first presented in White et al. (2006) and an improved formulation of the approach is addressed in White et al. (2008). It shows how certain parameters preserve the sign of the candidate Lyapunov function rate of change. There is a recent study on the DLA in White, Foss, Patenaude, and Garcia (2009) where all of these approaches rely on a matching equation solution method for tracking control.

1.3 New developments

The previous work using DLA, White et al. (2008), is taken as the starting point of this dissertation to investigate the problem of solving the first matching condition for the design of the stabilizing nonlinear control law of underactuated mechanical systems. The general approach with which this problem is addressed is the use of techniques that solve the matching equations such that the first matching condition is satisfied in terms of the generalized coordinates and certain restrictions. This work applies the method to solve the first matching condition and to stabilize a collection of underactuated mechanical systems. The attractiveness of the DLA used in the formulation is that this method offers a wider range of applications and the obtained LAEs, ODEs, and PDEs are more tractable than those obtained with early methods applied for the controller design of underactuated mechanical systems.

In this work three system examples are treated which are the ball and beam, the inverted pendulum cart, and, a more complicated system, the ball and arc. The methods to solve the matching equations emerging from the ball and beam system are easier to solve than those appearing in previous controller developments. A simple matching controller for the stabilization of the inverted pendulum cart is presented. There are previous publications for the ball and arc system controller design using linearized pole placement and LQR as reported by Sheng, Renner and Levine (2010), stabilizing under saturated actuator conditions presented in Aoustin et al. (2009), and using linear matrix inequalities as shown in Abhilash and Mahindrakar (2008). One contribution of this dissertation is a novel nonlinear controller design for the ball and arc system using the methodology presented here in. This is the first published nonlinear controller for such a system. For all the mentioned systems, simulation results are presented to illustrate the efficacy of the designed nonlinear control laws.

Chapter 2 - System Dynamic and Limiting Applications

2.1 Introduction

The tools and procedures for new design methods should evaluate both advantages and limitations in order to assess applicability. These procedures should also exploit available properties to expand the range of application. In this chapter, different methodologies for solving the first matching condition for underactuated mechanical systems are presented.

The main contribution in this chapter is to obtain a structured formulation that could be generally applied to underactuated systems as a general nonlinear control design method.

The chapter is organized as follows: In Section 2.2, the Lyapunov stability theory is discussed. In Section 2.3, the first matching condition formulation is presented. The methods to solve the first matching condition are shown in Section 2.4. Section 2.4.1 presents the first matching condition solvability when the control input $\mathbf{Fm_1}$ and the matrix $\mathbf{Fmc_1}$ are zero. Section 2.4.2 illustrates the strategy of considering $\mathbf{K_D}$ as a perfect differential. Section 2.4.3 presents the method of extracting the coefficients of each element of $\dot{\mathbf{q}}$ and solving the FMC. Section 2.4.4 presents the Lagrangian method derivation. Section 2.4.5 presents the first matching condition solvability with a control input $\mathbf{Fm_1}$ and the matrix $\mathbf{Fmc_1}$. Section 2.4.5.1 presents the strategy sequence of approaches to apply in developing the formulation. Section 2.4.6 illustrates the formulations when $\mathbf{K_D}$ is non-constant, $\mathbf{K_D}$ is constant, and for a simple $\mathbf{P}$. In Sections 2.5 and 2.6, the second and the third matching conditions are shown. The application of the linearization process for parameter selection is explained in Section 2.7 and the general strategy for the solvability of the first matching condition, as well as the limitations are listed in Section 2.8. The conclusion is addressed in Section 2.9.

2.2 Lyapunov

Before considering the first matching condition, which is used to find the $\mathbf{K_D}$ matrix, the details of the Lyapunov formulation will be presented in this section. The Lyapunov stability theory is the main tool used in the analysis and synthesis of this development. The Lyapunov candidate function is defined in Eq. 1.2 and the control input that satisfies the time derivative constraints of the Lyapunov candidate function along the system trajectories is designed such that $\dot{\mathbf{V}}(\mathbf{q},\dot{\mathbf{q}}) \leq 0$, $\mathbf{V}(\mathbf{q},\dot{\mathbf{q}})$ decreases to a constant, and $\mathbf{K_D} = \mathbf{K_D}^T > 0$. The matrices associated with the DLA formulation emerge from the dynamic equations. The dynamic equations contain the positive definite inertia matrix $\mathbf{M}(\mathbf{q}) \in \Re^{n \times n}$, a matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in \Re^{nxn}$ containing the Coriolis and centripetal coefficients, and the vector $\mathbf{G}(\mathbf{q}) \in \Re^{n}$ consisting of forces and/or moments stemming from gradients of conservative fields. The equations of motion are

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \boldsymbol{\tau} \\ 0 \end{bmatrix} \tag{2.1}$$

where the generalized coordinate vector is denoted by $\mathbf{q} \in \Re^{n}$. Eq. 2.1 is usually a nonlinear, ordinary differential equation. The quantity n represents the degrees of freedom of the mechanical system and $\tau \in \Re^{m}$ contains the forces or torques for the actuated degrees of freedom. Note that $m<n$ and it is assumed that the actuated axes are placed in the top of the equations as shown in Eq. 2.1. The actuation or control input is decomposed into three terms consisting of

$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{F_1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{F_2} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{F_3} \\ \mathbf{0} \end{bmatrix} \tag{2.2}$$

where $\boldsymbol{F}_i$ will be used with the i^{th} matching condition.

As presented earlier, the candidate Lyapunov function is expressed in Eq. 1.2. When Eq. 1.2 is differentiated with respect to time, $\dot{\mathbf{V}}$ is found to be

$$\dot{\mathbf{V}} = \dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\dot{\mathbf{K}}_{\mathbf{D}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathbf{T}}\nabla\boldsymbol{\Phi}(\mathbf{q}) = -\dot{\mathbf{q}}^{\mathbf{T}}(\mathbf{K}_v + \mathbf{Fmc_1})\dot{\mathbf{q}} \leq 0 \tag{2.3}$$

where the matrix $\mathbf{K_V} + \mathbf{Fmc_1} \in \Re^{n \times n}$ is symmetric and at least positive semi-definite while $\nabla\boldsymbol{\Phi}(\mathbf{q})$ is the gradient of the potential with respect to the generalized coordinates. The non-positive right hand side of Eq. 2.3 is a goal of the controller design. Substituting $\ddot{\mathbf{q}}$ from the dynamic equation into the derivative of the Lyapunov candidate function, the time derivative of $\mathbf{V}$ becomes

$$\begin{aligned} \dot{\mathbf{V}} = & \ \dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\mathbf{M}(\mathbf{q})^{-1}((-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}))\dot{\mathbf{q}} - \mathbf{G}(\mathbf{q}) + \boldsymbol{\tau}) \\ & + \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\dot{\mathbf{K}}_{\mathbf{D}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathbf{T}}\nabla\boldsymbol{\Phi}(\mathbf{q}) = -\dot{\mathbf{q}}^{\mathbf{T}}(\mathbf{K}_v + \mathbf{Fmc_1})\dot{\mathbf{q}}. \end{aligned} \tag{2.4}$$

The procedure to solve Eq. 2.4 is to break it into three separate equations that will be called matching equations. Following a procedure similar to that of White et al. (2008), we decompose Eq. 2.4 into these three matching equations. Examination of Eq. 2.4 shows that it can be grouped into portions having the same types of terms. Grouping the terms which are quadratic in the velocities, a linear ODE with $\mathbf{K_D}$, $\mathbf{F_1}$, and $\mathbf{Fmc_1}$ as unknowns is found to be

$$\dot{\mathbf{q}}^T\mathbf{K}_D\mathbf{M}(\mathbf{q})^{-1}\left(-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+\begin{bmatrix}\mathbf{F}_1\\ \mathbf{0}\end{bmatrix}\right)+\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{K}}_D\dot{\mathbf{q}}=-\dot{\mathbf{q}}^T\mathbf{Fmc}_1\dot{\mathbf{q}}. \tag{2.5}$$

This last result will be the first matching condition (FMC). Grouping the terms which are linear in the velocities results in a linear algebraic equation with unknowns $\mathbf{F_2}$ and $\mathbf{K_v}$. The equation is

$$\dot{\mathbf{q}}^T\mathbf{K}_D\mathbf{M}(\mathbf{q})^{-1}\left(\begin{bmatrix}\mathbf{F}_2\\ \mathbf{0}\end{bmatrix}\right)=-\dot{\mathbf{q}}^T\mathbf{K}_v\dot{\mathbf{q}} \tag{2.6}$$

which will be the second matching condition (SMC). Note that if the problem involves viscous damping, the damping is included in Eq. 2.6. Grouping the terms that are only a function of the position coordinates results in a linear partial differential equation with $\mathbf{F_3}$ and $\Phi(\mathbf{q})$ as unknowns. This is called the third matching condition (TMC) and is given by

$$\dot{\mathbf{q}}^T\mathbf{K}_D\mathbf{M}(\mathbf{q})^{-1}\left(-\mathbf{G}(\mathbf{q})+\begin{bmatrix}\mathbf{F}_3\\ \mathbf{0}\end{bmatrix}\right)+\dot{\mathbf{q}}^T\nabla\Phi(\mathbf{q})=\mathbf{0}. \tag{2.7}$$

To be seen in a later development, the $\mathbf{C(q,\dot{q})}$ matrix from the first term of Eq. 2.5 plays an important role when calculating $\mathbf{K_D}$. The $\mathbf{C(q,\dot{q})}$ matrix is defined as

$$\mathbf{C}_{ij}(\mathbf{q},\dot{\mathbf{q}})=[\mathbf{j\ k},\mathbf{i}]\dot{\mathbf{q}}^{\mathbf{k}}=\frac{1}{2}\left[\frac{\partial m_{\mathbf{ij}}}{\partial\mathbf{q}^{\mathbf{k}}}+\frac{\partial m_{\mathbf{ki}}}{\partial\mathbf{q}^{\mathbf{j}}}-\frac{\partial m_{\mathbf{jk}}}{\partial\mathbf{q}^{\mathbf{i}}}\right]\dot{\mathbf{q}}^{\mathbf{k}} \tag{2.8}$$

where m_{ij} is the ijth element of the mass matrix $\mathbf{M(q)}$ and **[jk,i]** is the Christoffel symbol of the first kind (Hicks, 1965). Murray (1994) and Woolsey (2004) show and explain that

$$\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})=-\left(\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\right)^T \tag{2.9}$$

is skew-symmetric. This is a special case of the FMC when $\mathbf{K_D}=\mathbf{M(q)}$ for which

$$\dot{\mathbf{M}}(\mathbf{q})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^T=\mathbf{0}. \tag{2.10}$$

When the FMC is solved, the SMC can be solved by using the results from FMC solutions. In the third matching condition, the last *m-n* equations are used to determine the potential $\mathbf{\Phi(q)}$ and the first *m* equations are used to determine the generalized force or control law term $\mathbf{F_3}$. The solutions of these three matching equations leads to the control law which is achieved by satisfying Lyapunov's second method and La Salle's invariance principle, Khalil (1996). The symbolic manipulation software Maple is used to solve the matching equations.

2.3 The first matching condition

Solving the matching conditions is an important challenge. Once the solution is found, the parameters involved in the matching equations are selected in terms of the stabilization and performance.

The FMC is used for the determination of the matrix $\mathbf{K_D}$. The goal in solving the FMC is to find the matrix $\mathbf{K_D}$ such that this matrix is symmetric and positive definite together with the control input $\mathbf{F_1}$, White et al. (2008). Another unknown is the matrix $\mathbf{Fmc_1}$. In examining Eq. 2.6 and Eq. 2.7, it is seen that the matrix $\mathbf{K_D}$ plays a role in both second and third matching conditions. Thus, the nature of $\mathbf{K_D}$ will have an influence on all three matching conditions, exemplifying its importance.

By defining the first matching condition control input as

$$\begin{bmatrix} \mathbf{F_1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Fm_1} \\ \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \tag{2.11}$$

where $\mathbf{Fm_1} \in \Re^{m \times n}$ is a coefficient matrix yet to be determined. It can then be seen from Eq. 2.5 that each term in the first matching condition is pre and post multiplied by the vector of generalized velocities. By introducing Eq. 2.11 into Eq. 2.5 and stripping off the generalized velocities, the first matching condition becomes

$$\frac{1}{2}\dot{\mathbf{K}}_D + \mathbf{K}_D \mathbf{M}(\mathbf{q})^{-1}\left(\begin{bmatrix} \mathbf{Fm_1} \\ 0 \end{bmatrix} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \right) = -\mathbf{Fmc_1}. \tag{2.12}$$

This is a linear ordinary differential equation for the matrix $\mathbf{K_D}$. Because Eq. 2.5 has a quadratic form, the skew-symmetric part of that equation vanishes. Setting the symmetric part of Eq. 2.12 to zero automatically satisfies Eq. 2.5. This operation produces

$$\dot{\mathbf{K}}_D + \mathbf{K}_D \mathbf{M}(\mathbf{q})^{-1}\left(\begin{bmatrix} \mathbf{Fm_1} \\ 0 \end{bmatrix} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \right) + \left(\begin{bmatrix} \mathbf{Fm_1} \\ 0 \end{bmatrix} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \right)^T \mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_D = -\mathbf{Fmc_1}. \tag{2.13}$$

The different limiting conditions that result from the solvability of $\mathbf{K_D}$ by using Eq. 2.13 are presented in the sections to follow.

2.4 Methods for solving the first matching equation

In order to solve the first matching condition for $\mathbf{K_D}$ and to satisfy all the involved restrictions, different methods are employed. They are mainly separated into the methods not containing and containing the input $\mathbf{Fm_1}$ and the matrix $\mathbf{Fmc_1}$. One of the main necessities in trying to solve the first matching condition is the fact that the input $\mathbf{F_1}$ is not quadratic in the velocities. Furthermore, the controller cannot have any singularities for any position (generalized coordinates). Also the control terms consisting of even functions are undesirable. If this situation occurs, then a different controller is needed for positive and negative values of the coordinates and velocities. The solution of the FMC equation is not trivial. Also, one more constraint that needs to be mentioned is that $\mathbf{P}(\mathbf{q})$ must be at least lower triangular as will be demonstrated later.

2.4.1 First matching condition formulation when Fm_1 and Fmc_1 are zero

The first strategy in solving for $\mathbf{K_D}$, based on the above constraints, is to choose the elements of $\mathbf{Fm_1}$ and $\mathbf{Fmc_1}$ as zero which result is

$$\begin{aligned}&\dot{\mathbf{K}}_{\mathbf{D}} - \mathbf{K}_D\mathbf{M}(\mathbf{q})^{-1}(\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}))-\\&\quad(\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}))^T\mathbf{M}(\mathbf{q})^{-1}\mathbf{K}_D = 0.\end{aligned} \tag{2.14}$$

In order to satisfy Eq. 2.14, $n(n+1)/2$ equations can be written to determine the same number of unknown $\mathbf{K_D}$ elements. As mentioned before, in order to get general solutions, $\mathbf{K_D}$ is determined symbolically.

$\mathbf{K_D}$ is defined as a product in Eq. 1.3, where $\mathbf{P}(\mathbf{q})$ is the matrix that will have a significant role in the definition of the main controller characteristics and the potential. It is essential to have a full, non-singular or at least lower triangular $\mathbf{P}(\mathbf{q})$ matrix to achieve this because the first $\mathbf{m}$ columns of the $\mathbf{P}(\mathbf{q})$ matrix are used to determine the $\mathbf{K}_v$ matrix in the SMC. The lack of a full or a lower triangular $\mathbf{P}(\mathbf{q})$ matrix represents a problem for the design of the controller because with a full $\mathbf{P}(\mathbf{q})$ matrix, there are a larger number of parameters in the controller that can be chosen to allow better performance of the system. With fewer parameters, it may not be possible to place the poles of a linearized version of the system and controller in desired locations. This is an example of limited flexibility in the controller.

Solving for the $\mathbf{P}(\mathbf{q})$ matrix in Eq. 1.3 shows that

$$\mathbf{P}(\mathbf{q}) = \mathbf{K}_D \mathbf{M}(\mathbf{q})^{-1}. \tag{2.15}$$

$\mathbf{K_D}$ must be a symmetric and positive definite matrix. $\mathbf{M}(\mathbf{q})$ is inherent to the dynamics of the system and is a given for the problem. However, when no input $\mathbf{F_1}$ or matrix $\mathbf{Fmc_1}$ is added to the FMC, $\mathbf{K_D}$ can become a multiple of the $\mathbf{M}(\mathbf{q})$ matrix. One difficulty that can occur when $\mathbf{K_D}$ is a multiple of $\mathbf{M}(\mathbf{q})$, is that $\mathbf{P}(\mathbf{q})$ is a diagonal matrix. This situation presents a problem for obtaining a proper $\mathbf{K_v}$ matrix as well as for the controller design. This will be discussed further when the solution of the SMC is covered. In general, if $\mathbf{K_D}$ is an integer multiple of the mass matrix $\mathbf{M}(\mathbf{q})$, then a non-full $\mathbf{P}(\mathbf{q})$ matrix will result. Note that $\mathbf{K_D} = \mathbf{M}(\mathbf{q})$ satisfies Eq. 2.14 where $\mathbf{P}(\mathbf{q})=\mathbf{I}$, the identity matrix. This is also shown in Eq. 2.10.

In order to solve Eq. 2.14, some different methods are investigated.

2.4.2 Considering K_D as a perfect differential

One way to find a convenient solution is to directly solve the linear differential equations of Eq. 2.14. If the FMC in Eq. 2.14 is rearranged to be a perfect differential, the solution is provided by separation of the $\dot{\mathbf{q}}$ terms. Considering a configuration variable $\mathbf{q_i}$ and its corresponding velocity $\dot{\mathbf{q}}_\mathbf{i}$ and grouping the elements of the Eq. 2.14 that are multiples of $\dot{\mathbf{q}}_\mathbf{i}$, results in

$$\dot{\mathbf{K}}_\mathbf{D} - \sum_{i=1}^{n} \mathbf{m}_{\mathbf{q_i}} \dot{\mathbf{q}}_\mathbf{i} = \dot{\mathbf{K}}_\mathbf{D} - \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{q_i}} \mathbf{K_D} \dot{\mathbf{q}}_\mathbf{i} \tag{2.16}$$

where the terms associated with $\dot{\mathbf{q}}_\mathbf{i}$ are called $\mathbf{m}_{\mathbf{q_i}}$ where $\mathbf{m}_{\mathbf{q_i}} = \mathbf{m}_{\mathbf{q_i}}{}^\mathbf{T}$. Note that Eq. 2.16 is true only if the partial derivatives of $\mathbf{K_D}$, do in fact produces the matrices $\mathbf{m}_{\mathbf{q_i}}$. For square symmetric matrices, an analytical solution for $\mathbf{K_D}$ can be found in Eq. 2.16 such that it is a perfect differential of $\mathbf{K_D}$. For n=2, this is possible when the condition

$$\frac{\partial}{\partial \mathbf{q}_1}\mathbf{m}_{\mathbf{q}_2} = \frac{\partial}{\partial \mathbf{q}_2}\mathbf{m}_{\mathbf{q}_1} \tag{2.17}$$

is satisfied. During the application of Eq. 2.17, $n(n+1)/2$ partial differential equations owing to the symmetry of the $\mathbf{m}_{\mathbf{q}_i}$, are found. For two of the example systems presented in this dissertation where the solution of E. 2.14 is attempted, there are three PDEs. Solving Eq. 2.17 for $\mathbf{K_D}$ also implies that the constraints mentioned before are to be satisfied. The less complicated solution found for the ball and beam system by applying this method was a $\mathbf{K_D}$ matrix which is multiple of the mass matrix. It did result in an inconvenient $\mathbf{P}(\mathbf{q})$ matrix and will be discussed later. For the other systems, the obtained solutions for $\mathbf{K_D}$ are not multiples of their respective mass matrices.

2.4.3 Separation of generalized velocities coefficients in solving the FMC

The method of extracting the coefficients of $\dot{\mathbf{q}}$ and solving the FMC in terms of the independent generalized coordinates is explained in this section. Grouping all of the terms of Eq. 2.14 including $\dot{\mathbf{K}}_\mathbf{D}$ into matrices that multiply $\dot{\mathbf{q}}_\mathbf{i}$, results in

$$\sum_{i=1}^{n} \mathbf{m}_{\mathbf{q}_\mathbf{i}} \dot{\mathbf{q}}_\mathbf{i} = 0 \tag{2.18}$$

for which we require each $\mathbf{m}_{\mathbf{q}_\mathbf{i}} = 0$ and where the matrices associated with $\dot{\mathbf{q}}_\mathbf{i}$ are called $\mathbf{m}_{\mathbf{q}_\mathbf{i}}$ where the $\mathbf{m}_{\mathbf{q}_\mathbf{i}}$ are square symmetric matrices. Note that the partial derivatives of $\mathbf{K_D}$ appear in the $\mathbf{m}_{\mathbf{q}_\mathbf{i}}$. An analytical solution for $\mathbf{K_D}$ can be found in Eq. 2.18.

During the application of Eq. 2.18 to the FMC, $n(n+1)/2$ ordinary differential equations, owing to the symmetry of the $\mathbf{m}_{\mathbf{q}_i}$, are found if the symmetry condition of $\mathbf{K_D}$ matrix is considered. There are $n(n+1)/2$ equations associated with each $\dot{\mathbf{q}}_\mathbf{i}$ which means solving $n(n(n+1)/2\ )$ equations for the $\mathbf{K_D}$ elements. If such a situation over determines the problem, then additional unknowns must be introduced through $\mathbf{Fm_1}$ and $\mathbf{Fmc_1}$.

2.4.4 Solving for K_D using the Lagrangian method

There is another method to solve for $\mathbf{K_D}$ which is to make $\mathbf{K_D}$ Lagrangian. When the control signal corresponding to FMC and $\mathbf{Fmc_1}$ are set to zero, the first matching condition is

$$\left(\frac{1}{2}\dot{\mathbf{K}}_{\mathbf{D}} - \mathbf{K_D}\mathbf{M}^{-1}\mathbf{C}\right)^{\mathbf{T}} = -\left(\frac{1}{2}\dot{\mathbf{K}}_{\mathbf{D}} - \mathbf{K_D}\mathbf{M}^{-1}\mathbf{C}\right). \tag{2.19}$$

This last result indicates that the resulting matrix in parenthesis is skew symmetric. In Eq. 2.19, $\mathbf{M(q)}$ and $\mathbf{C(q,\dot{q})}$ are the same matrices specified before. Another expression for Eq. 2.19 is Eq. 2.14. It will be now shown that if $\mathbf{K_D}$ is Lagrangian, then Eq. 2.19 and Eq. 2.14 are automatically satisfied. The kinetic energy analog of the system appearing in the candidate Lyapunov function is

$$\mathrm{T} = \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\dot{\mathbf{q}} \tag{2.20}$$

where T is the analogous kinetic energy. Substituting T into Lagrange's equation shows

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathrm{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathrm{T}}{\partial \mathbf{q}} = \frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{K_D}\dot{\mathbf{q}}) - \frac{\mathbf{1}}{\mathbf{2}}\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\dot{\mathbf{q}}\right) = \mathbf{K_D}\ddot{\mathbf{q}} + \dot{\mathbf{K}}_{\mathbf{D}}\dot{\mathbf{q}} - \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right)\dot{\mathbf{q}}. \tag{2.21}$$

If $\boldsymbol{K_D}$ is Lagrangian, we have the result

$$\frac{1}{2}\dot{\mathbf{K}}_{\mathbf{D}} - \left(\dot{\mathbf{K}}_{\mathbf{D}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right)\right) = -\left[\frac{1}{2}\dot{\mathbf{K}}_{\mathbf{D}} - \left(\dot{\mathbf{K}}_{\mathbf{D}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right)\right)\right]^{\mathbf{T}}. \tag{2.22}$$

Solving Eq. 2.14 is one way to proceed as reported in the previous section. The condition in Eq. 2.22 is particularly convenient for satisfying the first matching condition. When comparing Eq. 2.22 and Eq. 2.19, it is observed that for $\boldsymbol{K_D}$ to be Lagrangian it is necessary that

$$\dot{\mathbf{K}}_{\mathbf{D}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right) = \mathbf{K_D}\mathbf{M}^{-1}\mathbf{C}. \tag{2.23}$$

From Eq. 2.23 it is appreciated that Eq. 2.14 could be rewritten in a different and more convenient form as

$$\dot{\mathbf{K}}_{\mathbf{D}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right)^{\mathbf{T}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right) = \mathbf{0}. \tag{2.24}$$

Transposing and subtracting Eq. 2.23 from Eq. 2.24 produces

$$-\mathbf{C}^{\mathbf{T}}\mathbf{M}^{-1}\mathbf{K_D} + \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{K_D}\right) = \mathbf{0}. \tag{2.25}$$

In solving Eq. 2.25, it is demonstrated that $\mathbf{K_D}$ is not Lagrangian for the ball and beam system, as well as for the ball and arc system. The result is different for the inverted pendulum cart, which has a Lagrangian $\mathbf{K_D}$. Because this method does not work for all the example systems, this procedure is not applicable as a general solvability method. For those systems for which $\mathbf{K_D}$ is Lagrangian, a convenient way of solving the FMC is obtained. Further information about these formulations are presented in Section A.2 of Appendix A for the ball and beam system, in Section B.2 of Appendix B for the ball and arc system, and in Section C.2 of Appendix C for the inverted pendulum cart.

2.4.5 The control input Fm₁ is non-zero

In this section, the matching conditions for the mechanical systems under study are presented with two degrees of freedom $\mathbf{q_1}$ and $\mathbf{q_2}$. The FMC is presented as it appears in Eq. 2.13.

The control input can be expressed for the generalized coordinates as

$$\mathbf{Fm_1} = \begin{bmatrix} -\mathbf{F}_{11}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_2 & \mathbf{F}_{11}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_1 \\ -\mathbf{F}_{22}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_2 & \mathbf{F}_{22}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_1 \end{bmatrix} \tag{2.26}$$

where $\mathbf{Fm_1}$ is re-defined as $\mathbf{Fm_1} \in \Re^{nxn}$. Substituting Eq. 2.26 into Eq. 2.11 are such that $\mathbf{F_1} \equiv 0$ because the rows of $\mathbf{Fm_1}$ are orthogonal to $\dot{\mathbf{q}}$. Therefore, there are two unknowns from $\mathbf{Fm_1}$ and three unknowns from the $\mathbf{K_D}$ matrix, considering symmetry. From this formulation two cases are examined before deciding on the FMC approach. They are:

2.4.5.1 Strategy sequence

1. If $\mathbf{K_D}$ is a constant and in order to satisfy the FMC, an additional input can be added to $\mathbf{Fm_1}$ because the solution of the FMC requires at least three unknowns in order to solve the equations. Eq. 2.26 will be, in this case,

$$\mathbf{Fm_1} = \begin{bmatrix} -\mathbf{F}_{11}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_2 + \boldsymbol{\mu}(\mathbf{q},\dot{\mathbf{q}}) & \mathbf{F}_{11}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_1 \\ -\mathbf{F}_{22}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_2 & \mathbf{F}_{22}(\mathbf{q}_1,\mathbf{q}_2)\dot{\mathbf{q}}_1 \end{bmatrix}. \tag{2.27}$$

   The problem with this approach is that now, the first matching condition is satisfied, but $\mathbf{F}_1$ is quadratic in the velocities.

2. If $\mathbf{K_D}$ is not a constant, in order to satisfy the FMC, at least one of the elements of $\mathbf{K_D}$ must be non-constant to solve the FMC using Eq. 2.26. A concern here, is that the resulting $\mathbf{P}$ matrix might be too complicated to be used in solving the TMC for the potential. This is the main problem using this solvability method.

2.4.6 Choosing the formulation

In this section, the FMC, as it appears in Eq. 2.13, is presented with two degrees of freedom $\mathbf{q_1}$ and $\mathbf{q_2}$. The $\mathbf{K_D}$ matrix is given as

$$\mathbf{K_D} = \begin{bmatrix} \mathbf{K_{D11}}(\mathbf{q_1},\mathbf{q_2}) & \mathbf{K_{D21}}(\mathbf{q_1},\mathbf{q_2}) \\ \mathbf{K_{D21}}(\mathbf{q_1},\mathbf{q_2}) & \mathbf{K_{D22}}(\mathbf{q_1},\mathbf{q_2}) \end{bmatrix}. \tag{2.28}$$

The first matching condition control input matrix is proposed as

$$\mathbf{Fm_1} = \begin{bmatrix} -\mathbf{F_{11}}(\mathbf{q_1},\mathbf{q_2})\dot{\mathbf{q}}_2 - \nu & \mathbf{F_{11}}(\mathbf{q_1},\mathbf{q_2})\dot{\mathbf{q}}_1 + \sigma \\ -\mathbf{F_{22}}(\mathbf{q_1},\mathbf{q_2})\dot{\mathbf{q}}_2 & \mathbf{F_{22}}(\mathbf{q_1},\mathbf{q_2})\dot{\mathbf{q}}_1 \end{bmatrix} \tag{2.29}$$

where $\mathbf{Fm_1}$ is a $n \times n$ real matrix. Notice that, there are four unknowns from $\mathbf{Fm_1}$ and three unknowns from the $\mathbf{K_D}$ matrix, considering symmetry. The quantities ν and σ are functions of $\mathbf{q}$ that will be defined through the linearization process by considering the properties of $\mathbf{K_D}$, the convex shape of the potential, the Lyapunov monotonic behavior in time, and the time derivative of the of the Lyapunov function being non-positive. Substituting Eq. 2.29 into Eq. 2.11, i.e., $\mathbf{Fm_1}$ time the vector of generalized velocities is

$$\mathbf{Fm_1}\dot{\mathbf{q}} = \begin{bmatrix} -\nu\dot{\mathbf{q}}_1 + \sigma\dot{\mathbf{q}}_2 \\ 0 \end{bmatrix}. \tag{2.30}$$

The Lyapunov FMC contribution is

$$\mathbf{Fmc_1} = \begin{bmatrix} \mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}}) & \mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}}) \\ \mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}}) & \mathbf{F_{55}}(\mathbf{q},\dot{\mathbf{q}}) \end{bmatrix}. \tag{2.31}$$

which is also a $n \times n$ real matrix. It should be mentioned that Eq. 2.31 could vary according to the particular system. The requirements on Eq. 2.31consist of non- singularities and the sum of $\mathbf{K_v} + \mathbf{Fcm_1}$ is positive semi-definite.

2.4.6.1 KD is non-constant

As was mentioned before, the desired controller should be designed avoiding quadratic velocities. Due to this, some strategies are needed in order to satisfy the FMC and not having quadratic velocity terms in $\mathbf{F_1}$. In solving the FMC from Eq. 2.13 recall that $\mathbf{m}_{\mathbf{q}i} = \mathbf{m}_{\mathbf{q}i}{}^{\mathbf{T}}$. When $\mathbf{K_D}$ is non-constant, the solvability process requires that the FMC is organized into $\sum_{i=1}^{n} \mathbf{m}_{\mathbf{q_i}} \dot{\mathbf{q}}_\mathbf{i} = 0$. Here six equation are found and the terms in $\mathbf{Fm_1}$, namely, $\mathbf{F_{11}}(\mathbf{q_1},\mathbf{q_2})$ and $\mathbf{F_{22}}(\mathbf{q_1},\mathbf{q_2})$ as well as the terms in $\mathbf{K_D}$, namely, $\mathbf{K_{D11}}(\mathbf{q_1},\mathbf{q_2}), \mathbf{K_{D21}}(\mathbf{q_1},\mathbf{q_2})$,and $\mathbf{K_{D22}}(\mathbf{q_1},\mathbf{q_2})$ will be determined to satisfy the FMC. These five unknowns were enough to solve the equations in the ball and beam example. However, for some cases it can be possible to find a constant $\mathbf{K_D}$ through this process. Once the solutions are found, they are substituted in the FMC so that other, undefined terms can be found. The terms $\mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}}), \mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}})$, and $\mathbf{F_{55}}(\mathbf{q},\dot{\mathbf{q}})$ will be found to satisfied FMC.

It is important at this point, to find the terms resulting for $\mathbf{Fmc_1}$ such that $\mathbf{Fmc_1}$ does not include singularities and the sum of $\mathbf{K_v} + \mathbf{Fcm_1}$ is positive semi-definite. Thus, considering the addition of extra input(s) can be a possibility.

2.4.6.2 KD is constant

To solve the FMC when $\mathbf{K_D}$ is chosen to be a constant, it is still important that $\mathbf{K_D} = \mathbf{K_D}^{\mathbf{T}}$ and $\mathbf{Fmc_1}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Fmc_1}(\mathbf{q},\dot{\mathbf{q}})^{\mathrm{T}}$. In this formulation, three equation are found and the terms in $\mathbf{Fm_1}$, namely, $\mathbf{F_{11}}(\mathbf{q_1},\mathbf{q_2})$ and $\mathbf{F_{22}}(\mathbf{q_1},\mathbf{q_2})$, and one of the terms in $\mathbf{Fmc_1}$, namely $\mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}})$, will be determined to satisfy the FMC. Then the solution will be substituted in the FMC to allow the determination of other undefined terms. The terms $\mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}}), \mathbf{F_{55}}(\mathbf{q},\dot{\mathbf{q}})$, and any extra term, if needed, will be determined to define $\mathbf{Fmc_1}$ so that it is singularity free and the sum of $\mathbf{K_v} + \mathbf{Fcm_1}$ is positive semi-definite.

2.4.6.3 P is almost constant

To solve the FMC when $\mathbf{P}$ is chosen to be almost a constant, it is still important that $\mathbf{K_D}(\mathbf{q_1},\mathbf{q_2}) = \mathbf{K_D}(\mathbf{q_1},\mathbf{q_2})^{\mathrm{T}}$, so the symmetry of $\mathbf{K_D}$ must be enforced making the $\mathbf{P}_{12}$ element of $\mathbf{P}$ non-constant. Also, the requirement of $\mathbf{Fmc}_1(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Fmc}_1(\mathbf{q},\dot{\mathbf{q}})^{\mathrm{T}}$ remains. The rest of the formulation follows according to the previous presentation. Three equation are found and solved to obtain $\mathbf{F}_{11}(\mathbf{q}_1,\mathbf{q}_2)$, $\mathbf{F}_{22}(\mathbf{q}_1,\mathbf{q}_2)$, and $\mathbf{F}_{33}(\mathbf{q},\dot{\mathbf{q}})$ to satisfy the FMC. Then the solution will be substituted in the FMC. The terms $\mathbf{F}_{44}(\mathbf{q},\dot{\mathbf{q}}), \mathbf{F}_{55}(\mathbf{q},\dot{\mathbf{q}})$, and any extra term, if needed, will be determined to satisfy $\mathbf{Fmc}_1$ avoiding any singularities.

Moreover, the solvability of the FMC is a process that particularly depends on the system under study, $\mathbf{K_D}$ must satisfy the requirements:

i. $\mathbf{K_D} = \mathbf{K_D}^T > 0$

ii. $\mathbf{K_V} \in \Re^{n\times n}$ is symmetric and at least positive semi-definite

iii. $\mathbf{Fmc}_1 \in \Re^{n\times n}$ is symmetric and the sum of $\mathbf{K}_{\mathrm{v}} + \mathbf{Fcm}_1$ is positive semi-definite.

There are several steps in finding $\mathbf{K_D}$. This could be done by applying the methods presented here. If $\mathbf{K_D}$ is too complicated, as it usually is, try setting $\mathbf{K_D}$ to a constant. If $\mathbf{K_D}$ is considered as a constant, the equations must be solved using the input $\mathbf{F}_1$ and, possible $\mathbf{Fmc}_1$ on the right hand side of the FMC. As a result a simple $\mathbf{F}_1$ is obtained. The values of ν and σ do not affect the solution of the FMC, however they will affect the SMC or the TMC when the linearization process is applied. Finally, solving for the $\mathbf{P}(\mathbf{q})$ matrix could be more convenient depending on the system being treated.

2.5 The second matching condition

This section will present a method to solve the second matching condition by using the results of the first matching condition solution. Let the SMC input presented in Eq.2.6 to be written as

$$\begin{bmatrix} \mathbf{F}_2 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Fm}_2 \\ \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \tag{2.32}$$

where $\mathbf{Fm}_2$ is a $m \times n$ real matrix. Now, by substituting Eq. 2.32 into Eq. 2.6 and recalling that $\mathbf{P}(\mathbf{q}) = \mathbf{K}_D \mathbf{M}(\mathbf{q})^{-1}$, after removing the pre and post multiplication terms by the generalized velocity vector and simplifying, the second matching condition is found to be

$$\mathbf{P}(\mathbf{q})\begin{bmatrix}\mathbf{Fm_2}\\ \mathbf{0}\end{bmatrix} = -\mathbf{K}_v. \tag{2.33}$$

The condition that $\mathbf{K}_v$ be symmetric provides $n(n$-1)/2 linear algebra equations. This last result in Eq. 2.33 can be recognized as Lyapunov's equation (Chen 1998). In order to solve for the matrix $\mathbf{Fm_2}$, Eq. 2.33 is multiply by $\mathbf{P}(\mathbf{q})^{-1}$ to get

$$\begin{bmatrix}\mathbf{Fm_2}\\ \mathbf{0}\end{bmatrix} = -\mathbf{P}(\mathbf{q})^{-1}\mathbf{K}_v \tag{2.34}$$

for which the $\mathbf{K}_v$ solution is

$$\mathbf{K}_v = \sum_{i=1}^{m}\alpha_i\mathbf{P}_i\mathbf{P}_i^T. \tag{2.35}$$

where $\mathbf{P}_i$ is the ith column of $\mathbf{P}(\mathbf{q})$ and α_i is a positive constant and functions as a tuning parameter. Substituting Eq. 2.35 into Eq. 2.34 allows $\mathbf{Fm_2}$ to be found. When the $\mathbf{P}(\mathbf{q})$ matrix is a diagonal matrix, it presents a problem. The $\mathbf{P}(\mathbf{q})$ matrix must be a full matrix or at least lower triangular to achieve a proper $\mathbf{K}_v$. By inspection of Eq. 2.35, notice that the first **m** columns of the $\mathbf{P}(\mathbf{q})$ matrix are used to determine the $\mathbf{K}_v$ matrix in the SMC. If this constraint is not satisfied, a SMC contribution to the control law will lack certain terms making it difficult to select the parameters necessary to place the poles of the linearized version of the system and controller..

Because $\mathbf{K}_v$ needs to be at least positive- semi definite, the eigenvalues of $\mathbf{K}_v$ are required to be

$$eig(\mathbf{K}_v) \geq 0. \tag{2.36}$$

2.6 The third matching condition

In order to solve the third matching condition, the first matching equation must be solved, because the $\mathbf{P}(\mathbf{q})$ matrix is needed. From Eq. 2.7, the third matching equation is stated as

$$-\mathbf{P}(\mathbf{q})\mathbf{G}(\mathbf{q}) + \mathbf{P}(\mathbf{q})\begin{bmatrix}\mathbf{F}_3\\ 0\end{bmatrix} + \nabla\Phi(\mathbf{q}) = 0 \tag{2.37}$$

where the first m equations in Eq. 2.37 are used to determine the control law contribution $\mathbf{F}_3$ while the last $n - m$ rows of the equation provide linear, first order partial differential equations for the potential.

In taking the time derivative of the candidate Lyapunov function, the potential $\Phi(\mathbf{q})$ is assumed to be a function of the generalized positions $\mathbf{q}$ alone. In examining Eq. 2.37, it is seen that $\mathbf{P}(\mathbf{q})$appears in the equation leading to the conclusion that $\Phi(\mathbf{q})$also depends on $\mathbf{P}(\mathbf{q})$. If $\boldsymbol{K_D}$ is a constant when applying equation Eq. 2.15, $\mathbf{P}(\mathbf{q})$is a function of the inverse of the mass matrix, which means that it is a function of $\mathbf{q}$ alone. The potential $\Phi(\mathbf{q})$ is also needed to assure the stability condition of the system. The Hessian of the potential denotes the second derivative of the potential with respect to the generalized coordinates. The determinant of the Hessian of the potential evaluated at $\mathbf{q} \equiv 0$ must be a positive. The Hessian $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 \Phi(\mathbf{q}_1, \mathbf{q}_2)}{\partial \mathbf{q}_1 \partial \mathbf{q}_1} & \dfrac{\partial^2 \Phi(\mathbf{q}_1, \mathbf{q}_2)}{\partial \mathbf{q}_1 \partial \mathbf{q}_2} \\ \dfrac{\partial^2 \Phi(\mathbf{q}_1, \mathbf{q}_2)}{\partial \mathbf{q}_2 \partial \mathbf{q}_1} & \dfrac{\partial^2 \Phi(\mathbf{q}_1, \mathbf{q}_2)}{\partial \mathbf{q}_2 \partial \mathbf{q}_2} \end{bmatrix} \tag{2.38}$$

and the necessary condition on $|\mathbf{H}|$ is

$$\det(\mathbf{H}) > 0. \tag{2.39}$$

In order to guarantee that Eq. 2.38 is a positive matrix, its eigenvalues are required to satisfy

$$eig(\mathbf{H}) > 0. \tag{2.40}$$

The method to solve the third matching equation is similar to the matching equations developed for stabilization as shown in White et al. (2007- 2009).

2.7 Linearization of the system

The different parameters are chosen such that the eigenvalues of the linearized system are the same as the ones found from the linearized equation of motion. The Hessian of the potential is tested so that the potential is concave upward at the equilibrium point. It is a convenient way to choose the parameters. The stabilization will be achieved, once all the mentioned constrains are satisfied. Lyapunov also needs to be tested. Testing the control law through simulation will verify the reliability of the process. To simulate the systems, the quantities of $\mathbf{K_D}$, $\mathbf{K}_v$, the potential, the control inputs, and the coefficients are copied directly from Maple to Matlab using the command "*convert* to *string"* in Maple.

2.8 General strategy

Because effective procedures for designing control laws are very important in nonlinear control theory, a general procedure for achieving stabilization using the matching equations is given as a series of steps. These are:

1. Develop the equation of motion for the mechanical system.
2. Determine the specific matrices from the equation of motion.
3. Solve for the $\mathbf{K_D}$ matrix in the first matching condition, considering constraints. The steps for finding $\mathbf{K_D}$ are:
   a) First of all, try to solve for $\mathbf{K_D}$ by applying some of the mentioned methods to solve the homogeneous, linear differential equations. The positive definiteness of $\mathbf{K_D}$ must be tested. Methods to determine $\mathbf{K_D}$ are:
      i. Perfect differential
      ii. $\mathbf{K_D}$ Lagrangian
      iii. $\mathbf{m}_{\mathbf{q_i}} = 0$
   b) If part (a) does not work, solving for $\mathbf{K_D}$ as a constant or non-constant together with a non-zero $\mathbf{Fm_1}$ and possibly a non-zero $\mathbf{Fmc_1}$ is the next option.
   c) Finally, solving directly for the $\mathbf{P}(\mathbf{q})$ matrix could be attempted; however, the symmetry of $\mathbf{K_D}$ must be enforced.

   Again, it is important to recall that indeed, it is absolutely necessary to test all the mentioned restrictions when each of the different methods is applied.
4. Calculate $\mathbf{K}_v$ and $\mathbf{F_2}$ in the second matching equation.
5. Find the potential $\Phi(\mathbf{q})$ and the force $\mathbf{F_3}$ in the third matching equation.

6. Use linearization or another strategy such as the Hessian of the potential or potential shape to find the unknown constants. The best strategy to use is to find the parameters such that $\mathbf{K_D}$ is positive definite. The matrix $\mathbf{K}_\nu$ must be positive definite (semi-definite). After that, the parameters could be tuned making use of the eigenvalues, the Hessian of the potential, and using the $\mathbf{Fmc_1}$ matrix, because it depends on ν,σ, the elements of $\mathbf{K_D}$, or the elements $\mathbf{P(q)}$. Another important constraint to take into account is that the Lyapunov function has monotonic behavior in time and its time derivative is non-positive.
7. Write the corresponding control law $\boldsymbol{\tau}$ from the matching equations.
8. Test the controller.

In addition, it is important to recall the fact that each system is different, so sometimes the procedure that works best for one system may not necessarily work for another. The important point here is the performance, while also considering the best solution for $\mathbf{K_D}$ that easily satisfies all the matching conditions.

2.9 Chapter Summary

In this chapter, the methods used to develop a formulation to solve the FMC are presented. A discussion of the problems and limiting conditions of the methods for solving the first matching equation are presented for the case when $\mathbf{Fm_1}$ is and is not zero. When $\mathbf{Fm_1} \equiv 0$, $\mathbf{K_D}$ is solved using either the perfect differential method, the Lagrangian method, or the $\mathbf{m}_{\mathbf{q_i}} = 0$ method. Considering the addition of an input $\mathbf{F_1}$, the equations are solved whether $\mathbf{K_D}$ is a constant or not. The matrix $\mathbf{Fmc_1}$ can also play a role in the solution. In addition, $\mathbf{P(q)}$ as almost constant is also an important consideration in order to make the potential easier to find.

The strategy for solving the first matching condition, considering the presence of a control law contribution is based on the $\mathbf{K_D}$ properties and constraints. To this end, assuming that $\mathbf{K_D}$ is not a constant will be the first choice. As a next option, $\mathbf{K_D}$ will be considered as a constant testing the corresponding restrictions. Finally, starting with $\mathbf{P(q)}$ matrix as almost constant could be a possibility. The selected solvability method, in the end, will be based on the properties of the system and the constraints of the proposed method.

Chapter 3 - Ball and Beam System

3.1 Introduction

This chapter presents the balancing control for the ball and beam system applying the direct Lyapunov approach. The physical and challenging problem consists of stabilizing the ball at the center of the actuated beam which is an equilibrium point. The system consists of a ball that rolls along a rotating beam, actuated at its mass center. The rolling of the ball is driven by gravitational forces when the beam is inclined. Because the acceleration of the ball cannot be directly controlled, the system is underactuated.

The chapter is organized in the following way. Section 3.2 presents the dynamic equations for the ball and beam system. In Section 3.3, the first matching condition is formulated. The control contribution of the FMC is presented in Section 3.3.1. In Section 3.4, the second matching condition formulation is shown. Section 3.5 presents the third matching condition. The linearized model which is used to select the undetermined parameters is presented in Section 3.6. Sections 3.7 and 3.8 introduce the potential and the Hessian of the potential respectively, for the system under study. Section 3.9 illustrates the efficacy of the proposed control law with simulation. Finally, Section 3.10 contains the conclusion of the Chapter.

This chapter will present a ball and beam control law where the first matching condition is exactly satisfied and the control law is linear in the velocities. The controller to be presented is an improvement over White et al. (2007) where approximations where necessary to satisfy the FMC.

The ball and beam system geometry and the definitions of the physical parameters are shown in Figure 3.1. The beam has a center located at O, the ball has radius R_o, and m is the mass of the ball. θ is the angle of inclination of the beam and r is the radial position of the ball center relative to beam center.

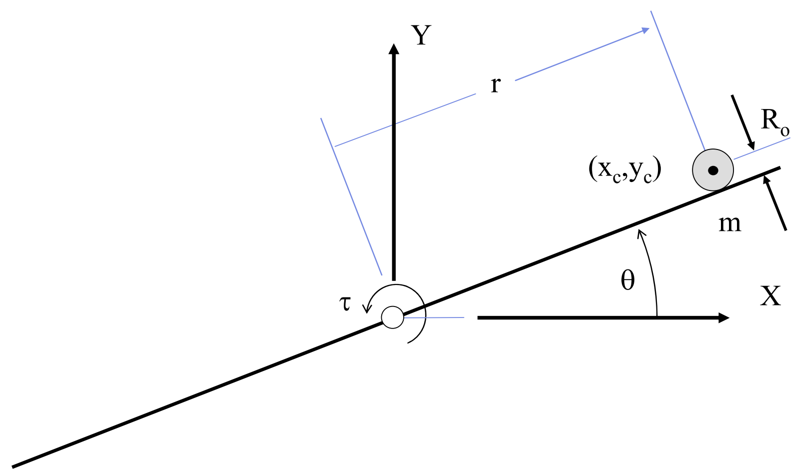

Figure 3.1: Ball and beam schematic

3.2 The Dynamic Equations

The matching method begins with the mechanical system having the unactuated degree of freedom *r*. The position components of the ball center is found in terms of the generalized coordinates as

$$\mathbf{x_c} = r\cos(\theta) - R_0 \sin(\theta) \tag{3.1}$$

and

$$\mathbf{y_c} = r\sin(\theta) + R_0 \cos(\theta) \tag{3.2}$$

for which the corresponding velocities are

$$\mathbf{Vx_c} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} - R_0\cos(\theta)\dot{\theta} \tag{3.3}$$

and

$$\mathbf{Vy_c} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} - R_0\sin(\theta)\dot{\theta}. \tag{3.4}$$

Note that Eq. 3.1 and Eq. 3.2 account for the ball radius.

As mentioned in Chapter 1, the Lagrangian is determined by the difference between the kinetic energy (KE) and the potential energy (PE). The Lagrangian is given by

$$\text{L= KE- PE} \tag{3.5}$$

where

$$\text{KE} = \tfrac{1}{2} m\left(\dot{x}_c^{\,2} + \dot{y}_c^{\,2}\right) + \tfrac{1}{2} I_b \dot{\theta}^2 + \tfrac{1}{2} J_{ball}\left(\dot{\theta} - \tfrac{\dot{r}}{R_0}\right)^2 \tag{3.6}$$

and

$$\text{PE} = mg\left(r\sin(\theta) + R_0\cos(\theta)\right). \tag{3.7}$$

Substituting Eq. 3.6 and Eq.3.7 into Eq. 3.5, the governing equations of the motion for the ball and beam are obtained by applying Lagrangian's equations. The dynamics of the ball and beam are given in Section A.1of Appendix A.

The matrices resulting from the particular dynamic system are $\mathbf{M}(\mathbf{q})$which is the inertia matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ containing the coriolis and centripetal coefficients, and the vector $\mathbf{G}(\mathbf{q})$containing the gravity terms. The dynamic equations of motion are

$$\begin{bmatrix} J_{ball}+\frac{7}{5}mR_0^{\ 2}+mr^2 & -\frac{7}{5}mR_0 \\ -\frac{7}{5}mR_0 & \frac{7}{5}m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix}+$$

$$\begin{bmatrix} mr\dot{r} & mr\dot{\theta} \\ -mr\dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}+$$

$$\begin{bmatrix} mg\cos(\theta)-R_0 mg\sin(\theta) \\ mg\sin(\theta) \end{bmatrix}=\begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{3.8}$$

where $J_{ball}=\frac{2}{5}mR_0^{\ 2}$.

$\mathbf{M}(\mathbf{q})$ has the property of $\mathbf{M}(\mathbf{q})=\mathbf{M}(\mathbf{q})^T>\mathbf{0}$. The coriolis forces and torques of the second term of the Eq. 3.8 are calculated using Christoffel symbols of the first kind to get

$$\mathbf{C}(\dot{\mathbf{q}},\mathbf{q})=\begin{bmatrix} m\,r\,\dot{r} & m\,r\,\dot{\theta} \\ -m\,r\,\dot{\theta} & 0 \end{bmatrix}. \tag{3.9}$$

The form of the dynamic equation matrix that contains coriolis and centripetal terms is such that Eq. 2.9 is a skew- symmetric matrix. Differentiating $\mathbf{M}(\mathbf{q})$with respect to time produces

$$\dot{\mathbf{M}}(\mathbf{q})=\begin{bmatrix} 2\,m\,r\,\dot{r} & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.10}$$

Putting Eq. 3.9 and Eq. 3.10 into Eq. 2.9 shows that

$$\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})=\begin{bmatrix} 0 & -m\,r\,\dot{\theta} \\ m\,r\,\dot{\theta} & 0 \end{bmatrix}. \tag{3.11}$$

The calculation to get Eq. 3.11 is shown in Section A.1 of Appendix A.

3.3 The first matching condition

The addition of a control input to the FMC, namely $\mathbf{F_1}=\mathbf{Fm_1}\dot{\mathbf{q}}$, can be necessary to improve the performance of the controller and to help satisfy the FMC. The addition of the control input to the FMC allows additional parameters for pole placement in the linearized version of the system and controller. The addition of this input also provides greater freedom to solve the FMC because by separating the coefficients of each generalized velocity six equations are found and must be solved for the $\mathbf{K_D}$ matrix and unspecific parameters. Another matrix, called $\mathbf{Fmc_1}$, which will be considered as a contribution of the FMC to the time derivative of the Lyapunov function. The FMC can be written as

$$\dot{\mathbf{K}}_D + \mathbf{K}_D \mathbf{M}(\mathbf{q})^{-1}(\mathbf{F}m_1 - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})) + (\mathbf{F}m_1 - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}))^T \mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_D + \mathbf{F}mc_1 = 0. \tag{3.12}$$

The $\mathbf{K_D}$ matrix is given as

$$\mathbf{K_D} = \begin{bmatrix} K_{D11}(\theta,r) & K_{D21}(\theta,r) \\ K_{D21}(\theta,r) & K_{D22}(\theta,r) \end{bmatrix}. \tag{3.13}$$

The elements of the matrix $\mathbf{K_D}$ are function of the generalized coordinates and the solution of Eq. 3.12 is provided by separation of terms involving $\dot{\theta}$ and $\dot{r}$ where the set of unknowns will be selected later such that the positive definiteness property of $\mathbf{K_D}$ is satisfied.

The first matching condition control input matrix is proposed as

$$\mathbf{Fm_1} = \begin{bmatrix} -\mathbf{F_{11}}(\theta,r)\dot{r} - \nu & \mathbf{F_{11}}(\theta,r)\dot{\theta} + \sigma \\ -\mathbf{F_{22}}(\theta,r)\dot{r} & \mathbf{F_{22}}(\theta,r)\dot{\theta} \end{bmatrix} \tag{3.14}$$

and the Lyapunov FMC contribution is

$$\mathbf{Fmc_1} = \begin{bmatrix} \mathbf{F}_{33}(\theta,r,\dot{\theta},\dot{r}) & \mathbf{F}_{44}(\theta,r,\dot{\theta},\dot{r}) \\ \mathbf{F}_{44}(\theta,r,\dot{\theta},\dot{r}) & \mathbf{F}_{55}(\theta,r,\dot{\theta},\dot{r}) \end{bmatrix}. \tag{3.15}$$

The input $\mathbf{Fm_1}$ is added to the FMC because more parameters, namely ν and σ, are needed to place the poles of the linearized system in desired locations. All of the terms of Eq. 3.15 and Eq. 3.14 are chosen so that $\mathbf{F_1}$, in the FMC, is linear in the velocities and $\mathbf{Fmc_1}$ is the FMC contribution to the control law. The negative of this matrix in Eq. 3.15 is added to the right hand side of the FMC in order to satisfy Eq. 3.12. Details about the solution of the terms leading to Eq. 3.12 are given in Section A.3 of Appendix A.

3.3.1 Actuation added to the first matching condition

The outline of the solution for the ball and beam system is presented considering that $\mathbf{K_D}$ is not a constant.

1. Organize FMC into $\mathbf{m}_r \dot{r} + \mathbf{m}_\theta \dot{\theta} + f(\mathbf{q}) = 0$
2. $\mathbf{m}_r = \mathbf{m}_\theta = 0$
3. Because of symmetry six ODEs are obtained, namely,

   $$\left.\begin{array}{l} \mathbf{m}_{r11} = \mathbf{m}_{r21} = \mathbf{m}_{r22} \\ \mathbf{m}_{\theta 11} = \mathbf{m}_{\theta 21} = \mathbf{m}_{\theta 22} \end{array}\right\} = 0.$$

4. Solve for $\mathbf{K_{D11}}(\theta,r), \mathbf{K_{D21}}(\theta,r), \mathbf{K_{D22}}(\theta,r), \mathbf{F_{11}}(\theta,r), \mathbf{F}_{22}(\theta,r)$. These five unknowns are enough to solve the equations for the ball and beam system. Substitute the solutions in FMC.
5. Solve resulting equations for $\mathbf{F_{33}}(\theta, r, \dot{\theta}, \dot{r}), \mathbf{F_{44}}(\theta, r, \dot{\theta}, \dot{r})$, and $\mathbf{F_{55}}(\theta, r, \dot{\theta}, \dot{r})$ to satisfied the FMC. It is important at this point, to find these results such that $\mathbf{Fmc_1}$ does not include singularities.

Solving for the elements of the $\mathbf{K_D}$ matrix together with the inputs $\mathbf{F_{11}}(\theta,r)$ and $\mathbf{F_{22}}(\theta,r)$, the $\mathbf{K_D}$ matrix is determined to be a function of *r*. The matrix $\mathbf{K_D}$ is

$$\mathbf{K_D} = \begin{bmatrix} C_1\left(5 I_b + 7 mR_o^2 + 5mr^2\right) + C_2\left(5 I_b + 7 mR_o^2 + 5mr^2\right)^2 & -35 R_o\, m\left(C_2\left(\frac{7}{5} mR_o^2 + r^2\right)m + \frac{1}{5}C_1 + C_2\right) \\ -35 R_o\, m\left(C_2\left(\frac{7}{5} mR_o^2 + r^2\right)m + \frac{1}{5}C_1 + C_2\right) & 49\left(C_2\, mR_o^2 + \frac{1}{7}C_1\right)m \end{bmatrix} \quad (3.16)$$

where the set of unknown constants, C_1 and C_2, will be selected later so that the nonlinear controller agrees with a linearized controller in a neighborhood of the equilibrium point and $\mathbf{K_D}$>0.

The result of $\mathbf{Fm_1}$ multiplied by the vector of velocities is

$$\mathbf{Fm_1\dot{q}} = \begin{bmatrix} -\nu\dot{\theta} + \sigma\dot{r} \\ 0 \end{bmatrix}. \quad (3.17)$$

The terms in $\mathbf{Fmc_1}$, namely, $\mathbf{F_{33}}(\theta, r, \dot{\theta}, \dot{r}), \mathbf{F_{44}}(\theta, r, \dot{\theta}, \dot{r})$, and $\mathbf{F_{55}}(\theta, r, \dot{\theta}, \dot{r})$, will be determined to satisfy the FMC. The result is

$$\mathbf{Fmc_1} = \begin{bmatrix} 50\nu C_2 Ib + 50\nu C_2 mr^2 + 10\nu C_1 + 70\nu 7_2 mRo^2 & -25\sigma C_2 Ib - 35C_2 mRo\,\nu - 35C_2 mRo^2\sigma - 25C_2\sigma mr^2 - 5\sigma C_1 \\ -25\sigma C_2 Ib - 35C_2 mRo\,\nu - 35C_2 mRo^2\sigma - 25C_2\sigma mr^2 - 5\sigma C_1 & 70mRo\sigma C_2 \end{bmatrix}. \tag{3.18}$$

The elements of the $\mathbf{Fmc_1}$ matrix are calculated such that it is always at least a positive semi-definite matrix, which is the reason to require its eigenvalues to satisfy

$$eig(\mathbf{K_v} + \mathbf{Fmc_1}) \geq 0. \tag{3.19}$$

The matrix $\mathbf{K_D}$ is already symmetric. Using Eq. 2.16, it is seen that

$$\mathbf{P(q)} = \begin{bmatrix} 5C_1 + 25C_2 Ib + 35C_2 mR_o^{\,2} + 25C_2 r^2 & 0 \\ -35C_2 mR_o & 5C1 \end{bmatrix} \tag{3.20}$$

and the eigenvalues of $\mathbf{P}(\mathbf{q})$ are calculated as

$$\mathbf{eig(P(q))} = \begin{bmatrix} 5C_1 \\ 5C_1 + 25C_2 Ib + 35C_2 mR_o^{\,2} + 25C_2 r^2 \end{bmatrix}. \tag{3.21}$$

Notice from Eq. 3.20 and Eq. 3.21 that the constant C_1 cannot be zero.

3.4 The second matching condition

This section will present the solution of the SMC by using the results of the FMC solution. From Eq. 2.27, if all of the eigenvalues of $\mathbf{K_D}$ have positive real parts, then $\mathbf{K}_v$ is symmetric with non-negative eigenvalues. The matrix $\mathbf{K}_v$ is always, at least, positive semi-definite. The matrix is evaluated by using Eq. 2.29 to get

$$\mathbf{K}_V = \begin{bmatrix} \alpha\left(5C_1 + 25C_2 Ib + 35C_2 mR_o^{\,2} + 25C_2 r^2\right)^2 & -35\alpha\left(5C_1 + 25C_2 Ib + 35C_2 mR_o^{\,2} + 25C_2 r^2\right)C_2\, m\, R_o \\ -35\alpha\left(5C_1 + 25C_2 Ib + 35C_2 mR_o^{\,2} + 25C_2 r^2\right)C_2\, m\, R_o & 1225\alpha C_2^{\,2}\, m^2 R_o^{\,2} \end{bmatrix} \tag{3.22}$$

and the eigenvalues of $\mathbf{K}_v$ are calculated as

$$\mathbf{eig(K_v)} = \begin{bmatrix} 0 \\ \lambda_{KvBB} \end{bmatrix} \tag{3.23}$$

where

$$\begin{aligned} \lambda_{KvBB} &= 0.12\alpha C_2^{\,2} m^2 R_o^{\,2} + 25\alpha C_1^{\,2} + 250\alpha C_1 C_2 I_b + 350\alpha C_1 C_2 mR_o^{\,2} \\ &+ 250\alpha C_1 C_2 mr^2 + 625\alpha C_2^{\,2} I_b^{\,2} + 1750\alpha C_2^{\,2} I_b mR_o^{\,2} + 1250\alpha C_2^{\,2} I_b mr^2 \\ &+ 1225\alpha C_2^{\,2} m^2 R_o^{\,4} + 1750\alpha C_2^{\,2} m^2 R_o^{\,2} r^2 + 1625\alpha C_2^{\,2} m^2 r^4. \end{aligned} \tag{3.24}$$

By using Eq. 2.25 the control law contribution is

$$\mathbf{F_2} = -5\left(5\, C_2\, \dot{\theta}\, I_b + 7\, C_2\, \dot{\theta}\, m\, R_o^{\,2} + 5\, C_2\, \dot{\theta}\, m\, r^2 - 7\, C_2\, m\, R_o\, \dot{r} + C_1\, \dot{\theta}\right)\alpha\,. \tag{3.25}$$

The parameters in Eq. 3.24 are chosen such that $\mathbf{K}_v$ is semi-positive definite. Section A.3 of Appendix A shows the determination of $\mathbf{K}_v$ and its corresponding eigenvalues.

3.5 The third matching equation

The partial differential equation that needs to be solved for the potential is complicated and in this case, some approximations are made. The details of this derivation can be found in Section A.3 of Appendix A.

From Eq. 2.30, the quantity $\mathbf{F}_3$ is determined by

$$\begin{bmatrix} \mathbf{F}_3 \\ 0 \end{bmatrix} = \mathbf{G}(\mathbf{q}) - \mathbf{P}(t)^{-1}\nabla\Phi(\mathbf{q}). \tag{3.26}$$

For the ball and beam, Eq. 3.26 consists of two equations, the first of which is

$$\mathbf{F}_3 - mrg\cos(\theta) + mRog\sin(\theta) + \frac{1}{5}\frac{\frac{\partial\Phi(\theta,r)}{\partial\theta}}{C_1 + 5C_2I_b + 7C_2mRo^2 + 5C_2mr^2} = 0 \tag{3.27}$$

while the second equation is

$$-mg\sin(\theta) + \frac{7}{5}\frac{C2mRo\left(\frac{\partial\Phi(\theta,r)}{\partial\theta}\right)}{\left(C_1 + 5C_2I_b + 7C_2mRo^2 + 5C_2mr^2\right)C_1} + \frac{1}{5}\frac{\frac{\partial\Phi(\theta,r)}{\partial r}}{C_1} = 0. \tag{3.28}$$

Solving Eq. 3.28 for $\Phi(\theta,r)$, shows the solution to be

$$\Phi(\theta,r) = \frac{5}{7}\int^{\theta}\frac{\eta\sin(a)}{\cos(a\delta-\beta)^2}\,da + F_5\left(\frac{\frac{1}{5}\left(\left(\begin{array}{l}-\sqrt{5}\sqrt{\left(C_1+5C_2\,Ib+7C_2\,m\,R_o^{\,2}\right)}C_2\,m\,\theta \\ +7\tan^{-1}\left(\frac{C_2\,m\,r\sqrt{5}}{\left(C_1+5C_2\,Ib+7C_2\,m\,R_o^{\,2}\right)}\right)C_2R_o m\end{array}\right)\sqrt{5}\right)}{\sqrt{\left(C_1+5C_2\,Ib+7C_2\,m\,R_o^{\,2}\right)}C_2\,m}\right)^2 \tag{3.29}$$

where

$$\eta = \frac{\sqrt{\left(C_1 + 5C_2\,Ib + 7C_2\,mR_o{}^2\right)C_1 g}}{R_o\,C_2},$$

$$\delta = \frac{1}{35}\frac{5\sqrt{\left(C_1 + 5C_2\,Ib + 7C_2\,mR_o{}^2\right)C_2 m}\sqrt{5}}{C_2\,R_o\,m},$$

and

$$\beta = \frac{1}{35}\frac{1}{C_2\,R_o\,m}\left(\left(\begin{array}{l} -5\sqrt{\left(C_1 + 5C_2\,Ib + 7C_2\,mR_o{}^2\right)C_2 m\theta} \\ +7\sqrt{5}\tan^{-1}\left(\dfrac{C_2 mr\sqrt{5}}{\sqrt{\left(C_1 + 5C_2\,Ib + 7C_2\,mR_o{}^2\right)C_2 m}}\right)C_2 R_o m \end{array}\right)\sqrt{5}\right).$$

In Eq. 3.29, the arbitrary function of the characteristic of the PDE in Eq. 3.28 is set equal to the parameter F_5 time the square of the characteristic. Once $\Phi(\theta, r)$ is found, it can be substituted into the Eq. 3.27 to find $\mathbf{F_3}$. Solving for $\mathbf{F_3}$, the result is

$$\mathbf{F_3} = \frac{\frac{1}{5}\left(\begin{array}{l} 5\,r\,m\,g\cos(\theta)C_1 + 25\,r\,m\,g\cos(\theta)C_2\,I_b + 35\,r\,m^2 g\cos(\theta)C_2\,R_o{}^2 - \\ 25\,r^3 m^2 g\cos(\theta)C_2 - 5\,m\,R_o g\sin(\theta)C_1 - 25\,m\,R_o g\sin(\theta)C_2\,I_b \\ -35\,m^2\,R_o{}^3 g\sin(\theta)C_2 - 25\,m^2 R_o g\sin(\theta)C_2\,r^2 - \left(\dfrac{\partial}{\partial\theta}\Phi(\theta,r)\right) \end{array}\right)}{\left(C_1 + 5\,C_2\,Ib + 7\,C_2\,m\,R_o{}^2 + 52\,m\,r^2\right)}. \tag{3.30}$$

3.6 Selection of the unknown parameters

By inspection of Eq. 3.8, it is seen that

$$\begin{bmatrix}\ddot{\theta} \\ \ddot{r}\end{bmatrix} = \begin{bmatrix} \dfrac{1}{J_{ball} + mr^2} & \dfrac{R_o}{J_{ball} + mr^2} \\ \dfrac{R_o}{J_{ball} + mr^2} & \dfrac{5J_{ball} + 7mR_0{}^2 + 5mr^2}{\left(J_{ball} + mr^2\right)m} \end{bmatrix}\left(\begin{bmatrix}\tau \\ 0\end{bmatrix} - \begin{bmatrix} mr\dot{r} & mr\dot{\theta} \\ -mr\dot{\theta} & 0\end{bmatrix}\begin{bmatrix}\dot{\theta} \\ \dot{r}\end{bmatrix} - \begin{bmatrix} mg\cos(\theta) - R_0 mg\sin(\theta) \\ mg\sin(\theta)\end{bmatrix}\right). \tag{3.31}$$

Therefore, after some simple manipulations, Eq. 3.31 can be rewritten as

$$\begin{bmatrix}\ddot{\theta} \\ \ddot{r}\end{bmatrix} = \begin{bmatrix} \dfrac{-\tau + 2mr\dot{r}\dot{\theta} + rmg\cos(\theta) - mR_o r\dot{\theta}^2}{J_{ball} + mr^2} \\ -\dfrac{1}{7}\dfrac{\left(-7R_o\tau + 14R_o mr\dot{r}\dot{\theta} + 7R_o rmg\cos(\theta) - 5J_{ball} r\dot{\theta}^2 + 5J_{ball} g\sin(\theta) - 7mR_o{}^2 r\dot{\theta}^2 - 5mr^3\dot{\theta}^2 - 5mr^2 g\sin(\theta)\right)}{J_{ball} + mr^2}\end{bmatrix}. \tag{3.32}$$

From Eq. 3.32, the linearized state equations for the ball and beam are found to be

$$\begin{bmatrix}\dot{\theta}\\ \dot{r}\\ \ddot{\theta}\\ \ddot{r}\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -\dfrac{mg}{I_b} & 0 & 0\\ -\dfrac{5}{7}g & -\dfrac{R_o mg}{I_b} & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ r\\ \dot{\theta}\\ \dot{r}\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \dfrac{1}{I_b+mr^2}\\ \dfrac{R_o}{I_b+mr^2}\end{bmatrix}\boldsymbol{\tau}. \tag{3.33}$$

The values of the physical parameters of the ball and beam system are listed in Table 3.1.

Table 3.1: Physical parameters of the ball and beam system

Parameter	Explanation	Given value
I_b	Beam moment of inertia	0.4 Kg m^2
m	Mass of ball	1.5 Kg
R_o	Radius of ball	0.02 m
g	Acceleration of gravity	9.81 m s^{-2}

Through the same process used to produce Eq. 3.33, the control input is linearized and after substituting the physical parameter values, the linearized control input becomes

$$\left.\begin{aligned}\boldsymbol{\tau}_L = &-\frac{1.428571429(98.51C_2C_1+49.05C_1^2+.4129C_2^2+.28C_2F_5)}{C_2(C_1+2.00420C_2)}\theta\\ &+\frac{.30(197.02C_2^2+.28C_2F_5+196.61C_2C_1+49.05C_1^2)}{(C_1+2.00C_2)^2}r\\ &+\left(-10.02C_2\alpha-5\alpha C_1-v\right)\dot{\theta}\\ &+\left(1.05C_2\alpha+\sigma\right)\dot{r}\end{aligned}\right\} \tag{3.34}$$

where $\boldsymbol{\tau}_L$ is the linear control.

A full state feedback control law is applied to the linearized model given by Eq. 3.33 and the control law is

$$\boldsymbol{\tau}=-\mathbf{K}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}=\begin{bmatrix}-K1 & -K2 & -K3 & -K4\end{bmatrix}\begin{bmatrix}\theta\\ r\\ \dot{\theta}\\ \dot{r}\end{bmatrix}. \tag{3.35}$$

Comparing Eq. 3.34 and Eq. 3.35 and equating like variables, four equations are found. The matrix **K** is determined such that the eigenvalues of the linearized closed loop system are -11.950

$\pm$ j12.049, -1.586, and -0.513 which are based on White et. al (2007) and the positive definiteness property of $\mathbf{K_D}$. The gain matrix is

$$\mathbf{K} = \begin{bmatrix}135.86 & -47.59 & 11.11 & -35.62\end{bmatrix},$$

and the equations are

$$K1 - \frac{1}{C_2(C_1 + 2C_2)}\left(1.43\left(98.51C_2C_1 + 197.02C_2^{\ 2} + 49.05C_1^{\ 2} + 0.28C_2F_5\right)\right) = 0,$$

$$K2 - \frac{1}{(C_1 + 2C_2)^2}\left(0.30\left(197.02C_2^{\ 2} + 196.61C_2C_1 + 49.05C_1^{\ 2} + 0.28C_2F_5\right)\right) = 0,$$

$$K3 - 10.02C_2\alpha + 20.04C_2\nu + 10\nu C_1 - 5\alpha C_1 = 0,$$

and

$$K4 - 1.05C_2(-2\nu + \alpha) = 0.$$

These equations are solved for F_5, α and ν. C_1 and C_2 are picked based on the positive definiteness property of $\mathbf{K_D}$. This property of $\mathbf{K_D}$ is satisfied provide the principal minors of $\mathbf{K_D}$ satisfy the inequalities of

$$C_1\left(5\,I_b + 7\,mR_o^{\ 2} + 5mr^2\right) + C_2\left(5\,I_b + 7\,mR_o^{\ 2} + 5m\,r^2\right)^2 > 0$$

and

$$\left(C_1\left(5\,I_b + 7\,mR_o^{\ 2} + 5mr^2\right) + C_2\left(5\,I_b + 7\,mR_o^{\ 2} + 5m\,r^2\right)^2\right)49\left(C_2\,mR_o^{\ 2} + \frac{1}{7}C_1\right)m$$

$$--35\,R_o\,m\left(C_2\left(\frac{7}{5}mR_o^{\ 2} + r^2\right)m + \frac{1}{5}C_1 + C_2\right) > 0.$$

The unknown ball and beam parameters values used in the pole placement are listed in Table 3.2.

Table 3.2: Values of the control system

Parameter	Explanation	Identified value
C_1	Coefficient of K_D	0.5
C_2	Coefficient of K_D	8.5
F_5	Coeff. of the $\Phi(\theta,r)$	5762.2946
σ	Coefficient from Fm_1	35.62-8.92α
ν	Coefficient from Fm_1	-87.68α+11.11

3.7 The potential

The potential $\Phi(\theta, r)$ is composed of the homogeneous and particular solutions of the governing PDE, namely

$$\Phi(\theta, r) = HS + PS \tag{3.36}$$

where the homogenous solution is

$$HS = 0.99\,F_5\theta^2 - 0.11\,F_5\theta\tan^{-1}(1.91r) + 0.28e\text{-}2\,F_5\tan^{-1}(1.91r)^2. \tag{3.37}$$

and the particular solution is

$$\begin{aligned}
PS = {} & 1.78\cos(\theta - 0.5e^{-1}a\arctan(1.91r))\arctan(-1.73+1620.53\theta^2\text{-}86.51\theta\arctan(1.91r) \\
& +1.15\arctan(1.91r)^2)\ln(1403.42\vartheta^2\text{-}74.92\theta\arctan(1.91r)+.99\arctan(1.91r)^2 \\
& +95.25\theta 9554\arctan(1.91r)+1.73)+24.47\sin(\theta - 0.5e\text{-}1\arctan(1.91r))\arctan(109.98\theta \\
& \text{-}2.93\arctan(1.91r)\text{-}3.73)\text{-}.367e^{-3}\cos(\theta\theta\text{-}.53^{-1}\arctan(1.91r))\ln(3.\text{-}4210.25\theta^2 \\
& +224.77\theta 22\arctan(1.91r)\text{-}2.99\arctan(1.91r)^2+1969580.31\theta^4 \\
& \text{-}210300.48\theta^3\arctan(1.91r)+8420.504193\theta^2\arctan(1.91r)^2 \\
& \text{-}149.85\theta 14\arctan(1.91r)^3+.99\arctan(1.91r)^4)\text{-}3.29\sin(\theta \\
& \text{-}.53e^{-1}\arctan(1.91r))\ln(1403.42\theta^2\text{-}74.92\theta\arctan(1.91r)+.99\arctan(1.91r)^2 \\
& \text{-}95.25\theta 95.\ 54\arctan(1.91r)+1.73)+.73e^{-3}\sin(\theta\theta\text{-}.53^{-1}\arctan(1.91r))\theta \\
& \text{-}.196e^{-4}\sin(\theta\theta\text{-}.53^{-1}\arctan(1.91r))\arctan(1.91r)24.47\sin(\theta \\
& \text{-}.53e^{-1}\arctan(1.91r))\arctan(109.98\theta 10.93\arctan(1.91r)+3.73) \\
& +.147e^{-3}cos(\theta os(\theta\text{--}1\arctan(1.91r))\theta^2\text{-}.78e^{-5}\cos(\theta \\
& \text{-}.53e^{-1}\arctan(1.91r))\theta\arctan(1.91r)+.10e^{-6}\cos(\theta \\
& \text{-}.53e^{-1}\arctan(1.91r))\arctan(1.91r)^2
\end{aligned} \tag{3.38}$$

obtained from a Taylor series expansion about the point **q**=0 of Eq.3.29. The number of terms in the series is six. Figure 3.2 shows a plot of the potential for the ball and beam.

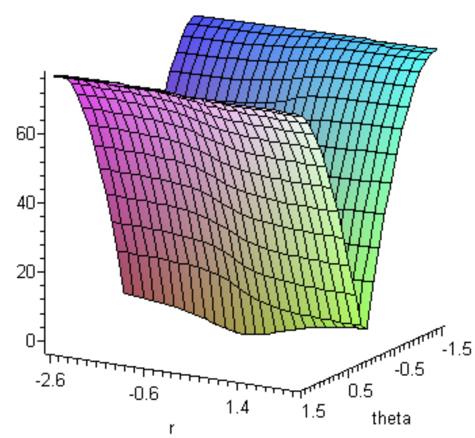

Figure 3.2: The Potential

The inflection points of the potential are the points on the graph at which the concavity of the potential changes. The value of the second derivative just to the left and to the right of each inflection point helps to determine the intervals of the plot. From the plot in Figure 3.2, it is seen that θ needs to be confined to the interval (-1.2, 1.2), while the ball position *r* needs to be restricted to the interval (-0.6, 0.6) so that $\Phi(\theta, r)$ is a convex function.

3.8 The Hessian

The Hessian of the potential denotes the second derivative of the potential $\Phi(\theta, r)$ with respect to θ and r. This matrix, evaluated at the equilibrium point, must be a positive definite matrix and, thus the eigenvalues have to be positive. These values result from the appropriate selection of the parameters considering that the potential has to be concave upward with the Hessian > 0 and a local minimum at the origin. The Hessian is defined as

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 \Phi(\theta, r)}{\partial \theta^2} & \dfrac{\partial^2 \Phi(\theta, r)}{\partial \theta \, \partial r} \\ \dfrac{\partial^2 \Phi(\theta, r)}{\partial \theta \, \partial r} & \dfrac{\partial^2 \Phi(\theta, r)}{\partial r^2} \end{bmatrix} \tag{3.39}$$

and is

$$\mathbf{H} = \begin{bmatrix} 130.6675 & -1357.2425 \\ -1357.2425 & 14417.6368 \end{bmatrix}. \tag{3.40}$$

The corresponding eigenvalues of $\mathbf{H}$ are

$$\boldsymbol{\lambda} = \begin{bmatrix} 2.8744 \\ 14545.4299 \end{bmatrix}. \tag{3.41}$$

3.9 Simulation

The control law design method is applied to the ball and beam system in order to drive the states from a given initial condition to the origin and stabilizing them at that point. Numerical simulation, done using Matlab, confirms that the nonlinear control law stabilizes the system. The details of the Matlab files used for the simulation are in Section A.4 and A.5 of Appendix A. The simulation results are presented in the plots for the angular displacement and angular velocity of the beam together with the position and the linear velocity of the ball. The selected control parameters accommodate a large initial angle for the beam as well as an initial velocity of the ball. In this example, the initial angle of the beam is chosen as 1.5 radians (almost vertical) and the initial velocity of the ball is chosen as 0.4 m/sec, while the initial angular velocity of the beam and the corresponding initial position of the ball are equal to zero.

Figure 3.3 illustrates the ball position and velocity as well as the beam angle and angular velocity as a function of time.

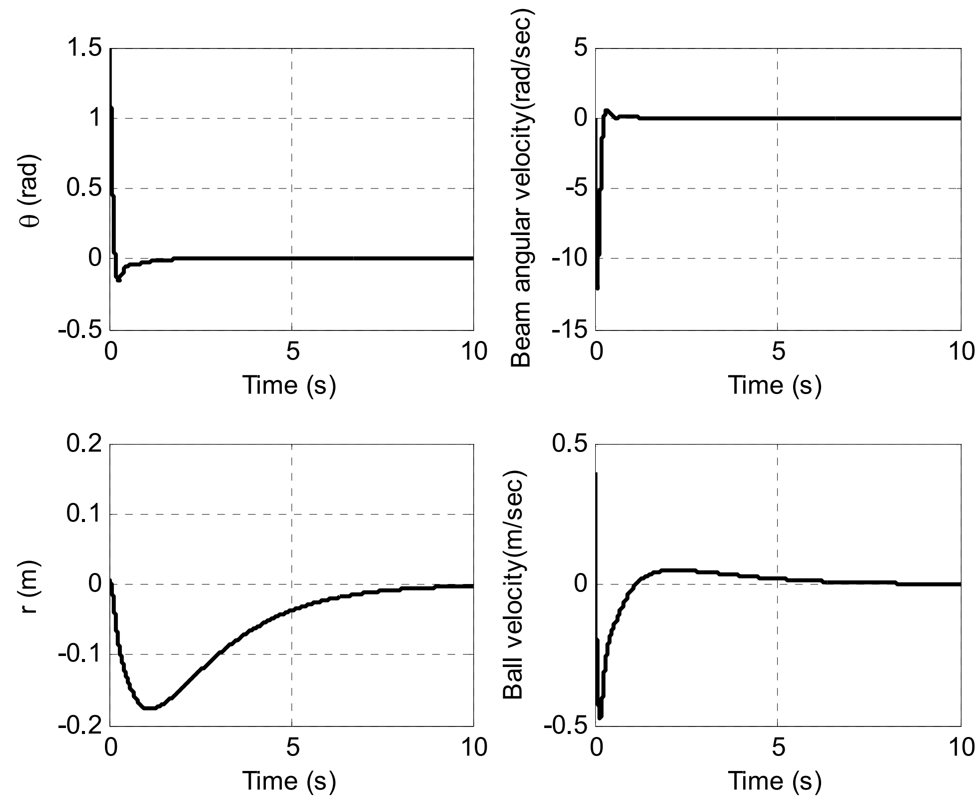

Figure 3.3: Stabilization of the ball and beam system

Figure 3.4 shows that the elements of ***P*** remaining constant after stabilization.

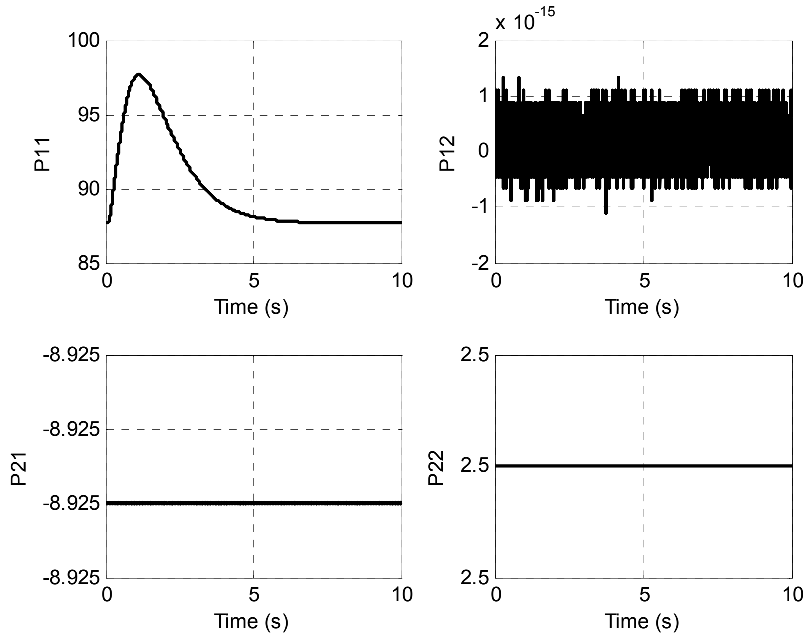

Figure 3.4: Time history for P elements

In Figure 3.5, it can be appreciated that the determinant of the ***P*** matrix is positive.

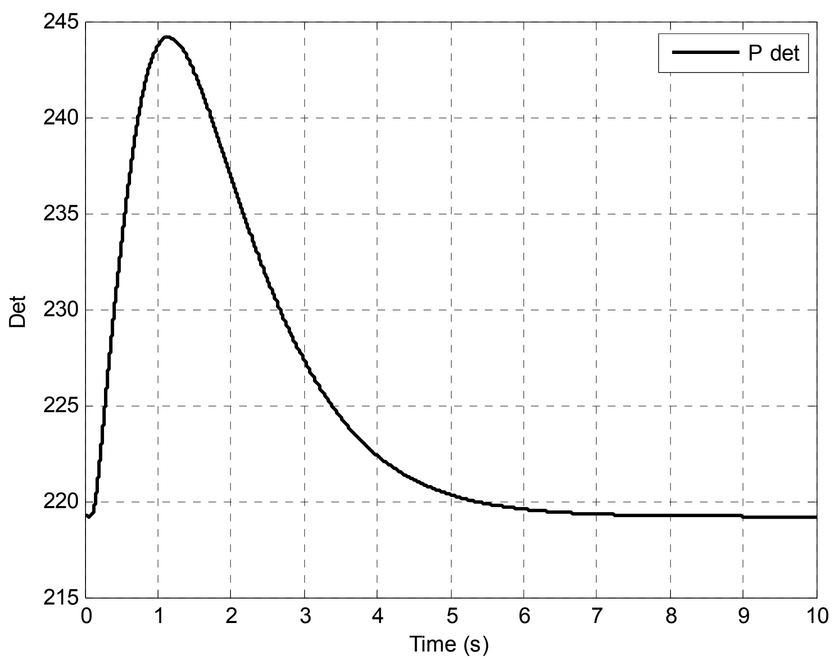

Figure 3.5: Determinant of the P matrix

The Lyapunov function should be monotonically decreasing. The first derivative of the Lyapunov function needs to be non-positive. Figure 3.6 shows the Lyapunov function value and its first time derivative.

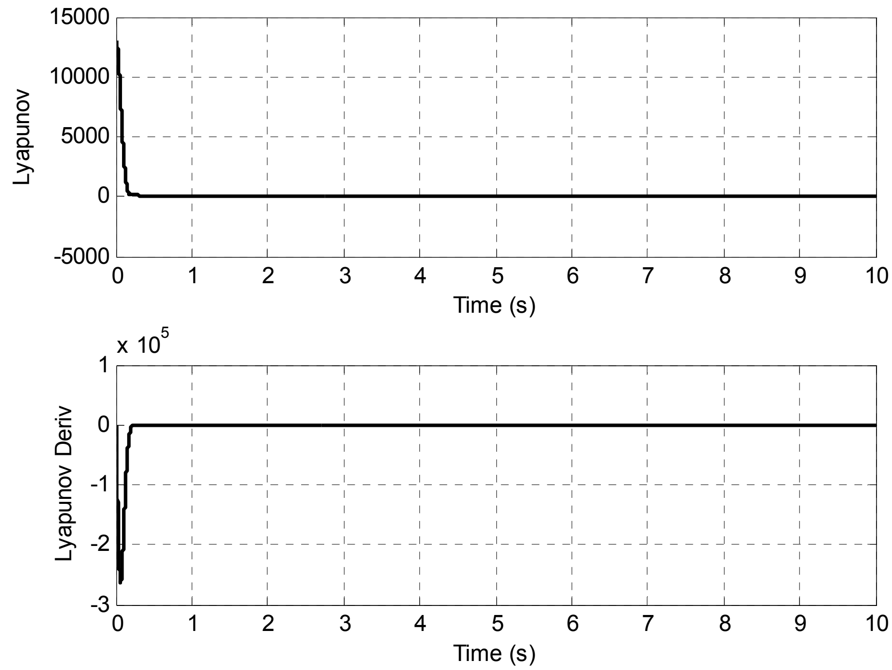

Figure 3.6: Lyapunov Time History and its Time Derivative

Figure 3.7 shows the control law. The behavior shown in Figures 3.3 through 3.6 demonstrates the validity of the Lyapunov candidate function and the derived control law.

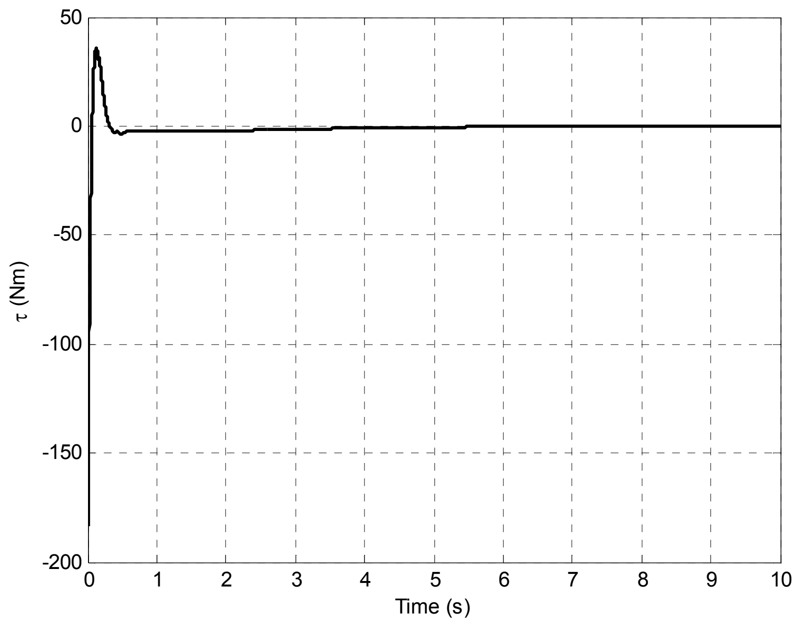

Figure 3.7: Control law

3.10 Chapter Summary

In this chapter, a problem of the stabilization of the underactuated nonlinear ball and beam system is considered. The procedure is based on the direct Lyapunov approach for the design of the controller.

There were several attempts in trying to find a controller for the ball and beam which failed before obtaining the results of the previous section. In the failed attempts, the determined $\mathbf{P}(\mathbf{q})$ matrix was diagonal because the $\mathbf{K_D}$ matrix was a multiple of the mass matrix. This situation is a problem because the first *m* columns of the $\mathbf{P}(\mathbf{q})$ matrix are used to determine the $\mathbf{K}_v$ matrix in the SMC. If $\mathbf{P}(\mathbf{q})$ is not at least lower triangular, the control law of the SMC will have some of the generalized velocity terms missing. At other times, complicated solutions with singularities were found for $\mathbf{K_D}$. In order to be realistic, the elements of $\mathbf{K_D}$ needs to be even functions of the generalized coordinates.

The solvability of the FMC is not trivial. The FMC is solved for $\mathbf{K_D}$ and the control input $\mathbf{F_1}$, considering the design restrictions and properties of the $\mathbf{K_D}$ matrix. The SMC is solved for $\mathbf{K}_v$ and for the contribution $\mathbf{F}_2$. The partial differential equation from the TMC is solved for the potential and the control law contribution $\mathbf{F}_3$. The derived control law is based on positive definiteness and symmetry properties of $\mathbf{K_D}$, the potential shape, and the Lyapunov monotonic behavior in time. As $\dot{\mathbf{V}}(\mathbf{q},\dot{\mathbf{q}})$ is negative semi-definite, the stability in the sense of Lyapunov is assured. The simulation results show that the nonlinear controller design can stabilize the system effectively even when the initial conditions of the system are relatively large.

The presented controller is an improvement over White et al. (2007) where approximations were necessary to satisfy the FMC. This controller, determined here in a straight forward fashion, satisfies the FMC exactly without introducing control law terms that are quadratic in the velocities or approximations.

Chapter 4 - Ball and Arc System

4.1 Introduction

In this Chapter, a simple system consisting of a ball rolling on an arc is considered. The ball and arc system geometry together with the physical parameters are shown in Figure 4.1. The arc has a center located at **O**, the arc radius is r, and the ball has radius R_o. The arc is actuated at **O'**, the arc mass center, located a distance d from the center of the arc. M is the mass of the arc and m is the mass of the ball. The angle θ measures the rotation of the arc from the vertical as shown in Figure 4.1. The angle ϕ measures angular position of the ball from the midpoint of the arc. The pivot is not placed at the center of the circle to avoid an uncontrollable system, Wellstead (1983).

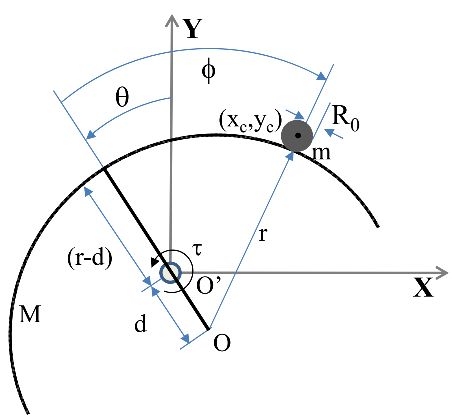

Figure 4.1 Ball and arc schematic

The formulation is presented for two cases. One considers the $\mathbf{K_D}$ matrix as a constant. The other case considers an almost constant $\mathbf{P}$ matrix. The purpose of presenting the two formulations is to demonstrate that one formulation is more complicated when solving for the potential. The second formulation is simpler than the first one.

This chapter is organized in five main sections. Section 4.2 presents the dynamic equations for the ball and arc system. In Section 4.3 a formulation is presented considering a constant $\mathbf{K_D}$ matrix. The first matching condition, applied to this case, appears in Section 4.3.1. The system control signals are presented in Section 4.3.1.1. The second matching condition formulation is shown in Section 4.3.2. Section 4.3.3 presents the third matching condition. The selection of the unknown parameters using the linearized model is presented in Section 4.3.4. Sections 4.3.5 and 4.3.6 introduce the potential and the Hessian for this system. Section 4.3.7 illustrates the efficacy of the proposed control law with simulation. Section 4.4 presents the other formulation considering an almost constant $\mathbf{P}$ matrix. For this part, the first matching condition can be found in Section 4.4.1. In Sections 4.4.2 and 4.4.3, the second and the third matching conditions are shown. The linearized model and the selection of parameters are in Section 4.4.4 and the potential and the Hessian of which are in Sections 4.4.5 and 4.4.6, respectively. Some numerical simulations are illustrated in Section 4.4.7. The conclusion of this chapter is in Section 4.5.

4.2 The Dynamic Equations

The equation of motion describing the ball and arc system can be found by means of Lagrange's equations. The kinetic energy is given by

$$\begin{aligned} KE &= m r\cos(\varphi-\theta)\dot{\varphi} d\cos(\theta)\dot{\theta} - m r\cos(\varphi-\theta)\dot{\theta}^2 d\cos(\theta) \\ &+\frac{1}{2}m r^2\dot{\theta}^2 - m r^2\dot{\varphi}\,\dot{\theta} - m r\sin(\varphi-\theta)\dot{\varphi} d\sin(\theta)\dot{\theta} \\ &+\frac{1}{2}m r^2\dot{\theta}^2 + m r\sin(\varphi-\theta)\dot{\theta}^2 d\sin(\theta) \\ &+\frac{1}{2}m d^2\dot{\theta}^2 + \frac{1}{2}I_A\dot{\theta}^2 \end{aligned} \tag{4.1}$$

where I_A is the inertia of the arc and the potential energy of the system is

$$PE = m g\left(r\cos(\varphi-\theta) - d\cos(\theta)\right). \tag{4.2}$$

Substituting Eq. 4.1 and Eq.4.2 in the Euler-Lagrange equations, the governing equations of the motion for the ball and arc are obtained. These are

$$\begin{aligned} &\begin{bmatrix} m r^2 + m d^2 + I_A - 2m\, rd\cos(\varphi) & mr(d\cos(\varphi)-r)) \\ mr(d\cos(\varphi)-r)) & mr^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta} \\ \ddot{\varphi}\end{bmatrix} + \\ &\begin{bmatrix} \dot{\varphi}mrd\sin(\varphi) - \dot{\theta}mrd\sin(\varphi) & \dot{\theta}mrd\sin(\varphi) + \dot{\varphi}mrd\sin(\varphi) \\ \dot{\theta}mrd\sin(\varphi) & 0 \end{bmatrix}\begin{bmatrix}\dot{\theta} \\ \dot{\varphi}\end{bmatrix} + \\ &\begin{bmatrix} mg(r\sin(\varphi)\cos(\theta) - r\cos(\varphi)\sin(\theta) + d\sin(\theta)) \\ mgr(\sin(\varphi)\cos(\theta) - \cos(\varphi)\sin(\theta)) \end{bmatrix} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) \end{aligned} \tag{4.3}$$

where $\mathbf{M(q)}$ is the inertia matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is the matrix containing the coriolis and centripetal coefficients, and the vector $\mathbf{G}(\mathbf{q})$ contains the gravity terms.

The $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ matrix is

$$\mathbf{C}(\dot{\mathbf{q}},\mathbf{q})=\begin{bmatrix} \dot{\varphi}mrd\sin(\varphi)-\dot{\theta}mrd\sin(\varphi) & \dot{\theta}mrd\sin(\varphi)+\dot{\varphi}mrd\sin(\varphi) \\ \dot{\theta}mrd\sin(\varphi) & 0 \end{bmatrix}. \tag{4.4}$$

The skew-symmetry formulation is tested. To do this the time derivative of the mass matrix for this particular system is computed as

$$\dot{\mathbf{M}}(\mathbf{q})=\begin{bmatrix} 2\,m\,r\,d\sin(\varphi)\dot{\varphi}\varphi & -m\,r\,d\sin(\varphi)\dot{\varphi} \\ -m\,r\,d\sin(\varphi)\dot{\varphi} & 0 \end{bmatrix}. \tag{4.5}$$

The mentioned skew-symmetry condition from Eq. 2.9 is

$$\frac{\mathbf{1}}{\mathbf{2}}\dot{\mathbf{M}}(\mathbf{q})-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})=\begin{bmatrix} 0 & \frac{1}{2}m\,r\,d\sin(\varphi)\dot{\varphi}-m\,r\,d\sin(\varphi)\dot{\varphi}\,\dot{\theta} \\ -\frac{1}{2}m\,r\,d\sin(\varphi)\dot{\varphi}+m\,r\,d\sin(\varphi)\dot{\varphi}\,\dot{\theta} & 0 \end{bmatrix}. \tag{4.6}$$

4.3 Development considering a constant $\mathbf{K_D}$ matrix

In order to apply the DLA, a candidate Lyapunov function will be selected with $\mathbf{K_D}$ as a constant, symmetric, positive definite matrix. This will be part of the candidate Lyapunov function.

4.3.1 The first matching condition

As mentioned in Chapter 2, the FMC provides a contribution for the control law. It will be seen that for the ball and arc case, $\mathbf{P}(\mathbf{q})$ is a full matrix, however because $\mathbf{P}(\mathbf{q})$ is almost constant some manipulations of the FMC are necessary in order to provide a solution.

4.3.1.1 Ball and arc system with $\mathbf{Fm_1}$ *and* $\mathbf{Fmc_1}$

The control input $\mathbf{Fm_1}$ is now utilized as well as $\mathbf{Fmc_1}$, however more unknowns appear in $\mathbf{Fmc_1}$ compared to the other examples. In this example the control term of the FMC is

$$\begin{bmatrix} \mathbf{F_1} \\ \mathbf{0} \end{bmatrix}=\mathbf{Fm_1}\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{4.7}$$

where

$$\mathbf{Fm_1}=\begin{bmatrix} -\mathbf{F_{11}}(qf,qdotf)\dot{\phi}-\nu & \mathbf{F_{11}}(qf,qdotf)\dot{\theta}+\sigma \\ -\mathbf{F_{22}}(qf,qdotf)\dot{\phi} & \mathbf{F_{22}}(qf,qdotf)\dot{\theta} \end{bmatrix} \tag{4.8}$$

and because more terms are necessary to avoid singularities for $\mathbf{Fmc_1}$, $\mathbf{Fmc_1}p$ is defined as

$$\mathbf{Fmc_1}p = \begin{bmatrix} \mathbf{F}_{33}(qf, qdotf) + \mathbf{F}_{55}(qf, qdotf) + \mathbf{F}_{66}(qf, qdotf) & \mathbf{F}_{77}(qf, qdotf) \\ \mathbf{F}_{88}(qf, qdotf) & \mathbf{F}_{44}(qf, qdotf) + \mathbf{F}_{66}(qf, qdotf) + \mathbf{F}_{55}(qf, qdotf) \end{bmatrix} \tag{4.9}$$

and because $\mathbf{Fmc_1}$ must be symmetric

$$\mathbf{Fmc_1} = \mathbf{Fmc_1}p + \mathbf{Fmc_1}p^T. \tag{4.10}$$

The terms ν and σ are included for this particular system in $\mathbf{Fm_1}$ in order to reach a positive semi definite $\mathbf{Fmc_1}$ matrix.

The matrix $\mathbf{Fm_1}$ when multiplied by the vector of velocities is found to be

$$\mathbf{Fm_1\dot{q}} = \begin{bmatrix} \dot{\theta}\nu - \dot{\phi}\sigma \\ 0 \end{bmatrix} \tag{4.11}$$

and $\mathbf{Fmc_1}$, after some manipulations, can be seen in Section B.3 of Appendix B. The formulation used in the solution for the control law, the manipulations to avoid singularities in $\mathbf{Fmc_1}$, and the matrix $\mathbf{Fmc_1}$ itself is show in Section B.3 (Appendix B).

The elements of the $\mathbf{Fmc_1}$ matrix are calculated such that it is always a positive semi-definite matrix. The eigenvalues are

$$eig(\mathbf{K_v} + \mathbf{Fmc_1}) \geq 0. \tag{4.12}$$

The elements of the matrix $\mathbf{K_D}$ are chosen to be constants so that $\mathbf{K_D}$ is positive definite and to get an easily solved partial differential equation for the potential. The FMC will be solved for two terms of $\mathbf{Fm_1}$ and one term of $\mathbf{Fmc_1}$. The $\mathbf{K_D}$ matrix for this problem is

$$\mathbf{K_D} = \begin{bmatrix} K_{D11} & K_{D21} \\ K_{D21} & K_{D22} \end{bmatrix}. \tag{4.13}$$

The set of unknown constants for $\mathbf{K_D}$ and $\mathbf{Fm_1}$ will be selected later so that $\mathbf{K_D}$ is positive definite, $\mathbf{K}_v$ is positive semi-definite, and $\mathbf{K_v} + \mathbf{Fmc_1}$ is positive semi-definite.

The following solution outline for the ball and arc system is presented considering that $\mathbf{K_D}$ is constant.

1. Consider $\mathbf{Fmc_1(q,\dot{q})} = \mathbf{Fmc_1(q,\dot{q})^T}$
2. Because of symmetry three LAEs are obtained from the FMC.
3. Solve for $\mathbf{F_{11}(q,\dot{q})}$, $\mathbf{F_{22}(q,\dot{q})}$, and one of the terms in $\mathbf{Fmc_1}$, namely $\mathbf{F_{33}(q,\dot{q})}$
4. Substitute solutions into the FMC.

5. The terms $\mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{55}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{66}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{77}}(\mathbf{q},\dot{\mathbf{q}})$ and, $\mathbf{F_{88}}(\mathbf{q},\dot{\mathbf{q}})$, are added to $\mathbf{Fmc_1}$ in order to avoid singularities and make $\mathbf{Fmc_1}$ at least positive semi-definite.

The resulting $\mathbf{Fmc_1}$ is

$$\mathbf{Fmc_1} = \begin{bmatrix} Fc_{11}(qf, qdotf) & Fc_{21}(qf, qdotf) \\ Fc_{21}(qf, qdotf) & Fc_{22}(qf, qdotf) \end{bmatrix} \tag{4.14}$$

where

$$Fmc_{11}(\theta,\phi) = \frac{2\begin{pmatrix} 2Ib\sin(\phi)K_{D22}d\dot{\phi} + 2Ib\sin(\phi)K_{D21}d\dot{\phi} + 2\sin(\phi)K_{D22}d^3\dot{\phi}m - \cos(\phi)\nu dK_{D21} \\ +\nu K_{D21}r - 4\cos(\phi)m\dot{\phi}rK_{D21}d^2\sin(\phi) + 2\cos(\phi)m\dot{\phi}rK_{D21}d^2\sin(\phi) \\ -4\cos(\phi)m\dot{\phi}rK_{D22}d^2\sin(\phi) + 2\sin(\phi)K_{D21}d^3\dot{\theta}m + 2\sin(\phi)K_{D11}d\dot{\theta}r^2m \\ +2\sin(\phi)K_{D21}d\dot{\theta}r^2m - 4\sin(\phi)m\dot{\phi}dK_{D11}r^2 + 2\sin(\phi)m\dot{\phi}dK_{D22}r^2 \\ -2\cos(\phi)\dot{\theta}mK_{D11}rd^2\sin(\phi) - 2\dot{\phi}mK_{D21}rd\sin(\phi) + \nu K_{D11}r \end{pmatrix}}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)},$$

$$Fmc_{21}(\theta,\phi) = -\frac{\begin{pmatrix} 2\cos(\phi)m\dot{\phi}rd^2K_{D21}\sin(\phi) + \nu K_{D21}r + 4\cos(\phi)m\dot{\phi}rK_{D22}d^2\sin(\phi) \\ +2\sin(\phi)dm\dot{\phi}r^2K_{D11} - \sigma K_{D21}r - 4\dot{\phi}r^2\sin(\phi)K_{D22}d + \cos(\phi)\sigma dK_{D21} \\ -\cos(\phi)\nu K_{D22}d + \nu K_{D22}r - \sigma K_{D11}r - 2m\dot{\phi}r^2dK_{D21}\sin(\phi) \end{pmatrix}}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)},$$

and

$$Fmc_{22}(\theta,\phi) = \frac{2\left(K_{D21}r + K_{D22}r - \cos(\phi)K_{D22}d\right)\left(2\dot{\phi}mrd\sin(\phi) - \sigma\right)}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)}.$$

The elements of the $\mathbf{F}mc_1$ matrix are calculated such that $\mathbf{K_v} + \mathbf{Fmc_1}$ is always a positive semi-definite matrix, which is the reason to require its eigenvalues to satisfy

$$eig(\mathbf{K_v} + \mathbf{Fmc_1}) \geq 0. \tag{4.15}$$

By using of Eq.2.16 is seen that $\mathbf{P}(\mathbf{q})$ is

$$\mathbf{P}(\mathbf{q}) = \begin{bmatrix} P_{11}(\theta,\phi) & P_{12}(\theta,\phi) \\ P_{21}(\theta,\phi) & P_{22}(\theta,\phi) \end{bmatrix} = \mathbf{M}^{-1}(\mathbf{q})\mathbf{K_D} \tag{4.16}$$

where

$$P_{11}(\theta,\phi)=\frac{KD_{11}r+KD_{21}r-KD_{21}d\cos(\phi)}{r\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$
$$P_{12}(\theta,\phi)=\frac{KD_{11}mr^2-KD_{11}mrd\cos(\phi)+KD_{21}mr^2-2KD_{21}d\cos(\phi)+KD_{21}md^2+KD_{21}I_b}{mr^2\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$
$$P_{21}(\theta,\phi)=-\frac{KD_{21}r-KD_{22}r-KD_{22}d\cos(\phi)}{r\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$
$$P_{22}(\theta,\phi)=\frac{KD_{21}mr^2-KD_{21}mrd\cos(\phi)+KD_{22}mr^2-2KD_{22}d\cos(\phi)+KD_{22}md^2+KD_{22}I_b}{mr^2\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$

and the determinant of $\mathbf{P}(\mathbf{q})$ is

$$\det(\mathbf{P}(\mathbf{q}))=\frac{KD_{22}KD_{11}-KD_{21}{}^2}{r^2m\left(md^2+Ib-md^2\cos(\varphi)^2\right)}. \tag{4.15}$$

4.3.2 The second matching condition

From Eq. 2.28, the control law contribution $\mathbf{F}m_2$ can be found. If all of the eigenvalues of $\mathbf{K_D}$ have positive real parts, then $\mathbf{K}_v$ is symmetric with non-negative eigenvalues. The $\mathbf{K}_v$ matrix is evaluated using Eq. 2.29 to get

$$\mathbf{K}_V=\begin{bmatrix} Kv_{11}(\theta,\phi) & Kv_{21}(\theta,\phi) \\ Kv_{21}(\theta,\phi) & Kv_{22}(\theta,\phi)\end{bmatrix}. \tag{4.16}$$

where

$$Kv_{11}(\theta,\phi)=\frac{\alpha\left(KD_{11}r+KD_{21}r-KD_{21}d\cos(\phi)\right)}{mr^2\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$
$$Kv_{21}(\theta,\phi)=-\frac{\alpha\left(KD_{11}r+KD_{21}r-KD_{21}d\cos(\phi)\right)\left(-KD_{21}r-KD_{22}r+KD_{22}d\cos(\phi)\right)}{mr^2\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$
$$Kv_{22}(\theta,\phi)=\frac{\alpha\left(-KD_{21}r-KD_{22}r+KD_{22}d\cos(\phi)\right)^2}{mr^2\left(md^2+Ib-md^2\cos(\phi)^2\right)},$$

and the eigenvalues of $\mathbf{K}_v$ are calculated as

$$eig(K_v)=\begin{bmatrix}0\\ \lambda_{KvBA}\end{bmatrix} \tag{4.17}$$

where

$$\lambda_{KvBA} = \alpha + \begin{pmatrix} d^2 KD_{22}{}^2 \cos(\phi) - KD_{22} d\cos(\phi) KD_{21} r - 2KD_{22}{}^2 d\cos(\phi) r \\ + 2KD_{21}{}^2 r^2 + 2KD_{21} r^2 KD_{22} + r^2 KD_{22} + KD_{21}{}^2 d^2 \cos(\phi)^2 \\ - 2KD_{21} d\cos(\phi) KD_{11} r - 2KD_{21}{}^2 d\cos(\phi) r + KD_{11}{}^2 r^2 \\ + 2KD_{11}{}^2 r^2 KD_{21} \end{pmatrix}. \tag{4.18}$$

The second matching condition provides the contribution F_2 for the control law which is

$$\mathbf{F_2} = \frac{\alpha\left(-F_2\,\dot{\theta}\,rF_2\dot{\phi} + F_2\,\dot{\theta}\,d\cos(\phi) - \dot{\theta}\,r\,KD_{11} - \dot{\phi}\,KD_{22}\,r + \dot{\phi}\,KD_{22}\,d\cos(\phi)\right)}{r^2\left(m\,d^4 + 2\,m\,dI_A - 2\,m\,d^4\cos(\phi)^2 + I_A^2 - 2\,I_A m\,d^2\cos(\phi)^2 + m^2\,d^4\cos(\phi)^4\right)}. \tag{4.19}$$

4.3.3 The third matching condition

The first row of the TMC provides the equation that is used to determine the control law contribution $\mathbf{F}_3$. From Eq. 2.35, two equations are found which are

$$\begin{aligned} &\mathbf{F}_3 - mg\left(r\sin(\phi)\cos(\theta)\right) - r\cos(\phi)\sin(\theta) + d\sin(\phi) \\ &+\frac{1}{5}\frac{\left(KD_{21}mr^2 - KD_{21}mrd\cos(\phi) + KD_{22}mr^2 - 2KD_{22}d\cos(\phi) + KD_{22}md^2 + KD_{22}I\right)\frac{\partial\Phi(\theta,\phi)}{\partial\theta}}{KD_{22}KD_{11} - KD_{21}{}^2} \\ &-\frac{\left(KD_{11}mr^2 - KD_{11}mrd\cos(\phi) + KD_{21}mr^2 - 2KD_{21}d\cos(\phi) + KD_{21}md^2 + KD_{21}I\right)\frac{\partial\Phi(\theta,\phi)}{\partial\phi}}{KD_{22}KD_{11} - KD_{21}{}^2} = 0 \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &-mgr\left(\sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta)\right) + \frac{\left(-KD_{21}r - KD_{22}r + 2KD_{22}d\cos(\phi)\right)rm\left(\frac{\partial\Phi(\theta,\phi)}{\partial\theta}\right)}{KD_{22}KD_{11} - KD_{21}{}^2} \\ &+\left(KD_{11}r - KD_{21}r - 2KD_{21}d\cos(\phi)\right)rm\left(\frac{\partial\Phi(\theta,\phi)}{\partial\phi}\right) = 0. \end{aligned} \tag{4.21}$$

Solving Eq.(4.21) for the potential $\mathbf{\Phi}(\theta,\phi)$ results in

$$\mathbf{\Phi}(\theta,\phi) = \frac{1}{-KD_{11}r - 2KD_{21}r + KD_{21}d - KD_{22}r + KD_{22}d}\left(g\cos(\mathbf{\Omega}(\theta,\phi))\right) \tag{4.22}$$

where

$$\Omega(\theta,\varphi)=\frac{1}{(-\mathrm{KD}_{11}\mathrm{r}-\mathrm{KD}_{21}\mathrm{r}+\mathrm{KD}_{21}\mathrm{d})KD_{21}\Psi}\begin{pmatrix}\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{11}\mathrm{r}\text{-}2\varphi\mathrm{KD}_{21}{}^{2}\Psi\mathrm{r}\text{-}\varphi\mathrm{KD}_{21}{}^{2}\Psi\mathrm{d}+\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{22}\mathrm{r}-\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{22}\,d\\ -\theta\,\mathrm{KD}_{21}\Psi\mathrm{KD}_{11}\mathrm{r}-\theta\,\mathrm{KD}_{21}{}^{2}\Psi\mathrm{r}+\theta\,\mathrm{KD}_{21}{}^{2}\Psi\,d+2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{11}\,r\\ +2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{21}\,r-2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{21}\,d\\ +2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}{}^{2}+2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}\mathrm{KD}_{21}\\ -2r\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}\mathrm{KD}_{21}\,d-2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{11}\\ -2r^{2}\,\mathrm{KD}_{21}{}^{3}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)+-2r\,\mathrm{KD}_{21}{}^{3}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)d\end{pmatrix}\mathrm{KD}_{22}\mathrm{KD}_{11}$$

$$-\frac{1}{(-\mathrm{KD}_{11}\mathrm{r}-\mathrm{KD}_{21}\mathrm{r}+\mathrm{KD}_{21}\mathrm{d})KD_{21}\Psi}\begin{pmatrix}\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{11}\mathrm{r}\text{-}2\varphi\mathrm{KD}_{21}{}^{2}\Psi\mathrm{r}\text{-}\varphi\mathrm{KD}_{21}{}^{2}\Psi\mathrm{d}+\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{22}\mathrm{r}-\varphi\mathrm{KD}_{21}\Psi\mathrm{KD}_{22}\,d\\ -\theta\,\mathrm{KD}_{21}\Psi\mathrm{KD}_{11}\mathrm{r}-\theta\,\mathrm{KD}_{21}{}^{2}\Psi\mathrm{r}+\theta\,\mathrm{KD}_{21}{}^{2}\Psi\,d+2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{11}\,r\\ +2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{21}\,r-2\,\mathrm{KD}_{22}\arctan(\varsigma(\phi))\Psi\,\mathrm{KD}_{21}\,d\\ +2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}{}^{2}+2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}\mathrm{KD}_{21}\\ -2r\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{22}\mathrm{KD}_{11}\mathrm{KD}_{21}\,d-2r^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\mathrm{KD}_{11}\\ -2r^{2}\,\mathrm{KD}_{21}{}^{3}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)+2r\,\mathrm{KD}_{21}{}^{3}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)d\end{pmatrix}\mathrm{KD}_{21}{}^{2}$$

$$-F_7\cosh\Xi(\theta,\phi)\mathrm{KD}_{11}\,r-2F_7\cosh\Xi(\theta,\phi)\mathrm{KD}_{21}\,r+2F_7\cosh\Xi(\theta,\phi)\mathrm{KD}_{21}\,d-F_7\cosh\Xi(\theta,\phi)\mathrm{KD}_{22}\,r+F_7\cosh\Xi(\theta,\phi)\mathrm{KD}_{22}\,d$$

and where

$$\Psi=\sqrt{(-KD_{11}r-KD_{21}r+KD_{21}d)(-KD_{11}r-KD_{21}r+KD_{21}d)},$$

$$\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)=\operatorname{arctanh}\left(\frac{(-KD_{11}r-KD_{21}r+KD_{21}d)(\cos(\varphi)-1)}{\sqrt{(-KD_{11}r-KD_{21}r+KD_{21}d)(-KD_{11}r-KD_{21}r+KD_{21}d)}\sin(\phi)}\right),$$

$$\arctan(\varsigma(\phi))=\arctan\left(\frac{\cos(\varphi)-1}{\sin(\varphi)}\right),$$

and

$$\cosh\Xi(\theta,\phi)=\cosh\left(\frac{\begin{pmatrix}-2rK_{D21}{}^{2}\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)\\ -2K_{D22}\arctan(\varsigma(\phi))\Psi+\theta K_{D21}\Psi\\ +2r\operatorname{arctanh}\left(\frac{\Gamma(\theta,\phi)}{\Psi}\right)K_{D11}K_{D22}\end{pmatrix}}{\left(K_{D21}\sqrt{(K_{D11}r+K_{D21}r-K_{D21}d)(K_{D11}r+K_{D21}r+K_{D21}d)}\right)}\right).$$

By substituting $\mathbf{\Phi}(\theta,\phi)$ from Eq. 4.22 into Eq.4.20, the contribution from the TMC to the control law is

$$\mathbf{F_3} = \frac{\frac{1}{5}\left(\begin{array}{l} r\,m\,g\sin(\phi)\cos(\theta)KD_{21}{}^2 - r\,m\,g\sin(\phi)\cos(\theta)KD_{22}\;KD_{11} - r\,m\,g\cos(\phi)\sin(\theta)KD_{21}{}^2 \\ + r\,m\,g\,cos(\phi)\sin(\theta)KD_{22}\;KD_{11} + m\,g\,d\,sin(\theta)KD_{21}{}^2 - m\,g\,d\sin(\theta)KD_{22}\;KD_{11} - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{21}\,m\,r^2 \\ - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{21}\,m\,r^2 + \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{21}\,m\,r\,d\cos(\phi) - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{22}\;m\,r^2 \\ + 2\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{22}\,m\,r\,d\cos(\phi) - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{21}\,m\,d^{\,2} - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD_{22}\,I_b \\ + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{22}\;m\,r^2 - \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{11}\,m\,r\,d\cos(\phi) + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{21}\,m\,r^2 \\ - 2\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{21}\,m\,r\,d\cos(\phi) + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{21}\,m\,d^{\,2} + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD_{22}\,I_b \end{array}\right)}{\left(KD_{22}\;KD_{11} - KD_{21}{}^2\right)} . \tag{4.23}$$

4.3.4 Selection of the unknown parameters

By inspection of Eq. 4.3 it is seen that

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix} \frac{1}{md^2 + Ib - md^2\cos(\phi)^2} & \frac{R_o}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)} \\ \frac{R_o}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)} & \frac{5J_{ball} + 7mR_0{}^2 + 5mr^2}{mr^2\left(md^2 + Ib - md^2\cos(\phi)^2\right)} \end{bmatrix} \tag{4.24}$$

$$\left(\begin{bmatrix}\tau\\0\end{bmatrix} - \begin{bmatrix} 2\dot{\phi}mrd\sin(\phi)\dot{r} & 2\dot{\theta}mrd\sin(\phi) - 2\dot{\phi}mrd\sin(\phi) \\ -2\dot{\theta}mrd\sin(\phi) & 0 \end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\end{bmatrix} - \begin{bmatrix} mg(r\sin(\phi)\cos(\theta)) - r\cos(\phi)\sin(\theta) + d\sin(\theta) \\ -mgr(\sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta)) \end{bmatrix}\right).$$

Thus, after some simple manipulations, Eq. 4.24 can be rewritten as

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix} \frac{1}{md^2 + Ib - md^2\cos(\phi)^2}\left(\begin{array}{l} \tau - 4\dot{\phi}mrd\sin(\phi)\dot{\theta} + 2\dot{\phi}^2mrd\sin(\phi) + mg\sin(\theta) \\ + 2\dot{\theta}^2mrd\sin(\phi) - 2md^2\cos(\phi)\dot{\theta}\sin(\phi) \\ + d\cos(\phi)mg\sin(\phi)\cos(\theta) - d\cos(\phi)^2mg\sin(\theta) \end{array}\right) \\ \frac{1}{r\left(md^2 + Ib - md^2cos(\varphi\sigma^2\right)}\left(\begin{array}{l} r\tau - 4\dot{\phi}mr^2d\sin(\phi)\dot{\theta} + 2\dot{\phi}^2mr^2d\sin(\phi) + rmg\sin(\theta) \\ - d\cos(\phi)\tau + 4md^2\cos(\phi)\dot{\phi}mr\sin(\phi)\dot{\theta} \\ - 2d^2\cos(\phi)\dot{\phi}^2mr\sin(\phi)\cos(\theta) \\ + d\cos(\phi)mgr\sin(\phi)\cos(\theta) - d\cos(\phi)^2mg\sin(\theta) \\ + 2mr^2\dot{\theta}^2d\sin(\phi) - 4mrd^2\cos(\phi)\dot{\theta}^2\sin(\phi) \\ + 2md^3\dot{\theta}^2\sin(\phi) - md^2g\sin(\phi)\cos(\theta) \\ + 2Ib\dot{\theta}^2d\sin(\phi) - Ibg\sin(\phi)\cos(\theta) + Ibg\cos(\phi)\sin(\theta) \end{array}\right) \end{bmatrix}. \tag{4.25}$$

The linearized equation of motion of the ball and arc is

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{d\,m\,g}{I_b} & 0 & 0 \\ \dfrac{g}{r} & \dfrac{d\,m\,g\,r - d^2\,m\,g - I_b\,g}{I_b} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{I_b + m\,d^2 - m\,d^2\cos(\phi)^2} \\ \dfrac{r - d\cos(\phi)}{r\left(I_b + m\,d^2 - m\,d^2\cos(\phi)^2\right)} \end{bmatrix} \tau. \tag{4.26}$$

The details of the linearization can be seen in Section B.3 of Appendix B.

Eq. 4.26 also shows the variables of the state vector. The values of the physical parameters of the ball and arc system are listed in Table 4.1 and are taken from Sheng, Renner, and Levine (2010).

Table 4.1: Physical parameters of the ball and arc system

Parameter	Explanation	Given value
I_b	Arc moment of inertia	0.05 Kg m^2
m	Mass of ball	0.02 Kg
d	Distance to the pivot	0.5 m
r	Radius of ball	0.75 m
g	Acceleration of gravity	9.81 m s^{-2}

By using Eq. 2.2, the control input is

$$\tau = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3\,. \tag{4.27}$$

Linearizing and substituting for the physical parameters of the system, the linearized control input becomes

$$\boldsymbol{\tau}_L = \begin{bmatrix} \left(\dfrac{\tau_1(KD)}{\tau_2(KD)}\right)\theta + \left(\dfrac{\tau_3(KD)}{\tau_4(KD)}\right)\phi \\ + \left(20\alpha KD_{11} - \nu + 6.66\alpha KD_{21}\right)\dot{\theta} \\ + \left(20\alpha KD_{21} + 6.66\alpha KD_{22} + \sigma\right)\dot{\phi} \end{bmatrix}. \tag{4.28}$$

where

$$\boldsymbol{\tau}_1(\mathbf{KD}) = 0.5\begin{pmatrix} 0.75KD_{21}F_6 + 0.188KD_{22}F_6 + 0.562KD_{11}F_6 + 2.45KD_{21}{}^2 \\ + 2.45KD_{22}KD_{21} + 7.35KD_{11}KD_{22} + 7.35KD_{11}KD_{21} \end{pmatrix}, \tag{4.29}$$

$$\boldsymbol{\tau}_2(\mathbf{KD}) = \begin{pmatrix} 0.56KD_{11}^{2}+0.937KD_{11}KD_{21}+0.188KD_{11}KD_{22}+0.25KD_{21}^{2} \\ +0.063KD_{22}KD_{21} \end{pmatrix}, \tag{4.30}$$

$$\tau_3(\mathbf{KD}) = \begin{pmatrix} -0.257KD_{11}^{2}KD_{22}+0.056F_6KD_{21}^{2}+0.076KD_{21}^{3} \\ +0.032F_6KD_{22}KD_{21}+0.211KD_{21}KD_{11}KD_{22} \\ +0.042F_6KD_{11}KD_{21}+0.167KD_{21}^{2}KD_{11} \\ +0.172KD_{21}KD_{11}^{2}+0.031KD_{22}^{2}KD_{21} \\ +0.094KD_{11}KD_{22}^{2}+0.098KD_{21}^{2}KD_{22} \\ +0.041KD_{11}^{3}+0.032F_6KD_{11}KD_{22}+0.004F_6KD_{22}^{2} \end{pmatrix}, \tag{4.31}$$

and

$$\boldsymbol{\tau}_4(\mathbf{KD}) = \begin{pmatrix} 0.063KD_{21}^{3}+0.422KD_{11}^{3}+0.844KD_{21}KD_{11}^{2} \\ +0.422KD_{21}^{2}KD_{11}+0.016KD_{21}^{2}KD_{22} \\ +0.146KD_{11}^{2}KD_{22}+0.094KD_{21}KD_{11}KD_{22} \end{pmatrix}. \tag{4.32}$$

A linear state feedback control law is applied to the linearized model described in Eq. 4.26. The control law is

$$\boldsymbol{\tau} = -\mathbf{K}\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} -K1 & -K2 & -K3 & -K4 \end{bmatrix}\begin{bmatrix} \theta \\ \phi \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}. \tag{4.33}$$

Comparing Eq. 4.28 and Eq. 4.33 and equating the coefficients on like state variables, four equations are found. The gain matrix **K** is determined such that the eigenvalues of the linearized closed loop system are -76.7918, -4.7457, -11.6366 and, -13.5405 based on the positive definiteness property of $\mathbf{K_D}$. The corresponding gain matrix is

$$\mathbf{K} = \begin{bmatrix} 82.2743 & 138.2016 & -34.1319 & 118.4029 \end{bmatrix}$$

which satisfies the equations

$$K_1 + \frac{0.1\left(5.4e^{30}K_{D21}^{3}+5.4e^{30}K_{D21}^{2}K_{D22}+1.8e^{30}e^{30}K_{D22}^{2}K_{D21}+1.8e^{29}K_{D22}^{3}\right)}{\left(2.26e^{6}K_{D21}+7.52e^{5}K_{D22}\right)\left(2.36e^{21}K_{D21}^{2}+1.54e^{21}K_{D22}K_{D21}+2.56e^{20}K_{D22}^{2}\right)} = 0,$$

$$K_2 + \frac{0.02\begin{pmatrix} 1.56e^{41}K_{D22}^{4}+1.76e^{43}K_{D22}K_{D21}^{3}+8.57e^{42}K_{D21}^{2}K_{D22}^{2} \\ +1.89e^{42}K_{D22}^{3}K_{D21}+1.30e^{43}K_{D21}^{4} \end{pmatrix}}{\left(2.26e^{6}K_{D21}+7.52e^{5}K_{D22}\right)\left(2.36e^{21}K_{D21}^{2}+1.54e^{21}K_{D22}K_{D21}+2.56e^{20}K_{D22}^{2}\right)} = 0,$$

$$K_3 + 6.e + 9\alpha K_{D21} + 34.13 = 0,$$

and

$$K_4 + 118.40 = 0.$$

The four equations can be solved for F_6, K_{D11}, σ, and ν. K_{D21}, α, and K_{D22} are chosen based on the positive definiteness property of $\mathbf{K_D}$, the potential shape, and the Hessian of the potential. These values are listed in Table 4.2

Table 4.2: Values of the control system

Parameter	**Explanation**	**Identified value**
K_{D11}	Element of K_D	$0.8537774058 KD21 + 0.3925497477 KD22$
σ	Term of F_1	$118.4029000 - 20.\,\alpha\, KD21 - 6.666666668\alpha\, KD22$
F_6	Coefficient of $\Phi(\theta,\phi)$	$\dfrac{0..15e^{-8}\left(3.8e^{17} K_{D21}{}^2 + 2.51e^{17} K_{D21} K_{D22} + 4.14e^{16} K_{D22}{}^2\right)}{2.26e^{6} K_{D21} + 7.52e^{5} K_{D22}}$
ν	Term of F_1	$34.13190000 + 23.74221478\alpha\, KD21 + 7.850994954\alpha\, KD22$

4.3.5 The potential

Once the solution is found for the potential $\mathbf{\Phi}(\theta,\phi)$, it is seen that it is composed of a homogeneous and a particular solution which can be written as

$$\mathbf{\Phi}(\theta,\phi) = \mathrm{HS} + \mathrm{PS} \tag{4.34}$$

where the homogenous solution is

$$\mathrm{HS} = F_5 \cosh\left(-1.46\operatorname{arctanh}\left(\frac{5.71(\cos(\phi)-1)}{\sin(\phi)}\right) + \theta + 6.25\arctan\left(\frac{(\cos(\phi)-1)}{\sin(\phi)}\right)\right) \tag{4.35}$$

and the particular solution is

$$\mathrm{PS} = -212.78\cos\left(2.02\phi - 0.999\theta - 6.25\arctan\left(\frac{(\cos(\phi)-1)}{\sin(\phi)}\right) + 1.46\operatorname{arctanh}\left(\frac{5.71(\cos(\phi)-1)}{\sin(\phi)}\right)\right). \tag{4.36}$$

Figure 4.2 presents a plot of the potential for the first formulation of the ball and arc system. The potential for the ball and arc system is plotted in the interval (-0.2, 0.2) for θ and for ϕ.

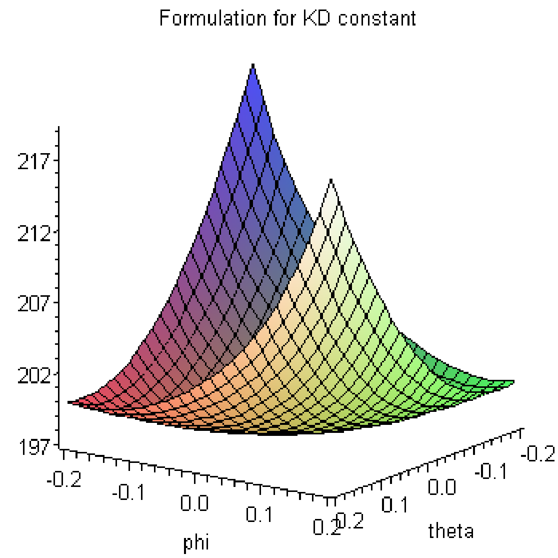

Figure 4.2: Potential for the ball and arc system

4.3.6 The Hessian

When taking the second derivative of the potential $\mathbf{\Phi}(\theta,\phi)$ with respect to θ and ϕ, the Hessian of the potential is found. This matrix is evaluated at the equilibrium point. The Hessian is

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 \mathbf{\Phi}(\theta,\phi)}{\partial \theta^2} & \dfrac{\partial^2 \mathbf{\Phi}(\theta,\phi)}{\partial \theta \, \partial \phi} \\ \dfrac{\partial^2 \mathbf{\Phi}(\theta,\phi)}{\partial \theta \, \partial \phi} & \dfrac{\partial^2 \mathbf{\Phi}(\theta,\phi)}{\partial \phi^2} \end{bmatrix} \tag{4.37}$$

and substituting for the derivatives in Eq. 4.36, the matrix becomes

$$\mathbf{H} = \begin{bmatrix} 727.9019 & 221.6720 \\ 221.6720 & 625.9347 \end{bmatrix}. \tag{4.38}$$

The eigenvalues of Eq.4.38 are

$$\boldsymbol{\lambda} = \begin{bmatrix} 904.3778 \\ 449.4588 \end{bmatrix}. \tag{4.39}$$

From Eq. 4.37 and Eq. 4.38 it is noticed that H is a positive definite matrix in the neighborhood of the origin.

4.3.7 Simulation

The corresponding accelerations $\ddot{\theta}$ and $\ddot{\phi}$ are calculated to numerically simulated the ball and arc system. Figure 4.4 presents the position and the velocity transient behaviors for the ball and arc. In this example, the initial angle and angular velocity of the arc is chosen as 0.6 radians and 1.2 rad/sec, respectively. The initial velocity of the ball is chosen as zero, while the position is 0.2 radians. Figure 4.4 illustrates the ball position and the arc angle, as well as also the corresponding velocities as a function of time.

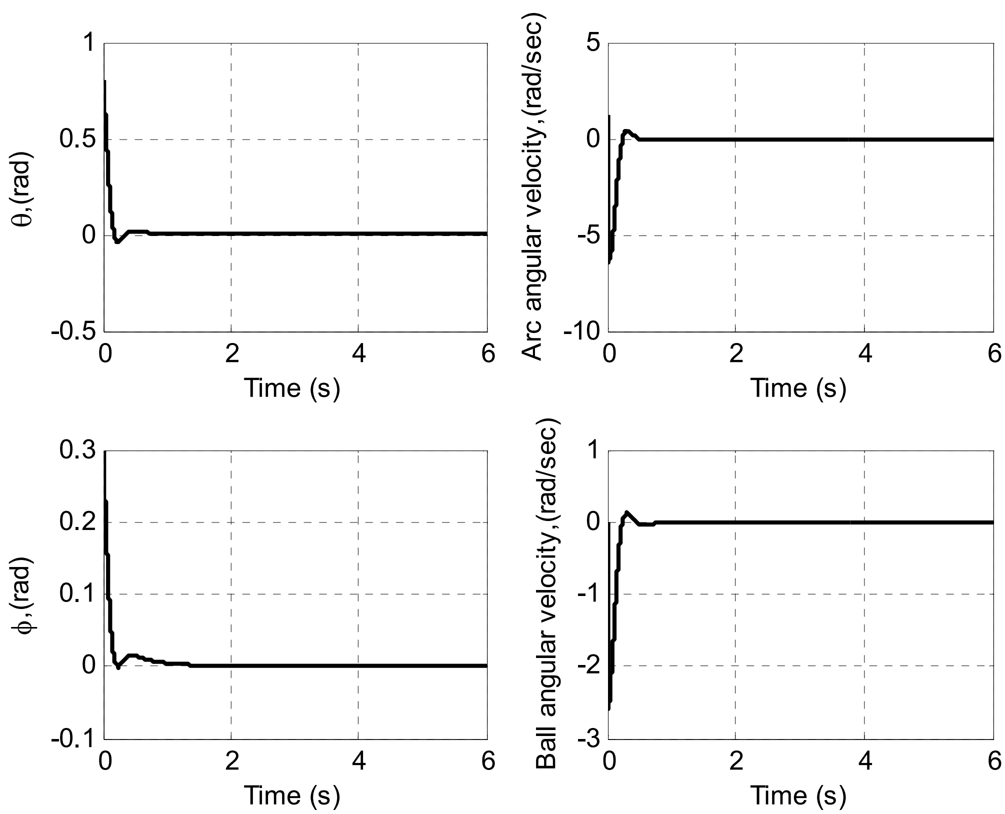

Figure 4.3: Simulation result with initial condition [0.8, 0.3, 1.2, 0]

Figure 4.5 shows that the elements of $\mathbf{P}$ remain constant after stabilization.

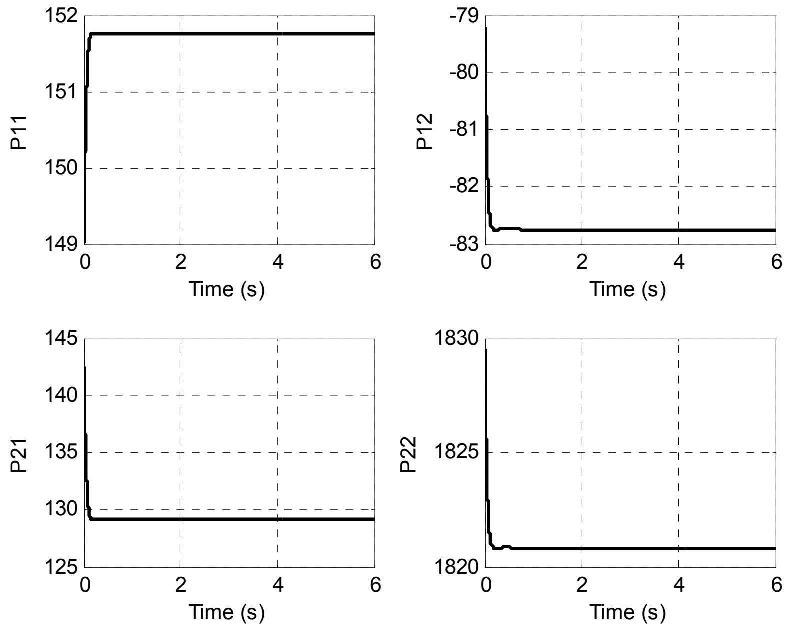

Figure 4.4: Simulation for P elements

Figure 4.6 shows that the determinant of the **P** matrix is positive.

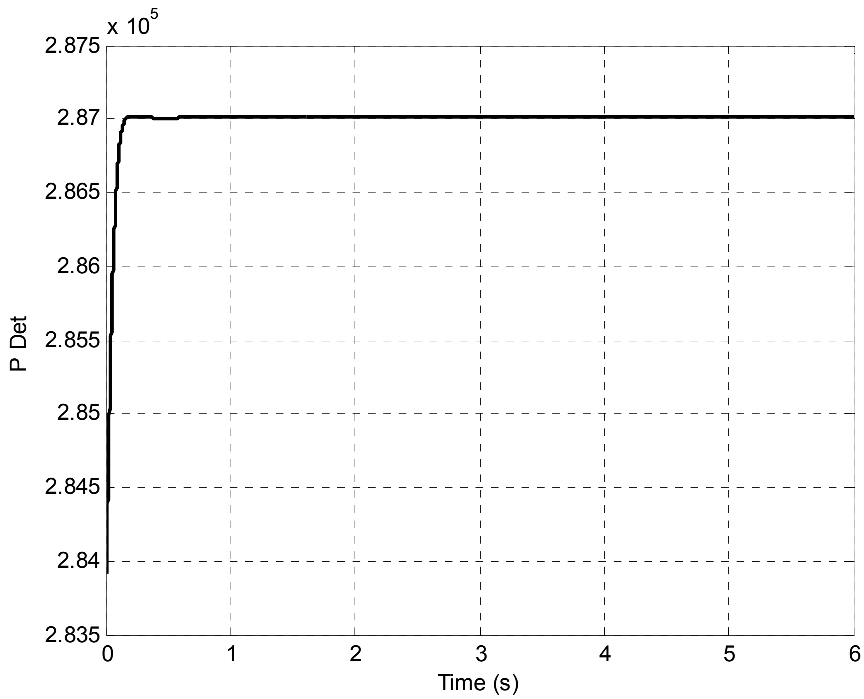

Figure 4.5: Determinant of the P matrix

Figure 4.7 shows the Lyapunov function value and its first time derivative. Notice that the Lyapunov function is monotonically decreasing. The first derivative of the Lyapunov function should be non-positive as illustrated in Figure 4.7.

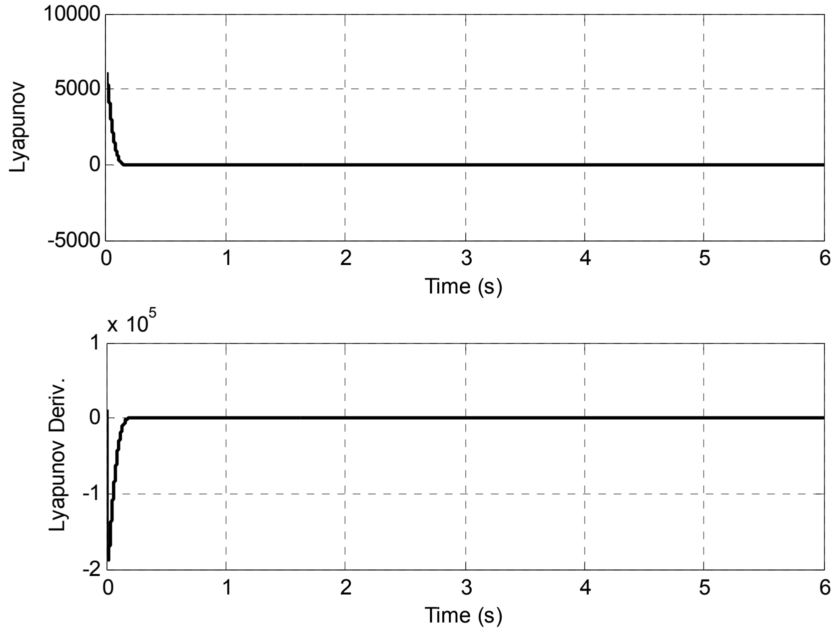

Figure 4.6: Lyapunov Time History and its Time Derivative

By using Eq. 4.27 the control law can be calculated, as

$$\mathbf{F}_1 = \nu\dot{\theta} - \sigma\dot{\phi},$$

$$\mathbf{F}_2 = \frac{-\alpha\left(-\dot{\theta}K_{D11}r + \dot{\theta}K_{D21}r - \dot{\theta}K_{D21}d\cos(\phi) + \dot{\phi}K_{D21}r + \dot{\phi}K_{D22}r - \dot{\phi}K_{D22}d\cos(\phi) + \dot{\phi}K_{D21}r\right)}{r\left(md^2 + Ib - md^2\cos(\phi)^2\right)},$$

and

$$
\begin{aligned}
\mathbf{F}_3 = \frac{1}{\mathrm{I}(\phi)}\Big(0.001\Big(&-495K_{D11}K_{D22}F_6 \operatorname{atan} J(\phi)+135K_{D11}{}^2\cos(\phi)^2-165K_{D21}K_{D22}F_6 \operatorname{atan} J(\phi)\Big)\Big) \\
&+3969K_{D11}{}^3\cos(\phi)\sin(\theta)-1764K_{D11}{}^2K_{D21}\sin(\Xi(\theta,\phi))\cos(\phi)^2 \\
&+34986K_{D11}K_{D21}{}^2\sin(\Xi(\theta,\phi))\cos(\phi)-15582K_{D11}K_{D21}\sin(\Xi(\theta,\phi))K_{D22} \\
&+1764K_{D11}{}^2K_{D22}\sin(\Xi(\theta,\phi))\cos(\phi)^2+6958K_{D21}{}^2\sin(\Xi(\theta,\phi))K_{D22}\cos(\phi) \\
&-7644K_{D11}\cos(\phi)^2\sin(\Xi(\theta,\phi))K_{D21}{}^2-1176\cos(\phi)^2\sin(\Xi(\theta,\phi))K_{D22}K_{D21}{}^2 \\
&+14112K_{D11}{}^2K_{D21}\cos(\phi)\sin(\Xi(\theta,\phi))+2548\cos(\phi)\sin(\theta)K_{D21}{}^3-882K_{D11}{}^2\sin(\theta)K_{D22} \\
&-7056K_{D11}{}^2\sin(\theta)K_{D21}+882K_{D11}{}^2K_{D22}\sin(\Xi(\theta,\phi))\cos(\phi)-2475K_{D11}K_{D21}F_6 \operatorname{atan} J(\phi) \\
&-8379K_{D11}\cos(\theta)\sin(\phi)K_{D21}{}^2-4410K_{D11}\cos(\phi)^2\sin(\theta)K_{D21}{}^2 \\
&-882K_{D11}K_{D22}\sin(\theta)K_{D21}\cos(\phi)^2-1764K_{D11}K_{D22}\cos(\phi)\sin(\theta)K_{D21} \\
&+882K_{D11}K_{D22}\cos(\phi)\sin(\phi)\cos(\theta)K_{D21}+1176\cos(\theta)\sin(\phi)\cos(\phi)K_{D21}{}^3\cos(\phi) \\
&+637\sin(\theta)K_{D22}K_{D21}{}^2\cos(\phi)-294\sin(\theta)K_{D22}K_{D21}{}^2\cos(\phi)^2 \\
&+11319K_{D11}\cos(\phi)\sin(\theta)K_{D21}{}^2-1176K_{D11}\sin(\theta)K_{D22}K_{D21} \\
&-2646K_{D11}{}^2\cos(\phi)^2\sin(\theta)K_{D21}+12348K_{D11}{}^2\cos(\phi)\sin(\theta)K_{D21} \\
&-10584K_{D11}{}^2\sin(\phi)\cos(\theta)K_{D21}-1323K_{D11}{}^2\sin(\phi)\cos(\theta)K_{D22} \\
&+1323K_{D11}{}^2\cos(\phi)\sin(\theta)K_{D22}+2646K_{D11}{}^2\sin(\phi)\cos(\theta)\cos(\phi)K_{D21} \\
&-32634K_{D11}{}^2K_{D21}\sin(\Xi(\theta,\phi))+27832K_{D21}{}^3\sin(\Xi(\theta,\phi))\cos(\phi) \\
&+2646K_{D11}{}^3\cos(\phi)\sin(\Xi(\theta,\phi))-20727K_{D11}{}^2K_{D22}\sin(\Xi(\theta,\phi)) \\
&-24696K_{D21}{}^3\sin(\Xi(\theta,\phi))-3969K_{D11}{}^3\sin(\Xi(\theta,\phi))+3528K_{D11}K_{D21}\cos(\phi)\sin(\Xi(\theta,\phi))K_{D22} \\
&+60K_{D21}{}^2F_6 \operatorname{atan} J(\phi)\cos(\phi)^2+1176\sin(\theta)K_{D21}{}^3\cos(\phi)^2-5586\sin(\theta)K_{D21}{}^2 \\
&-1764\sin(\phi)\cos(\theta)K_{D21}{}^3-1176\sin(\theta)K_{D21}{}^3-2646\sin(\theta)K_{D21}{}^3-7791K_{D21}{}^2\sin(\Xi(\theta,\phi))K_{D22} \\
&-5292\cos(\phi)^2\sin(\Xi(\theta,\phi))K_{D21}{}^3-40425K_{D11}K_{D21}{}^2\sin(\Xi(\theta,\phi))+W(\theta,\phi)
\end{aligned}
$$

where

$$
\begin{aligned}
W(\theta,\phi) = &\ 294\sin(\phi)\cos(\theta)K_{D22}K_{D21}{}^2\cos(\phi)-294\sin(\theta)K_{D22}K_{D21}{}^2 \\
&-3969K_{D21}{}^3\sin(\phi)\cos(\theta)+4410K_{D11}\sin(\phi)\cos(\theta)K_{D21}{}^2\cos(\phi) \\
&+45K_{D11}\cos(\phi)^2K_{D22}F_6 \operatorname{atan} J(\phi)-1485K_{D11}{}^2F_6 \operatorname{atan} J(\phi) \\
&+225K_{D11}K_{D21}F_6 \operatorname{atan} J(\phi)\cos(\phi)^2+15K_{D22}\cos(\phi)^2F_6 \operatorname{atan} J(\phi)K_{D21} \\
&-600K_{D21}{}^2F_6 \operatorname{atan} J(\phi)+2352K_{D11}\cos(\phi)\sin(\theta)K_{D22}K_{D21} \\
&-441\sin(\phi)\cos(\theta)K_{D22}K_{D21}{}^2
\end{aligned}
,
$$

$$
\mathrm{I}(\phi) = \frac{\left(9K_{D11}{}^2+15K_{D11}K_{D21}+3K_{D11}K_{D22}+4K_{D21}{}^2+K_{D22}K_{D21}\right)}{\left(-2\cos(\phi)K_{D21}+3K_{D11}+3K_{D21}\right)},
$$

$$\Xi(\theta,\phi)=\frac{1}{W(\theta,\phi)(3K_{D11}+K_{D21})K_{D21}}\begin{pmatrix}-3\phi K_{D21}-4\phi K_{D21}{}^{2}W(\theta,\phi)-\phi K_{D21}W(\theta,\phi)K_{D22}-3\theta K_{D21}W(\theta,\phi)K_{D11}\\+\theta K_{D21}{}^{2}W(\theta,\phi)-6K_{D22}\operatorname{atan}(\Upsilon(\theta,\phi))W(\theta,\phi)K_{D21}\\+18\operatorname{atan}(Z(\theta,\phi))K_{D11}{}^{2}K_{D22}+6\operatorname{atan}(Z(\theta,\phi))K_{D11}K_{D22}K_{D21}\\-18\operatorname{atan}(Z(\theta,\phi))K_{D11}-6K_{D21}{}^{3}\operatorname{atan}(Z(\theta,\phi))\end{pmatrix},$$

$$\sinh(J(\theta,\phi))=\sinh\left(\frac{6\operatorname{atan}(Z(\theta,\phi))K_{D11}K_{D22}-6K_{D21}{}^{2}\operatorname{atan}(Z(\theta,\phi))+\theta K_{D21}M(\theta,\phi)-2K_{D22}\operatorname{atan}(Z(\theta,\phi))M(\theta,\phi)}{M(\theta,\phi)K_{D21}}\right).$$

$$\operatorname{atan}\Upsilon(\phi)=\left(\frac{(\cos(\phi)-1)}{\sin(\phi)}\right),$$

$$\operatorname{atan}Z(\theta,\phi)=\frac{\left(5K_{D21}r+3K_{D11}\right)(\cos(\phi)-1)}{\left((3K_{D11}+K_{D21})(5K_{D21}+3K_{D11})\right)^{\frac{1}{2}}\sin(\phi)},$$

and

$$M(\theta,\phi)=\left((3K_{D11}+K_{D21})(5K_{D21}+3K_{D11})\right)^{\frac{1}{2}}.$$

See Section B.3 of Appendix B for details about these results. The behavior of the control law to stabilize the system can be seen in Figure 4.8.

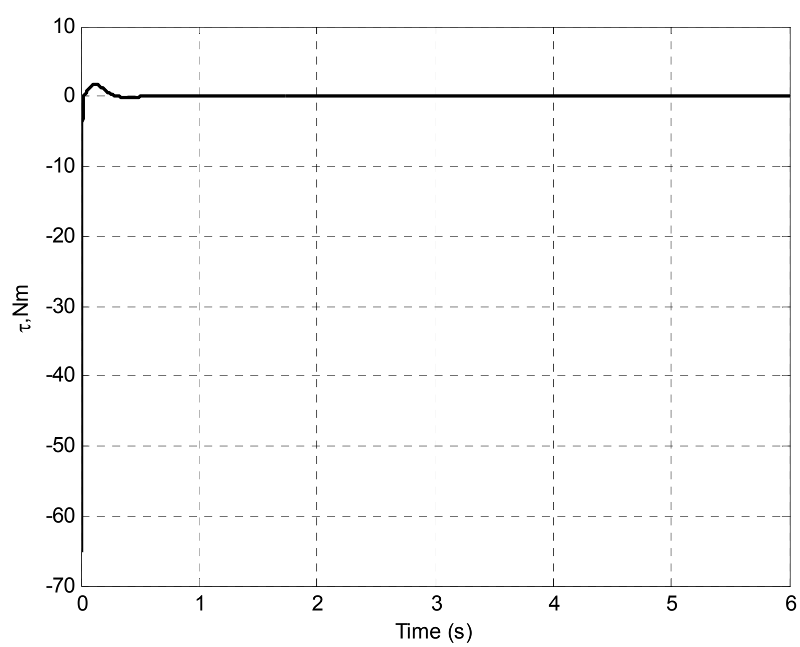

Figure 4.7: Control law

4.4 Development considering an almost constant P matrix

Considering the difficulties of trying to find the potential when $\mathbf{K_D}$ is assumed a constant matrix and the $\mathbf{P}$ matrix is a function of $\mathbf{q}$, a new strategy to develop the controller is used here. It will be seen that it is possible to use *n*(*n-1*)/*2* algebraic equations to accomplish the symmetry condition of the $\mathbf{K_D}$. Here the p_{12} element of $\mathbf{P}$ enforces symmetry. In this part, the formulation for the contribution $\mathbf{F}_1$ to the control law is kept the same as previously developed when $\mathbf{K_D}$ is constant. Regarding the control inputs for this formulation, there is a difference when compared to the previous formulation. In this case, solving for the potential is easier and simpler; also this formulation allows accommodation of a larger range of initial conditions. Details of the control law results can be found in the Section B.5 of Appendix B. The $\mathbf{K_D}$ matrix here is chosen as a constant matrix times the mass matrix, such that the $\mathbf{P}(\mathbf{q})$ matrix will almost be a constant matrix.

4.4.1 The first matching condition

Assume that $\mathbf{P}(\mathbf{q})$ is given as

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{4.40}$$

for the ball and arc system. If Eq. 4.40 is considered to accomplish $\mathbf{K_D}(\mathbf{q_1},\mathbf{q_2}) = \mathbf{K_D}(\mathbf{q_1},\mathbf{q_2})^{\mathbf{T}}$, the symmetry of $\mathbf{K_D}$ must be enforced. Then the upper right hand element of the $\mathbf{P}$ matrix is given by

$$p_{12} = \frac{\begin{pmatrix} p_{11}mr^2 - p_{11}mrd\cos(\phi) + p_{21}mr^2 - 2p_{21}mrd\cos(\phi) \\ + p_{21}md^2 + p_{21}Ib - p_{22}mr^2 + p_{22}mrd\cos(\phi) \end{pmatrix}}{mr^2} \tag{4.41}$$

which is not a constant, however, the other elements of the $\mathbf{P}$ matrix are constant.

Eq. 4.40 and Eq. 4.41 lead to the elements of the matrix $\mathbf{K_D}$ being functions of the mass matrix and to the symmetry of $\mathbf{K_D}$.

Under the above considerations, the $\mathbf{K_D}$ matrix is

$$\mathbf{K_D} = \mathbf{P(q)M(q)} = \begin{bmatrix} KD_{11}(\theta,\phi) & KD_{21}(\theta,\phi) \\ KD_{21}(\theta,\phi) & KD_{22}(\theta,\phi) \end{bmatrix} \tag{4.42}$$

where

$$KD_{11}(\theta,\phi)=\frac{1}{r}\begin{pmatrix} p_{11}md^2r+p_{11}Ibr-p_{21}mr3+3p_{21}mr^2d\cos(\phi)-p_{21}md^2r-p_{21}Ibr \\ +p_{22}mr^3-2p_{22}mr^2d\cos(\phi)-d^2\cos(\phi)^2p_{11}mr-2d^2\cos(\phi)^2p_{21}mr+d^3\cos(\phi)p_{21}m \\ +d\cos(\phi)p_{21}Ib+d^2\cos(\phi)p_{21}Ib+d2\cos(\phi)^2p_{22}mr \end{pmatrix},$$

$$KD_{21}(\theta,\phi)=p_{21}mr^2-2p_{21}mrd\cos(\phi)+p_{22}md^2+p_{21}Ib-p_{22}mr^2+p_{22}mrd\cos(\phi),$$

$$KD_{22}(\theta,\phi)=mr(-p_{21}r+p_{21}d\cos(\phi)+p_{22}r),$$

and the determinant of $\mathbf{K_D}$ is

$$\det(KD)=\left.\begin{array}{l} -p_{21}{}^2m2d^4-p_{21}{}^2Ib^2-mp_{21}{}^2Ibr^2-p_{21}{}^2m^2d^2r^2-2p_{21}{}^2md^2Ib \\ +d^4\cos(\phi)^2p_{21}{}^2m2-mp_{11}Ibr^2p_{22}+mp_{21}Ibr^2p_{22}-p_{11}m^2d^2r^2p_{21} \\ +p_{11}m^2d^2r^2p_{22}+p_{21}m^2d^2r^2p_{22}+md^2\cos(\phi)^2p_{21}{}^2Ib \\ +p_{11}m^2d^3rp_{21}\cos(\phi)+mp_{11}Ibrp_{21}d\cos(\phi)+p_{21}{}^2m^2d^2r^2\cos(\phi)^2 \\ +2p_{21}{}^2m^2d^3r\cos(\phi)-2d^3\cos(\phi)^3p_{21}{}^2m^2r+2mp_{21}{}^2Ibrd\cos(\phi) \\ +p_{22}{}^2m^2d^2r^2\cos(\phi)^2p_{21}+d^2\cos(\phi)^2p_{11}m^2r^2p_{21}-d^3\cos(\phi)^3p_{11}m^2rp_{21} \\ +d^2\cos(\phi)^2p_{11}{}^2m^2r^2p_{22}-d^3\cos(\phi)p_{21}m^2p_{22}r-md\cos(\phi)p_{21}Ibp_{22}r \\ +d^3\cos(\phi)^3p_{22}m^2rp_{21} \end{array}\right\}. \tag{4.43}$$

The control input $\mathbf{Fm_1}$ is now utilized as well as $\mathbf{Fmc_1}$, however only one unknown appear in $\mathbf{Fmc_1}$ compared to the other examples. The following solution outline for the ball and arc system is presented considering that $\mathbf{P}$ is almost a constant.

1. Consider $\mathbf{Fmc_1}(\mathbf{q},\dot{\mathbf{q}})=\mathbf{Fmc_1}(\mathbf{q},\dot{\mathbf{q}})^{\mathbf{T}}$
2. Because of symmetry three LAEs are obtained from the FMC.
3. Solve for $\mathbf{F_{11}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{22}}(\mathbf{q},\dot{\mathbf{q}})$, and one of the terms in $\mathbf{Fmc_1}$, namely $\mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}})$
4. Substitute the solutions into FMC.
5. The terms $\mathbf{F_{44}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{55}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{66}}(\mathbf{q},\dot{\mathbf{q}})$, $\mathbf{F_{77}}(\mathbf{q},\dot{\mathbf{q}})$ and, $\mathbf{F_{88}}(\mathbf{q},\dot{\mathbf{q}})$, are not necessary because $\mathbf{Fmc_1}$ only has $\mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}})$ in it where all the elements of $\mathbf{Fmc_1}$ are set equal to $\mathbf{F_{33}}(\mathbf{q},\dot{\mathbf{q}})$.

The resulting $\mathbf{Fmc_1}$ is

$$\mathbf{Fmc_1}=\begin{bmatrix} Fc_{11}(\mathbf{q},\dot{\mathbf{q}}) & Fc_{11}(\mathbf{q},\dot{\mathbf{q}}) \\ Fc_{11}(\mathbf{q},\dot{\mathbf{q}}) & Fc_{11}(\mathbf{q},\dot{\mathbf{q}}) \end{bmatrix} \tag{4.44}$$

where

$$
Fmc_{11}(\mathbf{q},\dot{\mathbf{q}}) = \frac{\left(\begin{array}{l}
4\dot{\theta}^2 mp_{11}r^2 d\sin(\phi)\dot{\phi} + 2\dot{\theta}p_{11}r\dot{\phi}\sigma - 2\dot{\theta}^2\dot{\phi}d^2\sin(\phi)rmp_{11}\cos(\phi) \\
-2\dot{\theta}^3 p_{22}mrd^2\cos(\phi)\sin(\phi) - 2\dot{\theta}mp_{11}r^2 d\sin(\phi)\dot{\phi}^2 \\
+\dot{\theta}^2 p_{21}d\sin(\phi)\dot{\phi}Ib + 3p_{21}\dot{\theta}^2\dot{\phi}mr^2 d\sin(\phi) \\
-4p_{21}\dot{\theta}^2 md^2\cos(\phi)r\sin(\phi)\dot{\phi} + 2\dot{\theta}^2\dot{\phi}d^2\sin(\phi)rmp_{22}\cos(\phi) \\
+2\dot{\theta}^3 p_{11}mrd^2\cos(\phi)\sin(\phi) - 4\dot{\theta}^2\dot{\phi}mr^2 d\sin(\phi)p_{22} \\
+2\dot{\theta}^3 r^2 mp_{22}d\sin(\phi) - 2p_{21}\dot{\theta}^3 Ibd\sin(\phi) \\
-2\dot{\theta}^3 mp_{11}r^2 d\sin(\phi) + p_{21}\dot{\theta}^2\dot{\phi}md^3\sin(\phi) \\
+4p_{21}\dot{\theta}^3 md^2\cos(\phi)r\sin(\phi) - 2p_{21}\dot{\theta}^3 mr^2 d\sin(\phi) \\
-r^2 m\dot{\phi}^3 p_{21}d\sin(\phi) + 2r^2 m\dot{\phi}^2\dot{\theta}d\sin(\phi)p_{22} + 2r\dot{\phi}p_{21}\dot{\theta}\nu \\
+2r\dot{\phi}^2 p_{21}\sigma + 2\dot{\theta}^2 p_{11}r\nu - 2p_{21}\dot{\theta}^3 md^3\sin(\phi)
\end{array}\right)}{\left(r\dot{\phi}^2 + 2r\dot{\phi}\dot{\theta} + \dot{\theta}^2 r\right)}.
$$

4.4.2 The second matching condition

The $\mathbf{K}_v$ matrix is evaluated by using Eq. 2.34 to get

$$
\mathbf{K}_v = \begin{bmatrix} \alpha p_{11}{}^2 & \alpha p_{11}p_{21} \\ \alpha p_{11}p_{21} & \alpha p_{21}{}^2 \end{bmatrix} \tag{4.45}
$$

Notice that $\mathbf{K}_v$ is a constant matrix due the fact that the lower elements of $\mathbf{P}$ are constants. The corresponding eigenvalues of $\mathbf{K}_v$ are

$$
eig(\mathbf{K}_v) = \begin{bmatrix} 0 \\ \alpha p_{21}{}^2 + \alpha p_{11}{}^2 \end{bmatrix}. \tag{4.46}
$$

The SMC provides the contribution $\mathbf{F}_2$ for the control law. The corresponding result is

$$
\mathbf{F}_2 = -\alpha\left(p_{11}\dot{\theta} + p_{21}\dot{\phi}\right). \tag{4.47}
$$

4.4.3 The third matching condition

The control law contribution $\mathbf{F}_3$ is determining from the first row of the TMC and the second row provides the equation that is used to determine the potential. The first row of the TMC provides the equation that is used to determine the control law contribution $\mathbf{F}_3$. These two equations are

$$
\begin{aligned}
&\mathbf{F}_3 + mg\left(r\sin(\phi)\cos(\theta)\right) - r\cos(\phi)\sin(\theta) + d\sin(\phi) \\
&- \frac{p_{22}mr^2 \dfrac{\partial \mathbf{\Phi}(\theta,\phi)}{\partial\theta}}{mp_{21}{}^2 r^2 - mp_{11}rd\cos(\phi)p_{21} + mrd\cos(\phi)p_{21}p_{22} - 2mp_{21}{}^2 rd\cos(\phi) + mp_{21}{}^2 d^2 + p_{11}mr^2 p_{21} + p_{21}{}^2 Ib - mp_{11}r^2 p_{22} - p_{21}mr^2 p_{22}} \\
&+ \frac{\left(p_{11}mr^2 - p_{11}mrd\cos(\phi) + p_{21}mr^2 - 2p_{21}mr\cos(\phi) + p_{21}md^2 + p_{21}Ib - p_{22}mr^2 + p_{22}mrd\cos(\phi)\right)\dfrac{\partial \mathbf{\Phi}(\theta,\phi)}{\partial\phi}}{mp_{21}{}^2 r^2 - mp_{11}rd\cos(\phi)p_{21} + mrd\cos(\phi)p_{21}p_{22} - 2mp_{21}{}^2 rd\cos(\phi) + mp_{21}{}^2 d^2 + p_{11}mr^2 p_{21} + p_{21}{}^2 Ib - mp_{11}r^2 p_{22} - p_{21}mr^2 p_{22}} = 0
\end{aligned} \tag{4.48}
$$

and

$$
\begin{aligned}
&-mgr\left(\sin(\phi)\cos(\theta)-\cos(\phi)\sin(\theta)\right)\\
&+\frac{p_{21}mr^{2}\dfrac{\partial\mathbf{\Phi}(\theta,\phi)}{\partial\theta}}{mp_{21}{}^{2}r^{2}-mp_{11}rd\cos(\phi)p_{21}+mrd\cos(\phi)p_{21}p_{22}-2mp_{21}{}^{2}rd\cos(\phi)+mp_{21}{}^{2}d^{2}+p_{11}mr^{2}p_{21}+p_{21}{}^{2}Ib-mp_{11}r^{2}p_{22}-p_{21}mr^{2}p_{22}}\\
&-\frac{p_{11}mr^{2}\dfrac{\partial\mathbf{\Phi}(\theta,\phi)}{\partial\phi}}{mp_{21}{}^{2}r^{2}-mp_{11}rd\cos(\phi)p_{21}+mrd\cos(\phi)p_{21}p_{22}-2mp_{21}{}^{2}rd\cos(\phi)+mp_{21}{}^{2}d^{2}+p_{11}mr^{2}p_{21}+p_{21}{}^{2}Ib-mp_{11}r^{2}p_{22}-p_{21}mr^{2}p_{22}}=0\cdot
\end{aligned}
\tag{4.49}
$$

Solving for the potential in Eq. 4.49, the result is

$$
\mathbf{\Phi}(\theta,\phi)=\frac{1}{r(p_{21}+p_{11})(p_{21}+2p_{11})}\begin{pmatrix} 6F_{5}\left(\dfrac{\phi p_{21}+p_{11}\theta}{p_{21}}\right)^{2}rp_{21}p_{11} \\ +4F_{5}\left(\dfrac{\phi p_{21}+p_{11}\theta}{p21}\right)^{2}rp_{11}{}^{2} \\ +4F_{5}\left(\dfrac{\phi p_{21}+p_{11}\theta}{p_{21}}\right)^{2}rp_{21}{}^{2}+\Lambda(\theta,\phi) \end{pmatrix}.
\tag{4.50}
$$

where

$$
\begin{aligned}
\Lambda(\theta,\varphi)=\;&2\mathrm{g}\cos(-\theta+\varphi)\mathrm{p}_{21}{}^{3}\mathrm{Ib}-6\mathrm{g}\cos(-\theta+\varphi)\mathrm{p}_{21}\mathrm{mr}^{2}\mathrm{p}_{22}\mathrm{p}_{11}\\
&+4\mathrm{g}\cos(-\theta+\varphi)\mathrm{p}_{11}{}^{2}\mathrm{mr}^{2}\mathrm{p}_{21}+3d\cos(\theta)\mathrm{mp}_{22}\mathrm{rp}_{21}\mathrm{p}_{11}\\
&+2\mathrm{gd}\cos(\theta)\mathrm{mp}_{22}\mathrm{rp}_{11}{}^{2}-2\mathrm{g}\,d\cos(\theta)\mathrm{mp}_{11}{}^{3}\mathrm{r}\\
&-2\mathrm{gp}_{21}{}^{3}\,d\cos(\theta)\mathrm{mr}-7\mathrm{gp}_{21}{}^{2}\,d\cos(\theta)\mathrm{mrp}_{11}\\
&-7\mathrm{gp}_{21}\,d\cos(\theta)\mathrm{mrp}_{11}{}^{2}+\mathrm{g}\,d\cos(\theta)\mathrm{mrp}_{22}\mathrm{rp}_{21}{}^{2}\\
&-4\mathrm{g}\cos(-\theta+\varphi)\mathrm{mp}_{11}{}^{2}\mathrm{r}^{2}\mathrm{p}_{22}-2\mathrm{gmr}\,d\,\mathrm{p}_{21}{}^{3}\cos(-\theta+2\varphi)\\
&-3\mathrm{gmrdp}_{21}{}^{2}\cos(-\theta+2\varphi)\mathrm{p}_{11}+\mathrm{gmr}\,d\,\mathrm{p}_{21}{}^{2}\cos(-\theta+2\varphi)\mathrm{p}_{22}\\
&+\mathrm{gmr}\,d\,\mathrm{p}_{21}\cos(-\theta+2\varphi)\mathrm{p}_{22}\mathrm{p}_{11}-\mathrm{gmr}\,d\,\mathrm{p}_{21}\cos(-\theta+2\varphi)\mathrm{p}_{11}{}^{2}\\
&+2\mathrm{g}\cos(-\theta+2\varphi)\mathrm{mp}_{21}{}^{3}\mathrm{r}^{2}+6\mathrm{g}\cos(-\theta+2\varphi)\mathrm{mp}_{21}{}^{2}\mathrm{r}^{2}\mathrm{p}_{11}\\
&+2\mathrm{g}\cos(-\theta+2\varphi)\mathrm{mp}_{21}{}^{3}\,d^{2}+4\mathrm{g}\cos(-\theta+\varphi)\mathrm{mp}_{21}{}^{2}\mathrm{d}^{2}\mathrm{p}_{11}\\
&+4\mathrm{g}\cos(-\theta+\varphi)\mathrm{p}_{21}{}^{2}\mathrm{Ibp}_{11}-2\mathrm{g}\cos(-\theta+2\varphi)\mathrm{p}_{21}{}^{2}\mathrm{mr}^{2}\mathrm{p}_{22}.
\end{aligned}
$$

By substituting $\mathbf{\Phi}(\theta,\phi)$ from Eq. 4.50 into Eq.4.48, the contribution for the control law is F_3. Details of this contribution for the control law can be seen in Appendix B.

4.4.4 Selection of the unknown parameters

By using Eq. 4.27 when substituting the parameters of the system, the linearized control law becomes

$$
\boldsymbol{\tau}_{L}=\left[\tau_{1}\theta+\tau_{2}r+(-\alpha p_{11})\dot{\theta}-p_{21}\alpha\dot{\varphi}\right]
\tag{4.51}
$$

where $\boldsymbol{\tau}_{L}$ is the linearized control law,

$$\boldsymbol{\tau}_1 = -\frac{1}{\left(p_{21}{}^2 + 3p_{21}p_{11} + 2p_{11}{}^2\right)p_{21}{}^2 r}\begin{pmatrix} 2Ibgp_{21}{}^4 + 4Ibp11gp_{21}{}^3 - 3p_{21}{}^4 gmrd \\ + 2p_{21}{}^4 gmd^2 + p_{21}{}^4 mr^2 g + 2p_{21}{}^3 mrdp_{22} g \\ + 3p_{21}{}^3 p_{11} mr^2 g - 3p_{21}{}^3 p_{11} gmrd \\ + 4p_{21}{}^3 p_{11} gmd^2 + 2p_{21}{}^2 p_{11}{}^2 gmr^2 \\ + 4p_{21}{}^2 p_{11} F_5 r + 2p_{21}{}^2 p_{11} gmrdp_{22} \\ + 8F_5 rp_{11}{}^3 \end{pmatrix}, \tag{4.52}$$

and

$$\boldsymbol{\tau}_2 = -\frac{1}{\left(p_{21}{}^2 + 3p_{21}p_{11} + 2p_{11}{}^2\right)p_{21} r}\begin{pmatrix} -2Ibgp_{21}{}^3 - 4Ibp_{11}gp_{21}{}^2 + 8p_{21}{}^3 gmrd \\ -2p_{21}{}^3 gmd^2 - p_{21}{}^3 mr^2 g + 4p_{21}{}^2 F_5 r \\ -4p_{21}{}^2 mrdp_{22} g - 3p_{21}{}^2 p_{11} mr^2 g \\ -4p_{21}{}^2 p_{11} gmd^2 + 12p_{21}{}^2 p_{11} gmd \\ -4p_{21}p_{11} gmrdp_{22} - 2p_{21}p_{11}{}^2 gmr^2 \\ +12p_{21}p_{11}F_5 r + 4p_{21}p_{11}{}^2 gmrd + 8p_{11}{}^2 F_5 r \end{pmatrix}. \tag{4.53}$$

Comparing Eq. 4.33 and Eq. 4.54 and extracting coefficients of like variables, four equations are found. The gain matrix **K** is determined such that the eigenvalues of the linearized closed loop system are -76.7918, -4.7457, -11.6366 and, -13.5405, the same as used previously. The physical parameters and the corresponding gain matrix **K** are also the same as before. The equations are

$$K_1 + \frac{1}{\left(p_{21}{}^2 + 3p_{21}p_{11} + 2p_{11}{}^2\right)p_{21}{}^2}\left(1.33\begin{pmatrix} 0.96p_{21}{}^4 + 2.26p_{11}p_{21}{}^3 \\ + 0.14p_{21}{}^3 p_{22} + 1.5p_{21}{}^2 p_{11}F_5 \\ + 0.14p_{21}{}^2 p_{11}p_{22} + 0.22p_{21}{}^2 p_{11} \\ + 4.5p_{11}{}^2 F_5 p_{21} + 3F_5 p_{11}{}^3 \end{pmatrix}\right) = 0$$

$$K_2 + \frac{1}{\left(p_{21}{}^2 + 3p_{21}p_{11} + 2p_{11}{}^2\right)p_{21}{}^2}\left(1.33\begin{pmatrix} -0.6p_{21}{}^3 - 1.6p_{11}p_{21}{}^2 \\ -0.29p_{21}{}^2 p_{22} + 1.5p_{21}{}^2 F_5 \\ + 4.5p_{21}p_{11}F_5 + 0.07p_{21}p_{11}{}^2 \\ -0.29p_{21}p_{11}p_{22} + 3p_{11}{}^2 F_5 \end{pmatrix}\right) = 0$$

$$K_3 + \alpha p_{11} - \nu = 0$$

and

$$K_4 + p_{21}\alpha = 0.$$

The four equations are solved for F_6, α, σ, and ν. The constant α is chosen in order to accomplish $\mathbf{K}_\nu + \mathbf{Fmc_1} > 0$. It is important that $\mathbf{K_D} > 0$, so $\mathbf{p_{21}}$ is chosen such that $\mathbf{p_{21}}=1.75\mathbf{p_{11}}$, $\mathbf{p_{22}}$ is chosen to satisfied the eigenvalues of the Hessian being positive. The constants $\mathbf{p_{22}}$ and $\mathbf{p_{11}}$ are chosen based on the Hessian, the shape of the potential, and in the positive definiteness property of $\mathbf{K_D}$. The values of the parameters appear in Table 4.3.

Table 4.3: Values of the control system

Parameter	Explanation	Identified value
p_{21}	Element of $\boldsymbol{P}$	$1.75p_{11}$
p_{22}	Element of $\boldsymbol{P}$	$4e^{-8}$
F_6	Coefficient of	$6.25e^{-14}p_{21}^{\;2}\left(1.2e^{15}p_{11}+1.2e^{15}p_{21}\right)/\left(p_{21}+p_{11}\right)\left(p_{21}+2p_{11}\right)$
σ	Coefficient of $\mathbf{Fm_1}$	$-\alpha p_{21}+118.4$
ν	Element of $\mathbf{Fm_1}$	$-0.58\alpha p_{21}+118.4$

4.4.5 The potential

Once the solution is evaluated for the potential $\Phi(\theta,\phi)$, it is noticed that it is composed of a homogeneous solution which is

$$HS = 2F_6\phi^2 + 0.000023F_6\theta^2 + 0.0135F_6\phi\theta \tag{4.54}$$

and a particular solution which is

$$PS = 0.0008\cos(\theta) - 0.0025\cos(-\theta+\phi) + 0.0008\cos(-\theta+2\phi). \tag{4.55}$$

Figure 4.8 presents a 3D plot of the potential for this second formulation of the ball and arc system. The potential for the ball and arc system is plotted in the interval (-1.8, 1.8) for θ and in the interval (-0.8, 0.8) for ϕ.

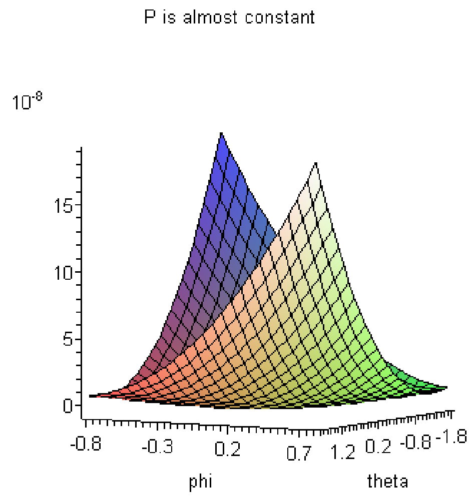

Figure 4.8: 3D Plot of the potential for the ball and arc system

4.4.6 The Hessian

From Eq.4.36 the Hessian is found. This is evaluated at the equilibrium point and the result is

$$\mathbf{H} = \begin{bmatrix} 3.99e^{-8} & 6.97e^{-8} \\ 6.97e^{-8} & 1.22e^{-7} \end{bmatrix} \tag{4.56}$$

and its corresponding eigenvalues are

$$\boldsymbol{\lambda} = \begin{bmatrix} 1.29e^{-10} \\ 1.62e^{-7} \end{bmatrix}. \tag{4.57}$$

Eq. 4.56 and Eq. 4.57 prove that H is a positive definite matrix.

4.4.7 Simulation

In the second simulation, the same initial conditions of the previous simulation are used in order to compare both formulations namely when $\mathbf{K_D}$ is constant and when $\mathbf{P}$ is almost constant. In the simulation, the initial angle of the arc is chosen as 0.8 radians, the ball angular displacement is 0.3 radians. The initial angular velocity of the arc is 1.2 rad/sec, while the angular velocity of the ball position is zero. Figure 4.9 illustrates both formulation for the ball angular displacement and angular velocity, as well as the arc angle and its angular velocity as a function of time.

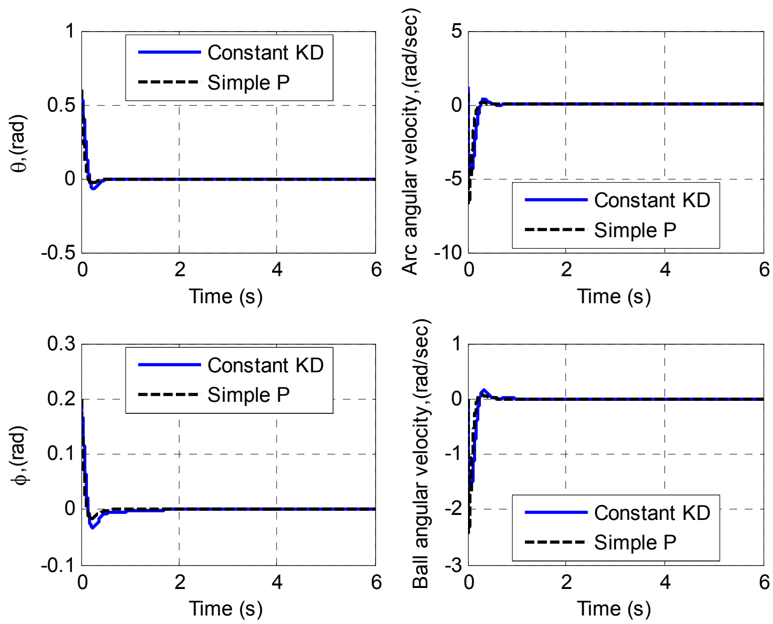

Figure 4.9: Comparison of performance for K_D constant and P almost constant

Figure 4.10 shows that the elements of $\mathbf{P}$ remain constant after stabilization. Notice the different stabilization values for both formulations.

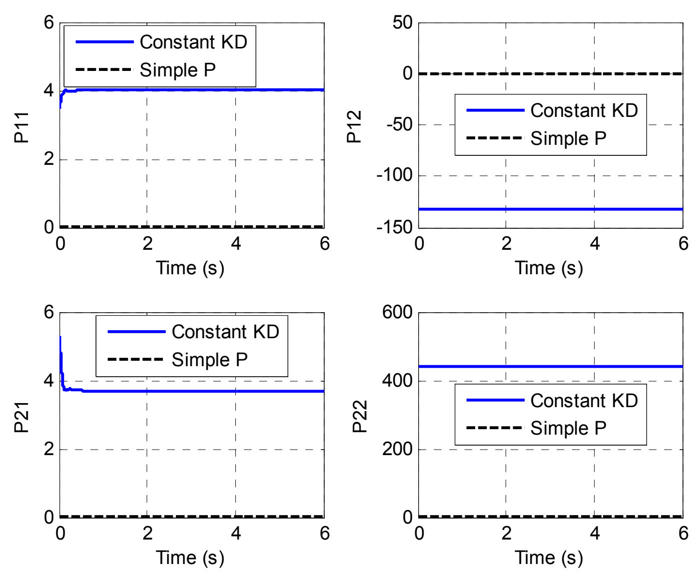

Figure 4.10: P elements Comparison

Figure 4.11 shows that the determinant of $\mathbf{P}$ matrix is positive. Notice that it is bigger for the case where $\mathbf{K_D}$ constant.

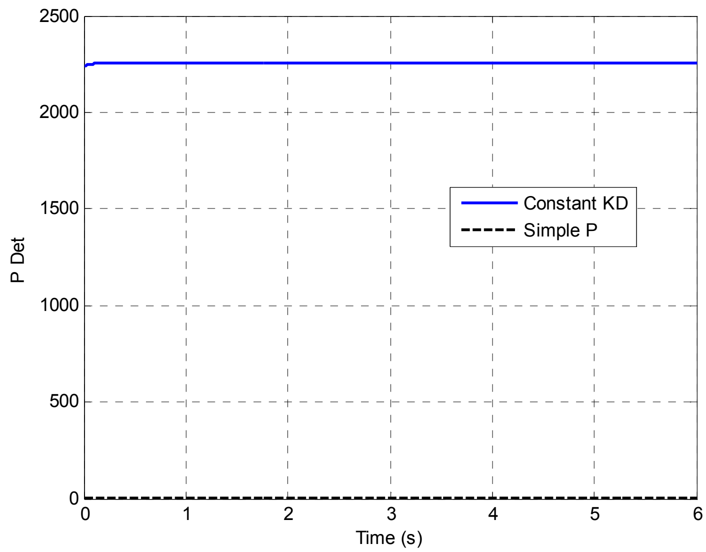

Figure 4.11: Determinant of the K_D and P

Figure 4.12 shows the Lyapunov function performance and its first time derivative. The behavior shown in Figures 4.12 and 4.9 demonstrates the validity of the Lyapunov candidate function and the control law.

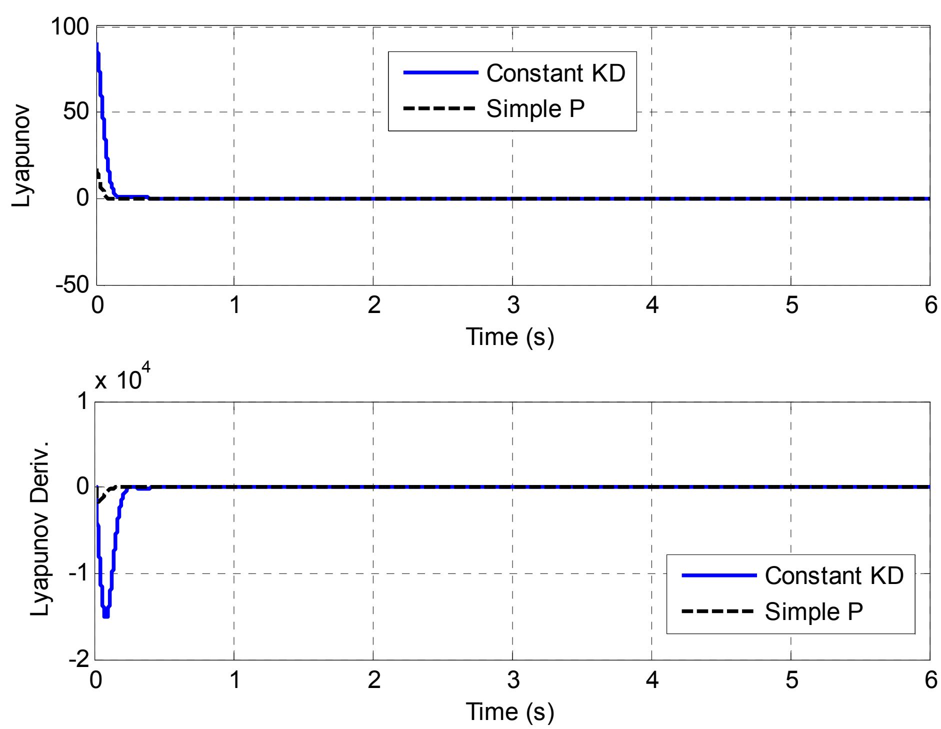

Figure 4.12: Lyapunov Time History and its Time Derivative

Considering Eq. 4.27 and using $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, the control law can be calculated. The details for the calculation of $\mathbf{F}_3$ can be found in Section B.4 of Appendix B. The behavior of the control law to stabilize the system for both formulations is shown in Figure 4.13.

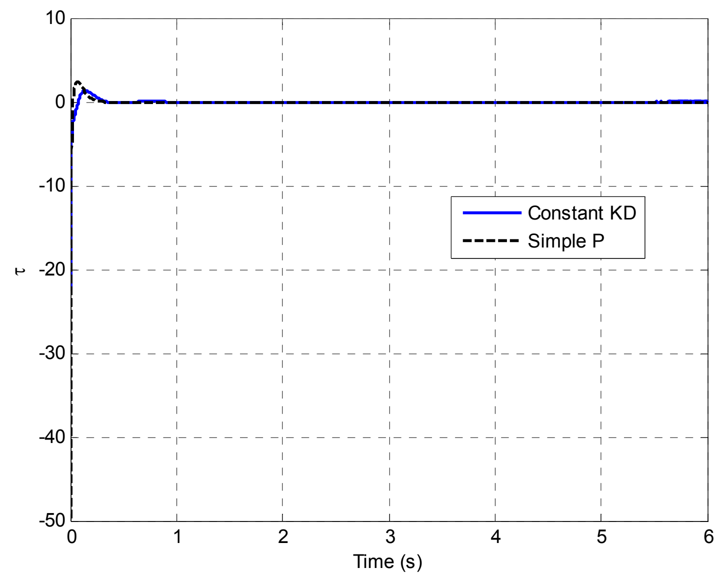

Figure 4.13: Control law

4.4.8 Simulation

The second control law formulation just presented will be tested in order to demonstrate the large range of initial states possible. The initial angle of the arc is chosen as 1.5 radians and the ball angle as 0.6 radians. The initial angular velocity of the arc is 2 rad/sec, while the angular velocity of the ball location is zero. Figure 4.14 illustrates the ball angular displacement and the arc angle, as well as the arc angular velocity and ball location angular velocity as a function of time. The control law stabilizes at the chosen equilibrium point.

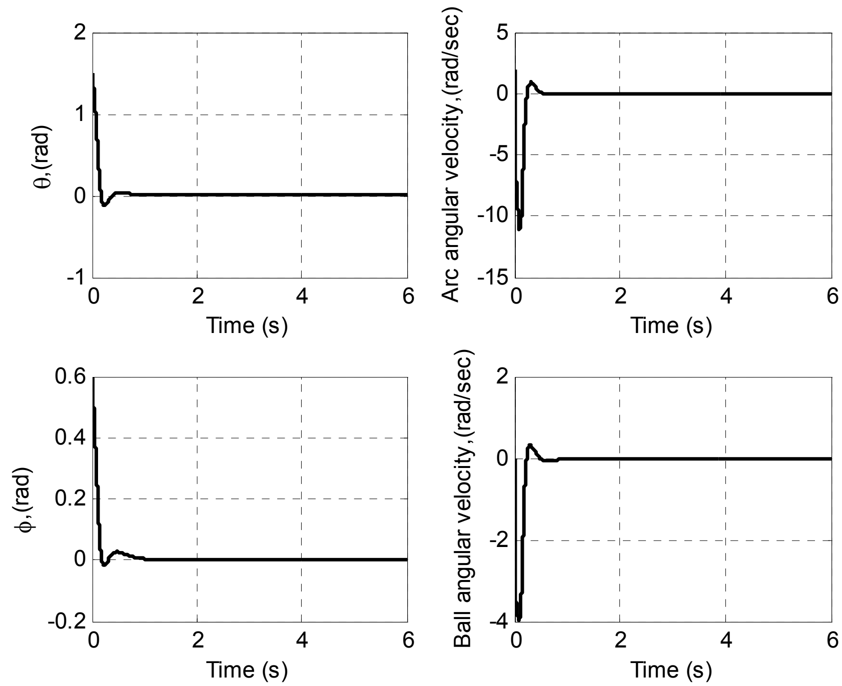

Figure 4.14: Stabilization of the ball and arc system.

Figure 4.15 shows the elements of $\mathbf{K_D}$ during the stabilization process.

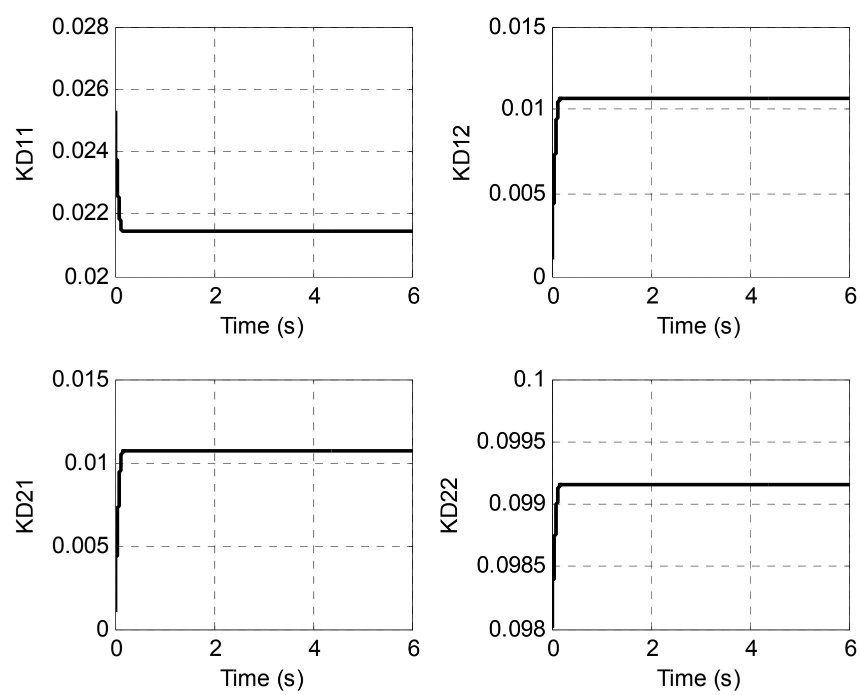

Figure 4.15: $\mathbf{K_D}$ elements I.C. [1.5, 0.6, 1.5, 0]

Figure 4.16 shows that the determinant of $\mathbf{K_D}$ is always positive.

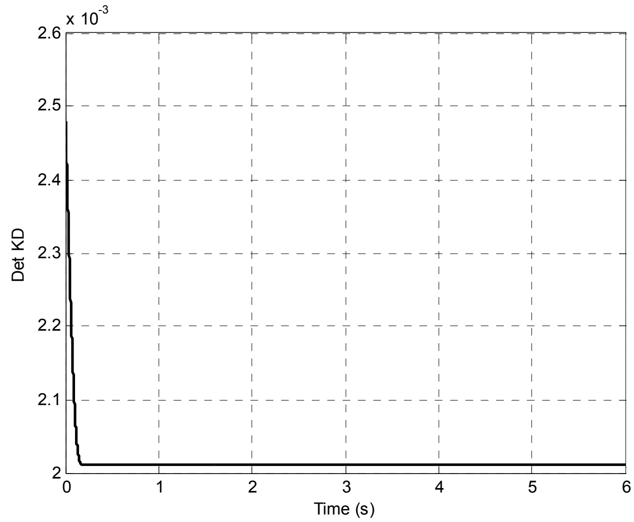

Figure 4.16: Determinant of the K_D

Figure 4.17 shows the Lyapunov function value and its time derivative. This behavior demonstrates the validity of the Lyapunov candidate function because it is monotonically decreasing with time.

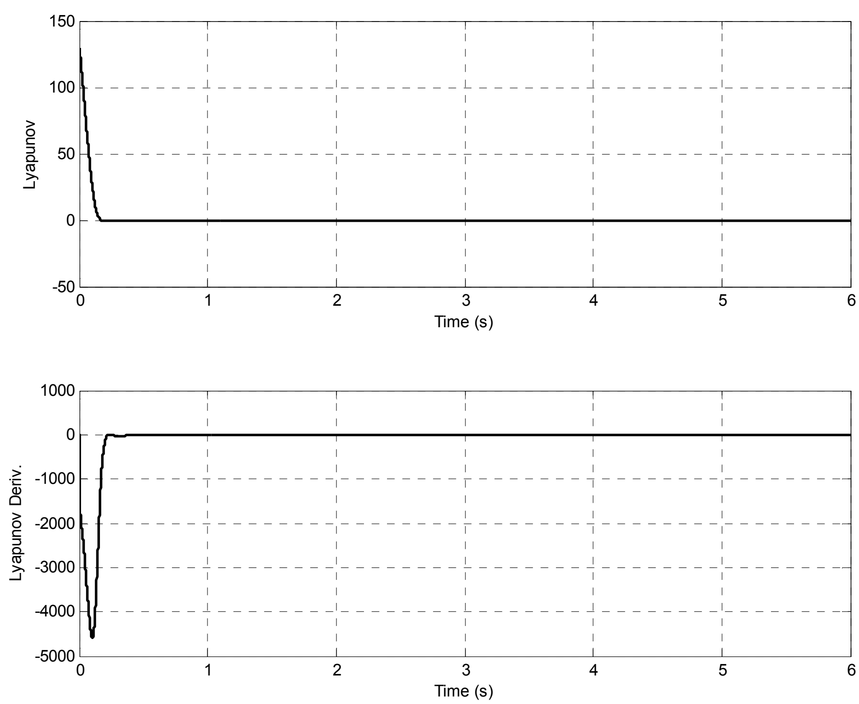

Figure 4.17 : Lyapunov Time History and its Time Derivative

Figure 4.18 shows the performance of the control law.

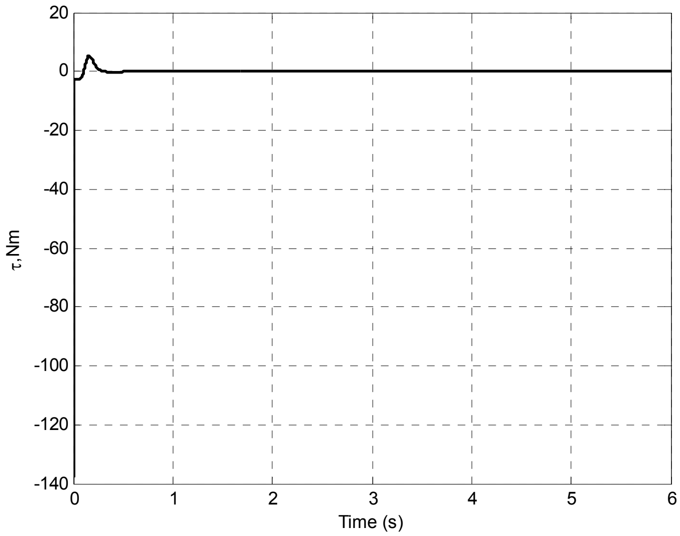

Figure 4.18: Control law

4.5 Chapter Summary

In this chapter, the ball and arc system is presented. The problem of stabilization of the unstable equilibrium of this system is studied.

The control strategy for the stabilization problem has been solved by the direct Lyapunov method considering the presence of a contribution from the FMC. For the first formulation $\mathbf{K_D}$ is a constant and in the second formulation, the $\mathbf{P}(\mathbf{q})$ matrix is almost a constant. The FMC is solved based on the positive definiteness property of $\mathbf{K_D}$, the proper shape of the potential, and the monotonically decreasing in time Lyapunov function behavior. All the parameters are selected under these criteria. The formulation is presented for two cases: considering a constant $\mathbf{K_D}$ and an almost constant $\mathbf{P}(\mathbf{q})$ matrix. The second formulation is simpler than the first one and it is easier to solve for a potential. The performance of the controllers appears to be very similar for both formulations.

The presented control strategy is significant because it is the first nonlinear stabilizing control law for the ball and arc system. Simulation results for this system are shown making the origin a point of asymptotic stability.

Chapter 5 - Inverted Pendulum Cart System

5.1 Introduction

Using a simple and effective formulation to find a control law for the inverted pendulum cart, such that the origin is an asymptotically stable equilibrium point, is the challenge presented in this chapter. The cart can move horizontally by means of a horizontal force which is the only control input. Because the angular acceleration of the vertical pendulum cannot be directly controlled, it becomes an interesting underactuated mechanical example.

The state variables x and θ are defined in the inverted pendulum cart schematic of Figure 5.1. In the figure, M_c is the mass of the cart, M_p and l are the mass and length of the pendulum, respectively, and g is the acceleration due to gravity.

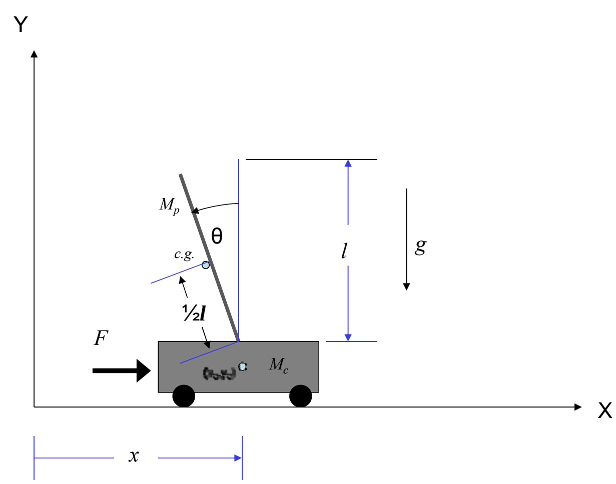

Figure 5.1 Inverted pendulum cart system

The formulation is presented for two cases. One case considers the control input from the FMC to be non-zero. The other case considers the control input from the FMC to be zero. Indeed, these two formulations are presented in order to demonstrate that for this particular system the application of a control signal from the FMC is not significant for the purpose of stabilization.

The chapter is organized as described in this paragraph. Section 5.2 shows the mathematical model for the inverted pendulum cart system. Section 5.3 presents the first matching condition. Section 5.4 shows the second matching condition. Section 5.5 presents the third matching condition. The selection of parameters from the linear model is shown in Section 5.6. The zero $\mathbf{F_1}$formulation is presented in Section 5.7. Sections 5.7.1 and 5.7.2 introduce the potential and the Hessian for the system potential under study. Section 5.7.3 illustrates the efficacy of the proposed control law with simulation. The non-zero $\mathbf{F_1}$ formulation is presented in Section 5.8. Sections 5.8.1 and 5.8.2 introduce the potential and the Hessian for this case. Section 5.8.3 shows the corresponding simulations. Section 5.9 presents the conclusion of the chapter.

5.2 The Dynamic Equations

From the Lagrange's equations it is seen that for the inverted pendulum cart system (IPC) there is an input $\mathbf{F}$ acting on the cart and there is no input acting on the pendulum.

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right)-\frac{\partial L}{\partial \mathbf{x}}=\mathbf{F};\quad \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right)-\frac{\partial L}{\partial \boldsymbol{\theta}}=\mathbf{0}. \tag{5.1}$$

The Lagrangian is written in terms of the kinetic energy

$$\begin{aligned} KE &= \tfrac{1}{2}M_c\,\dot{x}^2+\tfrac{1}{2}M_p\dot{x}^2-\tfrac{1}{2}M_p\dot{x}^2\dot{\theta}\,l\cos(\theta)\\ &+\tfrac{1}{2}I_p\,\dot{\theta}^2+\tfrac{1}{2}M_p\,\dot{\theta}^2 l^2, \end{aligned} \tag{5.2}$$

where I_p is the centroidal mass moment of inertia of the pendulum given by $I_p=\dfrac{1}{12}ml^2$ and the potential energy of the system which is

$$PE=\frac{1}{2}m\,g\,l\cos(\theta) \tag{5.3}$$

where $m = M_p$. By substituting Eq. 5.3 and Eq. 5.2 into Lagrange's equations, the governing equations of motion for the IPC are obtained.

The inertia matrix $\mathbf{M}(\mathbf{q})$, the $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$matrix containing the coriolis and centripetal coefficients, and the vector $\mathbf{G}(\mathbf{q})$containing the gravity terms are shown in the equation of motion which are

$$\begin{bmatrix} M_c+M_p & -\frac{1}{2}M_p l\cos(\theta)\\ -\frac{1}{2}M_p l\,cos(\theta) & \frac{1}{4}M_p l^2+I_p \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}+$$

$$\begin{bmatrix} 0 & -\frac{1}{2}M_p l\sin(\theta)\dot{\theta}\\ 0 & 0 \end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix}+$$

$$\begin{bmatrix} 0 \\ -\frac{1}{2}M_p l\sin(\theta) \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}. \tag{5.4}$$

The $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ matrix is

$$\mathbf{C}(\dot{\mathbf{q}},\mathbf{q}) = \begin{bmatrix} 0 & -\frac{1}{2}ml\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}. \tag{5.5}$$

Indeed, to demonstrate the skew-symmetry, the time derivative of $\mathbf{M}(\mathbf{q})$ is

$$\dot{\mathbf{M}}(\mathbf{q}) = \begin{bmatrix} 0 & -\frac{1}{4}ml\sin(\theta)\dot{\theta} \\ \frac{1}{4}ml\sin(\theta)\dot{\theta} & 0 \end{bmatrix}, \tag{5.6}$$

and Eq. 2.9 shows

$$\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} 0 & -\frac{1}{2}ml\sin(\theta)\dot{\theta} \\ \frac{1}{2}ml\sin(\theta)\dot{\theta} & 0 \end{bmatrix}. \tag{5.7}$$

Further information on this derivation is given in section C.1 of Appendix C.

5.3 The first matching condition

For the inverted pendulum cart system, $\mathbf{P}(\mathbf{q})$ is a full matrix. Additional actuation from $\mathbf{F}_1$ is not necessary in order to introduce more parameters for the placement of poles of the linearized system in desired locations. Thus, actuation could be considered or not depending on the performance of the controller.

The FMC is given by

$$\dot{\mathbf{K}}_D + \mathbf{K}_D\mathbf{M}(\mathbf{q})^{-1}(\mathbf{F}m_1 - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})) + (\mathbf{F}m_1 - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}))^T\mathbf{M}(\mathbf{q})^{-1}\mathbf{K}_D + \mathbf{F}mc_1 = 0. \tag{5.8}$$

The solution of the FMC is provided by separation of terms involving $\dot{x}$ and $\dot{\theta}$. After some simplifications, the $\mathbf{K}_\mathbf{D}$ matrix can be shown to be constant. The details of this simplification can be seen in Section C.3 of Appendix C.

The elements of the matrix $\mathbf{K}_\mathbf{D}$ are constants and the partial differential equation that needs to be solved for the potential is not complicated. The $\mathbf{K}_\mathbf{D}$ matrix is given as

$$\mathbf{K}_\mathbf{D} = \begin{bmatrix} C_3 & C_5 \\ C_5 & C_6 \end{bmatrix} \tag{5.9}$$

where the set of unknown constants will be selected later such that the positive definiteness property of $\mathbf{K_D}$ is satisfied.

The first matching condition control input matrix is proposed as

$$\mathbf{Fm_1} = \begin{bmatrix} -\mathbf{F_{11}}(x,\theta)\dot{\theta} + \nu & \mathbf{F_{11}}(x,\theta)\dot{x} + \sigma \\ -\mathbf{F_{22}}(x,\theta)\dot{\theta} & \mathbf{F_{22}}(x,\theta)\dot{x} \end{bmatrix} \tag{5.10}$$

Multiplying $\mathbf{Fm_1}$ by the vector of generalized velocities produces the result

$$\mathbf{Fm_1}\dot{\mathbf{q}} = \begin{bmatrix} \nu\dot{x} + \sigma\dot{\theta} \\ 0 \end{bmatrix} \tag{5.11}$$

and the Lyapunov FMC contribution is

$$\mathbf{Fmc_1} = \begin{bmatrix} \mathbf{F}_{33}\left(x,\theta,\dot{x},\dot{\theta}\right) & \mathbf{F}_{44}\left(x,\theta,\dot{x},\dot{\theta}\right) \\ \mathbf{F}_{44}\left(x,\theta,\dot{x},\dot{\theta}\right) & \mathbf{F}_{55}\left(x,\theta,\dot{x},\dot{\theta}\right) \end{bmatrix}. \tag{5.12}$$

The negative of this matrix is added to the right hand side of the FMC in order to satisfy the Eq. 5.8. $\mathbf{Fmc_1}$ will be solved for $\mathbf{F_{33}}\left(x,\theta,\dot{x},\dot{\theta}\right), \mathbf{F_{44}}\left(x,\theta,\dot{x},\dot{\theta}\right)$,and $\mathbf{F_{55}}\left(x,\theta,\dot{x},\dot{\theta}\right)$ to satisfy the FMC after determining the elements of the $\mathbf{K_D}$ matrix and the inputs $\mathbf{F_{11}}(x,\theta)$ and $\mathbf{F_{22}}(x,\theta)$.

The outline of the solution for the inverted pendulum cart system is presented considering that $\mathbf{K_D}$ is not a constant, however after some simplifications it became a constant as can be noticed in Section C.3 of Appendix C.

1. Organize FMC into $\mathbf{m}_\theta\dot{\theta} + \mathbf{m}_x\dot{x} + f(\mathbf{q}) = 0$
2. $\mathbf{m}_\theta = \mathbf{m}_x = 0$
3. Because of symmetry, six equations are obtained which are
$$\left.\begin{matrix} \mathbf{m}_{\theta 11} = \mathbf{m}_{\theta 21} = \mathbf{m}_{\theta 22} \\ \mathbf{m}_{x11} = \mathbf{m}_{x21} = \mathbf{m}_{x22} \end{matrix}\right\} = 0.$$
4. Solve the equations of step 3 for $\mathbf{K_{D11}}(x,\theta), \mathbf{K_{D21}}(x,\theta), \mathbf{K_{D22}}(x,\theta), \mathbf{F_{11}}(x,\theta)$, and $\mathbf{F}_{22}(x,\theta)$. These five unknowns are sufficient to solve the equations. Substitute the solutions into the FMC.
5. Solve resulting equations for $\mathbf{F_{33}}\left(x,\theta,\dot{x},\dot{\theta}\right), \mathbf{F_{44}}\left(x,\theta,\dot{x},\dot{\theta}\right)$, and $\mathbf{F_{55}}\left(x,\theta,\dot{x},\dot{\theta}\right)$ to satisfied FMC.

The resulting $\mathbf{Fmc_1}$ is

$$\mathbf{Fmc_1} = \begin{bmatrix} Fmc_{11}(x,\theta,\dot{x},\dot{\theta}) & Fmc_{21}(x,\theta,\dot{x},\dot{\theta}) \\ Fmc_{21}(x,\theta,\dot{x},\dot{\theta}) & Fmc_{22}(x,\theta,\dot{x},\dot{\theta}) \end{bmatrix} \tag{5.13}$$

where

$$Fmc_{11}(x,\theta,\dot{x},\dot{\theta}) = \frac{4\nu(2lC_3 + 3\cos(\theta)C_5)}{l(-4mb + 3m\cos(\theta)^2)},$$

$$Fmc_{21}(x,\theta,\dot{x},\dot{\theta}) = -\frac{2C_3ml^2\sin(\theta) - 4C_3l\sigma - 6\cos(\theta)\sigma C_5 - 4l\nu C_5 + 3ml\cos(\theta)\sin(\theta)\dot{\theta}C_5 - 6\cos(\theta)\nu C_6}{l(-4mb + 3m\cos(\theta)^2)},$$

and

$$Fmc_{22}(x,\theta,\dot{x},\dot{\theta}) = \frac{2(3\cos(\theta)C_6 + 2lC_5)(ml\sin(\theta)\dot{\theta} - 2\sigma)}{l(-4mb + 3m\cos(\theta)^2)}.$$

It is important at this point, to find these results such that they do not include singularities and $\mathbf{K_v} + \mathbf{Fmc_1}$ is positive semi-definite, so its eigenvalues requires satisfying

$$eig(\mathbf{K_v} + \mathbf{Fmc_1}) \geq 0. \tag{5.14}$$

By using Eq. 2.15, it is seen that $\mathbf{P}(\mathbf{q})$ is

$$\mathbf{P} = \begin{bmatrix} -\dfrac{2(2C_3l + 3C_5\cos(\theta))}{l(-4mb + 3m\cos(\theta)^2)} & -\dfrac{6(C_3\cos(\theta)ml + 2C_5mb)}{lm^2(-4mb + 3m\cos(\theta)^2)} \\ -\dfrac{2(2lC_5 + 3\cos(\theta)C_6)}{l(-4mb + 3m\cos(\theta)^2)} & -\dfrac{6(C_5\cos(\theta)ml + 2C_6mb)}{ml^2(-4mb + 3m\cos(\theta)^2)} \end{bmatrix} \tag{5.15}$$

and the determinant of $\mathbf{P}$ is

$$\det(\mathbf{P}) = -\frac{12(C_3C_6 - C_5^2)}{l^2m(-4mb + 3m\cos(\theta)^2)} \tag{5.16}$$

where $m = M_p$ and $mb = M_c + M_p$.

5.4 The second matching condition

This section will present the solution of the second matching condition. From Eq. 2.28 the matrix $\mathbf{F}m_2$ will be determined. If $\mathbf{K_D}$ is positive definite, then $\mathbf{K}_v$ is symmetric with non-negative eigenvalues. The matrix $\mathbf{K}_v$ is always positive definite (semi-definite). The matrix is evaluated by using Eq. 2.29 to get

$$\mathbf{K}_v = \begin{bmatrix} \dfrac{4\alpha(2C_3l + 3C_5\cos(\theta))^2}{l^2(-4mb + 3m\cos(\theta))^2} & \dfrac{4\alpha(2C_3l + 3C_5\cos(\theta))(2lC_5 + 3\cos(\theta)C_6)}{l^2(-4mb + 3m\cos(\theta))^2} \\ \dfrac{4\alpha(2C_3l + 3C_5\cos(\theta))(2lC_5 + 3\cos(\theta)C_6)}{l^2(-4mb + 3m\cos(\theta))^2} & \dfrac{4\alpha(2C_3l + 3C_5\cos(\theta))^2}{l^2(-4mb + 3m\cos(\theta))^2} \end{bmatrix} \tag{5.17}$$

and the eigenvalues of $\mathbf{K}_v$ are calculated as

$$eig(K_v) = \begin{bmatrix} 0 \\ \lambda_{KvIPC} \end{bmatrix} \tag{5.18}$$

where

$$\lambda_{KvIPC} = \frac{4\alpha\left(4l^2 C_5^{\,2} + 12l\cos(\theta)C_6 C_5 + 9\cos(\theta)^2 C_6^{\,2} + 4C_3^{\,2} l^2 + 12C_3 l C_5 \cos(\theta) + 9C_5^{\,2}\cos(\theta)^2\right)}{l^2\left(16mb^2 - 24mbm\cos(\theta)^2 + 9m^2\cos(\theta)^4\right)}.$$

The contribution to the control law from the second matching condition is obtained with the corresponding $\mathbf{P}(\mathbf{q})$. From Eq. 2.28 and multiplying by the generalized velocity vector, the solution for the control signal $\mathbf{F_2}$ is

$$\mathbf{F_2} = \frac{2\alpha\left(2l\dot{x}C_3 + 3\cos(\theta)C_5\dot{x} + 2l\dot{\theta}C_5 + 3\cos(\theta)\dot{\theta}C_6\right)}{\left(-4mb + 3m\cos(\theta)^2\right)l}. \tag{5.19}$$

5.5 The third matching equation

The TMC provides two equations. They are

$$F_3 + \frac{1}{2}\frac{(C_5\cos(\theta)ml + 2C_6 mb)\left(\frac{\partial}{\partial x}\Phi(\mathrm{x},\theta)\right)}{C_3C_6 - C_5^{\,2}} - \frac{1}{2}\frac{(C_3\cos(\theta)ml + 2C_5 mb)\left(\frac{\partial}{\partial\theta}\Phi(\mathrm{x},\theta)\right)}{C_3C_6 - C_5^{\,2}} = 0 \tag{5.20}$$

and

$$\frac{1}{2}mgl\sin(\theta) - \frac{1}{6}\frac{(2lC_5 + 3\cos(\theta)C_6)lm\left(\frac{\partial}{\partial x}\Phi(\mathrm{x},\theta)\right)}{C_3C_6 - C_5^{\,2}} + \frac{1}{6}\frac{(2C_3 l + 3C_5\cos(\theta))lm\left(\frac{\partial}{\partial\theta}\Phi(\mathrm{x},\theta)\right)}{C_3C_6 - C_5^{\,2}} = 0. \tag{5.21}$$

Solving Eq.(5.21), the corresponding potential is

$$\Phi(\mathrm{x},\theta) = \frac{1}{C_5}\left(-gC_5^{\,2}\ln(2C_3 l + 3C_5\cos(\theta)) + C_6 ln(2C_3 l + 3C_5\cos(\theta))C_3 g + \Gamma(\mathrm{x},\theta)\right) \tag{5.22}$$

where

$$\Gamma(\mathrm{x},\theta) = F_7\left(\left(-\frac{1}{C_5\sqrt{4C_3^{\,2}l^2 - 9C_5^{\,2}}}\left(\begin{aligned}&-xC_5\sqrt{4C_3^{\,2}l^2 - 9C_5^{\,2}} + 2C_6\arctan\left(\frac{\cos(\theta) - 1}{\sin(\theta)}\right)\sqrt{4C_3^{\,2}l^2 - 9C_5^{\,2}} \\ &+ 4\arctan\left(\frac{(2C_3 l - 3C_5)(\cos(\theta) - 1)}{\sqrt{4C_3^{\,2}l^2 - 9C_5^{\,2}}\sin(\theta)}\right)lC_5^{\,2} - 4\arctan\left(\frac{(2C_3 l - 3C_5)(\cos(\theta) - 1)}{\sqrt{4C_3^{\,2}l^2 - 9C_5^{\,2}}\sin(\theta)}\right)lC_3C_6\end{aligned}\right)\right)^2 C_5\right)$$

where the arbitrary function of the characteristic of the PDE is chosen as F7 time the square of the characteristic.

The necessary equation to find $\mathbf{F}_3$ is provided by the Eq. 5.20. When substituting Eq. 5.22 into Eq. 5.20 the result is

$$\mathbf{F_3} = \frac{1}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\,C_5(2C_3l + 3C_5\cos(\theta))}\left(12C_3C_6\cos(\theta)^2 ml^2 F_7 \arctan(\mathrm{N}(x,\theta)) - \right)$$
$$-\frac{1}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\,C_5(2C_3l + 3C_5\cos(\theta))}\left(16C_3C_6F_7ml^2\arctan(\mathrm{N}(x,\theta))\right) - \frac{3}{2}\frac{C_3\cos(\theta)mg\sin(\theta)l}{2C_3l + 3C_5\cos(\theta)}$$
$$+\frac{1}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\,C_5(2C_3l + 3C_5\cos(\theta))}\left(16C_5l^2F_7mb\arctan(\mathrm{N}(x,\theta))\right)$$
$$+\frac{1}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\,C_5(2C_3l + 3C_5\cos(\theta))}\left(12C_5l^2\cos(\theta)^2F_7m\arctan(\mathrm{N}(x,\theta))\right) - \frac{4lF_7mbx}{2C_3l + 3C_5\cos(\theta)} \tag{5.23}$$
$$-\left(\frac{6C_6\cos(\theta)^2 lmF_7\arctan\left(-\frac{1}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right)}{C_5(2C_3l + 3C_5\cos(\theta))}\right) + \left(\frac{8C_6lF_7mb\arctan\left(-\frac{1}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right)}{C_5(2C_3l + 3C_5\cos(\theta))}\right)$$
$$+\frac{3l\cos(\theta)^2mF_7x}{2C_3l + 3C_5\cos(\theta)} - \frac{3C5mbg\sin(\theta)}{2C_3l + 3C_5\cos(\theta)}$$

where

$$\mathrm{N}(x,\theta) = \arctan\left(\frac{2\cos(\theta)C_3l}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\sin(\theta)} - \frac{3C_5\cos(\theta)}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\sin(\theta)} - \frac{2C_3l\cos(\theta)}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\sin(\theta)} + \frac{3C_5}{\sqrt{4C_3{}^2l^2 - 9C_5{}^2}\sin(\theta)}\right).$$

5.6 Selection of the unknown parameters

By inspection of the Eq. 5.4 it is seen that

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} -\frac{4}{-4mb + 3m\cos(\theta)^2} & -\frac{6\cos(\theta)}{l(4mb + 3m\cos(\theta)^2)} \\ -\frac{6\cos(\theta)}{l(4mb + 3m\cos(\theta)^2)} & -\frac{12mb}{ml^2(4mb + 3m\cos(\theta)^2)}\end{bmatrix}\left(\begin{bmatrix}\tau\\0\end{bmatrix} - \begin{bmatrix}0 & -\frac{1}{2}M_p l\sin(\theta)\dot{\theta}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} - \begin{bmatrix}0\\ -\frac{1}{2}M_p l\sin(\theta)\end{bmatrix}\right). \tag{5.24}$$

Eq. 5.24 can be rewritten as

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} -\frac{4\tau - 2l\dot{\theta}^2 + 3\cos(\theta)mg\sin(\theta)}{md^2 + Ib - md^2\cos(\theta)^2} \\ \frac{3(-2\cos(\theta)\tau + \cos(\theta)m\sin(\theta)l\dot{\theta}^2 - 2mbg\sin(\theta))}{l(-4mb + 3m\cos(\theta)^2)}\end{bmatrix}. \tag{5.25}$$

The linearized state equations for the IPC are found to be

$$\begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -\frac{3mg}{-4m_p + 3m} & 0 & 0\\ 0 & -\frac{6m_pg}{l(-4m_p + 3m)} & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ \theta\\ \dot{x}\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -\frac{4}{-4m_p + 3m\cos(\theta)^2}\\ -\frac{6\cos(\theta)}{l(-4m_p + 3m\cos(\theta)^2)}\end{bmatrix}\mathbf{F}. \tag{5.26}$$

The values of the physical parameters for the inverted pendulum cart system are listed in Table 5.1.

Table 5.1: Physical parameters of the inverted pendulum cart

Parameter	Explanation	Given value
M_p	Mass of the pendulum	1.0 Kg
M_c	Mass of the cart	5.0 Kg
l	Length of pendulum	0.7 m
g	Acceleration of gravity	9.81 m s^{-2}

5.7 Considering F_1 as zero

As mentioned before, for the inverted pendulum cart system no additional parameters are needed to place the poles in the desired locations because $\mathbf{P}(\mathbf{q})$ is a full matrix. Then ν and σ are chosen to be zero for this particular section and $\mathbf{Fmc}_1$ is zero too.

Through the same process used to produce Eq. 5.26, the linearized control input, after substituting the parameter values, becomes

$$\mathbf{F}_L = \begin{bmatrix} \dfrac{11.9\,F_7}{1.4\,C_3+3\,C_5)}x - \dfrac{1}{2}\dfrac{1-28.84\,C_3^{\;2}-33.32\,C_5\,F_7-473.82\,C_5\,C_3-71.40\,C_6\,F_7-882.90\,C_5^{\;2}}{1.96\,C_3^{\;2}+8.40C_5\,C_3+9.00\,C_5^{\;2}}\theta \\ +(0.50\,C_5\,\alpha+0.23\,\alpha\,C_3\,)\dot{x} \\ +\left(0.50\,C_6\,\alpha+0.24\,\alpha\,C_5\right)\dot{\theta} \\ \end{bmatrix} \tag{5.27}$$

where $\mathbf{F}_L$ is the linear control. A full state feedback control is applied to the linearized model described in Eq. 5.26 and is given by

$$\mathbf{F}_L = -\mathbf{K}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}-K1 & -K2 & -K3 & -K4\end{bmatrix}\begin{bmatrix}x\\ \theta\\ \dot{x}\\ \dot{\theta}\end{bmatrix}. \tag{5.28}$$

Comparing like terms of Eq. 5.27 and Eq. 5.28, four equations are found. K is determined such that the eigenvalues of the linearized closed loop system are -5, -3, -2-2i and -2+2i based on the positive definiteness property of $\mathbf{K_D}$. The corresponding gain matrix is

$$\mathbf{K} = \begin{bmatrix}\textit{-24.2610} & \textit{169.4551} & \textit{-25.0697} & \textit{35.4992}\end{bmatrix}$$

and the corresponding equations from which the elements of $\mathbf{K}$ are found are

$$
\left.\begin{aligned}
&K_1+\frac{11.9F_7}{1.4C_3+3C_5}=0\\
&K_2-\frac{1}{2}\frac{-28.8414C_3^{\,2}-33.32C_5F_7-473.823C_3C_5-882.90C_5^{\,2}-71.4C_6F_7}{1.96C_3^{\,2}+8.4C_3C_5+9C_5^{\,2}}=0\\
&K_3+0.5042C_5\alpha+0.2353\alpha C_3=0\\
&K_4+0.17\left(1.4C_5\alpha+3C_6\alpha\right)=0
\end{aligned}\right\}.
$$

The four equations are solved for F_7, C_3, C_5 and C_6. The constant α is picked based on the positive definiteness property of $\mathbf{K_D}$.

The values of the control parameters for the inverted pendulum system are shown in the Table 5.2

Table 5.2 : Values of the control parameters for the inverted pendulum cart

C_3	Coefficient of K_D	219.9056/α
C_5	Coefficient of K_D	-152.3442/α
C_6	Coefficient of K_D	141.5007/α
F_7	Coefficient for the potential	304.1079/α
α	Coefficient of K_v	3

5.7.1 The potential

From the last row of the TMC, the partial differential equation for the potential $\Phi(x,\theta)$ has been solved and the potential is given by Eq. 5.22. The corresponding potential $\Phi(x,\theta)$ is evaluated and it is seen that it is composed of a homogeneous and a particular solution. The homogenous solution is

$$
\begin{aligned}
\mathrm{HS}=\;&3.0411x^2+2.6171x\,\mathrm{arctanh}\left(\frac{2.2645(\cos(\theta)-1)}{\sin(\theta)}\right)+11.2985x\arctan\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right)\\
&+0.5631\,\mathrm{arctanh}\left(\frac{2.2645(\cos(\theta)-1)}{\sin(\theta)}\right)^2+4.8617x\,\mathrm{arctanh}\left(\frac{2.2645(\cos(\theta)-1)}{\sin(\theta)}\right)\arctan\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right)\\
&+10.4943\arctan\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right)^2
\end{aligned}
\qquad (5.29)
$$

and the particular solution is

$$
\mathrm{PS}=-\frac{1}{C_5^{\,2}\left(4C_3^{\,2}l^2-9C_5^{\,2}\right)}\begin{pmatrix}4gC_5^{\,3}\ln(2C_3l+3C_5\cos(\theta)C_3^{\,2}l^2-9gC_5^{\,2}\ln(2C_3l+3C_5\cos(\theta)\\ -4g\ln(2C_3l+3C_5\cos(\theta)C_6C_3^{\,3}C_5l^2+9g\ln(2C_3l+3C_5\cos(\theta)C_6C_3C_5^{\,3}\end{pmatrix}.
\qquad (5.30)
$$

The plot of the potential is shown in Fig. 5.2. The potential for the inverted pendulum system is plotted on the interval (-1.5,1.5), for x and on the interval (-0.8, 0.8) for θ.

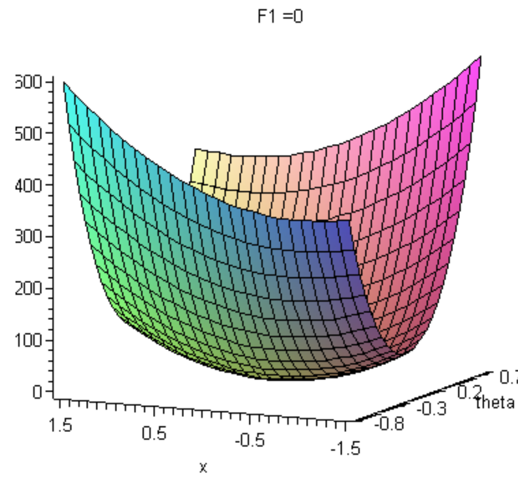

Figure 5.2: The potential

5.7.2 The Hessian

Taking the second derivative of the potential $\Phi(x,\theta)$ with respect to x and θ the Hessian of the potential is found. This Hessian matrix **H** is

$$\mathbf{H}=\begin{bmatrix}\dfrac{\partial^2\Phi(x,\theta)}{\partial x^2} & \dfrac{\partial^2\Phi(x,\theta)}{\partial x\,\partial\theta}\\ \dfrac{\partial^2\Phi(x,\theta)}{\partial x\,\partial\theta} & \dfrac{\partial^2\Phi(x,\theta)}{\partial\theta^2}\end{bmatrix}. \tag{5.31}$$

When evaluating the Hessian at the equilibrium point, the result is

$$\mathbf{H}=\begin{bmatrix}202.7387 & -287.0820\\ -287.0820 & 926.5961\end{bmatrix}, \tag{5.32}$$

and the corresponding eigenvalues of Eq. 5.32 are

$$\boldsymbol{\lambda}=\begin{bmatrix}102.7058\\ 1026.6289\end{bmatrix}. \tag{5.33}$$

From Eq. 5.32 and Eq. 5.33 it is noticed that the Hessian is a positive definite matrix.

5.7.3 Simulation when F_1 is zero

In this section some simulations are presented for the inverted pendulum cart. First, the plots where no actuation is considered are shown. The closed loop system response was simulated using MATLAB Simulink for a period of 6 seconds. The cart position and velocity were initialized at 0.5 m and 1.5 m/s, respectively, while the initial pendulum angle and pendulum angular velocity were set to 0.8 rad and zero, respectively. Figure 5.3 shows the stabilization of the states for the indicated initial conditions.

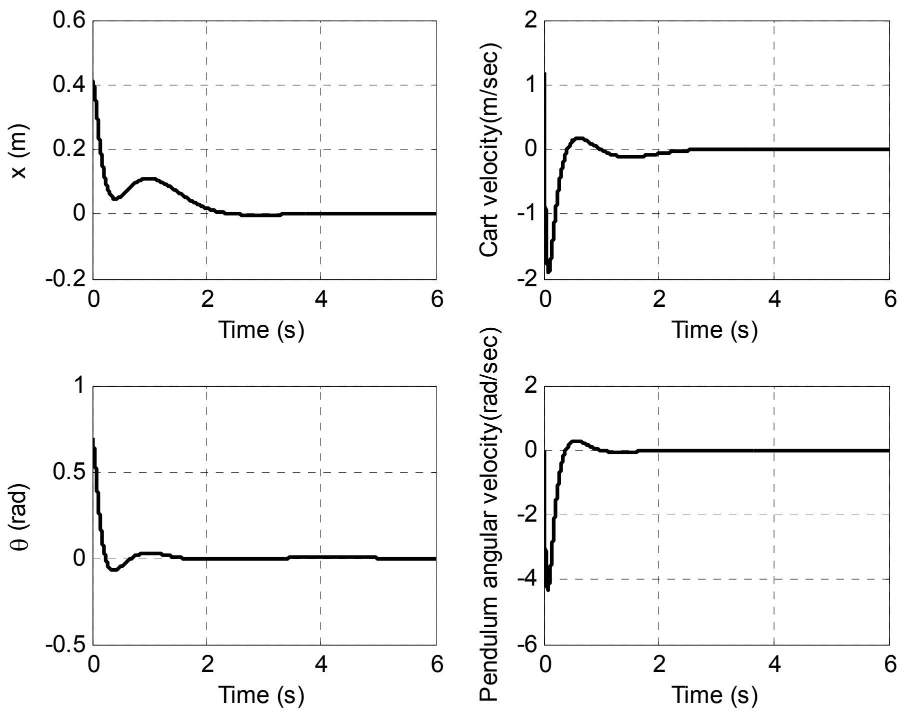

Figure 5.3: Simulation result with initial condition [0.4, 0.7, 1.2, 0]

Figure 5.4 shows the simulation result for the elements of the $\mathbf{P}(\mathbf{q})$ matrix.

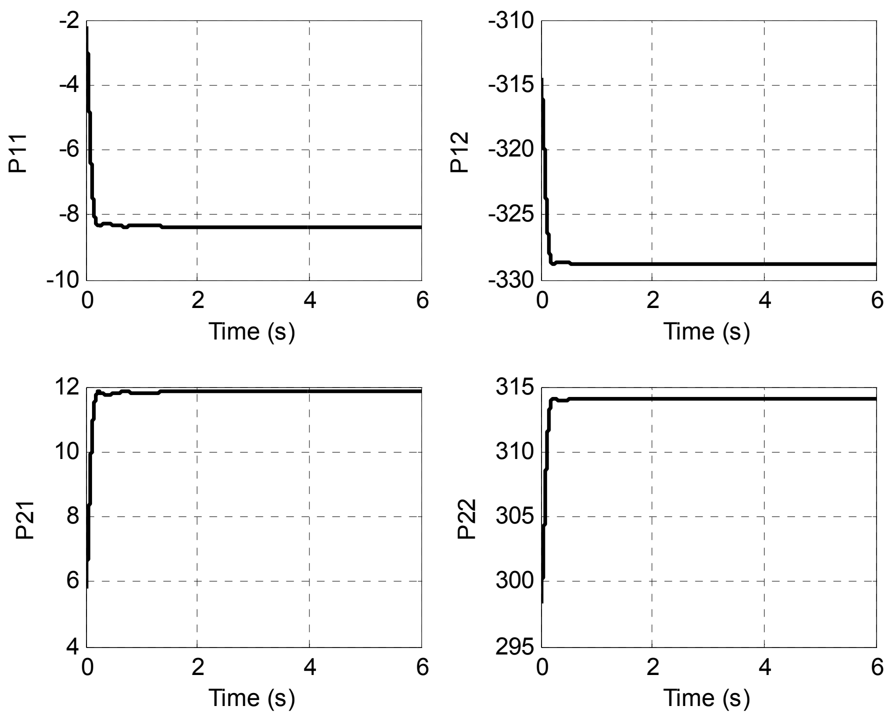

Figure 5.4: Simulation for P elements

Figure 5.5 shows that the determinant of the $\mathbf{P}(\mathbf{q})$ matrix is positive.

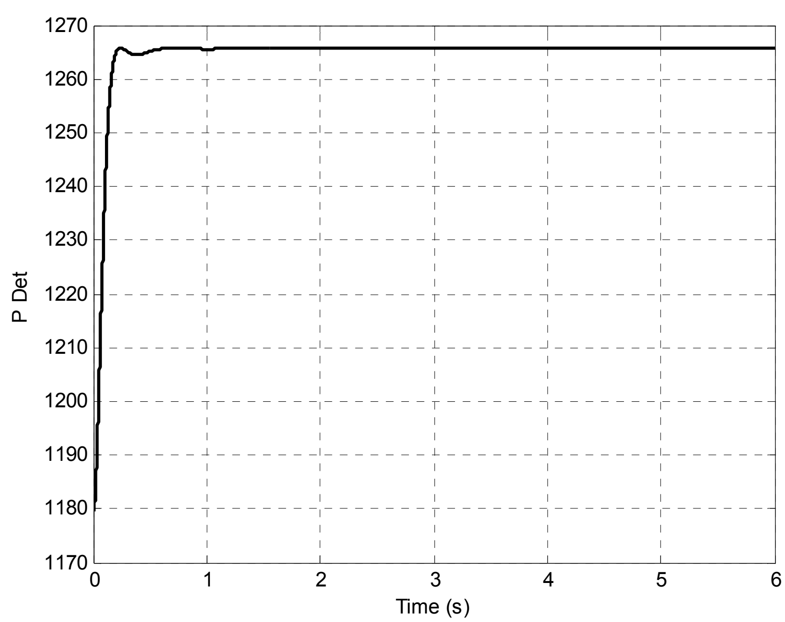

Figure 5.5: Determinant of the $\mathbf{P}(\mathbf{q})$ Matrix

Figure 5.6 shows the Lyapunov function as a function of time. Notice that the Lyapunov function is monotonically decreasing.

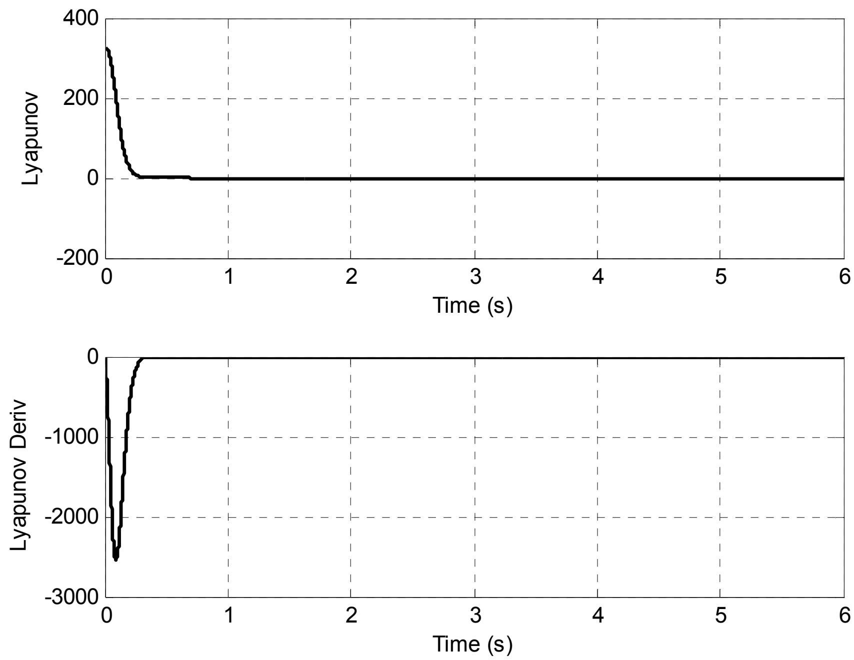

Figure 5.6: Lyapunov Time History and its Time Derivative

Finally, Figure 5.7 shows the control law to stabilize the IPC when F_1 is zero.

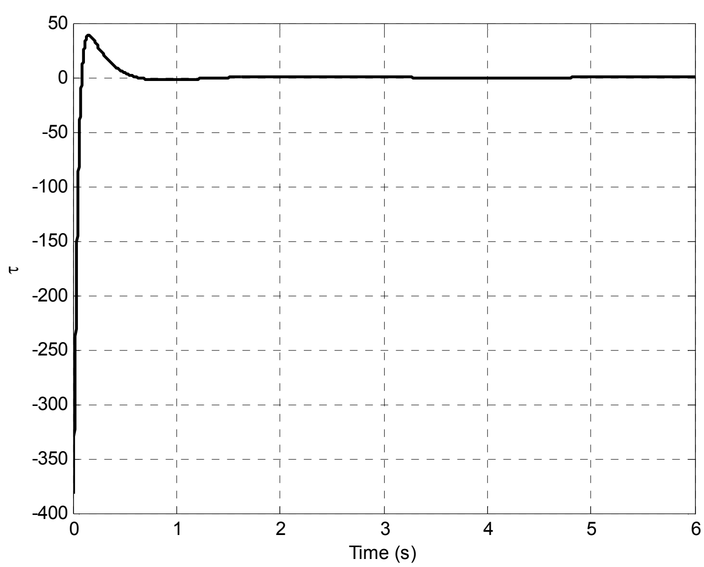

Figure 5.7: Control law

5.8 Considering $\mathbf{F_1}$ to be non-zero

An alternative formulation for the IPC is considered. The arbitrary input $\mathbf{F_1}$ given by

$$\mathbf{F_1} = v\dot{x} \tag{5.34}$$

was chosen where, from Eq. 5.14, σ was selected to be zero.

The addition of this control signal may change some values of the parameters. The physical parameters and the corresponding gain matrix **K** are the same as the previous formulation, when $\mathbf{F_1}$ is zero. The gain equations from the linearized closed loop system are

$$K_1 + \frac{59.50F_7}{7C_3 + 15C_5} = 0,$$

$$K_2 + 0.0025\frac{\left(1.44e^5C_3^{\,2} + 1.66e^5C_5F_7 + 2.37e^6C_3C_5 + 3.57e^5C_6F_7 + 4.41e^6C_5^{\,2}\right)}{49C_3^{\,2}+210C_3C_5+225C_5^{\,2}} = 0,$$

$$K_3 + v + 0.5042C_5\alpha + 0.2353\alpha C_3 = 0,$$

and

$$K_4 + 0.033\left(7C_5\alpha + 15C_6\alpha\right) = 0.$$

The four equations were solved for F_7 C_3, C_5 and ν. The constants C_6 and α are picked based on the positive definiteness property of $\mathbf{K_D}$. The values of the parameters appear in table 5.3.

Table 5.3 : Control values of the IPC system, actuator considered

Parameter	Explanation	Identified value
C_3	Element of K_D	1.12e-15(3.03e15αC_6-2.34e17)/ α
C_5	Element of K_D	-3.39e-9(6.30e8αC_6-4.43e11)/ α
F_7	Coefficient of characteristic	3.21e-15(1.05e15αC_6-5.39e16)/ α
α	Coefficient of K_v	3
C_6	Element of K_D	28
v	Coefficient of the force $\mathbf{F_1}$	0.28C_6α-39.32

5.8.1 The potential

The homogenous solution of the PDE governing the potential is

$$
\begin{aligned}
\mathrm{HS} = & -4900 F_7 x^2 C_5^{\,3} + 22500 F_7 x^2 C_5^{\,4} + 5.48e^5 F_7 x C_5^{\,3} \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right) C_3^{\,2} \\
& - 2.52e^6 F_7 x C_5^{\,3} \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right) C_3^{\,2} \\
& + 2800 F_7 x C_5^{\,3} \sqrt{49C_3^{\,2} - 255C_5^{\,2}} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right) \\
& - 78400 F_7 x C_5 \sqrt{49C_3^{\,2} - 255C_5^{\,2}} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right) C_3 \\
& - 1.53e^7 F_7 \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right)^2 C_3^{\,2} + 7.05e^7 F_7 \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right)^2 C_5^{\,2} \\
& - 1.56e^5 F_7 \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right) \sqrt{49C_3^{\,2} - 255C_5^{\,2}} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right) \\
& - 4.39e^5 F_7 \operatorname{arctanh}\left(\frac{\cos(\theta)-1}{\sin(\theta)}\right) \sqrt{49C_3^{\,2} - 255C_5^{\,2}} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right) C_3 \\
& - 19600 F_7 C_5^{\,4} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right)^2 \\
& + 1.09e^6 F_7 C_5^{\,2} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right)^2 C_3 \\
& - 1.53e^7 F_7 C_5^{\,2} \arctan\left(\frac{(7C_3 - 1.5C_5)(\cos(\theta)-1)}{\sqrt{49C_3^{\,2} - 255C_5^{\,2}}\sin(\theta)}\right)^2 C_3^{\,2}
\end{aligned}
\tag{5.35}
$$

and the particular solution is

$$
\mathrm{PS} = -\frac{1}{C_5^{\,2}\left(4C_3^{\,2} l^2 - 9C_5^{\,2}\right)}\left(\mathrm{P}(x,\theta)\right) \tag{5.36}
$$

where

$$
\begin{aligned}
\mathrm{P}(x,\theta) = & \; 0.01\left(48069 C_5^{\,3} \ln(1.4C_3 + 3\cos(\theta)C_5) C_3^{\,2}\right) - 2.2e^5 C_5^{\,5} \ln(1.4C_3 + 3\cos(\theta)C_5) \\
& - 1.34e^6 C_5^{\,3} \ln(1.4C_3 + 3\cos(\theta)C_5) C_3^{\,3} C_5 + 6.18e^6 C_5^{\,5} \ln(1.4C_3 + 3\cos(\theta)C_5) C_3^{\,3} C_5 .
\end{aligned}
$$

The plot of the potential when F_1 is non-zero is shown in Fig. 5.8. The potential for the inverted pendulum system is plotted in the interval (-1.5,1.5), for x and in the interval (-0.8, 0.8) for θ.

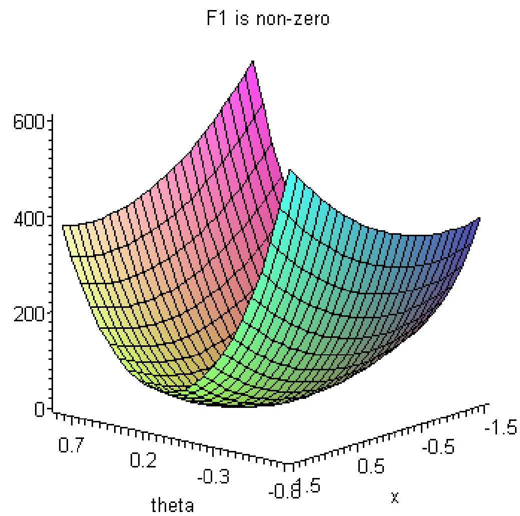

Figure 5.8: The potential

5.8.2 The Hessian

The Hessian at the equilibrium point is

$$\mathbf{H} = \begin{bmatrix} 73.51 & -287.08 \\ -287.08 & 1329.51 \end{bmatrix}, \tag{5.32}$$

and the corresponding eigenvalues of the matrix in Eq. 5.32 are

$$\boldsymbol{\lambda} = \begin{bmatrix} 11.0084 \\ 1392.0189 \end{bmatrix}. \tag{5.33}$$

From Eq. 5.32 and Eq. 5.33 it is noticed that the Hessian is a positive definite matrix.

5.8.4 Simulation considering F_1

In this section, some simulations are presented for the inverted pendulum cart. In this simulation, the same initial conditions are used in order to compare both formulations: when F1 is zero and non-zero. The plots where $\mathbf{F_1}$ is zero and non-zero are presented here. Figure 5.9 shows the plot of the angular position of the pendulum and the displacement of the cart versus time starting from the initial conditions x= 0.2 m, $\dot{x}$=1.5 m/sec, θ=0.8 rad, and $\dot{\theta}$=0. Notice that the control stabilizes the system before 3 seconds for both cases of $\mathbf{F_1}$ being zero and non-zero.

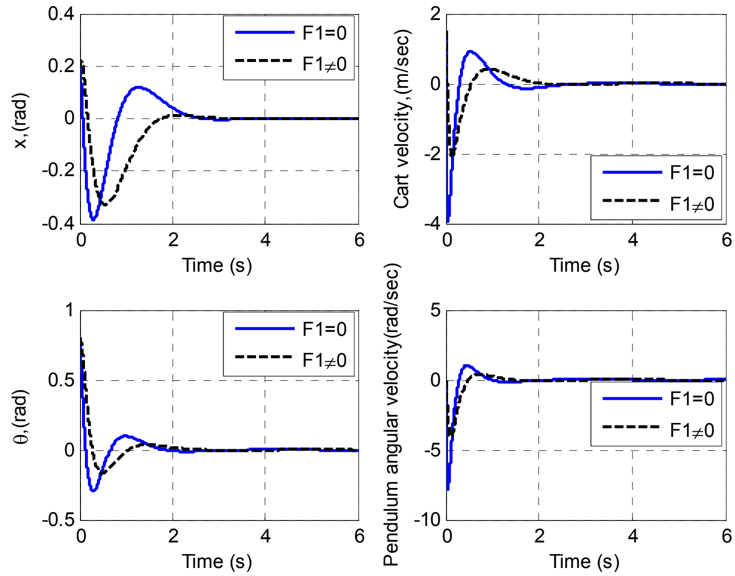

Figure 5.9: Comparison of state with F_1 zero and non-zero

Figure 5.10 plots the simulation results for the elements of the **P** matrix. The difference in the values of the elements of **P** are shown in decreasing order for the elements p_{12}, p_{22}, p_{11}, and p_{21}, however, it is not significant for stabilization.

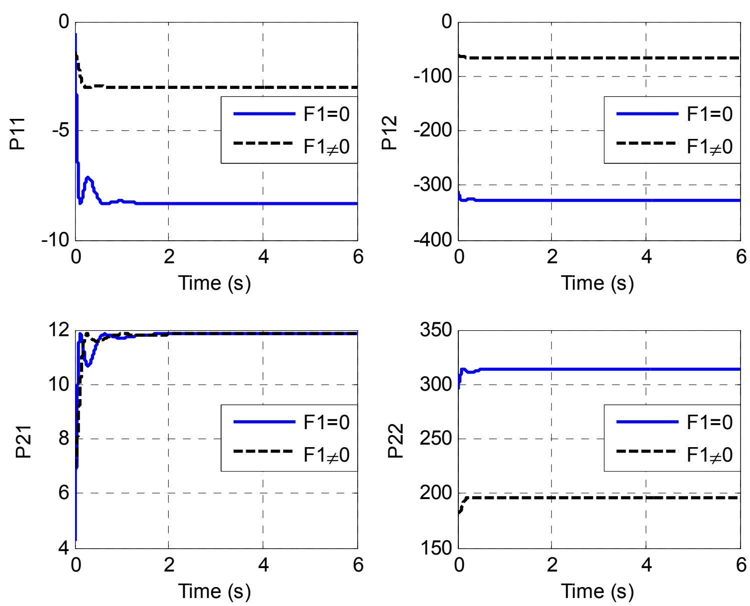

Figure 5.10: P elements Comparison

Figure 5.11 shows that the determinant of the $\mathbf{P}$ matrix is positive. Notice that the determinant for the $\mathbf{F_1}$ non-zero case is larger.

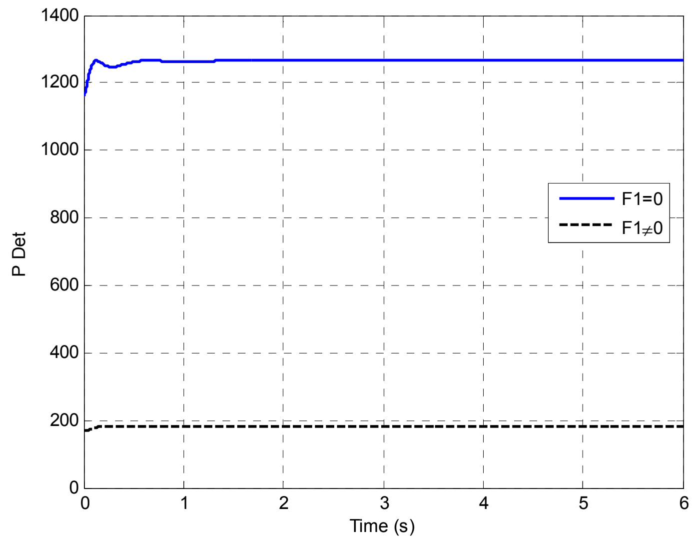

Figure 5.11: Determinant of the P Matrix

Figure 5.12 shows the Lyapunov time history decreasing monotonically. The plot shows a similar performance for the Lyapunov function and the first time derivative for both formulations.

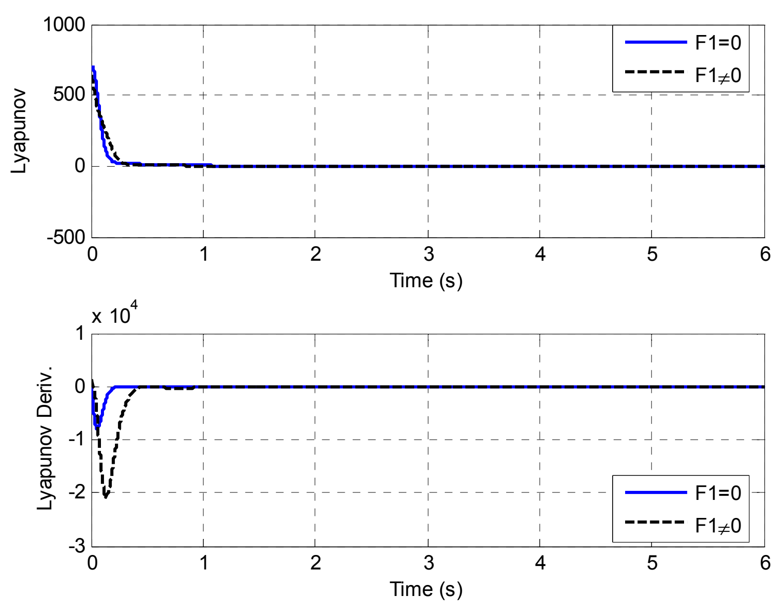

Figure 5.12: Lyapunov Time History and its Time Derivative

Finally, Figure 5.13 shows the control law stabilizes the IPC. In order to compare both controller designs, the control law is plotted for both formulations. The performance of the control law with and without $\mathbf{F_1}$ is practically the same.

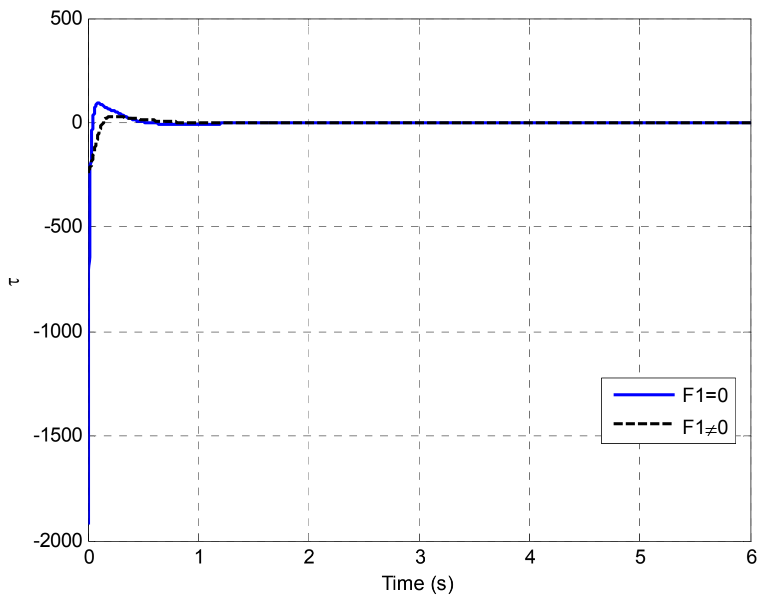

Figure 5.13: Control law

5.9 Chapter Summary

In this chapter a scheme has been applied to the inverted pendulum cart. The control design is based on the direct Lyapunov approach to stabilize the system at the equilibrium point under the constraints of a positive definite $\mathbf{K_D}$, convex potential shape, and Lyapunov time behavior so that the asymptotic stability of the state space origin is obtained. The formulation is presented for two cases, $\mathbf{F_1}$=0 and $\mathbf{F_1} \neq 0$. To carry out this approach, three matching conditions needed to be solved and the parameter values change for each case.

The feasibility of a simple control technique is presented and information of the linearized system is obtained. In making the selection of the parameters, the main difficulty is accomplishing all the required constraints at the same time. After satisfying these constrains, the controller works well. The performance of the controller is very similar for both formulations for the purpose of stabilization. It satisfies the desired objective for different initial conditions.

The simulation results show the advantages of this method with respect to recent studies because the controller is faster, with a large range of initial states possible and simpler, White et al. (2006).

Chapter 6 - Conclusion and Recommendations

6.1 Conclusion

A direct Lyapunov method for underactuated systems has been presented. The matching method, resulting in three equations, was developed to design the nonlinear system controller. The FMC is solved for $\mathbf{K_D}$, $\mathbf{F_1}$ and $\mathbf{Fmc_1}$. This step allows the solution of the SMC for $\mathbf{K}_v$ and $\mathbf{F_2}$ as well as the TMC for $\mathbf{\Phi(q)}$ and $\mathbf{F}_3$. The control law requires that the matrix $\mathbf{K_D}$ is positive definite, the potential has the proper shape, and the Lyapunov candidate function has a non-positive time derivative. Three benchmark examples are given. The controller development details are presented in the Appendices.

The $\mathbf{K_D}$ matrix for the ball and beam is not constant. Finding a non-constant $\mathbf{K_D}$ for the ball and arc and the inverted pendulum cart causes problems as seen in the difficulty of solving the PDE associated with the TMC.

The control coefficient matrix $\mathbf{Fm_1}$ is applied to facilitate the solution of the FMC. For the particular case of the ball and beam, $\mathbf{P}(\mathbf{q})$ is a diagonal matrix when $\mathbf{Fm_1}$ is zero. Also an additional term $\mathbf{Fmc_1\dot{q}}$ is included for the ball and beam application in order to improve the performance of the controller. These terms facilitate the controller development.

From the two formulations presented for the ball and arc, it is seen that by using an almost constant $\mathbf{P}(\mathbf{q})$ matrix a simpler solution method is found. Both formulations appear to be very similar. However, a less complicated potential and a bigger range in regard to initial conditions for θ are possible. The presented controller design procedure is significant because it has produced the first of nonlinear controller for the ball and arc system and it avoids the use of esoteric control formulations.

The controller formulation is presented for the inverted pendulum cart. Two situations are considered: the term $\mathbf{F_1}$ is zero in one formulation and non-zero in the other. For both situations, the performance of the controller is very similar. This development presents a convenient controller design method compared to recent studies because of the ease in solving the matching equations. For the inverted pendulum cart, the addition of $\mathbf{Fmc_1}$ is not significant for stabilization.

In solving the first matching condition when $\mathbf{F_1}$ and $\mathbf{Fmc_1}$ are zero, attempts to find a Lagrangian $\mathbf{K_D}$ were tried for the three examples. It was found that only the inverted pendulum cart had a Lagrangian $\mathbf{K_D}$.

All the simulation results, obtained in this dissertation for the nonlinear systems, validate the effective controller development. The positive definiteness property of $\mathbf{K_D}$, the proper shape of the potential, and the monotonic behavior of the Lyapunov time derivative is demonstrated for the stabilization examples.

6.2 Recommendations

Further study is needed to determine the $\mathbf{K_D}$ matrix and how it will affect the controller design, as well to the determination of the potential so that the basin of attraction can be expanded.

The ball and arc and the inverted pendulum cart examples motivate further studies because the control term $\mathbf{F_1}$ modifies the system performance in some cases. Guidelines for the control determination would be useful. Improvements and new strategies for selecting the parameters for better performance would also be welcome.

Is rather important to formulate and address what is the determining factor for $\mathbf{K_D}$ of some systems to be Lagrangian, i.e. satisfying Eq. 2.25, whereas for other systems not.

Applications to problems involving a larger number of degrees of freedom with the condition of a lack of symmetry or non-holonomic constraints are areas for further investigation.

The method presented consists in a systematic procedure that could be applied to the controller design for other systems which would allow a study of parameter variation.

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Appendix A - Ball and beam system

The presentation of this Appendix is organized into five major sections. These are:

A.1 Dynamics of the ball and beam system

A.2 Lagrangian KD for the ball and beam system, solving Eq.2.25

A.3 Direct Lyapunov Approach formulation for the ball and beam system

A.4 MATLAB code for running the simulations of the B&B system

A.5 Simulink file for the ball and beam system

A.1 Dynamics of the ball and beam system
B&B_dynamics.mw

Lagrangian Approach

The Lagrangian (L) is defined as KE-PE, where KE is the kinetic energy and PE is the potential energy of the system expressed in a minimum set of generalized coordinates. In our case the generalized coordinates were chosen as θ and r.

> restart;

> with(LinearAlgebra) :

>

> $x_c := r(t)\cdot\cos(\theta(t)) - R_o\sin(\theta(t));$

$$x_c := r(t)\cos(\theta(t)) - R_o\sin(\theta(t)) \quad \textbf{(1)}$$

> $y_c := r(t)\cdot\sin(\theta(t)) + R_o\cos(\theta(t));$

$$y_c := r(t)\sin(\theta(t)) + R_o\cos(\theta(t)) \quad \textbf{(2)}$$

> $\mathrm{KE} := \frac{1}{2}\cdot m\cdot\left(diff(x_c, t)^2 + diff(y_c, t)^2\right) + \frac{1}{2}\cdot I_{beam}\cdot diff(\theta(t), t)^2 + \frac{1}{2}\cdot J_{ball}\cdot\left(\frac{-diff(r(t), t)}{R_o} + diff(\theta(t), t)\right)^2;$

$$KE := \frac{1}{2}\, m\left(\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\, r(t)\right)\cos(\theta(t)) - r(t)\sin(\theta(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right) - R_o\cos(\theta(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right)\right)^2 + \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\, r(t)\right)\sin(\theta(t)) + r(t)\cos(\theta(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right) - R_o\sin(\theta(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right)\right)^2\right) + \frac{1}{2}\,\mathrm{I}_{beam}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right)^2 + \frac{1}{2}\, J_{ball}\left(-\frac{\frac{\mathrm{d}}{\mathrm{d}t}\, r(t)}{R_o} + \frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t)\right)^2 \quad \textbf{(3)}$$

>

> $PE := m\cdot g\cdot y;$

$$PE := m\, g\, y \quad \textbf{(4)}$$

> $PE := simplify(eval(PE, [y = y_c]));$

$$PE := m\, g\,(r(t)\sin(\theta(t)) + R_o\cos(\theta(t))) \quad \textbf{(5)}$$

> with(VariationalCalculus)

[ConjugateEquation, Convex, EulerLagrange, Jacobi, Weierstrass] **(6)**

> $L := simplify(\mathrm{KE} - \mathrm{PE});$

$$L := \frac{1}{2}\,\frac{1}{R_o^2}\left(m\,R_o^2\,r(t)^2\left(\frac{d}{dt}\,\theta(t)\right)^2 + m\,R_o^2\left(\frac{d}{dt}\,r(t)\right)^2 - 2\,m\,R_o^3\left(\frac{d}{dt}\,r(t)\right)\left(\frac{d}{dt}\,\theta(t)\right) + m\,R_o^4\left(\frac{d}{dt}\,\theta(t)\right)^2 + I_{beam}\left(\frac{d}{dt}\,\theta(t)\right)^2 R_o^2 + J_{ball}\left(\frac{d}{dt}\,r(t)\right)^2 - 2\,J_{ball}\left(\frac{d}{dt}\,r(t)\right)\left(\frac{d}{dt}\,\theta(t)\right)R_o + J_{ball}\left(\frac{d}{dt}\,\theta(t)\right)^2 R_o^2 - 2\,m\,g\,R_o^2\,r(t)\,\sin(\theta(t)) - 2\,m\,g\,R_o^3\cos(\theta(t))\right) \tag{7}$$

Defining temporary substitutions and variables

> $temp1 := \left[\frac{d^2}{dt^2}\,r(t) = ddr,\ diff(r(t),t) = dr,\ r(t) = r,\ \theta(t) = \theta,\ diff(\theta(t),t) = dtheta,\ diff(\theta(t),t,t) = ddtheta\right];$

$$temp1 := \left[\frac{d^2}{dt^2}\,r(t) = ddr,\ \frac{d}{dt}\,r(t) = dr,\ r(t) = r,\ \theta(t) = \theta,\ \frac{d}{dt}\,\theta(t) = dtheta,\ \frac{d^2}{dt^2}\,\theta(t) = ddtheta\right] \tag{8}$$

> $temp2 := [ddr = diff(r(t),t,t),\ dr = diff(r(t),t),\ r = r(t),\ \theta = \theta(t),\ dtheta = diff(\theta(t),t),\ ddtheta = diff(\theta(t),t,t)];$

$$temp2 := \left[ddr = \frac{d^2}{dt^2}\,r(t),\ dr = \frac{d}{dt}\,r(t),\ r = r(t),\ \theta = \theta(t),\ dtheta = \frac{d}{dt}\,\theta(t),\ ddtheta = \frac{d^2}{dt^2}\,\theta(t)\right] \tag{9}$$

> $variables := [diff(\theta(t),t,t),\ diff(r(t),t,t),\ diff(\theta(t),t),\ diff(r(t),t),\ \theta(t),\ r(t),\ g]:$

$Ltemp := \mathrm{eval}(L, \mathrm{temp1})$

$$\frac{1}{2}\,\frac{1}{R_o^2}\left(m\,R_o^2\,r^2\,dtheta^2 + m\,R_o^2\,dr^2 - 2\,m\,R_o^3\,dr\,dtheta + m\,R_o^4\,dtheta^2 + I_{beam}\,dtheta^2\,R_o^2 + J_{ball}\,dr^2 - 2\,J_{ball}\,dr\,dtheta\,R_o + J_{ball}\,dtheta^2\,R_o^2 - 2\,m\,g\,R_o^2\,r\sin(\theta) - 2\,m\,g\,R_o^3\cos(\theta)\right) \tag{10}$$

$DL[dr] := diff(Ltemp, dr)$

> $Eq1 := simplify(eval(diff(eval(DL[dr], temp2), t), temp1) - diff(Ltemp, r) = 0)$

$$Eq1 := \tag{11}$$

$$\frac{1}{R_o^2}\left(m R_o^2\, ddr - m R_o^3\, ddtheta + J_{ball}\, ddr - J_{ball}\, ddtheta\, R_o - m R_o^2\, r\, dtheta^2 + m\, g\, R_o^2 \sin(\theta)\right) = 0$$

The first governing equation is

```
> Eq1 := collect(simplify(eval(lhs(Eq1) - rhs(Eq1), temp2)), variables)
```

$$Eq1 := \frac{\left(-J_{ball} R_o - m R_o^3\right)\left(\frac{d^2}{dt^2}\theta(t)\right)}{R_o^2} + \frac{\left(m R_o^2 + J_{ball}\right)\left(\frac{d^2}{dt^2} r(t)\right)}{R_o^2} - m\, r(t)\left(\frac{d}{dt}\theta(t)\right)^2 + m\, g \sin(\theta(t)) \tag{12}$$

Following the same procedure for theta

```
> DL[dtheta] := diff(Ltemp, dtheta)
```

$$DL_{dtheta} := \frac{1}{2}\,\frac{1}{R_o^2}\left(2\, m R_o^2 r^2\, dtheta - 2\, m R_o^3\, dr + 2\, m R_o^4\, dtheta + 2\, I_{beam}\, dtheta\, R_o^2 - 2\, J_{ball}\, dr\, R_o + 2\, J_{ball}\, dtheta\, R_o^2\right) \tag{13}$$

```
> Eq2 := simplify(eval(diff(eval(DL[dtheta], temp2), t), temp1) -diff(Ltemp, θ)) = τ
```

$$Eq2 := \frac{1}{R_o}\left(2\, m R_o\, r\, dtheta\, dr + m R_o r^2\, ddtheta - m R_o^2\, ddr + m R_o^3\, ddtheta + I_{beam}\, ddtheta\, R_o - J_{ball}\, ddr + J_{ball}\, ddtheta\, R_o + m\, g\, R_o\, r \cos(\theta) - m\, g\, R_o^2 \sin(\theta)\right) = \tau \tag{14}$$

The second governing equation is

```
> Eq2 := collect(simplify(eval(lhs(Eq2) - rhs(Eq2), temp2)), variables)
```

$$Eq2 := \left(m\, r(t)^2 - \frac{-J_{ball} R_o - m R_o^3 - I_{beam} R_o}{R_o}\right)\left(\frac{d^2}{dt^2}\theta(t)\right) - \frac{\left(m R_o^2 + J_{ball}\right)\left(\frac{d^2}{dt^2} r(t)\right)}{R_o} + 2\, m\, r(t)\left(\frac{d}{dt} r(t)\right)\left(\frac{d}{dt}\theta(t)\right) + m\, g \cos(\theta(t))\, r(t) - m R_o \sin(\theta(t))\, g - \tau \tag{15}$$

```
>
```

(16)

Summarizing then we have the governing dynamic equations with $J_{ball} = \frac{2}{5} \cdot m \cdot R_o^2$ **are**

Equation 1

> $Eq1c := collect\left(eval\left(Eq1, \left[J_{ball} = \frac{2}{5} \cdot m \cdot R_o^2\right]\right), variables\right) = 0$

$$Eq1c := -\frac{7}{5}\, m\, R_o \left(\frac{d^2}{dt^2}\, \theta(t)\right) + \frac{7}{5}\, m \left(\frac{d^2}{dt^2}\, r(t)\right) - m\, r(t) \left(\frac{d}{dt}\, \theta(t)\right)^2 + m\, g \sin(\theta(t)) = 0 \quad (17)$$

Equation 2

> $Eq2c := collect\left(eval\left(Eq2, \left[J_{ball} = \frac{2}{5} \cdot m \cdot R_o^2\right]\right), variables\right) = 0$

$$Eq2c := \left(m\, r(t)^2 - \frac{-\frac{7}{5}\, m\, R_o^3 - I_{beam}\, R_o}{R_o}\right) \left(\frac{d^2}{dt^2}\, \theta(t)\right) - \frac{7}{5}\, m\, R_o \left(\frac{d^2}{dt^2}\, r(t)\right) + 2\, m\, r(t) \left(\frac{d}{dt}\, r(t)\right) \left(\frac{d}{dt}\, \theta(t)\right) + m\, g \cos(\theta(t))\, r(t) - m\, R_o \sin(\theta(t))\, g - \tau = 0 \quad (18)$$

>

> $masst := Matrix\left(2, 2, \left[m\, r(t)^2 + \frac{\frac{7}{5}\, m\, R_o^3 + I_{beam}\, R_o}{R_o}, -\frac{7}{5}\, m\, R_o, -\frac{7}{5}\, m\, R_o, \frac{7}{5}\, m\right]\right):$

> $mass := Matrix\left(2, 2, \left[m\, r^2 + \frac{\frac{7}{5}\, m\, R_o^3 + I_{beam}\, R_o}{R_o}, -\frac{7}{5}\, m\, R_o, -\frac{7}{5}\, m\, R_o, \frac{7}{5}\, m\right]\right):$

> $Mdot := map(diff, masst, t)$

$$Mdot := \begin{bmatrix} 2\, m\, r(t) \left(\frac{d}{dt}\, r(t)\right) & 0 \\ 0 & 0 \end{bmatrix} \quad (19)$$

> $\mathrm{Mdot} := Matrix(2, 2, [2\, \mathrm{m}\, \mathrm{r} \cdot \mathrm{rdot}, 0, 0, 0]);$

$$Mdot := \begin{bmatrix} 2\, m\, r\, rdot & 0 \\ 0 & 0 \end{bmatrix} \quad (20)$$

$$> \; Cmatrix := Matrix(2, 2, [\, m\, r \cdot rdot, m \cdot r \cdot \theta dot, -m \cdot r \cdot \theta dot, 0\,]);$$

$$Cmatrix := \begin{bmatrix} m\, r\, rdot & m\, r\, \theta dot \\ -m\, r\, \theta dot & 0 \end{bmatrix} \tag{21}$$

Cmatrix check

$$\boldsymbol{C}_{ij}(\boldsymbol{q}, \boldsymbol{qdot}) = [\boldsymbol{jk}, \boldsymbol{i}]qdot^{\boldsymbol{k}} = \frac{1}{2}\left[\frac{\partial}{\partial \boldsymbol{q}^{k}}\, m_{\boldsymbol{i}, \boldsymbol{j}} + \frac{\partial}{\partial \boldsymbol{q}^{j}}\, m_{\boldsymbol{k}, \boldsymbol{i}} - \frac{\partial}{\partial \boldsymbol{q}^{i}}\, m_{\boldsymbol{j}, \boldsymbol{k}}\right] qdot^{\boldsymbol{k}}$$

where m_{ij} is the ij^{th} component of the mass matrix and [jk,i] is the Christoffel symbol of the first kind.

$$> \; C11 := \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{1,1} + \frac{\partial}{\partial \theta}\, mass_{1,1} - \frac{\partial}{\partial \theta}\, mass_{1,1}\right) \cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r}\, mass_{1,1} + \frac{\partial}{\partial \theta}\, mass_{2,1} - \frac{\partial}{\partial \theta}\, mass_{1,2}\right) \cdot rdot;$$

$$C11 := m\, r\, rdot \tag{22}$$

$$> \; C12 := \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{2,1} + \frac{\partial}{\partial r}\, mass_{1,1} - \frac{\partial}{\partial \theta}\, mass_{2,1}\right) \cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r}\, mass_{2,1} + \frac{\partial}{\partial r}\, mass_{2,1} - \frac{\partial}{\partial \theta}\, mass_{2,2}\right) \cdot rdot;$$

$$C12 := m\, r\, \theta dot \tag{23}$$

$$> \; C21 := \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{1,2} + \frac{\partial}{\partial \theta}\, mass_{1,2} - \frac{\partial}{\partial r}\, mass_{1,1}\right) \cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r}\, mass_{1,2} + \frac{\partial}{\partial \theta}\, mass_{2,2} - \frac{\partial}{\partial r}\, mass_{1,2}\right) \cdot rdot;$$

$$C21 := -m\, r\, \theta dot \tag{24}$$

$$> \; C22 := \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{2,2} + \frac{\partial}{\partial r}\, mass_{1,2} - \frac{\partial}{\partial r}\, mass_{2,1}\right) \cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r}\, mass_{2,2} + \frac{\partial}{\partial r}\, mass_{2,2} - \frac{\partial}{\partial r}\, mass_{2,2}\right) \cdot rdot;$$

$$C22 := 0 \tag{25}$$

$$> \; Cmatrix := Matrix(2, 2, [C11, C12, C21, C22]);$$

$$Cmatrix := \begin{bmatrix} m\, r\, rdot & m\, r\, \theta dot \\ -m\, r\, \theta dot & 0 \end{bmatrix} \tag{26}$$

Differentiating the mass matrix with respect to time produces

$$> \; \mathrm{Mdot} := simplify\left(eval\left(Mdot, \left[\left(\frac{d^2}{dt^2}\, \theta(t)\right) = \theta\mathrm{ddot}, \theta(t) = \theta, \left(\frac{d^2}{dt^2}\, r(t)\right) = rddot, \left(\frac{d}{dt}\, \theta(t)\right) = xdot, \left(\frac{d}{dt}\, r(t)\right) = rdot, r(t) = r\right]\right)\right);$$

$$Mdot := \begin{bmatrix} 2\, m\, r\, rdot & 0 \\ 0 & 0 \end{bmatrix} \tag{27}$$

Because M is Lagrangian $\frac{1}{2} \cdot (\, Mdot - \; Cmatrix)$ is skew-symmetric. The result is

> $FMCskew := \left(\frac{1}{2} \cdot Mdot - \; Cmatrix \right)$

$$FMCskew := \begin{bmatrix} 0 & -m\, r\, \theta dot \\ m\, r\, \theta dot & 0 \end{bmatrix} \tag{28}$$

This is a special case of the FMC when KD=M, for which

> $FMC := (\, Mdot - Cmatrix - Transpose(Cmatrix) \,);$

$$FMC := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{29}$$

End of the file

A.2 Lagrangian KD_Ball and beam system

LagrangianKD_B&B.mw

Eq. 2.25 Solution - $-\boldsymbol{C}^T\boldsymbol{M}^{-1}\boldsymbol{K}_D + \frac{1}{2}\left(\frac{\partial}{\partial q}\boldsymbol{q}^{\cdot T}\boldsymbol{K}_D\right) = 0$

> restart :

> with(LinearAlgebra) :

Definitions

> $mass := \begin{bmatrix} Ibar + \frac{7}{5}\, m\, R_o^2 + m\, r^2 & -\frac{7}{5}\, m\, R_o \\ -\frac{7}{5}\, m\, R_o & \frac{7}{5}\, m \end{bmatrix};$

$$mass := \begin{bmatrix} Ibar + \frac{7}{5}\, m\, R_o^2 + m\, r^2 & -\frac{7}{5}\, m\, R_o \\ -\frac{7}{5}\, m\, R_o & \frac{7}{5}\, m \end{bmatrix} \quad \textbf{(1.1)}$$

> $q := \begin{bmatrix} \theta \\ r \end{bmatrix};$

$$q := \begin{bmatrix} \theta \\ r \end{bmatrix} \quad \textbf{(1.2)}$$

> $qdot := \begin{bmatrix} \theta dot \\ rdot \end{bmatrix}:$

> $C := Matrix(2, 2, [m\, r \cdot rdot, m\, r \cdot \theta dot, -m\, r \cdot \theta dot, 0]);$

$$C := \begin{bmatrix} m\, r\, rdot & m\, r\, \theta dot \\ -m\, r\, \theta dot & 0 \end{bmatrix} \quad \textbf{(1.3)}$$

> $KD := Matrix(2, 2, [KD11(\theta, r), KD12(\theta, r), KD21(\theta, r), KD22(\theta, r)]);$

$$KD := \begin{bmatrix} KD11(\theta, r) & KD12(\theta, r) \\ KD21(\theta, r) & KD22(\theta, r) \end{bmatrix} \quad \textbf{(1.4)}$$

Term 1

$> KDq := Transpose\left(Multiply\left(\frac{1}{2}\cdot KD, qdot\right)\right);$

$$KDq := \left[\frac{1}{2}\,KD11(\theta, r)\,\theta dot + \frac{1}{2}\,KD12(\theta, r)\,rdot,\ \frac{1}{2}\,KD21(\theta, r)\,\theta dot + \frac{1}{2}\,KD22(\theta, r)\,rdot\right] \tag{2.1}$$

$> Term1 := \begin{bmatrix} diff(KDq[1], \theta) & diff(KDq[2], \theta) \\ diff(KDq[1], r) & diff(KDq[2], r) \end{bmatrix};$

$$Term1 := \left[\left[\frac{1}{2}\left(\frac{\partial}{\partial\theta} KD11(\theta, r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD12(\theta, r)\right) rdot,\ \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD21(\theta, r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD22(\theta, r)\right) rdot\right],\ \left[\frac{1}{2}\left(\frac{\partial}{\partial r} KD11(\theta, r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r} KD12(\theta, r)\right) rdot,\ \frac{1}{2}\left(\frac{\partial}{\partial r} KD21(\theta, r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r} KD22(\theta, r)\right) rdot\right]\right] \tag{2.2}$$

Term 2

$> Term2 := Transpose(Multiply(Multiply(KD, MatrixInverse(mass)), C));$

$$Term2 := \left[\left[\left(\frac{KD11(\theta, r)}{Ibar + m\,r^2} + \frac{KD12(\theta, r)\,R_o}{Ibar + m\,r^2}\right) m\,r\,rdot - \left(\frac{KD11(\theta, r)\,R_o}{Ibar + m\,r^2} + \frac{1}{7}\,\frac{KD12(\theta, r)\,(5\,Ibar + 7\,m\,R_o^2 + 5\,m\,r^2)}{m\,(Ibar + m\,r^2)}\right) m\,r\,\theta dot,\ \left(\frac{KD21(\theta, r)}{Ibar + m\,r^2} + \frac{KD22(\theta, r)\,R_o}{Ibar + m\,r^2}\right) m\,r\,rdot - \left(\frac{KD21(\theta, r)\,R_o}{Ibar + m\,r^2} + \frac{1}{7}\,\frac{KD22(\theta, r)\,(5\,Ibar + 7\,m\,R_o^2 + 5\,m\,r^2)}{m\,(Ibar + m\,r^2)}\right) m\,r\,\theta dot\right],\ \left[\left(\frac{KD11(\theta, r)}{Ibar + m\,r^2} + \frac{KD12(\theta, r)\,R_o}{Ibar + m\,r^2}\right) m\,r\,\theta dot,\ \left(\frac{KD21(\theta, r)}{Ibar + m\,r^2} + \frac{KD22(\theta, r)\,R_o}{Ibar + m\,r^2}\right) m\,r\,\theta dot\right]\right] \tag{3.1}$$

The Eq. 2.25 is expresed as the following Eq. This is the equation to be solved

$> Eq := MatrixAdd(Term1, -Term2);$

$$Eq := \left[\left[\frac{1}{2}\left(\frac{\partial}{\partial\theta} KD11(\theta,r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD12(\theta,r)\right) rdot - \left(\frac{KD11(\theta,r)}{Ibar+m\,r^2} + \frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r\,rdot + \left(\frac{KD11(\theta,r)\,R_o}{Ibar+m\,r^2} + \frac{1}{7}\,\frac{KD12(\theta,r)\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\left(Ibar+m\,r^2\right)}\right) m\,r\,\theta dot, \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD21(\theta,r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\theta} KD22(\theta,r)\right) rdot - \left(\frac{KD21(\theta,r)}{Ibar+m\,r^2} + \frac{KD22(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r\,rdot + \left(\frac{KD21(\theta,r)\,R_o}{Ibar+m\,r^2} + \frac{1}{7}\,\frac{KD22(\theta,r)\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\left(Ibar+m\,r^2\right)}\right) m\,r\,\theta dot\right], \left[\frac{1}{2}\left(\frac{\partial}{\partial r} KD11(\theta,r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r} KD12(\theta,r)\right) rdot - \left(\frac{KD11(\theta,r)}{Ibar+m\,r^2} + \frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r\,\theta dot, \frac{1}{2}\left(\frac{\partial}{\partial r} KD21(\theta,r)\right)\theta dot + \frac{1}{2}\left(\frac{\partial}{\partial r} KD22(\theta,r)\right) rdot - \left(\frac{KD21(\theta,r)}{Ibar+m\,r^2} + \frac{KD22(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r\,\theta dot\right]\right] \quad \textbf{(4.1)}$$

Coefficient Matrices of θdot and $rdot$

> $Eq1 := Matrix(2, map(coeff, Eq, \theta dot));$

$$Eq1 := \left[\left[\frac{1}{2}\,\frac{\partial}{\partial\theta} KD11(\theta,r) + \left(\frac{KD11(\theta,r)\,R_o}{Ibar+m\,r^2} + \frac{1}{7}\,\frac{KD12(\theta,r)\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\left(Ibar+m\,r^2\right)}\right) m\,r, \frac{1}{2}\,\frac{\partial}{\partial\theta} KD21(\theta,r) + \left(\frac{KD21(\theta,r)\,R_o}{Ibar+m\,r^2} + \frac{1}{7}\,\frac{KD22(\theta,r)\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\left(Ibar+m\,r^2\right)}\right) m\,r\right], \left[\frac{1}{2}\,\frac{\partial}{\partial r} KD11(\theta,r) - \left(\frac{KD11(\theta,r)}{Ibar+m\,r^2} + \frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r, \frac{1}{2}\,\frac{\partial}{\partial r} KD21(\theta,r) - \left(\frac{KD21(\theta,r)}{Ibar+m\,r^2} + \frac{KD22(\theta,r)\,R_o}{Ibar+m\,r^2}\right) m\,r\right]\right] \quad \textbf{(5.1)}$$

> $Eq2 := Matrix(2, map(coeff, Eq, rdot));$

$$Eq2 := \left[\left[\frac{1}{2}\frac{\partial}{\partial\theta}KD12(\theta,r) - \left(\frac{KD11(\theta,r)}{Ibar+m\,r^2} + \frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right)m\,r, \frac{1}{2}\frac{\partial}{\partial\theta}KD22(\theta,r) - \left(\frac{KD21(\theta,r)}{Ibar+m\,r^2} + \frac{KD22(\theta,r)\,R_o}{Ibar+m\,r^2}\right)m\,r\right], \left[\frac{1}{2}\frac{\partial}{\partial r}KD12(\theta,r), \frac{1}{2}\frac{\partial}{\partial r}KD22(\theta,r)\right]\right] \quad \textbf{(5.2)}$$

Coefficient Matrices of Derivatives and K_D Elements

> $KD\theta := Matrix(2, map(diff, KD, \theta));$

$$KD\theta := \begin{bmatrix} \frac{\partial}{\partial\theta}KD11(\theta,r) & \frac{\partial}{\partial\theta}KD12(\theta,r) \\ \frac{\partial}{\partial\theta}KD21(\theta,r) & \frac{\partial}{\partial\theta}KD22(\theta,r) \end{bmatrix} \quad \textbf{(6.1)}$$

> $KDr := Matrix(2, map(diff, KD, r));$

$$KDr := \begin{bmatrix} \frac{\partial}{\partial r}KD11(\theta,r) & \frac{\partial}{\partial r}KD12(\theta,r) \\ \frac{\partial}{\partial r}KD21(\theta,r) & \frac{\partial}{\partial r}KD22(\theta,r) \end{bmatrix} \quad \textbf{(6.2)}$$

> $DKD := Matrix(8, 1, [\ ]):$

> $Eqsc := Matrix(8, 1, [\ ]):$

> $LSCM := Matrix(8, [\ ]):$

> **for** i **from** 1 **to** 4 **do**
for j **from** 1 **to** 4 **do**
$DKD_i := KD\theta(i):$
$DKD_{i+4} := KDr(i);$
$Eqsc_i := Eq1(i);$
$Eqsc_{i+4} := Eq2(i);$
$k := i+4;$
$l := j+4;$
$LSCM_{i,j} := coeff(Eq1(i), KD\theta(j));$
$LSCM_{i,l} := coeff(Eq1(i), KDr(j));$
$LSCM_{k,j} := coeff(Eq2(i), KD\theta(j));$
$LSCM_{k,l} := coeff(Eq2(i), KDr(j));$
end do
end do

> $map(eval, LSCM);$

$$
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix} \tag{6.3}
$$

The above matrix contents the coefficients of $\frac{\partial}{\partial \theta} KD$ and $\frac{\partial}{\partial r} KD$

> $map(eval, Eqsc);$

$$
\left[\left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD11(\theta,r)+\left(\frac{KD11(\theta,r)\,R_o}{Ibar+m\,r^2}+\frac{1}{7}\,\frac{KD12(\theta,r)\,\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\,\left(Ibar+m\,r^2\right)}\right)m\,r\right],\right. \tag{6.4}
$$

$$
\left[\frac{1}{2}\,\frac{\partial}{\partial r}\,KD11(\theta,r)-\left(\frac{KD11(\theta,r)}{Ibar+m\,r^2}+\frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right)m\,r\right],
$$

$$
\left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD21(\theta,r)+\left(\frac{KD21(\theta,r)\,R_o}{Ibar+m\,r^2}+\frac{1}{7}\,\frac{KD22(\theta,r)\,\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)}{m\,\left(Ibar+m\,r^2\right)}\right)m\,r\right],
$$

$$
\left[\frac{1}{2}\,\frac{\partial}{\partial r}\,KD21(\theta,r)-\left(\frac{KD21(\theta,r)}{Ibar+m\,r^2}+\frac{KD22(\theta,r)\,R_o}{Ibar+m\,r^2}\right)m\,r\right],
$$

$$
\left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD12(\theta,r)-\left(\frac{KD11(\theta,r)}{Ibar+m\,r^2}+\frac{KD12(\theta,r)\,R_o}{Ibar+m\,r^2}\right)m\,r\right],
$$

$$\left[\frac{1}{2}\ \frac{\partial}{\partial r}\ KD12(\theta, r)\right],$$

$$\left[\frac{1}{2}\ \frac{\partial}{\partial \theta}\ KD22(\theta, r) - \left(\frac{KD21(\theta, r)}{Ibar + m\,r^2} + \frac{KD22(\theta, r)\ R_o}{Ibar + m\,r^2}\right) m\,r\right],$$

$$\left[\frac{1}{2}\ \frac{\partial}{\partial r}\ KD22(\theta, r)\right]\bigg]$$

> $Eqcm := MatrixAdd(Eqsc, Multiply(-LSCM, DKD));$

$$Eqcm := \left[\begin{array}{c} \left(\frac{KD11(\theta, r)\ R_o}{Ibar + m\,r^2} + \frac{1}{7}\ \frac{KD12(\theta, r)\ \left(5\ Ibar + 7\ m\,R_o^2 + 5\ m\,r^2\right)}{m\ \left(Ibar + m\,r^2\right)}\right) m\,r \\ -\left(\frac{KD11(\theta, r)}{Ibar + m\,r^2} + \frac{KD12(\theta, r)\ R_o}{Ibar + m\,r^2}\right) m\,r \\ \left(\frac{KD21(\theta, r)\ R_o}{Ibar + m\,r^2} + \frac{1}{7}\ \frac{KD22(\theta, r)\ \left(5\ Ibar + 7\ m\,R_o^2 + 5\ m\,r^2\right)}{m\ \left(Ibar + m\,r^2\right)}\right) m\,r \\ -\left(\frac{KD21(\theta, r)}{Ibar + m\,r^2} + \frac{KD22(\theta, r)\ R_o}{Ibar + m\,r^2}\right) m\,r \\ -\left(\frac{KD11(\theta, r)}{Ibar + m\,r^2} + \frac{KD12(\theta, r)\ R_o}{Ibar + m\,r^2}\right) m\,r \\ 0 \\ -\left(\frac{KD21(\theta, r)}{Ibar + m\,r^2} + \frac{KD22(\theta, r)\ R_o}{Ibar + m\,r^2}\right) m\,r \\ 0 \end{array}\right] \quad (6.5)$$

> $RSCM := Matrix(8, 4, [\ ]):$

> **for** i **from** 1 **to** 8 **do**
for j **from** 1 **to** 4 **do**
$RSCM_{i, j} := coeff(Eqcm(i), KD(j));$
end do
end do

> $map(eval, RSCM);$

$$\left[\left[\frac{R_o\,m\,r}{Ibar + m\,r^2}, 0, \frac{1}{7}\ \frac{\left(5\ Ibar + 7\ m\,R_o^2 + 5\ m\,r^2\right) r}{Ibar + m\,r^2}, 0\right],\right. \quad (6.6)$$

$$\left[-\frac{m\,r}{Ibar + m\,r^2}, 0, -\frac{R_o\,m\,r}{Ibar + m\,r^2}, 0\right],$$

$$\left[0, \frac{R_o\,m\,r}{Ibar+m\,r^2}, 0, \frac{1}{7}\,\frac{\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)\,r}{Ibar+m\,r^2}\right],$$

$$\left[0, -\frac{m\,r}{Ibar+m\,r^2}, 0, -\frac{R_o\,m\,r}{Ibar+m\,r^2}\right],$$

$$\left[-\frac{m\,r}{Ibar+m\,r^2}, 0, -\frac{R_o\,m\,r}{Ibar+m\,r^2}, 0\right],$$

$$\left[0, 0, 0, 0\right],$$

$$\left[0, -\frac{m\,r}{Ibar+m\,r^2}, 0, -\frac{R_o\,m\,r}{Ibar+m\,r^2}\right],$$

$$\left[0, 0, 0, 0\right]\Big]$$

Extract Coefficient Matrices

> $Derivs := Multiply(MatrixInverse(LSCM), -RSCM);$

$$Derivs := \left[\left[-\frac{2\,R_o\,m\,r}{Ibar+m\,r^2}, 0, -\frac{2}{7}\,\frac{\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)\,r}{Ibar+m\,r^2}, 0\right],\right. \tag{7.1}$$

$$\left[0, -\frac{2\,R_o\,m\,r}{Ibar+m\,r^2}, 0, -\frac{2}{7}\,\frac{\left(5\,Ibar+7\,m\,R_o^2+5\,m\,r^2\right)\,r}{Ibar+m\,r^2}\right],$$

$$\left[\frac{2\,m\,r}{Ibar+m\,r^2}, 0, \frac{2\,R_o\,m\,r}{Ibar+m\,r^2}, 0\right],$$

$$\left[0, \frac{2\,m\,r}{Ibar+m\,r^2}, 0, \frac{2\,R_o\,m\,r}{Ibar+m\,r^2}\right],$$

$$\left[\frac{2\,m\,r}{Ibar+m\,r^2}, 0, \frac{2\,R_o\,m\,r}{Ibar+m\,r^2}, 0\right],$$

$$\left[0, \frac{2\,m\,r}{Ibar+m\,r^2}, 0, \frac{2\,R_o\,m\,r}{Ibar+m\,r^2}\right],$$

$$\left[0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0\right]\Big]$$

> $M\theta := Derivs[[1..4], [1..4]];$

$$M\theta := \left[\left[-\frac{2 R_o m r}{Ibar + m r^2}, 0, -\frac{2}{7}\frac{\left(5\, Ibar + 7 m R_o^2 + 5 m r^2\right) r}{Ibar + m r^2}, 0\right], \left[0, -\frac{2 R_o m r}{Ibar + m r^2}, 0, -\frac{2}{7}\frac{\left(5\, Ibar + 7 m R_o^2 + 5 m r^2\right) r}{Ibar + m r^2}\right], \left[\frac{2 m r}{Ibar + m r^2}, 0, \frac{2 R_o m r}{Ibar + m r^2}, 0\right], \left[0, \frac{2 m r}{Ibar + m r^2}, 0, \frac{2 R_o m r}{Ibar + m r^2}\right]\right] \quad \textbf{(7.2)}$$

$$> KDS := \begin{bmatrix} KD11(\theta, r) \\ KD12(\theta, r) \\ KD21(\theta, r) \\ KD22(\theta, r) \end{bmatrix}:$$

$$> M\theta KD := Multiply(M\theta, KDS);$$

$$M\theta KD := \begin{bmatrix} -\frac{2 R_o m r\, KD11(\theta, r)}{Ibar + m r^2} - \frac{2}{7}\frac{\left(5\, Ibar + 7 m R_o^2 + 5 m r^2\right) r\, KD21(\theta, r)}{Ibar + m r^2} \\ -\frac{2 R_o m r\, KD12(\theta, r)}{Ibar + m r^2} - \frac{2}{7}\frac{\left(5\, Ibar + 7 m R_o^2 + 5 m r^2\right) r\, KD22(\theta, r)}{Ibar + m r^2} \\ \frac{2 m r\, KD11(\theta, r)}{Ibar + m r^2} + \frac{2 R_o m r\, KD21(\theta, r)}{Ibar + m r^2} \\ \frac{2 m r\, KD12(\theta, r)}{Ibar + m r^2} + \frac{2 R_o m r\, KD22(\theta, r)}{Ibar + m r^2} \end{bmatrix} \quad \textbf{(7.3)}$$

$$> Mr := Derivs[[5..8], [1..4]];$$

$$Mr := \begin{bmatrix} \frac{2 m r}{Ibar + m r^2} & 0 & \frac{2 R_o m r}{Ibar + m r^2} & 0 \\ 0 & \frac{2 m r}{Ibar + m r^2} & 0 & \frac{2 R_o m r}{Ibar + m r^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(7.4)}$$

$$> MrKD := Multiply(Mr, KDS);$$

$$MrKD := \begin{bmatrix} \frac{2\, m\, r\, KD11(\theta, r)}{Ibar + m\, r^2} + \frac{2\, R_o\, m\, r\, KD21(\theta, r)}{Ibar + m\, r^2} \\ \frac{2\, m\, r\, KD12(\theta, r)}{Ibar + m\, r^2} + \frac{2\, R_o\, m\, r\, KD22(\theta, r)}{Ibar + m\, r^2} \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(7.5)}$$

Generate Differential Equations

> dKD := simplify(map(diff, MθKD, r) + map(diff, -MrKD, θ));

$$dKD := \Bigg[\Bigg[-\frac{2}{7} \frac{1}{(Ibar + m\, r^2)^2} \Big(-7\, R_o\, m^2\, r^2\, KD11(\theta, r) + 7\, R_o\, m\, KD11(\theta, r)\, Ibar \qquad \textbf{(8.1)}$$

$$+ 7\, R_o\, m\, r \left(\frac{\partial}{\partial r} KD11(\theta, r)\right) Ibar + 7\, R_o\, m^2\, r^3 \left(\frac{\partial}{\partial r} KD11(\theta, r)\right)$$

$$+ 10\, m\, r^2\, KD21(\theta, r)\, Ibar + 5\, m^2\, r^4\, KD21(\theta, r) - 7\, m^2\, r^2\, KD21(\theta, r)\, R_o^2$$

$$+ 5\, KD21(\theta, r)\, Ibar^2 + 7\, KD21(\theta, r)\, m\, R_o^2\, Ibar + 5\, r \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) Ibar^2$$

$$+ 10\, r^3 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) Ibar\, m + 7\, r \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m\, R_o^2\, Ibar$$

$$+ 7\, r^3 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m^2\, R_o^2 + 5\, r^5 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m^2$$

$$+ 7\, m\, r \left(\frac{\partial}{\partial \theta} KD11(\theta, r)\right) Ibar + 7\, m^2\, r^3 \left(\frac{\partial}{\partial \theta} KD11(\theta, r)\right)$$

$$+ 7\, R_o\, m\, r \left(\frac{\partial}{\partial \theta} KD21(\theta, r)\right) Ibar + 7\, R_o\, m^2\, r^3 \left(\frac{\partial}{\partial \theta} KD21(\theta, r)\right)\Big)\Bigg],$$

$$\Bigg[-\frac{2}{7} \frac{1}{(Ibar + m\, r^2)^2} \Big(-7\, R_o\, m^2\, r^2\, KD12(\theta, r) + 7\, R_o\, m\, KD12(\theta, r)\, Ibar$$

$$+ 7\, R_o\, m\, r \left(\frac{\partial}{\partial r} KD12(\theta, r)\right) Ibar + 7\, R_o\, m^2\, r^3 \left(\frac{\partial}{\partial r} KD12(\theta, r)\right)$$

$$+ 10\, m\, r^2\, KD22(\theta, r)\, Ibar + 5\, m^2\, r^4\, KD22(\theta, r) - 7\, m^2\, r^2\, KD22(\theta, r)\, R_o^2$$

$$+ 5\, KD22(\theta, r)\, Ibar^2 + 7\, KD22(\theta, r)\, m\, R_o^2\, Ibar + 5\, r \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) Ibar^2$$

$$+ 10\, r^3 \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) Ibar\, m + 7\, r \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) m\, R_o^2\, Ibar$$

$$+7\,r^3\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)m^2R_o^2+5\,r^5\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)m^2$$
$$+7\,m\,r\left(\frac{\partial}{\partial\theta}\,KD12(\theta,r)\right)Ibar+7\,m^2r^3\left(\frac{\partial}{\partial\theta}\,KD12(\theta,r)\right)$$
$$+7\,R_o\,m\,r\left(\frac{\partial}{\partial\theta}\,KD22(\theta,r)\right)Ibar+7\,R_o\,m^2r^3\left(\frac{\partial}{\partial\theta}\,KD22(\theta,r)\right)\Bigg)\Bigg]\Bigg],$$
$$\Bigg[\frac{1}{(Ibar+m\,r^2)^2}\Bigg(2\,m\Bigg(-m\,r^2\,KD11(\theta,r)+KD11(\theta,r)\,Ibar$$
$$+r\left(\frac{\partial}{\partial r}\,KD11(\theta,r)\right)Ibar+r^3\left(\frac{\partial}{\partial r}\,KD11(\theta,r)\right)m-R_o\,m\,r^2\,KD21(\theta,r)$$
$$+R_o\,KD21(\theta,r)\,Ibar+R_o\,r\left(\frac{\partial}{\partial r}\,KD21(\theta,r)\right)Ibar+R_o\,r^3\left(\frac{\partial}{\partial r}\,KD21(\theta,$$
$$r)\right)m\Bigg)\Bigg)\Bigg],$$
$$\Bigg[\frac{1}{(Ibar+m\,r^2)^2}\Bigg(2\,m\Bigg(-m\,r^2\,KD12(\theta,r)+KD12(\theta,r)\,Ibar$$
$$+r\left(\frac{\partial}{\partial r}\,KD12(\theta,r)\right)Ibar+r^3\left(\frac{\partial}{\partial r}\,KD12(\theta,r)\right)m-R_o\,m\,r^2\,KD22(\theta,r)$$
$$+R_o\,KD22(\theta,r)\,Ibar+R_o\,r\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)Ibar+R_o\,r^3\left(\frac{\partial}{\partial r}\,KD22(\theta,$$
$$r)\right)m\Bigg)\Bigg)\Bigg]\Bigg]$$

> $dKD_2 - dKD_3;$

$$-\frac{2}{7}\,\frac{1}{(Ibar+m\,r^2)^2}\Bigg(-7\,R_o\,m^2r^2\,KD12(\theta,r)+7\,R_o\,m\,KD12(\theta,r)\,Ibar \tag{8.2}$$
$$+7\,R_o\,m\,r\left(\frac{\partial}{\partial r}\,KD12(\theta,r)\right)Ibar+7\,R_o\,m^2r^3\left(\frac{\partial}{\partial r}\,KD12(\theta,r)\right)$$
$$+10\,m\,r^2\,KD22(\theta,r)\,Ibar+5\,m^2r^4\,KD22(\theta,r)-7\,m^2r^2\,KD22(\theta,r)\,R_o^2$$
$$+5\,KD22(\theta,r)\,Ibar^2+7\,KD22(\theta,r)\,m\,R_o^2\,Ibar+5\,r\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)Ibar^2$$
$$+10\,r^3\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)Ibar\,m+7\,r\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)m\,R_o^2\,Ibar$$
$$+7\,r^3\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)m^2R_o^2+5\,r^5\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)m^2$$
$$+7\,m\,r\left(\frac{\partial}{\partial\theta}\,KD12(\theta,r)\right)Ibar+7\,m^2r^3\left(\frac{\partial}{\partial\theta}\,KD12(\theta,r)\right)$$

$$+7 R_o m r \left(\frac{\partial}{\partial \theta} KD22(\theta, r)\right) Ibar + 7 R_o m^2 r^3 \left(\frac{\partial}{\partial \theta} KD22(\theta, r)\right)\Bigg)$$

$$- \frac{1}{(Ibar + m r^2)^2} \Bigg(2 m \Bigg(-m r^2 KD11(\theta, r) + KD11(\theta, r) \; Ibar$$

$$+ r \left(\frac{\partial}{\partial r} KD11(\theta, r)\right) Ibar + r^3 \left(\frac{\partial}{\partial r} KD11(\theta, r)\right) m - R_o m r^2 KD21(\theta, r)$$

$$+ R_o KD21(\theta, r) \; Ibar + R_o r \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) Ibar + R_o r^3 \Bigg(\frac{\partial}{\partial r} KD21(\theta,$$

$$r)\Bigg) m\Bigg)\Bigg)$$

> $deq1 := dKD_1;$

$$deq1 := -\frac{2}{7} \frac{1}{(Ibar + m r^2)^2} \Bigg(-7 R_o m^2 r^2 KD11(\theta, r) + 7 R_o m \, KD11(\theta, r) \; Ibar \tag{8.3}$$

$$+ 7 R_o m r \left(\frac{\partial}{\partial r} KD11(\theta, r)\right) Ibar + 7 R_o m^2 r^3 \left(\frac{\partial}{\partial r} KD11(\theta, r)\right)$$

$$+ 10 m r^2 KD21(\theta, r) \; Ibar + 5 m^2 r^4 KD21(\theta, r) - 7 m^2 r^2 KD21(\theta, r) \; R_o^2$$

$$+ 5 KD21(\theta, r) \; Ibar^2 + 7 KD21(\theta, r) \; m R_o^2 Ibar + 5 r \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) Ibar^2$$

$$+ 10 r^3 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) Ibar \, m + 7 r \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m R_o^2 Ibar$$

$$+ 7 r^3 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m^2 R_o^2 + 5 r^5 \left(\frac{\partial}{\partial r} KD21(\theta, r)\right) m^2$$

$$+ 7 m r \left(\frac{\partial}{\partial \theta} KD11(\theta, r)\right) Ibar + 7 m^2 r^3 \left(\frac{\partial}{\partial \theta} KD11(\theta, r)\right)$$

$$+ 7 R_o m r \left(\frac{\partial}{\partial \theta} KD21(\theta, r)\right) Ibar + 7 R_o m^2 r^3 \left(\frac{\partial}{\partial \theta} KD21(\theta, r)\right)\Bigg)$$

> $deq2 := dKD_2;$

$$deq2 := -\frac{2}{7} \frac{1}{(Ibar + m r^2)^2} \Bigg(-7 R_o m^2 r^2 KD12(\theta, r) + 7 R_o m \, KD12(\theta, r) \; Ibar \tag{8.4}$$

$$+ 7 R_o m r \left(\frac{\partial}{\partial r} KD12(\theta, r)\right) Ibar + 7 R_o m^2 r^3 \left(\frac{\partial}{\partial r} KD12(\theta, r)\right)$$

$$+ 10 m r^2 KD22(\theta, r) \; Ibar + 5 m^2 r^4 KD22(\theta, r) - 7 m^2 r^2 KD22(\theta, r) \; R_o^2$$

$$+ 5 KD22(\theta, r) \; Ibar^2 + 7 KD22(\theta, r) \; m R_o^2 Ibar + 5 r \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) Ibar^2$$

$$+ 10 r^3 \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) Ibar \, m + 7 r \left(\frac{\partial}{\partial r} KD22(\theta, r)\right) m R_o^2 Ibar$$

$$+7r^3\left(\frac{\partial}{\partial r}KD22(\theta,r)\right)m^2R_o^2+5r^5\left(\frac{\partial}{\partial r}KD22(\theta,r)\right)m^2$$
$$+7mr\left(\frac{\partial}{\partial\theta}KD12(\theta,r)\right)Ibar+7m^2r^3\left(\frac{\partial}{\partial\theta}KD12(\theta,r)\right)$$
$$+7R_omr\left(\frac{\partial}{\partial\theta}KD22(\theta,r)\right)Ibar+7R_om^2r^3\left(\frac{\partial}{\partial\theta}KD22(\theta,r)\right)\bigg)$$

> $deq3 := dKD_3;$

$$deq3 := \frac{1}{(Ibar+mr^2)^2}\Bigg(2m\bigg(-mr^2KD11(\theta,r)+KD11(\theta,r)\,Ibar+r\left(\frac{\partial}{\partial r}KD11(\theta,r)\right)Ibar+r^3\left(\frac{\partial}{\partial r}KD11(\theta,r)\right)m-R_omr^2KD21(\theta,r)+R_oKD21(\theta,r)\,Ibar+R_or\left(\frac{\partial}{\partial r}KD21(\theta,r)\right)Ibar+R_or^3\left(\frac{\partial}{\partial r}KD21(\theta,r)\right)m\bigg)\Bigg) \tag{8.5}$$

> $deq4 := dKD_4;$

$$deq4 := \frac{1}{(Ibar+mr^2)^2}\Bigg(2m\bigg(-mr^2KD12(\theta,r)+KD12(\theta,r)\,Ibar+r\left(\frac{\partial}{\partial r}KD12(\theta,r)\right)Ibar+r^3\left(\frac{\partial}{\partial r}KD12(\theta,r)\right)m-R_omr^2KD22(\theta,r)+R_oKD22(\theta,r)\,Ibar+R_or\left(\frac{\partial}{\partial r}KD22(\theta,r)\right)Ibar+R_or^3\left(\frac{\partial}{\partial r}KD22(\theta,r)\right)m\bigg)\Bigg) \tag{8.6}$$

Temporary variables

> $Temp := \left[\frac{\partial}{\partial\theta}KD11(\theta,r), \frac{\partial}{\partial\theta}KD12(\theta,r), \frac{\partial}{\partial\theta}KD21(\theta,r), \frac{\partial}{\partial\theta}KD22(\theta,r), \frac{\partial}{\partial r}KD11(\theta,r), \frac{\partial}{\partial r}KD12(\theta,r), \frac{\partial}{\partial r}KD21(\theta,r), \frac{\partial}{\partial r}KD22(\theta,r), KD11(\theta,r), KD12(\theta,r), KD21(\theta,r), KD22(\theta,r)\right]:$

> $deq1 := collect(deq1, Temp);$

$$deq1 := -\frac{2}{7}\,\frac{(7mr\,Ibar+7m^2r^3)\left(\frac{\partial}{\partial\theta}KD11(\theta,r)\right)}{(Ibar+mr^2)^2} \tag{10.1}$$

$$- \frac{2}{7} \frac{\left(7 R_o m r Ibar + 7 R_o m^2 r^3\right) \left(\frac{\partial}{\partial \theta} KD21(\theta, r)\right)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{\left(7 R_o m r Ibar + 7 R_o m^2 r^3\right) \left(\frac{\partial}{\partial r} KD11(\theta, r)\right)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{1}{\left(Ibar + m r^2\right)^2} \left(\left(10 r^3 Ibar m + 7 r m R_o^2 Ibar + 5 r Ibar^2 + 7 r^3 m^2 R_o^2 + 5 r^5 m^2\right) \left(\frac{\partial}{\partial r} KD21(\theta, r)\right)\right)$$

$$- \frac{2}{7} \frac{\left(-7 R_o m^2 r^2 + 7 R_o m Ibar\right) KD11(\theta, r)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{1}{\left(Ibar + m r^2\right)^2} \left(\left(10 m r^2 Ibar + 5 m^2 r^4 - 7 m^2 r^2 R_o^2 + 5 Ibar^2 + 7 m R_o^2 Ibar\right) KD21(\theta, r)\right)$$

> $deq2 := collect(deq2, Temp);$

$$deq2 := -\frac{2}{7} \frac{\left(7 m r Ibar + 7 m^2 r^3\right) \left(\frac{\partial}{\partial \theta} KD12(\theta, r)\right)}{\left(Ibar + m r^2\right)^2} \tag{10.2}$$

$$- \frac{2}{7} \frac{\left(7 R_o m r Ibar + 7 R_o m^2 r^3\right) \left(\frac{\partial}{\partial \theta} KD22(\theta, r)\right)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{\left(7 R_o m r Ibar + 7 R_o m^2 r^3\right) \left(\frac{\partial}{\partial r} KD12(\theta, r)\right)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{1}{\left(Ibar + m r^2\right)^2} \left(\left(10 r^3 Ibar m + 7 r m R_o^2 Ibar + 5 r Ibar^2 + 7 r^3 m^2 R_o^2 + 5 r^5 m^2\right) \left(\frac{\partial}{\partial r} KD22(\theta, r)\right)\right)$$

$$- \frac{2}{7} \frac{\left(-7 R_o m^2 r^2 + 7 R_o m Ibar\right) KD12(\theta, r)}{\left(Ibar + m r^2\right)^2}$$

$$- \frac{2}{7} \frac{1}{\left(Ibar + m r^2\right)^2} \left(\left(10 m r^2 Ibar + 5 m^2 r^4 - 7 m^2 r^2 R_o^2 + 5 Ibar^2 + 7 m R_o^2 Ibar\right) KD22(\theta, r)\right)$$

> $deq3 := collect(deq3, Temp);$

$$deq3 := \frac{2\, m\, \left(r\, Ibar + r^3\, m\right) \left(\frac{\partial}{\partial r}\, KD11(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(R_o\, r\, Ibar + R_o\, r^3\, m\right) \left(\frac{\partial}{\partial r}\, KD21(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(-m\, r^2 + Ibar\right) KD11(\theta, r)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(-R_o\, m\, r^2 + R_o\, Ibar\right) KD21(\theta, r)}{\left(Ibar + m\, r^2\right)^2} \tag{10.3}$$

> deq4 := collect(deq4, Temp);

$$deq4 := \frac{2\, m\, \left(r\, Ibar + r^3\, m\right) \left(\frac{\partial}{\partial r}\, KD12(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(R_o\, r\, Ibar + R_o\, r^3\, m\right) \left(\frac{\partial}{\partial r}\, KD22(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(-m\, r^2 + Ibar\right) KD12(\theta, r)}{\left(Ibar + m\, r^2\right)^2} + \frac{2\, m\, \left(-R_o\, m\, r^2 + R_o\, Ibar\right) KD22(\theta, r)}{\left(Ibar + m\, r^2\right)^2} \tag{10.4}$$

PDE Solution

> sys4 := [deq1, deq2, deq3, deq4];

$$sys4 := \left[-\frac{2}{7}\, \frac{\left(7\, m\, r\, Ibar + 7\, m^2\, r^3\right) \left(\frac{\partial}{\partial \theta}\, KD11(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} - \frac{2}{7}\, \frac{\left(7\, R_o\, m\, r\, Ibar + 7\, R_o\, m^2\, r^3\right) \left(\frac{\partial}{\partial \theta}\, KD21(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} - \frac{2}{7}\, \frac{\left(7\, R_o\, m\, r\, Ibar + 7\, R_o\, m^2\, r^3\right) \left(\frac{\partial}{\partial r}\, KD11(\theta, r)\right)}{\left(Ibar + m\, r^2\right)^2} - \frac{2}{7}\, \frac{1}{\left(Ibar + m\, r^2\right)^2} \left(\left(10\, r^3\, Ibar\, m + 7\, r\, m\, R_o^2\, Ibar + 5\, r\, Ibar^2 + 7\, r^3\, m^2\, R_o^2 + 5\, r^5\, m^2\right) \left(\frac{\partial}{\partial r}\, KD21(\theta, r)\right) \right) \right. \tag{11.1}$$

$$-\frac{2}{7}\frac{\left(-7 R_o m^2 r^2+7 R_o m\, Ibar\right) KD11(\theta, r)}{\left(Ibar+m r^2\right)^2}$$

$$-\frac{2}{7}\frac{1}{\left(Ibar+m r^2\right)^2}\left(\left(10 m r^2 Ibar+5 m^2 r^4-7 m^2 r^2 R_o^2+5 Ibar^2\right.\right.$$

$$\left.\left.+7 m R_o^2 Ibar\right) KD21(\theta, r)\right), -\frac{2}{7}\frac{\left(7 m r\, Ibar+7 m^2 r^3\right)\left(\frac{\partial}{\partial\theta} KD12(\theta, r)\right)}{\left(Ibar+m r^2\right)^2}$$

$$-\frac{2}{7}\frac{\left(7 R_o m r\, Ibar+7 R_o m^2 r^3\right)\left(\frac{\partial}{\partial\theta} KD22(\theta, r)\right)}{\left(Ibar+m r^2\right)^2}$$

$$-\frac{2}{7}\frac{\left(7 R_o m r\, Ibar+7 R_o m^2 r^3\right)\left(\frac{\partial}{\partial r} KD12(\theta, r)\right)}{\left(Ibar+m r^2\right)^2}$$

$$-\frac{2}{7}\frac{1}{\left(Ibar+m r^2\right)^2}\left(\left(10 r^3 Ibar\, m+7 r m R_o^2 Ibar+5 r\, Ibar^2+7 r^3 m^2 R_o^2\right.\right.$$

$$\left.\left.+5 r^5 m^2\right)\left(\frac{\partial}{\partial r} KD22(\theta, r)\right)\right)$$

$$-\frac{2}{7}\frac{\left(-7 R_o m^2 r^2+7 R_o m\, Ibar\right) KD12(\theta, r)}{\left(Ibar+m r^2\right)^2}$$

$$-\frac{2}{7}\frac{1}{\left(Ibar+m r^2\right)^2}\left(\left(10 m r^2 Ibar+5 m^2 r^4-7 m^2 r^2 R_o^2+5 Ibar^2\right.\right.$$

$$\left.\left.+7 m R_o^2 Ibar\right) KD22(\theta, r)\right), \frac{2 m\left(r\, Ibar+r^3 m\right)\left(\frac{\partial}{\partial r} KD11(\theta, r)\right)}{\left(Ibar+m r^2\right)^2}$$

$$+\frac{2 m\left(R_o r\, Ibar+R_o r^3 m\right)\left(\frac{\partial}{\partial r} KD21(\theta, r)\right)}{\left(Ibar+m r^2\right)^2}$$

$$+\frac{2 m\left(-m r^2+Ibar\right) KD11(\theta, r)}{\left(Ibar+m r^2\right)^2}$$

$$+\frac{2\,m\,\left(-R_o\,m\,r^2+R_o\,Ibar\right)\,KD21(\theta,r)}{\left(Ibar+m\,r^2\right)^2},$$

$$\frac{2\,m\,\left(r\,Ibar+r^3\,m\right)\,\left(\frac{\partial}{\partial r}\,KD12(\theta,r)\right)}{\left(Ibar+m\,r^2\right)^2}$$

$$+\frac{2\,m\,\left(R_o\,r\,Ibar+R_o\,r^3\,m\right)\,\left(\frac{\partial}{\partial r}\,KD22(\theta,r)\right)}{\left(Ibar+m\,r^2\right)^2}$$

$$+\frac{2\,m\,\left(-m\,r^2+Ibar\right)\,KD12(\theta,r)}{\left(Ibar+m\,r^2\right)^2}+\frac{2\,m\,\left(-R_o\,m\,r^2+R_o\,Ibar\right)\,KD22(\theta,r)}{\left(Ibar+m\,r^2\right)^2}\Bigg]$$

> $sol4 := pdsolve(sys4, [KD11(\theta, r), KD12(\theta, r), KD21(\theta, r), KD22(\theta, r)]);$

$$sol4 := \left\{KD11(\theta,r)=\frac{\left(Ibar+m\,r^2\right)\,\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r}\right.$$

$$-\frac{\left(\_F3(\theta)\,r+\_F4(\theta)\right)\,R_o}{r},\,KD12(\theta,r)$$

$$=\frac{\left(Ibar+m\,r^2\right)\,\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)}{r}$$

$$-\frac{\left(\_F1(\theta)\,r+\_F2(\theta)\right)\,R_o}{r},\,KD21(\theta,r)=\_F3(\theta)+\frac{\_F4(\theta)}{r},\,KD22(\theta,r)$$

$$\left.=\_F1(\theta)+\frac{\_F2(\theta)}{r}\right\} \tag{11.2}$$

> $pdetest(sol4, sys4);$

$$[0, 0, 0, 0] \tag{11.3}$$

> $KDF := \begin{bmatrix} rhs(sol4_1) & rhs(sol4_2) \\ rhs(sol4_3) & rhs(sol4_4) \end{bmatrix};$

$$KDF := \left[\left[\frac{\left(Ibar+m\,r^2\right)\,\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r}\right.\right. \tag{11.4}$$

$$-\frac{(\_F3(\theta)\ r+\_F4(\theta))\ R_o}{r},$$

$$\frac{(Ibar+m\ r^2)\left(\int\left(-\frac{5}{7}\ \frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)}{r}-\frac{(\_F1(\theta)\ r+\_F2(\theta))\ R_o}{r}$$

$$\Bigg],$$

$$\left[\_F3(\theta)+\frac{\_F4(\theta)}{r},\ \_F1(\theta)+\frac{\_F2(\theta)}{r}\right]\Bigg]$$

$$> KDF := \Bigg[\Bigg[\frac{(Ibar+m\ r^2)\left(\int\left(-\frac{5}{7}\ \frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r}$$

$$-\frac{(\_F3(\theta)\ r+\_F4(\theta))\ R_o}{r},$$

$$\frac{(Ibar+m\ r^2)\left(\int\left(-\frac{5}{7}\ \frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)}{r}$$

$$-\frac{(\_F1(\theta)\ r+\_F2(\theta))\ R_o}{r}\Bigg],$$

$$\left[\_F3(\theta)+\frac{\_F4(\theta)}{r},\ \_F1(\theta)+\frac{\_F2(\theta)}{r}\right]\Bigg];$$

$$KDF := \Bigg[\Bigg[\frac{(Ibar+m\ r^2)\left(\int\left(-\frac{5}{7}\ \frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r} \qquad \textbf{(11.5)}$$

$$-\frac{(\_F3(\theta)\ r+\_F4(\theta))\ R_o}{r},$$

$$\frac{(Ibar+m\ r^2)\left(\int\left(-\frac{5}{7}\ \frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)}{r}-\frac{(\_F1(\theta)\ r+\_F2(\theta))\ R_o}{r}$$

$$\Bigg],$$

$$\left[\_F3(\theta) + \frac{\_F4(\theta)}{r}, \_F1(\theta) + \frac{\_F2(\theta)}{r}\right]\Bigg]$$

> $KDFdot := Matrix(map(diff, KDF, \theta)) \cdot \theta dot + Matrix(map(diff, KDF, r)) \cdot rdot;$

$$KDFdot := \Bigg[\Bigg[\theta dot\left(-\frac{5}{7}\frac{(Ibar + m\,r^2)\,\_F3(\theta)}{r\,m} - \frac{\left(\left(\frac{d}{d\theta}\_F3(\theta)\right)r + \frac{d}{d\theta}\_F4(\theta)\right)R_o}{r}\right) + rdot\Bigg(2\,m\left(\int\left(-\frac{5}{7}\frac{\_F3(\theta)}{m}\right)d\theta + \_C2\right) - \frac{(Ibar + m\,r^2)\left(\int\left(-\frac{5}{7}\frac{\_F3(\theta)}{m}\right)d\theta + \_C2\right)}{r^2} - \frac{\_F3(\theta)\,R_o}{r} + \frac{(\_F3(\theta)\,r + \_F4(\theta))\,R_o}{r^2}\Bigg), \theta dot\left(-\frac{5}{7}\frac{(Ibar + m\,r^2)\,\_F1(\theta)}{r\,m} - \frac{\left(\left(\frac{d}{d\theta}\_F1(\theta)\right)r + \frac{d}{d\theta}\_F2(\theta)\right)R_o}{r}\right) + rdot\Bigg(2\,m\left(\int\left(-\frac{5}{7}\frac{\_F1(\theta)}{m}\right)d\theta + \_C1\right) - \frac{(Ibar + m\,r^2)\left(\int\left(-\frac{5}{7}\frac{\_F1(\theta)}{m}\right)d\theta + \_C1\right)}{r^2} - \frac{\_F1(\theta)\,R_o}{r} + \frac{(\_F1(\theta)\,r + \_F2(\theta))\,R_o}{r^2}\Bigg)\Bigg], \tag{12.1}$$

$$\left[\theta dot\left(\frac{d}{d\theta}\_F3(\theta) + \frac{\frac{d}{d\theta}\_F4(\theta)}{r}\right) - \frac{rdot\,\_F4(\theta)}{r^2}, \theta dot\left(\frac{d}{d\theta}\_F1(\theta)\right.\right.$$

$$+\frac{\frac{d}{d\theta}\_F2(\theta)}{r}\Bigg) - \frac{rdot\ \_F2(\theta)}{r^2}\Bigg]\Bigg]$$

> $QKD := Transpose(Multiply(KDF, qdot));$

$$QKD := \Bigg[\Bigg(\frac{(Ibar + m\,r^2)\left(\int\left(-\frac{5}{7}\frac{\_F3(\theta)}{m}\right)d\theta + \_C2\right)}{r} - \frac{(\_F3(\theta)\,r + \_F4(\theta))\,R_o}{r}\Bigg)\theta dot + \Bigg(\frac{(Ibar + m\,r^2)\left(\int\left(-\frac{5}{7}\frac{\_F1(\theta)}{m}\right)d\theta + \_C1\right)}{r} - \frac{(\_F1(\theta)\,r + \_F2(\theta))\,R_o}{r}\Bigg)rdot, \left(\_F3(\theta) + \frac{\_F4(\theta)}{r}\right)\theta dot + \left(\_F1(\theta) + \frac{\_F2(\theta)}{r}\right)rdot\Bigg] \quad \textbf{(12.2)}$$

The differentiation of $\mathbf{q}^T\mathbf{K}_D$ **with respect to q is going to be called** ***dKDIq***

> $dKDIq := Matrix(2, 2, [\mathrm{diff}(QKD_1, \theta), \mathrm{diff}(QKD_2, \theta), \mathrm{diff}(QKD_1, r), diff(QKD_2, r)]);$

$$dKDIq := \Bigg[\Bigg[\theta dot\Bigg(-\frac{5}{7}\frac{(Ibar + m\,r^2)\,\_F3(\theta)}{r\,m} - \frac{\left(\left(\frac{d}{d\theta}\_F3(\theta)\right)r + \frac{d}{d\theta}\_F4(\theta)\right)R_o}{r}\Bigg) + \Bigg(-\frac{5}{7}\frac{(Ibar + m\,r^2)\,\_F1(\theta)}{r\,m} - \frac{\left(\left(\frac{d}{d\theta}\_F1(\theta)\right)r + \frac{d}{d\theta}\_F2(\theta)\right)R_o}{r}\Bigg)rdot, \theta dot\Bigg(\frac{d}{d\theta}\_F3(\theta) + \frac{\frac{d}{d\theta}\_F4(\theta)}{r}\Bigg) + \Bigg(\frac{d}{d\theta}\_F1(\theta) + \frac{\frac{d}{d\theta}\_F2(\theta)}{r}\Bigg)rdot\Bigg], \quad \textbf{(12.3)}$$

$$\left[\left[\left(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)\right.\right.\right.$$

$$-\frac{(Ibar+m\,r^2)\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r^2}-\frac{\_F3(\theta)\,R_o}{r}$$

$$\left.+\frac{(\_F3(\theta)\,r+\_F4(\theta))\,R_o}{r^2}\right)\theta dot+rdot\left(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta\right.\right.$$

$$\left.+\_C1\right)-\frac{(Ibar+m\,r^2)\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)}{r^2}-\frac{\_F1(\theta)\,R_o}{r}$$

$$\left.\left.\left.+\frac{(\_F1(\theta)\,r+\_F2(\theta))\,R_o}{r^2}\right),-\frac{\_F4(\theta)\,\theta dot}{r^2}-\frac{rdot\,\_F2(\theta)}{r^2}\right]\right]$$

> $Ltest := KDFdot-\frac{1}{2}\cdot Transpose(dKDIq)-\frac{1}{2}\cdot dKDIq;$

$$Ltest := \left[\left[rdot\left(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)\right.\right.\right. \tag{12.4}$$

$$-\frac{(Ibar+m\,r^2)\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\mathrm{d}\theta+\_C2\right)}{r^2}-\frac{\_F3(\theta)\,R_o}{r}$$

$$\left.+\frac{(\_F3(\theta)\,r+\_F4(\theta))\,R_o}{r^2}\right)-\left(-\frac{5}{7}\,\frac{(Ibar+m\,r^2)\,\_F1(\theta)}{r\,m}\right.$$

$$\left.-\frac{\left(\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\,\_F1(\theta)\right)r+\frac{\mathrm{d}}{\mathrm{d}\theta}\,\_F2(\theta)\right)R_o}{r}\right)rdot,\ \theta dot\left(\right.$$

$$\left.-\frac{5}{7}\,\frac{(Ibar+m\,r^2)\,\_F1(\theta)}{r\,m}-\frac{\left(\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\,\_F1(\theta)\right)r+\frac{\mathrm{d}}{\mathrm{d}\theta}\,\_F2(\theta)\right)R_o}{r}\right)$$

$$+\frac{1}{2}\,rdot\left(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)\mathrm{d}\theta+\_C1\right)\right.$$

$$-\frac{\left(Ibar+m\,r^{2}\right)\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)d\theta+\_C1\right)}{r^{2}}-\frac{\_F1(\theta)\,R_{o}}{r}$$

$$+\frac{\left(\_F1(\theta)\,r+\_F2(\theta)\right)R_{o}}{r^{2}}\Bigg)-\frac{1}{2}\,\Bigg(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)d\theta+\_C2\right)$$

$$-\frac{\left(Ibar+m\,r^{2}\right)\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)d\theta+\_C2\right)}{r^{2}}-\frac{\_F3(\theta)\,R_{o}}{r}$$

$$+\frac{\left(\_F3(\theta)\,r+\_F4(\theta)\right)R_{o}}{r^{2}}\Bigg)\,\theta dot-\frac{1}{2}\,\theta dot\,\Bigg(\frac{d}{d\theta}\,\_F3(\theta)$$

$$+\frac{\frac{d}{d\theta}\,\_F4(\theta)}{r}\Bigg)-\frac{1}{2}\left(\frac{d}{d\theta}\,\_F1(\theta)+\frac{\frac{d}{d\theta}\,\_F2(\theta)}{r}\right)rdot\Bigg],$$

$$\Bigg[\frac{1}{2}\,\theta dot\left(\frac{d}{d\theta}\,\_F3(\theta)+\frac{\frac{d}{d\theta}\,\_F4(\theta)}{r}\right)-\frac{rdot\,\_F4(\theta)}{r^{2}}$$

$$-\frac{1}{2}\left(\frac{d}{d\theta}\,\_F1(\theta)+\frac{\frac{d}{d\theta}\,\_F2(\theta)}{r}\right)rdot-\frac{1}{2}\,\Bigg(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)\right.$$

$$\left.d\theta+\_C2\right)-\frac{\left(Ibar+m\,r^{2}\right)\left(\int\left(-\frac{5}{7}\,\frac{\_F3(\theta)}{m}\right)d\theta+\_C2\right)}{r^{2}}-\frac{\_F3(\theta)\,R_{o}}{r}$$

$$+\frac{\left(\_F3(\theta)\,r+\_F4(\theta)\right)R_{o}}{r^{2}}\Bigg)\,\theta dot-\frac{1}{2}\,rdot\,\Bigg(2\,m\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)d\theta\right.$$

$$\left.+\_C1\right)-\frac{\left(Ibar+m\,r^{2}\right)\left(\int\left(-\frac{5}{7}\,\frac{\_F1(\theta)}{m}\right)d\theta+\_C1\right)}{r^{2}}-\frac{\_F1(\theta)\,R_{o}}{r}$$

$$+\frac{\left(\_F1(\theta)\,r+\_F2(\theta)\right)R_{o}}{r^{2}}\Bigg),\ \theta dot\left(\frac{d}{d\theta}\,\_F1(\theta)+\frac{\frac{d}{d\theta}\,\_F2(\theta)}{r}\right)$$

$$+\frac{\_F4(\theta)\ \theta dot}{r^2}\Bigg]\Bigg]$$

> Ltest := simplify(eval(Ltest, [_F2(θ) = F2, _F1(θ) = F1, _F4(θ) = 0, _F3(θ) = 0]));

$$Ltest := \left[\left[\frac{1}{7}\ \frac{rdot\ (7\ m^2\ r^2\ \_C2 - 7\ Ibar\ \_C2\ m + 5\ r\ F1\ Ibar + 5\ F1\ m\ r^3)}{m\ r^2},\right.\right. \tag{12.5}$$

$$\frac{1}{14}\ \frac{1}{m\ r^2}\Big(-10\ \theta dot\ r\ F1\ Ibar - 10\ \theta dot\ r^3\ F1\ m - 5\ rdot\ F1\ \left(\int 1\, d\theta\right) m\ r^2$$

$$+ 7\ rdot\ \_C1\ m^2\ r^2 + 5\ rdot\ F1\ \left(\int 1\, d\theta\right) Ibar - 7\ rdot\ \_C1\ m\ Ibar$$

$$\left. + 7\ rdot\ m\ R_o\ F2 - 7\ \theta dot\ m^2\ r^2\ \_C2 + 7\ \theta dot\ Ibar\ \_C2\ m\Big)\right],$$

$$\Big[-\frac{1}{14}\ \frac{1}{m\ r^2}\Big(7\ \theta dot\ m^2\ r^2\ \_C2 - 7\ \theta dot\ Ibar\ \_C2\ m - 5\ rdot\ F1\ \left(\int 1\right.$$

$$\left. d\theta\right) m\ r^2 + 7\ rdot\ \_C1\ m^2\ r^2 + 5\ rdot\ F1\ \left(\int 1\, d\theta\right) Ibar - 7\ rdot\ \_C1\ m\ Ibar$$

$$\left.\left. + 7\ rdot\ m\ R_o\ F2\Big), 0\right]\right]$$

> solve(Ltest[1, 1])

$$\{F1 = F1, Ibar = Ibar, \_C2 = \_C2, m = m, r = r, rdot = 0\}, \Bigg\{F1 \tag{12.6}$$

$$= \frac{7}{5}\ \frac{\_C2\ m\ (-m\ r^2 + Ibar)}{(Ibar + m\ r^2)\ r}, Ibar = Ibar, \_C2 = \_C2, m = m, r = r, rdot = rdot\Bigg\}$$

$$, \{F1 = F1, Ibar = -m\ r^2, \_C2 = 0, m = m, r = r, rdot = rdot\}$$

> Ltest := simplify(eval(Ltest, [F1 = 0, _C2 = 0]));

$$Ltest := \left[\left[0, \frac{1}{2}\ \frac{rdot\ (\_C1\ m\ r^2 - \_C1\ Ibar + R_o\ F2)}{r^2}\right],\right. \tag{12.7}$$

$$\left.\left[-\frac{1}{2}\ \frac{rdot\ (\_C1\ m\ r^2 - \_C1\ Ibar + R_o\ F2)}{r^2}, 0\right]\right]$$

> Ltest := simplify(eval(Ltest, [_C1 = 0, F2 = 0]));

$$Ltest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.8}$$

> KDtest := simplify(eval(KDF, [_F2(θ) = F2, _F1(θ) = F1, _F4(θ) = 0, _F3(θ) = 0]));

(12.9)

$$KDtest := \left[\left[\frac{\_C2\,(Ibar + m\,r^2)}{r}, -\frac{1}{7}\,\frac{1}{r\,m}\Big(5\,Ibar\,F1\,\left(\int 1\,\mathrm{d}\theta\right) - 7\,Ibar\,m\,\_C1 + 5\,m\,r^2\,F1\,\left(\int 1\,\mathrm{d}\theta\right) - 7\,m^2\,r^2\,\_C1 + 7\,F1\,R_o\,m\,r + 7\,m\,R_o\,F2\Big)\right], \left[0, \frac{F1\,r + F2}{r}\right]\right] \tag{12.9}$$

> $KDtest := simplify(eval(KDtest, [F1 = 0, \_C2 = 0]));$

$$KDtest := \begin{bmatrix} 0 & -\dfrac{-\_C1\,Ibar - \_C1\,m\,r^2 + R_o\,F2}{r} \\ 0 & \dfrac{F2}{r} \end{bmatrix} \tag{12.10}$$

> $KDtest := simplify(eval(\ KDtest, [\_C1 = 0, \mathrm{F2} = 0]));$

$$KDtest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.11}$$

As a conclusion KD is not Lagrangian* for *the ball* and *beam system.

A.3 Direct Lyapunov Aproach for the B&B System

B&B_DLA_Fm1.mw

> *restart* :

> *with*(*LinearAlgebra*) :

Definitions

q is a vector of generalized coordinates

> $q := \begin{bmatrix} \theta \\ \mathrm{r} \end{bmatrix}$

$$q := \begin{bmatrix} \theta \\ r \end{bmatrix} \tag{1.1}$$

qdot is a vector of generalized velocites

> $qdot := \begin{bmatrix} \theta dot \\ rdot \end{bmatrix}$:

Centripetal **and** ***coriolis matrix***

> $C := \begin{bmatrix} m \cdot r \cdot rdot & m \cdot r \cdot \theta dot \\ -m\, r \cdot \theta dot & 0 \end{bmatrix}$:

Define the mass matrix

> $mass := Matrix\left(2, 2, \left[Ib + \frac{7}{5} \cdot m \cdot Ro^2 + m \cdot r^2, -\frac{7}{5} \cdot m \cdot Ro, -\frac{7}{5} \cdot m \cdot Ro, \frac{7}{5} \cdot m\right]\right)$;

$$mass := \begin{bmatrix} Ib + \frac{7}{5}\, m\, Ro^2 + m\, r^2 & -\frac{7}{5}\, m\, Ro \\ -\frac{7}{5}\, m\, Ro & \frac{7}{5}\, m \end{bmatrix} \tag{1.2}$$

Gravitational and Forces vector

> $G := \begin{bmatrix} r\, m\, g \cos(\theta) - Ro\, m\, g \sin(\theta) \\ m\, g \sin(\theta) \end{bmatrix}$:

Because more parameters are needed in order to place the poles in desired location ν and σ are added to *Fm1*

> $Fm1 := \begin{bmatrix} -F11(\theta, r) \cdot \mathrm{rdot} - \nu & F11(\theta, r) \cdot \theta\mathrm{dot} + \sigma \\ -F22(\theta, r) \cdot \mathrm{rdot} & F22(\theta, r) \cdot \theta dot \end{bmatrix}$:

This matrix is added in order to *satisfy FMC*

> $Fmc1 := \begin{bmatrix} F33(qf, qdotf) & F44(qf, qdotf) \\ F44(qf, qdotf) & F55(qf, qdotf) \end{bmatrix}$:

First Matching Condition

In this section the linear partial differential equations that determine the elements of the K_D matrix will be determined.

Starting form of the matrix K_D. K_D is not a constant

> $KDT := \begin{bmatrix} KD11(\theta, r) & KD21(\theta, r) \\ KD21(\theta, r) & KD22(\theta, r) \end{bmatrix}$:

The time derivative of KD is

> $KDdotT := \mathrm{simplify}(map(diff, \theta dot \cdot KDT, \theta) + map(diff, rdot \cdot KDT, r));$

$$KDdotT := \left[\left[\theta dot\left(\frac{\partial}{\partial\theta} KD11(\theta, r)\right) + rdot\left(\frac{\partial}{\partial r} KD11(\theta, r)\right), \theta dot\left(\frac{\partial}{\partial\theta} KD21(\theta, r)\right) + rdot\left(\frac{\partial}{\partial r} KD21(\theta, r)\right)\right], \left[\theta dot\left(\frac{\partial}{\partial\theta} KD21(\theta, r)\right) + rdot\left(\frac{\partial}{\partial r} KD21(\theta, r)\right), \theta dot\left(\frac{\partial}{\partial\theta} KD22(\theta, r)\right) + rdot\left(\frac{\partial}{\partial r} KD22(\theta, r)\right)\right]\right] \quad \textbf{(2.1)}$$

First Matching Condition with inputs Fm1 and matrix Fmc1

> $FMCsim := simplify(\mathrm{KDdotT} + \mathrm{Multiply}(KDT, Multiply(MatrixInverse(mass), (Fm1 - C))) + Transpose(\mathrm{Multiply}(KDT, Multiply(MatrixInverse(mass), (Fm1 - C)))) + Fmc1)$:

Symmetry proof

> $SymTest := FMCsim_{1,2} - FMCsim_{2,1};$

$$SymTest := 0 \quad \textbf{(2.1.1)}$$

> $mth := simplify(Matrix(2, 2, [coeff(FMCsim_{1,1}, \theta dot), coeff(FMCsim_{1,2}, \theta dot), coeff(FMCsim_{2,1}, \theta dot), coeff(FMCsim_{2,2}, \theta dot)]))$:

> $\mathrm{mR} := \mathrm{simplify}(\mathrm{Matrix}(2, 2, [coeff(FMCsim_{1,1}, rdot), coeff(FMCsim_{1,2}, rdot), coeff(FMCsim_{2,1}, rdot), coeff(FMCsim_{2,2}, rdot)]))$:

> $sys := simplify([mR_{1,1}, mR_{2,1}, mR_{2,2}, mth_{1,1}, mth_{2,1}, mth_{2,2}])$:

Solving for the KD elements and orthogonal control inputs in sys the result is going to be called *Forsol*

> $Forsol := pdsolve(sys, [F11(\theta, r), F22(\theta, r), KD22(\theta, r), \mathrm{KD21}(\theta, \mathrm{r}),$

$KD11(\theta, r)\,]);$

$$Forsol := \{F11(\theta,r) = F11(\theta,r), F22(\theta,r) = F22(\theta,r), KD11(\theta,r) = 0, KD21(\theta,r) = 0, KD22(\theta,r) = 0\}, \{F11(\theta,r) = 0, F22(\theta,r) = 0, KD11(\theta,r) = \_C1\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2), KD21(\theta,r) = -7\,m\,\_C1\,Ro, KD22(\theta,r) = 7\,m\,\_C1\}, \left\{F11(\theta,r) = m\,r, F22(\theta,r) = F22(\theta,r), KD11(\theta,r) = \_C1\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)^2, KD21(\theta,r) = -7\,m\,\_C1\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)\,Ro, KD22(\theta,r) = \frac{49}{25}\,\frac{m^2\,Ro^2\,\_C1\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)^2}{\left(\left(\frac{7}{5}\,Ro^2 + r^2\right)m + Ib\right)^2}\right\}, \left\{F11(\theta,r) = \left(25\,m\,\_C2\left(\left(\frac{7}{5}\,Ro^2 + r^2\right)m + Ib\right)^2 r\right) \Big/ \left((5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)\,(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2)\right), F22(\theta,r) = -\left(175\,m^2\,\_C2\left(\left(\frac{7}{5}\,Ro^2 + r^2\right)m + Ib\right)^2 r\,Ro\right) \Big/ \left((5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)^2\,(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2)\right), KD11(\theta,r) = \_C1\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2) + \_C2\,(5\,Ib + 7\,m\,Ro^2 + 5\,m\,r^2)^2, KD21(\theta,r) = -35\,m\left(\left(\frac{7}{5}\,Ro^2 + r^2\right)\_C2\,m + \frac{1}{5}\,\_C1 + \_C2\,Ib\right)Ro, KD22(\theta,r) = 49\,m\left(\_C2\,m\,Ro^2 + \frac{1}{7}\,\_C1\right)\right\} \quad (2.1.2)$$

Soluition 4 is selected to avoid KD being a multiple of the mass matrix.
The result *when substituing Forsol* in FMC is going to be called *FMCtest*

> $FMCTest := simplify(eval(FMCsim, [Forsol_{4,1}, Forsol_{4,2}, Forsol_{4,3}, Forsol_{4,4}, Forsol_{4,5}]));$

$$FMCTest := [[-50\,\nu\,\_C2\,Ib - 10\,\nu\,\_C1 + F33(qf, qdotf) - 70\,\nu\,\_C2\,m\,Ro^2 - 50\,\nu\,\_C2\,m\,r^2, 25\,\sigma\,\_C2\,Ib + 25\,\_C2\,\sigma\,m\,r^2 + 35\,\_C2\,m\,Ro\,\nu + 35\,\_C2\,m\,Ro^2\,\sigma + 5\,\sigma\,\_C1 + F44(qf, qdotf)], [25\,\sigma\,\_C2\,Ib + 25\,\_C2\,\sigma\,m\,r^2 + 35\,\_C2\,m\,Ro\,\nu + 35\,\_C2\,m\,Ro^2\,\sigma + 5\,\sigma\,\_C1 + F44(qf, qdotf), -70\,m\,Ro\,\sigma\,\_C2 + F55(qf, qdotf)]] \quad \textbf{(2.1.3)}$$

Substituing the solution *Forsol* in *Fm1*

> $Fm1 := simplify(eval(Fm1, [Forsol_{4,1}, Forsol_{4,2}, Forsol_{4,3}, Forsol_{4,4}, Forsol_{4,5}]));$

$$Fm1 := \Big[\Big[-(5\,m\,\_C2\,r\,rdot\,Ib + 7\,m^2\,\_C2\,r\,rdot\,Ro^2 + 5\,m^2\,\_C2\,r^3\,rdot + \nu\,\_C1 + 5\,\nu\,\_C2\,Ib + 7\,\nu\,\_C2\,m\,Ro^2 + 5\,\nu\,\_C2\,m\,r^2) / (\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2), (5\,m\,\_C2\,r\,\theta dot\,Ib + 7\,m^2\,\_C2\,r\,\theta dot\,Ro^2 + 5\,m^2\,\_C2\,r^3\,\theta dot + \sigma\,\_C1 + 5\,\sigma\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2\,\sigma + 5\,\_C2\,\sigma\,m\,r^2) / (\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2)\Big], \Big[\frac{7\,rdot\,Ro\,r\,\_C2\,m^2}{\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2}, -\frac{7\,\theta dot\,Ro\,r\,\_C2\,m^2}{\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2}\Big]\Big] \quad \textbf{(2.1.4)}$$

Solving for the inputs the solution is called binputs

> $binputs := solve(\{FMCTest_{1,1}, FMCTest_{2,1}, FMCTest_{2,2}\}, [F33(qf, qdotf), F44(qf, qdotf), F55(qf, qdotf)]);$

$$binputs := [[F33(qf, qdotf) = 50\,\nu\,\_C2\,Ib + 10\,\nu\,\_C1 + 70\,\nu\,\_C2\,m\,Ro^2 + 50\,\nu\,\_C2\,m\,r^2, F44(qf, qdotf) = -25\,\sigma\,\_C2\,Ib - 25\,\_C2\,\sigma\,m\,r^2 - 35\,\_C2\,m\,Ro\,\nu - 35\,\_C2\,m\,Ro^2\,\sigma - 5\,\sigma\,\_C1, F55(qf, qdotf) = 70\,m\,Ro\,\sigma\,\_C2]] \quad \textbf{(2.1.5)}$$

Substituing the solution binputs in FMCTest

> $FMCTest := simplify(eval(FMCTest, [F33(qf, qdotf) = 50\,\nu\,\_C2\,Ib + 50\,\nu\,\_C2\,m\,r^2 + 10\,\nu\,\_C1 + 70\,\nu\,\_C2\,m\,Ro^2, F44(qf, qdotf) = -25\,\sigma\,\_C2\,Ib - 35\,\_C2\,Ro\,m\,\nu - 25\,\_C2\,\sigma\,m\,r^2 - 35\,\_C2\,Ro^2\,m\,\sigma$

$-5\,\sigma\,\_C1,\ F55(qf,\ qdotf) = 70\,Ro\,m\,\sigma\,\_C2]))$;

$$FMCTest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.1.6)$$

FMC is satisfied!

Multiplying *Fm1* times the velocity generalized coordinates

> $F1 := simplify(Multiply(Fm1,\ qdot))$;

$$F1 := \begin{bmatrix} -\theta dot\,\nu + rdot\,\sigma \\ 0 \end{bmatrix} \quad (2.1.7)$$

Substituing the solution binputs in *Fmc1*

> $Fmc1 := simplify(eval(Fmc1,\ [F33(qf,\ qdotf) = 50\,\nu\,\_C2\,Ib + 50\,\nu\,\_C2\,m\,r^2 + 10\,\nu\,\_C1 + 70\,\nu\,\_C2\,m\,Ro^2,\ F44(qf,\ qdotf) = -25\,\sigma\,\_C2\,Ib - 35\,\_C2\,Ro\,m\,\nu - 25\,\_C2\,\sigma\,m\,r^2 - 35\,\_C2\,Ro^2\,m\,\sigma - 5\,\sigma\,\_C1,\ F55(qf,\ qdotf) = 70\,Ro\,m\,\sigma\,\_C2]))$;

$$Fmc1 := [[50\,\nu\,\_C2\,Ib + 10\,\nu\,\_C1 + 70\,\nu\,\_C2\,m\,Ro^2 + 50\,\nu\,\_C2\,m\,r^2,\ -25\,\sigma\,\_C2\,Ib - 25\,\_C2\,\sigma\,m\,r^2 - 35\,\_C2\,m\,Ro\,\nu - 35\,\_C2\,m\,Ro^2\,\sigma - 5\,\sigma\,\_C1],\ [-25\,\sigma\,\_C2\,Ib - 25\,\_C2\,\sigma\,m\,r^2 - 35\,\_C2\,m\,Ro\,\nu - 35\,\_C2\,m\,Ro^2\,\sigma - 5\,\sigma\,\_C1,\ 70\,m\,Ro\,\sigma\,\_C2]] \quad (2.1.8)$$

Substituing the solution *Forsol* in KD matrix.
The solution 4 is selected in order tu avoid multiple of the mass matrix for KD

> $KDT := simplify(eval(KDT,\ [Forsol_{4,1},\ Forsol_{4,2},\ Forsol_{4,3},\ Forsol_{4,4},\ Forsol_{4,5}]))$;

$$KDT := [[5\,\_C1\,Ib + 7\,\_C1\,m\,Ro^2 + 5\,\_C1\,m\,r^2 + 25\,\_C2\,Ib^2 + 70\,\_C2\,Ib\,m\,Ro^2 + 50\,\_C2\,Ib\,m\,r^2 + 49\,\_C2\,m^2\,Ro^4 + 70\,\_C2\,m^2\,Ro^2\,r^2 + 25\,\_C2\,m^2\,r^4,\ -7\,m\,(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2)\,Ro],\ [-7\,m\,(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2 + 5\,\_C2\,m\,r^2)\,Ro,\ 7\,m\,(7\,\_C2\,m\,Ro^2 + \_C1)]] \quad (2.1.9)$$

going to matlab

> $convert(KDT,\ string)$

"Matrix(2, 2, [[5*_C1*Ib+7*_C1*m*Ro^2+5*_C1*m*r^2+25*_C2*Ib^2+70*_C2*Ib*m*Ro^2+50*_C2*Ib*m*r^2+49*_C2*m^2*Ro^4+70*_C2*m^2*Ro^2*r^2+25*_C2*m^2*r^4,-7*m*(_C1+5*_C2*Ib+7*_C2*m*Ro^2+5*_C2*m*r^2)*Ro],[-7*m*(_C1+5*_C2*Ib+7*_C2*m*Ro^2+5*_C2*m*r^2)*Ro,7*m*(7*_C2*m*Ro^2+_C1)]])" **(2.1.10)**

The determinant of the KD matrix is

> simplify(Determinant(KDT));

$$175\, m\, \_C2\, Ib^2\, \_C1 + 350\, \_C2\, Ib\, m^2\, r^2\, \_C1 + 175\, \_C2\, m^3\, r^4\, \_C1 + 245\, \_C1\, Ib\, \_C2\, m^2\, Ro^2 + 245\, \_C1\, m^3\, r^2\, \_C2\, Ro^2 + 35\, m\, \_C1^2\, Ib + 35\, \_C1^2\, m^2\, r^2 \tag{2.1.11}$$

The P matrix and its determinant are

> P := simplify(Multiply(KDT, MatrixInverse(mass)));

$$P := \begin{bmatrix} 5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2 & 0 \\ -35\,\_C2\, m\, Ro & 5\,\_C1 \end{bmatrix} \tag{2.1.12}$$

> Determinant(P);

$$25\,(\_C1 + 5\,\_C2\, Ib + 7\,\_C2\, m\, Ro^2 + 5\,\_C2\, m\, r^2)\,\_C1 \tag{2.1.13}$$

The P matrix eigenvalues are

> simplify(Eigenvalues(P));

$$\begin{bmatrix} 5\,\_C1 \\ 5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2 \end{bmatrix} \tag{2.1.14}$$

Second Matching Condition

Define the matrix Kv
The Kv matrix is obtain by the product of the first column of the P matrix and its transpose

> Kv := α·Multiply(Column(P, [1]), Transpose(Column(P, [1])));

$$Kv := [[\alpha\,(5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2)^2, -35\,\alpha\,(5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2)\,\_C2\, m\, Ro], [-35\,\alpha\,(5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2)\,\_C2\, m\, Ro, 1225\,\alpha\,\_C2^2\, m^2\, Ro^2]] \tag{3.1}$$

The vector F2 is obtain by the product of the inverse of the P matrix and the Kv matrix. This result is then multiply by the vector of generalized velocities

> F2f := -Multiply(MatrixInverse(P), Kv);

$$F2f := \left[\left[-\frac{1}{5}\,\frac{\alpha\,(5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2)^2}{\_C1 + 5\,\_C2\, Ib + 7\,\_C2\, m\, Ro^2 + 5\,\_C2\, m\, r^2}, \frac{7\,\alpha\,(5\,\_C1 + 25\,\_C2\, Ib + 35\,\_C2\, m\, Ro^2 + 25\,\_C2\, m\, r^2)\,\_C2\, m\, Ro}{\_C1 + 5\,\_C2\, Ib + 7\,\_C2\, m\, Ro^2 + 5\,\_C2\, m\, r^2}\right],\right. \tag{3.2}$$

$$\left[-\frac{7}{5}\,\frac{\_C2\,m\,Ro\,\alpha\,(5\,\_C1+25\,\_C2\,Ib+35\,\_C2\,m\,Ro^2+25\,\_C2\,m\,r^2)^2}{(\_C1+5\,\_C2\,Ib+7\,\_C2\,m\,Ro^2+5\,\_C2\,m\,r^2)\,\_C1} + \frac{7\,\alpha\,(5\,\_C1+25\,\_C2\,Ib+35\,\_C2\,m\,Ro^2+25\,\_C2\,m\,r^2)\,\_C2\,m\,Ro}{\_C1},\right.$$

$$\left.\frac{49\,\_C2^2\,m^2\,Ro^2\,\alpha\,(5\,\_C1+25\,\_C2\,Ib+35\,\_C2\,m\,Ro^2+25\,\_C2\,m\,r^2)}{(\_C1+5\,\_C2\,Ib+7\,\_C2\,m\,Ro^2+5\,\_C2\,m\,r^2)\,\_C1} - \frac{245\,\alpha\,\_C2^2\,m^2\,Ro^2}{\_C1}\right]\Bigg]$$

> $simplify(Eigenvalues(Kv));$

$$[[0], [1225\,\alpha\,\_C2^2\,m^2\,Ro^2+25\,\alpha\,\_C1^2+250\,\alpha\,\_C1\,\_C2\,Ib+350\,\alpha\,\_C1\,\_C2\,m\,Ro^2 + 250\,\alpha\,\_C1\,\_C2\,m\,r^2+625\,\alpha\,\_C2^2\,Ib^2+1750\,\alpha\,\_C2^2\,Ib\,m\,Ro^2 + 1250\,\alpha\,\_C2^2\,Ib\,m\,r^2+1225\,\alpha\,\_C2^2\,m^2\,Ro^4+1750\,\alpha\,\_C2^2\,m^2\,Ro^2\,r^2 + 625\,\alpha\,\_C2^2\,m^2\,r^4]] \quad \textbf{(3.3)}$$

> $Fm2 := simplify(Multiply(F2f, qdot));$

$$Fm2 := [[-5\,(\theta dot\,\_C1+5\,\theta dot\,\_C2\,Ib+7\,\theta dot\,\_C2\,m\,Ro^2+5\,\theta dot\,\_C2\,m\,r^2 - 7\,\_C2\,m\,Ro\,rdot)\,\alpha], [0]] \quad \textbf{(3.4)}$$

Third Matching Condition

Define the gradient of the $\Phi(\theta, r)$

> $PHM := Matrix\left(2, 1, \left[\frac{\partial}{\partial\theta}\Phi(\theta, r), \frac{\partial}{\partial r}\Phi(\theta, r)\right]\right);$

$$PHM := \begin{bmatrix} \frac{\partial}{\partial\theta}\Phi(\theta, r) \\ \frac{\partial}{\partial r}\Phi(\theta, r) \end{bmatrix} \quad \textbf{(4.1)}$$

Using the third matching condition two equations can be obtained

> $TMC := convert\left(\begin{bmatrix} F3 \\ 0 \end{bmatrix}, Matrix\right) - convert(G, Matrix) + Multiply(MatrixInverse(P), PHM);$

$$TMC := \Bigg[\Bigg[F3 - m\,r\,g\cos(\theta) + m\,Ro\,g\sin(\theta) \quad \textbf{(4.2)}$$

$$+\frac{1}{5}\frac{\frac{\partial}{\partial\theta}\Phi(\theta,r)}{\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2+5\_C2\,m\,r^2}\Bigg],$$

$$\left[-m\,g\sin(\theta)+\frac{7}{5}\frac{\_C2\,m\,Ro\left(\frac{\partial}{\partial\theta}\Phi(\theta,r)\right)}{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2+5\_C2\,m\,r^2)\,\_C1}\right.$$

$$\left.+\frac{1}{5}\frac{\frac{\partial}{\partial r}\Phi(\theta,r)}{\_C1}\right]\Bigg]$$

From the second row of TMC $\Phi(\theta, r)$ is found

> $TMCphi := TMC_{2,\,1};$

$$TMCphi := -m\,g\sin(\theta)+\frac{7}{5}\frac{\_C2\,m\,Ro\left(\frac{\partial}{\partial\theta}\Phi(\theta,r)\right)}{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2+5\_C2\,m\,r^2)\,\_C1}+\frac{1}{5}\frac{\frac{\partial}{\partial r}\Phi(\theta,r)}{\_C1} \quad \textbf{(4.3)}$$

> $solphi := pdsolve(TMCphi);$

$$solphi := \Phi(\theta,r)=\frac{5}{7}\int^{\theta}\Big(g\,\_C1\,(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\sin(\_a)\Big)\Bigg/\Bigg(\_C2\,Ro\cos\Bigg(\frac{1}{7}\frac{1}{\_C2\,m\,Ro}\Bigg(\Bigg(\_a \quad \textbf{(4.4)}$$

$$-\frac{1}{5}\left(\left(\sqrt{5}\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}\;\theta\right.\right.$$

$$\left.\left.-7\_C2\,m\,Ro\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}}\right)\right)\sqrt{5}\right)$$

$$\Bigg/\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}\Bigg)^2$$

$$\left.\sqrt{5}\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}\right)\Bigg)\Bigg)\,\mathrm{d}\_a+\_F1\Bigg($$

$$-\frac{1}{5}\left(\left(\sqrt{5}\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}\;\theta\right.\right.$$

$$\left.\left.-7\_C2\,m\,Ro\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}}\right)\right)\sqrt{5}\right)$$

$$\Bigg/\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\_C2\,m}\Bigg)$$

From the first row of TMC F3 is found

> $f3 := simplify\left(F3-\left(TMC_{1,\,1}\right)\right);$

$$f3 := -\frac{1}{5}\Big(-5\,m\,r\,g\cos(\theta)\_C1-25\,m\,r\,g\cos(\theta)\_C2\,Ib \tag{4.5}$$

$$-35\,m^2\,r\,g\cos(\theta)\_C2\,Ro^2-25\,m^2\,r^3\,g\cos(\theta)\_C2+5\,m\,Ro\,g\sin(\theta)\_C1$$

$$+25\,m\,Ro\,g\sin(\theta)\_C2\,Ib+35\,m^2\,Ro^3\,g\sin(\theta)\_C2$$

$$+25\,m^2\,Ro\,g\sin(\theta)\_C2\,r^2+\frac{\partial}{\partial\theta}\Phi(\theta,r)\Big)\Big/\big(\_C1+5\_C2\,Ib$$

$$+7\ \_C2\ m\ Ro^2+5\ \_C2\ m\ r^2)$$

> PhiIifii := simplify(rhs(solphi));

$$PhiIifii := \frac{5}{7}\int^{\theta}\left(g\ \_C1\ (\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \sin(\_a)\right)\Bigg/\left(\_C2\ Ro\cos\left(\frac{1}{35}\ \frac{1}{\_C2\ m\ Ro}\left(\left(5\ \_a\ \sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}-5\sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}\ \theta+7\sqrt{5}\ \_C2\ m\ Ro\arctan\left(\frac{\_C2\ m\ r\sqrt{5}}{\sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}}\right)\right)\sqrt{5}\right)\right)^2\right)\mathrm{d}\_a+\_F1\left(-\frac{1}{5}\left(\left(\sqrt{5}\sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}\ \theta-7\ \_C2\ m\ Ro\arctan\left(\frac{\_C2\ m\ r\sqrt{5}}{\sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}}\right)\right)\sqrt{5}\right)\Bigg/\sqrt{(\_C1+5\ \_C2\ Ib+7\ \_C2\ m\ Ro^2)\ \_C2\ m}\right) \quad \mathbf{(4.6)}$$

There are only 6 terms in theTaylor series because the length of F_3

$$> PhiIifi := taylor\left(\frac{5}{7}\int^{\theta} expand\left(\frac{\eta\sin(\_a)}{\cos(\ \_a\,\delta-\beta)^2}\right) d\_a, \theta, 6\right) + F5$$

$$\cdot\left(\frac{1}{5}\left(\left(\left(-\sqrt{5}\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}\;\theta\right.\right.\right.\right.$$

$$\left.\left.+7\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}}\right)\_C2\,Ro\,m\right)\sqrt{5}\right)^2$$

$$\left.\middle/\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}\right);$$

$$PhiIifi := \left(\frac{5}{7}\int^{0}\frac{\eta\sin(\_a)}{(\cos(\_a\,\delta)\cos(\beta)+\sin(\_a\,\delta)\sin(\beta))^2}\,d\_a\right. \tag{4.7}$$

$$+\frac{5}{14}\frac{\eta}{\cos(\beta)^2}\theta^2-\frac{10}{21}\frac{\eta\,\delta\sin(\beta)}{\cos(\beta)^3}\theta^3$$

$$+\frac{5}{168}\frac{\eta\left(18\,\delta^2\sin(\beta)^2-\cos(\beta)^2+6\,\delta^2\cos(\beta)^2\right)}{\cos(\beta)^4}\theta^4$$

$$\left.-\frac{1}{21}\frac{\eta\,\delta\sin(\beta)\left(-\cos(\beta)^2+12\,\delta^2\sin(\beta)^2+8\,\delta^2\cos(\beta)^2\right)}{\cos(\beta)^5}\theta^5+O(\theta^6)\right)$$

$$+\frac{1}{5}\frac{1}{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}\left(F5\left(\right.\right.$$

$$-\sqrt{5}\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}\;\theta$$

$$\left.\left.+7\_C2\,m\,Ro\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{(\_C1+5\_C2\,Ib+7\_C2\,m\,Ro^2)\,\_C2\,m}}\right)\right)^2\right)$$

$$> PhiIifi := \left(\frac{5}{7}\int^{0}\frac{\eta\sin(\_a)}{(\cos(\_a\,\delta)\cos(\beta)+\sin(\_a\,\delta)\sin(\beta))^2}\,d\_a$$

$$+\frac{5}{14}\frac{\eta}{\cos(\beta)^2}\theta^2-\frac{10}{21}\frac{\eta\,\delta\sin(\beta)}{\cos(\beta)^3}\theta^3$$

$$+\frac{5}{168}\frac{\eta\left(6\,\delta^2\cos(\beta)^2-\cos(\beta)^2+18\,\delta^2\sin(\beta)^2\right)}{\cos(\beta)^4}\theta^4$$

$$\left.-\frac{1}{21}\frac{\eta\,\delta\sin(\beta)\left(-\cos(\beta)^2+12\,\delta^2\sin(\beta)^2+8\,\delta^2\cos(\beta)^2\right)}{\cos(\beta)^5}\theta^5\right)$$

$$+\frac{1}{5}\frac{1}{\left(\_C1+5\,\_C2\,Ib+7\,\_C2\,m\,Ro^2\right)\_C2\,m}\left(F5\left(\right.\right.$$

$$-\sqrt{5}\sqrt{\left(\_C1+5\,\_C2\,Ib+7\,\_C2\,m\,Ro^2\right)\_C2\,m}\;\theta$$

$$\left.\left.+7\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{\left(\_C1+5\,\_C2\,Ib+7\,\_C2\,m\,Ro^2\right)\_C2\,m}}\right)\_C2\,Ro\,m\right)^2\right):$$

Input contribution from FMC, F_1

> $F1 := simplify(F1_1);$

$$F1 := -\theta dot\,\nu+rdot\,\sigma \tag{4.8}$$

Input contribution from SMC, F_2

> $F2 := simplify(Fm2_1);$

$$F2 := -5\left(\theta dot\,\_C1+5\,\theta dot\,\_C2\,Ib+7\,\theta dot\,\_C2\,m\,Ro^2+5\,\theta dot\,\_C2\,m\,r^2-7\,\_C2\,m\,Ro\,rdot\right)\alpha \tag{4.9}$$

Input contribution from TMC, F_3

> $F3 := simplify(eval(f3, \Phi(\theta, r) = PhiIifi)):$

Contribution from FMC to Lyapunov, Fmc_1

> $Fmc1 := Fmc1;$

$$Fmc1 := [[50\,\nu\,\_C2\,Ib+10\,\nu\,\_C1+70\,\nu\,\_C2\,m\,Ro^2+50\,\nu\,\_C2\,m\,r^2, -25\,\sigma\,\_C2\,Ib-25\,\_C2\,\sigma\,m\,r^2-35\,\_C2\,m\,Ro\,\nu-35\,\_C2\,m\,Ro^2\,\sigma-5\,\sigma\,\_C1], [-25\,\sigma\,\_C2\,Ib-25\,\_C2\,\sigma\,m\,r^2-35\,\_C2\,m\,Ro\,\nu-35\,\_C2\,m\,Ro^2\,\sigma-5\,\sigma\,\_C1, 70\,m\,Ro\,\sigma\,\_C2]] \tag{4.10}$$

The control input of the system is

Define the control law

> $\mathrm{Fc} := simplify(eval(\mathrm{F1}+\mathrm{F2}+\mathrm{F3})):$

$$> \ \mathrm{Fc} := simplify\left(eval\left(\mathrm{Fc}, \left[\eta = \frac{(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2)\,\_C1\,g}{Ro\,\_C2}, \delta = \frac{1}{35}\,\frac{\left(5\sqrt{(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2)\,\_C2\,m}\right)\sqrt{5}}{\_C2\,Ro\,m}, \beta = \frac{1}{35}\,\frac{1}{\_C2\,Ro\,m}\left(\left(\left(-5\sqrt{(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2)\,\_C2\,m}\;\theta + 7\sqrt{5}\,\arctan\left(\frac{\_C2\,m\,r\sqrt{5}}{\sqrt{(\_C1 + 5\,\_C2\,Ib + 7\,\_C2\,m\,Ro^2)\,\_C2\,m}}\right)\_C2\,Ro\,m\right)\sqrt{5}\right)\right)\right]\right)\right):$$

$$> \ \mathrm{Fc} := simplify(eval(\mathrm{Fc}, [\_C1 = 0.5, \_C2 = 8.5])):$$

Linearization of the control law

$$> \ \mathrm{T} := \left[\ diff(Fc, \theta)\ \ diff(Fc, r)\ \ diff(Fc, \theta dot)\ \ diff(Fc, rdot)\ \right]:$$

$$> \ T1 := simplify(eval(\mathrm{T}, [\theta dot = 0, rdot = 0])):$$

T2 is the linearized control law

$$> \ T2 := simplify(eval(T1, [r = 0, \theta = 0])):$$

$$> \ sysid := [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]:$$

$$> \ LCond := [\theta = 0, \theta dot = 0, r = 0, rdot = 0]:$$

The equation of motion of the ball and beam system is

$$> \ Eqs := simplify\left(Multiply\left(MatrixInverse(mass), \left(\left(\begin{bmatrix}\tau \\ 0\end{bmatrix} - Multiply(C, qdot)\right) - \mathrm{G}\right)\right)\right);$$

$$Eqs := \left[\left[-\frac{-\tau + 2\,m\,r\,rdot\,\theta dot + m\,r\,g\cos(\theta) - m\,Ro\,r\,\theta dot^2}{Ib + m\,r^2}\right], \left[-\frac{1}{7}\,\frac{1}{Ib + m\,r^2}\left(-7\,Ro\,\tau + 14\,Ro\,m\,r\,rdot\,\theta dot + 7\,Ro\,m\,r\,g\cos(\theta) - 5\,Ib\,r\,\theta dot^2 + 5\,Ib\,g\sin(\theta) - 7\,m\,Ro^2\,r\,\theta dot^2 - 5\,m\,r^3\,\theta dot^2 + 5\,m\,r^2\,g\sin(\theta)\right)\right]\right] \quad \textbf{(6.1)}$$

$$> \ A$$

$$:= [[0,0,1,0],$$
$$[0,0,0,1],$$
$$[\mathrm{diff}(Eqs[1],\theta), \mathrm{diff}(Eqs[1],r), \mathrm{diff}(Eqs[1],\theta dot), \mathrm{diff}(Eqs[1],rdot)],$$
$$[\mathrm{diff}(Eqs[2],\theta), \mathrm{diff}(Eqs[2],r), \mathrm{diff}(Eqs[2],\theta dot), \mathrm{diff}(Eqs[2],rdot)]];$$

$$A := \left[\left[\left[0,0,1,0\right]\right],\right. \qquad \textbf{(6.2)}$$

$$\left[0,0,0,1\right],$$

$$\left[\frac{m\,r\,g\sin(\theta)}{Ib+m\,r^2}, -\frac{2\,m\,rdot\,\theta dot+m\,g\cos(\theta)-m\,Ro\,\theta dot^2}{Ib+m\,r^2}\right.$$
$$+\frac{2\left(-\tau+2\,m\,r\,rdot\,\theta dot+m\,r\,g\cos(\theta)-m\,Ro\,r\,\theta dot^2\right)m\,r}{\left(Ib+m\,r^2\right)^2},$$
$$\left.-\frac{2\,m\,r\,rdot-2\,m\,Ro\,r\,\theta dot}{Ib+m\,r^2}, -\frac{2\,m\,r\,\theta dot}{Ib+m\,r^2}\right],$$

$$\left[-\frac{1}{7}\,\frac{-7\,Ro\,m\,r\,g\sin(\theta)+5\,Ib\,g\cos(\theta)+5\,m\,r^2\,g\cos(\theta)}{Ib+m\,r^2},\right.$$
$$-\frac{1}{7}\,\frac{1}{Ib+m\,r^2}\left(14\,Ro\,m\,rdot\,\theta dot+7\,Ro\,m\,g\cos(\theta)-5\,Ib\,\theta dot^2\right.$$
$$\left.-7\,m\,Ro^2\,\theta dot^2-15\,m\,r^2\,\theta dot^2+10\,m\,r\,g\sin(\theta)\right)+\frac{2}{7}\,\frac{1}{\left(Ib+m\,r^2\right)^2}\left(\left(\right.\right.$$
$$-7\,Ro\,\tau+14\,Ro\,m\,r\,rdot\,\theta dot+7\,Ro\,m\,r\,g\cos(\theta)-5\,Ib\,r\,\theta dot^2$$
$$\left.\left.+5\,Ib\,g\sin(\theta)-7\,m\,Ro^2\,r\,\theta dot^2-5\,m\,r^3\,\theta dot^2+5\,m\,r^2\,g\sin(\theta)\right)m\,r\right),$$
$$-\frac{1}{7}\,\frac{14\,Ro\,m\,r\,rdot-10\,Ib\,r\,\theta dot-14\,m\,Ro^2\,r\,\theta dot-10\,m\,r^3\,\theta dot}{Ib+m\,r^2},$$
$$\left.\left.-\frac{2\,m\,Ro\,r\,\theta dot}{Ib+m\,r^2}\right]\right]$$

> $A := map(eval, A, LCond);$

(6.3)

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m\,g}{Ib} & 0 & 0 \\ -\frac{5}{7}\,g & -\frac{Ro\,m\,g}{Ib} & 0 & 0 \end{bmatrix} \tag{6.3}$$

> $A := eval(\mathrm{A}, sysid);$

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -36.78750000 & 0 & 0 \\ -7.007142857 & -0.7357500000 & 0 & 0 \end{bmatrix} \tag{6.4}$$

> $B := \begin{bmatrix} 0 \\ 0 \\ \mathrm{diff}\left(Eqs[1], \tau\right) \\ \mathrm{diff}\left(Eqs[2], \tau\right) \end{bmatrix};$

$$B := \begin{bmatrix} 0 \\ 0 \\ \frac{1}{Ib + m\,r^2} \\ \frac{Ro}{Ib + m\,r^2} \end{bmatrix} \tag{6.5}$$

> $B := simplify(map(eval, B, LCond)) :$

> $B := simplify(map(eval, B, sysid));$

$$B := \begin{bmatrix} 0 \\ 0 \\ 2.500000000 \\ 0.05000000000 \end{bmatrix} \tag{6.6}$$

> $LinCi := T2 :$

> $LinC3 := map(eval, LinCi, sysid);$

$$LinC3 := [-4.416148739 - 0.02281060922\,F5,\ 14.71500000 + 0.002321945366\,F5,\ -87.67850000\,\alpha - 1.000000000\,\nu,\ 1.000000000\,\sigma + 8.925000000\,\alpha] \tag{6.7}$$

> $\mathrm{k} := [\ 135.8576\ ,\ -47.5957\ ,\ 11.1124\ ,\ -35.6208];$

$$k := [135.8576,\ -47.5957,\ 11.1124,\ -35.6208] \tag{6.8}$$

> $LinC3 := -map(eval, LinC3, [\_C1 = 0.5, \_C2 = 8.5]);$

$$LinC3 := [4.416148739 + 0.02281060922\, F5,\ -14.71500000 - 0.002321945366\, F5,\ 87.67850000\,\alpha + 1.000000000\,\nu,\ -1.000000000\,\sigma - 8.925000000\,\alpha] \quad \textbf{(6.9)}$$

> $sol := solve(\{LinC3_1 - k_1, LinC3_3 - k_3, LinC3_4 - k_4\});$

$$sol := \{F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000000\,\alpha\} \quad \textbf{(6.10)}$$

> $K := map(eval, LinC3, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\,\alpha]);$

$$K := \begin{bmatrix} 135.8576000 & -28.09473334 & 11.11240000 & -35.62080000 + 4.\,10^{-9}\,\alpha \end{bmatrix} \quad \textbf{(6.11)}$$

The gain matrix is

> $K := map(eval, K, [\alpha = 0.001]);$

$$K := \begin{bmatrix} 135.8576000 & -28.09473334 & 11.11240000 & -35.62080000 \end{bmatrix} \quad \textbf{(6.12)}$$

> $Acl := eval(A - Multiply(B, K));$

$$Acl := [[0., 0., 1., 0.], [0., 0., 0., 1.], [-339.644000000000005, 33.4493333499999963, -27.7809999999999988, 89.0520000000000067], [-13.8000228570000001, 0.668986667000000090, -0.555620000000000003, 1.78104000000000018]] \quad \textbf{(6.13)}$$

> $EACLi := Eigenvalues(Acl);$

$$EACLi := \begin{bmatrix} -11.9503753772407784 + 12.0489417753864583\,\mathrm{I} \\ -11.9503753772407784 - 12.0489417753864583\,\mathrm{I} \\ -1.58607741179245587 + 0.\,\mathrm{I} \\ -0.513131833725887887 + 0.\,\mathrm{I} \end{bmatrix} \quad \textbf{(6.14)}$$

> $KDf := simplify(eval(KDT, [\_C1 = 0.5, \_C2 = 8.5]));$

$$KDf := [[2.500000000\,Ib + 3.500000000\,m\,Ro^2 + 2.500000000\,m\,r^2 + 212.5000000\,Ib^2 + 595.\,Ib\,m\,Ro^2 + 425.\,Ib\,m\,r^2 + 416.5000000\,m^2\,Ro^4 + 595.\,m^2\,Ro^2\,r^2 + 212.5000000\,m^2\,r^4, -3.500000000\,m\,(1. + 85.\,Ib + 119.\,m\,Ro^2 + 85.\,m\,r^2)\,Ro], [-3.500000000\,m\,(1. + 85.\,Ib + 119.\,m\,Ro^2 + 85.\,m\,r^2)\,Ro, 3.500000000\,m\,(119.\,m\,Ro^2 + 1.)]] \quad \textbf{(6.15)}$$

> $KDf := simplify(eval(KDf, [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]));$

$$KDf := [[35.14504994 + 259.2855000\,r^2 + 478.1250000\,r^4, -3.682497000 - 13.38750000\,r^2], [-3.682497000 - 13.38750000\,r^2, 5.624850000]] \quad \textbf{(6.16)}$$

> $Determinant(KDf);$
$Kvf := simplify(eval(Kv, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\,\alpha])):$

$$184.1248500 + 1359.843188\, r^2 + 2510.156250\, r^4 \tag{6.17}$$

> $Kvf := simplify(eval(Kvf, [\_C1 = 0.5, \_C2 = 8.5])):$

> $Kvf := simplify(eval(Kvf, [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02, \alpha = 0.001]));$

$$Kvf := [[2.500000000\,10^{-10}\,(1.75357\,10^5 + 6.37500\,10^5\, r^2)^2, -0.7825306125 - 2.844843750\, r^2], [-0.7825306125 - 2.844843750\, r^2, 0.07965562500]] \tag{6.18}$$

> $P := simplify(Multiply(KDf, MatrixInverse(mass))):$

> $P := simplify(eval(P, [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]));$

$$P := \begin{bmatrix} 87.67850000 + 318.7500000\, r^2 & 0. \\ -8.925000000 & 2.500000000 \end{bmatrix} \tag{6.19}$$

> $PhiIif := simplify(eval(PhiIifii, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\,\alpha])):$

> $PhiIif := simplify(eval(PhiIif, [\_C1 = 0.5, \_C2 = 8.5, Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]));$

$$PhiIif := 0.7142857143 \left(\int^{\theta} \frac{1011913041}{2000000} (\sin(\_a)) \Big/ \cos\left(\frac{936555313}{50000000}\,\_a - \frac{936555313}{50000000}\,\theta + \frac{1999999999}{2000000000} \arctan\left(\frac{476670801}{250000000}\, r \right) \right)^2 d\_a \right) + \_F1\left(-\frac{4999999999}{5000000000}\,\theta + \frac{2669356487}{50000000000} \arctan\left(\frac{476670801}{250000000}\, r \right) \right) \tag{6.20}$$

> $PhiIif := simplify\Bigg(0.7142857143 \Bigg($

$$\int^{\theta} \frac{1011913041}{2000000} (\sin(\_a)) \Big/ \left(\cos\left(\frac{936555313}{50000000}\_a - \frac{936555313}{50000000}\theta + \frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}r\right)\right)^2\right) d\_a\right) + F5\cdot\left(-\frac{4999999999}{5000000000}\theta + \frac{2669356487}{50000000000}\arctan\left(\frac{476670801}{250000000}r\right)\right)^2\right);$$

$$PhiIif := 0.7142857143 \left(\int^{\theta} \frac{1011913041}{2000000} (\sin(\_a)) \Big/ \cos\left(\frac{936555313}{50000000}\_a - \frac{936555313}{50000000}\theta + \frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}r\right)\right)^2 d\_a \right) + 0.9999999996\, F5\, \theta^2 - 0.1067742595\, F5\, \theta \arctan(1.906683204\, r) + 0.002850185622\, F5 \arctan(1.906683204\, r)^2 \tag{6.21}$$

> $intLf2 := 0.9999999996\, F5\, \theta^2 - 0.1067742595\, F5\, \theta \arctan(1.906683204\, r) + 0.002850185622\, F5 \arctan(1.906683204\, r)^2 :$

> $taylor\left(\sin\left(\frac{t}{\delta}\right), t = 0\right);$

$$\frac{1}{\delta} t - \frac{1}{6\,\delta^3} t^3 + \frac{1}{120\,\delta^5} t^5 + O(t^6) \tag{6.22}$$

> $taylor\left(\cos\left(\frac{t}{\delta}\right), t = 0\right);$

$$1 - \frac{1}{2\,\delta^2} t^2 + \frac{1}{24\,\delta^4} t^4 + O(t^6) \tag{6.23}$$

> $taylor\left((\cos(t))^2, t = 0\right);$

$$1 - t^2 + \frac{1}{3} t^4 + O(t^6) \tag{6.24}$$

$$> intT1 := simplify\left(0.7142857143\cdot\left(\frac{1011913041}{2000000}\right)\cdot\left(\frac{1}{\delta}\cdot\left(\cos\left(\frac{\beta}{\delta}\right)\right)\cdot\int\frac{\left(\frac{1}{\delta}t-\frac{1}{6\,\delta^3}t^3+\frac{1}{120\,\delta^5}t^5\right)}{\left(1-t^2+\frac{1}{3}t^4\right)}\,dt-\frac{1}{\delta}\cdot\left(\sin\left(\frac{\beta}{\delta}\right)\right)\cdot\int\frac{\left(1-\frac{1}{2\,\delta^2}t^2+\frac{1}{24\,\delta^4}t^4\right)}{\left(1-t^2+\frac{1}{3}t^4\right)}\,dt\right)\right):$$

$$> ti := simplify(\theta\cdot(\delta)-\beta);$$

$$ti := \theta\,\delta-\beta \tag{6.25}$$

$$> intL1 := simplify\left(eval\left(intT1, \left[\delta=\frac{936555313}{50000000}, \beta=-\frac{936555313}{50000000}\theta+\frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}r\right)\right]\right)\right);$$

$$\begin{aligned} intL1 :=\ & 1.782827863\cos(\theta-0.05338712971\arctan(1.906683204\,r))\arctan(\\ & -1.732050808+1.154700539\,t^2)+3.294107309\sin(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\ln(t^2+2.542459757\,t\\ & +1.732050808)-3.294107307\sin(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\ln(-1.\,t^2+2.542459757\,t\\ & -1.732050808)-0.0003668223189\cos(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\ln(3.-3.\,t^2+t^4)\\ & +24.46655598\sin(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\arctan(2.935779648\,t\\ & -3.732050805)+24.46655598\sin(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\arctan(2.935779648\,t\\ & +3.732050805)+0.00001959196682\sin(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\,t+1.045958875\,10^{-7}\cos(\theta\\ & -0.05338712971\arctan(1.906683204\,r))\,t^2 \end{aligned} \tag{6.26}$$

$$> ti := simplify\left(eval\left(ti, \left[\delta=\frac{936555313}{50000000}, \beta=-\frac{936555313}{50000000}\theta+\frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}r\right)\right]\right)\right);$$

$$ti := \frac{936555313}{25000000}\,\theta - \frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}\,r\right) \qquad \textbf{(6.27)}$$

> $intLf1 := simplify(eval(intL1, [\,t = \mathrm{ti}]))$;

$$intLf1 := 1.782827863\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(-1.732050808 + 1620.526790\,\theta^2 - 86.51527393\,\theta\arctan(1.906683204\,r) + 1.154700538\arctan(1.906683204\,r)^2) + 3.294107309\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(1403.417367\,\theta^2 - 74.92442500\,\theta\arctan(1.906683204\,r) + 0.9999999990\arctan(1.906683204\,r)^2 + 95.24616774\,\theta - 2.542459756\arctan(1.906683204\,r) + 1.732050808) - 3.294107307\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(-1403.417367\,\theta^2 + 74.92442500\,\theta\arctan(1.906683204\,r) - 0.9999999990\arctan(1.906683204\,r)^2 + 95.24616774\,\theta - 2.542459756\arctan(1.906683204\,r) - 1.732050808) - 0.0003668223189\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(3. - 4210.252101\,\theta^2 + 224.7732750\,\theta\arctan(1.906683204\,r) - 2.999999997\arctan(1.906683204\,r)^2 + 1.969580306\;10^6\,\theta^4 - 2.103004785\;10^5\,\theta^3\arctan(1.906683204\,r) + 8420.504193\,\theta^2\arctan(1.906683204\,r)^2 - 149.8488499\,\theta\arctan(1.906683204\,r)^3 + 0.9999999980\arctan(1.906683204\,r)^4) + 24.46655598\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(109.9808011\,\theta - 2.935779647\arctan(1.906683204\,r) - 3.732050805) + 24.46655598\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(109.9808011\,\theta - 2.935779647\arctan(1.906683204\,r) + 3.732050805) + 0.0007339584247\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\,\theta - 0.00001959196681\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(1.906683204\,r) + 0.0001467916850\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\,\theta^2 - 0.000007836786729\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\,\theta\arctan(1.906683204\,r) \qquad \textbf{(6.28)}$$

$$+ 1.045958874\ 10^{-7} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan(1.906683204\, r)^2$$

> $intLf1 := 1.782827863 \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan\big(-1.732050808 + 1620.526790\, \theta^2 - 86.51527393\, \theta \arctan(1.906683204\, r) + 1.154700538 \arctan(1.906683204\, r)^2\big) - 0.0003668223189 \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(3. - 4210.252101\, \theta^2 + 224.7732750\, \theta \arctan(1.906683204\, r) - 2.999999997 \arctan(1.906683204\, r)^2 + 1.969580306\ 10^6\, \theta^4 - 2.103004785\ 10^5\, \theta^3 \arctan(1.906683204\, r) + 8420.504193\, \theta^2 \arctan(1.906683204\, r)^2 - 149.8488499\, \theta \arctan(1.906683204\, r)^3 + 0.9999999980 \arctan(1.906683204\, r)^4\big) - 3.294107309 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(1403.417367\, \theta^2 - 74.92442500\, \theta \arctan(1.906683204\, r) + 0.9999999990 \arctan(1.906683204\, r)^2 - 95.24616774\, \theta + 2.542459756 \arctan(1.906683204\, r) + 1.732050808\big) + 3.294107309 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(1403.417367\, \theta^2 - 74.92442500\, \theta \arctan(1.906683204\, r) + 0.9999999990 \arctan(1.906683204\, r)^2 + 95.24616774\, \theta - 2.542459756 \arctan(1.906683204\, r) + 1.732050808\big) + 24.46655598 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan\big(109.9808011\, \theta - 2.935779647 \arctan(1.906683204\, r) - 3.732050805\big) + 24.46655598 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan\big(109.9808011\, \theta - 2.935779647 \arctan(1.906683204\, r) + 3.732050805\big) + 0.0007339584247 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta - 0.00001959196681 \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan(1.906683204\, r) + 0.0001467916850 \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^2 - 0.000007836786729 \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta \arctan(1.906683204\, r) + 1.045958874\ 10^{-7} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan(1.906683204\, r)^2 :$

$PhiIif := simplify(intLf1 + intLf2) :$

The Potential Φ

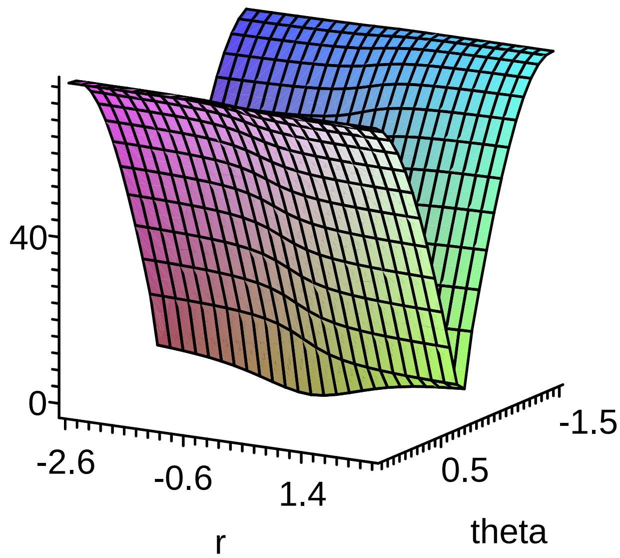
> $PhiGraphi := simplify(eval(PhiIif, [Ro = 0.02, g = 9.81, m = 1.5, Ib = 1.4])) :$

> $PhiGraph := simplify(eval(PhiGraphi, [F5 = 5762.294642])) :$

> $PhiGraph := intLf1 :$

> $f := (\theta, r) \rightarrow PhiGraph :$

> $plot3d(f(\theta, r), \theta = -1.5 .. 1.5, r = -2.6 .. 2.6, axes = FRAME, orientation = [30, 75], style = PATCH)$



The Hessian

> $Hessiani := simplify(eval(Matrix(2, 2, [diff(diff(\mathrm{PhiIif}, \mathrm{r}), \mathrm{r}), diff(diff(\mathrm{PhiIif}, \mathrm{r}), \theta), diff(diff(PhiIif, \theta), r), diff(diff(PhiIif, \theta), \theta)]), [r = 0, \theta = 0]));$

$$Hessiani := \begin{bmatrix} 0.02072336243\, F5 + 11.25340666 & -184.1275842 - 0.2035846872\, F5 \\ -184.1275842 - 0.2035846872\, F5 & 2893.047494 + 1.999999999\, F5 \end{bmatrix} \quad \textbf{(8.1)}$$

> $Hessiani := simplify(eval(Hessiani, F5 = 5762.294642));$

$$Hessiani := \begin{bmatrix} 130.6675270 & -1357.242536 \\ -1357.242536 & 14417.63677 \end{bmatrix} \quad \textbf{(8.2)}$$

> $Determinant(Hessiani);$

$$41809.640 \quad \textbf{(8.3)}$$

> $Eigenvalues(Hessiani);$

$$\begin{bmatrix} 2.87441765140283678 + 0.\ \mathrm{I} \\ 14545.4298793485959 + 0.\ \mathrm{I} \end{bmatrix} \quad \textbf{(8.4)}$$

> $intLf3 := simplify\left(eval\left(intT1, \left[\delta = \frac{936555313}{50000000}, \beta = -\frac{936555313}{50000000}\theta + \frac{1999999999}{2000000000}\arctan\left(\frac{476670801}{250000000}r\right), t = \frac{936555313}{50000000}\theta\right]\right)\right);$

$$intLf3 := 1.782827863\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(-1.732050808 + 405.1316975\,\theta^2) + 3.294107309\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(350.8543417\,\theta^2 + 47.62308387\,\theta + 1.732050808) - 3.294107307\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(-350.8543417\,\theta^2 + 47.62308387\,\theta - 1.732050808) - 0.0003668223189\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\ln(3. - 1052.563025\,\theta^2 + 1.230987691\,10^5\,\theta^4) + 24.46655598\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta - 3.732050805) + 24.46655598\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta + 3.732050805) + 0.0003669792124\sin(\theta - 0.05338712971\arctan(1.906683204\,r))\,\theta + 0.00003669792127\cos(\theta - 0.05338712971\arctan(1.906683204\,r))\,\theta^2 \quad \textbf{(8.5)}$$

> $intPHI2 := simplify\left(eval\left(-\frac{1}{5}\frac{\left(\frac{\partial}{\partial\theta}intLf3 + \frac{\partial}{\partial\theta}intLf2\right)}{\_C1 + 5\,\_C2\,Ib + 7\,Ro^2\,\_C2\,m + 5\,\_C2\,m\,r^2}, [\_C1 = 0.5, \_C2 = 8.5, Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]\right)\right):$

> $Fm3i := \frac{1}{5}(5\,r\,m\,g\cos(\theta)\,\_C1 + 25\,r\,m\,g\cos(\theta)\,\_C2\,Ib + 35\,r\,m^2\,g\cos(\theta)\,Ro^2\,\_C2 + 25\,r^3\,m^2\,g\cos(\theta)\,\_C2 - 5\,m\,Ro\,g\sin(\theta)\,\_C1 - 25\,m\,Ro\,g\sin(\theta)\,\_C2\,Ib - 35\,m^2\,Ro^3\,g\sin(\theta)\,\_C2 - 25\,m^2\,Ro\,g\sin(\theta)\,\_C2\,r^2)\Big/(\_C1 + 5\,\_C2\,Ib + 7\,Ro^2\,\_C2\,m + 5\,\_C2\,m\,r^2):$

> $Fm3i := simplify(eval(Fm3i, [\_C1 = 0.5, \_C2 = 8.5, Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]));$

$$Fm3i := -\frac{1}{1.75357\,10^5 + 6.37500\,10^5\,r^2}(0.0001000000000(\quad \textbf{(8.6)}$$

$-2.580378256\ 10^{10}\ r \cos(\theta) - 9.380812500\ 10^{10}\ r^3 \cos(\theta) + 5.16075651\ 10^{8} \sin(\theta) + 1.876162500\ 10^{9} \sin(\theta)\ r^2))$

> $F3 := simplify(Fm3i + intPHI2)$:

> $F2 := simplify(eval(F2, [\_C1 = 0.5, \_C2 = 8.5, \mathrm{Ib} = 0.4, m = 1.5, g = 9.81, Ro = 0.02]))$;

$$F2 := -0.0005000000000\ (1.75357\ 10^{5}\ \theta dot + 6.37500\ 10^{5}\ \theta dot\ r^2 - 17850.\ rdot)\ \alpha \quad \textbf{(8.7)}$$

> $F1 := simplify(eval(F1, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\ \alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\ \alpha]))$;

$$F1 := 87.67850000\ \theta dot\ \alpha - 11.11240000\ \theta dot + 35.62080000\ rdot - 8.925000004\ rdot\ \alpha \quad \textbf{(8.8)}$$

> $Fmc1 := simplify(eval(Fmc1, [\_C1 = 0.5, \_C2 = 8.5, \mathrm{Ib} = 0.4, m = 1.5, g = 9.81, Ro = 0.02]))$:

> $Fmc1 := simplify(eval(Fmc1, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\ \alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\ \alpha]))$;

$$Fmc1 := [[-15375.03872\ \alpha + 1948.637127 - 55895.04375\ r^2\ \alpha + 7084.155000\ r^2, -3222.356483 + 1565.061225\ \alpha - 11354.13000\ r^2 + 2844.843751\ r^2\ \alpha], [-3222.356483 + 1565.061225\ \alpha - 11354.13000\ r^2 + 2844.843751\ r^2\ \alpha, 635.8312800 - 159.3112501\ \alpha]] \quad \textbf{(8.9)}$$

> $F3 := simplify(eval(F3, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\ \alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\ \alpha]))$:

> $F3 := simplify(eval(F3, [\_C1 = 0.5, \_C2 = 8.5, \mathrm{Ib} = 0.4, m = 1.5, g = 9.81, Ro = 0.02]))$:

Variables to *matlab*

> *convert*(*F1*, *string*)

"87.67850000*`θdot`*alpha-11.11240000*`θdot`+35.62080000*rdot-8.925000004*rdot*alpha" **(9.1)**

> *convert*(*Fmc1*, *string*)

"Matrix(2, 2, [[-15375.03872*alpha+1948.637127-55895.04375*r^2*alpha+7084.155000*r^2,-3222.356483+1565.061225*alpha-11354.13000*r^2+2844.843751*r^2*alpha],[-3222.356483+1565.061225*alpha-11354.13000*r^2+2844.843751*r^2*alpha,635.8312800-159.3112501*alpha]])" **(9.2)**

> *convert*(*F2*, *string*)

"-.5000000000e-3*(175357.*`θdot`+637500.*`θdot`*r^2-17850.*rdot)*alpha" **(9.3)**

> $convert(F3, string)$:

> $convert(\text{PhiIif}, string)$

"1.782827863*cos(theta-.5338712971e-1*arctan(1.906683204*r))*arctan(-1.732050808+1620.526790*theta^2-86.51527393*theta*arctan(1.906683204*r)+1.154700538*arctan(1.906683204*r)^2)-.3668223189e-3*cos(theta-.5338712971e-1*arctan(1.906683204*r))*ln(3.-4210.252101*theta^2+224.7732750*theta*arctan(1.906683204*r)-2.999999997*arctan(1.906683204*r)^2+1969580.306*theta^4-210300.4785*theta^3*arctan(1.906683204*r)+8420.504193*theta^2*arctan(1.906683204*r)^2-149.8488499*theta*arctan(1.906683204*r)^3+.9999999980*arctan(1.906683204*r)^4)-3.294107309*sin(theta-.5338712971e-1*arctan(1.906683204*r))*ln(1403.417367*theta^2-74.92442500*theta*arctan(1.906683204*r)+.9999999990*arctan(1.906683204*r)^2-95.24616774*theta+2.542459756*arctan(1.906683204*r)+1.732050808)+3.294107309*sin(theta-.5338712971e-1*arctan(1.906683204*r))*ln(1403.417367*theta^2-74.92442500*theta*arctan(1.906683204*r)+.9999999990*arctan(1.906683204*r)^2+95.24616774*theta-2.542459756*arctan(1.906683204*r)+1.732050808)+24.46655598*sin(theta-.5338712971e-1*arctan(1.906683204*r))*arctan(109.9808011*theta-2.935779647*arctan(1.906683204*r)-3.732050805)+24.46655598*sin(theta-.5338712971e-1*arctan(1.906683204*r))*arctan(109.9808011*theta-2.935779647*arctan(1.906683204*r)+3.732050805)+.7339584247e-3*sin(theta-.5338712971e-1*arctan(1.906683204*r))*theta-.1959196681e-4*sin(theta-.5338712971e-1*arctan(1.906683204*r))*arctan(1.906683204*r)+.1467916850e-3*cos(theta-.5338712971e-1*arctan(1.906683204*r))*theta^2-.7836786729e-5*cos(theta-.5338712971e-1*arctan(1.906683204*r))*theta*arctan(1.906683204*r)+.1045958874e-6*cos(theta-.5338712971e-1*arctan(1.906683204*r))*arctan(1.906683204*r)^2+.9999999996*F5*theta^2-.1067742595*F5*theta*arctan(1.906683204*r)+.2850185622e-2*F5*arctan(1.906683204*r)^2" **(9.4)**

> $Fmc1 := simplify(eval(Fmc1, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\,\alpha, \_C2 = 0.8, \_C1 = 0.5, Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02]));$

$$Fmc1 := [[-15375.03872\,\alpha + 1948.637127 - 55895.04375\,r^2\,\alpha + 7084.155000\,r^2, -3222.356483 + 1565.061225\,\alpha - 11354.13000\,r^2 + 2844.843751\,r^2\,\alpha], [-3222.356483 + 1565.061225\,\alpha - 11354.13000\,r^2 + 2844.843751\,r^2\,\alpha, 635.8312800 - 159.3112501\,\alpha]] \quad (9.5)$$

α is chosen very small to garanty the positive definiteness of the Fmc1 matrix

> $Fmc1 := simplify(eval(Fmc1, [\alpha = 0.001]));$

$$Fmc1 := \begin{bmatrix} 1933.262088 + 7028.259956\, r^2 & -3220.791422 - 11351.28516\, r^2 \\ -3220.791422 - 11351.28516\, r^2 & 635.6719687 \end{bmatrix} \quad \textbf{(9.6)}$$

> convert(Fmc1, string)

"Matrix(2, 2, [[1933.262088+7028.259956*r^2,-3220.791422-11351.28516*r^2], [-3220.791422-11351.28516*r^2,635.6719687]])" **(9.7)**

The control law numerical value is

> $tauf := simplify(eval(F1 + F2 + F3, [\_C1 = 0.5, \_C2 = 8.5, Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02])) :$

> $tauf := simplify(eval(tauf, [F5 = 5762.294642, \alpha = \alpha, \nu = -87.67850000\,\alpha + 11.11240000, \sigma = 35.62080000 - 8.925000004\,\alpha])) :$

> $tauf := simplify(eval(tauf, [\alpha = 0.001]));$

$$\begin{aligned} tauf := -\big(&2.000000000\,10^{32}\,\big(-8.109795405\,10^{27}\,\theta^3 - 1.566009438\,10^{24}\,rdot \\ &+ 5.000000000\,10^{27}\,rdot\,r^2\,\theta^{13} - 5.000000000\,10^{22}\,\theta dot\,r^4\,\theta^7 \\ &+ 2.500000000\,10^{25}\,\theta dot\,r^4\,\theta^9 - 1.760226417\,10^{31}\,\theta dot\,r^4\,\theta^6 \\ &- 7.222723510\,10^{35}\,\theta dot\,r^4\,\theta^{10} - 5.000000000\,10^{13}\sin(\theta)\,\theta \\ &+ 2.000000000\,10^{17}\sin(\theta)\,\theta^3 - 4.470452874\,10^{30}\sin(\theta)\,\theta^6 \\ &+ 3.000000000\,10^{24}\sin(\theta)\,\theta^9 + 3.667843448\,10^{40}\sin(\theta)\,\theta^{16} \\ &+ 4.598891022\,10^{28}\,\theta dot\,r^4\,\theta^4 - 1.651715854\,10^{21}\cos(\theta \\ &- 0.05338712971\arctan(1.906683204\,r))\ln\big(-350.8543417\,\theta^2 + 47.62308387\,\theta \\ &- 1.732050808\big) + 1.839303287\,10^{17}\sin(\theta \\ &- 0.05338712971\arctan(1.906683204\,r))\ln\big(3. - 1052.563025\,\theta^2 \\ &+ 1.230987691\,10^5\,\theta^4\big) + 1.651715855\,10^{21}\cos(\theta \\ &- 0.05338712971\arctan(1.906683204\,r))\ln\big(350.8543417\,\theta^2 + 47.62308387\,\theta \\ &+ 1.732050808\big) + 5.000000000\,10^{14}\sin(\theta)\,r^2\,\theta - 2.500000000\,10^{19}\,r^3\cos(\theta)\,\theta^3 \\ &+ 8.126033480\,10^{32}\,r^3\cos(\theta)\,\theta^6 - 3.845404647\,10^{15}\,\theta\arctan(1.906683204\,r) \\ &+ 2.500000000\,10^{20}\sin(\theta)\,r^2\,\theta^5 - 2.000000000\,10^{17}\sin(\theta)\,r^2\,\theta^3 \\ &- 1.625206697\,10^{31}\sin(\theta)\,r^2\,\theta^6 + 1.537106441\,10^{35}\,\theta dot\,r^2\,\theta^8 \\ &- 2.537884557\,10^{37}\,\theta dot\,r^2\,\theta^{10} + 2.720751661\,10^{39}\,\theta dot\,r^2\,\theta^{12} \\ &+ 1.638147013\,10^{43}\,\theta^{17} + 5.000000000\,10^{26}\,\theta dot\,r^4\,\theta^{11} \\ &- 4.939490965\,10^{39}\,\theta dot\,r^4\,\theta^{14} + 7.743156370\,10^{37}\,\theta dot\,r^4\,\theta^{12} \\ &+ 1.444201541\,10^{41}\,\theta dot\,r^4\,\theta^{16} + 4.374546822\,10^{33}\,\theta dot\,r^4\,\theta^8 \end{aligned} \quad \textbf{(1)}$$

$$-2.250000000\ 10^{19}\ \theta dot\ \theta^{5} + 2.500000000\ 10^{18}\ rdot\ \theta^{5}$$
$$+ 5.094457950\ 10^{22}\ \theta dot\ r^{4} + 5.000000000\ 10^{29} \sin(\theta)\ \theta^{15}$$
$$-4.560602953\ 10^{39} \sin(\theta)\ r^{2}\ \theta^{14} + 2.058817068\ 10^{29} \cos(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \ln\left(350.8543417\ \theta^{2} + 47.62308387\ \theta\right.$$
$$\left.+ 1.732050808\right)\ \theta^{15} - 6.176451205\ 10^{27} \cos(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \ln\left(350.8543417\ \theta^{2} + 47.62308387\ \theta\right.$$
$$\left.+ 1.732050808\right)\ \theta^{13} + 4.682363872\ 10^{39} \cos(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \ln\left(350.8543417\ \theta^{2} + 47.62308387\ \theta\right.$$
$$\left.+ 1.732050808\right)\ \theta^{16} - 3.229726184\ 10^{35} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r))\ \theta^{10} + 1.268002548\ 10^{37} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r))\ \theta^{12} - 4.062500000\ 10^{25} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r))\ \theta^{9} - 6.273057500\ 10^{36} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r))\ \theta^{14} + 1.111005108\ 10^{33} \sin(\theta)\ \theta^{8}$$
$$-1.834357488\ 10^{35} \sin(\theta)\ \theta^{10} + 7.149210725\ 10^{37} \sin(\theta)\ r^{2}\ \theta^{12}$$
$$+ 5.519956695\ 10^{41}\ rdot\ r^{2}\ \theta^{14} + 4.885382496\ 10^{23}\ \theta dot$$
$$-4.736775320\ 10^{40}\ \theta dot\ \theta^{14} - 2.763680375\ 10^{30}\ \theta dot\ \theta^{13} + 5.778605565\ 10^{24}\ \theta$$
$$-5.262360750\ 10^{17}\ \theta dot\ \theta^{3} - 1.687987106\ 10^{32}\ \theta dot\ \theta^{6}$$
$$+ 6.250022940\ 10^{14} \sin(\theta - 0.05338712971 \arctan(1.906683204\ r))\ \theta$$
$$+ 1.790065221\ 10^{24}\ \theta dot\ r^{2} - 9.005547060\ 10^{14} \cos(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r))\ \theta^{2} + 5.000000000\ 10^{23}\ r \cos(\theta)\ \theta^{7}$$
$$-4.500000000\ 10^{24}\ r^{3} \cos(\theta)\ \theta^{7} - 2.019496676\ 10^{35}\ r^{3} \cos(\theta)\ \theta^{8}$$
$$-5.555025545\ 10^{34}\ r \cos(\theta)\ \theta^{8} + 2.228534829\ 10^{23} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \arctan\left(-1.732050808\right.$$
$$\left.+ 405.1316975\ \theta^{2}\right)\ \theta^{9} + 3.342802244\ 10^{25} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \arctan\left(-1.732050808\right.$$
$$\left.+ 405.1316975\ \theta^{2}\right)\ \theta^{11} + 8.667447530\ 10^{37} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \arctan\left(-1.732050808\right.$$
$$\left.+ 405.1316975\ \theta^{2}\right)\ \theta^{14} + 3.342802244\ 10^{27} \sin(\theta$$
$$-0.05338712971 \arctan(1.906683204\ r)) \arctan\left(-1.732050808\right.$$

$$+ 405.1316975\, \theta^{2})\, \theta^{13} - 2.534176331\, 10^{39} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \arctan(-1.732050808$$
$$+ 405.1316975\, \theta^{2})\, \theta^{16} - 1.114267414\, 10^{29} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \arctan(-1.732050808$$
$$+ 405.1316975\, \theta^{2})\, \theta^{15} + 1.254566984\, 10^{24} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \arctan(-1.732050808$$
$$+ 405.1316975\, \theta^{2})\, \theta^{2} + 3.088713030\, 10^{29} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \arctan(-1.732050808$$
$$+ 405.1316975\, \theta^{2})\, \theta^{6} - 7.676126000\, 10^{31} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \arctan(-1.732050808$$
$$+ 405.1316975\, \theta^{2})\, \theta^{8} + 1.647053654\, 10^{16} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{3} + 2.058817068\, 10^{13} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta + 1.491044050\, 10^{27} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{4} - 2.341738230\, 10^{34} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{10} + 2.510472020\, 10^{36} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{12} - 4.117634138\, 10^{23} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{9} - 6.176451205\, 10^{25} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{11} - 1.601472742\, 10^{38} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(350.8543417\, \theta^{2} + 47.62308387\, \theta$$
$$+ 1.732050808)\, \theta^{14} - 2.607692364\, 10^{30} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\, r))\, \ln(3. - 1052.563025\, \theta^{2}$$
$$+ 1.230987691\, 10^{5}\, \theta^{4})\, \theta^{10} + 2.795589462\, 10^{32} \sin(\theta$$

$$
\begin{aligned}
& - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(3. - 1052.563025\, \theta^{2} \\
& + 1.230987691\ 10^{5}\, \theta^{4}\big)\, \theta^{12} + 3.125000000\ 10^{30} \sin\big(\theta \\
& - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{15} - 5.000000000\ 10^{22} \sin(\theta)\, r^{2}\, \theta^{7} \\
& + 4.038993350\ 10^{33} \sin(\theta)\, r^{2}\, \theta^{8} + 5.000000000\ 10^{30} \sin(\theta)\, r^{2}\, \theta^{15} \\
& - 6.926304495\ 10^{36}\, \theta dot\, \theta^{10} + 2.220227018\ 10^{37}\, rdot\, \theta^{10} \\
& - 2.380205328\ 10^{39}\, rdot\, \theta^{12} + 3.561187500\ 10^{26}\, rdot\, \theta^{9} \\
& - 5.344281250\ 10^{28}\, rdot\, \theta^{11} + 7.425378905\ 10^{38}\, \theta dot\, \theta^{12} \\
& - 6.667113940\ 10^{42}\, r^{3} \cos(\theta)\, \theta^{16} - 1.833921724\ 10^{42}\, r \cos(\theta)\, \theta^{16} \\
& - 2.000000000\ 10^{32}\, r \cos(\theta)\, \theta^{15} + 3.334349065\ 10^{37}\, r^{3} \cos(\theta)\, \theta^{10} \\
& + 5.000000000\ 10^{24} \sin(\theta)\, r^{2}\, \theta^{9} - 6.668698135\ 10^{35} \sin(\theta)\, r^{2}\, \theta^{10} \\
& - 2.352389072\ 10^{31} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{6} \\
& - 3.125000000\ 10^{20} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{5} \\
& + 3.672989381\ 10^{33} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{8} \\
& + 1.250000000\ 10^{23} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{7} \\
& + 5.234217095\ 10^{35} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{16} \\
& + 8.925899105\ 10^{28} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{5} \\
& - 3.217540288\ 10^{35} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{11} \\
& + 1.257313590\ 10^{37} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{13} \\
& + 1.803402052\ 10^{25} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{10} \\
& - 3.125000003\ 10^{17} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{3} \\
& + 8.932240930\ 10^{28} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{4} \\
& - 5.216375930\ 10^{34} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{18} \\
& - 2.293620080\ 10^{24} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{17} \\
& + 1.148521622\ 10^{15} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{4} \\
& - 2.107067714\ 10^{36} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{15} \\
& + 4.585278988\ 10^{17} \cos\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{6} \\
& - 4.585278988\ 10^{19} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(3. \\
& - 1052.563025\, \theta^{2} + 1.230987691\ 10^{5}\, \theta^{4}\big)\, \theta^{9} - 6.877918475\ 10^{21} \sin\big(\theta \\
& - 0.05338712971 \arctan(1.906683204\, r)\big) \ln\big(3. - 1052.563025\, \theta^{2}
\end{aligned}
$$

$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{11} - 1.783354000\ 10^{34} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{14} - 6.877918475\ 10^{23} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{13} + 5.214145785\ 10^{35} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{16} + 2.292639493\ 10^{25} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{15} - 2.581310175\ 10^{20} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{2} - 6.355122100\ 10^{25} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{6} + 1.579386546\ 10^{28} \sin(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \ln\Big(3. - 1052.563025\ \theta^{2}$$
$$+ 1.230987691\ 10^{5}\ \theta^{4})\ \theta^{8} + 1.223327799\ 10^{17} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{3} + 1.529159749\ 10^{14} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta + 1.107453682\ 10^{28} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{4} - 1.739295782\ 10^{35} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{10} + 1.864620622\ 10^{37} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{12} - 3.058319498\ 10^{24} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{9} - 4.587479247\ 10^{26} \cos(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r))\ \arctan(54.99040054\ \theta$$
$$- 3.732050805)\ \theta^{11} + 2.752344093\ 10^{25} \cos(\theta$$

$$
\begin{aligned}
&- 0.05338712971 \arctan(1.906683204\, r))\ \theta^{16} + 1.834111594\ 10^{12} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(3. - 1052.563025\ \theta^{2} \\
&+ 1.230987691\ 10^{5}\ \theta^{4}\big)\ \theta^{3} + 2.292639493\ 10^{9} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(3. - 1052.563025\ \theta^{2} \\
&+ 1.230987691\ 10^{5}\ \theta^{4}\big)\ \theta + 1.660383784\ 10^{23} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(3. - 1052.563025\ \theta^{2} \\
&+ 1.230987691\ 10^{5}\ \theta^{4}\big)\ \theta^{4} - 2.318046715\ 10^{24} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&+ 1.732050808\big)\ \theta^{2} - 5.706973945\ 10^{29} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&+ 1.732050808\big)\ \theta^{6} + 1.418307583\ 10^{32} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&+ 1.732050808\big)\ \theta^{8} - 1.647053654\ 10^{16} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{3} - 2.058817067\ 10^{13} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta - 1.491044049\ 10^{27} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{4} + 2.341738228\ 10^{34} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{10} - 2.510472019\ 10^{36} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{12} + 4.117634134\ 10^{23} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{9} + 6.176451200\ 10^{25} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{11} + 1.601472741\ 10^{38} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\ \ln\big(-350.8543417\ \theta^{2} + 47.62308387\ \theta \\
&- 1.732050808\big)\ \theta^{14} + 6.176451200\ 10^{27} \cos(\theta
\end{aligned}
$$

$$
\begin{aligned}
&- 0.05338712971 \arctan(1.906683204\, r)) \ln(-350.8543417\, \theta^2 + 47.62308387\, \theta \\
&- 1.732050808)\, \theta^{13} - 4.682363869\, 10^{39} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \ln(-350.8543417\, \theta^2 + 47.62308387\, \theta \\
&- 1.732050808)\, \theta^{16} - 2.058817067\, 10^{29} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \ln(-350.8543417\, \theta^2 + 47.62308387\, \theta \\
&- 1.732050808)\, \theta^{15} + 2.318046714\, 10^{24} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \ln(-350.8543417\, \theta^2 + 47.62308387\, \theta \\
&- 1.732050808)\, \theta^{2} + 5.706973945\, 10^{29} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \ln(-350.8543417\, \theta^2 + 47.62308387\, \theta \\
&- 1.732050808)\, \theta^{6} + 6.259651110\, 10^{35} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r))\, \theta^{17} - 1.254484160\, 10^{39} \sin(\theta)\, \theta^{14} \\
&+ 1.333422788\, 10^{41} \sin(\theta)\, r^2\, \theta^{16} - 2.784933059\, 10^{29} \arctan(1.906683204\, r)\, \theta^{4} \\
&+ 9.171787435\, 10^{36}\, r \cos(\theta)\, \theta^{10} - 9.832659965\, 10^{38}\, r \cos(\theta)\, \theta^{12} \\
&- 1.418307582\, 10^{32} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r)) \ln(\\
&-350.8543417\, \theta^2 + 47.62308387\, \theta - 1.732050808)\, \theta^{8} - 1.358710888\, 10^{36} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(-1.732050808 \\
&+ 405.1316975\, \theta^2)\, \theta^{12} - 8.914139310\, 10^{15} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(-1.732050808 \\
&+ 405.1316975\, \theta^2)\, \theta^{3} - 1.114267414\, 10^{13} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(-1.732050808 \\
&+ 405.1316975\, \theta^2)\, \theta - 8.069788345\, 10^{26} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(-1.732050808 \\
&+ 405.1316975\, \theta^2)\, \theta^{4} + 1.267389241\, 10^{34} \sin(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(-1.732050808 \\
&+ 405.1316975\, \theta^2)\, \theta^{10} - 1.189473166\, 10^{39} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\, \theta^{14} - 4.587479247\, 10^{28} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\, \theta^{13} - 8.257032275\, 10^{23} \cos(\theta
\end{aligned}
$$

$$
\begin{aligned}
&- 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{14} + 2.500000000\ 10^{26}\, r \cos(\theta)\, \theta^{9} \\
&- 4.000000000\ 10^{28}\, r \cos(\theta)\, \theta^{11} - 3.574605363\ 10^{39}\, r^{3} \cos(\theta)\, \theta^{12} \\
&- 1.500000000\ 10^{27} \sin(\theta)\, r^{2}\, \theta^{11} + 2.137037500\ 10^{27}\, rdot\, r^{2}\, \theta^{9} \\
&- 3.561437500\ 10^{29}\, rdot\, r^{2}\, \theta^{11} - 6.664832900\ 10^{26}\, \theta dot\, r^{2}\, \theta^{9} \\
&+ 1.094972150\ 10^{29}\, \theta dot\, r^{2}\, \theta^{11} + 8.071503985\ 10^{37}\, rdot\, r^{2}\, \theta^{10} \\
&- 8.653095670\ 10^{39}\, rdot\, r^{2}\, \theta^{12} - 1.344712564\ 10^{35}\, rdot\, \theta^{8} \\
&+ 1.958903125\ 10^{24}\, rdot\, \theta^{7} + 1.293841177\ 10^{22} \sin(\theta) \\
&+ 5.074563335\ 10^{42}\, \theta dot\, r^{2}\, \theta^{16} + 5.562360750\ 10^{32}\, \theta dot\, r^{2}\, \theta^{15} \\
&+ 5.762294640\ 10^{19}\, \theta^{4} + 1.226790588\ 10^{22} \cos\big(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)\big) \arctan(54.99040054\, \theta - 3.732050805) \\
&- 8.939371950\ 10^{20} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big) \arctan\big(\\
&-1.732050808 + 405.1316975\, \theta^{2}\big) + 1.167981879\ 10^{28} \sin(\theta)\, \theta^{4} \\
&+ 1.966531992\ 10^{37} \sin(\theta)\, \theta^{12} - 1.815799177\ 10^{25} \sin(\theta)\, \theta^{2} \\
&+ 5.000000000\ 10^{30}\, \theta dot\, r^{4}\, \theta^{15} + 2.500000000\ 10^{14}\, \theta dot\, r^{4}\, \theta \\
&+ 1.226790588\ 10^{22} \cos\big(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)\big) \arctan(54.99040054\, \theta + 3.732050805) \\
&+ 6.272420795\ 10^{40}\, r \cos(\theta)\, \theta^{14} + 2.280301475\ 10^{41}\, r^{3} \cos(\theta)\, \theta^{14} \\
&- 5.000000000\ 10^{30}\, r^{3} \cos(\theta)\, \theta^{13} - 2.123064514\ 10^{30}\, r^{3} \cos(\theta)\, \theta^{4} \\
&+ 9.078995890\ 10^{26}\, r \cos(\theta)\, \theta^{2} + 4.246129026\ 10^{28} \sin(\theta)\, r^{2}\, \theta^{4} \\
&- 5.000000000\ 10^{15}\, r \cos(\theta)\, \theta - 6.601230490\ 10^{25} \sin(\theta)\, r^{2}\, \theta^{2} \\
&- 5.839909400\ 10^{29}\, r \cos(\theta)\, \theta^{4} + 3.300615248\ 10^{27}\, r^{3} \cos(\theta)\, \theta^{2} \\
&+ 2.235226439\ 10^{32}\, r \cos(\theta)\, \theta^{6} + 5.000000000\ 10^{21}\, r \cos(\theta)\, \theta^{5} \\
&+ 5.000000000\ 10^{21}\, r^{3} \cos(\theta)\, \theta^{5} + 1.641208225\ 10^{28}\, \theta dot\, \theta^{11} \\
&+ 1.518373406\ 10^{41}\, rdot\, \theta^{14} + 8.905468750\ 10^{30}\, rdot\, \theta^{13} \\
&- 1.908258125\ 10^{26} \sin\big(\theta - 0.05338712971 \arctan(1.906683204\, r)\big)\, \theta^{2} \\
&+ 1.153621394\ 10^{28} \arctan(1.906683204\, r)\, \theta^{11} + 1.780093750\ 10^{22}\, rdot\, r^{2}\, \theta^{5} \\
&+ 1.783093750\ 10^{24}\, rdot\, r^{2}\, \theta^{7} - 5.362360750\ 10^{23}\, \theta dot\, r^{2}\, \theta^{7} \\
&- 1.089972150\ 10^{26}\, \theta dot\, \theta^{9} - 2.351841528\ 10^{24}\, r^{3} \cos(\theta) \\
&- 1.440573660\ 10^{27}\, \theta^{10} - 2.160860489\ 10^{29}\, \theta^{12} + 4.962015753\ 10^{35}\, \theta^{9}
\end{aligned}
$$

$$
\begin{aligned}
&- 9.017120355\ 10^{26} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r))\ \theta^{12} \\
&- 1.907656678\ 10^{26} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r))\ \theta^{3} \\
&- 2.349420562\ 10^{31} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r))\ \theta^{7} \\
&+ 3.665175788\ 10^{33} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r))\ \theta^{9} \\
&+ 9.038359330\ 10^{22} \cos(\theta - 0.05338712971 \arctan(1.906683204\, r))\ \theta^{8} \\
&+ 3.477765203\ 10^{40} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\ \theta^{16} + 1.529159749\ 10^{30} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\ \theta^{15} - 1.721699216\ 10^{25} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\ \theta^{2} - 4.238781088\ 10^{30} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\ \theta^{6} + 1.053429613\ 10^{33} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&- 3.732050805)\ \theta^{8} + 1.223327799\ 10^{17} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta^{3} + 1.529159749\ 10^{14} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta + 1.107453682\ 10^{28} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta^{4} - 1.739295782\ 10^{35} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta^{10} + 1.864620622\ 10^{37} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta^{12} - 3.058319498\ 10^{24} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta \\
&+ 3.732050805)\ \theta^{9} - 4.587479247\ 10^{26} \cos(\theta \\
&- 0.05338712971 \arctan(1.906683204\, r)) \arctan(54.99040054\, \theta
\end{aligned}
$$

$$+3.732050805)\,\theta^{11}-1.189473166\;10^{39}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{14}-4.587479247\;10^{28}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{13}+3.477765203\;10^{40}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{16}+1.529159749\;10^{30}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{15}-1.721699216\;10^{25}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{2}-4.238781088\;10^{30}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{6}+1.053429613\;10^{33}\cos(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))\arctan(54.99040054\,\theta$$
$$+3.732050805)\,\theta^{8}-6.469205885\;10^{23}\,r\cos(\theta)+1.384931822\;10^{42}\,\theta dot\,\theta^{16}$$
$$+1.646208225\;10^{32}\,\theta dot\,\theta^{15}-7.149650760\;10^{25}\,\theta dot\,r^{4}\,\theta^{2}$$
$$+2.500000000\;10^{17}\,\theta dot\,r^{4}\,\theta^{3}-1.735613694\;10^{41}\,\theta dot\,r^{2}\,\theta^{14}$$
$$-1.613917310\;10^{43}\,rdot\,r^{2}\,\theta^{16}-1.780593750\;10^{33}\,rdot\,r^{2}\,\theta^{15}$$
$$-8.192681280\;10^{37}\,\theta^{11}-2.160860489\;10^{31}\,\theta^{14}+8.783004380\;10^{39}\,\theta^{13}$$
$$+7.202868300\;10^{32}\,\theta^{16}-5.602827675\;10^{41}\,\theta^{15}$$
$$-3.085031654\;10^{23}\arctan(1.906683204\,r)+4.195016366\;10^{34}\,\theta dot\,\theta^{8}$$
$$-5.988596825\;10^{23}\,\theta dot\,\theta^{7}+7.202868300\;10^{16}\,\theta^{2}+1.812102485\;10^{23}\sin(\theta$$
$$-0.05338712971\arctan(1.906683204\,r))-4.888622978\;10^{35}\,rdot\,r^{2}\,\theta^{8}$$
$$+4.703683056\;10^{22}\sin(\theta)\,r^{2}-5.487360750\;10^{21}\,\theta dot\,r^{2}\,\theta^{5}$$
$$+8.902968750\;10^{16}\,rdot\,r^{2}\,\theta-3.563687500\;10^{19}\,rdot\,r^{2}\,\theta^{3}$$
$$+1.967079942\;10^{33}\,rdot\,r^{2}\,\theta^{6}+4.329586998\;10^{26}\,\theta^{2}\arctan(1.906683204\,r)$$
$$+1.812104329\;10^{23}\cos(\theta-0.05338712971\arctan(1.906683204\,r))\,\theta$$
$$+1.780093750\;10^{16}\,rdot\,\theta+1.780593750\;10^{18}\,rdot\,\theta^{3}$$
$$+5.410842940\;10^{32}\,rdot\,\theta^{6}-5.342531250\;10^{32}\,rdot\,\theta^{15}$$

$$- 4.439399179\ 10^{42}\ rdot\ \theta^{16} + 1.153621394\ 10^{30} \arctan(1.906683204\ r)\ \theta^{13}$$
$$- 3.000000000\ 10^{29} \sin(\theta)\ r^2\ \theta^{13} + 5.216487700\ 10^{30}\ \theta^5 - 1.996611664\ 10^{33}\ \theta^7$$
$$+ 2.991188879\ 10^{40} \arctan(1.906683204\ r)\ \theta^{14}$$
$$- 3.076323717\ 10^{18}\ \theta^3 \arctan(1.906683204\ r)$$
$$- 8.745596720\ 10^{41} \arctan(1.906683204\ r)\ \theta^{16}$$
$$- 3.845404647\ 10^{31} \arctan(1.906683204\ r)\ \theta^{15}$$
$$+ 1.065933660\ 10^{32} \arctan(1.906683204\ r)\ \theta^6$$
$$- 2.649077790\ 10^{34} \arctan(1.906683204\ r)\ \theta^8$$
$$+ 4.373837387\ 10^{36} \arctan(1.906683204\ r)\ \theta^{10} - 2.512208622\ 10^{27}\ \theta dot\ r^2\ \theta^2$$
$$+ 1.615935386\ 10^{30}\ \theta dot\ r^2\ \theta^4 - 5.693134670\ 10^{24}\ rdot\ r^2$$
$$- 6.856230640\ 10^{26}\ \theta dot\ \theta^2 + 4.410153532\ 10^{29}\ \theta dot\ \theta^4$$
$$+ 2.197764843\ 10^{27}\ rdot\ \theta^2 - 1.413674786\ 10^{30}\ rdot\ \theta^4$$
$$- 5.537360750\ 10^{15}\ \theta dot\ \theta + 7.989844080\ 10^{27}\ rdot\ r^2\ \theta^2$$
$$- 5.139331055\ 10^{30}\ rdot\ r^2\ \theta^4 - 2.756180375\ 10^{16}\ \theta dot\ r^2\ \theta$$
$$+ 1.102472150\ 10^{19}\ \theta dot\ r^2\ \theta^3 - 6.184995780\ 10^{32}\ \theta dot\ r^2\ \theta^6$$
$$+ 7.690809295\ 10^{25} \arctan(1.906683204\ r)\ \theta^9$$
$$- 4.688993945\ 10^{38} \arctan(1.906683204\ r)\ \theta^{12})) \Big/ ((1.75357\ 10^5$$
$$+ 6.37500\ 10^5\ r^2)\ \left(4.000000001\ 10^9 - 1.403417368\ 10^{12}\ \theta^2\right.$$
$$\left.+ 1.641316923\ 10^{14}\ \theta^4\right)\ \left(4.385679270\ 10^{10}\ \theta^2 + 5.952885485\ 10^9\ \theta\right.$$
$$\left.+ 2.16506351\ 10^8\right)\ \left(4.385679270\ 10^{10}\ \theta^2 - 5.952885485\ 10^9\ \theta\right.$$
$$\left.+ 2.16506351\ 10^8\right)\ \left(1.20000\ 10^5 - 4.2102521\ 10^7\ \theta^2\right.$$
$$\left.+ 4.923950764\ 10^9\ \theta^4\right)\ \left(1.492820321\ 10^9 + 3.023944152\ 10^{11}\ \theta^2\right.$$
$$\left.- 4.104539372\ 10^{10}\ \theta\right)\ \left(1.492820321\ 10^9 + 3.023944152\ 10^{11}\ \theta^2\right.$$
$$\left.+ 4.104539372\ 10^{10}\ \theta\right))$$

The potential numerical value is

> *phif* := *PhiGraph*;

$$phif := 1.782827863 \cos(\theta - 0.05338712971 \arctan(1.906683204\ r)) \arctan\Big($$
$$-1.732050808 + 1620.526790\ \theta^2 - 86.51527393\ \theta \arctan(1.906683204\ r)$$
$$+ 1.154700538 \arctan(1.906683204\ r)^2\Big) - 0.0003668223189 \cos\Big(\theta$$
$$- 0.05338712971 \arctan(1.906683204\ r)\Big) \ln\Big(3. - 4210.252101\ \theta^2 \qquad \mathbf{(2)}$$

$$
\begin{aligned}
&+ 224.7732750\,\theta\,\arctan(1.906683204\,r) - 2.999999997\,\arctan(1.906683204\,r)^2 \\
&+ 1.969580306\;10^6\,\theta^4 - 2.103004785\;10^5\,\theta^3\,\arctan(1.906683204\,r) \\
&+ 8420.504193\,\theta^2\,\arctan(1.906683204\,r)^2 \\
&- 149.8488499\,\theta\,\arctan(1.906683204\,r)^3 \\
&+ 0.9999999980\,\arctan(1.906683204\,r)^4\Big) - 3.294107309\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\ln\Big(1403.417367\,\theta^2 \\
&- 74.92442500\,\theta\,\arctan(1.906683204\,r) + 0.9999999990\,\arctan(1.906683204\,r)^2 \\
&- 95.24616774\,\theta + 2.542459756\,\arctan(1.906683204\,r) + 1.732050808\Big) \\
&+ 3.294107309\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\ln\Big(1403.417367\,\theta^2 \\
&- 74.92442500\,\theta\,\arctan(1.906683204\,r) + 0.9999999990\,\arctan(1.906683204\,r)^2 \\
&+ 95.24616774\,\theta - 2.542459756\,\arctan(1.906683204\,r) + 1.732050808\Big) \\
&+ 24.46655598\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\arctan\Big(109.9808011\,\theta \\
&- 2.935779647\,\arctan(1.906683204\,r) - 3.732050805\Big) + 24.46655598\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\arctan\Big(109.9808011\,\theta \\
&- 2.935779647\,\arctan(1.906683204\,r) + 3.732050805\Big) + 0.0007339584247\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\theta - 0.00001959196681\,\sin\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\arctan(1.906683204\,r) \\
&+ 0.0001467916850\,\cos\big(\theta - 0.05338712971\,\arctan(1.906683204\,r)\big)\,\theta^2 \\
&- 0.000007836786729\,\cos\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\theta\,\arctan(1.906683204\,r) \\
&+ 1.045958874\;10^{-7}\,\cos\Big(\theta \\
&- 0.05338712971\,\arctan(1.906683204\,r)\Big)\,\arctan(1.906683204\,r)^2
\end{aligned}
$$

>

```matlab
%%%%%%%%%%%%%%  A.4 BALL AND BEAM SIMULATION  %%%%%%%%%%%%%%
%%%%%%%%%%%%%%       B_B_Nonlinear_Sys.m      %%%%%%%%%%%%%%
%%
function dxdt = B_B_NonLinear_Sys(u)

%% Main Vectors
theta       = u(1);                 % feedback array
r           = u(2);
thetadot    = u(3);
rdot        = u(4);

%% Generalized quantities
q           = [theta r]';           % Generalized coordinates
qdot        = [thetadot rdot]';     % Generalized velocities

%% Physical parameter values
Ro          = 0.02;                 % meters   - radius of ball
Ib          = 0.4;                  % kg m^2   - inertia of beam
m           = 1.5;                  % kg       - mass of ball
g           = 9.81;                 % m/sec^2 - acceleration of gravity

%% Linear model parameters
C2          = 8.5;
C1          = 0.5;
F5          = 5762.294642;
alpha       = 0.001;
nu          = -87.67850000*alpha+11.11240000;
sigma       = 35.62080000-8.925000004*alpha;

%% The G, M, C, P and KD matrices
            %gravity terms
G           =   [m*g*r*cos(theta)-...
            m*g*Ro*sin(theta); m*g*sin(theta)];
            %mass matrix
mass        =   [Ib+7/5*m*Ro^2+m*r^2 ...
           -7/5*m*Ro; -7/5*m*Ro 7/5*m];
            %centripetal and coriolis forces matrix
C           =   [m*r*rdot, m*r*thetadot; -m*r*thetadot, 0];
%%  FMC


KD          =↙
[5*C1*Ib+7*C1*m*Ro^2+5*C1*m*r^2+25*C2*Ib^2+70*C2*Ib*m*Ro^2+50*C2*Ib*m*r^2+49*↙
C2*m^2*Ro^4+...
            70*C2*m^2*Ro^2*r^2+25*C2*m^2*r^4,-7*↙
(C1+5*C2*Ib+7*Ro^2*C2*m+5*C2*m*r^2)*m*Ro;-7*(C1+5*C2*Ib+...
            7*Ro^2*C2*m+5*C2*m*r^2)*m*Ro,7*(7*Ro^2*C2*m+C1)*m];

P           =  KD*inv(mass);
DETP        =  det(P);
%% SMC
```

```
            % Kv Matrix

Kv          = [alpha*(5*C1+25*C2*Ib+35*Ro^2*C2*m+25*C2*m*r^2)^2,-35*alpha*
(5*C1+25*C2*Ib+35*Ro^2*C2*m+...
            25*C2*m*r^2)*C2*m*Ro;-35*alpha*
(5*C1+25*C2*Ib+35*Ro^2*C2*m+25*C2*m*r^2)*C2*m*Ro,1225*alpha*C2^2*m^2*Ro^2];

%% Evaluating the control law
            %FMC input
F1         = 87.67850000*thetadot*alpha-11.11240000*thetadot+35.
62080000*rdot-8.925000004*rdot*alpha ;

            %SMC input
F2          = -5*
(thetadot*C1+5*thetadot*C2*Ib+7*thetadot*C2*m*Ro^2+5*thetadot*C2*m*r^2-
7*C2*m*Ro*rdot)*alpha  ;
            %TMC input
F3          = -.1000000000e39*(.3845404647e21*atan(1.906683204*r)*theta^9+.
3678606574e12*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025
*theta^2+123098.7691*theta^4)-.3303431709e16*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-
47.62308387*theta+1.732050808)+.2453581176e17*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta-
3.732050805)-.1000000000e24*sin(theta)*theta^13-...
            .3000000000e12*sin(theta)*theta^3-.1000000000e15*sin(theta)
*theta^5-.3668714974e30*sin(theta)*theta^10+...
            .1200000000e20*sin(theta)*theta^9-.5000000000e17*sin(theta)
*theta^7+.3624208657e18*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta-.3202945482e33*cos
(theta-.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^14-.8235268275e15*cos(theta-
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^7-.1141394790e25*cos(theta-.
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^6+.2836615166e27*cos(theta-.
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^8+.1235290241e14*cos(theta-.
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^5-.4636093434e19*cos(theta-.
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^2+.4117634136e24*cos(theta-.
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^15+.5020944041e31*cos(theta-
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)*theta^12+.2214907366e23*cos(theta-
.5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta-...
            3.732050805)*theta^4+.6955530406e35*cos(theta-.5338712971e-1*atan
(1.906683204*r))*atan(54.99040054*theta-3.732050805)*theta^16+...
            .6116638995e23*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^13-...
```

```
            .3478591564e30*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^10-...
            .1529159749e20*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^9-...
            .1223327799e22*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^11-...
            .2378946331e34*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^14-...
            .6116638995e16*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan
(54.99040054*theta-3.732050805)*theta^7+...
            .1440573660e28*theta^16+.1756600876e35*theta^13-.
7202868297e22*theta^10+.9924031511e30*theta^9-.1638536256e33*theta^11+...
            625000000.*sin(theta-.5338712971e-1*atan(1.906683204*r))*theta-
1801114918.*cos(theta-.5338712971e-1*atan(1.906683204*r))*theta^2+...
            .7335686894e35*sin(theta)*theta^16+.8492258060e23*sin(theta)
*r^2*theta^4-.1320246099e21*sin(theta)*r^2*theta^2+...
            .1815799179e22*r*cos(theta)*theta^2-.1167981880e25*r*cos(theta)
*theta^4+.6601230494e22*r^3*cos(theta)*theta^2-...
            .3250413394e26*sin(theta)*r^2*theta^6+.8077986701e28*sin(theta)
*r^2*theta^8+.2000000000e19*r*cos(theta)*theta^7+...
            .1000000000e11*r*cos(theta)*theta+2000000000.*sin(theta)
*r^2*theta-.2000000000e15*sin(theta)*r^2*theta^5-...
            .1000000000e13*sin(theta)*r^2*theta^3+.1000000000e14*r*cos(theta)
*theta^3+.1000000000e17*r*cos(theta)*theta^5+...
            .1000000000e18*sin(theta)*r^2*theta^7-.1333739626e31*sin(theta)
*r^2*theta^10+.2000000000e20*sin(theta)*r^2*theta^9-...
            .3000000000e17*r^3*cos(theta)*theta^5+.1834357487e32*r*cos(theta)
*theta^10+.6668698128e32*r^3*cos(theta)*theta^10-...
            .2000000000e22*r^3*cos(theta)*theta^9-.9121205900e34*sin(theta)
*r^2*theta^14-.1000000000e24*sin(theta)*r^2*theta^13+...
            .2587682354e17*sin(theta)+.2881147320e26*theta^14+.
5982377759e35*atan(1.906683204*r)*theta^14+.1046843419e31*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^16-.4587240159e19*sin
(theta-.5338712971e-1*atan(1.906683204*r))*theta^17-...
            .1254611546e32*sin(theta-.5338712971e-1*atan(1.906683204*r))
*theta^14+.1250000000e18*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^7-.4704778144e26*sin
(theta-.5338712971e-1*atan(1.906683204*r))*theta^6+...
            .7345978756e28*sin(theta-.5338712971e-1*atan(1.906683204*r))
*theta^8+.1250001376e16*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^5+.1875000000e13*sin
(theta-.5338712971e-1*atan(1.906683204*r))*theta^3+...
            .2536005094e32*sin(theta-.5338712971e-1*atan(1.906683204*r))
*theta^12-.6459452369e30*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^10+.1733489506e33*sin
(theta-.5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^14+.4457069658e15*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^7+.6177426062e24*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^6-.1535225201e27*sin(theta-.
```

```
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^8-.6685604488e13*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^5+.2509133970e19*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^2-.2228534829e24*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^15-.2717421776e31*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^12+.3320767568e18*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^4+.1042829157e31*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^16+.9170557975e18*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^13-.5215384728e25*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^10-.2292639493e15*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^9-.1834111594e17*sin(theta-.
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^11-.7690809294e26*atan(1.906683204*r)
*theta^15-.5569866119e24*atan(1.906683204*r)*theta^4-...
            .1749119343e37*atan(1.906683204*r)*theta^16-.1538161859e25*atan
(1.906683204*r)*theta^13+...
            .3076323717e23*atan(1.906683204*r)*theta^11+.1254484159e36*r*cos
(theta)*theta^14+.4560602950e36*r^3*cos(theta)*theta^14-...
            .1000000000e26*r^3*cos(theta)*theta^13-.4246129029e25*r^3*cos
(theta)*theta^4-.1111005109e30*r*cos(theta)*theta^8+...
            .1625206696e28*r^3*cos(theta)*theta^6-.4038993351e30*r^3*cos
(theta)*theta^8+.1785179822e24*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^5-.6435080575e30*cos
(theta-.5338712971e-1*atan(1.906683204*r))*theta^11-...
            .1805727720e20*cos(theta-.5338712971e-1*atan(1.906683204*r))
*theta^10-.1808014854e22*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^12-.4698841125e26*cos
(theta-.5338712971e-1*atan(1.906683204*r))*theta^7+...
            .7330351569e28*cos(theta-.5338712971e-1*atan(1.906683204*r))
*theta^9-.3815313354e21*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^3+.2514627181e32*cos
(theta-.5338712971e-1*atan(1.906683204*r))*theta^13-...
            .4214135429e31*cos(theta-.5338712971e-1*atan(1.906683204*r))
*theta^15+.9187072038e12*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^6-2292639493.*cos
(theta-.5338712971e-1*atan(1.906683204*r))*theta^4+...
            .1786448188e24*sin(theta-.5338712971e-1*atan(1.906683204*r))
*theta^4+.1251930222e31*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^17+.1100937638e19*cos
(theta-.5338712971e-1*atan(1.906683204*r))*theta^14-...
            .1100937638e12*cos(theta-.5338712971e-1*atan(1.906683204*r))
*theta^8+.5504688186e20*cos(theta-...
```

```
            .5338712971e-1*atan(1.906683204*r))*theta^16-.1043275186e30*sin
(theta-.5338712971e-1*atan(1.906683204*r))*theta^18-...
            .3993223329e28*theta^7+.4321720978e17*theta^6-.
2881147320e19*theta^8+.1043297540e26*theta^5-.3816516251e21*sin(theta-...
            .5338712971e-1*atan(1.906683204*r))*theta^2+.3303431709e16*cos
(theta-.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2+...
            47.62308387*theta+1.732050808)+.2453581176e17*cos(theta-.
5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.732050805)-...
            .6170063307e18*atan(1.906683204*r)-.1787874390e16*sin(theta-.
5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)+.1155721113e20*theta+.
8659174001e21*theta^2*atan(1.906683204*r)-.1120565535e37*theta^15+...
            .3624204971e18*sin(theta-.5338712971e-1*atan(1.906683204*r))-.
1000000000e27*r*cos(theta)*theta^15+...
            .1429842145e33*sin(theta)*r^2*theta^12-.6000000000e22*sin(theta)
*r^2*theta^11-.1966531992e34*r*cos(theta)*theta^12-...
            .7149210723e34*r^3*cos(theta)*theta^12+.1000000000e24*r^3*cos
(theta)*theta^11+.4470452877e27*r*cos(theta)*theta^6-...
            .1333422788e38*r^3*cos(theta)*theta^16+.2666845576e36*sin(theta)
*r^2*theta^16-.1000000000e26*sin(theta)*r^2*theta^15-...
            .3667843448e37*r*cos(theta)*theta^16+.1538161859e18*atan
(1.906683204*r)*theta^7+.2131867321e27*atan(1.906683204*r)*theta^6+...
            .8747674774e31*atan(1.906683204*r)*theta^10-.9377987891e33*atan
(1.906683204*r)*theta^12-.4117634136e24*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-
47.62308387*theta+1.732050808)*theta^15-.3443398436e20*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta-
3.732050805)*theta^2+.3058319498e25*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta-
3.732050805)*theta^15+.3729241246e32*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta-
3.732050805)*theta^12+.2214907366e23*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^4+.6955530406e35*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^16+.6116638995e23*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^13-.3478591564e30*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^10-.1529159749e20*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^9-.1223327799e22*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^11-.2378946331e34*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^14-.6116638995e16*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^7-.8477562175e25*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
732050805)*theta^6+.2106859226e28*cos(theta-...
            .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.
```

```
732050805)*theta^8+.9174958494e14*cos(theta-...
             .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.↙
732050805)*theta^5-.3443398436e20*cos(theta-...
             .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.↙
732050805)*theta^2+.3058319498e25*cos(theta-...
             .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.↙
732050805)*theta^15+.3729241246e32*cos(theta-...
             .5338712971e-1*atan(1.906683204*r))*atan(54.99040054*theta+3.↙
732050805)*theta^12-.5298155580e29*atan(1.906683204*r)*theta^8+...
             .2534778482e29*sin(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(-1.732050808+405.1316975*theta^2)*theta^10+...
             .1114267414e19*sin(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(-1.732050808+405.1316975*theta^2)*theta^9+...
             .8914139312e20*sin(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(-1.732050808+405.1316975*theta^2)*theta^11-...
             .5020944041e31*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2-47.62308387*theta+1.732050808)*theta^12+...
             .2982088099e22*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^4+...
             .9364727744e34*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^16-...
             .5068352662e34*sin(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(-1.732050808+405.1316975*theta^2)*theta^16+...
             .8235268275e22*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^13-...
             .4683476459e29*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^10-...
             .2058817068e19*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^9-...
             .8477562175e25*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(54.99040054*theta-3.732050805)*theta^6+...
             .2106859226e28*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(54.99040054*theta-3.732050805)*theta^8+...
             .9174958494e14*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(54.99040054*theta-3.732050805)*theta^5-...
             .1271024421e21*sin(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(3.-1052.563025*theta^2+123098.7691*theta^4)*theta^6+...
             .3158773092e23*sin(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(3.-1052.563025*theta^2+123098.7691*theta^4)*theta^8+...
             1375583696.*sin(theta-.5338712971e-1*atan(1.906683204*r))*log(3.↙
-1052.563025*theta^2+123098.7691*theta^4)*theta^5-...
             .1647053654e21*cos(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(350.8543417*theta^2+47.62308387*theta+1.732050808)*theta^11-...
             .1621959083e23*theta^3-.1293841177e19*r*cos(theta)-.↙
4703683057e19*r^3*cos(theta)+.9407366115e17*sin(theta)*r^2-...
             .2307242788e16*atan(1.906683204*r)*theta^5-.3566708000e29*sin↙
(theta-.5338712971e-1*atan(1.906683204*r))*log(3.-...
             1052.563025*theta^2+123098.7691*theta^4)*theta^14-.↙
9170557975e11*sin(theta-.5338712971e-1*atan(1.906683204*r))*log(3.-...
             1052.563025*theta^2+123098.7691*theta^4)*theta^7-.↙
4457069658e22*sin(theta-.5338712971e-1*atan(1.906683204*r))*atan(-...
```

```
            1.732050808+405.1316975*theta^2)*theta^13-.1613957669e22*sin↙
(theta-.5338712971e-1*atan(1.906683204*r))*atan(-1.732050808+...
            405.1316975*theta^2)*theta^4-.5162620354e15*sin(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^2+.4585278986e20*sin(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^15+.5591178924e27*sin(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(3.-1052.563025*theta^2+...
            123098.7691*theta^4)*theta^12-.2982088099e22*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^4-.9364727744e34*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^16-.8235268275e22*cos(theta-↙
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^13+.4683476459e29*cos(theta-↙
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^10+.2058817068e19*cos(theta-↙
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^9+.1647053654e21*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^11+.3202945482e33*cos(theta-↙
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^14+.8235268275e15*cos(theta-↙
.5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^7+.1141394790e25*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^6-.2836615166e27*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^8-.1235290241e14*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^5+.4636093434e19*cos(theta-.↙
5338712971e-1*atan(1.906683204*r))*log(350.8543417*theta^2-...
            47.62308387*theta+1.732050808)*theta^2+.3276294025e38*theta^17-.↙
5762294639e24*theta^12-.3000000000e25*sin(theta)*theta^15+...
            200000000.*sin(theta)*theta-.3631598357e20*sin(theta)*theta^2+.↙
2335963760e23*sin(theta)*theta^4+.2222010218e28*sin(theta)*theta^8-...
            .8940905754e25*sin(theta)*theta^6+.3933063986e32*sin(theta)↙
*theta^12-.2508968320e34*sin(theta)*theta^14)/(175357.+...
            637500.*r^2)/(4000000001.-.1403417368e13*theta^2+.↙
1641316923e15*theta^4)/(120000.-42102521.*theta^2+...
            4923950764.*theta^4)/(.4385679270e11*theta^2-5952885485.↙
*theta+216506351.)/(.4385679270e11*theta^2+5952885485.*theta+...
            216506351.)/(1492820321.+.3023944152e12*theta^2-.↙
4104539372e11*theta)/(1492820321.+.3023944152e12*theta^2+.↙
4104539372e11*theta);

     tau    = F1+F2+F3;
%% Lyapunov
 Fmc1        =[1933.262088+7028.259956*r^2,-3220.791422-11351.28516*r^2;↙
-3220.791422-11351.28516*r^2,635.6719687];
```

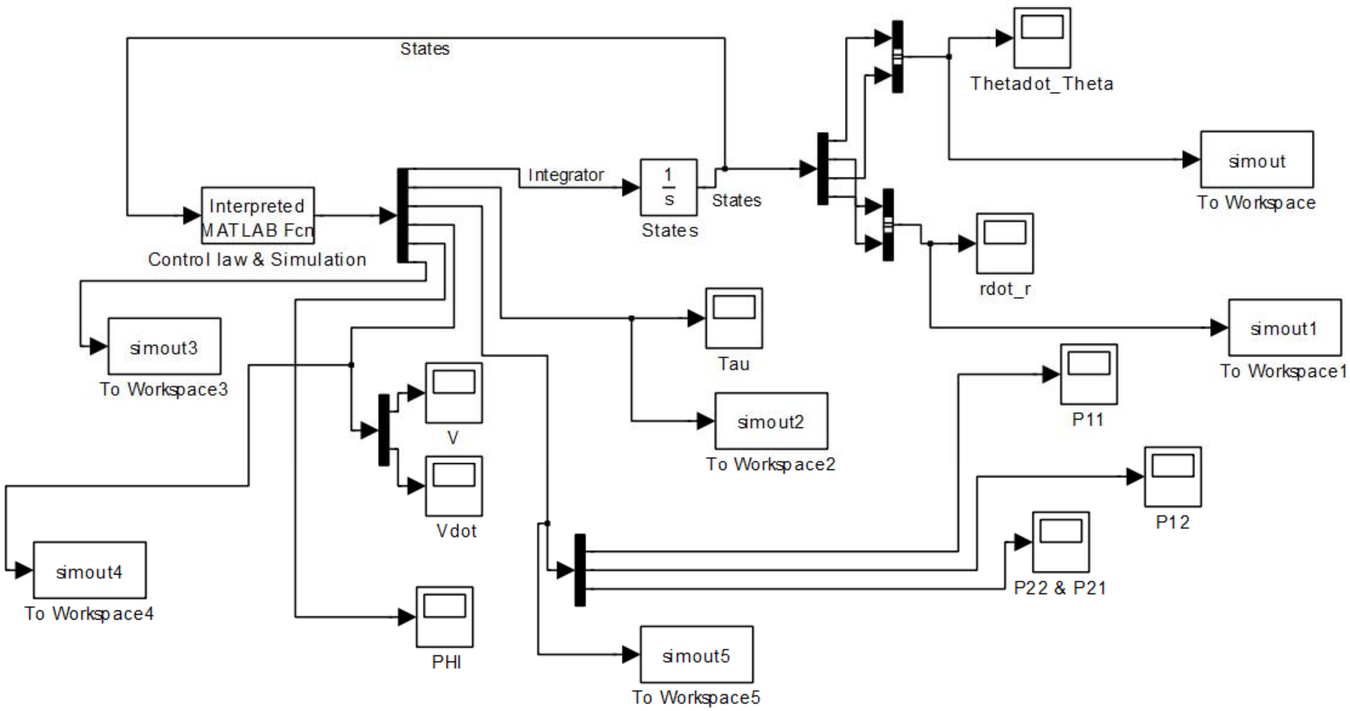
```
 PHi        =1.782827863*cos(theta-.5338712971e-1*atan(1.906683204*r))*atan↙
(-1.732050808+1620.526790*theta^2-...
                86.51527393*theta*atan(1.906683204*r)+1.154700538*atan↙
(1.906683204*r)^2)-.3668223189e-3*cos(theta-...
                .5338712971e-1*atan(1.906683204*r))*log(3.-4210.252101↙
*theta^2+224.7732750*theta*atan(1.906683204*r)-...
                2.999999997*atan(1.906683204*r)^2+1969580.306*theta^4-↙
210300.4785*theta^3*atan(1.906683204*r)+...
                8420.504193*theta^2*atan(1.906683204*r)^2-149.8488499↙
*theta*atan(1.906683204*r)^3+.9999999980*atan(1.906683204*r)^4)-...
                3.294107309*sin(theta-.5338712971e-1*atan(1.906683204*r))*log↙
(1403.417367*theta^2-74.92442500*theta*atan(1.906683204*r)+...
                .9999999990*atan(1.906683204*r)^2-95.24616774*theta+2.↙
542459756*atan(1.906683204*r)+1.732050808)+3.294107309*sin(theta-...
                .5338712971e-1*atan(1.906683204*r))*log(1403.417367*theta^2-↙
74.92442500*theta*atan(1.906683204*r)+...
                .9999999990*atan(1.906683204*r)^2+95.24616774*theta-↙
2.542459756*atan(1.906683204*r)+1.732050808)+...
                24.46655598*sin(theta-.5338712971e-1*atan(1.906683204*r))↙
*atan(109.9808011*theta-2.935779647*atan(1.906683204*r)-...
                3.732050805)+24.46655598*sin(theta-.5338712971e-1*atan↙
(1.906683204*r))*atan(109.9808011*theta-...
                2.935779647*atan(1.906683204*r)+3.732050805)+.7339584247e-↙
3*sin(theta-.5338712971e-1*atan(1.906683204*r))*theta-...
                .1959196681e-4*sin(theta-.5338712971e-1*atan(1.906683204*r))↙
*atan(1.906683204*r)+.1467916850e-3*cos(theta-...
                .5338712971e-1*atan(1.906683204*r))*theta^2-.7836786729e-↙
5*cos(theta-...
                .5338712971e-1*atan(1.906683204*r))*theta*atan(1.906683204*r)↙
+.1045958874e-6*cos(theta-...
                .5338712971e-1*atan(1.906683204*r))*atan(1.906683204*r)^2+.↙
9999999996*F5*theta^2-...
                .1067742595*F5*theta*atan(1.906683204*r)+.2850185622e-↙
2*F5*atan(1.906683204*r)^2;
 Vc         =  0;
 V          =  0.5*qdot'*KD*qdot+PHi+Vc;

 Vdot       =  -qdot'*(Kv+Fmc1)*qdot;


%% Evaluate the Dynamics

qdotdot     = inv(mass)*([tau;0]-C*qdot-G);
ddtheta     = qdotdot(1);
ddr         = qdotdot(2);
%%  M-File Output
dxdt        =[thetadot;rdot;ddtheta;ddr;tau(1);P(1,1);P(1,2);P(2,1);P(2,2);V;↙
Vdot;PHi;DETP];
%% End of the Function
```

A.5 Simulink file for the ball and beam system



Appendix B - Ball and arc system

The presentation of this Appendix is organized in eight major sections. These are:

B.1 Dynamics of the ball and arc system

B.2 Lagrangian KD for the ball and arc system, solving Eq.2.25

B.3 Direct Lyapunov Approach formulation for the ball and arc system, case when K_D is a constant

B.4 MATLAB code for running the simulations of the Ball and Arc system

B.5 Simulink file, case when K_D is a constant

B.6 Direct Lyapunov Approach formulation for the ball and arc system when the lower triangular part of the matrix P elements are constant

B.7 MATLAB code for the simulations of the Ball and Arc system

B.8 Simulink file, case when P is almost constant

B.1 Dynamics of the ball and Arc system
Ball and Arc_dynamics.mw

Lagrangian Approach

The Lagrangian (L) is defined as KE-PE, where KE is the kinetic energy and PE is the potential energy of the system in terms of the generalized coordinates. In our case the generalized coordinates are expresed in terms of as φ, and θ.

> restart;

> with(LinearAlgebra) :

> $KE := \frac{1}{2} \cdot m \cdot (Velx^2 + Vely^2) + \frac{1}{2} \cdot Ib \cdot (diff(\theta(t), t))^2;$

$$KE := \frac{1}{2} m \left(Velx^2 + Vely^2\right) + \frac{1}{2} Ib \left(\frac{d}{dt} \theta(t)\right)^2 \quad \textbf{(1)}$$

> $x_c := r \cdot \sin(\phi(t) - \theta(t)) + d \cdot \sin(\theta(t));$

$$x_c := r \sin(\phi(t) - \theta(t)) + d \sin(\theta(t)) \quad \textbf{(2)}$$

> $y_c := r \cdot \cos(\phi(t) - \theta(t)) - d \cdot \cos(\theta(t));$

$$y_c := r \cos(\phi(t) - \theta(t)) - d \cos(\theta(t)) \quad \textbf{(3)}$$

> $v_x := diff(x_c, t);$

$$v_x := r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt} \phi(t) - \left(\frac{d}{dt} \theta(t)\right)\right) + d \cos(\theta(t)) \left(\frac{d}{dt} \theta(t)\right) \quad \textbf{(4)}$$

> $v_y := diff(y_c, t);$

$$v_y := -r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt} \phi(t) - \left(\frac{d}{dt} \theta(t)\right)\right) + d \sin(\theta(t)) \left(\frac{d}{dt} \theta(t)\right) \quad \textbf{(5)}$$

> $KE := simplify(eval(KE, [Velx = v_x, Vely = v_y]));$

$$\begin{aligned} KE := {} & m\, r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt} \phi(t)\right) d \cos(\theta(t)) \left(\frac{d}{dt} \theta(t)\right) - m\, r \cos(\phi(t) \\ & - \theta(t)) \left(\frac{d}{dt} \theta(t)\right)^2 d \cos(\theta(t)) + \frac{1}{2} m\, r^2 \left(\frac{d}{dt} \phi(t)\right)^2 \\ & - m\, r^2 \left(\frac{d}{dt} \phi(t)\right) \left(\frac{d}{dt} \theta(t)\right) - m\, r \sin(\phi(t) \\ & - \theta(t)) \left(\frac{d}{dt} \phi(t)\right) d \sin(\theta(t)) \left(\frac{d}{dt} \theta(t)\right) + \frac{1}{2} m\, r^2 \left(\frac{d}{dt} \theta(t)\right)^2 \\ & + m\, r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt} \theta(t)\right)^2 d \sin(\theta(t)) + \frac{1}{2} m\, d^2 \left(\frac{d}{dt} \theta(t)\right)^2 \\ & + \frac{1}{2} Ib \left(\frac{d}{dt} \theta(t)\right)^2 \end{aligned} \quad \textbf{(6)}$$

> $PE := m \cdot g \cdot y$

(7)

$$PE := m\, g\, y \qquad \textbf{(7)}$$

> $PE := simplify(eval(PE, [y = y_c]))$;

$$PE := m\, g\, (r \cos(\phi(t) - \theta(t)) - d \cos(\theta(t))) \qquad \textbf{(8)}$$

> $with(VariationalCalculus)$

$$[ConjugateEquation, Convex, EulerLagrange, Jacobi, Weierstrass] \qquad \textbf{(9)}$$

> $L := simplify(KE - PE)$;

$$L := m\, r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \cos(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) - m\, r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \cos(\theta(t)) + \frac{1}{2} m\, r^2 \left(\frac{d}{dt}\phi(t)\right)^2 - m\, r^2 \left(\frac{d}{dt}\phi(t)\right)\left(\frac{d}{dt}\theta(t)\right) - m\, r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) + \frac{1}{2} m\, r^2 \left(\frac{d}{dt}\theta(t)\right)^2 + m\, r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \sin(\theta(t)) + \frac{1}{2} m\, d^2 \left(\frac{d}{dt}\theta(t)\right)^2 + \frac{1}{2} Ib \left(\frac{d}{dt}\theta(t)\right)^2 - m\, g\, r \cos(\phi(t) - \theta(t)) + m\, g\, d \cos(\theta(t)) \qquad \textbf{(10)}$$

Defining temporary substitutions and variables

> $temp1 := \left[\left(\frac{d^2}{dt^2}\theta(t)\right) = \theta ddot, \theta(t) = \theta, \left(\frac{d^2}{dt^2}\phi(t)\right) = \phi ddot, \left(\frac{d}{dt}\theta(t)\right) = \theta dot, \left(\frac{d}{dt}\phi(t)\right) = \phi dot, \phi(t) = \phi\right]$;

$$temp1 := \left[\frac{d^2}{dt^2}\theta(t) = \theta ddot, \theta(t) = \theta, \frac{d^2}{dt^2}\phi(t) = \phi ddot, \frac{d}{dt}\theta(t) = \theta dot, \frac{d}{dt}\phi(t) = \phi dot, \phi(t) = \phi\right] \qquad \textbf{(11)}$$

> $temp2 := [\theta ddot, \phi ddot, \theta dot, \phi dot, \phi dot^2]$;

$$temp2 := [\theta ddot, \phi ddot, \theta dot, \phi dot, \phi dot^2] \qquad \textbf{(12)}$$

> $EL1 := EulerLagrange(L, t, [\theta(t), \phi(t)])[1] = 0$

$$EL1 := -m\, r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \cos(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) + 2\, m\, r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \cos(\theta(t)) - m\, r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) + 2\, m\, r \cos(\phi(t) \qquad \textbf{(13)}$$

$$- \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \sin(\theta(t)) + m g r \sin(\phi(t) - \theta(t)) + m r \sin(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\phi(t) - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\theta(t)\right) d \cos(\theta(t)) - m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d^2}{dt^2}\theta(t)\right) d \cos(\theta(t)) - m r^2 \left(\frac{d^2}{dt^2}\phi(t)\right) + m r^2 \left(\frac{d^2}{dt^2}\theta(t)\right)$$

$$+ m r \cos(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t) - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\theta(t)\right) d \sin(\theta(t))$$

$$+ m r \sin(\phi(t) - \theta(t)) \left(\frac{d^2}{dt^2}\theta(t)\right) d \sin(\theta(t)) = 0$$

> $EL2 := EulerLagrange(L, t, [\theta(t), \phi(t)])[2] = 0$

$$EL2 := -m g r \sin(\phi(t) - \theta(t)) - m g d \sin(\theta(t)) + m r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right. \tag{14}$$

$$\left. - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\phi(t)\right) d \cos(\theta(t)) - m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d^2}{dt^2}\phi(t)\right) d \cos(\theta(t)) + m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) - 2 m r \sin(\phi(t) - \theta(t)) \left(\frac{d}{dt}\phi(t)\right.$$

$$\left. - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\theta(t)\right) d \cos(\theta(t)) + 2 m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d^2}{dt^2}\theta(t)\right) d \cos(\theta(t)) - 2 m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \sin(\theta(t)) + m r^2 \left(\frac{d^2}{dt^2}\phi(t)\right) + m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\phi(t) - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\phi(t)\right) d \sin(\theta(t)) + m r \sin(\phi(t)$$

$$- \theta(t)) \left(\frac{d^2}{dt^2}\phi(t)\right) d \sin(\theta(t)) + m r \sin(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\phi(t)\right) d \cos(\theta(t)) \left(\frac{d}{dt}\theta(t)\right) - m r^2 \left(\frac{d^2}{dt^2}\theta(t)\right) - 2 m r \cos(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\phi(t) - \left(\frac{d}{dt}\theta(t)\right)\right) \left(\frac{d}{dt}\theta(t)\right) d \sin(\theta(t)) - 2 m r \sin(\phi(t)$$

$$- \theta(t)) \left(\frac{d^2}{dt^2}\theta(t)\right) d \sin(\theta(t)) - 2 m r \sin(\phi(t)$$

$$- \theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2 d \cos(\theta(t)) - m d^2 \left(\frac{d^2}{dt^2}\theta(t)\right) - Ib \left(\frac{d^2}{dt^2}\theta(t)\right) = 0$$

The first governing equation is

Equation1

$> \; EQ\theta := expand(eval(EL1, temp1));$

$$EQ\theta := m\,r\,\theta dot^2\,d\cos(\theta)^2\sin(\phi) + m\,r\,\theta dot^2\,d\sin(\theta)^2\sin(\phi) + m\,g\,r\sin(\phi)\cos(\theta) - m\,g\,r\cos(\phi)\sin(\theta) - m\,r\,\theta ddot\,d\cos(\theta)^2\cos(\phi) - m\,r^2\,\phi ddot + m\,r^2\,\theta ddot - m\,r\,\theta ddot\,d\sin(\theta)^2\cos(\phi) = 0 \quad \textbf{(15)}$$

$> \; EQ\theta := collect(EQ\theta, temp2);$

$$EQ\theta := \left(-m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi)\right)\theta ddot - m\,r^2\,\phi ddot + \left(m\,r\,d\cos(\theta)^2\sin(\phi) + m\,r\,d\sin(\theta)^2\sin(\phi)\right)\theta dot^2 + m\,g\,r\sin(\phi)\cos(\theta) - m\,g\,r\cos(\phi)\sin(\theta) = 0 \quad \textbf{(16)}$$

The second governing equation is

Equation2

$> \; EQ\phi := simplify(eval(\mathrm{EL2})):$

$> \; EQ\phi := expand(eval(EQ\phi, temp1)):$

$> \; EQ\phi := collect(EQ\phi, temp2);$

$$EQ\phi := \left(2\,m\,r\,d\cos(\theta)^2\cos(\phi) - m\,r^2 + 2\,m\,r\,d\sin(\theta)^2\cos(\phi) - m\,d^2 - Ib\right)\theta ddot + \left(-m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi)\right)\phi ddot + \left(-2\,m\,r\,d\cos(\theta)^2\sin(\phi) - 2\,m\,r\,d\sin(\theta)^2\sin(\phi)\right)\phi dot\,\theta dot + \left(m\,r\,d\cos(\theta)^2\sin(\phi) + m\,r\,d\sin(\theta)^2\sin(\phi)\right)\phi dot^2 - m\,g\,r\sin(\phi)\cos(\theta) + m\,g\,r\cos(\phi)\sin(\theta) - m\,g\,d\sin(\theta) = 0 \quad \textbf{(17)}$$

$> \; G := -simplify(Matrix(2, 1, [-m\,g\,r\sin(\phi)\cos(\theta) + m\,g\,r\cos(\phi)\sin(\theta) - m\,g\,d\sin(\theta), m\,g\,r\sin(\phi)\cos(\theta) - m\,g\,r\cos(\phi)\sin(\theta)]));$

$$G := \begin{bmatrix} -m\,g\,(-r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta) - d\sin(\theta)) \\ m\,g\,r\,(-\sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta)) \end{bmatrix} \quad \textbf{(18)}$$

$> \; mass := -simplify\big(Matrix\big(2, 2, \big[2\,m\,r\,d\cos(\theta)^2\cos(\phi) - m\,r^2 + 2\,m\,r\,d\sin(\theta)^2\cos(\phi) - m\,d^2 - Ib, -m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi), -m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi), -m\,r^2]));$

$$mass := \begin{bmatrix} m\,r^2 - 2\,m\,r\,d\cos(\phi) + m\,d^2 + Ib & m\,r\,(-r + d\cos(\phi)) \\ m\,r\,(-r + d\cos(\phi)) & m\,r^2 \end{bmatrix} \quad \textbf{(19)}$$

$> \; masst := -simplify\big(Matrix\big(2, 2, \big[2\,m\,r\,d\cos(\theta(t))^2\cos(\phi(t)) - m\,r^2 + 2\,m\,r\,d\sin(\theta(t))^2\cos(\phi(t)) - m\,d^2 - Ib, -m\,r\,d\cos(\theta(t))^2\cos(\phi(t)) + m\,r^2 - m\,r\,d\sin(\theta(t))^2\cos(\phi(t)), -m\,r\,d\cos(\theta(t))^2\cos(\phi(t)) + m\,r^2 - m\,r\,d\sin(\theta(t))^2\cos(\phi(t)), -m\,r^2])):$

> $masstdot := map(diff, masst, t)$

$$masstdot := \begin{bmatrix} 2\,m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) & -m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) \\ -m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) & 0 \end{bmatrix} \quad \textbf{(20)}$$

> $Mdot := map(diff, masst, t);$

$$Mdot := \begin{bmatrix} 2\,m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) & -m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) \\ -m\,r\,d\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right) & 0 \end{bmatrix} \quad \textbf{(21)}$$

Cmatrix **is defined as**

$$C_{ij}(\boldsymbol{q}, \boldsymbol{qdot}) = [\boldsymbol{jk, i}]qdot^{\boldsymbol{k}} = \frac{1}{2}\left[\frac{\partial}{\partial \boldsymbol{q}^{k}} m_{i,j} + \frac{\partial}{\partial \boldsymbol{q}^{j}} m_{k,i} - \frac{\partial}{\partial \boldsymbol{q}^{i}} m_{j,k}\right]qdot^{\boldsymbol{k}}$$

where m_{ij} is the ijth **component of the mass matrix and [jk,i] is the Christoffel symbol of the first kind.**

> $C11 := \frac{1}{2}\left(\frac{\partial}{\partial\theta} mass_{1,1} + \frac{\partial}{\partial\theta} mass_{1,1} - \frac{\partial}{\partial\theta} mass_{1,1}\right)\cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\phi} mass_{1,1} + \frac{\partial}{\partial\theta} mass_{2,1} - \frac{\partial}{\partial\theta} mass_{1,2}\right)\cdot \phi dot;$

$$C11 := m\,r\,d\sin(\phi)\ \phi dot \quad \textbf{(22)}$$

> $C12 := \frac{1}{2}\left(\frac{\partial}{\partial\theta} mass_{2,1} + \frac{\partial}{\partial\phi} mass_{1,1} - \frac{\partial}{\partial\theta} mass_{2,1}\right)\cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\phi} mass_{2,1} + \frac{\partial}{\partial\phi} mass_{2,1} - \frac{\partial}{\partial\theta} mass_{2,2}\right)\cdot \phi dot;$

$$C12 := m\,r\,d\sin(\phi)\ \theta dot - m\,r\,d\sin(\phi)\ \phi dot \quad \textbf{(23)}$$

> $C21 := \frac{1}{2}\left(\frac{\partial}{\partial\theta} mass_{1,2} + \frac{\partial}{\partial\theta} mass_{1,2} - \frac{\partial}{\partial\phi} mass_{1,1}\right)\cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\phi} mass_{1,2} + \frac{\partial}{\partial\theta} mass_{2,2} - \frac{\partial}{\partial\phi} mass_{1,2}\right)\cdot \phi dot;$

$$C21 := -m\,r\,d\sin(\phi)\ \theta dot \quad \textbf{(24)}$$

> $C22 := \frac{1}{2}\left(\frac{\partial}{\partial\theta} mass_{2,2} + \frac{\partial}{\partial\phi} mass_{1,2} - \frac{\partial}{\partial\phi} mass_{2,1}\right)\cdot \theta dot + \frac{1}{2}\left(\frac{\partial}{\partial\phi} mass_{2,2} + \frac{\partial}{\partial\phi} mass_{2,2} - \frac{\partial}{\partial\phi} mass_{2,2}\right)\cdot \phi dot;$

$$C22 := 0 \quad \textbf{(25)}$$

> $Cmatrix := Matrix(2, 2, [C11, C12, C21, C22]);$

$$Cmatrix := \begin{bmatrix} m\,r\,d\sin(\phi)\ \phi dot & m\,r\,d\sin(\phi)\ \theta dot - m\,r\,d\sin(\phi)\ \phi dot \\ -m\,r\,d\sin(\phi)\ \theta dot & 0 \end{bmatrix} \tag{26}$$

Differentiating the mass matrix with respect to time produces

> $mdot := Matrix(2, 2, [-2\,m\,r\,d\sin(\phi(t))\ \phi dot, m\,r\,d\sin(\phi(t))\ \phi dot, m\,r\,d\sin(\phi(t))\ \phi dot\ ])$

$$mdot := \begin{bmatrix} -2\,m\,r\,d\sin(\phi(t))\ \phi dot & m\,r\,d\sin(\phi(t))\ \phi dot \\ m\,r\,d\sin(\phi(t))\ \phi dot & 0 \end{bmatrix} \tag{27}$$

> $\mathrm{Mdot} := simplify(eval(Mdot, temp1\ ));$

$$Mdot := \begin{bmatrix} 2\,m\,r\,d\sin(\phi)\ \phi dot & -m\,r\,d\sin(\phi)\ \phi dot \\ -m\,r\,d\sin(\phi)\ \phi dot & 0 \end{bmatrix} \tag{28}$$

>

Because M is Lagrangian $\frac{1}{2}\cdot(\ Mdot - \ Cmatrix)$ is skew-symmetric. The result is

> $FMCskew := \left(\frac{1}{2}\cdot Mdot - Cmatrix\right);$

$$FMCskew := \tag{29}$$

$$\left[\left[0, \frac{1}{2}\ m\,r\,d\sin(\phi)\ \phi dot - m\,r\,d\sin(\phi)\ \theta dot\right],\right.$$

$$\left.\left[-\frac{1}{2}\ m\,r\,d\sin(\phi)\ \phi dot + m\,r\,d\sin(\phi)\ \theta dot, 0\right]\right]$$

This is a special case of the FMC when KD =M, for which

> $FMC := (\ Mdot - Cmatrix - Transpose(Cmatrix));$

$$FMC := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

End of the file

B.2 Lagrangian KD_Ball and Arc system

LagrangianKD_B&Arc.mw

Eq. 2.25 Solution - $-\boldsymbol{C}^T\boldsymbol{M}^{-1}\boldsymbol{K}_D + \frac{1}{2}\left(\frac{\partial}{\partial q}\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\right) = 0$

> restart :

> with(LinearAlgebra) :

Definitions

> $mass := simplify\big(Matrix\big(2, 2, \big[2\,m\,r\,d\cos(\theta)^2\cos(\phi) - m\,r^2 + 2\,m\,r\,d\sin(\theta)^2\cos(\phi) - m\,d^2 - Ib, -m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi), -m\,r\,d\cos(\theta)^2\cos(\phi) + m\,r^2 - m\,r\,d\sin(\theta)^2\cos(\phi), -m\,r^2\big]\big)\big);$

$$mass := \begin{bmatrix} -m\,r^2 + 2\,m\,r\,d\cos(\phi) - m\,d^2 - Ib & -m\,r\,(-r + d\cos(\phi)) \\ -m\,r\,(-r + d\cos(\phi)) & -m\,r^2 \end{bmatrix} \quad \textbf{(1.1)}$$

> $q := \begin{bmatrix} \theta \\ r \end{bmatrix}:$

> $qdot := \begin{bmatrix} \theta dot \\ \phi dot \end{bmatrix}:$

> $C := Matrix(2, 2, [-2\,\phi dot\,m\,r\,d\sin(\phi), -2\,\theta dot\,m\,r\,d\sin(\phi) + 2\,\phi dot\,m\,r\,d\sin(\phi), 2\,\theta dot\,m\,r\,d\sin(\phi), 0]);$

$$C := \begin{bmatrix} -2\,\phi dot\,m\,r\,d\sin(\phi) & -2\,\theta dot\,m\,r\,d\sin(\phi) + 2\,\phi dot\,m\,r\,d\sin(\phi) \\ 2\,\theta dot\,m\,r\,d\sin(\phi) & 0 \end{bmatrix} \quad \textbf{(1.2)}$$

> $KD := Matrix(2, 2, [KD11(\theta, \phi), KD12(\theta, \phi), KD21(\theta, \phi), KD22(\theta, \phi)]);$

$$KD := \begin{bmatrix} KD11(\theta, \phi) & KD12(\theta, \phi) \\ KD21(\theta, \phi) & KD22(\theta, \phi) \end{bmatrix} \quad \textbf{(1.3)}$$

Term 1

> $KDq := Transpose\left(Multiply\left(\frac{1}{2} \cdot KD, qdot\right)\right);$

$$KDq := \left[\frac{1}{2}\,KD11(\theta, \phi)\,\theta dot + \frac{1}{2}\,KD12(\theta, \phi)\,\phi dot, \frac{1}{2}\,KD21(\theta, \phi)\,\theta dot \right. \quad \textbf{(2.1)}$$

$$+\frac{1}{2}\,KD22(\theta,\phi)\,\phi dot\Big]$$

> $Term1 := \begin{bmatrix} diff(KDq[1],\theta) & diff(KDq[2],\theta) \\ diff(KDq[1],\phi) & diff(KDq[2],\phi) \end{bmatrix};$

$$Term1 := \left[\left[\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD11(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD12(\theta,\phi)\right)\phi dot,\right.\right.$$ **(2.2)**

$$\left.\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)\phi dot\right],$$

$$\left[\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD11(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD12(\theta,\phi)\right)\phi dot,\right.$$

$$\left.\left.\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)\phi dot\right]\right]$$

Term 2

> $Term2 := Transpose(Multiply(Multiply(KD, MatrixInverse(mass)), C));$

$$Term2 := \left[\left[-2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right.\right.\right.$$ **(3.1)**

$$\left.+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)\phi dot\,m\,r\,d\sin(\phi)$$

$$+2\left(\frac{KD11(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right.$$

$$\left.-\frac{KD12(\theta,\phi)\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)\theta dot\,m\,r\,d\sin(\phi),\ -2\Bigg($$

$$-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)$$

$$\phi dot\,m\,r\,d\sin(\phi)+2\left(\frac{KD21(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right.$$

$$\left.\left.-\frac{KD22(\theta,\phi)\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)\theta dot\,m\,r\,d\sin(\phi)\right],$$

$$\left[\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)\Big(\right.$$

$$-2\,\theta dot\,m\,r\,d\sin(\phi)+2\,\phi dot\,m\,r\,d\sin(\phi)\Big),\ \Bigg(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\,(-2\,\theta dot\,m\,r\,d\sin(\phi)$$

$$+2\,\phi dot\,m\,r\,d\sin(\phi))\Big]\Big]$$

The Eq. 2.25 is expresed as the following Eq. This is the equation to be Solved

```
> Eq := MatrixAdd(Term1,-Term2);
```

$$Eq := \Bigg[\Bigg[\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD11(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD12(\theta,\phi)\right)\phi dot+2\Bigg(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\phi dot\,m\,r\,d\sin(\phi)-2\Bigg(\frac{KD11(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}-\frac{KD12(\theta,\phi)\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\theta dot\,m\,r\,d\sin(\phi),$$
$$\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)\phi dot+2\Bigg(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\phi dot\,m\,r\,d\sin(\phi)-2\Bigg(\frac{KD21(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}-\frac{KD22(\theta,\phi)\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\theta dot\,m\,r\,d\sin(\phi)\Bigg],$$
$$\Bigg[\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD11(\theta,\phi)\right)\theta dot+\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD12(\theta,\phi)\right)\phi dot-\Bigg(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\,(-2\,\theta dot\,m\,r\,d\sin(\phi)+2\,\phi dot\,m\,r\,d\sin(\phi)),\ \frac{1}{2}\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)\theta dot$$
$$+\frac{1}{2}\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)\phi dot-\Bigg(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\qquad(4.1)$$

$$+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,(-2\,\theta dot\,m\,r\,d\sin(\phi)$$
$$+2\,\phi dot\,m\,r\,d\sin(\phi))\Bigg]\Bigg]$$

Coefficient Matrices of θdot and ϕdot

> $Eq1 := Matrix(2, map(coeff, Eq, \theta dot));$

$$Eq1 := \Bigg[\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD11(\theta,\phi)-2\left(\frac{KD11(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}-\frac{KD12(\theta,\phi)\,(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib)}{m\,r^2\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi),$$
$$\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD21(\theta,\phi)-2\left(\frac{KD21(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}-\frac{KD22(\theta,\phi)\,(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib)}{m\,r^2\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\Bigg],$$
$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD11(\theta,\phi)+2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi),\ \frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD21(\theta,\phi)+2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\Bigg]\Bigg] \quad (5.1)$$

> $Eq2 := Matrix(2, map(coeff, Eq, \phi dot));$

$$Eq2 := \Bigg[\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD12(\theta,\phi)+2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi),\ \frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD22(\theta,\phi)+2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\Bigg], \quad (5.2)$$
$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD12(\theta,\phi)-2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi),\ \frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD22(\theta,\phi)-2\,\Bigg($$

$$-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg]\Bigg]$$

Coefficient Matrices of Derivatives and K_D Elements

> $KD\theta := Matrix(2, map(diff, KD, \theta));$

$$KD\theta := \begin{bmatrix} \frac{\partial}{\partial\theta} KD11(\theta,\phi) & \frac{\partial}{\partial\theta} KD12(\theta,\phi) \\ \frac{\partial}{\partial\theta} KD21(\theta,\phi) & \frac{\partial}{\partial\theta} KD22(\theta,\phi) \end{bmatrix} \quad \textbf{(6.1)}$$

> $KD\phi := Matrix(2, map(diff, KD, \phi));$

$$KD\phi := \begin{bmatrix} \frac{\partial}{\partial\phi} KD11(\theta,\phi) & \frac{\partial}{\partial\phi} KD12(\theta,\phi) \\ \frac{\partial}{\partial\phi} KD21(\theta,\phi) & \frac{\partial}{\partial\phi} KD22(\theta,\phi) \end{bmatrix} \quad \textbf{(6.2)}$$

> $DKD := Matrix(8, 1, [\ ]):$

> $Eqsc := Matrix(8, 1, [\ ]):$

> $LSCM := Matrix(8, [\ ]):$

> **for** i **from** 1 **to** 4 **do**
for j **from** 1 **to** 4 **do**
$DKD_i := KD\theta(i):$
$DKD_{i+4} := KD\phi(i);$
$Eqsc_i := Eq1(i);$
$Eqsc_{i+4} := Eq2(i);$
$k := i+4;$
$l := j+4;$
$LSCM_{i,j} := coeff(Eq1(i), KD\theta(j));$
$LSCM_{i,l} := coeff(Eq1(i), KD\phi(j));$
$LSCM_{k,j} := coeff(Eq2(i), KD\theta(j));$
$LSCM_{k,l} := coeff(Eq2(i), KD\phi(j));$
end do
end do

> $map(eval, LSCM);$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{6.3}$$

The above matrix contents the coefficients of $\frac{\partial}{\partial\,\theta}$ *KD* and $\frac{\partial}{\partial\,\phi}$ *KD*

> $map(eval, Eqsc);$

$$\left[\left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD11(\theta,\phi)-2\left(\frac{KD11(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right.\right.\right. \tag{6.4}$$

$$\left.\left.-\frac{KD12(\theta,\phi)\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD11(\theta,\phi)+2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right.\right.$$

$$\left.\left.+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD21(\theta,\phi)-2\left(\frac{KD21(\theta,\phi)\,(-r+d\cos(\phi))}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right.\right.$$

$$\left.\left.-\frac{KD22(\theta,\phi)\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)}{m\,r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD21(\theta,\phi)+2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right.\right.$$

$$+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD12(\theta,\phi)+2\,\Bigg(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD12(\theta,\phi)-2\,\Bigg(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\theta}\,KD22(\theta,\phi)+2\,\Bigg(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[\frac{1}{2}\,\frac{\partial}{\partial\phi}\,KD22(\theta,\phi)-2\,\Bigg(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg]\Bigg]$$

> $Eqcm := MatrixAdd(Eqsc, Multiply(-LSCM, DKD));$

$$Eqcm := \Bigg[\Bigg[-2\,\Bigg(\frac{KD11(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)} \tag{6.5}$$

$$-\frac{KD12(\theta,\phi)\,(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib)}{m\,r^2\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[2\,\Bigg(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}$$

$$+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\Bigg[-2\,\Bigg(\frac{KD21(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}$$

$$-\frac{KD22(\theta,\phi)\,(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib)}{m\,r^2\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\Bigg)\,m\,r\,d\sin(\phi)\Bigg],$$

$$\left[2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[-2\left(-\frac{KD11(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD12(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\right],$$

$$\left[-2\left(-\frac{KD21(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{KD22(\theta,\phi)\,(-r+d\cos(\phi))}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)}\right)m\,r\,d\sin(\phi)\right]\Bigg]$$

```
> RSCM := Matrix(8, 4, [ ]) :
> for i from 1 to 8 do
  for j from 1 to 4 do
  RSCM[i, j] := coeff(Eqcm(i), KD(j));
  end do
  end do
> map(eval, RSCM);
```

$$\left[\left[-\frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2},0,\frac{2\,(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib)\,d\sin(\phi)}{r\,(m\,d^2+Ib-m\,d^2\cos(\phi)^2)},0\right],\right. \tag{6.6}$$

$$\left[-\frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2},0,\frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2},0\right],$$

$$\left[0, -\frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, \frac{2\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)\,d\sin(\phi)}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right],$$

$$\left[0, -\frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, \frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right],$$

$$\left[-\frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, \frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0\right],$$

$$\left[\frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0\right],$$

$$\left[0, -\frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, \frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right],$$

$$\left[0, \frac{2\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{2\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right]\Bigg]$$

Extract Coefficient Matrices

> Derivs := Multiply(MatrixInverse(LSCM), -RSCM);

$$Derivs := \Bigg[\Bigg[\frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{4\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)\,d\sin(\phi)}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}, 0\Bigg], \tag{7.1}$$

$$\left[0, \frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{4\,\left(m\,r^2-2\,m\,r\,d\cos(\phi)+m\,d^2+Ib\right)\,d\sin(\phi)}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right],$$

$$\left[\frac{4\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0\right],$$

$$\left[0, \frac{4\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right],$$

$$\left[\frac{4\,m\,r\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0, -\frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}, 0\right],$$

$$\left[0, \frac{4\, m\, r\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, -\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}\right],$$

$$\left[-\frac{4\, m\, r\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, \frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0\right],$$

$$\left[0, -\frac{4\, m\, r\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, \frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}\right]\Bigg]$$

> $M\theta := Derivs[[1..4], [1..4]];$

$$M\theta := \left[\left[\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, -\frac{4\left(m\, r^2 - 2\, m\, r\, d\cos(\phi) + m\, d^2 + Ib\right) d \sin(\phi)}{r\left(m\, d^2 + Ib - m\, d^2 \cos(\phi)^2\right)}, 0\right],\right. \tag{7.2}$$

$$\left[0, \frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, -\frac{4\left(m\, r^2 - 2\, m\, r\, d\cos(\phi) + m\, d^2 + Ib\right) d \sin(\phi)}{r\left(m\, d^2 + Ib - m\, d^2 \cos(\phi)^2\right)}\right],$$

$$\left[\frac{4\, m\, r\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, -\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0\right],$$

$$\left.\left[0, \frac{4\, m\, r\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}, 0, -\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}\right]\right]$$

> $KDS := \begin{bmatrix} KD11(\theta, \phi) \\ KD12(\theta, \phi) \\ KD21(\theta, \phi) \\ KD22(\theta, \phi) \end{bmatrix}:$

> $M\theta KD := Multiply(M\theta, KDS);$

$$M\theta KD := \left[\left[\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)\, KD11(\theta, \phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2} - \frac{4\left(m\, r^2 - 2\, m\, r\, d\cos(\phi) + m\, d^2 + Ib\right) d \sin(\phi)\, KD21(\theta, \phi)}{r\left(m\, d^2 + Ib - m\, d^2 \cos(\phi)^2\right)}\right],\right. \tag{7.3}$$

$$\left[\frac{4\,(-r + d\cos(\phi))\, m\, d \sin(\phi)\, KD12(\theta, \phi)}{m\, d^2 + Ib - m\, d^2 \cos(\phi)^2}\right.$$

$$-\frac{4\left(m r^2-2 m r d \cos(\phi)+m d^2+Ib\right) d \sin(\phi)\, KD22(\theta,\phi)}{r\left(m d^2+Ib-m d^2 \cos(\phi)^2\right)}\Bigg],$$

$$\left[\frac{4 m r d \sin(\phi)\, KD11(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}-\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)\, KD21(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right],$$

$$\left[\frac{4 m r d \sin(\phi)\, KD12(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}-\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)\, KD22(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right]\Bigg]$$

> $M\phi := Derivs[[5..8],[1..4]];$

$$M\phi := \left[\left[\frac{4 m r d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0, -\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0\right],\right. \tag{7.4}$$

$$\left[0, \frac{4 m r d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0, -\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right],$$

$$\left[-\frac{4 m r d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0, \frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0\right],$$

$$\left.\left[0, -\frac{4 m r d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}, 0, \frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right]\right]$$

> $M\phi KD := Multiply(M\phi, KDS);$

$$M\phi KD := \tag{7.5}$$

$$\left[\left[\frac{4 m r d \sin(\phi)\, KD11(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}-\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)\, KD21(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right],\right.$$

$$\left[\frac{4 m r d \sin(\phi)\, KD12(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}-\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)\, KD22(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right],$$

$$\left[-\frac{4 m r d \sin(\phi)\, KD11(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}+\frac{4\left(-r+d\cos(\phi)\right) m d \sin(\phi)\, KD21(\theta,\phi)}{m d^2+Ib-m d^2\cos(\phi)^2}\right.$$

$$\left[\left[-\frac{4\,m\,r\,d\sin(\phi)\,KD12(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}+\frac{4\,(-r+d\cos(\phi))\,m\,d\sin(\phi)\,KD22(\theta,\phi)}{m\,d^2+Ib-m\,d^2\cos(\phi)^2}\right]\right]$$

Generate Differential Equations

> $dKD := simplify(map(diff, M\theta KD, \phi) + map(diff, -M\phi KD, \theta));$

$$dKD := \Bigg[\Bigg[-\Bigg(4\,d\,\Bigg(m^2 r^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) d^2 + m\,r^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) Ib - m^2 r^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) d^2\cos(\phi)^2 + m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r^2 d^2 + m\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r^2\,Ib - m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r^2 d^2\cos(\phi)^2 - m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r\,d^3\cos(\phi) - m\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r\,d\cos(\phi)\,Ib + m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r\,d^3\cos(\phi)^3 + 2\,d\,m\,KD21(\theta,\phi)\,r\,Ib + d\,m\,KD11(\theta,\phi)\,r\,Ib + m^2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r^2 d^2 + m\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r^2\,Ib - m^2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r^2 d^2\cos(\phi)^2 + d^3 m^2 KD11(\theta,\phi)\,r + 2\,d^3 m^2 KD21(\theta,\phi)\,r - m^2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r\,d^3\cos(\phi) - m\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r\,d\cos(\phi)\,Ib + m^2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) r\,d^3\cos(\phi)^3 - \cos(\phi)\,KD21(\theta,\phi)\,m^2 r^2 d^2 + \cos(\phi)\,KD21(\theta,\phi)\,m\,r^2\,Ib + \cos(\phi)^3 KD21(\theta,\phi)\,m^2 r^2 d^2 - \cos(\phi)\,KD21(\theta,\phi)\,m^2 d^4 \tag{8.1}$$

$$+\cos(\phi)^3 KD21(\theta,\phi)\, m^2 d^4 + \sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m^2 d^4$$
$$-2\cos(\phi)^2 KD21(\theta,\phi)\, m^2 r d^3 - 4\cos(\phi)^2 KD21(\theta,\phi)\, m r d\, Ib$$
$$+\cos(\phi)^3 KD21(\theta,\phi)\, Ib\, m d^2 + \sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m^2 r^2 d^2$$
$$+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m r^2 Ib - \sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,$$
$$\phi)\right) m^2 r^2 d^2 \cos(\phi)^2 - 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m^2 r d^3 \cos(\phi)$$
$$-2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m r d \cos(\phi)\, Ib + 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,$$
$$\phi)\right) m^2 r d^3 \cos(\phi)^3 + 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m d^2 Ib$$
$$-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) m^2 d^4 \cos(\phi)^2 - \sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,$$
$$\phi)\right) Ib\, m d^2 \cos(\phi)^2 - m^2\cos(\phi)\, KD11(\theta,\phi)\, r^2 d^2 + m\cos(\phi)\, KD11(\theta,$$
$$\phi)\, r^2 Ib + m^2\cos(\phi)^3 KD11(\theta,\phi)\, r^2 d^2 - m^2\cos(\phi)^2 KD11(\theta,\phi)\, r d^3$$
$$-2 m\cos(\phi)^2 KD11(\theta,\phi)\, r d\, Ib + \cos(\phi)\, KD21(\theta,\phi)\, Ib^2$$
$$+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) Ib^2\Big)\Big)\Big/\Big(r\,\big(m^2 d^4 + 2 m d^2 Ib - 2 m^2 d^4\cos(\phi)^2$$
$$+ Ib^2 - 2\, Ib\, m d^2\cos(\phi)^2 + m^2 d^4\cos(\phi)^4\big)\Big)\Big],$$
$$\Big[-\Big(4 d\,\Big(d\, m\, KD12(\theta,\phi)\, r\, Ib + 2\, d\, m\, KD22(\theta,\phi)\, r\, Ib$$
$$-m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,\phi)\right) r d^3\cos(\phi) + m^2\sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,$$
$$\phi)\right) r d^3\cos(\phi)^3 + \sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m r^2 Ib$$
$$-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 r^2 d^2\cos(\phi)^2 - 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,$$
$$\phi)\right) m^2 r d^3\cos(\phi) - 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m r d\cos(\phi)\, Ib$$
$$+2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 r d^3\cos(\phi)^3 + 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,$$

$$\phi)\Big) m\, d^2\, Ib - \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD22(\theta, \phi)\right) m^2\, d^4 \cos(\phi)^2$$

$$- \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD22(\theta, \phi)\right) Ib\, m\, d^2 \cos(\phi)^2 + m^2\, r^2 \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD12(\theta,\right.$$

$$\left.\phi)\right) d^2 + m\, r^2 \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD12(\theta, \phi)\right) Ib - m^2\, r^2 \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD12(\theta,\right.$$

$$\left.\phi)\right) d^2 \cos(\phi)^2 + m^2 \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD22(\theta, \phi)\right) r^2\, d^2$$

$$+ m \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD22(\theta, \phi)\right) r^2\, Ib - m^2 \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD22(\theta,\right.$$

$$\left.\phi)\right) r^2\, d^2 \cos(\phi)^2 - m \sin(\phi) \left(\frac{\partial}{\partial \theta}\, KD22(\theta, \phi)\right) r\, d \cos(\phi)\, Ib$$

$$+ m \cos(\phi)\, KD12(\theta, \phi)\, r^2\, Ib - 2\, m \cos(\phi)^2\, KD12(\theta, \phi)\, r\, d\, Ib$$

$$+ m^2 \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta, \phi)\right) r^2\, d^2 + m \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta, \phi)\right) r^2\, Ib$$

$$- m^2 \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta, \phi)\right) r^2\, d^2 \cos(\phi)^2 - m^2 \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta,\right.$$

$$\left.\phi)\right) r\, d^3 \cos(\phi) - m \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta, \phi)\right) r\, d \cos(\phi)\, Ib$$

$$+ m^2 \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD12(\theta, \phi)\right) r\, d^3 \cos(\phi)^3 + m^2 \cos(\phi)^3\, KD12(\theta, \phi)\, r^2\, d^2$$

$$- m^2 \cos(\phi)^2\, KD12(\theta, \phi)\, r\, d^3 + \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD22(\theta, \phi)\right) Ib^2$$

$$- \cos(\phi)\, KD22(\theta, \phi)\, m^2\, r^2\, d^2 + \cos(\phi)\, KD22(\theta, \phi)\, m\, r^2\, Ib$$

$$+ \cos(\phi)^3\, KD22(\theta, \phi)\, m^2\, r^2\, d^2 - 2 \cos(\phi)^2\, KD22(\theta, \phi)\, m^2\, r\, d^3$$

$$- 4 \cos(\phi)^2\, KD22(\theta, \phi)\, m\, r\, d\, Ib - m^2 \cos(\phi)\, KD12(\theta, \phi)\, r^2\, d^2$$

$$+ d^3\, m^2\, KD12(\theta, \phi)\, r + 2\, d^3\, m^2\, KD22(\theta, \phi)\, r - \cos(\phi)\, KD22(\theta, \phi)\, m^2\, d^4$$

$$+ \cos(\phi)^3\, KD22(\theta, \phi)\, m^2\, d^4 + \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD22(\theta, \phi)\right) m^2\, d^4$$

$$+ \cos(\phi)^3\, KD22(\theta, \phi)\, Ib\, m\, d^2 + \sin(\phi) \left(\frac{\partial}{\partial \phi}\, KD22(\theta, \phi)\right) m^2\, r^2\, d^2$$

$$\left.\left. + \cos(\phi)\, KD22(\theta, \phi)\, Ib^2\right)\right) \Big/ \Big(r\, \big(m^2\, d^4 + 2\, m\, d^2\, Ib - 2\, m^2\, d^4 \cos(\phi)^2 + Ib^2$$

$$-2\,Ib\,m\,d^2\cos(\phi)^2+m^2\,d^4\cos(\phi)^4)\big)\Big],$$

$$\Big[\Big(4\,m\,d\,\Big(-\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)d\cos(\phi)\,Ib-2\cos(\phi)^2 KD21(\theta,\phi)\,d\,Ib+\cos(\phi)\,KD21(\theta,\phi)\,r\,Ib+\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)r\,Ib$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD11(\theta,\phi)\right)Ib+\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)r\,Ib$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD11(\theta,\phi)\right)Ib+r\cos(\phi)\,KD11(\theta,\phi)\,Ib$$

$$-\cos(\phi)^2 KD21(\theta,\phi)\,d^3\,m-\cos(\phi)\,KD21(\theta,\phi)\,r\,m\,d^2+\cos(\phi)^3 KD21(\theta,\phi)\,r\,m\,d^2+\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)r\,m\,d^2-\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)r\,m\,d^2\cos(\phi)^2+d\,KD21(\theta,\phi)\,Ib+d^3\,KD21(\theta,\phi)\,m$$

$$-\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)d^3\cos(\phi)\,m+\sin(\phi)\left(\frac{\partial}{\partial\phi}KD21(\theta,\phi)\right)d^3\cos(\phi)^3\,m+r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD11(\theta,\phi)\right)m\,d^2$$

$$-r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD11(\theta,\phi)\right)m\,d^2\cos(\phi)^2+\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)r\,m\,d^2-\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)r\,m\,d^2\cos(\phi)^2$$

$$-\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)d^3\cos(\phi)\,m-\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)d\cos(\phi)\,Ib+\sin(\phi)\left(\frac{\partial}{\partial\theta}KD21(\theta,\phi)\right)d^3\cos(\phi)^3\,m$$

$$-r\cos(\phi)\,KD11(\theta,\phi)\,m\,d^2+r\cos(\phi)^3 KD11(\theta,\phi)\,m\,d^2$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD11(\theta,\phi)\right)m\,d^2-r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD11(\theta,\phi)\right)m\,d^2\cos(\phi)^2\Big)\Big)\Big/\big(m^2\,d^4+2\,m\,d^2\,Ib-2\,m^2\,d^4\cos(\phi)^2+Ib^2$$

$$-2\,Ib\,m\,d^2\cos(\phi)^2+m^2\,d^4\cos(\phi)^4\big)\Big],$$

$$\Big[\Big(4\,m\,d\,\Big(-\sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)d^3\cos(\phi)\,m+\cos(\phi)\,KD22(\theta,$$

$$\phi)\ r\,Ib + r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD12(\theta,\phi)\right)Ib + \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)r\,Ib$$

$$+ r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD12(\theta,\phi)\right)Ib - 2\cos(\phi)^2 KD22(\theta,\phi)\,d\,Ib$$

$$- r\cos(\phi)\,KD12(\theta,\phi)\,m\,d^2 + r\cos(\phi)^3 KD12(\theta,\phi)\,m\,d^2$$

$$+ r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD12(\theta,\phi)\right)m\,d^2 - r\sin(\phi)\left(\frac{\partial}{\partial\phi}KD12(\theta,\right.$$

$$\left.\phi)\right)m\,d^2\cos(\phi)^2 - \cos(\phi)\,KD22(\theta,\phi)\,r\,m\,d^2 + \cos(\phi)^3 KD22(\theta,\phi)\,r\,m\,d^2$$

$$+ \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)r\,m\,d^2 - \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\right.$$

$$\left.\phi)\right)r\,m\,d^2\cos(\phi)^2 - \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)d\cos(\phi)\,Ib$$

$$+ \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)d^3\cos(\phi)^3 m + r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD12(\theta,\right.$$

$$\left.\phi)\right)m\,d^2 - r\sin(\phi)\left(\frac{\partial}{\partial\theta}KD12(\theta,\phi)\right)m\,d^2\cos(\phi)^2$$

$$+ \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)r\,m\,d^2 - \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\right.$$

$$\left.\phi)\right)r\,m\,d^2\cos(\phi)^2 - \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)d^3\cos(\phi)\,m$$

$$- \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)d\cos(\phi)\,Ib + \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\right.$$

$$\left.\phi)\right)d^3\cos(\phi)^3 m + r\cos(\phi)\,KD12(\theta,\phi)\,Ib + \sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)r\,Ib$$

$$\left.\left. - \cos(\phi)^2 KD22(\theta,\phi)\,d^3 m + d^3 KD22(\theta,\phi)\,m + d\,KD22(\theta,\phi)\,Ib\right)\right)\Big/$$

$$\left(m^2 d^4 + 2\,m\,d^2 Ib - 2\,m^2 d^4\cos(\phi)^2 + Ib^2 - 2\,Ib\,m\,d^2\cos(\phi)^2\right.$$

$$\left. + m^2 d^4\cos(\phi)^4\right)\Big]\Big]$$

> $dKD_2 - dKD_3;$

$$-\left(4\,d\left(d\,m\,KD12(\theta,\phi)\,r\,Ib + 2\,d\,m\,KD22(\theta,\phi)\,r\,Ib - m^2\sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\right.\right.\right. \qquad \mathbf{(8.2)}$$

$$\left.\phi)\right)r\,d^3\cos(\phi) + m^2\sin(\phi)\left(\frac{\partial}{\partial\theta}KD22(\theta,\phi)\right)r\,d^3\cos(\phi)^3$$

$$+ \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\phi)\right)m\,r^2\,Ib - \sin(\phi)\left(\frac{\partial}{\partial\phi}KD22(\theta,\right.$$

$$\phi)\Big) m^2 r^2 d^2 \cos(\phi)^2 - 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 r d^3 \cos(\phi)$$

$$-2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m r d \cos(\phi)\, Ib + 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\right.$$

$$\phi)\Big) m^2 r d^3 \cos(\phi)^3 + 2\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m d^2\, Ib$$

$$-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 d^4 \cos(\phi)^2 - \sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\right.$$

$$\phi)\Big) Ib\, m d^2 \cos(\phi)^2 + m^2 r^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD12(\theta,\phi)\right) d^2$$

$$+m r^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD12(\theta,\phi)\right) Ib - m^2 r^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD12(\theta,\right.$$

$$\phi)\Big) d^2 \cos(\phi)^2 + m^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,\phi)\right) r^2 d^2$$

$$+m \sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,\phi)\right) r^2\, Ib - m^2 \sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,\right.$$

$$\phi)\Big) r^2 d^2 \cos(\phi)^2 - m \sin(\phi)\left(\frac{\partial}{\partial\theta} KD22(\theta,\phi)\right) r d \cos(\phi)\, Ib$$

$$+m \cos(\phi)\, KD12(\theta,\phi)\, r^2\, Ib - 2 m \cos(\phi)^2\, KD12(\theta,\phi)\, r d\, Ib$$

$$+m^2 \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\phi)\right) r^2 d^2 + m \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\phi)\right) r^2\, Ib$$

$$-m^2 \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\phi)\right) r^2 d^2 \cos(\phi)^2 - m^2 \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\right.$$

$$\phi)\Big) r d^3 \cos(\phi) - m \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\phi)\right) r d \cos(\phi)\, Ib$$

$$+m^2 \sin(\phi)\left(\frac{\partial}{\partial\phi} KD12(\theta,\phi)\right) r d^3 \cos(\phi)^3 + m^2 \cos(\phi)^3\, KD12(\theta,\phi)\, r^2 d^2$$

$$-m^2 \cos(\phi)^2\, KD12(\theta,\phi)\, r d^3 + \sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) Ib^2$$

$$-\cos(\phi)\, KD22(\theta,\phi)\, m^2 r^2 d^2 + \cos(\phi)\, KD22(\theta,\phi)\, m r^2\, Ib$$

$$+\cos(\phi)^3\, KD22(\theta,\phi)\, m^2 r^2 d^2 - 2\cos(\phi)^2\, KD22(\theta,\phi)\, m^2 r d^3$$

$$-4\cos(\phi)^2\, KD22(\theta,\phi)\, m r d\, Ib - m^2 \cos(\phi)\, KD12(\theta,\phi)\, r^2 d^2$$

$$+d^3 m^2\, KD12(\theta,\phi)\, r + 2 d^3 m^2\, KD22(\theta,\phi)\, r - \cos(\phi)\, KD22(\theta,\phi)\, m^2 d^4$$

$$+\cos(\phi)^3 KD22(\theta,\phi)\, m^2 d^4+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 d^4$$

$$+\cos(\phi)^3 KD22(\theta,\phi)\, Ib\, m\, d^2+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD22(\theta,\phi)\right) m^2 r^2 d^2$$

$$+\cos(\phi)\, KD22(\theta,\phi)\, Ib^2\Big)\Big) \Big/ \Big(r\,\big(m^2 d^4+2\, m\, d^2\, Ib-2\, m^2 d^4\cos(\phi)^2+Ib^2$$

$$-2\, Ib\, m\, d^2\cos(\phi)^2+m^2 d^4\cos(\phi)^4\big)\Big)-\Big(4\, m\, d\,\Big($$

$$-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) d\cos(\phi)\, Ib-2\cos(\phi)^2 KD21(\theta,\phi)\, d\, Ib$$

$$+\cos(\phi)\, KD21(\theta,\phi)\, r\, Ib+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) r\, Ib$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) Ib+\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r\, Ib$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) Ib+r\cos(\phi)\, KD11(\theta,\phi)\, Ib$$

$$-\cos(\phi)^2 KD21(\theta,\phi)\, d^3 m-\cos(\phi)\, KD21(\theta,\phi)\, r\, m\, d^2+\cos(\phi)^3 KD21(\theta,$$

$$\phi)\, r\, m\, d^2+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) r\, m\, d^2-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\right.$$

$$\left.\phi)\right) r\, m\, d^2\cos(\phi)^2+d\, KD21(\theta,\phi)\, Ib+d^3 KD21(\theta,\phi)\, m$$

$$-\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\phi)\right) d^3\cos(\phi)\, m+\sin(\phi)\left(\frac{\partial}{\partial\phi} KD21(\theta,\right.$$

$$\left.\phi)\right) d^3\cos(\phi)^3 m+r\sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) m\, d^2$$

$$-r\sin(\phi)\left(\frac{\partial}{\partial\theta} KD11(\theta,\phi)\right) m\, d^2\cos(\phi)^2+\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\right.$$

$$\left.\phi)\right) r\, m\, d^2-\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) r\, m\, d^2\cos(\phi)^2$$

$$-\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) d^3\cos(\phi)\, m-\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\right.$$

$$\left.\phi)\right) d\cos(\phi)\, Ib+\sin(\phi)\left(\frac{\partial}{\partial\theta} KD21(\theta,\phi)\right) d^3\cos(\phi)^3 m$$

$$-r\cos(\phi)\, KD11(\theta,\phi)\, m\, d^2+r\cos(\phi)^3 KD11(\theta,\phi)\, m\, d^2$$

$$+r\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\phi)\right) m\, d^2-r\sin(\phi)\left(\frac{\partial}{\partial\phi} KD11(\theta,\right.$$

$$\phi)\Big)\, m\, d^2 \cos(\phi)^2\Big)\Big) \Big/ \Big(m^2 d^4 + 2\, m\, d^2\, Ib - 2\, m^2 d^4 \cos(\phi)^2 + Ib^2 - 2\, Ib\, m\, d^2 \cos(\phi)^2 + m^2 d^4 \cos(\phi)^4\Big)$$

> $deq1 := dKD_1 :$

> $deq2 := dKD_2 :$

> $deq3 := dKD_3 :$

> $deq4 := dKD_4 :$

Temporary variables

> $$Temp := \left[\frac{\partial}{\partial\theta} KD11(\theta,\phi), \frac{\partial}{\partial\theta} KD12(\theta,\phi), \frac{\partial}{\partial\theta} KD21(\theta,\phi), \frac{\partial}{\partial\theta} KD22(\theta,\phi), \frac{\partial}{\partial\phi} KD11(\theta,\phi), \frac{\partial}{\partial\phi} KD12(\theta,\phi), \frac{\partial}{\partial\phi} KD21(\theta,\phi), \frac{\partial}{\partial\phi} KD22(\theta,\phi), KD11(\theta,\phi), KD12(\theta,\phi), KD21(\theta,\phi), KD22(\theta,\phi)\right] :$$

> $deq1 := collect(deq1, Temp) :$

> $deq2 := collect(deq2, Temp) :$

> $deq3 := collect(deq3, Temp) :$

> $deq4 := collect(deq4, Temp) :$

PDE Solution

> $sys4 := [deq1, deq2, deq3, deq4] :$

> $sol4 := pdsolve(sys4, [KD11(\theta,\phi), KD12(\theta,\phi), KD21(\theta,\phi), KD22(\theta,\phi)]) :$

> $pdetest(sol4, sys4);$

$$[0, 0, 0, 0] \tag{11.1}$$

> $$KDF := \begin{bmatrix} rhs(sol4_1) & rhs(sol4_2) \\ rhs(sol4_3) & rhs(sol4_4) \end{bmatrix} :$$

> $KDFdot := Matrix(map(diff, KDF, \theta)) \cdot \theta dot + Matrix(map(diff, KDF, \phi)) \cdot \phi dot :$

> $QKD := Transpose(Multiply(KDF, qdot)) :$

> $dKDIq := Matrix\left(2, 2, \left[\mathrm{diff}\left(QKD_1, \theta\right), \mathrm{diff}\left(QKD_2, \theta\right), \mathrm{diff}\left(QKD_1, \phi\right), diff\left(QKD_2, \phi\right)\right]\right) :$

> $Ltest := KDFdot - \frac{1}{2} \cdot Transpose(dKDIq) - \frac{1}{2} \cdot dKDIq :$

> $Ltest := simplify(eval(\, Ltest, [\_C1 = 0]));$

$$Ltest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.1}$$

> $KDtest := simplify(eval(\, KDF, [\_C1 = 0]));$

$$KDtest := \left[\left[0, \frac{1}{r\, d\, m \sin(\phi)} \left(\mathrm{e}^{-\_c_1 (-\theta + \phi)} \_C1 \left(-\mathrm{e}^{-\frac{1}{2} \_c_1 \pi} \_C3 \cos(\phi)\, d\, m\, r + \_C2\, \mathrm{e}^{\_c_1 \phi} \cos(\phi)\, m\, d^2 + \mathrm{e}^{-\frac{1}{2} \_c_1 \pi} \_C3\, Ib - \_C2\, \mathrm{e}^{\_c_1 \phi}\, d\, m\, r + \mathrm{e}^{-\frac{1}{2} \_c_1 \pi} \_C3\, d^2\, m\right)\right)\right], \left[0, \frac{\_C1\, \mathrm{e}^{-\_c_1 (-\theta + \phi)} \left(\mathrm{e}^{-\frac{1}{2} \_c_1 \pi} \_C3 \cos(\phi) + \_C2\, \mathrm{e}^{\_c_1 \phi}\right)}{\sin(\phi)}\right]\right] \tag{12.2}$$

***As a conclusion KD is* not *Lagrangian* for the ball and arc system.**

B.3 Direct Lyapunov Aproach for the B&Arc System KD

B&Arc_DLA_Fm1_KD.mw

> *restart* :

> *with*(*LinearAlgebra*) :

q is a vector of generalized coordinates

> $q := \begin{bmatrix} \theta \\ \phi \end{bmatrix}$:

qdot is a vector of generalized velocites

> $qdot := \begin{bmatrix} \theta dot \\ \phi dot \end{bmatrix}$:

Cmatrix is the matrix of Centripetal* and *coriolis forces

> $Cmatrix := -Matrix(2, 2, [-2\ \phi dot\ m\ r\ d \sin(\phi), -2\ \theta dot\ m\ r\ d \sin(\phi) + 2\ \phi dot\ m\ r\ d \sin(\phi), 2\ \theta dot\ m\ r\ d \sin(\phi), 0]);$

$$Cmatrix := \begin{bmatrix} 2\ \phi dot\ m\ r\ d \sin(\phi) & 2\ \theta dot\ m\ r\ d \sin(\phi) - 2\ \phi dot\ m\ r\ d \sin(\phi) \\ -2\ \theta dot\ m\ r\ d \sin(\phi) & 0 \end{bmatrix} \tag{1.1}$$

Define the mass matrix and its inverse

> $Mass := -simplify(Matrix(2, 2, [2\ m\ r\ d \cos(\theta)^2 \cos(\phi) - m\ r^2 + 2\ m\ r\ d \sin(\theta)^2 \cos(\phi) - m\ d^2 - Ib, -m\ r\ d \cos(\theta)^2 \cos(\phi) + m\ r^2 - m\ r\ d \sin(\theta)^2 \cos(\phi), -m\ r\ d \cos(\theta)^2 \cos(\phi) + m\ r^2 - m\ r\ d \sin(\theta)^2 \cos(\phi), -m\ r^2]));$

$$Mass := \begin{bmatrix} m\ r^2 - 2\ m\ r\ d \cos(\phi) + m\ d^2 + Ib & m\ r\ (-r + d \cos(\phi)) \\ m\ r\ (-r + d \cos(\phi)) & m\ r^2 \end{bmatrix} \tag{1.2}$$

> $IMass := MatrixInverse(Mass);$

$$IMass := \begin{bmatrix} \frac{1}{m\ d^2 + Ib - m\ d^2 \cos(\phi)^2} & -\frac{-r + d \cos(\phi)}{r\left(m\ d^2 + Ib - m\ d^2 \cos(\phi)^2\right)} \\ -\frac{-r + d \cos(\phi)}{r\left(m\ d^2 + Ib - m\ d^2 \cos(\phi)^2\right)} & \frac{m\ r^2 - 2\ m\ r\ d \cos(\phi) + m\ d^2 + Ib}{m\ r^2 \left(m\ d^2 + Ib - m\ d^2 \cos(\phi)^2\right)} \end{bmatrix} \tag{1.3}$$

G is the Gravitational Forces vector

> $G := -\begin{bmatrix} m\ g\ (r \sin(\phi) \cos(\theta) - r \cos(\phi) \sin(\theta) + d \sin(\theta)) \\ -m\ g\ r\ (\sin(\phi) \cos(\theta) - \cos(\phi) \sin(\theta)) \end{bmatrix}$:

Fm1 Control law matrix for FMC, Eq. 4.8

$$> \ Fm1 := \begin{bmatrix} -F11(\mathrm{qf}, \mathrm{qdotf}) \cdot \phi dot + \nu & F11(\mathrm{qf}, \mathrm{qdotf}) \cdot \theta dot - \sigma \\ -F22(\mathrm{qf}, \mathrm{qdotf}) \cdot \phi dot & F22(\mathrm{qf}, \mathrm{qdotf}) \cdot \theta\mathrm{dot} \end{bmatrix}:$$

Fmc1 control law matrix for FMC, Eq. 4.9 and Eq. 4.10

$$> \ Fmc1p := [[F33(\mathrm{qf}, \mathrm{qdotf}) + F55(\mathrm{qf}, \mathrm{qdotf}) + \mathrm{F66}(\mathrm{qf}, \mathrm{qdotf}), \ \mathrm{F77}(\mathrm{qf}, \mathrm{qdotf})], [\mathrm{F88}(\mathrm{qf}, \mathrm{qdotf}), F44(\mathrm{qf}, \mathrm{qdotf}) + F66(\mathrm{qf}, \mathrm{qdotf}) + \mathrm{F55}(\mathrm{qf}, \mathrm{qdotf})]];$$

$$Fmc1p := [[F33(qf, qdotf) + F55(qf, qdotf) + F66(qf, qdotf), F77(qf, qdotf)], [F88(qf, qdotf), F44(qf, qdotf) + F66(qf, qdotf) + F55(qf, qdotf)]] \tag{1.4}$$

$$> \ Fmc1 := Fmc1p + Transpose(Fmc1p);$$

$$Fmc1 := [[2\,F33(qf, qdotf) + 2\,F55(qf, qdotf) + 2\,F66(qf, qdotf), F77(qf, qdotf) + F88(qf, qdotf)], [F77(qf, qdotf) + F88(qf, qdotf), 2\,F44(qf, qdotf) + 2\,F66(qf, qdotf) + 2\,F55(qf, qdotf)]] \tag{1.5}$$

First Matching Condition

KD is a constant

$$> \ KDT := \begin{bmatrix} KD11 & \mathrm{KD21} \\ \mathrm{KD21} & KD22 \end{bmatrix}:$$

The time derivative of KD is

$$> \ KDdot := \mathrm{simplify}(map(diff, \theta dot \cdot KDT, \theta) + map(diff, \phi dot \cdot KDT, \phi));$$

$$KDdot := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

First Matching Condition with inputsFm1 and Fmc1

$$> \ FMCsim := simplify(\mathrm{KDdot} + \mathrm{Multiply}(KDT, \mathrm{Multiply}(IMass, (Fm1 - Cmatrix))) + Transpose(\mathrm{Multiply}(KDT, \mathrm{Multiply}(IMass, (Fm1 - Cmatrix)))) + Fmc1):$$

$$> \ F1 := simplify(Multiply(Fm1, qdot));$$

$$F1 := \begin{bmatrix} \theta dot\, \nu - \phi dot\, \sigma \\ 0 \end{bmatrix} \tag{2.2}$$

Simtest represents the symmetry proof for the FMC

$$> \ Simtest := simplify(FMCsim_{2,\,1} - FMCsim_{1,\,2});$$

$$Simtest := 0 \tag{2.3}$$

$$> \ sys := simplify(\{FMCsim_{1,\,1}, FMCsim_{2,\,1}, FMCsim_{2,\,2}\}):$$

Solving for the forces on sys the result is going to be called *forsol*

$$> \ forsol := solve(sys, [F11(\mathrm{qf}, \mathrm{qdotf}), F22(\mathrm{qf}, \mathrm{qdotf}), F33(\mathrm{qf}, \mathrm{qdotf})]);$$

(2.4)

$$forsol := \Bigg[\Bigg[F11(qf, qdotf) = \Big(-F88(qf, qdotf)\ \theta dot\ m^2\ r^3\ d^2\ KD21 \sin(\phi)^2 \qquad \textbf{(2.4)}$$

$$- F88(qf, qdotf)\ \theta dot\ m^2\ KD22 \sin(\phi)^2\ d^4\ r - m\ \theta dot\ F88(qf,$$

$$qdotf)\ Ib\ r\ KD22\ d^2 \sin(\phi)^2 + m\ \phi dot\ F44(qf, qdotf)\ Ib\ KD21\ d \cos(\phi)\ r^2$$

$$- m\ KD11\ \theta dot\ F44(qf, qdotf)\ Ib\ d \cos(\phi)\ r^2 - F44(qf,$$

$$qdotf)\ \theta dot\ KD11\ m^2\ d^3 \cos(\phi) \sin(\phi)^2\ r^2 - F44(qf,$$

$$qdotf)\ \phi dot\ m^2\ r^3\ KD22\ d^2 \sin(\phi)^2 - F44(qf, qdotf)\ \phi dot\ m^2\ r^3\ d^2\ KD21 \sin(\phi)^2$$

$$- F44(qf, qdotf)\ \phi dot\ m^2\ KD22 \sin(\phi)^2\ d^4\ r - 2\ KD22^2\ d \sin(\phi)\ \theta dot^2\ Ib^2$$

$$- KD22\ r\ F77(qf, qdotf)\ \theta dot\ Ib^2 + KD21\ r\ F55(qf, qdotf)\ Ib^2\ \theta dot$$

$$- KD22\ r\ F44(qf, qdotf)\ \phi dot\ Ib^2 - 2\ F66(qf,$$

$$qdotf)\ \theta dot\ m^2\ r^2\ d^3\ KD21 \cos(\phi) \sin(\phi)^2 + 2\ m\ \phi dot\ F66(qf,$$

$$qdotf)\ Ib\ KD22\ r^2\ d \cos(\phi) - F77(qf, qdotf)\ \theta dot\ m^2\ r^3\ d^2\ KD21 \sin(\phi)^2$$

$$- KD22\ r\ F55(qf, qdotf)\ \phi dot\ Ib^2 + KD21\ r\ F44(qf, qdotf)\ Ib^2\ \theta dot + F44(qf,$$

$$qdotf)\ \theta dot\ m^2\ r^3\ d^2\ KD21 \sin(\phi)^2 + F44(qf, qdotf)\ \theta dot\ m^2\ KD21 \sin(\phi)^2\ d^4\ r$$

$$+ 2\ F44(qf, qdotf)\ \phi dot\ m^2\ r^2\ d^3 \cos(\phi) \sin(\phi)^2\ KD22 + F55(qf,$$

$$qdotf)\ \theta dot\ m^2\ r^3\ d^2\ KD21 \sin(\phi)^2 + F66(qf, qdotf)\ \theta dot\ m^2\ KD21 \sin(\phi)^2\ d^4\ r$$

$$
\begin{aligned}
&+F66(qf, qdotf)\ \phi dot\ m^2\ r^2\ d^3\ KD21\cos(\phi)\ \sin(\phi)^2 - m\ \phi dot\ F66(qf, \\
&qdotf)\ Ib\ r\ KD22\ d^2\sin(\phi)^2 + m\ \theta dot\ F66(qf, qdotf)\ Ib\ r\ d^2\ KD21\sin(\phi)^2 \\
&+F66(qf, qdotf)\ \theta dot\ KD11\ m^2\ d^2\sin(\phi)^2\ r^3 - 2\ \theta dot^2\ m^2\ KD22^2\sin(\phi)\ d^5 \\
&-2\ F55(qf, qdotf)\ \theta dot\ m^2\ r^2\ d^3\ KD21\cos(\phi)\ \sin(\phi)^2 - 2\ m\ \theta dot\ F55(qf, \\
&qdotf)\ Ib\ KD21\ d\cos(\phi)\ r^2 + 2\ m\ \phi dot\ F55(qf, qdotf)\ Ib\ KD22\ r^2\ d\cos(\phi) \\
&+m\ \phi dot\ F55(qf, qdotf)\ Ib\ KD21\ d\cos(\phi)\ r^2 - F55(qf, \\
&qdotf)\ \theta dot\ KD11\ m^2\ d^3\cos(\phi)\ \sin(\phi)^2\ r^2 - m\ KD11\ \theta dot\ F55(qf, \\
&qdotf)\ Ib\ d\cos(\phi)\ r^2 - F55(qf, qdotf)\ \phi dot\ m^2\ r^3\ KD22\ d^2\sin(\phi)^2 - F55(qf, \\
&qdotf)\ \phi dot\ m^2\ r^3\ d^2\ KD21\sin(\phi)^2 - F55(qf, qdotf)\ \phi dot\ m^2\ KD22\sin(\phi)^2\ d^4\ r \\
&-m\ \theta dot\ F77(qf, qdotf)\ Ib\ KD22\ r^3 - m\ \theta dot\ F77(qf, qdotf)\ Ib\ KD21\ r^3 \\
&-m\ \theta dot\ F77(qf, qdotf)\ Ib\ r\ d^2\ KD22 - KD22\ r\ F88(qf, qdotf)\ \theta dot\ Ib^2 \\
&+F55(qf, qdotf)\ \theta dot\ m^2\ KD21\sin(\phi)^2\ d^4\ r + 2\ F55(qf, \\
&qdotf)\ \phi dot\ m^2\ r^2\ d^3\cos(\phi)\ \sin(\phi)^2\ KD22 + F55(qf, \\
&qdotf)\ \phi dot\ m^2\ r^2\ d^3\ KD21\cos(\phi)\ \sin(\phi)^2 + m\ \theta dot\ F55(qf, \\
&qdotf)\ Ib\ r\ d^2\ KD21\sin(\phi)^2 - m\ \phi dot\ F55(qf, qdotf)\ Ib\ r\ KD22\ d^2\sin(\phi)^2
\end{aligned}
$$

$$+ F55(qf, qdotf)\ \theta dot\ KD11\ m^2 d^2 \sin(\phi)^2 r^3 - 2\ \theta dot^2 m^2 r^4 KD22^2 d \sin(\phi)$$

$$- 4\ \theta dot^2 m^2 r^4 KD21\ d \sin(\phi)\ KD22 - 2\ \theta dot^2 m^2 r^4 d\ KD21^2 \sin(\phi)$$

$$- 12\ \theta dot^2 m^2 r^2 d^3 KD21 \sin(\phi)\ KD22 + 8\ \theta dot^2 m^2 r^2 d^3 KD22^2 \sin(\phi)^3$$

$$- 12\ \theta dot^2 m^2 r^2 d^3 KD22^2 \sin(\phi) + 8\ \theta dot^2 m^2 r^2 d^3 KD22\ KD21 \sin(\phi)^3$$

$$- 2\ \theta dot^2 m^2 r^2 d^3 KD21^2 \sin(\phi) + 2\ F88(qf,$$

$$qdotf)\ \theta dot\ m^2 r^2 d^3 \cos(\phi) \sin(\phi)^2 KD22 + F88(qf,$$

$$qdotf)\ \theta dot\ m^2 r^2 d^3 KD21 \cos(\phi) \sin(\phi)^2$$

$$+ 2 \sin(\phi)^3 KD22\ d^3 r^2 KD11\ \theta dot^2 m^2 - KD21^2 \sin(\phi)^2 d^2 r\ m\ \theta dot\ \sigma$$

$$+ r\ KD22\ d^2 \sin(\phi)^2 KD11\ m\ \theta dot\ \sigma - 3\ m\ \phi dot\ \sigma\ r^2 KD22^2 d \cos(\phi)$$

$$- KD22\ r\ F66(qf, qdotf)\ \phi dot\ Ib^2 - 4\ m\ \phi dot\ \sigma\ r^2 KD21\ d \cos(\phi)\ KD22$$

$$- m\ \phi dot\ \sigma\ r^2 KD21^2 d \cos(\phi) - m\ \phi dot\ \sigma\ d^3 \cos(\phi)\ KD22^2$$

$$+ \nu\ \theta dot\ Ib\ KD22^2 d \cos(\phi) + 2\ \phi dot^2 m\ Ib\ d^2 \sin(\phi) \cos(\phi)\ KD22^2 r$$

$$- 4\ Ib\ \theta dot\ \phi dot\ m\ d^2 \sin(\phi) \cos(\phi)\ KD22^2 r + 3\ m\ \theta dot\ \nu\ r^2 KD22^2 d \cos(\phi)$$

$$+ 4\ m\ \theta dot\ \nu\ r^2 KD21\ d \cos(\phi)\ KD22 + m\ \theta dot\ \nu\ r^2 KD21^2 d \cos(\phi)$$

$$+ m\ \theta dot\ \nu\ d^3 \cos(\phi)\ KD22^2 - \sigma\ \phi dot\ Ib\ KD22^2 d \cos(\phi)$$

$+6\ \phi dot^2\ m^2\ r^3\ d^2 \sin(\phi)\ \cos(\phi)\ KD22^2$

$+8\ \phi dot^2\ m^2\ r^3\ d^2\ KD21 \cos(\phi)\ \sin(\phi)\ KD22$

$+2\ \phi dot^2\ m^2\ r^3\ d^2\ KD21^2 \cos(\phi)\ \sin(\phi)\ +2\ \phi dot^2\ m^2\ d^4 \cos(\phi)\ KD22^2 \sin(\phi)\ r$

$-12\ \theta dot\ \phi dot\ m^2\ r^3\ d^2 \sin(\phi)\ \cos(\phi)\ KD22^2$

$-16\ \theta dot\ \phi dot\ m^2\ r^3\ d^2\ KD21 \cos(\phi)\ \sin(\phi)\ KD22$

$-4\ \theta dot\ \phi dot\ m^2\ r^3\ d^2\ KD21^2 \cos(\phi)\ \sin(\phi)\ -2\ m\ \theta dot\ F44(qf,$

$qdotf)\ Ib\ KD21\ d \cos(\phi)\ r^2 +2\ m\ \phi dot\ F44(qf, qdotf)\ Ib\ KD22\ r^2\ d \cos(\phi)$

$+F44(qf, qdotf)\ \phi dot\ m^2\ r^2\ d^3\ KD21 \cos(\phi)\ \sin(\phi)^2 + m\ \theta dot\ F44(qf,$

$qdotf)\ Ib\ r\ d^2\ KD21 \sin(\phi)^2 - m\ \phi dot\ F44(qf, qdotf)\ Ib\ r\ KD22\ d^2 \sin(\phi)^2$

$+F44(qf, qdotf)\ \theta dot\ KD11\ m^2\ d^2 \sin(\phi)^2\ r^3 +2\ m\ \theta dot\ F77(qf,$

$qdotf)\ Ib\ KD22\ r^2\ d \cos(\phi)\ + m\ \theta dot\ F77(qf, qdotf)\ Ib\ KD21\ d \cos(\phi)\ r^2$

$-8\ \theta dot\ \phi dot\ m^2\ r^2\ d^3\ KD22^2 \sin(\phi)^3$

$-4\ \theta dot\ \phi dot\ m^2\ r^2\ d^3\ KD22\ KD21 \sin(\phi)^3$

$+8\ \theta dot\ \phi dot\ m^2\ r^2\ d^3\ KD21 \sin(\phi)\ KD22 +2\ \theta dot\ \phi dot\ m^2\ r^2\ d^3 \sin(\phi)^3\ KD21^2$

$-4\ Ib\ \theta dot^2\ m\ r^2\ KD22^2\ d \sin(\phi)\ -4\ Ib\ \theta dot^2\ m\ r^2\ KD21\ d \sin(\phi)\ KD22$

$$- 2\ Ib\ \theta dot^2\ m\ r^2\ d\ KD21^2 \sin(\phi) - 4\ Ib\ \theta dot^2\ m\ d^3\ KD22^2 \sin(\phi)$$

$$- 2\ m\ \phi dot\ \sigma\ r\ d^2\ KD22^2 \sin(\phi)^2 - m\ \phi dot\ \sigma\ r\ KD21\ d^2\ KD22 \sin(\phi)^2$$

$$- 2\ \phi dot^2\ m\ Ib\ r^2\ KD22^2\ d \sin(\phi) + 8\ \theta dot^2\ m^2\ r^3\ d^2 \sin(\phi) \cos(\phi)\ KD22^2$$

$$- m\ \phi dot\ F66(qf, qdotf)\ Ib\ KD22\ r^3 - m\ \theta dot\ F88(qf, qdotf)\ Ib\ KD22\ r^3$$

$$- m\ \theta dot\ F88(qf, qdotf)\ Ib\ KD21\ r^3 - m\ \theta dot\ F88(qf, qdotf)\ Ib\ r\ d^2\ KD22$$

$$- 2\ \phi dot^2\ m\ Ib\ r^2\ KD21\ d \sin(\phi)\ KD22$$

$$+ 12\ \theta dot^2\ m^2\ r^3\ d^2\ KD21 \cos(\phi) \sin(\phi)\ KD22$$

$$+ 4\ \theta dot^2\ m^2\ r^3\ d^2\ KD21^2 \cos(\phi) \sin(\phi) + 8\ \theta dot^2\ m^2\ d^4 \cos(\phi)\ KD22^2 \sin(\phi)\ r$$

$$+ 4\ \theta dot^2\ m^2\ r\ d^4\ KD21 \cos(\phi) \sin(\phi)\ KD22$$

$$- 4\ \theta dot\ \phi dot\ m^2\ d^4 \cos(\phi)\ KD22^2 \sin(\phi)\ r$$

$$+ 8\ Ib\ \theta dot^2\ m\ d^2 \sin(\phi) \cos(\phi)\ KD22^2\ r$$

$$+ 4\ Ib\ \theta dot^2\ m\ r\ d^2\ KD21 \cos(\phi) \sin(\phi)\ KD22$$

$$+ 4\ Ib\ \theta dot\ \phi dot\ m\ r^2\ KD22^2\ d \sin(\phi) - m\ \phi dot\ F66(qf, qdotf)\ Ib\ KD21\ r^3$$

$$- m\ \phi dot\ F66(qf, qdotf)\ Ib\ r\ d^2\ KD22 + m\ \theta dot\ F66(qf, qdotf)\ Ib\ KD21\ r^3$$

$$+ m\ \theta dot\ F66(qf, qdotf)\ Ib\ r\ KD21\ d^2 + KD21\ r\ F66(qf, qdotf)\ Ib^2\ \theta dot$$

$+ m\,\theta dot\,F55(qf, qdotf)\,Ib\,KD21\,r^3 + m\,\phi dot\,\sigma\,KD22^2\,r^3 + m\,\theta dot\,F55(qf,$

$qdotf)\,Ib\,r\,KD21\,d^2 - m\,\phi dot\,F55(qf, qdotf)\,Ib\,KD22\,r^3 - m\,\phi dot\,F55(qf,$

$qdotf)\,Ib\,KD21\,r^3 - m\,\phi dot\,F55(qf, qdotf)\,Ib\,r\,d^2\,KD22$

$+ 4\,Ib\,\theta dot\,\phi dot\,m\,r^2\,KD21\,d\,\sin(\phi)\,KD22$

$+ 2\,Ib\,\theta dot\,\phi dot\,m\,r^2\,d\,KD21^2\,\sin(\phi) + 2\,m\,\theta dot\,\nu\,r\,d^2\,KD22^2\,\sin(\phi)^2$

$+ m\,\theta dot\,\nu\,r\,KD21\,d^2\,KD22\,\sin(\phi)^2 - 2\,\phi dot^2\,m^2\,r^4\,KD22^2\,d\,\sin(\phi)$

$- 4\,\phi dot^2\,m^2\,r^4\,KD21\,d\,\sin(\phi)\,KD22 - 2\,\phi dot^2\,m^2\,r^4\,d\,KD21^2\,\sin(\phi)$

$- 6\,\phi dot^2\,m^2\,r^2\,d^3\,KD22^2\,\sin(\phi) + 4\,\phi dot^2\,m^2\,r^2\,d^3\,KD22^2\,\sin(\phi)^3$

$+ 2\,\phi dot^2\,m^2\,r^2\,d^3\,KD22\,KD21\,\sin(\phi)^3 - 4\,\phi dot^2\,m^2\,r^2\,d^3\,KD21\,\sin(\phi)\,KD22$

$+ 4\,\theta dot\,\phi dot\,m^2\,r^4\,KD22^2\,d\,\sin(\phi) + 8\,\theta dot\,\phi dot\,m^2\,r^4\,KD21\,d\,\sin(\phi)\,KD22$

$+ 4\,\theta dot\,\phi dot\,m^2\,r^4\,d\,KD21^2\,\sin(\phi) + 12\,\theta dot\,\phi dot\,m^2\,r^2\,d^3\,KD22^2\,\sin(\phi)$

$+ 2\,\sin(\phi)\,KD22\,d\,r^2\,Ib\,KD11\,\theta dot^2\,m$

$- 2\,\sin(\phi)^3\,KD22\,d^3\,r^2\,KD11\,\theta dot\,\phi dot\,m^2$

$- 2\,\sin(\phi)\,KD22\,d\,r^2\,Ib\,KD11\,\theta dot\,\phi dot\,m + m\,KD11\,\theta dot\,F66(qf, qdotf)\,Ib\,r^3$

$+ m\,KD11\,\theta dot\,F55(qf, qdotf)\,Ib\,r^3 - KD21^2\,r\,\sigma\,\theta dot\,Ib + m\,\theta dot\,F44(qf,$

$qdotf)\ Ib\, KD21\, r^3 + m\ \theta dot\ F44(qf, qdotf)\ Ib\, r\, KD21\, d^2 - m\ \phi dot\ F44(qf,$

$qdotf)\ Ib\, KD22\, r^3 - m\ \phi dot\ F44(qf, qdotf)\ Ib\, KD21\, r^3 - m\ \phi dot\ F44(qf,$

$qdotf)\ Ib\, r\, d^2\, KD22 + m\, KD11\ \theta dot\ F44(qf, qdotf)\ Ib\, r^3 - F77(qf,$

$qdotf)\ \theta dot\ m^2\, r^3\, KD22\, d^2 \sin(\phi)^2 - F77(qf, qdotf)\ \theta dot\ m^2\, KD22 \sin(\phi)^2\, d^4\, r$

$+ 2\, F77(qf, qdotf)\ \theta dot\ m^2\, r^2\, d^3 \cos(\phi) \sin(\phi)^2\, KD22 + F77(qf,$

$qdotf)\ \theta dot\ m^2\, r^2\, d^3\, KD21 \cos(\phi) \sin(\phi)^2 - m\ \theta dot\ F77(qf,$

$qdotf)\ Ib\, r\, KD22\, d^2 \sin(\phi)^2 - 2\, F44(qf,$

$qdotf)\ \theta dot\ m^2\, r^2\, d^3\, KD21 \cos(\phi) \sin(\phi)^2 + m\ \phi dot\ F66(qf,$

$qdotf)\ Ib\, KD21\, d \cos(\phi)\, r^2 - 2\, m\ \theta dot\ F66(qf, qdotf)\ Ib\, KD21\, d \cos(\phi)\, r^2$

$- F66(qf, qdotf)\ \theta dot\ KD11\, m^2\, d^3 \cos(\phi) \sin(\phi)^2\, r^2$

$+ 2\, m\ \phi dot\ \sigma\, r^3\, KD22\, KD21 + m\ \phi dot\ \sigma\, r^3\, KD21^2 + 3\, m\ \phi dot\ \sigma\, r\, d^2\, KD22^2$

$+ 2\, m\ \phi dot\ \sigma\, r\, KD21\, d^2\, KD22 - \nu\ \theta dot\ Ib\, KD22\, KD21\, r - \nu\ \theta dot\ Ib\, KD22^2\, r$

$- m\ \theta dot\ \nu\, r^3\, KD21^2 - m\ \theta dot\ \nu\, KD22^2\, r^3 - 2\, m\ \theta dot\ \nu\, r^3\, KD22\, KD21$

$- 3\, m\ \theta dot\ \nu\, r\, d^2\, KD22^2 - 2\, m\ \theta dot\ \nu\, r\, KD21\, d^2\, KD22$

$+ \sigma\ \phi dot\ Ib\, KD22\, KD21\, r + \sigma\ \phi dot\ Ib\, KD22^2\, r + KD22\, r\, Ib\, KD11\ \theta dot\ \sigma$

$$- m\, KD11\, \theta dot\, F66(qf, qdotf)\, Ib\, d \cos(\phi)\, r^2 - F66(qf,$$

$$qdotf)\, \phi dot\, m^2\, r^3\, KD22\, d^2 \sin(\phi)^2 - F66(qf, qdotf)\, \phi dot\, m^2\, r^3\, d^2\, KD21 \sin(\phi)^2$$

$$- F66(qf, qdotf)\, \phi dot\, m^2\, KD22 \sin(\phi)^2\, d^4\, r + F66(qf,$$

$$qdotf)\, \theta dot\, m^2\, r^3\, d^2\, KD21 \sin(\phi)^2 + 2\, F66(qf,$$

$$qdotf)\, \phi dot\, m^2\, r^2\, d^3 \cos(\phi) \sin(\phi)^2\, KD22 + 2\, m\, \theta dot\, F88(qf,$$

$$qdotf)\, Ib\, KD22\, r^2\, d \cos(\phi) + m\, \theta dot\, F88(qf, qdotf)\, Ib\, KD21\, d \cos(\phi)\, r^2$$

$$- F88(qf, qdotf)\, \theta dot\, m^2\, r^3\, KD22\, d^2 \sin(\phi)^2 \Big) \Big/ \Big(\theta dot^2\, r \Big(-KD21^2\, Ib$$

$$+ KD11\, KD22\, Ib + d^2\, m\, KD22\, KD11 \sin(\phi)^2 - KD21^2\, d^2\, m \sin(\phi)^2 \Big) \Big),$$

$$F22(qf, qdotf) = \Big(\cos(\phi)\, m \Big(-2\, KD21\, r\, \phi dot\, KD22\, d\, \sigma$$

$$+ 2\, \theta dot\, KD22^2\, r\, \nu\, d + \theta dot\, F77(qf, qdotf)\, r\, Ib\, KD22\, d$$

$$- KD21\, d\, \theta dot\, F66(qf, qdotf)\, r\, Ib + \cos(\phi)\, KD11\, r^2\, \theta dot\, F55(qf,$$

$$qdotf)\, m\, d^2 \sin(\phi)^2 + \cos(\phi)\, \theta dot\, KD21\, r^2\, F55(qf, qdotf)\, m\, d^2 \sin(\phi)^2$$

$$- 2\, KD21^2\, d^2\, m\, r^2\, \theta dot^2 \sin(\phi)^3 - \theta dot\, F77(qf, qdotf)\, m\, r\, d^3 \sin(\phi)^4\, KD22$$

$$- \theta dot\, F77(qf, qdotf)\, r\, Ib\, KD22\, d \sin(\phi)^2 - 2\, \theta dot\, KD22^2\, r\, \nu\, d \sin(\phi)^2$$

$$+ 4\, KD22^2\, r^2\, \phi dot^2\, m\, d^2 \sin(\phi) - 4\, KD22^2\, r^2\, \phi dot^2\, m\, d^2 \sin(\phi)^3$$

$$- \cos(\phi)\ \theta dot\ F77(qf, qdotf)\ r^2\ Ib\ KD22 - \cos(\phi)\ \theta dot\ F77(qf,$$

$$qdotf)\ m\ r^2\ d^2 \sin(\phi)^2\ KD22 + \theta dot\ F77(qf, qdotf)\ m\ r\ d^3 \sin(\phi)^2\ KD22$$

$$+ 4\cos(\phi)\ KD22^2\ r^3\ \phi dot\ m\ \theta dot\ d \sin(\phi)$$

$$- 2\cos(\phi)\ KD22^2\ r^3\ \phi dot^2\ m\ d \sin(\phi) + KD22\ d^3\ \phi dot\ F44(qf,$$

$$qdotf)\ m\ r \sin(\phi)^2 - KD22\ d^3\ \phi dot\ F44(qf, qdotf)\ m\ r \sin(\phi)^4$$

$$+ 2\ KD21\ r\ \phi dot\ KD22\ d\ \sigma \sin(\phi)^2 + 4\ KD21\ r^2\ \phi dot^2\ KD22\ m\ d^2 \sin(\phi)$$

$$- 4\ KD21\ r^2\ \phi dot^2\ KD22\ m\ d^2 \sin(\phi)^3 - \cos(\phi)\ \theta dot\ F88(qf,$$

$$qdotf)\ m\ r^2\ d^2 \sin(\phi)^2\ KD21 + \cos(\phi)\ KD11\ r^2\ \theta dot\ F55(qf, qdotf)\ Ib$$

$$- \cos(\phi)\ KD21\ r^2\ \phi dot\ F55(qf, qdotf)\ Ib - \cos(\phi)\ KD22^2\ d^2\ \phi dot\ \sigma \sin(\phi)^2$$

$$- 2\cos(\phi)\ KD21^2\ r^3\ m\ \theta dot^2\ d \sin(\phi)$$

$$+ 4\cos(\phi)\ KD22^2\ d^3\ \phi dot\ m\ r\ \theta dot \sin(\phi) + KD21\ d^3\ \theta dot\ F44(qf,$$

$$qdotf)\ m\ r \sin(\phi)^4 + 2\ KD21^2\ d^2\ m\ r^2\ \theta dot^2 \sin(\phi)$$

$$- 8\ KD21\ r^2\ \phi dot\ KD22\ m\ d^2\ \theta dot \sin(\phi)$$

$$+ 8\ KD21\ r^2\ \phi dot\ KD22\ m\ d^2\ \theta dot \sin(\phi)^3 + 8\ KD21\ r^2\ KD22\ m\ d^2\ \theta dot^2 \sin(\phi)$$

$$- 2\ KD22^2\ m\ d^4\ \theta dot^2 \sin(\phi)^3 - 4\cos(\phi)\ KD22^2\ d^3\ \phi dot\ m\ r\ \theta dot \sin(\phi)^3$$

$$-4\cos(\phi)\ KD21\ r^3\ \phi dot^2\ KD22\ m\ d\ \sin(\phi)\ +2\ KD22^2\ m\ d^4\ \theta dot^2\ \sin(\phi)$$

$$+\ \theta dot\ F88(qf, qdotf)\ r\ Ib\ KD22\ d$$

$$+8\cos(\phi)\ KD21\ r^3\ \phi dot\ KD22\ m\ \theta dot\ d\ \sin(\phi)$$

$$+4\cos(\phi)\ KD21^2\ r^3\ \phi dot\ m\ \theta dot\ d\ \sin(\phi)\ -\cos(\phi)\ KD21\ r^2\ \phi dot\ F66(qf,$$

$$qdotf)\ Ib-8\ KD21\ r^2\ KD22\ m\ d^2\ \theta dot^2\ \sin(\phi)^3+\cos(\phi)\ KD11\ r^2\ \theta dot\ F66(qf,$$

$$qdotf)\ Ib+\cos(\phi)\ KD11\ r^2\ \theta dot\ F66(qf, qdotf)\ m\ d^2\ \sin(\phi)^2$$

$$+2\cos(\phi)\ KD21\ d^3\ KD22\ m\ r\ \theta dot^2\ \sin(\phi)^3$$

$$-4\cos(\phi)\ KD21\ d^3\ KD22\ m\ r\ \theta dot^2\ \sin(\phi)$$

$$-4\cos(\phi)\ KD21\ r^3\ KD22\ m\ \theta dot^2\ d\ \sin(\phi)\ -KD21\ d\ \theta dot\ F44(qf, qdotf)\ r\ Ib$$

$$-2\cos(\phi)\ KD22^2\ Ib\ \theta dot^2\ r\ d\ \sin(\phi)\ +\cos(\phi)\ KD22^2\ d^2\ \phi dot\ \sigma$$

$$-2\cos(\phi)\ KD21\ Ib\ \theta dot^2\ r\ d\ \sin(\phi)\ KD22+\cos(\phi)\ KD21^2\ r^2\ \phi dot\ \sigma$$

$$-\cos(\phi)\ \theta dot\ F77(qf, qdotf)\ m\ r^2\ d^2\ \sin(\phi)^2\ KD21-\cos(\phi)\ \theta dot\ F88(qf,$$

$$qdotf)\ r^2\ Ib\ KD21-\ \theta dot\ F88(qf, qdotf)\ m\ r\ d^3\ \sin(\phi)^4\ KD22$$

$$-\cos(\phi)\ \theta dot\ F88(qf, qdotf)\ r^2\ Ib\ KD22-\cos(\phi)\ \theta dot\ F88(qf,$$

$$qdotf)\ m\ r^2\ d^2\ \sin(\phi)^2\ KD22-\cos(\phi)\ KD22\ r^2\ \phi dot\ F55(qf, qdotf)\ m\ d^2\ \sin(\phi)^2$$

$$- \cos(\phi)\ KD22\, r^2\, \phi dot\, F66(qf, qdotf)\, m\, d^2 \sin(\phi)^2$$

$$- 2\, KD22^2\, Ib\, \theta dot^2\, d^2 \sin(\phi)^3 - \cos(\phi)\ \theta dot\, KD21^2\, r^2\, \nu$$

$$+ \cos(\phi)\ \theta dot\, KD22^2\, d^2\, \nu \sin(\phi)^2 - 2\cos(\phi)\ KD22^2\, m\, r^3\, \theta dot^2\, d \sin(\phi)$$

$$- \cos(\phi)\ \theta dot\, KD22^2\, d^2\, \nu + \cos(\phi)\ KD11\, r^2\, \theta dot\, F44(qf, qdotf)\, Ib$$

$$- 6\cos(\phi)\ KD22^2\, m\, d^3\, \theta dot^2\, r \sin(\phi) + 4\cos(\phi)\ KD22^2\, m\, d^3\, \theta dot^2\, r \sin(\phi)^3$$

$$- 2\cos(\phi)\ KD22^2\, d^3\, \phi dot^2\, m\, r \sin(\phi) + 2\cos(\phi)\ KD21\, r^2\, \phi dot\, KD22\, \sigma$$

$$- 2\cos(\phi)\ KD21^2\, r^3\, \phi dot^2\, m\, d \sin(\phi) - 2\cos(\phi)\ \theta dot\, KD21\, r^2\, \nu\, KD22$$

$$+ 2\cos(\phi)\ KD22^2\, d^3\, \phi dot^2\, m\, r \sin(\phi)^3 + \cos(\phi)\ KD21\, r^2\, \theta dot\, F44(qf,$$

$$qdotf)\, Ib + \cos(\phi)\ KD22^2\, r^2\, \phi dot\, \sigma + \cos(\phi)\ KD11\, r^2\, \theta dot\, F44(qf,$$

$$qdotf)\, m\, d^2 \sin(\phi)^2 - KD21\, d^3\, \theta dot\, F44(qf, qdotf)\, m\, r \sin(\phi)^2$$

$$+ KD21\, d\ \theta dot\, F44(qf, qdotf)\, r\, Ib \sin(\phi)^2 + 2\, KD22^2\, r\, \phi dot\, d\, \sigma \sin(\phi)^2$$

$$+ 2\, KD22^2\, Ib\, \theta dot^2\, d^2 \sin(\phi) - 8\, KD22^2\, r^2\, \phi dot\, m\, d^2\, \theta dot \sin(\phi)$$

$$+ 8\, KD22^2\, r^2\, \phi dot\, m\, d^2\, \theta dot \sin(\phi)^3 + 6\, KD22^2\, m\, r^2\, \theta dot^2\, d^2 \sin(\phi)$$

$$- 6\, KD22^2\, m\, r^2\, \theta dot^2\, d^2 \sin(\phi)^3 - \cos(\phi)\ \theta dot\, KD22^2\, r^2\, \nu$$

$$- \cos(\phi)\ KD21\, r^2\, \phi dot\, F55(qf, qdotf)\, m\, d^2 \sin(\phi)^2 - KD21\, d^3\, \theta dot\, F55(qf,$$

$qdotf)\ m\, r \sin(\phi)^2 + KD21\, d^3\ \theta dot\ F55(qf, qdotf)\ m\, r \sin(\phi)^4$

$+ \cos(\phi)\ \theta dot\ KD21\, r^2\, F66(qf, qdotf)\ m\, d^2 \sin(\phi)^2$

$- \cos(\phi)\ KD22\, r^2\ \phi dot\ F66(qf, qdotf)\ Ib + \cos(\phi)\ \theta dot\ KD21\, r^2\, F66(qf,$

$qdotf)\ Ib + KD22\, d\ \phi dot\ F66(qf, qdotf)\ r\, Ib - KD22\, d\ \phi dot\ F66(qf,$

$qdotf)\ r\, Ib \sin(\phi)^2 + KD22\, d^3\ \phi dot\ F66(qf, qdotf)\ m\, r \sin(\phi)^2$

$- KD22\, d^3\ \phi dot\ F66(qf, qdotf)\ m\, r \sin(\phi)^4 + KD21\, d\ \theta dot\ F66(qf,$

$qdotf)\ r\, Ib \sin(\phi)^2 - KD21\, d^3\ \theta dot\ F66(qf, qdotf)\ m\, r \sin(\phi)^2$

$- \cos(\phi)\ KD21\, r^2\ \phi dot\ F66(qf, qdotf)\ m\, d^2 \sin(\phi)^2 + KD21\, d^3\ \theta dot\ F66(qf,$

$qdotf)\ m\, r \sin(\phi)^4 - \theta dot\ F88(qf, qdotf)\ r\, Ib\, KD22\, d \sin(\phi)^2 + \theta dot\ F88(qf,$

$qdotf)\ m\, r\, d^3 \sin(\phi)^2\, KD22 - \cos(\phi)\ KD22\, r^2\ \phi dot\ F44(qf, qdotf)\ Ib$

$+ KD22\, d\ \phi dot\ F44(qf, qdotf)\ r\, Ib - KD21\, d\ \theta dot\ F55(qf, qdotf)\ r\, Ib$

$+ KD22\, d\ \phi dot\ F55(qf, qdotf)\ r\, Ib - 2\, KD22^2\, r\ \phi dot\ d\, \sigma$

$- \cos(\phi)\ KD21\, r^2\ \phi dot\ F44(qf, qdotf)\ m\, d^2 \sin(\phi)^2$

$+ \cos(\phi)\ KD21\, r^2\ \theta dot\ F44(qf, qdotf)\ m\, d^2 \sin(\phi)^2$

$- \cos(\phi)\ KD22\, r^2\ \phi dot\ F44(qf, qdotf)\ m\, d^2 \sin(\phi)^2$

$-\cos(\phi)\ KD21\, r^2\ \phi dot\ F44(qf, qdotf)\ Ib - KD22\ d\ \phi dot\ F44(qf,$

$qdotf)\ r\ Ib \sin(\phi)^2 + KD21\ d\ \theta dot\ F55(qf, qdotf)\ r\ Ib \sin(\phi)^2$

$-\cos(\phi)\ KD22\, r^2\ \phi dot\ F55(qf, qdotf)\ Ib + \cos(\phi)\ \theta dot\ KD21\, r^2\ F55(qf,$

$qdotf)\ Ib - KD22\ d^3\ \phi dot\ F55(qf, qdotf)\ m\ r \sin(\phi)^4 + KD22\ d^3\ \phi dot\ F55(qf,$

$qdotf)\ m\ r \sin(\phi)^2 + 2\ \theta dot\ KD21\ r\ \nu\ KD22\ d - KD22\ d\ \phi dot\ F55(qf,$

$qdotf)\ r\ Ib \sin(\phi)^2 - 2\ \theta dot\ KD21\ r\ \nu\ KD22\ d \sin(\phi)^2 - \cos(\phi)\ \theta dot\ F77(qf,$

$qdotf)\ r^2\ Ib\ KD21)\Big) \Big/ \Big(\big(-1 + \sin(\phi)^2\big)\ \theta dot^2\ \big(-KD21^2\ Ib + KD11\ KD22\ Ib$

$+ d^2\ m\ KD22\ KD11 \sin(\phi)^2 - KD21^2\ d^2\ m \sin(\phi)^2\big)\Big), F33(qf, qdotf) =$

$-\frac{1}{r\left(Ib + m\, d^2 \sin(\phi)^2\right)\ \theta dot^2}\Big(2 \sin(\phi)\ d\ r^2\ \theta dot\ m\ \phi dot^2\ KD11$

$-2\cos(\phi)\ \theta dot^3\ m\ KD11\ r\ d^2 \sin(\phi) + 2\ d\ KD21 \sin(\phi)\ \theta dot^3\ Ib$

$+ \sin(\phi)^2\ d^2\ r\ F55(qf, qdotf)\ \phi dot^2\ m + \sin(\phi)^2\ d^2\ r\ F77(qf,$

$qdotf)\ \theta dot\ \phi dot\ m + 2\ \theta dot^3\ m\ r^2\ d\ KD21 \sin(\phi)$

$-2\ \theta dot\ m\ \phi dot^2\ r^2\ d\ KD21 \sin(\phi) + 2\ \theta dot^2\ m\ \phi dot\ r^2\ d \sin(\phi)\ KD22$

$+2\ \theta dot^3\ m\ KD21\ d^3 \sin(\phi) + \sin(\phi)^2\ d^2\ r\ F88(qf, qdotf)\ \theta dot\ \phi dot\ m$

$-4 \sin(\phi)\ d\ r^2\ \theta dot^2\ m\ \phi dot\ KD11 + \cos(\phi)\ \phi dot^2\ \sigma\ KD22\ d$

$-4\cos(\phi)\ \theta dot^3\ m\ r\ d^2\ KD21 \sin(\phi) - \cos(\phi)\ \phi dot\ \theta dot\ \nu\ KD22\ d$

$-\cos(\phi)\ \theta dot^2\ \nu\ d\ KD21 + \cos(\phi)\ \phi dot\ \sigma\ \theta dot\ d\ KD21$

$+4\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ d^2\ KD22 \sin(\phi) + \sin(\phi)^2\ d^2\ r\ m\ F55(qf,$

$qdotf)\ \theta dot^2 + 2\ \theta dot^3\ m\ KD11 \sin(\phi)\ d\ r^2 + 2\ m\ \phi dot^3\ r^2\ d \sin(\phi)\ KD22$

$+2\ m\ \phi dot^3\ r^2\ d\ KD21 \sin(\phi) + 2\ \theta dot^2\ m\ \phi dot\ KD22\ d^3 \sin(\phi)$

$-4\ \theta dot\ m\ \phi dot^2\ r^2\ d \sin(\phi)\ KD22 + \sin(\phi)^2\ d^2\ r\ F44(qf, qdotf)\ \phi dot^2\ m$

$$+\sin(\phi)^2 d^2 r\, m\, F66(qf, qdotf)\ \theta dot^2$$
$$-2\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ d^2\ KD21 \sin(\phi) + \sin(\phi)^2 d^2 r\, F66(qf,$$
$$qdotf)\ \phi dot^2\ m - 4\cos(\phi)\ \theta dot^2\ m\ \phi dot\ r\ d^2\ KD22 \sin(\phi)$$
$$+2\cos(\phi)\ \theta dot^2\ m\ \phi dot\ r\ d^2\ KD21 \sin(\phi) + 2\, d \sin(\phi)\ KD22\ \theta dot^2\ \phi dot\ Ib$$
$$-2\cos(\phi)\ m\ \phi dot^3\ r\ d^2\ KD22 \sin(\phi) - 2\ \theta dot^2\ m\ \phi dot\ r^2\ d\ KD21 \sin(\phi)$$
$$+\theta dot^2\ \nu\ KD21\ r - \phi dot^2\ \sigma\ KD21\ r + \phi dot\ \theta dot\ \nu\ KD21\ r - r\ \theta dot\ \sigma\ \phi dot\ KD11$$
$$+r\ Ib\ F66(qf, qdotf)\ \theta dot^2 + r\ F66(qf, qdotf)\ \phi dot^2\ Ib - \phi dot\ \sigma\ \theta dot\ KD21\ r$$
$$+r\ F88(qf, qdotf)\ \theta dot\ \phi dot\ Ib + r\ F77(qf, qdotf)\ \theta dot\ \phi dot\ Ib$$
$$-\phi dot^2\ \sigma\ KD22\ r + r\ F55(qf, qdotf)\ \phi dot^2\ Ib + r\ Ib\ F55(qf, qdotf)\ \theta dot^2$$
$$+r\ F44(qf, qdotf)\ \phi dot^2\ Ib + \phi dot\ \theta dot\ \nu\ KD22\ r + r\ \nu\ KD11\ \theta dot^2\big)\big]\big]$$

$forsol_1$ **are the solutions to Eq. 2.3**

Testing the forces on the First matching condition equation the result is going to be called FMCtest

> FMCtest := simplify(*eval*(*FMCsim*, [*forsol*$_{1,1}$, *forsol*$_{1,2}$, *forsol*$_{1,3}$]));

$$FMCtest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \textbf{(2.5)}$$

FMC is satisfied!

> *F1* := *simplify*(*eval*(*F1*, [*forsol*$_{1,1}$, *forsol*$_{1,2}$, *forsol*$_{1,3}$]));

$$F1 := \begin{bmatrix} \theta dot\ \nu - \phi dot\ \sigma \\ 0 \end{bmatrix} \qquad \textbf{(2.6)}$$

> *Fc1* := *simplify*(*eval*(*Fmc1*, [*forsol*$_{1,1}$, *forsol*$_{1,2}$, *forsol*$_{1,3}$]));

$$Fc1 := \Big[\Big[\qquad \textbf{(2.7)}$$
$$-\frac{1}{\left(m\, d^2 + Ib - m\, d^2 \cos(\phi)^2\right)\ r\ \theta dot^2}\Big(2\ \big(2\sin(\phi)\ d\ r^2\ \theta dot\ m\ \phi dot^2\ KD11$$
$$-2\cos(\phi)\ \theta dot^3\ m\ KD11\ r\ d^2 \sin(\phi) + 2\, d\ KD21 \sin(\phi)\ \theta dot^3\ Ib$$
$$+2\ \theta dot^3\ m\ r^2\ d\ KD21 \sin(\phi) - 2\ \theta dot\ m\ \phi dot^2\ r^2\ d\ KD21 \sin(\phi)$$
$$+2\ \theta dot^2\ m\ \phi dot\ r^2\ d \sin(\phi)\ KD22 + 2\ \theta dot^3\ m\ KD21\ d^3 \sin(\phi)$$
$$-4\sin(\phi)\ d\ r^2\ \theta dot^2\ m\ \phi dot\ KD11 + \cos(\phi)\ \phi dot^2\ \sigma\ KD22\ d$$
$$-4\cos(\phi)\ \theta dot^3\ m\ r\ d^2\ KD21 \sin(\phi) - \cos(\phi)\ \phi dot\ \theta dot\ \nu\ KD22\ d$$

$$
\begin{aligned}
&-\cos(\phi)\ \theta dot^2\ \nu\ d\ KD21+\cos(\phi)\ \phi dot\ \sigma\ \theta dot\ d\ KD21\\
&+4\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ d^2\ KD22\sin(\phi)+2\ \theta dot^3\ m\ KD11\sin(\phi)\ d\ r^2\\
&+2\ m\ \phi dot^3\ r^2\ d\sin(\phi)\ KD22+2\ m\ \phi dot^3\ r^2\ d\ KD21\sin(\phi)\\
&+2\ \theta dot^2\ m\ \phi dot\ KD22\ d^3\sin(\phi)-4\ \theta dot\ m\ \phi dot^2\ r^2\ d\sin(\phi)\ KD22\\
&-2\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ d^2\ KD21\sin(\phi)\\
&-4\cos(\phi)\ \theta dot^2\ m\ \phi dot\ r\ d^2\ KD22\sin(\phi)\\
&+2\cos(\phi)\ \theta dot^2\ m\ \phi dot\ r\ d^2\ KD21\sin(\phi)+2\ d\sin(\phi)\ KD22\ \theta dot^2\ \phi dot\ Ib\\
&-2\cos(\phi)\ m\ \phi dot^3\ r\ d^2\ KD22\sin(\phi)-2\ \theta dot^2\ m\ \phi dot\ r^2\ d\ KD21\sin(\phi)\\
&+\theta dot^2\ \nu\ KD21\ r-\phi dot^2\ \sigma\ KD21\ r+\phi dot\ \theta dot\ \nu\ KD21\ r-r\ \theta dot\ \sigma\ \phi dot\ KD11\\
&-d^2\ r\ F88(qf, qdotf)\ \theta dot\ \phi dot\ m\cos(\phi)^2+r\ F66(qf, qdotf)\ \phi dot^2\ Ib\\
&-\phi dot\ \sigma\ \theta dot\ KD21\ r+r\ F88(qf, qdotf)\ \theta dot\ \phi dot\ Ib+r\ F77(qf,\\
&qdotf)\ \theta dot\ \phi dot\ Ib-\phi dot^2\ \sigma\ KD22\ r+r\ F55(qf, qdotf)\ \phi dot^2\ Ib+r\ F44(qf,\\
&qdotf)\ \phi dot^2\ Ib+\phi dot\ \theta dot\ \nu\ KD22\ r+r\ \nu\ KD11\ \theta dot^2+d^2\ r\ F44(qf,\\
&qdotf)\ \phi dot^2\ m-d^2\ r\ F44(qf, qdotf)\ \phi dot^2\ m\cos(\phi)^2+d^2\ r\ F66(qf,\\
&qdotf)\ \phi dot^2\ m-d^2\ r\ F66(qf, qdotf)\ \phi dot^2\ m\cos(\phi)^2+d^2\ r\ F55(qf,\\
&qdotf)\ \phi dot^2\ m-d^2\ r\ F55(qf, qdotf)\ \phi dot^2\ m\cos(\phi)^2+d^2\ r\ F77(qf,\\
&qdotf)\ \theta dot\ \phi dot\ m-d^2\ r\ F77(qf, qdotf)\ \theta dot\ \phi dot\ m\cos(\phi)^2+d^2\ r\ F88(qf,\\
&qdotf)\ \theta dot\ \phi dot\ m)\Big),\ F77(qf, qdotf)+F88(qf, qdotf)\Big],\\
&[F77(qf, qdotf)+F88(qf, qdotf),\ 2\ F44(qf, qdotf)+2\ F66(qf, qdotf)\\
&+2\ F55(qf, qdotf)]]
\end{aligned}
$$

> $F1 := simplify(\mathrm{eval}(F1_1));$

$$F1 := \theta dot\ \nu-\phi dot\ \sigma \tag{2.8}$$

tempfT1 is calculated* in *order* to *avoid singularities on Fmc1

> $$
\begin{aligned}
&tempfT1 := solve\big(-\phi dot^2\ \sigma\ KD21\ r+\phi dot\ \theta dot\ \nu\ KD22\ r+\phi dot\ \theta dot\ \nu\ KD21\ r\\
&\quad-r\ \theta dot\ \sigma\ \phi dot\ KD11-2\cos(\phi)\ m\ \phi dot^3\ r\ KD22\ d^2\sin(\phi)\\
&\quad-2\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ d^2\ KD21\sin(\phi)+2\ m\ \phi dot^3\ r^2\ KD22\ d\sin(\phi)\\
&\quad-2\ \theta dot\ m\ \phi dot^2\ KD21\ d\sin(\phi)\ r^2-4\ \theta dot\ m\ \phi dot^2\ r^2\ KD22\ d\sin(\phi)\\
&\quad+2\sin(\phi)\ d\ r^2\ \theta dot\ m\ \phi dot^2\ KD11+2\ m\ \phi dot^3\ KD21\ d\sin(\phi)\ r^2\\
&\quad-d^2\ r\ F88(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ m\cos(\phi)^2+d^2\ r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m\\
&\quad-d^2\ r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m\cos(\phi)^2+d^2\ r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m\\
&\quad+r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ Ib-\phi dot\ \sigma\ \theta dot\ KD21\ r
\end{aligned}
$$

$$+\cos(\phi)\ \phi dot\ \sigma\ \theta dot\ KD21\ d+\cos(\phi)\ \phi dot^2\ \sigma\ KD22\ d + 4\cos(\phi)\ \theta dot\ m\ \phi dot^2\ r\ KD22\ d^2 \sin(\phi) - d^2\ r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m\cos(\phi)^2 + d^2\ r\ F77(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ m - d^2\ r\ F77(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ m\cos(\phi)^2 + d^2\ r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m - d^2\ r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ m\cos(\phi)^2 + d^2\ r\ F88(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ m + r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ Ib + r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot^2\ Ib + r\ F88(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ Ib + r\ F77(\mathrm{qf}, \mathrm{qdotf})\ \theta dot\ \phi dot\ Ib - \phi dot^2\ \sigma\ KD22\ r - \cos(\phi)\ \phi dot\ \theta dot\ \nu\ KD22\ d, F77(\mathrm{qf}, \mathrm{qdotf})\Big)\ :$$

Fmc1 is the Lyapunov contribution from FMC. $F77(\mathrm{qf}, \mathrm{qdotf})$ **and** $F44(\mathrm{qf}, \mathrm{qdotf})$ **are the parameters to reach Fmc1 at least positive semi-definite**

$> \ Fc1 := simplify(\mathrm{eval}(Fc1, F77(\mathrm{qf}, \mathrm{qdotf}) = tempfT1))\ :$

tempfT2 is calculated* in *order* to *avoid singularities on Fmc1

$$>\ tempfT2 := solve\Big(-\phi dot\ \sigma\ KD21\ r - \phi dot\ \sigma\ KD22\ r + d^2\ r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m + d^2\ r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m + d^2\ r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m + \cos(\phi)\ \phi dot\ \sigma\ KD22\ d + r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ Ib + r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ Ib + r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ Ib - 2\cos(\phi)\ m\ \phi dot^2\ r\ KD22\ d^2 \sin(\phi) - d^2\ r\ F44(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m\cos(\phi)^2 - d^2\ r\ F66(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m\cos(\phi)^2 + 2\ m\ \phi dot^2\ r^2\ KD22\ d\sin(\phi) + 2\ m\ \phi dot^2\ KD21\sin(\phi)\ d\ r^2 - d^2\ r\ F55(\mathrm{qf}, \mathrm{qdotf})\ \phi dot\ m\cos(\phi)^2, F44(\mathrm{qf}, \mathrm{qdotf})\Big)\ :$$

$> \ Fc1 := simplify(\mathrm{eval}(Fc1, F44(\mathrm{qf}, \mathrm{qdotf}) = tempfT2));$

$$Fc1 := \Bigg[\Bigg[-\frac{1}{r\left(m\ d^2 + Ib - m\ d^2\cos(\phi)^2\right)}\Big(2\ \big(2\ Ib\ d\sin(\phi)\ KD22\ \phi dot + 2\ Ib\ d\ KD21\sin(\phi)\ \theta dot - 2\ m\ \phi dot\ r^2\ d\ KD21\sin(\phi) + 2\cos(\phi)\ m\ \phi dot\ r\ d^2\ KD21\sin(\phi) + \nu\ KD21\ r - 4\sin(\phi)\ d\ r^2\ m\ \phi dot\ KD11 + 2\ m\ \phi dot\ KD22\ d^3\sin(\phi) + 2\ \theta dot\ m\ r^2\ d\ KD21\sin(\phi) - \cos(\phi)\ \nu\ d\ KD21 - 4\cos(\phi)\ \theta dot\ m\ r\ d^2\ KD21\sin(\phi) + r\ \nu\ KD11 + 2\ \theta dot\ m\ KD11\sin(\phi)\ d\ r^2 + 2\ d^3\ KD21\sin(\phi)\ \theta dot\ m - 2\cos(\phi)\ \theta dot\ m\ KD11\ r\ d^2\sin(\phi)$$

(2.9)

$$-4\cos(\phi)\, m\, \phi dot\, r\, d^2\, KD22 \sin(\phi) + 2\, m\, \phi dot\, r^2\, d \sin(\phi)\, KD22)),$$

$$-\frac{1}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\,(-2\cos(\phi)\, m\, \phi dot\, r\, d^2\, KD21 \sin(\phi)$$
$$+\nu\, KD21\, r - r\,\sigma\, KD11 - \sigma\, KD21\, r + \nu\, KD22\, r - \cos(\phi)\,\nu\, KD22\, d$$
$$-2\, m\, \phi dot\, r^2\, d\, KD21 \sin(\phi) + 2\sin(\phi)\, d\, r^2\, m\, \phi dot\, KD11 + \cos(\phi)\,\sigma\, d\, KD21$$
$$+4\cos(\phi)\, m\, \phi dot\, r\, d^2\, KD22 \sin(\phi) - 4\, m\, \phi dot\, r^2\, d \sin(\phi)\, KD22)],$$

$$\left[-\frac{1}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\,(-2\cos(\phi)\, m\, \phi dot\, r\, d^2\, KD21 \sin(\phi)\right.$$
$$+\nu\, KD21\, r - r\,\sigma\, KD11 - \sigma\, KD21\, r + \nu\, KD22\, r - \cos(\phi)\,\nu\, KD22\, d$$
$$-2\, m\, \phi dot\, r^2\, d\, KD21 \sin(\phi) + 2\sin(\phi)\, d\, r^2\, m\, \phi dot\, KD11 + \cos(\phi)\,\sigma\, d\, KD21$$
$$+4\cos(\phi)\, m\, \phi dot\, r\, d^2\, KD22 \sin(\phi) - 4\, m\, \phi dot\, r^2\, d \sin(\phi)\, KD22),$$

$$\left.-\frac{2\,(KD22\, r - \cos(\phi)\, KD22\, d + KD21\, r)\,(2\,\phi dot\, m\, r\, d \sin(\phi) - \sigma)}{r\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)}\right]\Bigg]$$

> $tempfT3 := simplify(solve(tempfT2 - tempfT1, F55(\mathrm{qf}, \mathrm{qdotf})));$

$$tempfT3 := -\big(r\,F44(qf, qdotf)\,\phi dot\, Ib + r\,F66(qf, qdotf)\,\phi dot\, Ib + r\,F88(qf, qdotf)\,\theta dot\, Ib - \phi dot\,\sigma\, KD22\, r - \phi dot\,\sigma\, KD21\, r + \theta dot\,\sigma\, KD22\, r \tag{2.10}$$
$$+\theta dot\,\nu\, KD21\, r - r\,\theta dot\,\sigma\, KD11 + \theta dot\,\nu\, KD22\, r - \theta dot\, r\, F66(qf, qdotf)\, Ib$$
$$-\theta dot\, d^2\, r\, F66(qf, qdotf)\, m - \theta dot\cos(\phi)\,\sigma\, KD22\, d + \theta dot\, d^2\, r\, F66(qf, qdotf)\, m\cos(\phi)^2 - 2\cos(\phi)\, m\, \phi dot^2\, r\, d^2\, KD22 \sin(\phi)$$
$$-4\,\theta dot\, m\, \phi dot\, r^2\, d\, KD21 \sin(\phi) + 2\, m\, \phi dot^2\, r^2\, d \sin(\phi)\, KD22 - d^2\, r\, F88(qf, qdotf)\,\theta dot\, m\cos(\phi)^2 + 2\, m\, \phi dot^2\, r^2\, d\, KD21 \sin(\phi) - \cos(\phi)\,\theta dot\,\nu\, KD22\, d$$
$$+2\sin(\phi)\, d\, r^2\,\theta dot\, m\, \phi dot\, KD11 + \cos(\phi)\,\sigma\,\theta dot\, d\, KD21$$
$$+6\cos(\phi)\,\theta dot\, m\, \phi dot\, r\, d^2\, KD22 \sin(\phi) + \cos(\phi)\,\phi dot\,\sigma\, KD22\, d$$
$$+d^2\, r\, F44(qf, qdotf)\,\phi dot\, m + d^2\, r\, F66(qf, qdotf)\,\phi dot\, m - d^2\, r\, F66(qf, qdotf)\,\phi dot\, m\cos(\phi)^2 - 6\,\theta dot\, m\, \phi dot\, r^2\, d \sin(\phi)\, KD22$$
$$-2\cos(\phi)\,\theta dot\, m\, \phi dot\, r\, d^2\, KD21 \sin(\phi) - d^2\, r\, F44(qf, qdotf)\,\phi dot\, m\cos(\phi)^2$$
$$+d^2\, r\, F88(qf, qdotf)\,\theta dot\, m\big)\Big/\big(r\,(d^2\,\phi dot\, m - d^2\,\phi dot\, m\cos(\phi)^2 - \theta dot\, Ib$$

$+ \phi dot\, Ib + \theta dot\, d^2\, m \cos(\phi)^2 - \theta dot\, d^2\, m))$

> $FMCforsol := simplify(\mathrm{eval}(forsol, [F55(\mathrm{qf}, \mathrm{qdotf}) = tempfT3])):$

> $\mathrm{FMCtest1} := \mathrm{simplify}(eval(FMCsim, FMCforsol_1));$

$$FMCtest1 := [[(2\,(r\,F44(qf, qdotf)\,\phi dot\,Ib + r\,F66(qf, qdotf)\,\phi dot\,Ib + r\,F55(qf, qdotf)\,\phi dot\,Ib + r\,F88(qf, qdotf)\,\theta dot\,Ib - \phi dot\,\sigma\,KD22\,r - \phi dot\,\sigma\,KD21\,r + \theta dot\,\sigma\,KD22\,r + \theta dot\,\nu\,KD21\,r - r\,\theta dot\,\sigma\,KD11 + \theta dot\,\nu\,KD22\,r - F55(qf, qdotf)\,m\,r\,d^2\,\theta dot - \theta dot\,r\,F66(qf, qdotf)\,Ib + F55(qf, qdotf)\,m\,r\,d^2\cos(\phi)^2\,\theta dot - Ib\,F55(qf, qdotf)\,r\,\theta dot - \theta dot\,d^2\,r\,F66(qf, qdotf)\,m - \theta dot\cos(\phi)\,\sigma\,KD22\,d + \theta dot\,d^2\,r\,F66(qf, qdotf)\,m\cos(\phi)^2 - 2\cos(\phi)\,m\,\phi dot^2\,r\,d^2\,KD22\sin(\phi) - 4\,\theta dot\,m\,\phi dot\,r^2\,d\,KD21\sin(\phi) + 2\,m\,\phi dot^2\,r^2\,d\sin(\phi)\,KD22 - d^2\,r\,F88(qf, qdotf)\,\theta dot\,m\cos(\phi)^2 + 2\,m\,\phi dot^2\,r^2\,d\,KD21\sin(\phi) - \cos(\phi)\,\theta dot\,\nu\,KD22\,d + 2\sin(\phi)\,d\,r^2\,\theta dot\,m\,\phi dot\,KD11 + \cos(\phi)\,\sigma\,\theta dot\,d\,KD21 + 6\cos(\phi)\,\theta dot\,m\,\phi dot\,r\,d^2\,KD22\sin(\phi) + \cos(\phi)\,\phi dot\,\sigma\,KD22\,d + d^2\,r\,F44(qf, qdotf)\,\phi dot\,m + d^2\,r\,F66(qf, qdotf)\,\phi dot\,m + d^2\,r\,F55(qf, qdotf)\,\phi dot\,m - d^2\,r\,F66(qf, qdotf)\,\phi dot\,m\cos(\phi)^2 - 6\,\theta dot\,m\,\phi dot\,r^2\,d\sin(\phi)\,KD22 - 2\cos(\phi)\,\theta dot\,m\,\phi dot\,r\,d^2\,KD21\sin(\phi) - d^2\,r\,F55(qf, qdotf)\,\phi dot\,m\cos(\phi)^2 - d^2\,r\,F44(qf, qdotf)\,\phi dot\,m\cos(\phi)^2 + d^2\,r\,F88(qf, qdotf)\,\theta dot\,m))\Big/(r\,(d^2\,\phi dot\,m - d^2\,\phi dot\,m\cos(\phi)^2 - \theta dot\,Ib + \phi dot\,Ib + \theta dot\,d^2\,m\cos(\phi)^2 - \theta dot\,d^2\,m)), 0], \tag{2.11}$$

$$[0, (2\,(r\,F44(qf, qdotf)\,\phi dot\,Ib + r\,F66(qf, qdotf)\,\phi dot\,Ib + r\,F55(qf, qdotf)\,\phi dot\,Ib + r\,F88(qf, qdotf)\,\theta dot\,Ib - \phi dot\,\sigma\,KD22\,r - \phi dot\,\sigma\,KD21\,r + \theta dot\,\sigma\,KD22\,r + \theta dot\,\nu\,KD21\,r - r\,\theta dot\,\sigma\,KD11 + \theta dot\,\nu\,KD22\,r - F55(qf, qdotf)\,m\,r\,d^2\,\theta dot - \theta dot\,r\,F66(qf, qdotf)\,Ib + F55(qf, qdotf)\,m\,r\,d^2\cos(\phi)^2\,\theta dot - Ib\,F55(qf, qdotf)\,r\,\theta dot - \theta dot\,d^2\,r\,F66(qf, qdotf)\,m - \theta dot\cos(\phi)\,\sigma\,KD22\,d + \theta dot\,d^2\,r\,F66(qf, qdotf)\,m\cos(\phi)^2 - 2\cos(\phi)\,m\,\phi dot^2\,r\,d^2\,KD22\sin(\phi) - 4\,\theta dot\,m\,\phi dot\,r^2\,d\,KD21\sin(\phi) + 2\,m\,\phi dot^2\,r^2\,d\sin(\phi)\,KD22 - d^2\,r\,F88(qf, qdotf)\,\theta dot\,m\cos(\phi)^2 + 2\,m\,\phi dot^2\,r^2\,d\,KD21\sin(\phi) - \cos(\phi)\,\theta dot\,\nu\,KD22\,d$$

$+ 2 \sin(\phi)\ d\ r^2\ \theta dot\ m\ \phi dot\ KD11 + \cos(\phi)\ \sigma\ \theta dot\ d\ KD21$
$+ 6 \cos(\phi)\ \theta dot\ m\ \phi dot\ r\ d^2\ KD22 \sin(\phi) + \cos(\phi)\ \phi dot\ \sigma\ KD22\ d$
$+ d^2\ r\ F44(qf, qdotf)\ \phi dot\ m + d^2\ r\ F66(qf, qdotf)\ \phi dot\ m + d^2\ r\ F55(qf,$
$qdotf)\ \phi dot\ m - d^2\ r\ F66(qf, qdotf)\ \phi dot\ m \cos(\phi)^2$
$- 6\ \theta dot\ m\ \phi dot\ r^2\ d \sin(\phi)\ KD22 - 2 \cos(\phi)\ \theta dot\ m\ \phi dot\ r\ d^2\ KD21 \sin(\phi)$
$- d^2\ r\ F55(qf, qdotf)\ \phi dot\ m \cos(\phi)^2 - d^2\ r\ F44(qf, qdotf)\ \phi dot\ m \cos(\phi)^2$
$+ d^2\ r\ F88(qf, qdotf)\ \theta dot\ m)) \Big/ \left(r \left(d^2\ \phi dot\ m - d^2\ \phi dot\ m \cos(\phi)^2 - \theta dot\ Ib\right.\right.$
$\left.\left.+ \phi dot\ Ib + \theta dot\ d^2\ m \cos(\phi)^2 - \theta dot\ d^2\ m\right)\right)]]$

> Forsol88 := solve(FMCtest1$_{1,1}$, $F88(qf, qdotf)$);

$$Forsol88 := -\frac{1}{r\ \theta dot\ \left(m\ d^2 + Ib - m\ d^2 \cos(\phi)^2\right)} \left(r\ F44(qf, qdotf)\ \phi dot\ Ib\right. \quad \textbf{(2.12)}$$
$+ r\ F66(qf, qdotf)\ \phi dot\ Ib + r\ F55(qf, qdotf)\ \phi dot\ Ib - \phi dot\ \sigma\ KD22\ r$
$- \phi dot\ \sigma\ KD21\ r + \theta dot\ \sigma\ KD22\ r + \theta dot\ \nu\ KD21\ r - r\ \theta dot\ \sigma\ KD11$
$+ \theta dot\ \nu\ KD22\ r - F55(qf, qdotf)\ m\ r\ d^2\ \theta dot - \theta dot\ r\ F66(qf, qdotf)\ Ib$
$+ F55(qf, qdotf)\ m\ r\ d^2 \cos(\phi)^2\ \theta dot - Ib\ F55(qf, qdotf)\ r\ \theta dot$
$- \theta dot\ d^2\ r\ F66(qf, qdotf)\ m - \theta dot \cos(\phi)\ \sigma\ KD22\ d + \theta dot\ d^2\ r\ F66(qf,$
$qdotf)\ m \cos(\phi)^2 - 2 \cos(\phi)\ m\ \phi dot^2\ r\ d^2\ KD22 \sin(\phi)$
$- 4\ \theta dot\ m\ \phi dot\ r^2\ d\ KD21 \sin(\phi) + 2\ m\ \phi dot^2\ r^2\ d \sin(\phi)\ KD22$
$+ 2\ m\ \phi dot^2\ r^2\ d\ KD21 \sin(\phi) - \cos(\phi)\ \theta dot\ \nu\ KD22\ d$
$+ 2 \sin(\phi)\ d\ r^2\ \theta dot\ m\ \phi dot\ KD11 + \cos(\phi)\ \sigma\ \theta dot\ d\ KD21$
$+ 6 \cos(\phi)\ \theta dot\ m\ \phi dot\ r\ d^2\ KD22 \sin(\phi) + \cos(\phi)\ \phi dot\ \sigma\ KD22\ d$
$+ d^2\ r\ F44(qf, qdotf)\ \phi dot\ m + d^2\ r\ F66(qf, qdotf)\ \phi dot\ m + d^2\ r\ F55(qf,$
$qdotf)\ \phi dot\ m - d^2\ r\ F66(qf, qdotf)\ \phi dot\ m \cos(\phi)^2$
$- 6\ \theta dot\ m\ \phi dot\ r^2\ d \sin(\phi)\ KD22 - 2 \cos(\phi)\ \theta dot\ m\ \phi dot\ r\ d^2\ KD21 \sin(\phi)$
$\left.- d^2\ r\ F55(qf, qdotf)\ \phi dot\ m \cos(\phi)^2 - d^2\ r\ F44(qf, qdotf)\ \phi dot\ m \cos(\phi)^2\right)$

> FMCtest1 := simplify(*eval*(FMCtest1, $F88(qf, qdotf)$ = Forsol88));

$$FMCtest1 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(2.13)}$$

***FMC is satisfied*!**

> $KDT := simplify(\mathrm{eval}(KDT, [KD11(\theta, \phi) = KD11, KD21(\theta, \phi) = KD21, KD22(\theta, \phi) = KD22]))$;

$$KDT := \begin{bmatrix} KD11 & KD21 \\ KD21 & KD22 \end{bmatrix} \tag{2.14}$$

> Detti := Determinant(KDT);

$$Detti := KD11\,KD22 - KD21^2 \tag{2.15}$$

> Eigenvalues(KDT)

$$\begin{bmatrix} \frac{1}{2}\,KD22 + \frac{1}{2}\,KD11 + \frac{1}{2}\sqrt{KD22^2 - 2\,KD11\,KD22 + KD11^2 + 4\,KD21^2} \\ \frac{1}{2}\,KD22 + \frac{1}{2}\,KD11 - \frac{1}{2}\sqrt{KD22^2 - 2\,KD11\,KD22 + KD11^2 + 4\,KD21^2} \end{bmatrix} \tag{2.16}$$

> P := simplify(Multiply(KDT, IMass));

$$P := \left[\left[\frac{KD11\,r + KD21\,r - KD21\cos(\phi)\,d}{r\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)}, \frac{1}{m\,r^2\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)}\left(KD11\,m\,r^2 - KD11\,m\,r\cos(\phi)\,d + KD21\,m\,r^2 - 2\,KD21\,m\,r\cos(\phi)\,d + KD21\,m\,d^2 + KD21\,Ib\right)\right], \left[\frac{KD22\,r - \cos(\phi)\,KD22\,d + KD21\,r}{r\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)}, \frac{1}{m\,r^2\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)}\left(KD21\,m\,r^2 - KD21\,m\,r\cos(\phi)\,d + KD22\,m\,r^2 - 2\,KD22\,m\,r\,d\cos(\phi) + KD22\,m\,d^2 + KD22\,Ib\right)\right]\right] \tag{2.17}$$

> Determinant(P);

$$\frac{KD11\,KD22 - KD21^2}{r^2\,m\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)} \tag{2.18}$$

Second Matching Condition

> Kv := α·Multiply(Column(P, [1]), Transpose(Column(P, [1])));

$$Kv := \left[\left[\frac{\alpha\left(KD11\,r + KD21\,r - KD21\cos(\phi)\,d\right)^2}{r^2\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)^2}, \right.\right. \tag{3.1}$$

$$\frac{1}{r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)^2}\left(\alpha\left(KD11\,r+KD21\,r-KD21\cos(\phi)\,d\right)\left(KD22\,r-\cos(\phi)\,KD22\,d+KD21\,r\right)\right)\Bigg],$$

$$\left[\frac{1}{r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)^2}\left(\alpha\left(KD11\,r+KD21\,r-KD21\cos(\phi)\,d\right)\left(KD22\,r-\cos(\phi)\,KD22\,d+KD21\,r\right)\right),\right.$$

$$\left.\frac{\alpha\left(KD22\,r-\cos(\phi)\,KD22\,d+KD21\,r\right)^2}{r^2\left(m\,d^2+Ib-m\,d^2\cos(\phi)^2\right)^2}\right]\Bigg]$$

> *convert*(*Kv*, *string*);

"Matrix(2, 2, [[alpha*(KD11*r+KD21*r-KD21*cos(phi)*d)^2/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2,alpha*(KD11*r+KD21*r-KD21*cos(phi)*d)/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2*(KD22*r-cos(phi)*KD22*d+KD21*r)],[alpha*(KD11*r+KD21*r-KD21*cos(phi)*d)/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2*(KD22*r-cos(phi)*KD22*d+KD21*r),alpha*(KD22*r-cos(phi)*KD22*d+KD21*r)^2/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2]])" **(3.2)**

> $F2f := Multiply(MatrixInverse(P), Kv)$:

> *simplify*(*Eigenvalues*(*Kv*));

$$\left[\left[0\right],\left[\left(\alpha\left(2\,KD21^2\,r^2+2\,KD21\,r^2\,KD22-2\,KD21\,r\,KD22\cos(\phi)\,d+KD22^2\,r^2-2\,KD22^2\,r\cos(\phi)\,d+KD22^2\cos(\phi)^2\,d^2+KD11^2\,r^2+2\,KD11\,r^2\,KD21-2\,KD11\,r\,KD21\cos(\phi)\,d-2\,KD21^2\,r\cos(\phi)\,d+KD21^2\cos(\phi)^2\,d^2\right)\right)\Big/\left(r^2\left(m^2\,d^4+2\,m\,d^2\,Ib-2\,m^2\,d^4\cos(\phi)^2+Ib^2-2\,Ib\,m\,d^2\cos(\phi)^2+m^2\,d^4\cos(\phi)^4\right)\right)\right]\right] \quad \textbf{(3.3)}$$

> $F2m := -Multiply(F2f, qdot)$:

> $F2 := simplify(F2m_1)$:

Input contribution from SMC, F_2

> *convert*(*F2*, *string*)

"-alpha*(`θdot`*KD11*r+`θdot`*KD21*r-`θdot`*KD21*cos(phi)*d+`ϕdot`*KD21*r+`ϕdot`*KD22*r-`ϕdot`*KD22*cos(phi)*d)/r/(m*d^2+Ib-m*d^2*cos(phi)^2)" **(3.4)**

Third Matching Condition

> $PHM := Matrix\left(2, 1, \left[\frac{\partial}{\partial\theta}\Phi(\theta,\phi), \frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right]\right):$

> $TMC := convert\left(\begin{bmatrix} F3 \\ 0 \end{bmatrix}, Matrix\right) - convert(G, Matrix)$
$+Multiply(MatrixInverse(P), PHM);$

$$TMC := \left[\left[F3 + m\,g\,(r\sin(\phi)\cos(\theta) - r\cos(\phi)\sin(\theta) + d\sin(\theta)) + \frac{1}{KD11\,KD22 - KD21^2}\left((KD21\,m\,r^2 - KD21\,m\,r\cos(\phi)\,d + KD22\,m\,r^2 - 2\,KD22\,m\,r\,d\cos(\phi) + KD22\,m\,d^2 + KD22\,Ib)\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)\right) - \frac{1}{KD11\,KD22 - KD21^2}\left((KD11\,m\,r^2 - KD11\,m\,r\cos(\phi)\,d + KD21\,m\,r^2 - 2\,KD21\,m\,r\cos(\phi)\,d + KD21\,m\,d^2 + KD21\,Ib)\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)\right)\right],\right.$$
$$\left.\left[-m\,g\,r\,(\sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta)) - \frac{(KD22\,r - \cos(\phi)\,KD22\,d + KD21\,r)\,r\,m\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)}{KD11\,KD22 - KD21^2} + \frac{(KD11\,r + KD21\,r - KD21\cos(\phi)\,d)\,r\,m\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)}{KD11\,KD22 - KD21^2}\right]\right] \quad \textbf{(4.1)}$$

> $TMCphi := simplify(eval(TMC_{2,\,1}, KD21 = KD21(\phi)));$

$$TMCphi := -\frac{1}{KD11\,KD22 - KD21(\phi)^2}\left(m\,r\left(g\sin(\phi)\cos(\theta)\,KD11\,KD22 - g\sin(\phi)\cos(\theta)\,KD21(\phi)^2 - g\cos(\phi)\sin(\theta)\,KD11\,KD22 + g\cos(\phi)\sin(\theta)\,KD21(\phi)^2 + \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD22\,r - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\right.\right.\right. \quad \textbf{(4.2)}$$

$$\phi)\Big)\cos(\phi)\,KD22\,d+\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD21(\phi)\,r-\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD11\,r$$
$$-\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21(\phi)\,r+\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21(\phi)\cos(\phi)\,d\Big)\Big)$$

> *solphi* := *pdsolve*(*TMCphi*);

$$solphi := \Phi(\theta,\phi) = -\Bigg(\int^{\phi}\Bigg(-\Bigg(g\sin\Bigg(\_b-\Bigg(\int\frac{-KD22\,r+\cos(\_b)\,KD22\,d-KD21(\_b)\,r}{KD11\,r+KD21(\_b)\,r-KD21(\_b)\cos(\_b)\,d}\,d\_b\Bigg)-\theta+\int\frac{-KD22\,r+\cos(\phi)\,KD22\,d-KD21(\phi)\,r}{KD11\,r+KD21(\phi)\,r-KD21(\phi)\cos(\phi)\,d}\,d\phi\Bigg)\left(KD11\,KD22-KD21(\_b)^2\right)\Bigg)\Bigg/\left(KD11\,r+KD21(\_b)\,r-KD21(\_b)\cos(\_b)\,d\right)\Bigg)d\_b\Bigg)+\_F1\left(\theta-\left(\int\frac{-KD22\,r+\cos(\phi)\,KD22\,d-KD21(\phi)\,r}{KD11\,r+KD21(\phi)\,r-KD21(\phi)\cos(\phi)\,d}\,d\phi\right)\right) \tag{4.3}$$

> *PhiIiF* := *simplify*(*rhs*(*solphi*));

$$PhiIiF := -\Bigg(\int^{\phi}\Bigg(-\Bigg(g\sin\Bigg(\_b-\Bigg(\int\frac{-KD22\,r+\cos(\_b)\,KD22\,d-KD21(\_b)\,r}{KD11\,r+KD21(\_b)\,r-KD21(\_b)\cos(\_b)\,d}\,d\_b\Bigg)-\theta+\int\frac{-KD22\,r+\cos(\phi)\,KD22\,d-KD21(\phi)\,r}{KD11\,r+KD21(\phi)\,r-KD21(\phi)\cos(\phi)\,d}\,d\phi\Bigg)\left(KD11\,KD22-KD21(\_b)^2\right)\Bigg)\Bigg/\left(KD11\,r+KD21(\_b)\,r-KD21(\_b)\cos(\_b)\,d\right)\Bigg)d\_b\Bigg)+\_F1\left(\theta-\left(\int\frac{-KD22\,r+\cos(\phi)\,KD22\,d-KD21(\phi)\,r}{KD11\,r+KD21(\phi)\,r-KD21(\phi)\cos(\phi)\,d}\,d\phi\right)\right) \tag{4.4}$$

> *PhiIif* := *simplify*(*eval*(*PhiIiF*, [*KD21*(_*b*) = *KD21*, *KD21*(φ) = *KD21*]));

$$PhiIif := -\Bigg(\int^{\phi}\Bigg(-\frac{1}{KD11\,r+KD21\,r-KD21\cos(\_b)\,d}\Bigg(g\sin\Bigg(\_b+\int\frac{KD22\,r-\cos(\_b)\,KD22\,d+KD21\,r}{KD11\,r+KD21\,r-KD21\cos(\_b)\,d}\,d\_b-\theta-\left(\int\frac{KD22\,r-\cos(\phi)\,KD22\,d+KD21\,r}{KD11\,r+KD21\,r-KD21\cos(\phi)\,d}\,d\phi\right)\Bigg)\left(KD11\,KD22-KD21^2\right)\Bigg)\Bigg)d\_b\Bigg)+\_F1\left(\theta+\int\frac{KD22\,r-\cos(\phi)\,KD22\,d+KD21\,r}{KD11\,r+KD21\,r-KD21\cos(\phi)\,d}\,d\phi\right) \tag{4.5}$$

> $PhiIif := simplify\left(int\left(\frac{1}{KD11\,r + KD21\,r - KD21\,d}\left(\left(-KD21^2 + KD22\,KD11\right) g \sin\left(\xi + int\left(\frac{KD21\,r + KD22\,r - KD22\,d}{KD11\,r + KD21\,r - KD21\,d}, \xi\right) - \theta - \left(int\left(\frac{KD21\,r + KD22\,r - KD22\,d\cos(\phi)}{KD11\,r + KD21\,r - KD21\,d\cos(\phi)}, \phi\right)\right)\right)\right), \xi\right) + F6\cdot\cosh\left(\theta + int\left(\frac{KD21\,r + KD22\,r - KD22\,d\cos(\phi)}{KD11\,r + KD21\,r - KD21\,d\cos(\phi)}, \phi\right)\right)\right):$

> $PhiIif := simplify(eval(PhiIif, [\xi = \phi])):$

> $f3 := simplify(F3 - (TMC_{1,1}));$

$$f3 := \frac{1}{KD11\,KD22 - KD21^2}\Big(-m\,g\,r\sin(\phi)\cos(\theta)\,KD11\,KD22 + m\,g\,r\sin(\phi)\cos(\theta)\,KD21^2 + m\,g\,r\cos(\phi)\sin(\theta)\,KD11\,KD22 - m\,g\,r\cos(\phi)\sin(\theta)\,KD21^2 - m\,g\,d\sin(\theta)\,KD11\,KD22 + m\,g\,d\sin(\theta)\,KD21^2 - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD21\,m\,r^2 + \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD21\,m\,r\cos(\phi)\,d - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD22\,m\,r^2 + 2\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD22\,m\,r\,d\cos(\phi) - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD22\,m\,d^2 - \left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)KD22\,Ib + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD11\,m\,r^2 - \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD11\,m\,r\cos(\phi)\,d + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21\,m\,r^2 - 2\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21\,m\,r\cos(\phi)\,d + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21\,m\,d^2 + \left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)KD21\,Ib\Big) \qquad (4.6)$$

Input contribution from TMC, F_3

> $F3 := simplify(eval(f3, [\Phi(\theta,\phi) = PhiIif])):$

The control input of the system is

> $Fc := simplify(F1 + F2 + F3):$

> $T1i := diff(Fc, \theta):$

> $T2i := diff(Fc, \phi)$:

> $T3i := diff(Fc, \theta dot)$:

> $T4i := diff(Fc, \phi dot)$:

> $T1 := simplify(eval(T1i, [\cos(\phi) = 1]))$:

> $T1 := simplify(eval(T1, [\cos(\theta) = 1]))$:

> $T1 := simplify(eval(T1, [\sin(\phi) = 0, \sin(\theta) = 0, \phi = 0, \theta = 0]))$:

> $T2 := simplify(eval(T2i, [\cos(\phi) = 1, \cos(\theta) = 1]))$:

> $T2 := simplify(eval(T2, [\cos(\theta) = 1, \sin(\phi) = 0, \sin(\theta) = 0, \phi = 0, \theta = 0]))$:

> $T3 := simplify(eval(T3i, [\phi = 0, \theta = 0]))$;

$$T3 := \frac{-\alpha\, r\, KD11 - r\,\alpha\, KD21 + r\,\nu\, Ib + \alpha\, KD21\, d}{Ib\, r} \tag{5.1}$$

> $T4 := simplify(eval(T4i, [\phi = 0, \theta = 0]))$;

$$T4 := -\frac{r\,\alpha\, KD22 + r\,\sigma\, Ib + r\,\alpha\, KD21 - \alpha\, KD22\, d}{Ib\, r} \tag{5.2}$$

> $T := simplify\left(\left[\begin{array}{cccc} T1 & T2 & T3 & T4 \end{array}\right]\right)$:

Linearization of control law

> $sysid1 := [R_o = 0, ro = 0, Ia = Ib]$:
$sysid2 := [m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5]$:

> $LCond := [\theta = 0, \theta dot = 0, \phi = 0, \phi dot = 0]$:

> $Eqs := simplify\left(Multiply\left(IMass, \left(\left[\begin{array}{c} \tau \\ 0 \end{array}\right] - Multiply(Cmatrix, qdot) - G\right)\right)\right)$;

$$Eqs := \left[\left[-\frac{1}{m\,d^2 + Ib - m\,d^2\cos(\phi)^2}\left(-\tau + 4\,\phi dot\, m\, r\, d\sin(\phi)\,\theta dot - 2\,\phi dot^2\, m\, r\, d\sin(\phi) - m\, g\, d\sin(\theta) - 2\,\theta dot^2\, m\, r\, d\sin(\phi) + 2\, m\, d^2\cos(\phi)\,\theta dot^2\sin(\phi) - d\cos(\phi)\, m\, g\sin(\phi)\cos(\theta) + d\cos(\phi)^2\, m\, g\sin(\theta)\right)\right], \left[\frac{1}{r\left(m\,d^2 + Ib - m\,d^2\cos(\phi)^2\right)}\left(r\,\tau - 4\,\phi dot\, m\, r^2\, d\sin(\phi)\,\theta dot + 2\,\phi dot^2\, m\, r^2\, d\sin(\phi) + r\, m\, g\, d\sin(\theta) - d\cos(\phi)\,\tau + 4\, d^2\cos(\phi)\,\phi dot\, m\, r\sin(\phi)\,\theta dot - 2\, d^2\cos(\phi)\,\phi dot^2\, m\, r\sin(\phi) + d\cos(\phi)\, m\, g\, r\sin(\phi)\cos(\theta) - d\cos(\phi)^2\, m\, g\, r\sin(\theta) + 2\, m\, r^2\,\theta dot^2\, d\sin(\phi) - 4\, m\, r\, d^2\cos(\phi)\,\theta dot^2\sin(\phi) + 2\, m\, d^3\,\theta dot^2\sin(\phi)\right.\right.\tag{6.1}$$

$$-m\,d^2\,g\,\sin(\phi)\,\cos(\theta)+2\,Ib\,\theta dot^2\,d\,\sin(\phi)-Ib\,g\,\sin(\phi)\,\cos(\theta)+Ib\,g\,\cos(\phi)\,\sin(\theta)\big)\big]\big]$$

```
> A := [[0, 0, 1, 0],
  [0, 0, 0, 1],
  [diff(Eqs[1], θ), diff(Eqs[1], ϕ), diff(Eqs[1], θdot), diff(Eqs[1], ϕdot)],
  [diff(Eqs[2], θ), diff(Eqs[2], ϕ), diff(Eqs[2], θdot), diff(Eqs[2], ϕdot)]]:
> A := map(eval, A, LCond) :
> A := eval(A, sysid1) :
> A := eval(A, sysid2);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.960000000 & 0 & 0 \\ 13.06666666 & -12.41333333 & 0 & 0 \end{bmatrix} \tag{6.2}$$

$$> B := \begin{bmatrix} 0 \\ 0 \\ \mathrm{diff}(Eqs[1], \tau) \\ \mathrm{diff}(Eqs[2], \tau) \end{bmatrix}:$$

```
> B := simplify(map(eval, B, LCond)) :
> B := simplify(map(eval, B, sysid1)) :
> B := simplify(map(eval, B, sysid2));
```

$$B := \begin{bmatrix} 0 \\ 0 \\ 20. \\ 6.666666666 \end{bmatrix} \tag{6.3}$$

```
> LinCi := -T:
> LinC := map(eval, LinCi, sysid1):
> LinC1 := map(eval, LinC, sysid2):
> LinC3 := map(eval, LinC1, LCond):
> k := [ 82.2743  138.2016  -34.1319  118.4029 ];
```

$$k := [\,82.2743 \quad 138.2016 \quad -34.1319 \quad 118.4029\,] \tag{6.4}$$

$$> solve(\{\, k_1 - LinC3_1,\ k_2 - LinC3_2,\ k_3 - LinC3_3,\ k_4 - LinC3_4 \,\});$$

$$\{F6 \tag{6.5}$$

$$= \frac{1}{2.265359\ 10^{6}\ KD21 + 7.51853\ 10^{5}\ KD22} (1.158550537\ 10^{-8} (3.805640687\ 10^{17}\ KD21^{2} + 2.512230276\ 10^{17}\ KD22\ KD21 + 4.146003213\ 10^{16}\ KD22^{2})),$$

$KD11 = 0.8537774058\ KD21 + 0.3925497477\ KD22,\ KD21 = KD21,\ KD22 = KD22,\ \alpha = \alpha,\ \nu = 34.13190000 + 23.74221478\ \alpha\ KD21 + 7.850994954\ \alpha\ KD22,\ \sigma = 118.4029000 - 6.666666668\ \alpha\ KD22 - 20.\ \alpha\ KD21\}$

> $K := simplify\left(eval\left(LinC3, \left[F6 = \frac{1}{2.265359\ 10^{6}\ KD21 + 7.51853\ 10^{5}\ KD22} (1.158550537\ 10^{-8} (3.805640687\ 10^{17}\ KD21^{2} + 2.512230276\ 10^{17}\ KD22\ KD21 + 4.146003213\ 10^{16}\ KD22^{2})),\right.\right.\right.$
> $KD11 = 0.8537774058\ KD21 + 0.3925497477\ KD22,\ KD21 = KD21,\ KD22 = KD22,\ \alpha = \alpha,\ \nu = 34.13190000 + 23.74221478\ \alpha\ KD21 + 7.850994954\ \alpha\ KD22,\ \sigma = 118.4029000 - 6.666666668\ \alpha\ KD22 - 20.\ \alpha\ KD21])) :$

> $K := simplify\left(eval\left(K, \left[KD21 = -8, \alpha = r^{2} \cdot (m \cdot d^{2} + Ib - m \cdot d^{2} \cdot 0.01)^{2}, KD22 = 25\right]\right)\right) :$

> $K := simplify(eval(K, sysid2));$

$$K := [\ 82.27303888\quad 138.1617159\quad -34.13190000\quad 118.4029000\ ] \tag{6.6}$$

> $Acl := eval(A - Multiply(B, K));$

$Acl := [[0., 0., 1., 0.],$ **(6.7)**
$[0., 0., 0., 1.],$
$[-1645.46077600000003, -2761.27431799999977, 682.638000000000034, -2368.05799999999999],$
$[-535.420259151817959, -933.491439237891996, 227.545999977245401, -789.352666587731392]]$

> $EACLi := Eigenvalues(Acl);$

$$EACLi := \begin{bmatrix} -76.7102449890722369 + 0.\ I \\ -4.80474800020141224 + 0.\ I \\ -11.0097791129625779 + 0.\ I \\ -14.1898944854941575 + 0.\ I \end{bmatrix} \tag{6.8}$$

> $Dett := eval\left(Detti, \left[F6\right.\right.$

$$= \frac{1}{2.265359\ 10^{6}\ KD21 + 7.51853\ 10^{5}\ KD22} \left(1.158550537\ 10^{-8} \left(3.805640687\ 10^{17}\ KD21^{2} + 2.512230276\ 10^{17}\ KD22\ KD21 + 4.146003213\ 10^{16}\ KD22^{2}\right)\right),$$

$$KD11 = 0.8537774058\ KD21 + 0.3925497477\ KD22,\ KD21 = KD21,\ KD22 = KD22,\ \alpha = \alpha,\ \nu = 34.13190000 + 23.74221478\ \alpha\ KD21 + 7.850994954\ \alpha\ KD22,\ \sigma = 118.4029000 - 20.\ \alpha\ KD21 - 6.666666668\ \alpha\ KD22\big]\big);$$

$$Dett := (0.8537774058\ KD21 + 0.3925497477\ KD22)\ KD22 - KD21^{2} \quad \textbf{(6.9)}$$

> $Dettf := eval\left(Dett, \left[KD21 = -8,\ \alpha = r^{2} \cdot \left(m \cdot d^{2} + Ib - m \cdot d^{2} \cdot 0.01\right)^{2},\ KD22 = 25\right]\right);$

$$Dettf := 10.58811115 \quad \textbf{(6.10)}$$

where *Detti is denined as the determinant of KD*

The Potential Φ

This part is made in order to simplify the potential

> $PhiIif:$

> $PhiIifw := eval\left(PhiIif, \left[\sqrt{\sqrt{(KD11\ r + KD21\ r - KD21\ d)\ (KD11\ r + KD21\ r + KD21\ d)}} = \Psi\right]\right):$

> $PhiIifw1 := eval\left(PhiIifw, \left[\arctan\left(\left((KD11\ r + KD21\ r + KD21\ d)\ (-1 + \cos(\phi))\right) \Big/ \left(\sqrt{(KD11\ r + KD21\ r - KD21\ d)\ (KD11\ r + KD21\ r + KD21\ d)}\ \sin(\phi)\right)\right) = \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right]\right):$

> $PhiIifw2 := eval\left(PhiIifw1, \left[\arctan\left(\frac{-1 + \cos(\phi)}{\sin(\phi)}\right) = \mathrm{arct}(\varsigma(\phi)),\ \sqrt{(KD11\ r + KD21\ r - KD21\ d)\ (KD11\ r + KD21\ r + KD21\ d)} = \Psi\right]\right);$

$$PhiIifw2 := \Big(-g \cos\Big(\Big(\Big(-\phi\ KD21\ \Psi\ KD11\ r - 2\ \phi\ KD21^{2}\ \Psi\ r + \phi\ KD21^{2}\ \Psi\ d \quad \textbf{(7.1)}$$

$$- \phi\ KD21\ \Psi\ KD22\ r + \phi\ KD21\ \Psi\ KD22\ d + \theta\ KD21\ \Psi\ KD11\ r + \theta\ KD21^{2}\ \Psi\ r$$

$$- \theta\, KD21^2\, \Psi\, d + 2\, r^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11^2\, KD22$$

$$+ 2\, r^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\, KD21$$

$$- 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\, KD21\, d$$

$$- 2\, r^2\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11 - 2\, r^2\, KD21^3 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)$$

$$+ 2\, r\, KD21^3 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) d - 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD11\, r$$

$$- 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD21\, r + 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD21\, d\Big) \Big/$$

$$\big((KD11\, r + KD21\, r$$

$$- KD21\, d)$$

$$KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\,)\,)$$

$$KD11\, KD22 + g \cos\Big(\Big(-\phi\, KD21\, \Psi\, KD11\, r - 2\, \phi\, KD21^2\, \Psi\, r + \phi\, KD21^2\, \Psi\, d$$

$$- \phi\, KD21\, \Psi\, KD22\, r + \phi\, KD21\, \Psi\, KD22\, d + \theta\, KD21\, \Psi\, KD11\, r + \theta\, KD21^2\, \Psi\, r$$

$$- \theta\, KD21^2\, \Psi\, d + 2\, r^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11^2\, KD22$$

$$+ 2\, r^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\, KD21$$

$$- 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\, KD21\, d$$

$$- 2\, r^2\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11 - 2\, r^2\, KD21^3 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)$$

$$+ 2\, r\, KD21^3 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) d - 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD11\, r$$

$$- 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD21\, r + 2\, KD22\, arct(\varsigma(\phi))\, \Psi\, KD21\, d\Big) \Big/$$

$$\Big((KD11\, r + KD21\, r$$

$$- KD21\, d)$$

$$KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\,)\Big)$$

$$KD21^2 + F6 \cosh\left(\left(-2\, r\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right.\right.$$

$$\left. - 2\, KD22\, arct(\varsigma(\phi))\, \Psi + \theta\, KD21\, \Psi + 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\right)$$

$$\Big/ \left(KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\right)$$

$$KD11\, r + 2\, F6 \cosh\left(\left(-2\, r\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right.\right.$$

$$\left. - 2\, KD22\, arct(\varsigma(\phi))\, \Psi + \theta\, KD21\, \Psi + 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\right)$$

$$\Big/ \left(KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\right)$$

$$KD21\, r - F6 \cosh\left(\left(-2\, r\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right.\right.$$

$$\left. - 2\, KD22\, arct(\varsigma(\phi))\, \Psi + \theta\, KD21\, \Psi + 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\right)$$

$$\Big/ \left(KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\right)$$

$$KD21\, d + F6 \cosh\left(\left(-2\, r\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right.\right.$$

$$\left. - 2\, KD22\, arct(\varsigma(\phi))\, \Psi + \theta\, KD21\, \Psi + 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\right)$$

$$\Big/ \left(KD21 \sqrt{(KD11\, r + KD21\, r - KD21\, d)\, (KD11\, r + KD21\, r + KD21\, d)}\right)$$

$$KD22\, r - F6 \cosh\left(\left(-2\, r\, KD21^2 \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right)\right.\right.$$

$$\left. - 2\, KD22\, arct(\varsigma(\phi))\, \Psi + \theta\, KD21\, \Psi + 2\, r \arctan\left(\frac{\Gamma(\theta, \phi)}{\Psi}\right) KD11\, KD22\right)$$

$$\Big/ \left(KD21\sqrt{(KD11\,r+KD21\,r-KD21\,d)\,(KD11\,r+KD21\,r+KD21\,d)}\right)\Big)$$

$$KD22\,d\Big) \Big/ (KD11\,r+2\,KD21\,r-KD21\,d+KD22\,r-KD22\,d)$$

> $PhiGraphi := simplify(eval(PhiIif, [R_o = 0, ro = 0, Ia = Ib, m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5])) :$

> $PhiGraph := simplify\Big(eval\Big(PhiGraphi, \Big[F6 = \frac{1}{2.265359\,10^6\,KD21 + 7.51853\,10^5\,KD22}(1.158550537\,10^{-8}(3.805640687\,10^{17}\,KD21^2 + 2.512230276\,10^{17}\,KD22\,KD21 + 4.146003213\,10^{16}\,KD22^2)),$ $KD11 = 0.8537774058\,KD21 + 0.3925497477\,KD22, KD21 = KD21, KD22 = KD22, \alpha = \alpha, \nu = 34.13190000 + 23.74221478\,\alpha\,KD21 + 7.850994954\,\alpha\,KD22, \sigma = 118.4029000 - 6.666666668\,\alpha\,KD22 - 20.\,\alpha\,KD21\Big]\Big)\Big) :$

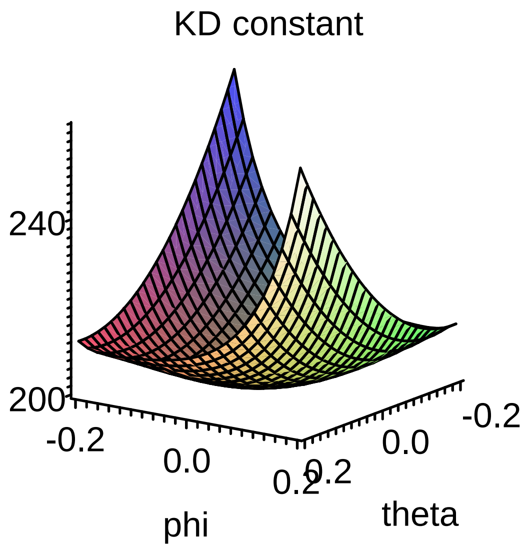
> $PhiGrapht := simplify(eval(PhiGraph, [\mathrm{KD21} = -8, \alpha = \mathrm{r}^2 \cdot (\mathrm{m} \cdot \mathrm{d}^2 + \mathrm{Ib} - \mathrm{m} \cdot \mathrm{d}^2 \cdot 0.01)^2, KD22 = 25])) :$

where KD21 and KD22 are the tunning parameters to get the convex shape of the potential.

α is the tunning parameter for the positive definiteness property of Fmc1

> $f := (\theta, r) \rightarrow PhiGrapht :$

> $plot3d(f(\theta, r), \theta = -0.2\,..\,0.2, \phi = -0.2\,..\,0.2, axes = FRAME, orientation = [35, 75], style = PATCH, title = \text{" KD constant"});$



The Hessian

> $Hessiani := simplify(eval(Matrix(2, 2, [\ diff\,(diff\,(\mathrm{PhiIif}, \phi), \phi), diff\,(diff\,(PhiIif, \phi), \theta), diff\,(diff\,(PhiIif, \theta), \phi), diff\,(diff\,(PhiIif, \theta), \theta)\,]))) :$

> $Hessian := simplify\left(eval\left(Hessiani, \left[\sin(\phi) = \sin\left(\phi + 0.1 * \left(10^{-15}\right)\right)\right]\right)\right) :$

> $Hessian := simplify(eval(Hessian, [\theta = 0, \phi = 0])) :$

> $Hessian := simplify(eval(Hessian, [m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5])) :$

> $Hessian := simplify\Bigg(eval\Bigg(Hessian, \Bigg[F6 = \frac{1}{2.265359\ 10^{6}\ KD21 + 7.51853\ 10^{5}\ KD22}\,(1.158550537\ 10^{-8}\,(3.805640687\ 10^{17}\ KD21^{2} + 2.512230276\ 10^{17}\ KD22\ KD21 + 4.146003213\ 10^{16}\ KD22^{2})), KD11 = 0.8537774058\ KD21 + 0.3925497477\ KD22, KD21 = KD21, KD22 = KD22, \alpha = \alpha, \nu = 34.13190000 + 23.74221478\ \alpha\ KD21 + 7.850994954\ \alpha\ KD22, \sigma = 118.4029000 - 6.666666668\ \alpha\ KD22 - 20.\ \alpha\ KD21\Bigg]\Bigg)\Bigg) :$

> $Hessian := simplify\left(eval\left(Hessian, \left[KD21 =-8, \alpha = \mathrm{r}^2 \cdot (\mathrm{m} \cdot \mathrm{d}^2 + \mathrm{Ib} - \mathrm{m} \cdot \mathrm{d}^2 \cdot 0.01)^2, KD22 = 25\right]\right)\right);$

(8.1)

$$Hessian := \begin{bmatrix} 727.9019701 & 221.2876183 \\ 221.2876183 & 625.9346949 \end{bmatrix} \quad \textbf{(8.1)}$$

> $Determinant(Hessian);$

$$4.066508876\ 10^5 \quad \textbf{(8.2)}$$

> $Eigenvalues(Hessian);$

$$\begin{bmatrix} 904.003209398076592 + 0.\ I \\ 449.833455601923333 + 0.\ I \end{bmatrix} \quad \textbf{(8.3)}$$

Variables to matlab

> $F3 := simplify(eval(F3, \mathrm{sysid2}))$:

> $Fmc1 := simplify(eval(\mathrm{Fmc1}, \mathrm{sysid2}))$:

> $convert(\mathrm{F1}, string)$;

"\`θdot\`*nu-\`ϕdot\`*sigma" **(9.1)**

> $convert(\mathrm{Fmc1}, string)$;

"Matrix(2, 2, [[2*F33(qf,qdotf)+2*F55(qf,qdotf)+2*F66(qf,qdotf),F77(qf,qdotf)+F88(qf,qdotf)],[F77(qf,qdotf)+F88(qf,qdotf),2*F44(qf,qdotf)+2*F66(qf,qdotf)+2*F55(qf,qdotf)]])" **(9.2)**

> $convert(F2, string)$

"-alpha*(\`θdot\`*KD11*r+\`θdot\`*KD21*r-\`θdot\`*KD21*cos(phi)*d+\`ϕdot\`*KD21*r+\`ϕdot\`*KD22*r-\`ϕdot\`*KD22*cos(phi)*d)/r/(m*d^2+Ib-m*d^2*cos(phi)^2)" **(9.3)**

> $convert(F3, string)$

".1000000000e-2*(225.*KD11*KD21*F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*cos(phi)^2+4410.*KD11*sin(phi)*cos(theta)*KD21^2*cos(phi)-294.*sin(theta)*KD22*KD21^2+3528.*KD11*KD21*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*

KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22*cos(phi)-3969.*KD11^3*sin
(phi)*cos(theta)-5586.*KD11*sin(theta)*KD21^2-1764.*sin(phi)*cos(theta)*
KD21^3+3969.*KD11^3*cos(phi)*sin(theta)-882.*KD11^2*KD22*sin(theta)
-441.*sin(phi)*cos(theta)*KD22*KD21^2-8379.*KD11*sin(phi)*cos(theta)*
KD21^2-1176.*KD11*KD22*sin(theta)*KD21+1176.*sin(phi)*cos(theta)*
KD21^3*cos(phi)-4410.*KD11*cos(phi)^2*sin(theta)*KD21^2+11319.*KD11*
sin(theta)*KD21^2*cos(phi)+637.*KD22*sin(theta)*KD21^2*cos(phi)-294.*
KD22*cos(phi)^2*sin(theta)*KD21^2+1323.*KD11^2*KD22*cos(phi)*sin
(theta)-2646.*KD11^2*cos(phi)^2*sin(theta)*KD21-1176.*cos(phi)^2*sin((-3.*
phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22*KD21^2-660.*KD21^2*F6*
sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*
KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)-1176.*
KD21^3*sin(theta)-2646.*KD11^3*sin(theta)+6958.*cos(phi)*sin((-3.*phi*
KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*

KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22*KD21^2-495.*KD11*KD22*
F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*
KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)-165.*KD22*
F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*
KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD21-2475.*
KD11*F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos
(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*
KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)
/KD21)*KD21+1764.*KD11^2*KD22*sin((-3.*phi*KD21*((3.*KD11+KD21)
*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11
-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.

*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*cos(phi)^2+34986.*KD11*cos(phi)*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD21^2-15582.*KD11*KD21*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22+60.*cos(phi)^2*F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21

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KD21))^(1/2)/KD21)*KD21^2-1764.*KD11^2*KD21*sin((-3.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)
*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*
arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.
*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi)
)*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*
arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*
KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)
/KD21)*cos(phi)^2+882.*KD11^2*KD22*sin((-3.*phi*KD21*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*
arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.
*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi)
)*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*
arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*
KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)
/KD21)*cos(phi)+135.*KD11^2*cos(phi)^2*F6*sinh((-6.*KD21^2*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)-7644.*KD11*cos(phi)^2*sin((-3.*
phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
```

KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD21^2+14112.*KD11^2*KD21*
cos(phi)*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*
KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*
phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*
KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/(
(3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21
-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-10584.*KD11^2*sin(phi)*cos(theta)*
KD21+45.*KD11*KD22*cos(phi)^2*F6*sinh((-6.*KD21^2*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)+294.*sin(phi)*cos(theta)*KD22*
KD21^2*cos(phi)-5292.*cos(phi)^2*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.
*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+

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KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11
-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.
*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)
-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*
KD21^3-40425.*KD11*KD21^2*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*
(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11
-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.
*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)
-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-1485.*
KD11^2*F6*sinh((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos
(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*
KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*
KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)/sin(phi))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)
/KD21)+2646.*KD11^2*sin(phi)*cos(theta)*cos(phi)*KD21+882.*KD11*
KD22*sin(phi)*cos(theta)*cos(phi)*KD21+2646.*KD11^3*cos(phi)*sin((-3.*
phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
```

KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-32634.*KD11^2*KD21*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-24696.*KD21^3*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-3969.*KD11^3*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*

KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*
arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos
(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*
KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*
arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-1323.*
KD11^2*sin(phi)*cos(theta)*KD22-7791.*KD21^2*sin((-3.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)
*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11^2*KD22+6.*
arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*KD21^2*arctan((3.*KD11+5.
KD21)(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi)
)*KD11-6.*KD21^3*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))-6.*KD22*arctan((cos(phi)-1.)/sin
(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*
arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*
KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)
/KD21)*KD22-7056.*KD11^2*sin(theta)*KD21-20727.*KD11^2*KD22*sin(
(-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*
phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*(
(3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+

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KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)+27832.*KD21^3*sin((-3.*phi*
KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*
KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*
KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*arctan((3.*KD11+5.*KD21)
*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*
KD11^2*KD22+6.*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*KD22*KD21-18.*
KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*
KD11+5.*KD21))^(1/2)/sin(phi))*KD11-6.*KD21^3*arctan((3.*KD11+5.*
KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))
-6.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+KD21)*(3.*KD11+5.*
KD21))^(1/2)*KD11-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*cos(phi)-1176.*sin(theta)*KD21^3*
cos(phi)^2+2548.*sin(theta)*KD21^3*cos(phi)+15.*KD22*cos(phi)^2*F6*sinh
((-6.*KD21^2*arctan((3.*KD11+5.*KD21)*(cos(phi)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi))-2.*KD22*arctan((cos(phi)-1.)/sin(phi))*(
(3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+
KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*arctan((3.*KD11+5.*KD21)*(cos
(phi)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi))*KD11*
KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD21+2352.
*KD11*cos(phi)*sin(theta)*KD22*KD21-1764.*KD11*sin(phi)*cos(theta)*
KD22*KD21-882.*KD11*KD22*sin(theta)*KD21*cos(phi)^2+12348.*
KD11^2*cos(phi)*sin(theta)*KD21)/(9.*KD11^2+15.*KD11*KD21+3.*
KD11*KD22+4.*KD21^2+KD22*KD21)/(3.*KD11+3.*KD21-2.*cos(phi)*
KD21)"
```

```
> convert(PhiIif, string);
```

```
"1/(KD11*r+2*KD21*r-KD21*d+KD22*r-KD22*d)*(-g*cos((-phi*KD21*(          (9.5)
(KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r-2*
phi*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^
(1/2)*r+phi*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+
KD21*d))^(1/2)*d-phi*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*
r+KD21*d))^(1/2)*KD22*r+phi*KD21*((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2)*KD22*d+theta*KD21*((KD11*r+
KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r+theta*
KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*
r-theta*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))
```

^(1/2)*d+2*r^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11^2*KD22+2*r^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22*KD21-2*r*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22*KD21*d-2*r^2*KD21^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11-2*r^2*KD21^3*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))+2*r*KD21^3*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*d-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD21*r+2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD21*d)/(KD11*r+KD21*r-KD21*d)/KD21/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2))*KD11*KD22+g*cos((-phi*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r-2*phi*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*r+phi*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*d-phi*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD22*r+phi*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD22*d+theta*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r+theta*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*r-theta*KD21^2*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*d+2*r^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11^2*KD22+2*r^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22*KD21-2*r*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22*KD21*d-2*r^2*KD21^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11-2*r^2*KD21^3*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))+2*r*KD21^3*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*d-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-

KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD11*r-2*KD22*arctan((cos
(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))
^(1/2)*KD21*r+2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-
KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*KD21*d)/(KD11*r+KD21*r-
KD21*d)/KD21/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^
(1/2))*KD21^2+F6*cosh((-2*r*KD21^2*arctan((KD11*r+KD21*r+KD21*d)/
((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)
-1)/sin(phi))-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*
d)*(KD11*r+KD21*r+KD21*d))^(1/2)+theta*KD21*((KD11*r+KD21*r-
KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)+2*r*arctan((KD11*r+KD21*
r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^
(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22)/KD21/((KD11*r+KD21*r-KD21*d)
*(KD11*r+KD21*r+KD21*d))^(1/2))*KD11*r+2*F6*cosh((-2*r*KD21^2*
arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+
KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))-2*KD22*arctan((cos(phi)-1)
/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)
+theta*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^
(1/2)+2*r*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22)
/KD21/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2))*
KD21*r-F6*cosh((-2*r*KD21^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*
r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin
(phi))-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2)+theta*KD21*((KD11*r+KD21*r-KD21*
d)*(KD11*r+KD21*r+KD21*d))^(1/2)+2*r*arctan((KD11*r+KD21*r+
KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*
(cos(phi)-1)/sin(phi))*KD11*KD22)/KD21/((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2))*KD21*d+F6*cosh((-2*r*KD21^2*
arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+
KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))-2*KD22*arctan((cos(phi)-1)
/sin(phi))*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)
+theta*KD21*((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^
(1/2)+2*r*arctan((KD11*r+KD21*r+KD21*d)/((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22)
/KD21/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2))*
KD22*r-F6*cosh((-2*r*KD21^2*arctan((KD11*r+KD21*r+KD21*d)/((KD11*
r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin
(phi))-2*KD22*arctan((cos(phi)-1)/sin(phi))*((KD11*r+KD21*r-KD21*d)*
(KD11*r+KD21*r+KD21*d))^(1/2)+theta*KD21*((KD11*r+KD21*r-KD21*
d)*(KD11*r+KD21*r+KD21*d))^(1/2)+2*r*arctan((KD11*r+KD21*r+

KD21*d)/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2)*(cos(phi)-1)/sin(phi))*KD11*KD22)/KD21/((KD11*r+KD21*r-KD21*d)*(KD11*r+KD21*r+KD21*d))^(1/2))*KD22*d)"

The control law numerical value is

> $tauf := simplify\Big(eval\Big(F1 + F2 + F3, \Big[F6 = \frac{1}{2.265359\ 10^{6}\ KD21 + 7.51853\ 10^{5}\ KD22}\big(1.158550537\ 10^{-8}\ \big(3.805640687\ 10^{17}\ KD21^{2} + 2.512230276\ 10^{17}\ KD22\ KD21 + 4.146003213\ 10^{16}\ KD22^{2}\big)\big),$
$KD11 = 0.8537774058\ KD21 + 0.3925497477\ KD22, KD21 = KD21, KD22 = KD22, \alpha = \alpha, \nu = 34.13190000 + 23.74221478\ \alpha\ KD21 + 7.850994954\ \alpha\ KD22, \sigma = 118.4029000 - 6.666666668\ \alpha\ KD22 - 20.\ \alpha\ KD21\Big]\Big)\Big):$

> $tauf := simplify\big(eval\big(tauf, \big[KD21 = -8, \alpha = r^{2}\cdot\big(m\cdot d^{2} + Ib - m\cdot d^{2}\cdot 0.01\big)^{2}, KD22 = 25\big]\big)\big):$

> $tauf := simplify(eval(tauf, sysid2));$

$$tauf := \Big(0.002000000000\ \Big(-1.51916500\ 10^{8}\sin(\phi)\cos(\theta) + 2.59624000\ 10^{8}\cos(\phi)\sin(\theta) + 3.545267777\ 10^{10}\ \theta dot - 1.229168756\ 10^{11}\ \phi dot - 1.52321000\ 10^{8}\sin(\theta)\cos(\phi)^{2} + 1.312049277\ 10^{11}\ \phi dot\cos(\phi) - 3.785894573\ 10^{10}\ \theta dot\cos(\phi) + 2.665670400\ 10^{10}\sin\Big(2.051996670\ \phi - 0.9999999808\ \theta + 1.461707192\ \operatorname{arctanh}\Big(\frac{5.715233488\ (\cos(\phi) - 1.)}{\sin(\phi)}\Big) - 6.250000032\ \arctan\Big(\frac{\cos(\phi) - 1.}{\sin(\phi)}\Big)\Big)\cos(\phi) - 8.50500000\ 10^{8}\cos(\phi)^{2}\sinh\Big(\tag{9.6}$$

$$-1.461706399 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi)-1.)}{\sin(\phi)}\right)$$

$$+6.250000000 \arctan\left(\frac{\cos(\phi)-1.}{\sin(\phi)}\right)+\theta\Bigg)+3.8720000\ 10^{7}\cos(\phi)^{4}\sinh\Bigg($$

$$-1.461706399 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi)-1.)}{\sin(\phi)}\right)$$

$$+6.250000000 \arctan\left(\frac{\cos(\phi)-1.}{\sin(\phi)}\right)+\theta\Bigg)+1.4684500\ 10^{7}\sin(\theta)\cos(\phi)^{4}$$

$$+3.410773433\ 10^{9}\ \theta dot \cos(\phi)^{3}-1.182667026\ 10^{10}\ \phi dot \cos(\phi)^{3}$$

$$+3.82233000\ 10^{8}\sin\Big(2.051996670\,\phi-0.9999999808\,\theta$$

$$+1.461707192 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi)-1.)}{\sin(\phi)}\right)$$

$$-6.250000032 \arctan\left(\frac{\cos(\phi)-1.}{\sin(\phi)}\right)\Bigg)\cos(\phi)^{4}$$

$$+1.055935072\ 10^{10}\ \phi dot \cos(\phi)^{2}-1.01277000\ 10^{8}\sin(\theta)$$

$$-2.423335000\ 10^{9}\sin\Big(2.051996670\,\phi-0.9999999808\,\theta$$

$$+1.461707192 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi)-1.)}{\sin(\phi)}\right)$$

$$-6.250000032 \arctan\left(\frac{\cos(\phi)-1.}{\sin(\phi)}\right)\Bigg)\cos(\phi)^{3}$$

$$-3.027374440\ 10^{9}\ \theta dot \cos(\phi)^{2}-2.3603500\ 10^{7}\cos(\phi)^{3}\sin(\theta)$$

$$+4.686000000\ 10^{9}\sinh\Bigg($$

$$-1.461706399 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi)-1.)}{\sin(\phi)}\right)$$

$$+6.250000000 \arctan\left(\frac{\cos(\phi)-1.}{\sin(\phi)}\right)+\theta\Bigg)$$

$$-2.357063500\ 10^{10}\sin\Big(2.051996670\,\phi-0.9999999808\,\theta$$

$$+ 1.461707192\,\mathrm{arctanh}\left(\frac{5.715233488\,(\cos(\phi) - 1.)}{\sin(\phi)}\right)$$

$$- 6.250000032\,\arctan\left(\frac{\cos(\phi) - 1.}{\sin(\phi)}\right)\Bigg)$$

$$+ 1.3810500\ 10^{7} \sin(\phi) \cos(\theta) \cos(\phi)^{2}$$

$$+ 1.61529500\ 10^{8} \sin(\phi) \cos(\theta) \cos(\phi)$$

$$- 1.4684400\ 10^{7} \sin(\phi) \cos(\theta) \cos(\phi)^{3}$$

$$- 2.061790000\ 10^{9} \sin\Bigg(2.051996670\,\phi - 0.9999999808\,\theta$$

$$+ 1.461707192\,\mathrm{arctanh}\left(\frac{5.715233488\,(\cos(\phi) - 1.)}{\sin(\phi)}\right)$$

$$- 6.250000032\,\arctan\left(\frac{\cos(\phi) - 1.}{\sin(\phi)}\right)\Bigg) \cos(\phi)^{2}\Bigg)\Bigg) \Bigg/ \big(-1.87946\ 10^{5} \cos(\phi)^{2}$$

$$+ 2.067450\ 10^{6} - 2.197801\ 10^{6} \cos(\phi) + 1.99800\ 10^{5} \cos(\phi)^{3}\big)$$

The potential numerical value is

> $phif := PhiGraph:$

> $phif := simplify\Big(eval\Big(phif, \Big[F6 = \frac{1}{2.265359\ 10^{6}\,KD21 + 7.51853\ 10^{5}\,KD22}\,(1.158550537\ 10^{-8}\,(3.805640687\ 10^{17}\,KD21^{2} + 2.512230276\ 10^{17}\,KD22\,KD21 + 4.146003213\ 10^{16}\,KD22^{2})),$
$KD11 = 0.8537774058\,KD21 + 0.3925497477\,KD22, KD21 = KD21, KD22 = KD22, \alpha = \alpha, \nu = 34.13190000 + 23.74221478\,\alpha\,KD21 + 7.850994954\,\alpha\,KD22, \sigma = 118.4029000 - 6.666666668\,\alpha\,KD22 - 20.\,\alpha\,KD21\Big]\Big)\Big):$

> $phif := simplify\big(eval\big(phif, \big[KD21 = -8, \alpha = r^{2}\cdot(m\cdot d^{2} + Ib - m\cdot d^{2}\cdot 0.01)^{2}, KD22 = 25\big]\big)\big):$

> $phif := simplify(eval(phif, sysid2));$

(9.7)

$$
\begin{aligned}
phif := & -212.7856499 \cos\Bigg(2.051996670\,\phi - 0.9999999808\,\theta \\
& + 1.461707192 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi) - 1.)}{\sin(\phi)}\right) \\
& - 6.250000032 \arctan\left(\frac{\cos(\phi) - 1.}{\sin(\phi)}\right)\Bigg) + 413.1093621 \cosh\Bigg(\\
& -1.461706399 \operatorname{arctanh}\left(\frac{5.715233488\,(\cos(\phi) - 1.)}{\sin(\phi)}\right) \\
& + 6.250000000 \arctan\left(\frac{\cos(\phi) - 1.}{\sin(\phi)}\right) + \theta\Bigg)
\end{aligned}
\tag{9.7}
$$

```
%%%%%%%%%%%%%%%%%%  B.4 BALL AND ARC SIMULATION KD %%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%       B&Arc_Nonlinear_Sys.m    %%%%%%%%%%%%%%%%%%
function dxdt = B_Arc_NonLinear_Sys(u)

%% Main Vectors
theta     = u(1);                 % feedback array
phi       = u(2);
thetadot  = u(3);
phidot    = u(4);

%% Generalized quantities
q         = [theta phi]';         % Generalized coordinates
qdot      = [thetadot phidot]';  % Generalized velocities

%% Physical parameter values
r         =   0.75;               % arc's radius
d         =   0.5;                % distance from the center of the arc
Ib        =   0.05;               % kg m^2  - inertia of beam
m         =   0.02;               % kg      - mass of ball
g         =   9.81;               % m/sec^2 - acceleration of gravity
%% Linear model parameters
KD21      = -8;
KD22      = 25;
alpha     = r^2*(m*d^2+Ib-m*d^2*0.01)^2;
F6        = 1.158550537*10^(-8)*(3.805640687*10^17*KD21^2+2.
512230276*10^17*KD22*KD21+...
            4.146003213*10^16*KD22^2)/(2.265359*10^6*KD21+7.51853*10^5*KD22);
KD11      = .8537774058*KD21+.3925497477*KD22;
nu        = 34.13190000+23.74221478*alpha*KD21+7.850994954*alpha*KD22;
sigma     = 118.4029000-20.*alpha*KD21-6.666666668*alpha*KD22;
epsilon   =0.1*(10^-15);

%% The G, M, C, P and KD matrices
          %gravity terms
G         =    -[m*g*r*sin(phi)*cos(theta)- m*g*r*cos(phi)*sin(theta)+...
          m*g*d*sin(theta);-m*g*r*sin(phi)*cos(theta)+m*g*r*cos(phi)*sin
(theta)];
          %mass matrix
mass      =   [m*r^2-2*m*r*d*cos(phi)-...
          m*d^2+Ib,m*r*(-r+...
          d*cos(phi));m*r*(-...
          r+d*cos(phi)),m*r^2];
          %centripetal and coriolis forces matrix
C         =  -[-phidot*m*r*d*sin(phi),-phidot*m*r*d*sin(phi)+...
          phidot*m*r*d*sin(phi);phidot*m*r*d*sin(phi),0];

%% FMC
          % KD matrix
KD        =  [KD11, KD21;KD21,KD22];
          % P matrix
P         =  KD*inv(mass);
```

```
          % Determinant of P matrix
DETP      =  det(P);
%% SMC
          % Kv Matrix
Kv        =  [alpha*(KD11*r+KD21*r-KD21*d*cos(phi))^2/r^2/(m*d^2+Ib-...
            m*d^2*cos(phi)^2)^2,alpha*(KD11*r+KD21*r-KD21*d*cos(phi))/r^2/↙
(m*d^2+...
            Ib-m*d^2*cos(phi)^2)^2*(KD21*r+KD22*r-KD22*d*cos(phi));alpha*↙
(KD11*r+...
            KD21*r-KD21*d*cos(phi))/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2*↙
(KD21*r+...
            KD22*r-KD22*d*cos(phi)),alpha*(KD21*r+KD22*r-...
            KD22*d*cos(phi))^2/r^2/(m*d^2+Ib-m*d^2*cos(phi)^2)^2];

%% Evaluating the control law
          %FMC input
F1        =thetadot*nu-phidot*sigma;
          %SMC input
F2        =66.66666665*alpha*(3.*thetadot*KD11+3.*KD21*thetadot-2.↙
*thetadot*KD21*cos(phi)+3.*phidot*KD21+3.*phidot*KD22-...
            2.*phidot*KD22*cos(phi))/(-11.+cos(phi)^2);
          %TMC input
F3        =.1000000000e-2*(-1176.*sin(theta)*KD21^3+637.*sin(theta)↙
*KD22*KD21^2*cos(phi+epsilon)-15582.*KD11*KD21*sin((-...
            3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.↙
*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^↙
(1/2)*KD22+3.*theta*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)*KD22-882.*KD11*sin(theta)*KD22*KD21*cos(phi+epsilon)↙
^2+4410.*KD11*sin(phi+epsilon)*cos(theta)*KD21^2*cos(phi+epsilon)+...
            2352.*KD11*sin(theta)*KD22*cos(phi+epsilon)*KD21-2646.*KD11^3*sin↙
(theta)-24696.*KD21^3*sin((-3.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos↙
```

```
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan↙
((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.↙
*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan↙
((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)/sin↙
(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/↙
(3.*KD11+KD21)/KD21)-1485.*KD11^2*F6*sinh((-...
            6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-2.*KD22*atan((cos↙
(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)-20727.*KD11^2*KD22*sin((-3.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))↙
*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos↙
(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/↙
(3.*KD11+KD21)/KD21)-4410.*KD11*sin(theta)*KD21^2*cos(phi+epsilon)^2-...
            1764.*KD21^3*sin(phi+epsilon)*cos(theta)-294.*sin(theta)↙
*KD22*KD21^2+2548.*cos(phi+epsilon)*sin(theta)*KD21^3-3969.*KD11^3*sin↙
(phi+epsilon)*cos(theta)+...
            15.*KD21*F6*sinh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-↙
...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.↙
*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD22*cos(phi+epsilon)^2-↙
```

```
1176.*cos(phi+epsilon)^2*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5. ↙
*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5. ↙
*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan↙
((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))↙
*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos↙
(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22*KD21^2-↙
2475.*KD11*F6*sinh((-6.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))↙
*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/KD21)*KD21+3528.*KD11*KD21*sin((-3.*phi*KD21*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.↙
*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan↙
((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)↙
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD22*cos(phi+epsilon)-882.↙
*KD11^2*KD22*sin(theta)+882.*KD11*KD22*sin(phi+...
            epsilon)*cos(theta)*cos(phi+epsilon)*KD21-1176.*KD11*KD22*sin↙
(theta)*KD21-10584.*KD11^2*sin(phi+epsilon)*cos(theta)*KD21-...
            2646.*KD11^2*sin(theta)*KD21*cos(phi+epsilon)^2-40425.↙
*KD11*KD21^2*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-↙
...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.↙
```

```
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)+882.*KD11^2*KD22*sin((-3.*phi*KD21*((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.↙
*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan↙
((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)↙
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)/sin↙
(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*cos↙
(phi+epsilon)+1764.*KD11^2*KD22*cos(phi+epsilon)^2*sin((-3.*phi*KD21*((3.↙
*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)↙
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)↙
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)+2646.*KD11^2*sin(phi+epsilon)*cos(theta)*cos↙
```

```
(phi+epsilon)*KD21+225.*KD11*KD21*F6*sinh((-6.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+...
            epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/KD21)*cos(phi+epsilon)^2+2646.*KD11^3*cos↙
(phi+epsilon)*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)↙
/sin(phi+...
            epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-...
            660.*KD21^2*F6*sinh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))-2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)/KD21)-...
            7791.*KD21^2*KD22*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+...
            epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.↙
*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.↙
*KD11+...
```

```
            5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/
(3.*KD11+KD21)/KD21)-5586.*KD11*sin(theta)*KD21^2-...
            1176.*sin(theta)*KD21^3*cos(phi+epsilon)^2+45.*KD11*cos
(phi+epsilon)^2*F6*sinh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)
-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-2.*KD22*atan
((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD22-...
            7056.*KD11^2*sin(theta)*KD21+27832.*KD21^3*sin((-3.*phi*KD21*((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)
*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)*cos(phi+epsilon)+294.*KD22*sin(phi+epsilon)*cos
(theta)*KD21^2*cos(phi+epsilon)-3969.*KD11^3*sin((-3.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)
*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)
*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)
*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)+12348.*KD11^2*cos(phi+epsilon)*sin(theta)*KD21+1323.
*KD11^2*cos(phi+epsilon)*sin(theta)*KD22-1323.*KD11^2*KD22*sin(phi+...
            epsilon)*cos(theta)+1176.*sin(phi+epsilon)*cos(theta)*KD21^3*cos
(phi+epsilon)-1764.*KD11*sin(phi+epsilon)*cos(theta)*KD22*KD21-...
```

```
            32634.*KD11^2*KD21*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^↙
(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-...
            1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))↙
*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-↙
18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.↙
*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.↙
*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)↙
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)/sin↙
(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)-5292.↙
*KD21^3*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+...
            epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)*cos(phi+epsilon)^2+135.*KD11^2*cos(phi+epsilon)↙
^2*F6*sinh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+...
            epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin↙
(phi+epsilon))-2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+...
            epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/KD21)-7644.*KD11*cos(phi+epsilon)^2*sin((-3.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
```

```
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD21^2+60.
*KD21^2*F6*sinh((-6.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))
^(1/2)/sin(phi+epsilon))-2.*KD22*atan((cos(phi+...
            epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))
^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*cos(phi+epsilon)^2+6958.
*KD22*KD21^2*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan
((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos
(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/
(3.*KD11+KD21)/KD21)*cos(phi+epsilon)-...
            165.*KD22*KD21*F6*sinh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-2.*KD22*atan((cos(phi+epsilon)
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)-...
            1764.*KD11^2*KD21*sin((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.
*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^
(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.
```

```
*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+...
            epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin↙
(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+...
            epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin↙
(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)-...
            1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+...
            epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+...
            KD21)/KD21)*cos(phi+epsilon)^2+34986.*KD11*KD21^2*sin((-3.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            4.*phi*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)-1.↙
*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+...
            3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+...
            18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+...
            6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-...
            18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-...
            6.*KD21^3*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*cos↙
(phi+epsilon)+14112.*KD11^2*cos(phi+epsilon)*sin((-3.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)↙
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)↙
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos(phi+epsilon)-1.)/sin↙
(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD21+3969.↙
*KD11^3*cos(phi+epsilon)*sin(theta)-495.*KD11*KD22*F6*sinh((-...
```

```
            6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3. ↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3. ↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+...
            epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)↙
/KD21)-294.*KD22*cos(phi+epsilon)^2*sin(theta)*KD21^2-...
            441.*sin(phi+epsilon)*cos(theta)*KD22*KD21^2+11319.*KD11*sin↙
(theta)*KD21^2*cos(phi+epsilon)-8379.*KD11*sin(phi+...
            epsilon)*cos(theta)*KD21^2)/(9.*KD11^2+15.*KD11*KD21+3.↙
*KD11*KD22+4.*KD21^2+KD21*KD22)/(3.*KD11+3.*KD21-2.*cos(phi+epsilon)*KD21);

tau      = F1+F2+F3;

%%  Lyapunov
Fmc1     = [-.2000000000e-8*(-.1766666666e11*phidot*sin(phi)*KD22-.↙
1766666666e11*KD21*sin(phi)*thetadot+...
            2000000000.*cos(phi)*thetadot*KD11*sin(phi)+3999999999.*cos(phi)↙
*phidot*KD22*sin(phi)-...
            2999999999.*thetadot*KD11*sin(phi)+3999999999.*cos(phi+epsilon)↙
*thetadot*sin(phi)*KD21-2000000000.*cos(phi)*phidot*sin(phi)*KD21+...
            2999999999.*phidot*KD21*sin(phi)+6000000000.*sin(phi)↙
*phidot*KD11+.1333333333e12*cos(phi)*nu*KD21-...
            .2000000000e12*nu*KD21-.2000000000e12*nu*KD11)/(-11.+cos(phi)↙
^2),.1000000000e-6*(-30000000.*phidot*KD21*sin(phi)-...
            20000000.*cos(phi)*phidot*sin(phi)*KD21+40000000.*cos(phi)↙
*phidot*KD22*sin(phi)-60000000.*phidot*sin(phi)*KD22+...
            30000000.*sin(phi)*phidot*KD11+2000000000.*nu*KD22+1333333333.↙
*cos(phi)*sigma*KD21-1333333333.*cos(phi)*nu*KD22-...
            2000000000.*sigma*KD21+2000000000.*nu*KD21-2000000000.↙
*sigma*KD11)/(-11.+cos(phi)^2);.1000000000e-6*(-...
            30000000.*phidot*KD21*sin(phi)-20000000.*cos(phi)*phidot*sin(phi)↙
*KD21+40000000.*cos(phi)*phidot*KD22*sin(phi)-...
            60000000.*phidot*sin(phi)*KD22+30000000.*sin(phi)↙
*phidot*KD11+2000000000.*nu*KD22+1333333333.*cos(phi)*sigma*KD21-...
            1333333333.*cos(phi)*nu*KD22-2000000000.*sigma*KD21+2000000000.↙
*nu*KD21-2000000000.*sigma*KD11)/(-11.+...
            cos(phi)^2),.1000000000e-8*(5999999998.*phidot*KD21*sin(phi)↙
+5999999998.*phidot*sin(phi)*KD22-...
            3999999999.*cos(phi)*phidot*KD22*sin(phi)-.↙
3999999999e12*sigma*KD21-.3999999999e12*KD22*sigma+...
            .2666666666e12*cos(phi)*KD22*sigma)/(-11.+cos(phi)^2)];

PHI      =.2000000000*(-196.*cos((-3.*phi*KD21*((3.*KD11+KD21)*(3.*KD11+5.↙
*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+KD21)*↙
(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
```

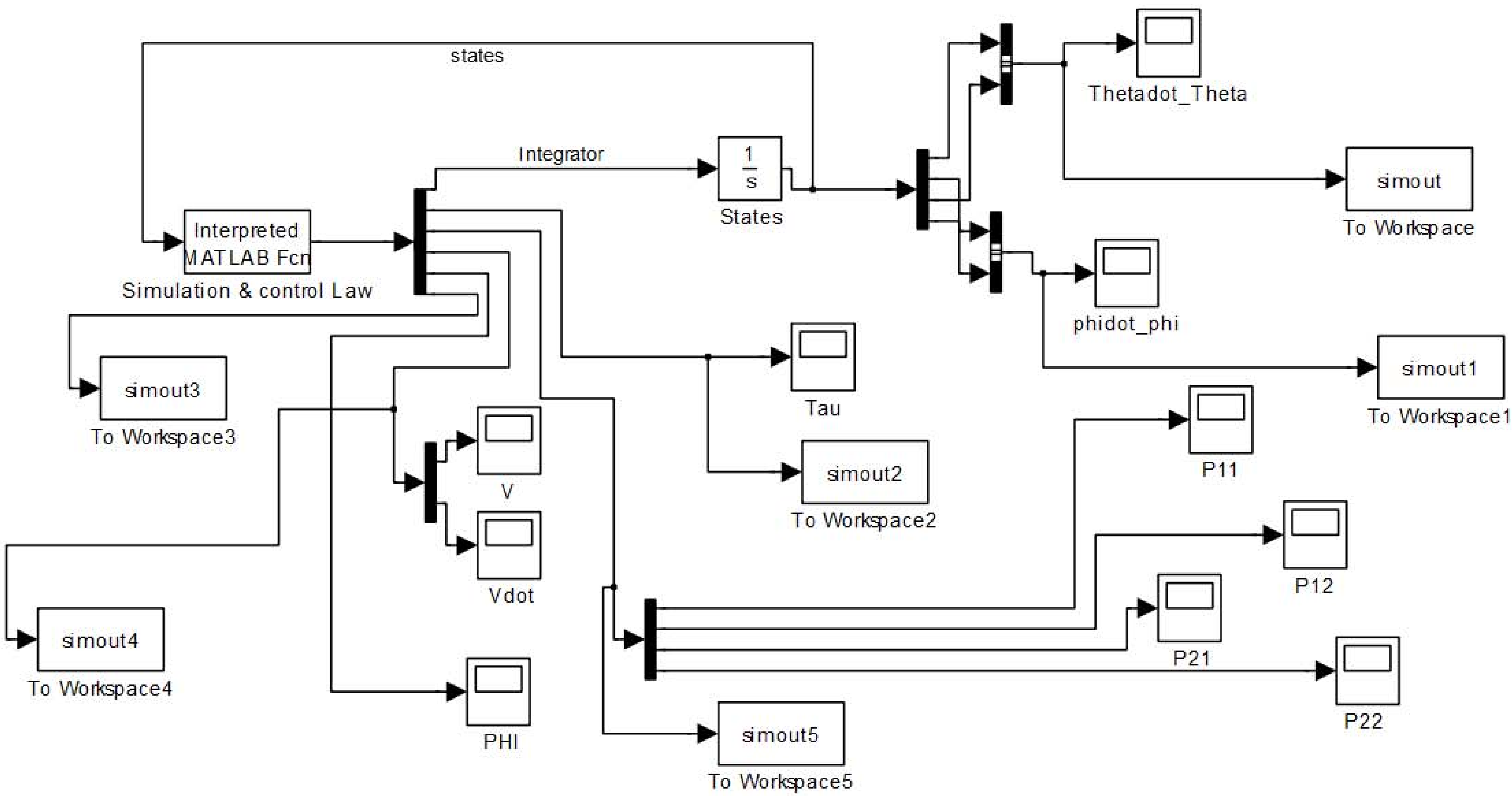
```
^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+...
            5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))↙
^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))↙
*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-2.*KD22*atan((cos↙
(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/↙
(3.*KD11+KD21)/KD21)*KD11*KD22+196.*cos((-3.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-4.*phi*KD21^2*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)-1.*phi*KD21*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)*KD22+3.*theta*KD21*((3.*KD11+KD21)↙
*(3.*KD11+5.*KD21))^(1/2)*KD11+theta*KD21^2*((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)+18.*atan((3.*KD11+5.*KD21)*(cos↙
(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11^2*KD22+6.*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22*KD21-18.*KD21^2*atan↙
((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11-6.*KD21^3*atan((3.*KD11+5.↙
*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))-6.*KD22*atan((cos(phi+epsilon)↙
-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD11-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)*KD21)/((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/(3.*KD11+KD21)/KD21)*KD21^2+15.*F6*cosh((-6.↙
*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-2.*KD22*atan↙
((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+5.*KD21))^↙
(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/((3.*KD11+...
            KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD11+...
            20.*F6*cosh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)↙
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.*KD11+KD21)*(3.↙
*KD11+5.*KD21))^(1/2)/KD21)*KD21+...
            5.*F6*cosh((-6.*KD21^2*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)↙
-1.)/((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))-...
            2.*KD22*atan((cos(phi+epsilon)-1.)/sin(phi+epsilon))*((3.↙
*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)+theta*KD21*((3.*KD11+KD21)*(3.*KD11+...
            5.*KD21))^(1/2)+6.*atan((3.*KD11+5.*KD21)*(cos(phi+epsilon)-1.)/↙
((3.*KD11+KD21)*(3.*KD11+5.*KD21))^(1/2)/sin(phi+epsilon))*KD11*KD22)/((3.↙
*KD11+...
```

```
            KD21)*(3.*KD11+5.*KD21))^(1/2)/KD21)*KD22)/(3.*KD11+4.↙
*KD21+KD22);
Vc       =  -210;
V        =  0.5*qdot'*KD*qdot+PHI+Vc;
Vdot     =  -qdot'*(Kv+Fmc1)*qdot;

%% Evaluate the dynamic

qdotdot  = inv(mass)*([tau;0]-C*qdot-G);
ddtheta  = qdotdot(1);
ddphi    = qdotdot(2);
%% M-File Output
dxdt     =[thetadot;phidot;ddtheta;ddphi;tau(1);P(1,1);P(1,2);P(2,1);P(2,2);↙
V;Vdot;PHI;DETP];
%% End of the Function
```

B.5 Simulink file for the ball and arc system, KD constant



B.6 Direct Lyapunov Aproach for the Ball andArc System P
B&Arc_DLA_Fm1_P.mw

> $restart:$

> $with(LinearAlgebra):$

> $q := \begin{bmatrix} \theta \\ \phi \end{bmatrix}:$

>

q is a vector of generalized coordinates

> $qdot := \begin{bmatrix} \theta dot \\ \phi dot \end{bmatrix}:$

qdot is a vector of generalized velocites

> $C := -Matrix(2, 2, [-m\, r\, d \sin(\phi)\ \phi dot, -m\, r\, d \sin(\phi)\ \theta dot + m\, r\, d \sin(\phi)\ \phi dot, m\, r\, d \sin(\phi)\ \theta dot, 0]);$

$$C := \begin{bmatrix} m\, r\, d \sin(\phi)\ \phi dot & m\, r\, d \sin(\phi)\ \theta dot - m\, r\, d \sin(\phi)\ \phi dot \\ -m\, r\, d \sin(\phi)\ \theta dot & 0 \end{bmatrix} \tag{1.1}$$

Cmatrix is the matrix of Centripetal* and *coriolis forces

Define the mass matrix

>

> $mass := -simplify\left(Matrix\left(2, 2, \left[2\, m\, r\, d \cos(\theta)^2 \cos(\phi) - m\, r^2 + 2\, m\, r\, d \sin(\theta)^2 \cos(\phi) - m\, d^2 - Ib, -m\, r\, d \cos(\theta)^2 \cos(\phi) + m\, r^2 - m\, r\, d \sin(\theta)^2 \cos(\phi), -m\, r\, d \cos(\theta)^2 \cos(\phi) + m\, r^2 - m\, r\, d \sin(\theta)^2 \cos(\phi), -m\, r^2\right]\right)\right);$

$$mass := \begin{bmatrix} m\, r^2 - 2\, m\, r\, d \cos(\phi) + m\, d^2 + Ib & m\, r\,(-r + d \cos(\phi)) \\ m\, r\,(-r + d \cos(\phi)) & m\, r^2 \end{bmatrix} \tag{1.2}$$

> $IMass := MatrixInverse(mass):$

> $G := -\begin{bmatrix} m\, g\,(r \sin(\phi) \cos(\theta) - r \cos(\phi) \sin(\theta) + d \sin(\theta)) \\ -m\, g\, r\,(\sin(\phi) \cos(\theta) - \cos(\phi) \sin(\theta)) \end{bmatrix}:$

> $Fm1 := \begin{bmatrix} -F11(qf, qdotf) \cdot \phi dot - \nu & F11(qf, qdotf) \cdot \theta dot - \sigma \\ -F22(qf, qdotf) \cdot \phi dot & F22(qf, qdotf) \cdot \theta dot \end{bmatrix}:$

Fmc1 **control law matrix for FMC, Eq. 4.14**

$$> Fmc1 := \begin{bmatrix} \frac{1}{F33(qf, qdotf)} & \frac{1}{F33(qf, qdotf)} \\ \frac{1}{F33(qf, qdotf)} & \frac{1}{F33(qf, qdotf)} \end{bmatrix}:$$

\>

\>

First Matching Condition

$$> KDT := simplify(Multiply(\mathrm{Matrix}(2, 2, [p11, \mathrm{p12}, \mathrm{p21}, \mathrm{p22}]), mass));$$

$$KDT := [[p11\, m\, r^2 - 2\, p11\, m\, r\, d\cos(\phi) + p11\, m\, d^2 + p11\, Ib - p12\, m\, r^2 + p12\, m\, r\, d\cos(\phi), m\, r\,(-p11\, r + p11\, d\cos(\phi) + p12\, r)], [p21\, m\, r^2 - 2\, p21\, m\, r\, d\cos(\phi) + p21\, m\, d^2 + p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d\cos(\phi), m\, r\,(-p21\, r + p21\, d\cos(\phi) + p22\, r)]] \tag{2.1}$$

Forcing the symmetry condition p12 is

$$> solve\left(KDT_{1,\,2} - KDT_{2,\,1}, p12\right)$$

$$\frac{1}{m\, r^2}\left(p11\, m\, r^2 - p11\, m\, r\, d\cos(\phi) + p21\, m\, r^2 - 2\, p21\, m\, r\, d\cos(\phi) + p21\, m\, d^2 + p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d\cos(\phi)\right) \tag{2.2}$$

$$> KDT := simplify\left(eval\left(KDT, \mathrm{p12} = \frac{1}{m\, r^2}\left(p11\, m\, r^2 - p11\, m\, r\, d\cos(\phi) + p21\, m\, r^2 - 2\, p21\, m\, r\, d\cos(\phi) + p21\, m\, d^2 + p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d\cos(\phi)\right)\right)\right);$$

$$KDT := \left[\left[\frac{1}{r}\left(p11\, m\, d^2\, r + p11\, Ib\, r - p21\, m\, r^3 + 3\, p21\, m\, r^2\, d\cos(\phi) - p21\, m\, d^2\, r - p21\, Ib\, r + p22\, m\, r^3 - 2\, p22\, m\, r^2\, d\cos(\phi) - d^2\cos(\phi)^2\, p11\, m\, r - 2\, d^2\cos(\phi)^2\, p21\, m\, r + d^3\cos(\phi)\, p21\, m + d\cos(\phi)\, p21\, Ib + d^2\cos(\phi)^2\, p22\, m\, r\right), p21\, m\, r^2 - 2\, p21\, m\, r\, d\cos(\phi) + p21\, m\, d^2 + p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d\cos(\phi)\right], [p21\, m\, r^2 - 2\, p21\, m\, r\, d\cos(\phi) + p21\, m\, d^2 + p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d\cos(\phi), m\, r\,(-p21\, r + p21\, d\cos(\phi) + p22\, r)]\right] \tag{2.3}$$

\>

\>

symmetry test for KD

> $simplify(KDT_{1,2} - KDT_{2,1});$

$$0 \tag{2.4}$$

> $KDdot := \text{simplify}(map(diff, \theta dot \cdot KDT, \theta) + map(diff, \phi dot \cdot KDT, \phi));$

$$KDdot := \left[\left[-\frac{1}{r}\left(\phi dot\, d \sin(\phi)\, (3\, p21\, m\, r^2 - 2\, p22\, m\, r^2 - 2\, p11\, m\, r\, d \cos(\phi) - 4\, p21\, m\, r\, d \cos(\phi) + p21\, m\, d^2 + p21\, Ib + 2\, p22\, m\, r\, d \cos(\phi))\right), \phi dot\, m\, r\, d \sin(\phi)\, (2\, p21 - p22)\right], \left[\phi dot\, m\, r\, d \sin(\phi)\, (2\, p21 - p22), -\phi dot\, m\, r\, p21\, d \sin(\phi)\right]\right] \tag{2.5}$$

First Matching Condition with inputFm1 and matrix Fmc1

> $FMCsim := simplify(\text{KDdot} + \text{Multiply}(KDT, \text{Multiply}(IMass, (Fm1 - C))) + Transpose(\text{Multiply}(KDT, \text{Multiply}(IMass, (Fm1 - C)))) + \text{Fmc1}) :$

> $Fm1 := simplify(Multiply(Fm1, qdot));$

$$Fm1 := \begin{bmatrix} -\theta dot\, \nu - \phi dot\, \sigma \\ 0 \end{bmatrix} \tag{2.6}$$

symmetry test for FMC

> $simplify(FMCsim_{1,2} - FMCsim_{2,1});$

$$0 \tag{2.7}$$

> $sys := simplify([FMCsim_{1,1}, FMCsim_{2,1}, FMCsim_{2,2}]) :$

Solving for the forces on sys the result is going to be called *forsol*

> $forsol := pdsolve(sys, [F11(qf, qdotf), F22(qf, qdotf), F33(qf, qdotf)]);$

$$forsol := \Big\{ F11(qf, qdotf) = \frac{1}{2}\Big(-d\Big(2\, d^2\, m^2\, \theta dot\, r^2\, (2\, p21 + p11 - p22)^2\, (-\theta dot + \phi dot) \cos(\phi)^2 - 3\, d\Big(\Big(\Big(\frac{2}{3}\, \theta dot\, (-3\, \theta dot + \phi dot)\, p21 - \frac{2}{3}\, ((-2\, p11 + 4\, p22)\, \theta dot + \phi dot\, p11)\, (-\theta dot + \phi dot)\Big) r^2 + d^2\, p21\, \theta dot\Big(-\frac{4}{3}\, \theta dot + \phi dot\Big)\Big) m + Ib\, p21\, \theta dot\Big(-\frac{4}{3}\, \theta dot + \phi dot\Big)\Big)(2\, p21 + p11 - p22)\, m\, r \cos(\phi) + \Big((-4\, \theta dot^2\, p21^2 + ((10\, p22 - 6\, p11)\, \theta dot^2 + 4\, \phi dot\, (p11 - 2\, p22)\, \theta dot - 2\, \phi dot^2\, (p22 + p11))\, p21 - 2\, (p11 - p22)\, (\tag{2.8}$$

$$-\theta dot+\phi dot)\ \left((-p11+4\,p22)\ \theta dot+\phi dot\,p11)\right)\ r^4-2\,d^2\left(-\frac{1}{2}\ \theta dot\ (\phi dot\right.$$

$$-6\ \theta dot)\ p21+(2\,p11-4\,p22)\ \theta dot^2-\frac{5}{2}\ \phi dot\ (p11-p22)\ \theta dot+\Bigg($$

$$-\frac{1}{2}\ p22+p11\Bigg)\ \phi dot^2\Bigg)\ p21\ r^2+d^4\ \theta dot\,p21^2\ (\phi dot-2\ \theta dot)\Bigg)\ m^2$$

$$+2\ Ib\left(\left(\frac{1}{2}\ \theta dot\ (\phi dot-6\ \theta dot)\ p21+(-2\,p11+4\,p22)\ \theta dot^2\right.\right.$$

$$+\frac{5}{2}\ \phi dot\ (p11-p22)\ \theta dot-\left(-\frac{1}{2}\ p22+p11\right)\ \phi dot^2\Bigg)\ r^2$$

$$+d^2\ \theta dot\,p21\ (\phi dot-2\ \theta dot)\Bigg)\ p21\ m+\theta dot\,p21^2\ Ib^2\ (\phi dot-2\ \theta dot)\Bigg)$$

$$\sin(\phi)-2\left(-d\ (2\,p21+p11-p22)\left(\left(-\sigma\ \theta dot-2\left(-\frac{1}{2}\ \nu\right.\right.\right.\right.$$

$$+\sigma\Big)\ \phi dot\Big)\ p21+p11\ (\phi dot\,\sigma+\theta dot\,\nu)\Big)\ m\,r\cos(\phi)+\Bigg(\Bigg(\Bigg(-\sigma\ \theta dot-2\Bigg($$

$$-\frac{1}{2}\ \nu+\sigma\Big)\ \phi dot\Big)\ p21^2+\Big(\big((-\sigma+\nu)\ p11+p22\ (\sigma+\nu)\big)\ \theta dot-\phi dot\Big((\sigma$$

$$-\nu)\ p11-3\,p22\left(\sigma-\frac{1}{3}\ \nu\right)\Big)\Big)\ p21+\big((p11\,\nu+p22\ (\sigma-3\,\nu)\big)\ \theta dot$$

$$+(\sigma\,p11-p22\ (\sigma+\nu))\ \phi dot)\ p11\Big)\ r^2+d^2\left(\left(-\sigma\ \theta dot-2\left(-\frac{1}{2}\ \nu\right.\right.\right.$$

$$+\sigma\Big)\ \phi dot\Big)\ p21+p11\ (\phi dot\,\sigma+\theta dot\,\nu)\Big)\ p21\Big)\ m+Ib\,p21\left(\left(-\sigma\ \theta dot-2\right.\right.\Bigg($$

$$-\frac{1}{2}\ \nu+\sigma\Big)\ \phi dot\Big)\ p21+p11\ (\phi dot\,\sigma+\theta dot\,\nu)\Big)\Big)\ r\Big)\Big/\Big((\phi dot+\theta dot)^2\ ($$

$$-m\,r\,d\,p21\ (2\,p21+p11-p22)\cos(\phi)+((p21-p22)\ (p21+p11)\ r^2$$

$$+p21^2\ d^2)\ m+p21^2\ Ib)\ r\Big),\ F22(qf,qdotf)=-\frac{1}{2}\ \Big(m\Big(d\Big(-2\,d\ \big(\big(\phi dot$$

$+ 2\ \theta dot)\ p21 - p11\ \theta dot)\ (2\ p21 + p11 - p22)\ m\ (-\theta dot + \phi dot)\ r\cos(\phi)$

$+ (((4\ \phi dot^2 + 4\ \theta dot\ \phi dot - 4\ \theta dot^2)\ p21^2 + ((6\ p22 - 2\ p11)\ \theta dot^2$

$+ 4\ \phi dot\ (p11 - p22)\ \theta dot + 2\ \phi dot^2\ (p11 - p22))\ p21 + 2\ p11\ (-\theta dot$

$+ \phi dot)^2\ (p11 - p22))\ r^2 + d^2\ (\phi dot - 2\ \theta dot)\ ((\phi dot + 2\ \theta dot)\ p21$

$- p11\ \theta dot)\ p21)\ m + (\phi dot - 2\ \theta dot)\ ((\phi dot + 2\ \theta dot)\ p21$

$- p11\ \theta dot)\ Ib\ p21)\ \sin(\phi) - 2\ r\ (-p21 + p11)^2\ (\phi dot\ \sigma + \theta dot\ \nu))\ r)\ /$

$((\phi dot + \theta dot)^2\ (-m\ r\ d\ p21\ (2\ p21 + p11 - p22)\cos(\phi) + ((p21$

$- p22)\ (p21 + p11)\ r^2 + p21^2\ d^2)\ m + p21^2\ Ib)),\ F33(qf, qdotf)$

$= (r\ \phi dot^2 + 2\ r\ \phi dot\ \theta dot + \theta dot^2\ r)\ /\ (4\ \theta dot^2\ m\ p11\ r^2\ d\sin(\phi)\ \phi dot + 2\ \theta dot\ p11\ r\ \phi dot\ \sigma$
$- 2\ \theta dot^2\ \phi dot\ d^2\sin(\phi)\ r\ m\ p11\cos(\phi) - 2\ \theta dot^3\ p22\ m\ r\ d^2\cos(\phi)\sin(\phi)$
$- 2\ \theta dot\ m\ p11\ r^2\ \phi dot^2\ d\sin(\phi) + p21\ \theta dot^2\ \phi dot\ d\sin(\phi)\ Ib$
$+ 3\ p21\ \theta dot^2\ \phi dot\ m\ r^2\ d\sin(\phi) - 4\ p21\ \theta dot^2\ m\ d^2\cos(\phi)\ r\sin(\phi)\ \phi dot$
$+ 2\ \theta dot^2\ \phi dot\ d^2\sin(\phi)\ r\ m\ p22\cos(\phi) + 2\ \theta dot^3\ m\ p11\ r\ d^2\cos(\phi)\sin(\phi)$
$- 4\ \theta dot^2\ \phi dot\ m\ r^2\ d\sin(\phi)\ p22 + 2\ \theta dot^3\ r^2\ m\ p22\ d\sin(\phi)$
$- 2\ p21\ \theta dot^3\ Ib\ d\sin(\phi) - 2\ \theta dot^3\ m\ p11\ r^2\ d\sin(\phi)$
$+ p21\ \theta dot^2\ \phi dot\ m\ d^3\sin(\phi) + 4\ p21\ \theta dot^3\ m\ d^2\cos(\phi)\ r\sin(\phi)$
$- 2\ p21\ \theta dot^3\ m\ r^2\ d\sin(\phi) - r^2\ m\ \phi dot^3\ p21\ d\sin(\phi)$
$+ 2\ r^2\ m\ \phi dot^2\ \theta dot\ d\sin(\phi)\ p22 + 2\ r\ \phi dot\ p21\ \theta dot\ \nu + 2\ r\ \phi dot^2\ p21\ \sigma$
$+ 2\ \theta dot^2\ p11\ r\ \nu - 2\ p21\ \theta dot^3\ m\ d^3\sin(\phi))\}$

Testing the forces on the First matching condition equation the result is going to be called *FMCtest*

> FMCtest := simplify(*eval*(*FMCsim*, *forsol*));

$$FMCtest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.9}$$

>

FMC is satisfied!

> $Fm1 := simplify(eval(Fm1, forsol));$

$$Fm1 := \begin{bmatrix} -\theta dot\,\nu - \phi dot\,\sigma \\ 0 \end{bmatrix} \tag{2.10}$$

> $Fmc1 := simplify(eval(Fmc1, forsol)) :$

> $\#f33sol := solve\big(-r\,\phi dot^2\,F33(qf, qdotf) - \theta dot^2\,F33(qf, qdotf)\,r + 2\,\theta dot^2\,p11\,r\,\nu + 2\,r\,\phi dot^2\,p21\,\sigma - \theta dot^2\,f33\,\phi dot^2\,r - 2\,p21\,\theta dot^3\,r^2\,m\,d\sin(\phi) - m\,r^2\,\phi dot^3\,p21\,d\sin(\phi) - 2\,p21\,\theta dot^3\,Ib\,d\sin(\phi) + 2\,\theta dot^3\,m\,r^2\,p22\,d\sin(\phi) - 2\,\theta dot^3\,p11\,r^2\,m\,d\sin(\phi) - 2\,p21\,\theta dot^3\,m\,d^3\sin(\phi) + 4\,p21\,\theta dot^3\,d^2\cos(\phi)\,m\,r\sin(\phi) - 2\,\theta dot^3\,d^2\cos(\phi)\,p22\,m\,r\sin(\phi) + 2\,\theta dot^3\,p11\,r\,d^2\cos(\phi)\,m\sin(\phi),\ F33(qf, qdotf)\big);$

> $\#Fmc1 := simplify(eval(Fmc1, F33(qf, qdotf) = f33sol));$

> $solve\big(Fmc1_{1,1} - r\,(\theta dot^2 + \phi dot^2), f33\big)$

> $solve\big(Fmc1_{2,2} - r\,(\theta dot^2 + \phi dot^2), f44\big)$

> $F1 := simplify(\mathrm{eval}(Fm1_1));$

$$F1 := -\theta dot\,\nu - \phi dot\,\sigma \tag{2.11}$$

> $convert(KDT, string)$

"Matrix(2, 2, [[(p11*m*d^2*r+p11*Ib*r-p21*m*r^3+3*p21*m*r^2*d*cos(phi)-p21*m*d^2*r-p21*Ib*r+p22*m*r^3-2*p22*m*r^2*d*cos(phi)-d^2*cos(phi)^2*p11*m*r-2*d^2*cos(phi)^2*p21*m*r+d^3*cos(phi)*p21*m+d*cos(phi)*p21*Ib+d^2*cos(phi)^2*p22*m*r)/r,p21*m*r^2-2*p21*m*r*d*cos(phi)+p21*m*d^2+p21*Ib-p22*m*r^2+p22*m*r*d*cos(phi)],[p21*m*r^2-2*p21*m*r*d*cos(phi)+p21*m*d^2+p21*Ib-p22*m*r^2+p22*m*r*d*cos(phi),m*r*(-p21*r+p21*d*cos(phi)+p22*r)]])" **(2.12)**

> $simplify(Determinant(KDT))$

$$-p21^2\,m^2\,d^4 - p21^2\,Ib^2 + m\,p21\,Ib\,r^2\,p22 - p11\,m^2\,d^2\,r^2\,p21 + p11\,m^2\,d^2\,r^2\,p22 + p21\,m^2\,d^2\,r^2\,p22 + m\,d^2\cos(\phi)^2\,p21^2\,Ib + p11\,m^2\,d^3\,r\,p21\cos(\phi) + m\,p11\,Ib\,r\,p21\,d\cos(\phi) + p21^2\,m^2\,r^2\,d^2\cos(\phi)^2 + 2\,p21^2\,m^2\,d^3\,r\cos(\phi) - 2\,d^3\cos(\phi)^3\,p21^2\,m^2\,r - p22\,m^2\,r^2\,d^2\cos(\phi)^2\,p21 + d^2\cos(\phi)^2\,p11\,m^2\,r^2\,p21 - d^3\cos(\phi)^3\,p11\,m^2\,r\,p21 - d^2\cos(\phi)^2\,p11\,m^2\,r^2\,p22 - m\,p21^2\,Ib\,r^2 \tag{2.13}$$

$$- p21^2 m^2 d^2 r^2 - 2\, p21^2 m\, d^2\, Ib + d^4 \cos(\phi)^2\, p21^2 m^2 - m\, p11\, Ib\, r^2\, p21 + m\, p11\, Ib\, r^2\, p22 - d^3 \cos(\phi)\, p21\, m^2\, p22\, r - m\, d \cos(\phi)\, p21\, Ib\, p22\, r + d^3 \cos(\phi)^3\, p22\, m^2\, r\, p21 + 2\, m\, p21^2\, Ib\, r\, d \cos(\phi)$$

> P := simplify(Multiply(KDT, MatrixInverse(mass)));

$$P := \left[\left[p11,\ \frac{1}{m\, r^2}\left(p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d \cos(\phi) + p11\, m\, r^2 - 2\, p21\, m\, r\, d \cos(\phi) + p21\, m\, d^2 + p21\, m\, r^2 - p11\, m\, r\, d \cos(\phi)\right)\right], \left[p21, p22\right]\right] \tag{2.14}$$

> Determinant(P);

$$-\frac{1}{m\, r^2}\left(-p22\, m\, p11\, r^2 + p21^2\, Ib - p21\, p22\, m\, r^2 + p21\, p22\, m\, r\, d \cos(\phi) + p21\, m\, p11\, r^2 - 2\, p21^2\, m\, r\, d \cos(\phi) + p21^2\, m\, d^2 + p21^2\, m\, r^2 - p21\, m\, p11\, r\, d \cos(\phi)\right) \tag{2.15}$$

> Determinant(KDT);

$$-p21^2 m^2 d^4 - p21^2 Ib^2 + m\, p21\, Ib\, r^2\, p22 - p11\, m^2 d^2 r^2\, p21 + p11\, m^2 d^2 r^2\, p22 + p21\, m^2 d^2 r^2\, p22 + m\, d^2 \cos(\phi)^2\, p21^2\, Ib + p11\, m^2 d^3\, r\, p21 \cos(\phi) + m\, p11\, Ib\, r\, p21\, d \cos(\phi) + p21^2 m^2 r^2 d^2 \cos(\phi)^2 + 2\, p21^2 m^2 d^3\, r \cos(\phi) - 2\, d^3 \cos(\phi)^3\, p21^2 m^2\, r - p22\, m^2 r^2 d^2 \cos(\phi)^2\, p21 + d^2 \cos(\phi)^2\, p11\, m^2 r^2\, p21 - d^3 \cos(\phi)^3\, p11\, m^2\, r\, p21 - d^2 \cos(\phi)^2\, p11\, m^2 r^2\, p22 - m\, p21^2\, Ib\, r^2 - p21^2 m^2 d^2 r^2 - 2\, p21^2 m\, d^2\, Ib + d^4 \cos(\phi)^2\, p21^2 m^2 - m\, p11\, Ib\, r^2\, p21 + m\, p11\, Ib\, r^2\, p22 - d^3 \cos(\phi)\, p21\, m^2\, p22\, r - m\, d \cos(\phi)\, p21\, Ib\, p22\, r + d^3 \cos(\phi)^3\, p22\, m^2\, r\, p21 + 2\, m\, p21^2\, Ib\, r\, d \cos(\phi) \tag{2.16}$$

> convert(KDT, string) ;

"Matrix(2, 2, [[(p11*m*d^2*r+p11*Ib*r-p21*m*r^3+3*p21*m*r^2*d*cos(phi)-p21*m*d^2*r-p21*Ib*r+p22*m*r^3-2*p22*m*r^2*d*cos(phi)-d^2*cos(phi)^2*p11*m*r-2*d^2*cos(phi)^2*p21*m*r+d^3*cos(phi)*p21*m+d*cos(phi)*p21*Ib+d^2*cos(phi)^2*p22*m*r)/r,p21*m*r^2-2*p21*m*r*d*cos(phi)+p21*m*d^2+p21*Ib-p22*m*r^2+p22*m*r*d*cos(phi)],[p21*m*r^2-2*p21*m*r*d*cos(phi)+p21*m*d^2+p21*Ib-p22*m*r^2+p22*m*r*d*cos(phi),m*r*(-p21*r+p21*d*cos(phi)+p22*r)]])" (2.17)

Second Matching Condition

> $Kv := \alpha \cdot Multiply(Column(P, [1]), Transpose(Column(P, [1])));$

$$Kv := \begin{bmatrix} \alpha\, p11^2 & \alpha\, p11\, p21 \\ \alpha\, p11\, p21 & \alpha\, p21^2 \end{bmatrix} \qquad \textbf{(3.1)}$$

> $convert(Kv, string)$

"Matrix(2, 2, [[alpha*p11^2,alpha*p11*p21],[alpha*p11*p21,alpha*p21^2]])" **(3.2)**

> $F2f := -Multiply(MatrixInverse(P), Kv);$

$$F2f := \Big[\Big[\big(p22\, m\, r^2\, \alpha\, p11^2\big) \big/ \big(-p22\, m\, p11\, r^2 + p21^2\, Ib - p21\, p22\, m\, r^2$$

$$+ p21\, p22\, m\, r\, d \cos(\phi) + p21\, m\, p11\, r^2 - 2\, p21^2\, m\, r\, d \cos(\phi) + p21^2\, m\, d^2$$

$$+ p21^2\, m\, r^2 - p21\, m\, p11\, r\, d \cos(\phi) \big) - \big(\big(p21\, Ib - p22\, m\, r^2$$

$$+ p22\, m\, r\, d \cos(\phi) + p11\, m\, r^2 - 2\, p21\, m\, r\, d \cos(\phi) + p21\, m\, d^2 + p21\, m\, r^2$$

$$- p11\, m\, r\, d \cos(\phi) \big)\, \alpha\, p11\, p21 \big) \big/ \big(-p22\, m\, p11\, r^2 + p21^2\, Ib - p21\, p22\, m\, r^2$$

$$+ p21\, p22\, m\, r\, d \cos(\phi) + p21\, m\, p11\, r^2 - 2\, p21^2\, m\, r\, d \cos(\phi) + p21^2\, m\, d^2$$

$$+ p21^2\, m\, r^2 - p21\, m\, p11\, r\, d \cos(\phi) \big),\ \big(p22\, m\, r^2\, \alpha\, p11\, p21\big) \big/ \big($$

$$-p22\, m\, p11\, r^2 + p21^2\, Ib - p21\, p22\, m\, r^2 + p21\, p22\, m\, r\, d \cos(\phi) + p21\, m\, p11\, r^2$$

$$- 2\, p21^2\, m\, r\, d \cos(\phi) + p21^2\, m\, d^2 + p21^2\, m\, r^2 - p21\, m\, p11\, r\, d \cos(\phi) \big)$$

$$- \big(\big(p21\, Ib - p22\, m\, r^2 + p22\, m\, r\, d \cos(\phi) + p11\, m\, r^2 - 2\, p21\, m\, r\, d \cos(\phi) + p21\, m\, d^2 + p21\, m\, r^2 -$$

$$+ p21^2\, m\, d^2 + p21^2\, m\, r^2 - p21\, m\, p11\, r\, d \cos(\phi) \big) \Big],$$

$$\Big[0, 0\Big]\Big]$$

> $simplify(Eigenvalues(Kv));$

$$\begin{bmatrix} 0 \\ \alpha\, p21^2 + \alpha\, p11^2 \end{bmatrix} \qquad \textbf{(3.4)}$$

> $F2 := simplify(Multiply(F2f, qdot));$

$$F2 := \begin{bmatrix} -\alpha\,(p11\ \theta dot + p21\ \phi dot) \\ 0 \end{bmatrix} \tag{3.5}$$

Third Matching Condition

> $PHM := Matrix\left(2, 1, \left[\frac{\partial}{\partial\theta}\Phi(\theta,\phi), \frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right]\right);$

$$PHM := \begin{bmatrix} \frac{\partial}{\partial\theta}\Phi(\theta,\phi) \\ \frac{\partial}{\partial\phi}\Phi(\theta,\phi) \end{bmatrix} \tag{4.1}$$

> $TMC := convert\left(\begin{bmatrix} F3 \\ 0 \end{bmatrix}, Matrix\right) - convert(G, Matrix)$
> $+ Multiply(MatrixInverse(P), PHM):$

> $TMCphi := TMC_{2,1};$

$$TMCphi := -m\,g\,r\,(\sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta)) + \left(p21\ m\ r^2\left(\frac{\partial}{\partial\theta}\Phi(\theta,\phi)\right)\right) \Big/ \left(-p22\ m\ p11\ r^2 + p21^2\ Ib - p21\ p22\ m\ r^2 + p21\ p22\ m\ r\ d\cos(\phi) + p21\ m\ p11\ r^2 - 2\ p21^2\ m\ r\ d\cos(\phi) + p21^2\ m\ d^2 + p21^2\ m\ r^2 - p21\ m\ p11\ r\ d\cos(\phi)\right) - \left(p11\ m\ r^2\left(\frac{\partial}{\partial\phi}\Phi(\theta,\phi)\right)\right) \Big/ \left(-p22\ m\ p11\ r^2 + p21^2\ Ib - p21\ p22\ m\ r^2 + p21\ p22\ m\ r\ d\cos(\phi) + p21\ m\ p11\ r^2 - 2\ p21^2\ m\ r\ d\cos(\phi) + p21^2\ m\ d^2 + p21^2\ m\ r^2 - p21\ m\ p11\ r\ d\cos(\phi)\right) \tag{4.2}$$

> $solphi := pdsolve(TMCphi);$

$$solphi := \Phi(\theta,\phi) = \frac{1}{p21\ r}\Bigg(g\ \Bigg(-\frac{1}{2}\ \frac{1}{p21 + 2\ p11}\Bigg(m\ r\ d\ p21^2\ (2\ p21 + p11 - p22)\cos\left(\frac{(p21 + 2\ p11)\ \theta}{p21} + \frac{-2\ \phi\ p21 - 2\ p11\ \theta}{p21}\right)\Bigg) \tag{4.3}$$

$$+\frac{1}{p21+p11}\Bigg(\Big(\big(\big((d^2+r^2)\,m+Ib\big)\,p21^2+m\,r^2\,(p11-p22)\,p21$$

$$-p22\,m\,p11\,r^2\big)\,p21\cos\left(\frac{(p21+p11)\,\theta}{p21}+\frac{-\phi\,p21-p11\,\theta}{p21}\right)\Big)$$

$$-\frac{1}{2}\,d\cos(\theta)\,m\,r\,p21\,(2\,p21+p11-p22)\Bigg)\Bigg)+\_F1\left(\frac{\phi\,p21+p11\,\theta}{p21}\right)$$

> $f3 := simplify\big(F3-(TMC_{1,\,1})\big);$

$$f3:=-\Bigg(m^2\,g\,d\sin(\theta)\,p21^2\,r^2+m^2\,g\,d\sin(\theta)\,p21\,p11\,r^2-\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p22\,m\,r^2+\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p11\,m\,r^2+\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p21\,m\,d^2$$ **(4.4)**

$$+\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p21\,m\,r^2+\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p21\,Ib+\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p22\,m\,r\,d\cos(\phi)-2\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p21\,m\,r\,d\cos(\phi)-\left(\frac{\partial}{\partial\phi}\,\Phi(\theta,\phi)\right)p11\,m\,r\,d\cos(\phi)+2\,m^2\,g\,r^2\cos(\phi)^2\sin(\theta)\,p21^2\,d$$

$$-3\,m^2\,g\,r\cos(\phi)\sin(\theta)\,p21^2\,d^2+m\,g\,d\sin(\theta)\,p21^2\,Ib+m^2\,g\,d^3\sin(\theta)\,p21^2$$

$$-m^2\,g\,d\sin(\theta)\,p22\,p11\,r^2-m^2\,g\,d\sin(\theta)\,p21\,p22\,r^2$$

$$-m^2\,g\,d^2\sin(\theta)\,p21\,p11\,r\cos(\phi)+m^2\,g\,d^2\sin(\theta)\,p21\,p22\,r\cos(\phi)$$

$$-m^2\,g\,r^3\sin(\phi)\cos(\theta)\,p22\,p11+m\,g\,r\sin(\phi)\cos(\theta)\,p21^2\,Ib$$

$$-m^2\,g\,r^3\sin(\phi)\cos(\theta)\,p21\,p22+m^2\,g\,r^2\sin(\phi)\cos(\theta)\,p21\,p22\,d\cos(\phi)$$

$$+m^2\,g\,r^3\sin(\phi)\cos(\theta)\,p21\,p11-2\,m^2\,g\,r^2\sin(\phi)\cos(\theta)\,p21^2\,d\cos(\phi)$$

$$+m^2\,g\,r\sin(\phi)\cos(\theta)\,p21^2\,d^2+m^2\,g\,r^3\sin(\phi)\cos(\theta)\,p21^2$$

$$-m^2\,g\,r^2\sin(\phi)\cos(\theta)\,p21\,p11\,d\cos(\phi)+m^2\,g\,r^3\cos(\phi)\sin(\theta)\,p22\,p11$$

$$-m\,g\,r\cos(\phi)\sin(\theta)\,p21^2\,Ib+m^2\,g\,r^3\cos(\phi)\sin(\theta)\,p21\,p22$$

$$-m^2\,g\,r^2\cos(\phi)^2\sin(\theta)\,p21\,p22\,d-m^2\,g\,r^3\cos(\phi)\sin(\theta)\,p21\,p11$$

$$- m^2 g r^3 \cos(\phi) \sin(\theta) p21^2 + m^2 g r^2 \cos(\phi)^2 \sin(\theta) p21\, p11\, d$$

$$- p22\, m r^2 \left(\frac{\partial}{\partial \theta} \Phi(\theta, \phi) \right) \Bigg) \Bigg/ \big(-p22\, m\, p11\, r^2 + p21^2 Ib - p21\, p22\, m r^2$$

$$+ p21\, p22\, m r d \cos(\phi) + p21\, m\, p11\, r^2 - 2\, p21^2 m r d \cos(\phi) + p21^2 m d^2$$

$$+ p21^2 m r^2 - p21\, m\, p11\, r d \cos(\phi) \big)$$

> $PhiIifii := simplify(rhs(solphi));$

$$PhiIifii := \frac{1}{2} \frac{1}{r (p21 + 2\, p11) (p21 + p11)} \Bigg(4\, \_F1\left(\frac{\phi\, p21 + p11\, \theta}{p21} \right) r\, p11^2 \qquad (4.5)$$

$$+ 6\, \_F1\left(\frac{\phi\, p21 + p11\, \theta}{p21} \right) r\, p21\, p11 + 2\, g \cos(-\theta + \phi)\, p21^3 m d^2 + 4\, g \cos($$

$$-\theta + \phi)\, p21^2 m d^2 p11 + 2\, g \cos(-\theta + \phi)\, p21^3 m r^2 + 6\, g \cos(-\theta$$

$$+ \phi)\, p21^2 m r^2 p11 + 4\, g \cos(-\theta + \phi)\, p21^2 Ib\, p11 + 4\, g \cos(-\theta$$

$$+ \phi)\, p21\, m\, p11^2 r^2 - 2\, g \cos(-\theta + \phi)\, p21^2 p22\, m r^2 - 6\, g \cos(-\theta$$

$$+ \phi)\, p21\, p22\, m r^2 p11 - 4\, g \cos(-\theta + \phi)\, p22\, m\, p11^2 r^2 - 2\, g m r d\, p21^3 \cos($$

$$-\theta + 2\, \phi) - 3\, g m r d\, p21^2 \cos(-\theta + 2\, \phi)\, p11 - g m r d\, p21 \cos(-\theta$$

$$+ 2\, \phi)\, p11^2 + g m r d\, p21^2 \cos(-\theta + 2\, \phi)\, p22 + g m r d\, p21 \cos(-\theta$$

$$+ 2\, \phi)\, p22\, p11 + 2\, \_F1\left(\frac{\phi\, p21 + p11\, \theta}{p21} \right) r\, p21^2 - 7\, g\, p21\, d \cos(\theta)\, m r\, p11^2$$

$$- 2\, g\, p21^3 d \cos(\theta)\, m r + g d \cos(\theta)\, m\, p22\, r\, p21^2$$

$$+ 3\, g d \cos(\theta)\, m\, p22\, r\, p21\, p11 - 7\, g\, p21^2 d \cos(\theta)\, m r\, p11$$

$$- 2\, g d \cos(\theta)\, m\, p11^3 r + 2\, g d \cos(\theta)\, m\, p22\, r\, p11^2 + 2\, g \cos(-\theta$$

$$+ \phi)\, p21^3 Ib \Bigg)$$

> $$\mathrm{PhiIifii} := \frac{1}{r (p21 + p11) (p21 + 2\, p11)} \Bigg(6\, F6 \cdot \left(\frac{\phi\, p21 + p11\, \theta}{p21} \right)^2 r\, p21\, p11$$

$$+ 4\, F6 \cdot \left(\frac{\phi\, p21 + p11\, \theta}{p21} \right)^2 r\, p11^2 + 2\, F6 \cdot \left(\frac{\phi\, p21 + p11\, \theta}{p21} \right)^2 r\, p21^2$$

$$+ 2\, g \cos(-\theta + \phi)\, p21^3 Ib - 6\, g \cos(-\theta + \phi)\, p21\, m r^2 p22\, p11 + 4\, g \cos(-\theta$$

$$+ \phi)\, p11^2 m r^2 p21 + 3\, g d \cos(\theta)\, m\, p22\, r\, p21\, p11$$

$$+2\,g\,d\cos(\theta)\,m\,p22\,r\,p11^2-2\,g\,d\cos(\theta)\,m\,p11^3\,r-2\,g\,p21^3\,d\cos(\theta)\,m\,r$$
$$-7\,g\,p21^2\,d\cos(\theta)\,m\,r\,p11-7\,g\,p21\,d\cos(\theta)\,m\,r\,p11^2$$
$$+g\,d\cos(\theta)\,m\,p22\,r\,p21^2-4\,g\cos(-\theta+\phi)\,m\,p11^2\,r^2\,p22$$
$$-2\,g\,m\,r\,d\,p21^3\cos(-\theta+2\,\phi)-3\,g\,m\,r\,d\,p21^2\cos(-\theta+2\,\phi)\,p11$$
$$+g\,m\,r\,d\,p21^2\cos(-\theta+2\,\phi)\,p22+g\,m\,r\,d\,p21\cos(-\theta+2\,\phi)\,p22\,p11$$
$$-g\,m\,r\,d\,p21\cos(-\theta+2\,\phi)\,p11^2+2\,g\cos(-\theta+\phi)\,m\,p21^3\,r^2+6\,g\cos(-\theta$$
$$+\phi)\,m\,p21^2\,r^2\,p11+2\,g\cos(-\theta+\phi)\,m\,p21^3\,d^2+4\,g\cos(-\theta$$
$$+\phi)\,m\,p21^2\,d^2\,p11+4\,g\cos(-\theta+\phi)\,p21^2\,Ib\,p11-2\,g\cos(-\theta$$
$$+\phi)\,p21^2\,m\,r^2\,p22):$$

> $F2 := simplify(F2_1);$

$$F2 := -\alpha\,(p11\,\theta dot+p21\,\phi dot) \tag{4.6}$$

> $F3 := simplify(eval(f3, \Phi(\theta,\phi)=PhiIifii)):$

The control input of the system is

> $\mathrm{Fc} := simplify(eval(F1+F2+F3)):$

> $\mathrm{T} := \left[\; diff(Fc,\theta)\;\; diff(Fc,\phi)\;\; diff(Fc,\theta dot)\;\; diff(Fc,\phi dot)\;\right]:$

> $T1 := simplify(eval(\mathrm{T}, [\theta dot=0, \phi dot=0])):$

> $T2 := simplify(eval(T1, [\phi=0, \theta=0]));$

$$T2 := \left[-\frac{1}{(p21^2+3\,p21\,p11+2\,p11^2)\,p21^2\,r}(2\,Ib\,g\,p21^4+4\,Ib\,p11\,g\,p21^3 \right. \tag{5.1}$$
$$-3\,p21^4\,g\,r\,m\,d+2\,p21^4\,g\,m\,d^2+p21^4\,g\,r^2\,m+3\,p21^3\,p11\,m\,g\,r^2$$
$$-3\,p21^3\,p11\,g\,r\,m\,d+4\,p21^3\,p11\,g\,m\,d^2+2\,p21^3\,g\,r\,p22\,m\,d$$
$$+2\,p21^2\,p11^2\,m\,g\,r^2+2\,p21^2\,p11\,g\,r\,p22\,m\,d+4\,p21^2\,p11\,F6\,r$$
$$+12\,p11^2\,F6\,r\,p21+8\,F6\,r\,p11^3),$$
$$-\frac{1}{(p21^2+3\,p21\,p11+2\,p11^2)\,p21\,r}(-2\,Ib\,g\,p21^3-4\,Ib\,p11\,g\,p21^2$$
$$-p21^3\,g\,r^2\,m-2\,p21^3\,g\,m\,d^2+8\,p21^3\,g\,r\,m\,d-4\,p21^2\,p11\,g\,m\,d^2$$
$$+12\,p21^2\,p11\,g\,r\,m\,d-4\,p21^2\,g\,r\,p22\,m\,d-3\,p21^2\,p11\,m\,g\,r^2+4\,p21^2\,F6\,r$$
$$-4\,p21\,p11\,g\,r\,p22\,m\,d-2\,p21\,p11^2\,m\,g\,r^2+12\,p21\,p11\,F6\,r$$
$$\left.+4\,p21\,p11^2\,g\,m\,r\,d+8\,p11^2\,F6\,r),\ -\alpha\,p11-\nu,\ -\alpha\,p21-\sigma\right]$$

Linearization

> $sysid := [R_o = 0, ro = 0, Ia = Ib, m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5]:$

> $LCond := [\theta = 0, \theta dot = 0, \phi = 0, \phi dot = 0]:$

> $Eqs := simplify\left(Multiply\left(MatrixInverse(mass), \left(\left[\begin{array}{c} \tau \\ 0 \end{array}\right] - Multiply(C, qdot) - \mathrm{G}\right)\right)\right);$

$$Eqs := \left[\left[\frac{1}{m\,d^2 + Ib - m\,d^2 \cos(\phi)^2}\left(\tau - 2\,m\,r\,d \sin(\phi)\,\phi dot\,\theta dot + m\,r\,d \sin(\phi)\,\phi dot^2 + m\,g\,d \sin(\theta) + m\,r\,d \sin(\phi)\,\theta dot^2 - m\,d^2 \cos(\phi) \sin(\phi)\,\theta dot^2 + d \cos(\phi)\,m\,g \sin(\phi) \cos(\theta) - d \cos(\phi)^2\,m\,g \sin(\theta)\right)\right], \left[-\frac{1}{r\left(m\,d^2 + Ib - m\,d^2 \cos(\phi)^2\right)}\left(-r\,\tau + 2\,m\,r^2\,d \sin(\phi)\,\phi dot\,\theta dot - m\,r^2\,d \sin(\phi)\,\phi dot^2 - r\,m\,g\,d \sin(\theta) + d \cos(\phi)\,\tau - 2\,d^2 \cos(\phi)\,m\,r \sin(\phi)\,\phi dot\,\theta dot + d^2 \cos(\phi)\,m\,r \sin(\phi)\,\phi dot^2 - d \cos(\phi)\,m\,g\,r \sin(\phi) \cos(\theta) + d \cos(\phi)^2\,m\,g\,r \sin(\theta) - m\,r^2\,d \sin(\phi)\,\theta dot^2 + 2\,m\,r\,d^2 \cos(\phi) \sin(\phi)\,\theta dot^2 - m\,d^3 \sin(\phi)\,\theta dot^2 + m\,d^2\,g \sin(\phi) \cos(\theta) - Ib\,d \sin(\phi)\,\theta dot^2 + Ib\,g \sin(\phi) \cos(\theta) - Ib\,g \cos(\phi) \sin(\theta)\right)\right]\right] \quad (6.1)$$

> $A := [[0, 0, 1, 0], [0, 0, 0, 1], [\mathrm{diff}(Eqs[1], \theta), \mathrm{diff}(Eqs[1], \phi), \mathrm{diff}(Eqs[1], \theta dot), \mathrm{diff}(Eqs[1], \phi dot)], [\mathrm{diff}(Eqs[2], \theta), \mathrm{diff}(Eqs[2], \phi), \mathrm{diff}(Eqs[2], \theta dot), \mathrm{diff}(Eqs[2], \phi dot)]]:$

> $A := map(eval, A, LCond);$

$$A := \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d\,m\,g}{Ib} & 0 & 0 \\ \frac{g}{r} & -\frac{-d\,m\,g\,r + m\,d^2\,g + Ib\,g}{r\,Ib} & 0 & 0 \end{array}\right] \quad (6.2)$$

> $A := eval(\mathrm{A}, sysid);$

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.960000000 & 0 & 0 \\ 13.06666666 & -12.41333333 & 0 & 0 \end{bmatrix} \quad \textbf{(6.3)}$$

> $B := \begin{bmatrix} 0 \\ 0 \\ \mathrm{diff}(Eqs[1], \tau) \\ \mathrm{diff}(Eqs[2], \tau) \end{bmatrix};$

$$B := \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m\,d^2 + Ib - m\,d^2 \cos(\phi)^2} \\ -\dfrac{-r + d\cos(\phi)}{r\,(m\,d^2 + Ib - m\,d^2\cos(\phi)^2)} \end{bmatrix} \quad \textbf{(6.4)}$$

> $B := simplify(map(eval, B, LCond)):$

> $B := simplify(map(eval, B, sysid));$

$$B := \begin{bmatrix} 0 \\ 0 \\ 20. \\ 6.666666664 \end{bmatrix} \quad \textbf{(6.5)}$$

> $LinCi := T2;$

$$LinCi := \Bigg[-\frac{1}{(p21^2 + 3\,p21\,p11 + 2\,p11^2)\,p21^2\,r}\,(2\,Ib\,g\,p21^4 + 4\,Ib\,p11\,g\,p21^3 \quad \textbf{(6.6)}$$
$$- 3\,p21^4\,g\,r\,m\,d + 2\,p21^4\,g\,m\,d^2 + p21^4\,g\,r^2\,m + 3\,p21^3\,p11\,m\,g\,r^2$$
$$- 3\,p21^3\,p11\,g\,r\,m\,d + 4\,p21^3\,p11\,g\,m\,d^2 + 2\,p21^3\,g\,r\,p22\,m\,d$$
$$+ 2\,p21^2\,p11^2\,m\,g\,r^2 + 2\,p21^2\,p11\,g\,r\,p22\,m\,d + 4\,p21^2\,p11\,F6\,r$$
$$+ 12\,p11^2\,F6\,r\,p21 + 8\,F6\,r\,p11^3),$$

$$-\frac{1}{(p21^2+3\,p21\,p11+2\,p11^2)\,p21\,r}\,(-2\,Ib\,g\,p21^3-4\,Ib\,p11\,g\,p21^2$$
$$-p21^3\,g\,r^2\,m-2\,p21^3\,g\,m\,d^2+8\,p21^3\,g\,r\,m\,d-4\,p21^2\,p11\,g\,m\,d^2$$
$$+12\,p21^2\,p11\,g\,r\,m\,d-4\,p21^2\,g\,r\,p22\,m\,d-3\,p21^2\,p11\,m\,g\,r^2+4\,p21^2\,F6\,r$$
$$-4\,p21\,p11\,g\,r\,p22\,m\,d-2\,p21\,p11^2\,m\,g\,r^2+12\,p21\,p11\,F6\,r$$
$$+4\,p21\,p11^2\,g\,m\,r\,d+8\,p11^2\,F6\,r),\ -\alpha\,p11-\nu,\ -\alpha\,p21-\sigma]$$

> $LinC3 := -map(eval, LinCi, sysid);$

$$LinC3 := \left[\frac{1}{(p21^2+3\,p21\,p11+2\,p11^2)\,p21^2}\,(1.333333333\,(0.9677500\,p21^4\right. \quad \textbf{(6.7)}$$
$$+2.2662500\,p11\,p21^3+0.147000\,p21^3\,p22+0.2205000\,p21^2\,p11^2$$
$$+0.147000\,p21^2\,p11\,p22+3.00\,p21^2\,p11\,F6+9.00\,p11^2\,F6\,p21$$
$$+6.00\,F6\,p11^3)),\ \frac{1}{(p21^2+3\,p21\,p11+2\,p11^2)\,p21}\,(1.333333333\,($$
$$-0.6002500\,p21^3-1.6047500\,p11\,p21^2-0.294000\,p21^2\,p22+3.00\,p21^2\,F6$$
$$-0.294000\,p21\,p11\,p22+0.0735000\,p21\,p11^2+9.00\,p21\,p11\,F6$$
$$\left.+6.00\,p11^2\,F6)),\ \alpha\,p11+\nu,\ \alpha\,p21+\sigma\right]$$

> $k := [\ 82.2743\ \ 138.2016\ \ -34.1319\ \ 118.4029\ ];$

$$k := [\ 82.2743\ \ 138.2016\ \ -34.1319\ \ 118.4029\ ] \quad \textbf{(6.8)}$$

> $sol := solve(\{LinC3_1-k_1, LinC3_2-k_2, LinC3_3-k_3, LinC3_4-k_4\}, [F6, \alpha, \nu, p22]);$

$$sol := \left[\left[F6 = \frac{6.250000002\;10^{-14}\,p21^2\,(1.209628800\;10^{15}\,p11+1.203879467\;10^{15}\,p21)}{p21^2+3.\,p21\,p11+2.\,p11^2},\right.\right. \quad \textbf{(6.9)}$$
$$\alpha = -\frac{0.0001000000000\,(10000.\,\sigma-1.184029\;10^6)}{p21},\ \nu = \frac{0.0001000000000\,(10000.\,\sigma\,p11-1.184029\;10^6\,p11-3.41319\;10^5\,p21)}{p21},$$
$$\left.\left.p22 = -704.8602043\,p11+413.1835035\,p21\right]\right]$$

> $K := map\Big(eval, LinC3, \Big[F6$

$$= \frac{6.250000002\ 10^{-14}\ p21^2\ (1.209628800\ 10^{15}\ p11 + 1.203879467\ 10^{15}\ p21)}{p21^2 + 3.\ p11\ p21 + 2.\ p11^2},$$

$$\alpha = -\frac{0.0001000000000\ (10000.\ \sigma - 1.184029\ 10^6)}{p21}, \nu$$

$$= \frac{0.0001000000000\ (10000.\ p11\ \sigma - 1.184029\ 10^6\ p11 - 3.41319\ 10^5\ p21)}{p21},$$

$$p22 = -704.8602043\ p11 + 413.1835035\ p21 \Big]\Big]\Big):$$

> $solve(-7.874677297\ p11 + 4.582064414\ p21 > 0)$

$$\{p11 = p11,\ 1.718587210\ p11 < p21\} \tag{6.10}$$

> $K := map(eval, K, [p21 = 1.75 \cdot p11]):$

> $sol2 := solve(\{K_2 - k_2\});$

$$sol2 := \tag{6.11}$$

> $K := map(eval, K, [p11 = 0.5]);$

$$K := \begin{bmatrix} 82.27430002 & 138.2016001 & -34.13190000 & 118.4029000 \end{bmatrix} \tag{6.12}$$

>

> $Acl := eval(A - Multiply(B, K));$

$$Acl := [[0., 0., 1., 0.], \tag{6.13}$$

$$[0., 0., 0., 1.],$$

$$[-1645.48600039999997, -2762.07200199999988, 682.638000000000034,$$

$$-2368.05799999999999],$$

$$[-535.428666587268481, -933.757333628129004, 227.545999908981571,$$

$$-789.352666350925574]]$$

> $EACLi := Eigenvalues(Acl);$

$$EACLi := \begin{bmatrix} -76.7044076387869040 + 0.\ I \\ -4.80892851458787796 + 0.\ I \\ -10.9746454501303639 + 0.\ I \\ -14.2266847474219578 + 0.\ I \end{bmatrix} \tag{6.14}$$

> $KDf := simplify\Big(eval\Big(KDT, \Big[F6$

$$= \frac{6.250000002\ 10^{-14}\ p21^2\ (1.209628800\ 10^{15}\ p11 + 1.203879467\ 10^{15}\ p21)}{p21^2 + 3.\ p11\ p21 + 2.\ p11^2},$$

$$\alpha = -\frac{0.0001000000000\ (10000.\ \sigma - 1.184029\ 10^6)}{p21}, \nu$$

$$= \frac{0.0001000000000\ (10000.\ p11\ \sigma - 1.184029\ 10^6\ p11 - 3.41319\ 10^5\ p21)}{p21},$$

$$p22 = -704.8602043\ p11 + 413.1835035\ p21 \Big]\Big]\Big)\Big):$$

> $KDff := simplify(eval(KDf, [R_o = 0, ro = 0, Ia = Ib, m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5]))$
:

$$KDff := [[-7.874677297\, p11 + 4.582064414\, p21 - 6.138585884\, p21 \cos(\phi) + 10.57290307\, p11 \cos(\phi) - 3.529301020 \cos(\phi)^2\, p11 + 2.055917517 \cos(\phi)^2\, p21, -4.582064414\, p21 + 3.083876276\, p21 \cos(\phi) + 7.929677298\, p11 - 5.286451532\, p11 \cos(\phi)], [-4.582064414\, p21 + 3.083876276\, p21 \cos(\phi) + 7.929677298\, p11 - 5.286451532\, p11 \cos(\phi), 4.637064414\, p21 + 0.007500000000\, p21 \cos(\phi) - 7.929677298\, p11]] \quad \textbf{(6.15)}$$

> $KDff := simplify(eval(KDff, [p21 = 1.75\ p11]))$:

> $KDff := simplify(eval(KDff, [p11 = 0.5]))$;

$$KDff := [[0.07196771350 - 0.08481111500 \cos(\phi) + 0.03427731750 \cos(\phi)^2, -0.04446771300 + 0.05516597550 \cos(\phi)], [-0.04446771300 + 0.05516597550 \cos(\phi), 0.09259271300 + 0.006562500000 \cos(\phi)]] \quad \textbf{(6.16)}$$

> $solve(Determinant(KDff) > 0)$;

Warning, solutions may have been lost

>

> $simplify(Determinant(KDff))$;

$$0.004686308342 - 0.002474393578 \cos(\phi) - 0.0004260279730 \cos(\phi)^2 + 0.0002249448961 \cos(\phi)^3 \quad \textbf{(6.17)}$$

> $simplify(Eigenvalues(KDff))$;

$$Kvf := simplify\left(eval\left(Kv, \left[F6 = \frac{6.250000002\ 10^{-14}\, p21^2\, (1.209628800\ 10^{15}\, p11 + 1.203879467\ 10^{15}\, p21)}{p21^2 + 3.\, p11\, p21 + 2.\, p11^2}, \alpha = -\frac{0.0001000000000\, (10000.\, \sigma - 1.184029\ 10^6)}{p21}, \nu = \frac{0.0001000000000\, (10000.\, p11\, \sigma - 1.184029\ 10^6\, p11 - 3.41319\ 10^5\, p21)}{p21}, p22 = -704.8602043\, p11 + 413.1835035\, p21\right]\right)\right);$$

$$[[0.08228021325 - 0.03912430750 \cos(\phi) + 0.01713865875 \cos(\phi)^2$$

$$+ 2.500000000\, 10^{-7} \left(3.333960241\, 10^{10} - 6.342270641\, 10^{10} \cos(\phi) + 7.643335046\, 10^{10} \cos(\phi)^2 - 2.505633930\, 10^{10} \cos(\phi)^3 + 4.699737980\, 10^{9} \cos(\phi)^4\right)^{1/2}\Big],$$

$$\Big[0.08228021325 - 0.03912430750 \cos(\phi) + 0.01713865875 \cos(\phi)^2 - 2.500000000\, 10^{-7} \left(3.333960241\, 10^{10} - 6.342270641\, 10^{10} \cos(\phi) + 7.643335046\, 10^{10} \cos(\phi)^2 - 2.505633930\, 10^{10} \cos(\phi)^3 + 4.699737980\, 10^{9} \cos(\phi)^4\right)^{1/2}\Big]\Big]$$

$$Kvf := \left[\left[-\frac{0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11^2}{p21}, -0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11\right], \left[-0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11, -0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p21\right]\right] \quad \textbf{(6.18)}$$

> $Kvf := simplify(eval(Kvf, [R_o = 0, ro = 0, Ia = Ib, m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5]));$

$$Kvf := \left[\left[-\frac{0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11^2}{p21}, -0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11\right], \left[-0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11, -0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p21\right]\right] \quad \textbf{(6.19)}$$

> $Kvf := simplify(eval(Kvf, [\mathrm{p21} = 1.75\, \mathrm{p11}]));$

$$Kvf := \left[\left[-0.00005714285714\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11, -0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11\right], \left[-0.0001000000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11, -0.0001750000000\,(10000.\,\sigma - 1.184029\, 10^{6})\, p11\right]\right] \quad \textbf{(6.20)}$$

> $Kvf := simplify(eval(Kvf, [\mathrm{p11} = 0.5]));$

$$Kvf := \begin{bmatrix} -0.2857142857\,\sigma + 33.82940000 & -0.5000000000\,\sigma + 59.20145000 \\ -0.5000000000\,\sigma + 59.20145000 & -0.8750000000\,\sigma + 103.6025375 \end{bmatrix} \quad \textbf{(6.21)}$$

The Hessian

```
> Hessiani := simplify(eval(Matrix(2, 2, [diff(diff(PhiIifii, θ), θ),
    diff(diff(PhiIifii, θ), ϕ), diff(diff(PhiIifii, ϕ), θ), diff(diff(PhiIifii, ϕ), ϕ)]
    ), [ϕ = 0, θ = 0])) :
```

> $Hessiani := simplify\Bigg(eval\Bigg(Hessiani, \Bigg[F6 = \dfrac{6.250000002\ 10^{-14}\ p21^2\ (1.209628800\ 10^{15}\ p11 + 1.203879467\ 10^{15}\ p21)}{p21^2 + 3.\ p21\ p11 + 2.\ p11^2}, \alpha = -\dfrac{0.0001000000000\ (10000.\ \sigma - 1.184029\ 10^6)}{p21}, \nu = \dfrac{0.0001000000000\ (10000.\ p11\ \sigma - 1.184029\ 10^6\ p11 - 3.41319\ 10^5\ p21)}{p21}, p22 = -704.8602043\ p11 + 413.1835035\ p21\Bigg]\Bigg)\Bigg) :$

```
> Hessiani := simplify(eval(Hessiani, sysid)) :
> Hessiani := simplify(eval(Hessiani, [p21 = 1.75 p11])) :
>
> Hessiani := simplify(eval(Hessiani, [p11 = 0.5, σ = 0]));
```

$$Hessiani := \begin{bmatrix} 40.83306874 & 69.35588125 \\ 69.35588125 & 123.4747812 \end{bmatrix} \quad \textbf{(7.1)}$$

```
> Eigenvalues(Hessiani);
```

$$\begin{bmatrix} 1.42195222348257744 + 0.\ I \\ 162.885897716517405 + 0.\ I \end{bmatrix} \quad \textbf{(7.2)}$$

The Potential Φ

> $PhiGraphi := simplify(eval(PhiIifii, [R_o = 0, ro = 0, Ia = Ib, m = 0.02, Ib = 0.05, g = 9.8, r = 0.75, d = 0.5])) :$

> $PhiGraph := simplify\Bigg(eval\Bigg(PhiGraphi, \Bigg[F6 = \dfrac{6.250000002\ 10^{-14}\ p21^2\ (1.209628800\ 10^{15}\ p11 + 1.203879467\ 10^{15}\ p21)}{p21^2 + 3.\ p21\ p11 + 2.\ p11^2}, \alpha = -\dfrac{0.0001000000000\ (10000.\ \sigma - 1.184029\ 10^6)}{p21}, \nu$

$$= \frac{0.0001000000000\left(10000.\,p11\,\sigma - 1.184029\,10^{6}\,p11 - 3.41319\,10^{5}\,p21\right)}{p21},$$

$$p22 = -704.8602043\,p11 + 413.1835035\,p21\Big]\Big)\Big);$$

$$PhiGraph := -\big(1.000000000\,10^{-8}\,\big(-1.726016000\,10^{10}\,p21^{4}\cos(\theta)\,p11 + 4.157141004\,10^{10}\cos(\theta)\,p11^{3}\,p21^{2} - 3.024072000\,10^{10}\,p11^{5}\,\theta^{2} - 1.504849334\,10^{10}\,p21^{5}\,\phi^{2} + 7.521728345\,10^{9}\,p21^{3}\cos(-1.\,\theta + 2.\,\phi)\,p11^{2} - 9.200963328\,10^{9}\,p21^{4}\cos(-1.\,\theta + 2.\,\phi)\,p11 + 3.204932827\,10^{10}\cos(-1.\,\theta + \phi)\,p21^{3}\,p11^{2} + 5.126761332\,10^{10}\cos(-1.\,\theta + \phi)\,p21^{4}\,p11 - 4.029598333\,10^{9}\,p21^{5}\cos(-1.\,\theta + 2.\,\phi) + 1.197446167\,10^{10}\cos(-1.\,\theta + \phi)\,p21^{5} - 1.088019832\,10^{10}\,p21^{3}\cos(\theta)\,p11^{2} - 6.048144000\,10^{10}\,p11^{4}\,\phi\,p21\,\theta - 1.509161334\,10^{11}\,p11^{3}\,\phi\,p21^{2}\,\theta - 1.205316800\,10^{11}\,p21^{3}\,\phi\,p11^{2}\,\theta - 3.009698668\,10^{10}\,p21^{4}\,\phi\,p11\,\theta - 2.002019000\,10^{11}\cos(-1.\,\theta + \phi)\,p21\,p11^{4} + 6.689076670\,10^{10}\,p21\cos(\theta)\,p11^{4} + 2.652795335\,10^{10}\,p21^{2}\cos(-1.\,\theta + 2.\,\phi)\,p11^{3} + 1.383486001\,10^{10}\,p21\cos(-1.\,\theta + 2.\,\phi)\,p11^{4} - 1.245541633\,10^{11}\cos(-1.\,\theta + \phi)\,p21^{2}\,p11^{3} - 4.029598333\,10^{9}\,p21^{5}\cos(\theta) - 6.026584001\,10^{10}\,p21^{2}\,p11^{3}\,\theta^{2} - 7.545806668\,10^{10}\,p21^{3}\,p11^{2}\,\phi^{2} - 6.026584001\,10^{10}\,p21^{4}\,p11\,\phi^{2} - 3.024072000\,10^{10}\,p11^{3}\,\phi^{2}\,p21^{2} - 1.504849334\,10^{10}\,p21^{3}\,p11^{2}\,\theta^{2} + 2.766972002\,10^{10}\cos(\theta)\,p11^{5} - 8.289156005\,10^{10}\cos(-1.\,\theta + \phi)\,p11^{5} - 7.545806668\,10^{10}\,p11^{4}\,\theta^{2}\,p21\big)\big)\Big/\big((p21 + 2.\,p11)\,(p21 + p11)\,(p21^{2} + 3.\,p21\,p11 + 2.\,p11^{2})\big) \tag{8.1}$$

\> $PhiGraph := simplify(eval(PhiGraph, [p21 = 1.75 \cdot p11]))$:

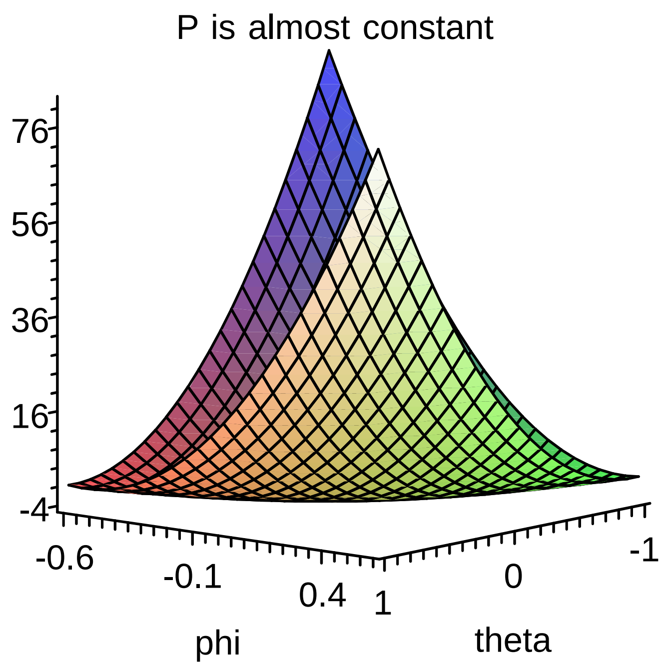
\> $PhiGraph := simplify(eval(PhiGraph, [p11 = 0.5, \sigma = 0]))$;

$$PhiGraph := 61.55472556\,\phi^{2} + 20.09950222\,\theta^{2} + 70.34825780\,\phi\,\theta + 0.6718354076\cos(\theta) - 1.619422904\cos(-1.\,\theta + \phi) + 0.3135231860\cos(-1.\,\theta + 2.\,\phi) \tag{8.2}$$

\> $PhiGraph := 70.34825780\,\phi\,\theta + 0.6718354080\cos(\theta) - 4.619422892\cos(-1.\,\theta + \phi) + 20.09950222\,\theta^{2} + 0.3135231860\cos(-1.\,\theta + 2.\,\phi) + 61.55472556\,\phi^{2}$:

> $f := (\theta, \phi) \rightarrow PhiGraph$:

> $plot3d(f(\theta, \phi), \theta = -1.0 .. 1.0, \phi = -0.6 .. 0.6, axes = FRAME, orientation = [40, 80], style = PATCH, title = \text{"P is almost constant"});$

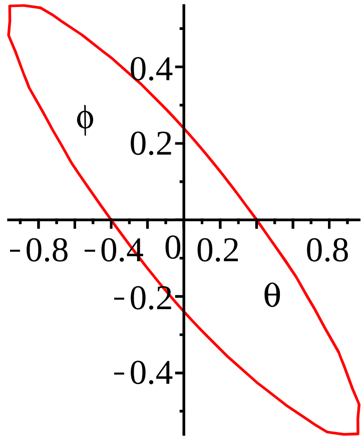


> $with(plots)$

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, graphplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] **(8.3)**

The following plot indicates one of the bigger levels of the potential

> $implicitplot(f(\theta, \phi), \theta = -2 .. 2, \phi = -1 .. 1)$




```
> convert(F1, string)
                         "-`θdot`*nu-`ϕdot`*sigma"                                  (8.4)
> convert(Fmc1, string)
"Matrix(2, 2, [[(4*`θdot`^2*m*p11*r^2*d*sin(phi)*`ϕdot`+2*`θdot`*p11*r*`ϕdot`*     (8.5)
    sigma-2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p11*cos(phi)-2*`θdot`^3*p22*m*r*
    d^2*cos(phi)*sin(phi)-2*`θdot`*m*p11*r^2*`ϕdot`^2*d*sin(phi)+p21*
    `θdot`^2*`ϕdot`*d*sin(phi)*Ib+3*p21*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)-4*
    p21*`θdot`^2*m*d^2*cos(phi)*r*sin(phi)*`ϕdot`+2*`θdot`^2*`ϕdot`*d^2*sin
    (phi)*r*m*p22*cos(phi)+2*`θdot`^3*m*p11*r*d^2*cos(phi)*sin(phi)-4*
    `θdot`^2*`ϕdot`*m*r^2*d*sin(phi)*p22+2*`θdot`^3*r^2*m*p22*d*sin(phi)-2*
    p21*`θdot`^3*Ib*d*sin(phi)-2*`θdot`^3*m*p11*r^2*d*sin(phi)+p21*`θdot`^2*
    `ϕdot`*m*d^3*sin(phi)+4*p21*`θdot`^3*m*d^2*cos(phi)*r*sin(phi)-2*p21*
    `θdot`^3*m*r^2*d*sin(phi)-r^2*m*`ϕdot`^3*p21*d*sin(phi)+2*r^2*m*
    `ϕdot`^2*`θdot`*d*sin(phi)*p22+2*r*`ϕdot`*p21*`θdot`*nu+2*r*`ϕdot`^2*p21*
    sigma+2*`θdot`^2*p11*r*nu-2*p21*`θdot`^3*m*d^3*sin(phi))/r/(`ϕdot`^2+2*
    `θdot`*`ϕdot`+`θdot`^2),(4*`θdot`^2*m*p11*r^2*d*sin(phi)*`ϕdot`+2*`θdot`*
    p11*r*`ϕdot`*sigma-2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p11*cos(phi)-2*
    `θdot`^3*p22*m*r*d^2*cos(phi)*sin(phi)-2*`θdot`*m*p11*r^2*`ϕdot`^2*d*sin
    (phi)+p21*`θdot`^2*`ϕdot`*d*sin(phi)*Ib+3*p21*`θdot`^2*`ϕdot`*m*r^2*d*sin
    (phi)-4*p21*`θdot`^2*m*d^2*cos(phi)*r*sin(phi)*`ϕdot`+2*`θdot`^2*`ϕdot`*
    d^2*sin(phi)*r*m*p22*cos(phi)+2*`θdot`^3*m*p11*r*d^2*cos(phi)*sin(phi)
    -4*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)*p22+2*`θdot`^3*r^2*m*p22*d*sin(phi)
    -2*p21*`θdot`^3*Ib*d*sin(phi)-2*`θdot`^3*m*p11*r^2*d*sin(phi)+p21*
    `θdot`^2*`ϕdot`*m*d^3*sin(phi)+4*p21*`θdot`^3*m*d^2*cos(phi)*r*sin(phi)
    -2*p21*`θdot`^3*m*r^2*d*sin(phi)-r^2*m*`ϕdot`^3*p21*d*sin(phi)+2*r^2*m*
    `ϕdot`^2*`θdot`*d*sin(phi)*p22+2*r*`ϕdot`*p21*`θdot`*nu+2*r*`ϕdot`^2*p21*
    sigma+2*`θdot`^2*p11*r*nu-2*p21*`θdot`^3*m*d^3*sin(phi))/r/(`ϕdot`^2+2*
    `θdot`*`ϕdot`+`θdot`^2)],[(4*`θdot`^2*m*p11*r^2*d*sin(phi)*`ϕdot`+2*`θdot`*
    p11*r*`ϕdot`*sigma-2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p11*cos(phi)-2*
```

`θdot`^3*p22*m*r*d^2*cos(phi)*sin(phi)-2*`θdot`*m*p11*r^2*`ϕdot`^2*d*sin(phi)+p21*`θdot`^2*`ϕdot`*d*sin(phi)*Ib+3*p21*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)-4*p21*`θdot`^2*m*d^2*cos(phi)*r*sin(phi)*`ϕdot`+2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p22*cos(phi)+2*`θdot`^3*m*p11*r*d^2*cos(phi)*sin(phi)-4*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)*p22+2*`θdot`^3*r^2*m*p22*d*sin(phi)-2*p21*`θdot`^3*Ib*d*sin(phi)-2*`θdot`^3*m*p11*r^2*d*sin(phi)+p21*`θdot`^2*`ϕdot`*m*d^3*sin(phi)+4*p21*`θdot`^3*m*d^2*cos(phi)*r*sin(phi)-2*p21*`θdot`^3*m*r^2*d*sin(phi)-r^2*m*`ϕdot`^3*p21*d*sin(phi)+2*r^2*m*`ϕdot`^2*`θdot`*d*sin(phi)*p22+2*r*`ϕdot`*p21*`θdot`*nu+2*r*`ϕdot`^2*p21*sigma+2*`θdot`^2*p11*r*nu-2*p21*`θdot`^3*m*d^3*sin(phi))/r/(`ϕdot`^2+2*`θdot`*`ϕdot`+`θdot`^2),(4*`θdot`^2*m*p11*r^2*d*sin(phi)*`ϕdot`+2*`θdot`*p11*r*`ϕdot`*sigma-2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p11*cos(phi)-2*`θdot`^3*p22*m*r*d^2*cos(phi)*sin(phi)-2*`θdot`*m*p11*r^2*`ϕdot`^2*d*sin(phi)+p21*`θdot`^2*`ϕdot`*d*sin(phi)*Ib+3*p21*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)-4*p21*`θdot`^2*m*d^2*cos(phi)*r*sin(phi)*`ϕdot`+2*`θdot`^2*`ϕdot`*d^2*sin(phi)*r*m*p22*cos(phi)+2*`θdot`^3*m*p11*r*d^2*cos(phi)*sin(phi)-4*`θdot`^2*`ϕdot`*m*r^2*d*sin(phi)*p22+2*`θdot`^3*r^2*m*p22*d*sin(phi)-2*p21*`θdot`^3*Ib*d*sin(phi)-2*`θdot`^3*m*p11*r^2*d*sin(phi)+p21*`θdot`^2*`ϕdot`*m*d^3*sin(phi)+4*p21*`θdot`^3*m*d^2*cos(phi)*r*sin(phi)-2*p21*`θdot`^3*m*r^2*d*sin(phi)-r^2*m*`ϕdot`^3*p21*d*sin(phi)+2*r^2*m*`ϕdot`^2*`θdot`*d*sin(phi)*p22+2*r*`ϕdot`*p21*`θdot`*nu+2*r*`ϕdot`^2*p21*sigma+2*`θdot`^2*p11*r*nu-2*p21*`θdot`^3*m*d^3*sin(phi))/r/(`ϕdot`^2+2*`θdot`*`ϕdot`+`θdot`^2)]])"

>

> *convert*(*F2*, *string*)

"-alpha*(p11*`θdot`+p21*`ϕdot`)" **(8.6)**

> *convert*(*F3*, *string*)

"-(-4*m^2*g*r^4*cos(phi)*sin(theta)*p21^5*p11-2*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^3*p22*d*p11^2-2*m^2*g*r^4*cos(phi)*sin(theta)*p21^3*p11^3+8*p21^2*p22*m*r^2*d*cos(phi)*F6*p11^2*phi-m^2*g*r^4*cos(phi)*sin(theta)*p21^6+4*p21^5*Ib*F6*r*phi+8*p21*p22*m*r^2*d*cos(phi)*F6*p11^3*theta-2*p21^5*p22*m^2*r*d^3*cos(phi)*g*sin(-theta+phi)-4*p21^4*p22*m^2*r*d^3*cos(phi)*g*sin(-theta+phi)*p11-6*p21^5*p22*m^2*r^3*d*cos(phi)*g*sin(-theta+phi)+12*p21^3*m*d^2*F6*r*p11^2*theta+8*p21^3*m*d^2*F6*r*p11^2*phi+4*p21^4*Ib*F6*r*p11*theta+12*p21^4*m*d^2*F6*r*p11*phi+8*p21^2*m*d^2*F6*r*p11^3*theta-4*p21^4*p22*m*r*d*cos(phi)*g*sin(-theta+phi)*Ib*p11+2*p21^4*p22^2*m^2*r^3*d*cos(phi)*g*sin(-theta+phi)+10*p22*m^2*r^4*g*sin(-theta+phi)*p21^3*p11^2-3*p22*m^2*r^3*g*d*p21^3*sin(-theta+2*phi)*p11^2-2*p21^6*m^2*r^4*g*sin(-theta+phi)+p22^2*m^2*r^3*g*d*p21^4*sin(-theta+2*phi)+p22^2*m^2*r^3*g*d* **(8.7)**

p21^3*sin(-theta+2*phi)*p11-4*p22*m^2*r^3*g*d*p21^5*sin(-theta+2*phi)
-7*p22*m^2*r^3*g*d*p21^4*sin(-theta+2*phi)*p11+8*p21^5*p22*m^2*r^2*
d^2*cos(phi)*g*sin(-theta+2*phi)+6*p21^3*p22^2*m^2*r^3*d*cos(phi)*g*
sin(-theta+phi)*p11+4*p21^2*p22^2*m^2*r^3*d*cos(phi)*g*sin(-theta+phi)*
p11^2-2*p21^4*p22^2*m^2*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)-2*p21^3*
p22^2*m^2*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)*p11+4*p21^3*Ib*g*sin(-
theta+phi)*p22*m*p11^2*r^2+4*p21^6*Ib*g*m*r*d*sin(-theta+2*phi)+6*
p21^5*Ib*g*m*r*d*sin(-theta+2*phi)*p11+2*p21^4*Ib*g*m*r*d*sin(-
theta+2*phi)*p11^2+4*p21^5*m*d^2*F6*r*phi+4*m^2*g*d*sin(theta)*
p21^5*r^3*p11-m*g*r^2*cos(phi)*sin(theta)*p21^6*Ib-3*m*g*r^2*cos(phi)*
sin(theta)*p21^5*Ib*p11-2*m*g*r^2*cos(phi)*sin(theta)*p21^4*Ib*p11^2+4*
p21^6*m^2*r^3*g*d*sin(-theta+2*phi)-2*p21^5*Ib*g*m*r*d*sin(-theta+2*
phi)*p22-2*p21^4*Ib*g*m*r*d*sin(-theta+2*phi)*p22*p11-4*p21^5*m^2*
d^4*g*sin(-theta+phi)*p11-4*p21^6*m^2*d^2*g*sin(-theta+phi)*r^2+2*m^2*
g*d^3*sin(theta)*p21^4*r*p11^2-4*p21^4*p22*m*r^3*F6*phi+4*m^2*g*r^4*
sin(phi)*cos(theta)*p21^5*p11+5*m^2*g*r^4*sin(phi)*cos(theta)*p21^4*
p11^2+2*m^2*g*r^4*sin(phi)*cos(theta)*p21^3*p11^3+3*m^2*g*r^3*sin
(phi)*cos(theta)*p21^4*p22*d*cos(phi)*p11+5*m^2*g*d*sin(theta)*p21^4*
r^3*p11^2-8*p22*m*r^3*F6*p11^4*theta-4*p21^3*p22*m*r^3*F6*p11*
theta+4*p21^3*m^2*d^2*g*sin(-theta+phi)*p22*p11^2*r^2+4*p21^6*m^2*
d^3*g*r*sin(-theta+2*phi)+6*p21^5*m^2*d^3*g*r*sin(-theta+2*phi)*
p11+4*p21^3*m^2*p11^3*r^3*d*cos(phi)*g*sin(-theta+phi)+3*m^2*g*d^3*
sin(theta)*p21^5*r*p11+4*p21^4*m^2*p11^2*r*d^3*cos(phi)*g*sin(-theta+
phi)+14*p21^5*m^2*p11*r^3*d*cos(phi)*g*sin(-theta+phi)+14*p21^4*m^2*
p11^2*r^3*d*cos(phi)*g*sin(-theta+phi)+4*p21^4*m*p11^2*r*d*cos(phi)*g*
sin(-theta+phi)*Ib-20*p21^4*m^2*p11*r^3*d*cos(phi)*g*sin(-theta+phi)*p22
-18*p21^3*m^2*p11^2*r^3*d*cos(phi)*g*sin(-theta+phi)*p22-16*p21^5*
m^2*p11*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)-4*p21^2*m^2*p11^3*r^3*d*
cos(phi)*g*sin(-theta+phi)*p22-10*p21^4*m^2*p11^2*r^2*d^2*cos(phi)*g*sin
(-theta+2*phi)-2*p21^3*m^2*p11^3*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)
+12*p21^4*m^2*p11*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)*p22+m^2*g*
d^3*sin(theta)*p21^6*r+m^2*g*r^4*sin(phi)*cos(theta)*p21^6+4*p21^3*
m^2*p11^2*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)*p22-28*p21^4*m*p11*
r^2*d*cos(phi)*F6*phi-28*p21^3*m*p11^2*r^2*d*cos(phi)*F6*theta+2*m^2*
g*r^3*sin(phi)*cos(theta)*p21^3*p22*d*cos(phi)*p11^2+m^2*g*d*sin(theta)*
p21^6*r^3+m^2*g*r^3*sin(phi)*cos(theta)*p21^5*p22*d*cos(phi)-12*p21^5*
m^2*p11*r^2*g*sin(-theta+phi)*d^2-8*p21^5*m^2*p11*r^4*g*sin(-theta+
phi)+10*p21^5*m*p11*r*d*cos(phi)*g*sin(-theta+phi)*Ib+2*p21^4*m^2*
d^3*g*r*sin(-theta+2*phi)*p11^2-2*p21^5*m^2*d^3*g*r*sin(-theta+2*phi)*
p22-2*p21^4*m^2*d^3*g*r*sin(-theta+2*phi)*p22*p11+4*p21^4*m*d^2*F6*

r*p11*theta+m*g*r^2*sin(phi)*cos(theta)*p21^6*Ib+3*m*g*r^2*sin(phi)*cos(theta)*p21^5*Ib*p11+2*m*g*r^2*sin(phi)*cos(theta)*p21^4*Ib*p11^2+m^2*g*d^2*sin(theta)*p21^5*p22*r^2*cos(phi)-2*p21^6*Ib^2*g*sin(-theta+phi)+3*m^2*g*d^2*sin(theta)*p21^4*p22*r^2*cos(phi)*p11+2*m^2*g*d^2*sin(theta)*p21^3*p22*r^2*cos(phi)*p11^2+4*m^2*g*r^4*cos(phi)*sin(theta)*p22*p11*p21^4+5*m^2*g*r^4*cos(phi)*sin(theta)*p22*p11^2*p21^3+2*p22*m*r^2*g*sin(-theta+phi)*p21^5*Ib-4*p21^5*Ib^2*g*sin(-theta+phi)*p11+2*m^2*g*r^4*cos(phi)*sin(theta)*p22*p11^3*p21^2-11*m^2*g*d*sin(theta)*p22*p11*r^3*p21^4-12*m^2*g*d*sin(theta)*p22*p11^2*r^3*p21^3-4*m^2*g*d*sin(theta)*p22*p11^3*r^3*p21^2-8*p21^4*m^2*p11^2*r^2*g*sin(-theta+phi)*d^2+4*p21^6*m^2*r*d^3*cos(phi)*g*sin(-theta+phi)+4*p21^4*p22*m*r^2*d*cos(phi)*F6*phi+4*p21^3*p22*m*r^2*d*cos(phi)*F6*p11*theta-8*p21^4*m*p11^2*r^2*g*sin(-theta+phi)*Ib-2*p21^5*p22*m*r*d*cos(phi)*g*sin(-theta+phi)*Ib-10*p21^4*m^2*p11^2*r^4*g*sin(-theta+phi)-4*p21^3*m^2*p11^3*r^4*g*sin(-theta+phi)+4*p21^2*m^2*p11^3*r^4*g*sin(-theta+phi)*p22+10*p21^5*m^2*p11*r^3*g*d*sin(-theta+2*phi)+m^2*g*r^4*cos(phi)*sin(theta)*p21^5*p22+2*p22^2*m^2*r^3*g*d*sin(theta)*p11^2*p21^2-10*m^2*g*d^2*sin(theta)*p21^5*p11*r^2*cos(phi)-9*m^2*g*d^2*sin(theta)*p21^4*p11^2*r^2*cos(phi)+8*p21^4*m^2*p11^2*r^3*g*d*sin(-theta+2*phi)+2*p21^3*m^2*p11^3*r^3*g*d*sin(-theta+2*phi)+4*p21^6*m^2*r^3*d*cos(phi)*g*sin(-theta+phi)-2*m^2*g*r^3*sin(phi)*cos(theta)*p21^6*d*cos(phi)+m^2*g*r^2*sin(phi)*cos(theta)*p21^6*d^2-16*p22*m*r^3*F6*p11*p21^3*phi-16*p22*m*r^3*F6*p11^2*p21^2*theta-20*p22*m*r^3*F6*p11^2*p21^2*phi-20*p22*m*r^3*F6*p11^3*p21*theta+20*p21^3*m*p11^2*r^3*F6*phi-12*p21^5*m*p11*r^2*g*sin(-theta+phi)*Ib+3*m^2*g*r^2*sin(phi)*cos(theta)*p21^5*d^2*p11+2*m^2*g*r^2*sin(phi)*cos(theta)*p21^4*d^2*p11^2-28*p21^3*m*p11^2*r^2*d*cos(phi)*F6*phi+2*m^2*g*d*sin(theta)*p21^3*p11^3*r^3-8*p22*m*r^3*F6*p11^3*phi*p21+20*p21^2*m*p11^3*r^3*F6*theta+8*p21^2*m*p11^3*r^3*F6*phi+8*p21*m*p11^4*r^3*F6*theta+2*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^3*p11^3*d-28*p21^2*m*p11^3*r^2*d*cos(phi)*F6*theta-8*p21^2*m*p11^3*r^2*d*cos(phi)*F6*phi-8*p21*m*p11^4*r^2*d*cos(phi)*F6*theta-4*p21^6*Ib*g*sin(-theta+phi)*m*d^2+4*p21^5*m*r^3*F6*phi+4*p21^4*m*r^3*F6*p11*theta-8*p21^4*m*r^2*d*cos(phi)*F6*p11*theta+4*p21^6*m*r*d*cos(phi)*g*sin(-theta+phi)*Ib+10*p21^5*m^2*p11*r*d^3*cos(phi)*g*sin(-theta+phi)-8*p21^6*m^2*r^2*d^2*cos(phi)*g*sin(-theta+2*phi)-8*p21^5*Ib*g*sin(-theta+phi)*m*d^2*p11-4*p21^6*Ib*g*sin(-theta+phi)*m*r^2-8*p21^5*m*r^2*d*cos(phi)*F6*phi-2*m^2*g*d^2*sin(theta)*p21^3*p11^3*r^2*cos(phi)-3*m^2*g*d*sin(theta)*p21^5*p22*r^3+12*p21^3*p22*m*r^2*d*cos(phi)*F6*p11*phi+2*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^6*d-2*p21^6*m^2*d^4*g*sin(-theta+phi)+16*p21^4*m*p11*r^3*

F6*phi+16*p21^3*m*p11^2*r^3*F6*theta+7*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^5*d*p11+7*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^4*d*p11^2+2*p22*m^2*r^4*g*sin(-theta+phi)*p21^5+8*p22*m^2*r^4*g*sin(-theta+phi)*p21^4*p11+6*p22*m*r^2*g*sin(-theta+phi)*p21^4*Ib*p11+p22^2*m^2*r^3*g*d*sin(theta)*p21^4+3*p22^2*m^2*r^3*g*d*sin(theta)*p21^3*p11+3*m*g*d*sin(theta)*p21^5*Ib*r*p11+2*m*g*d*sin(theta)*p21^4*Ib*r*p11^2+12*p21^2*p22*m*r^2*d*cos(phi)*F6*p11^2*theta-4*m^2*g*r^4*sin(phi)*cos(theta)*p22*p11*p21^4+2*p22*m^2*r^2*g*sin(-theta+phi)*p21^5*d^2+6*p22*m^2*r^2*g*sin(-theta+phi)*p21^4*d^2*p11-3*m^2*g*r^2*cos(phi)*sin(theta)*p21^6*d^2+m*g*d*sin(theta)*p21^6*Ib*r-m^2*g*r^4*sin(phi)*cos(theta)*p21^5*p22-5*m^2*g*r^4*sin(phi)*cos(theta)*p22*p11^2*p21^3-2*m^2*g*r^4*sin(phi)*cos(theta)*p22*p11^3*p21^2-7*m^2*g*r^3*sin(phi)*cos(theta)*p21^5*p11*d*cos(phi)-7*m^2*g*r^3*sin(phi)*cos(theta)*p21^4*p11^2*d*cos(phi)+12*p21^4*Ib*F6*r*p11*phi+12*p21^3*Ib*F6*r*p11^2*theta+8*p21^3*Ib*F6*r*p11^2*phi+8*p21^2*Ib*F6*r*p11^3*theta-2*m^2*g*r^3*sin(phi)*cos(theta)*p21^3*p11^3*d*cos(phi)-5*m^2*g*r^4*cos(phi)*sin(theta)*p21^4*p11^2-m^2*g*r^3*cos(phi)^2*sin(theta)*p21^5*p22*d-3*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^4*p22*d*p11)/p21^2/r/(p21^4*r^2*m+p21^4*Ib+3*p21^3*m*d^2*p11+2*p21^2*m*d^2*p11^2+2*p21*r^2*m*p11^3+p21^4*m*d^2-2*cos(phi)*p21^4*r*m*d-7*cos(phi)*p21^3*r*m*d*p11-7*cos(phi)*p21^2*r*m*d*p11^2+cos(phi)*p21^3*r*p22*m*d+3*cos(phi)*p21^2*r*p22*m*d*p11+2*cos(phi)*p21*r*p22*m*d*p11^2-2*cos(phi)*p21*r*m*p11^3*d-4*p21^2*r^2*p22*m*p11-5*p21*r^2*p22*m*p11^2-2*r^2*p22*m*p11^3+3*p21^3*Ib*p11+2*p21^2*Ib*p11^2-p21^3*r^2*p22*m+4*p21^3*r^2*m*p11+5*p21^2*r^2*m*p11^2)"

> *convert*(*PhiIifii*, *string*) ;

"(6*F6*(phi*p21+p11*theta)^2/p21*r*p11+4*F6*(phi*p21+p11*theta)^2/p21^2*r*p11^2+2*g*cos(-theta+phi)*p21^3*m*d^2+4*g*cos(-theta+phi)*p21^2*m*d^2*p11+2*g*cos(-theta+phi)*p21^3*m*r^2+6*g*cos(-theta+phi)*p21^2*m*r^2*p11+4*g*cos(-theta+phi)*p21^2*Ib*p11+4*g*cos(-theta+phi)*p21*m*p11^2*r^2-2*g*cos(-theta+phi)*p21^2*p22*m*r^2-6*g*cos(-theta+phi)*p21*p22*m*r^2*p11-4*g*cos(-theta+phi)*p22*m*p11^2*r^2-2*g*m*r*d*p21^3*cos(-theta+2*phi)-3*g*m*r*d*p21^2*cos(-theta+2*phi)*p11-g*m*r*d*p21*cos(-theta+2*phi)*p11^2+g*m*r*d*p21^2*cos(-theta+2*phi)*p22+g*m*r*d*p21*cos(-theta+2*phi)*p22*p11+2*F6*(phi*p21+p11*theta)^2*r-7*g*p21*d*cos(theta)*m*r*p11^2-2*g*p21^3*d*cos(theta)*m*r+g*d*cos(theta)*m*p22*r*p21^2+3*g*d*cos(theta)*m*p22*r*p21*p11-7*g*p21^2*d*cos(theta)*m*r*p11-2*g*d*cos(theta)*m*p11^3*r+2*g*d*cos(theta)*m*p22*r*p11^2+2*g*cos(-theta+phi)*p21^3*Ib)/(p21+2*p11)/r/(p21+p11)" **(8.8)**

The control law numerical value is

$$> \ tauf := simplify\Bigg(eval\Bigg(F1+F2+F3, \Bigg[F6 = \frac{6.250000002\ 10^{-14}\ p21^2\ (1.209628800\ 10^{15}\ p11 + 1.203879467\ 10^{15}\ p21)}{p21^2 + 3.\ p21\ p11 + 2.\ p11^2}, \alpha = -\frac{0.0001000000000\ (10000.\ \sigma - 1.184029\ 10^6)}{p21}, \nu = \frac{0.0001000000000\ (10000.\ p11\ \sigma - 1.184029\ 10^6\ p11 - 3.41319\ 10^5\ p21)}{p21}, p22 = -704.8602043\ p11 + 413.1835035\ p21\Bigg]\Bigg)\Bigg):$$

$$> \ tauf := simplify(eval(tauf, [p21 = 1.75 \cdot p11])):$$

$$> \ tauf := simplify(eval(tauf, [p11 = 0.5, \sigma = 0])):$$

$>$

$$> \ tauf := simplify(eval(tauf, sysid));$$

$$tauf := (0.1000000000\ (2.914079373\ 10^{12}\ \theta + 5.099638909\ 10^{12}\ \phi - 1.237133449\ 10^{12}\ \theta dot + 4.291591956\ 10^{12}\ \phi dot - 2.572350105\ 10^{10}\ \sin(\theta) - 4.380892814\ 10^{10}\ \sin(-1.\ \theta + \phi) + 3.301040038\ 10^9\ \sin(-1.\ \theta + 2.\ \phi) + 5.328112868\ 10^9\ \sin(\phi)\cos(\theta) - 7.203626066\ 10^9\ \cos(\phi)\sin(\theta) - 2.692634414\ 10^{12}\ \cos(\phi)\ \phi + 1.371467370\ 10^{10}\ \cos(\phi)\sin(-1.\ \theta + 2.\ \phi) - 3.541980758\ 10^{10}\ \cos(\phi)\sin(-1.\ \theta + \phi) + 6.532125366\ 10^{11}\ \theta dot\cos(\phi) - 2.265981651\ 10^{12}\ \phi dot\cos(\phi) - 1.538648234\ 10^{12}\ \cos(\phi)\ \theta + 2.813269799\ 10^9\ \cos(\phi)^2\sin(\theta) - 2.813269799\ 10^9\ \sin(\phi)\cos(\theta)\cos(\phi)))\Big/(-3.624566623\ 10^9 + 1.913788961\ 10^9\ \cos(\phi)) \qquad \textbf{(8.9)}$$

The potential numerical value is

$$> \ phif := PhiGraph;$$

$$phif := 70.34825780\ \phi\ \theta + 0.6718354080\ \cos(\theta) - 4.619422892\ \cos(-1.\ \theta + \phi) + 20.09950222\ \theta^2 + 0.3135231860\ \cos(-1.\ \theta + 2.\ \phi) + 61.55472556\ \phi^2 \qquad \textbf{(8.10)}$$

```
%%%%%%%%%%%%%%%%%%  B.7 BALL AND ARC SIMULATION P %%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%       B&Arc_Nonlinear_Sys.m    %%%%%%%%%%%%%%%%%%
function dxdt = B_AP_NonLinear_Sys(u)

%% Main Vectors
theta     = u(1);                % feedback array
phi       = u(2);
thetadot  = u(3);
phidot    = u(4);

%% Generalized quantities
q         = [theta phi]';        % Generalized coordinates
qdot      = [thetadot phidot]';  % Generalized velocities

%% Physical parameter values
r         = 0.75;                % arc's radius
d         = 0.5;                 % distance from the center of the arc
Ib        = 0.05;                % kg m^2  - inertia of arc
m         = 0.02;                % kg      - mass of ball
g         = 9.81;                % m/sec^2 - acceleration of gravity
%% Linear model parameters
sigma  =0;
p11       = 0.5;
p21       = 1.75*p11;
F6        = 6.250000002*10^(-14)*p21^2*(1.209628800*10^15*p11+1.↙
203879467*10^15*p21)/(p21^2+3.*p21*p11+2.*p11^2);
alpha     = -(0.1000000000e-3*(10000.*sigma-1.184029*10^6))/p21;
nu        = (0.1000000000e-3*(10000.*p11*sigma-1.184029*10^6*p11-3.41319↙
*10^5*p21))/p21;
p22       = -704.8602043*p11+413.1835035*p21;

%% The G, M, C, P and KD matrices
          %gravity terms
G         = -[m*g*r*sin(phi)*cos(theta)- m*g*r*cos(phi)*sin(theta)+...
            m*g*d*sin(theta);-m*g*r*sin(phi)*cos(theta)+m*g*r*cos(phi)*sin↙
(theta)];
          %mass matrix
mass      = [m*r^2-2*m*r*d*cos(phi)-...
            m*d^2+Ib,m*r*(-r+...
            d*cos(phi));m*r*(-...
            r+d*cos(phi)),m*r^2];
           %centripetal an coriolis matrix
C         = -[-phidot*m*r*d*sin(phi),-phidot*m*r*d*sin(phi)+...
          phidot*m*r*d*sin(phi);phidot*m*r*d*sin(phi),0];


%% FMC
          %KD matrix
KD        =[(p11*m*d^2*r+p11*Ib*r-p21*m*r^3+3*p21*m*r^2*d*cos(phi)-↙
p21*m*d^2*r-p21*Ib*r+p22*m*r^3-...
          2*p22*m*r^2*d*cos(phi)-d^2*cos(phi)^2*p11*m*r-2*d^2*cos(phi)↙
```

```
^2*p21*m*r+d^3*cos(phi)*p21*m+...
          d*cos(phi)*p21*Ib+d^2*cos(phi)^2*p22*m*r)/r,p21*m*r^2-↙
2*p21*m*r*d*cos(phi)+p21*m*d^2+...
          p21*Ib-p22*m*r^2+p22*m*r*d*cos(phi);p21*m*r^2-2*p21*m*r*d*cos(phi)↙
+p21*m*d^2+p21*Ib-...
          p22*m*r^2+p22*m*r*d*cos(phi),m*r*(-p21*r+p21*d*cos(phi)+p22*r)];
          %P matrix
P         = KD*inv(mass);
          %Determinant of KD matrix
KDdet     = det(KD);
%% SMC
          %Kv Matrix
Kv        = [alpha*p11^2,alpha*p11*p21;alpha*p11*p21,alpha*p21^2];

%% Evaluating the control law
          %FMC input
F1        = -thetadot*nu-phidot*sigma ;
          %SMC input
F2        = -alpha*(p11*thetadot+p21*phidot);
          %TMC input
F3        = -(m^2*g*d*sin(theta)*p21^6*r^3+m*g*d*sin(theta)↙
*p21^6*Ib*r+3*m*g*d*sin(theta)*p21^5*Ib*r*p11+...
            2*m*g*d*sin(theta)*p21^4*Ib*r*p11^2+m^2*g*r^4*sin(phi)*cos(theta)↙
*p21^6-...
            7*m^2*g*r^3*sin(phi)*cos(theta)*p21^5*d*cos(phi)*p11-↙
7*m^2*g*r^3*sin(phi)*cos(theta)*p21^4*d*cos(phi)*p11^2-...
            2*m^2*g*r^3*sin(phi)*cos(theta)*p21^3*d*cos(phi)*p11^3-↙
20*p21^4*d*cos(phi)*p22*m^2*r^3*g*sin(-theta+phi)*p11-...
            4*p21^4*d*cos(phi)*p22*m*r*g*sin(-theta+phi)*Ib*p11-↙
18*p21^3*d*cos(phi)*p22*m^2*r^3*g*sin(-theta+phi)*p11^2+...
            2*p21^4*d*cos(phi)*p22^2*m^2*r^3*g*sin(-theta+phi)-8*p21^2*d*cos↙
(phi)*p11^3*m*r^2*F6*phi-...
            4*p21^2*d*cos(phi)*p11^3*m^2*r^3*g*sin(-theta+phi)*p22-↙
8*p21*d*cos(phi)*p11^4*m*r^2*F6*theta-...
            28*p21^4*d*cos(phi)*p11*m*r^2*F6*phi-28*p21^3*d*cos(phi)↙
*p11^2*m*r^2*F6*theta+...
            10*p21^5*d*cos(phi)*p11*m*r*g*sin(-theta+phi)*Ib+m*g*r^2*sin(phi)↙
*cos(theta)*p21^6*Ib+...
            3*m*g*r^2*sin(phi)*cos(theta)*p21^5*Ib*p11+2*m*g*r^2*sin(phi)*cos↙
(theta)*p21^4*Ib*p11^2-...
            3*m^2*g*d*sin(theta)*p21^5*p22*r^3-11*m^2*g*d*sin(theta)↙
*p21^4*p22*r^3*p11-12*m^2*g*d*sin(theta)*p21^3*p22*r^3*p11^2-...
            2*m^2*g*d^2*sin(theta)*p21^3*cos(phi)*p11^3*r^2-↙
4*p21^6*m*d^2*g*sin(-theta+phi)*Ib+4*p21^6*m^2*d^3*g*r*sin(-theta+...
            2*phi)+6*p21^5*m^2*d^3*g*r*sin(-theta+2*phi)*p11-m^2*g*r^4*cos↙
(phi)*sin(theta)*p21^6+2*p21^5*p22*m*r^2*g*sin(-theta+...
            phi)*Ib+p21^4*p22^2*m^2*r^3*g*d*sin(-theta+2*phi)↙
+2*p21^5*p22*m^2*r^2*g*sin(-theta+phi)*d^2-...
            8*p21^4*p11^2*m^2*r^2*g*sin(-theta+phi)*d^2-↙
8*p21^4*p11^2*m*r^2*g*sin(-theta+phi)*Ib-4*p21^3*p11^3*m^2*r^4*g*sin(-↙
theta+...
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            phi)+m^2*g*d^3*sin(theta)*p21^6*r+3*m^2*g*d^3*sin(theta)↙
*p21^5*r*p11+2*m^2*g*d^3*sin(theta)*p21^4*r*p11^2+...
            6*p21^4*p22*m^2*r^2*g*sin(-theta+phi)↙
*d^2*p11+6*p21^4*p22*m*r^2*g*sin(-theta+phi)*Ib*p11+2*p21^4*m^2*d^3*g*r*sin(-↙
theta+...
            2*phi)*p11^2-2*p21^5*m^2*d^3*g*r*sin(-theta+2*phi)*p22-↙
2*p21^4*m^2*d^3*g*r*sin(-theta+2*phi)*p11*p22-...
            4*m^2*g*r^4*cos(phi)*sin(theta)*p21^5*p11-5*m^2*g*r^4*cos(phi)↙
*sin(theta)*p21^4*p11^2+2*m^2*g*r^3*cos(phi)^2*sin(theta)*d*p21^6+...
            7*m^2*g*r^3*cos(phi)^2*sin(theta)*d*p21^5*p11+7*m^2*g*r^3*cos↙
(phi)^2*sin(theta)*d*p21^4*p11^2+m^2*g*r^2*sin(phi)*cos(theta)*p21^6*d^2+...
            3*m^2*g*r^2*sin(phi)*cos(theta)*p21^5*d^2*p11+2*m^2*g*r^2*sin↙
(phi)*cos(theta)*p21^4*d^2*p11^2+4*m^2*g*r^4*sin(phi)*cos(theta)↙
*p21^5*p11+...
            5*m^2*g*r^4*sin(phi)*cos(theta)*p21^4*p11^2+2*m^2*g*r^4*sin(phi)↙
*cos(theta)*p21^3*p11^3+m^2*g*d^2*sin(theta)*p21^5*cos(phi)*p22*r^2+...
            3*m^2*g*d^2*sin(theta)*p21^4*cos(phi)*p22*r^2*p11+2*m^2*g*d^2*sin↙
(theta)*p21^3*cos(phi)*p22*r^2*p11^2-...
            8*d*cos(phi)*p21^5*m*r^2*F6*phi-8*d*cos(phi)↙
*p21^4*m*r^2*F6*theta*p11+4*d*cos(phi)*p21^6*m*r*g*sin(-theta+phi)*Ib-...
            28*p21^3*d*cos(phi)*p11^2*m*r^2*F6*phi-m^2*g*r^3*cos(phi)^2*sin↙
(theta)*p21^5*d*p22-3*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^4*d*p22*p11-...
            2*m^2*g*r^3*cos(phi)^2*sin(theta)↙
*p21^3*d*p22*p11^2+2*m^2*g*r^3*cos(phi)^2*sin(theta)*p21^3*d*p11^3-...
            2*m^2*g*r^3*sin(phi)*cos(theta)*d*cos(phi)*p21^6+m^2*g*r^3*sin↙
(phi)*cos(theta)*p21^5*d*cos(phi)*p22+...
            3*m^2*g*r^3*sin(phi)*cos(theta)*p21^4*d*cos(phi)↙
*p22*p11+2*m^2*g*r^3*sin(phi)*cos(theta)*p21^3*d*cos(phi)*p22*p11^2-...
            2*p21^6*m^2*r^4*g*sin(-theta+phi)+6*p21^3*d*cos(phi)↙
*p22^2*m^2*r^3*g*sin(-theta+phi)*p11+...
            4*p21^2*d*cos(phi)*p22^2*m^2*r^3*g*sin(-theta+phi)↙
*p11^2+8*p21^2*d*cos(phi)*p22*m*r^2*F6*p11^2*phi+...
            8*p21*d*cos(phi)*p22*m*r^2*F6*p11^3*theta+4*p21^4*d*cos(phi)↙
*p22*m*r^2*F6*phi+4*p21^3*d*cos(phi)*p22*m*r^2*F6*theta*p11-...
            2*p21^5*d*cos(phi)*p22*m*r*g*sin(-theta+phi)*Ib+12*p21^3*d*cos↙
(phi)*p22*m*r^2*F6*p11*phi+4*p21^3*m^2*d^2*g*sin(-theta+...
            phi)*p11^2*p22*r^2-m^2*g*r^4*sin(phi)*cos(theta)*p21^5*p22-↙
4*m^2*g*r^4*sin(phi)*cos(theta)*p21^4*p22*p11-...
            5*m^2*g*r^4*sin(phi)*cos(theta)*p21^3*p22*p11^2+m^2*g*r^4*cos↙
(phi)*sin(theta)*p21^5*p22+4*m^2*g*r^4*cos(phi)*sin(theta)*p21^4*p22*p11+...
            5*m^2*g*r^4*cos(phi)*sin(theta)*p21^3*p22*p11^2-3*m^2*g*r^2*cos↙
(phi)*sin(theta)*p21^6*d^2-10*m^2*g*r^2*cos(phi)*sin(theta)*p21^5*d^2*p11-...
            9*m^2*g*r^2*cos(phi)*sin(theta)*p21^4*d^2*p11^2-↙
8*p22*m*r^3*F6*p11^4*theta+4*p21^5*m*r^3*F6*phi+4*p21^5*Ib*F6*r*phi+20*p21^3*↙
m*r^3*F6*p11^2*phi-...
↙
8*p22*m*r^3*F6*p11^3*phi*p21+20*p21^2*m*r^3*F6*p11^3*theta+4*p21^4*m*r^3*F6*t↙
heta*p11+16*p21^4*m*r^3*F6*p11*phi+16*p21^3*m*r^3*F6*p11^2*theta-...
            8*d^2*cos(phi)*p21^6*m^2*r^2*g*sin(-theta+2*phi)+4*d^3*cos(phi)↙
*p21^6*m^2*r*g*sin(-theta+phi)+4*d*cos(phi)*p21^6*m^2*r^3*g*sin(-theta+phi)↙
+...
```

```
            12*p21^2*d*cos(phi)*p22*m*r^2*F6*p11^2*theta+8*p21^5*d^2*cos(phi)↙
*p22*m^2*r^2*g*sin(-theta+2*phi)+12*p21^4*d^2*cos(phi)*p22*m^2*r^2*g*sin(-↙
theta+...
            2*phi)*p11+4*p21^3*d^2*cos(phi)*p22*m^2*r^2*g*sin(-theta+2*phi)↙
*p11^2-2*p21^4*d^2*cos(phi)*p22^2*m^2*r^2*g*sin(-theta+2*phi)-...
            2*p21^3*d^2*cos(phi)*p22^2*m^2*r^2*g*sin(-theta+2*phi)*p11-↙
2*p21^6*m^2*d^4*g*sin(-theta+phi)-20*p21^2*p22*m*r^3*F6*p11^2*phi-...
            20*p21*p22*m*r^3*F6*p11^3*theta-4*p21^4*p22*m*r^3*F6*phi-↙
2*p21^5*d^3*cos(phi)*p22*m^2*r*g*sin(-theta+phi)-...
            4*p21^4*d^3*cos(phi)*p22*m^2*r*g*sin(-theta+phi)*p11-↙
6*p21^5*d*cos(phi)*p22*m^2*r^3*g*sin(-theta+phi)-...
            4*m^2*g*d*sin(theta)*p11^3*p22*r^3*p21^2-m*g*r^2*cos(phi)*sin↙
(theta)*p21^6*Ib-3*m*g*r^2*cos(phi)*sin(theta)*p21^5*Ib*p11-...
            2*m*g*r^2*cos(phi)*sin(theta)*p21^4*Ib*p11^2-2*m^2*g*r^4*sin(phi)↙
*cos(theta)*p11^3*p22*p21^2+2*m^2*g*r^4*cos(phi)*sin(theta)↙
*p11^3*p22*p21^2+...
            4*m^2*g*d*sin(theta)↙
*p21^5*p11*r^3+8*p21^3*Ib*F6*r*p11^2*phi+8*p21^2*Ib*F6*r*p11^3*theta+4*p21^4*↙
Ib*F6*r*theta*p11+12*p21^4*Ib*F6*r*p11*phi+...
↙
8*p21^2*p11^3*m*r^3*F6*phi+8*p21*p11^4*m*r^3*F6*theta+12*p21^3*Ib*F6*r*p11^2*↙
theta-2*p21^6*Ib^2*g*sin(-theta+phi)-...
            4*p21^3*p22*m*r^3*F6*theta*p11-16*p21^3*p22*m*r^3*F6*p11*phi-↙
16*p21^2*p22*m*r^3*F6*p11^2*theta+5*m^2*g*d*sin(theta)*p21^4*p11^2*r^3+...
            2*m^2*g*d*sin(theta)*p21^3*p11^3*r^3-4*p21^5*m^2*d^4*g*sin(-↙
theta+phi)*p11-8*p21^5*m*d^2*g*sin(-theta+phi)*Ib*p11-...
            28*p21^2*d*cos(phi)*p11^3*m*r^2*F6*theta-16*p21^5*d^2*cos(phi)↙
*p11*m^2*r^2*g*sin(-theta+2*phi)-10*p21^4*d^2*cos(phi)*p11^2*m^2*r^2*g*sin(-↙
theta+...
            2*phi)-2*p21^3*d^2*cos(phi)*p11^3*m^2*r^2*g*sin(-theta+2*phi)↙
+10*p21^5*d^3*cos(phi)*p11*m^2*r*g*sin(-theta+phi)+...
            4*p21^4*d^3*cos(phi)*p11^2*m^2*r*g*sin(-theta+phi)-4↙
*p21^5*Ib^2*g*sin(-theta+phi)*p11+14*p21^5*d*cos(phi)*p11*m^2*r^3*g*sin(-↙
theta+phi)+...
            14*p21^4*d*cos(phi)*p11^2*m^2*r^3*g*sin(-theta+phi)+4*p21^4*d*cos↙
(phi)*p11^2*m*r*g*sin(-theta+phi)*Ib+...
            4*p21^3*d*cos(phi)*p11^3*m^2*r^3*g*sin(-theta+phi)↙
+8*p21^4*m^2*r^4*g*sin(-theta+phi)*p22*p11+10*p21^3*m^2*r^4*g*sin(-theta+...
            phi)*p11^2*p22+3*p22^2*m^2*r^3*g*d*sin(theta)↙
*p21^3*p11+2*p22^2*m^2*r^3*g*d*sin(theta)*p11^2*p21^2+...
            p22^2*m^2*r^3*g*d*sin(theta)*p21^4-4*p21^6*m*r^2*g*sin(-↙
theta+phi)*Ib+8*p21^3*m*d^2*F6*r*p11^2*phi+4*p21^6*m^2*r^3*g*d*sin(-theta+...
            2*phi)+10*p21^5*m^2*r^3*g*d*sin(-theta+2*phi)↙
*p11+8*p21^4*m^2*r^3*g*d*sin(-theta+2*phi)*p11^2-4*p21^5*m^2*r^3*g*d*sin(-↙
theta+...
            2*phi)*p22-7*p21^4*m^2*r^3*g*d*sin(-theta+2*phi)*p11*p22-↙
4*p21^6*m^2*r^2*g*sin(-theta+phi)*d^2-12*p21^5*m^2*r^2*g*sin(-theta+...
            phi)*d^2*p11-2*m^2*g*r^4*cos(phi)*sin(theta)↙
*p21^3*p11^3+4*p21^3*Ib*g*sin(-theta+phi)*p11^2*p22*m*r^2+...
            2*p21^3*p11^3*m^2*r^3*g*d*sin(-theta+2*phi)-3↙
*p21^3*p11^2*m^2*r^3*g*d*sin(-theta+2*phi)*p22-8*p21^5*m^2*r^4*g*sin(-↙
```

```
theta+...
            phi)*p11-12*p21^5*m*r^2*g*sin(-theta+phi)*Ib*p11-↙
10*p21^4*m^2*r^4*g*sin(-theta+phi)*p11^2+2*p21^5*m^2*r^4*g*sin(-theta+phi)↙
*p22+...
            4*p21^2*p11^3*m^2*r^4*g*sin(-theta+phi)↙
*p22+4*p21^6*Ib*g*d*m*r*sin(-theta+2*phi)+6*p21^5*Ib*g*d*m*r*sin(-↙
theta+2*phi)*p11+...
            2*p21^4*Ib*g*d*m*r*sin(-theta+2*phi)*p11^2-2*p21^5*Ib*g*d*m*r*sin↙
(-theta+2*phi)*p22-2*p21^4*Ib*g*d*m*r*sin(-theta+2*phi)*p11*p22+...
            p21^3*p22^2*m^2*r^3*g*d*sin(-theta+2*phi)↙
*p11+8*p21^2*m*d^2*F6*r*p11^3*theta+4*p21^5*m*d^2*F6*r*phi+4*p21^4*m*d^2*F6*r↙
*theta*p11+...
            12*p21^4*m*d^2*F6*r*p11*phi+12*p21^3*m*d^2*F6*r*p11^2*theta)↙
/p21^2/r/(5*p21^2*r^2*p11^2*m+...
            4*p21^3*r^2*p11*m+3*cos(phi)*p21^2*r*d*p22*m*p11+2*cos(phi)↙
*p21*r*d*p22*m*p11^2-7*cos(phi)*p21^3*r*d*m*p11-2*cos(phi)*p21^4*r*d*m+...
            2*p21*r^2*p11^3*m+2*p21^2*Ib*p11^2+3*p21^3*Ib*p11-↙
p21^3*r^2*p22*m+p21^4*m*d^2+cos(phi)*p21^3*r*d*p22*m-2*cos(phi)↙
*p21*r*d*p11^3*m-...
            7*cos(phi)*p21^2*r*d*m*p11^2-5*p21*r^2*p11^2*p22*m-↙
4*p21^2*r^2*p11*p22*m-2*r^2*p11^3*p22*m+...
            3*p21^3*m*d^2*p11+2*p21^2*m*d^2*p11^2+p21^4*r^2*m+p21^4*Ib);
tau      = F1+F2+F3;

%% Lyapunov
Fmc1     = [1/3*(3*p21*thetadot^2*phidot*m*r^2*d*sin(phi)-4↙
*p21*thetadot^2*m*d^2*cos(phi)*r*sin(phi)↙
*phidot+2*thetadot*p11*r*phidot*sigma+...
            2*thetadot^2*p11*r*nu+2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos↙
(phi)*p22-2*p21*thetadot^3*Ib*d*sin(phi)+2*r*phidot*p21*thetadot*nu-...
            2*thetadot^3*d^2*cos(phi)*p22*m*r*sin(phi)-↙
r^2*m*phidot^3*p21*d*sin(phi)+2*r^2*m*phidot^2*thetadot*d*sin(phi)*p22+...
            2*thetadot^3*m*p11*r*d^2*cos(phi)*sin(phi)-2↙
*p21*thetadot^3*m*r^2*d*sin(phi)+p21*thetadot^2*phidot*m*d^3*sin(phi)-...
            2*thetadot^3*m*p11*r^2*d*sin(phi)-2↙
*thetadot*m*p11*r^2*phidot^2*d*sin(phi)+2*thetadot^3*r^2*m*p22*d*sin(phi)+...
            4*thetadot^2*m*p11*r^2*d*sin(phi)*phidot-↙
4*thetadot^2*phidot*m*r^2*d*sin(phi)*p22+p21*thetadot^2*phidot*d*sin(phi)*Ib-↙
...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p11-↙
2*p21*thetadot^3*m*d^3*sin(phi)+2*r*phidot^2*p21*sigma+...
            4*p21*thetadot^3*m*d^2*cos(phi)*r*sin(phi))/r/↙
(phidot^2+thetadot^2+2*phidot*thetadot),1/4*↙
(3*p21*thetadot^2*phidot*m*r^2*d*sin(phi)-...
            4*p21*thetadot^2*m*d^2*cos(phi)*r*sin(phi)↙
*phidot+2*thetadot*p11*r*phidot*sigma+2*thetadot^2*p11*r*nu+...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p22-↙
2*p21*thetadot^3*Ib*d*sin(phi)+2*r*phidot*p21*thetadot*nu-...
            2*thetadot^3*d^2*cos(phi)*p22*m*r*sin(phi)-↙
r^2*m*phidot^3*p21*d*sin(phi)+2*r^2*m*phidot^2*thetadot*d*sin(phi)*p22+...
            2*thetadot^3*m*p11*r*d^2*cos(phi)*sin(phi)-2↙
```

```
*p21*thetadot^3*m*r^2*d*sin(phi)+p21*thetadot^2*phidot*m*d^3*sin(phi)-...
            2*thetadot^3*m*p11*r^2*d*sin(phi)-2↙
*thetadot*m*p11*r^2*phidot^2*d*sin(phi)+2*thetadot^3*r^2*m*p22*d*sin(phi)+...
            4*thetadot^2*m*p11*r^2*d*sin(phi)*phidot-↙
4*thetadot^2*phidot*m*r^2*d*sin(phi)*p22+p21*thetadot^2*phidot*d*sin(phi)*Ib-↙
...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p11-↙
2*p21*thetadot^3*m*d^3*sin(phi)+2*r*phidot^2*p21*sigma+...
            4*p21*thetadot^3*m*d^2*cos(phi)*r*sin(phi))/r/↙
(phidot^2+thetadot^2+2*phidot*thetadot);1/4*↙
(3*p21*thetadot^2*phidot*m*r^2*d*sin(phi)-...
            4*p21*thetadot^2*m*d^2*cos(phi)*r*sin(phi)↙
*phidot+2*thetadot*p11*r*phidot*sigma+2*thetadot^2*p11*r*nu+...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p22-↙
2*p21*thetadot^3*Ib*d*sin(phi)+2*r*phidot*p21*thetadot*nu-...
            2*thetadot^3*d^2*cos(phi)*p22*m*r*sin(phi)-↙
r^2*m*phidot^3*p21*d*sin(phi)+2*r^2*m*phidot^2*thetadot*d*sin(phi)*p22+...
            2*thetadot^3*m*p11*r*d^2*cos(phi)*sin(phi)-2↙
*p21*thetadot^3*m*r^2*d*sin(phi)+p21*thetadot^2*phidot*m*d^3*sin(phi)-...
            2*thetadot^3*m*p11*r^2*d*sin(phi)-2↙
*thetadot*m*p11*r^2*phidot^2*d*sin(phi)+2*thetadot^3*r^2*m*p22*d*sin(phi)+...
            4*thetadot^2*m*p11*r^2*d*sin(phi)*phidot-↙
4*thetadot^2*phidot*m*r^2*d*sin(phi)*p22+p21*thetadot^2*phidot*d*sin(phi)*Ib-↙
...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p11-↙
2*p21*thetadot^3*m*d^3*sin(phi)+2*r*phidot^2*p21*sigma+...
            4*p21*thetadot^3*m*d^2*cos(phi)*r*sin(phi))/r/↙
(phidot^2+thetadot^2+2*phidot*thetadot),1/4*↙
(3*p21*thetadot^2*phidot*m*r^2*d*sin(phi)-...
            4*p21*thetadot^2*m*d^2*cos(phi)*r*sin(phi)↙
*phidot+2*thetadot*p11*r*phidot*sigma+2*thetadot^2*p11*r*nu+...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p22-↙
2*p21*thetadot^3*Ib*d*sin(phi)+2*r*phidot*p21*thetadot*nu-...
            2*thetadot^3*d^2*cos(phi)*p22*m*r*sin(phi)-↙
r^2*m*phidot^3*p21*d*sin(phi)+2*r^2*m*phidot^2*thetadot*d*sin(phi)*p22+...
            2*thetadot^3*m*p11*r*d^2*cos(phi)*sin(phi)-2↙
*p21*thetadot^3*m*r^2*d*sin(phi)+p21*thetadot^2*phidot*m*d^3*sin(phi)-...
            2*thetadot^3*m*p11*r^2*d*sin(phi)-2↙
*thetadot*m*p11*r^2*phidot^2*d*sin(phi)+2*thetadot^3*r^2*m*p22*d*sin(phi)+...
            4*thetadot^2*m*p11*r^2*d*sin(phi)*phidot-↙
4*thetadot^2*phidot*m*r^2*d*sin(phi)*p22+p21*thetadot^2*phidot*d*sin(phi)*Ib-↙
...
            2*thetadot^2*phidot*d^2*sin(phi)*r*m*cos(phi)*p11-↙
2*p21*thetadot^3*m*d^3*sin(phi)+2*r*phidot^2*p21*sigma+...
            4*p21*thetadot^3*m*d^2*cos(phi)*r*sin(phi))/r/↙
(phidot^2+thetadot^2+2*phidot*thetadot)];

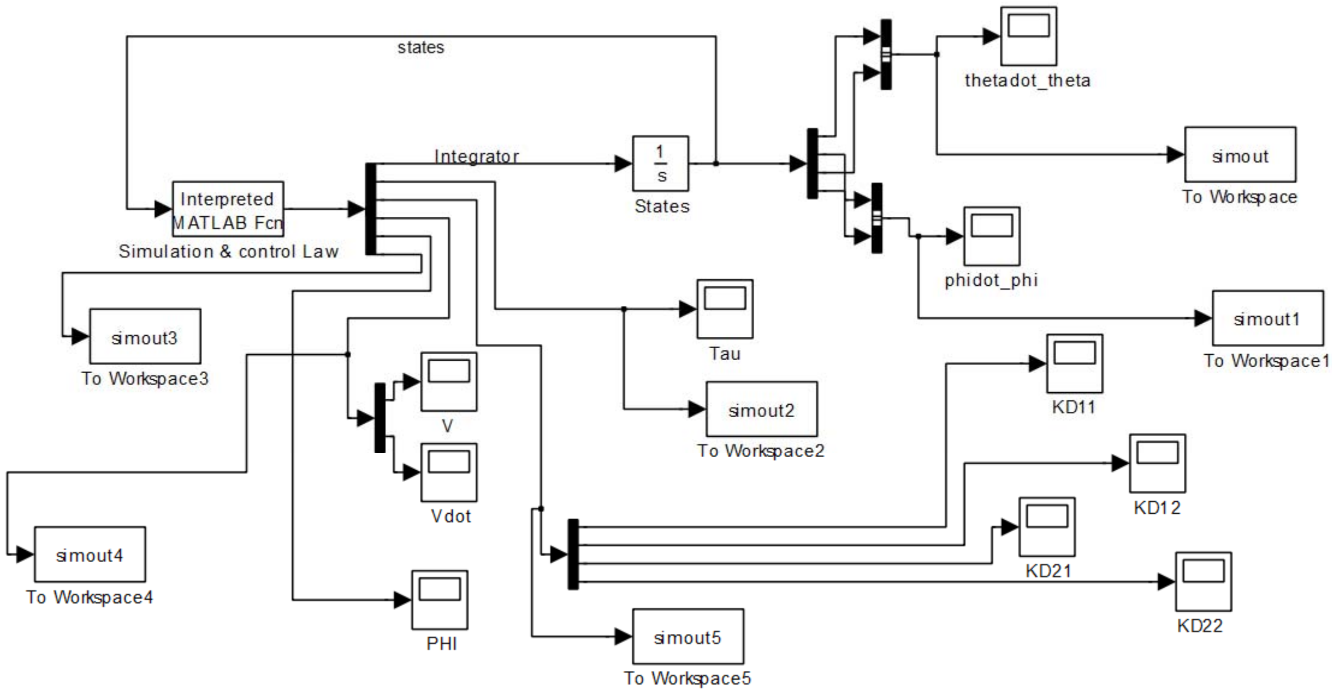
PHI      =(-6*g*cos(-theta+phi)*p21*p22*m*r^2*p11-4*g*cos(-theta+phi)↙
*p11^2*p22*m*r^2+...
            4*F6*(phi*p21+theta*p11)^2/p21^2*r*p11^2-2*g*p21^3*d*cos(theta)↙
*m*r-...
```

```
            7*g*p21^2*d*cos(theta)*m*r*p11-7*g*p21*d*cos(theta)*m*r*p11^2-↙
2*g*d*cos(theta)*m*p11^3*r+...
            g*d*cos(theta)*m*p22*r*p21^2+3*g*d*cos(theta)↙
*m*p22*r*p21*p11+2*g*d*cos(theta)*m*p22*r*p11^2+...
            2*F6*(phi*p21+theta*p11)^2*r+2*g*cos(-theta+phi)*p21^3*Ib+6*F6*↙
(phi*p21+theta*p11)^2/p21*r*p11-...
            2*g*d*m*r*p21^3*cos(-theta+2*phi)-3*g*d*m*r*p21^2*cos(-↙
theta+2*phi)*p11-g*d*m*r*p21*cos(-theta+...
            2*phi)*p11^2+g*d*m*r*p21^2*cos(-theta+2*phi)*p22+g*d*m*r*p21*cos↙
(-theta+2*phi)*p11*p22+...
            2*g*cos(-theta+phi)*p21^3*m*d^2+4*g*cos(-theta+phi)↙
*p21^2*m*d^2*p11+2*g*cos(-theta+...
            phi)*p21^3*m*r^2+6*g*cos(-theta+phi)*p21^2*m*r^2*p11+4*g*cos(-↙
theta+phi)*p21^2*Ib*p11+...
            4*g*cos(-theta+phi)*p21*p11^2*m*r^2-2*g*cos(-theta+phi)↙
*p21^2*p22*m*r^2)/(p21+2*p11)/r/(p21+p11);
 Vc      = 0;
 V       = 0.5*qdot'*KD*qdot+PHI+Vc;
 Vdot    = -qdot'*(Kv+Fmc1)*qdot;

%% Evaluate the Dynamics

qdotdot  = inv(mass)*([tau;0]-C*qdot-G);
ddtheta  = qdotdot(1);
ddphi    = qdotdot(2);
%% M-File Output
dxdt  =[thetadot;phidot;ddtheta;ddphi;tau(1);KD(1,1);KD(1,2);KD(2,1);KD(2,2);↙
V;Vdot;PHI;KDdet];
%% End of the Function
```

B.8 Simulink file for the ball and arc system, P almost constant



Appendix C - Inverted pendulum cart system

The presentation of this Appendix is organized in six major parts. These are:

C.1 Dynamics of the inverted pendulum cart system

C.2 Lagrangian KD for the inverted pendulum cart system, solving Eq.2.25

C.3 Direct Lyapunov Approach formulation for the inverted pendulum cart system, inverted pendulum cart system case when F_1 is and is not considered

C.4 MATLAB code for the simulations of the IPC system

C.5 MATLAB code for the simulations of the IPC system when F_1 non-zero

C.6 Simulink file for the inverted pendulum cart system when F_1 is zero and non-zero

C.1 Dynamics of the Inverted Pendulum Cart system IPC_dynamics.mw

Lagrangian Approach

The Lagrangian (L) is defined as KE-PE, where KE is the kinetic energy and PE is the potential energy of the system expressed in a minimum set of generalized coordinates. In the current case the generalized coordinates are x, and θ.

> restart;

> with(LinearAlgebra) :

$$> \mathrm{KE} := \frac{1}{2}\cdot Mc\cdot\left(\frac{d}{dt}\,x(t)\right)^2 + \frac{1}{2}\cdot Mp\cdot\left(\frac{d}{dt}\,x(t) - \frac{\left(\frac{d}{dt}\,\theta(t)\right)\cdot l}{2}\cdot\cos(\theta(t))\right)^2 + \frac{1}{2}\cdot Mp\cdot\left(-\frac{\left(\frac{d}{dt}\,\theta(t)\right)\cdot l}{2}\cdot\sin(\theta(t))\right)^2 + \frac{1}{2}\cdot(Ip)\cdot\left(\frac{d}{dt}\,\theta(t)\right)^2 :$$

$$> PE := \frac{m\cdot g\cdot l}{2}\cdot\cos(\theta(t)) :$$

$$PE := \frac{1}{2}\,m\,g\,l\cos(\theta(t)) \quad \textbf{(1)}$$

> with(VariationalCalculus)

$$[ConjugateEquation, Convex, EulerLagrange, Jacobi, Weierstrass] \quad \textbf{(2)}$$

> L := -simplify(KE − PE);

$$L := -\frac{1}{2}\,Mc\left(\frac{d}{dt}\,x(t)\right)^2 - \frac{1}{2}\,Mp\left(\frac{d}{dt}\,x(t)\right)^2 + \frac{1}{2}\,Mp\left(\frac{d}{dt}\,x(t)\right)\left(\frac{d}{dt}\,\theta(t)\right)l\cos(\theta(t)) - \frac{1}{8}\,Mp\left(\frac{d}{dt}\,\theta(t)\right)^2 l^2 - \frac{1}{2}\,Ip\left(\frac{d}{dt}\,\theta(t)\right)^2 + \frac{1}{2}\,m\,g\,l\cos(\theta(t)) \quad \textbf{(3)}$$

Defining temporary substitutions and variables

$$> temp1 := \left[\left(\frac{d^2}{dt^2}\,x(t)\right) = xddot, x(t) = x, \left(\frac{d^2}{dt^2}\,\theta(t)\right) = \theta ddot, \left(\frac{d}{dt}\,x(t)\right) = xdot, \left(\frac{d}{dt}\,\theta(t)\right) = \theta dot, \theta(t) = \theta\right];$$

$$temp1 := \left[\frac{d^2}{dt^2}\,x(t) = xddot, x(t) = x, \frac{d^2}{dt^2}\,\theta(t) = \theta ddot, \frac{d}{dt}\,x(t) = xdot, \frac{d}{dt}\,\theta(t) = \theta dot, \theta(t) = \theta\right] \quad \textbf{(4)}$$

$$> temp2 := [xddot, \theta ddot, xdot, \theta dot, xdot^2, \theta dot^2];$$

$$temp2 := [xddot, \theta ddot, xdot, \theta dot, xdot^2, \theta dot^2] \quad \textbf{(5)}$$

> $EL1 := EulerLagrange(L, t, [x(t), \theta(t)])[1] = 0$

$$EL1 := Mc \left(\frac{d^2}{dt^2} x(t)\right) + Mp \left(\frac{d^2}{dt^2} x(t)\right) - \frac{1}{2} Mp \left(\frac{d^2}{dt^2} \theta(t)\right) l \cos(\theta(t)) + \frac{1}{2} Mp \left(\frac{d}{dt} \theta(t)\right)^2 l \sin(\theta(t)) = 0 \quad \textbf{(6)}$$

> $EL2 := EulerLagrange(L, t, [x(t), \theta(t)])[2] = 0$

$$EL2 := -\frac{1}{2} m g l \sin(\theta(t)) - \frac{1}{2} Mp \left(\frac{d^2}{dt^2} x(t)\right) l \cos(\theta(t)) + \frac{1}{4} Mp \left(\frac{d^2}{dt^2} \theta(t)\right) l^2 + Ip \left(\frac{d^2}{dt^2} \theta(t)\right) = 0 \quad \textbf{(7)}$$

The first governing equation is

Equation1

> $EQx := expand(eval(EL1, temp1));$

$$EQx := Mc\, xddot + Mp\, xddot - \frac{1}{2} Mp\, \theta ddot\, l \cos(\theta) + \frac{1}{2} Mp\, \theta dot^2\, l \sin(\theta) = 0 \quad \textbf{(8)}$$

> $EQx := collect(EQx, temps2);$

$$EQx := Mc\, xddot + Mp\, xddot - \frac{1}{2} Mp\, \theta ddot\, l \cos(\theta) + \frac{1}{2} Mp\, \theta dot^2\, l \sin(\theta) = 0 \quad \textbf{(9)}$$

The second governing equation is

Equation2

> $EQ\theta := simplify(eval(\mathrm{EL2}))$:

> $EQ\theta := expand(eval(EQ\theta, temp1))$:

> $EQ\theta := collect(EQ\theta, temp2);$

$$EQ\theta := -\frac{1}{2} Mp\, xddot\, l \cos(\theta) + \left(\frac{1}{4} Mp\, l^2 + Ip\right) \theta ddot - \frac{1}{2} m g l \sin(\theta) = 0 \quad \textbf{(10)}$$

> $mass := simplify\left(Matrix\left(2, 2, \left[(Mc + Mp), -\frac{1}{2} Mp\, l \cos(\theta), -\frac{1}{2} Mp \cdot l \cos(\theta), \left(\frac{1}{4} Mp\, l^2 + Ip\right)\right]\right)\right);$

$$mass := \begin{bmatrix} Mc + Mp & -\frac{1}{2} Mp\, l \cos(\theta) \\ -\frac{1}{2} Mp\, l \cos(\theta) & \frac{1}{4} Mp\, l^2 + Ip \end{bmatrix} \quad \textbf{(11)}$$

> $mass := simplify\left(eval\left(mass, \left[Ip = \frac{1}{12} \cdot m \cdot l^2, (Mc + Mp) = mb, Mp = m\right]\right)\right);$

(12)

$$mass := \begin{bmatrix} mb & -\frac{1}{2}\, m\, l \cos(\theta) \\ -\frac{1}{2}\, m\, l \cos(\theta) & \frac{1}{3}\, m\, l^2 \end{bmatrix} \quad (12)$$

$$> masst := simplify\left(Matrix\left(2, 2, \left[(Mc + Mp), -\frac{1}{2}\, Mp\, l \cos(\theta(t)), -\frac{1}{2}\, Mp \cdot l \cos(\theta(t)), \left(\frac{1}{4}\, Mp\, l^2 + Ip\right)\right]\right)\right):$$

$$> masst := simplify\left(eval\left(masst, \left[Ip = \frac{1}{12} \cdot m \cdot l^2, (Mc + Mp) = mb, Mp = m\right]\right)\right):$$

$$> Mdot := map(diff, masst, t);$$

$$Mdot := \begin{bmatrix} 0 & \frac{1}{2}\, m\, l \sin(\theta(t)) \left(\frac{d}{dt}\, \theta(t)\right) \\ \frac{1}{2}\, m\, l \sin(\theta(t)) \left(\frac{d}{dt}\, \theta(t)\right) & 0 \end{bmatrix} \quad (13)$$

Cmatrix check

$$C_{ij}(\boldsymbol{q}, \boldsymbol{qdot}) = [\boldsymbol{jk}, \boldsymbol{i}]qdot^{\boldsymbol{k}} = \frac{1}{2}\left[\frac{\partial}{\partial \boldsymbol{q}^{\boldsymbol{k}}}\, m_{\boldsymbol{i},\boldsymbol{j}} + \frac{\partial}{\partial \boldsymbol{q}^{\boldsymbol{j}}}\, m_{\boldsymbol{k},\boldsymbol{i}} - \frac{\partial}{\partial \boldsymbol{q}^{\boldsymbol{i}}}\, m_{\boldsymbol{j},\boldsymbol{k}}\right]qdot^{\boldsymbol{k}}$$

where m_{ij} is the ijth component of the mass matrix and [jk,i] is the Christoffel symbol of the first kind.

$$> C11 := \frac{1}{2}\left(\frac{\partial}{\partial x}\, mass_{1,1} + \frac{\partial}{\partial x}\, mass_{1,1} - \frac{\partial}{\partial x}\, mass_{1,1}\right) \cdot xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{1,1} + \frac{\partial}{\partial x}\, mass_{2,1} - \frac{\partial}{\partial x}\, mass_{1,2}\right) \cdot \theta dot;$$

$$C11 := 0 \quad (14)$$

$$> C12 := \frac{1}{2}\left(\frac{\partial}{\partial x}\, mass_{2,1} + \frac{\partial}{\partial \theta}\, mass_{1,1} - \frac{\partial}{\partial x}\, mass_{2,1}\right) \cdot xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{2,1} + \frac{\partial}{\partial \theta}\, mass_{2,1} - \frac{\partial}{\partial x}\, mass_{2,2}\right) \cdot \theta dot;$$

$$C12 := \frac{1}{2}\, m\, l \sin(\theta)\, \theta dot \quad (15)$$

$$> C21 := \frac{1}{2}\left(\frac{\partial}{\partial x}\, mass_{1,2} + \frac{\partial}{\partial x}\, mass_{1,2} - \frac{\partial}{\partial \theta}\, mass_{1,1}\right) \cdot xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta}\, mass_{1,2} + \frac{\partial}{\partial x}\, mass_{2,2} - \frac{\partial}{\partial \theta}\, mass_{1,2}\right) \cdot \theta dot;$$

$$C21 := 0 \tag{16}$$

> $C22 := \frac{1}{2}\left(\frac{\partial}{\partial x} mass_{2,2} + \frac{\partial}{\partial \theta} mass_{1,2} - \frac{\partial}{\partial \theta} mass_{2,1}\right) \cdot xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta} mass_{2,2} + \frac{\partial}{\partial \theta} mass_{2,2} - \frac{\partial}{\partial \theta} mass_{2,2}\right) \cdot \theta dot;$

$$C22 := 0 \tag{17}$$

> $Cmatrix := Matrix(2, 2, [C11, C12, C21, C22]);$

$$Cmatrix := \begin{bmatrix} 0 & \frac{1}{2}\, m\, l \sin(\theta)\ \theta dot \\ 0 & 0 \end{bmatrix} \tag{18}$$

Differentiating the mass matrix with respect to time produces

> $\mathrm{Mdot} := simplify\left(eval\left(Mdot, \left[\left(\frac{d^2}{dt^2} x(t)\right) = xddot, x(t) = x, \left(\frac{d^2}{dt^2} \theta(t)\right) = \theta ddot, \left(\frac{d}{dt} x(t)\right) = xdot, \left(\frac{d}{dt} \theta(t)\right) = \theta dot, \theta(t) = \theta\right]\right)\right);$

$$Mdot := \begin{bmatrix} 0 & \frac{1}{2}\, m\, l \sin(\theta)\ \theta dot \\ \frac{1}{2}\, m\, l \sin(\theta)\ \theta dot & 0 \end{bmatrix} \tag{19}$$

Because M is Lagrangian $\frac{1}{2} \cdot$ *Mdot* $-$ *Cmatrix* is skew-symmetric. The result is

> $FMCskew := \left(\frac{1}{2} \cdot Mdot - Cmatrix\right);$

$$FMCskew := \begin{bmatrix} 0 & -\frac{1}{4}\, m\, l \sin(\theta)\ \theta dot \\ \frac{1}{4}\, m\, l \sin(\theta)\ \theta dot & 0 \end{bmatrix} \tag{20}$$

This is a special case of the FMC when KD =M, for which

> $FMC := (\,Mdot - Cmatrix - Transpose(Cmatrix)\,);$

$$FMC := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

End of the file

C.2 Lagrangian KD_Inverted Pendulum Cart system

LagrangianKD_IPC.mw

Eq. 2.25 Solution - $-\boldsymbol{C}^T\boldsymbol{M}^{-1}\boldsymbol{K}_D + \frac{1}{2}\left(\frac{\partial}{\partial q}\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\right) = 0$

```
> restart :
```

```
> with(LinearAlgebra) :
```

Definitions

$> mass := Matrix\left(2, 2, \left[mb, -\frac{m \cdot l \cdot \cos(\theta)}{2}, -\frac{m \cdot l \cdot \cos(\theta)}{2}, \frac{m \cdot l^2}{3}\right]\right);$

$$mass := \begin{bmatrix} mb & -\frac{1}{2}\, m\, l \cos(\theta) \\ -\frac{1}{2}\, m\, l \cos(\theta) & \frac{1}{3}\, m\, l^2 \end{bmatrix} \quad \textbf{(1.1)}$$

$> q := \begin{bmatrix} x \\ \theta \end{bmatrix}:$

$> qdot := \begin{bmatrix} xdot \\ \theta dot \end{bmatrix}:$

$> C := Matrix\left(2, 2, \left[0, \frac{m \cdot l \cdot \sin(\theta) \cdot \theta dot}{2}, 0, 0\right]\right);$

$$C := \begin{bmatrix} 0 & \frac{1}{2}\, m\, l \sin(\theta)\ \theta dot \\ 0 & 0 \end{bmatrix} \quad \textbf{(1.2)}$$

$>$

$> KD := Matrix(2, 2, [KD11(x, \theta), KD12(x, \theta), KD21(x, \theta), KD22(x, \theta)]) :$

Term 1

$> KDq := Transpose\left(Multiply\left(\frac{1}{2} \cdot KD, qdot\right)\right);$

$$KDq := \left[\frac{1}{2}\, KD11(x, \theta)\ xdot + \frac{1}{2}\, KD12(x, \theta)\ \theta dot, \frac{1}{2}\, KD21(x, \theta)\ xdot + \frac{1}{2}\, KD22(x, \theta)\ \theta dot\right] \quad \textbf{(2.1)}$$

> $Term1 := \begin{bmatrix} diff(KDq[1], x) & diff(KDq[2], x) \\ diff(KDq[1], \theta) & diff(KDq[2], \theta) \end{bmatrix};$

$$Term1 := \left[\left[\frac{1}{2}\left(\frac{\partial}{\partial x} KD11(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial x} KD12(x,\theta)\right) \theta dot, \frac{1}{2}\left(\frac{\partial}{\partial x} KD21(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial x} KD22(x,\theta)\right) \theta dot\right], \left[\frac{1}{2}\left(\frac{\partial}{\partial \theta} KD11(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD12(x,\theta)\right) \theta dot, \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD21(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD22(x,\theta)\right) \theta dot\right]\right] \quad (2.2)$$

Term 2

> $Term2 := Transpose(Multiply(Multiply(KD, MatrixInverse(mass)), C));$

$$Term2 := \left[[0, 0], \left[\frac{1}{2}\left(-\frac{4\,KD11(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{6\,KD12(x,\theta)\cos(\theta)}{l\left(-4\,mb + 3\,m\cos(\theta)^2\right)}\right) m\,l\sin(\theta)\,\theta dot, \frac{1}{2}\left(-\frac{4\,KD21(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{6\,KD22(x,\theta)\cos(\theta)}{l\left(-4\,mb + 3\,m\cos(\theta)^2\right)}\right) m\,l\sin(\theta)\,\theta dot\right]\right] \quad (3.1)$$

The Eq. 2.25 is expresed as the following Eq. This is the equation to be Solved

> $Eq := MatrixAdd(Term1, -Term2);$

$$Eq := \left[\left[\frac{1}{2}\left(\frac{\partial}{\partial x} KD11(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial x} KD12(x,\theta)\right) \theta dot, \frac{1}{2}\left(\frac{\partial}{\partial x} KD21(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial x} KD22(x,\theta)\right) \theta dot\right], \left[\frac{1}{2}\left(\frac{\partial}{\partial \theta} KD11(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD12(x,\theta)\right) \theta dot - \frac{1}{2}\left(-\frac{4\,KD11(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{6\,KD12(x,\theta)\cos(\theta)}{l\left(-4\,mb + 3\,m\cos(\theta)^2\right)}\right) m\,l\sin(\theta)\,\theta dot, \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD21(x,\theta)\right) xdot + \frac{1}{2}\left(\frac{\partial}{\partial \theta} KD22(x,\theta)\right) \theta dot - \frac{1}{2}\left(\right.\right.\right. \quad (4.1)$$

$$-\frac{4\,KD21(x,\theta)}{-4\,mb+3\,m\cos(\theta)^2}-\frac{6\,KD22(x,\theta)\cos(\theta)}{l\left(-4\,mb+3\,m\cos(\theta)^2\right)}\Bigg)\,m\,l\sin(\theta)\ \theta dot\Bigg]\Bigg]$$

Coefficient Matrices of *xdot* and *θdot*

> Eq1 := Matrix(2, map(coeff, Eq, xdot));

$$Eq1 := \begin{bmatrix} \frac{1}{2}\frac{\partial}{\partial x}KD11(x,\theta) & \frac{1}{2}\frac{\partial}{\partial x}KD21(x,\theta) \\ \frac{1}{2}\frac{\partial}{\partial \theta}KD11(x,\theta) & \frac{1}{2}\frac{\partial}{\partial \theta}KD21(x,\theta) \end{bmatrix} \quad \textbf{(5.1)}$$

> Eq2 := Matrix(2, map(coeff, Eq, θdot));

$$Eq2 := \Bigg[\Bigg[\frac{1}{2}\frac{\partial}{\partial x}KD12(x,\theta), \frac{1}{2}\frac{\partial}{\partial x}KD22(x,\theta)\Bigg], \quad \textbf{(5.2)}$$

$$\Bigg[\frac{1}{2}\frac{\partial}{\partial \theta}KD12(x,\theta)-\frac{1}{2}\Bigg(-\frac{4\,KD11(x,\theta)}{-4\,mb+3\,m\cos(\theta)^2}$$

$$-\frac{6\,KD12(x,\theta)\cos(\theta)}{l\left(-4\,mb+3\,m\cos(\theta)^2\right)}\Bigg)\,m\,l\sin(\theta), \frac{1}{2}\frac{\partial}{\partial \theta}KD22(x,\theta)-\frac{1}{2}\Bigg($$

$$-\frac{4\,KD21(x,\theta)}{-4\,mb+3\,m\cos(\theta)^2}-\frac{6\,KD22(x,\theta)\cos(\theta)}{l\left(-4\,mb+3\,m\cos(\theta)^2\right)}\Bigg)\,m\,l\sin(\theta)\Bigg]\Bigg]$$

Coefficient Matrices of Derivatives and K_D Elements

> KDx := Matrix(2, map(diff, KD, x));

$$KDx := \begin{bmatrix} \frac{\partial}{\partial x}KD11(x,\theta) & \frac{\partial}{\partial x}KD12(x,\theta) \\ \frac{\partial}{\partial x}KD21(x,\theta) & \frac{\partial}{\partial x}KD22(x,\theta) \end{bmatrix} \quad \textbf{(6.1)}$$

> KDθ := Matrix(2, map(diff, KD, θ));

$$KD\theta := \begin{bmatrix} \frac{\partial}{\partial \theta}KD11(x,\theta) & \frac{\partial}{\partial \theta}KD12(x,\theta) \\ \frac{\partial}{\partial \theta}KD21(x,\theta) & \frac{\partial}{\partial \theta}KD22(x,\theta) \end{bmatrix} \quad \textbf{(6.2)}$$

> DKD := Matrix(8, 1, [ ]) :

> Eqsc := Matrix(8, 1, [ ]) :

> LSCM := Matrix(8, [ ]) :

> **for** *i* **from** 1 **to** 4 **do**

for j **from** 1 **to** 4 **do**

$DKD_i := KDx(i)$:

$DKD_{i+4} := KD\theta(i)$;

$Eqsc_i := Eq1(i)$;

$Eqsc_{i+4} := Eq2(i)$;

$k := i + 4$;

$l := j + 4$;

$LSCM_{i,j} := coeff(Eq1(i), KDx(j))$;

$LSCM_{i,l} := coeff(Eq1(i), KD\theta(j))$;

$LSCM_{k,j} := coeff(Eq2(i), KDx(j))$;

$LSCM_{k,l} := coeff(Eq2(i), KD\theta(j))$;

end do

end do

> $map(eval, LSCM)$;

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{6.3}$$

The above matrix contents the coefficients of $\frac{\partial}{\partial x}$ *KD* and $\frac{\partial}{\partial \theta}$ *KD*

> $map(eval, Eqsc)$;

$$\left[\left[\frac{1}{2}\frac{\partial}{\partial x} KD11(x, \theta)\right], \tag{6.4}$$

$$\left[\frac{1}{2}\frac{\partial}{\partial \theta} KD11(x, \theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial x}\,KD21(x,\theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial \theta}\,KD21(x,\theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial x}\,KD12(x,\theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial \theta}\,KD12(x,\theta) - 4\left(-\frac{4\,KD11(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{3}{4}\,\frac{KD12(x,\theta)\cos(\theta)}{-4\,mb + 3\,m\cos(\theta)^2}\right) m\sin(\theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial x}\,KD22(x,\theta)\right],$$

$$\left[\frac{1}{2}\,\frac{\partial}{\partial \theta}\,KD22(x,\theta) - 4\left(-\frac{4\,KD21(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{3}{4}\,\frac{KD22(x,\theta)\cos(\theta)}{-4\,mb + 3\,m\cos(\theta)^2}\right) m\sin(\theta)\right]\Bigg]$$

> $Eqcm := MatrixAdd(Eqsc, Multiply(-LSCM, DKD));$

$$Eqcm := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4\left(-\frac{4\,KD11(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{3}{4}\,\frac{KD12(x,\theta)\cos(\theta)}{-4\,mb + 3\,m\cos(\theta)^2}\right) m\sin(\theta) \\ 0 \\ -4\left(-\frac{4\,KD21(x,\theta)}{-4\,mb + 3\,m\cos(\theta)^2} - \frac{3}{4}\,\frac{KD22(x,\theta)\cos(\theta)}{-4\,mb + 3\,m\cos(\theta)^2}\right) m\sin(\theta) \end{bmatrix} \quad \textbf{(6.5)}$$

> $RSCM := Matrix(8, 4, [\ ]):$

```
> for i from 1 to 8 do
  for j from 1 to 4 do
   RSCM[i, j] := coeff(Eqcm(i), KD(j));
   end do
  end do
```

> $map(eval, RSCM);$

$$\left[\left[0,0,0,0\right],\right.$$ **(6.6)**

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[\frac{16\,m\sin(\theta)}{-4\,mb+3\,m\cos(\theta)^2},0,\frac{3\cos(\theta)\,m\sin(\theta)}{-4\,mb+3\,m\cos(\theta)^2},0\right],$$

$$\left[0,0,0,0\right],$$

$$\left.\left[0,\frac{16\,m\sin(\theta)}{-4\,mb+3\,m\cos(\theta)^2},0,\frac{3\cos(\theta)\,m\sin(\theta)}{-4\,mb+3\,m\cos(\theta)^2}\right]\right]$$

Extract Coefficient Matrices

> $Derivs := Multiply(MatrixInverse(LSCM), -RSCM);$

$$Derivs := \left[\left[0,0,0,0\right],\right.$$ **(7.1)**

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[0,0,0,0\right],$$

$$\left[-\frac{32\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0, -\frac{6 \cos(\theta)\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0\right],$$
$$\left[0, -\frac{32\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0, -\frac{6 \cos(\theta)\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}\right]\Bigg]$$

> $Mx := Derivs[[1..4], [1..4]];$

$$Mx := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.2}$$

> $KDS := \begin{bmatrix} KD11(x, \theta) \\ KD12(x, \theta) \\ KD21(x, \theta) \\ KD22(x, \theta) \end{bmatrix};$

$$KDS := \begin{bmatrix} KD11(x, \theta) \\ KD12(x, \theta) \\ KD21(x, \theta) \\ KD22(x, \theta) \end{bmatrix} \tag{7.3}$$

> $MxKD := Multiply(Mx, KDS);$

$$MxKD := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.4}$$

> $M\theta := Derivs[[5..8], [1..4]];$

$$M\theta := \Bigg[\left[0, 0, 0, 0\right], \tag{7.5}$$
$$\left[0, 0, 0, 0\right],$$
$$\left[-\frac{32\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0, -\frac{6 \cos(\theta)\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0\right],$$
$$\left[0, -\frac{32\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}, 0, -\frac{6 \cos(\theta)\, m \sin(\theta)}{-4\, mb + 3\, m \cos(\theta)^2}\right]\Bigg]$$

> $M\theta KD := Multiply(M\theta, KDS);$

$$M\theta KD := \begin{bmatrix} 0 \\ 0 \\ -\dfrac{32\, m \sin(\theta)\, KD11(x,\theta)}{-4\, mb + 3\, m \cos(\theta)^2} - \dfrac{6 \cos(\theta)\, m \sin(\theta)\, KD21(x,\theta)}{-4\, mb + 3\, m \cos(\theta)^2} \\ -\dfrac{32\, m \sin(\theta)\, KD12(x,\theta)}{-4\, mb + 3\, m \cos(\theta)^2} - \dfrac{6 \cos(\theta)\, m \sin(\theta)\, KD22(x,\theta)}{-4\, mb + 3\, m \cos(\theta)^2} \end{bmatrix} \quad \textbf{(7.6)}$$

Generate Differential Equations

> $dKD := simplify(map(diff, MxKD, \theta) + map(\text{diff}, -\text{M}\theta\text{KD}, \text{x}));$

$$dKD := \begin{bmatrix} 0 \\ 0 \\ \dfrac{2\, m \sin(\theta) \left(16 \left(\frac{\partial}{\partial x} KD11(x,\theta)\right) + 3 \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \\ \dfrac{2\, m \sin(\theta) \left(16 \left(\frac{\partial}{\partial x} KD12(x,\theta)\right) + 3 \cos(\theta) \left(\frac{\partial}{\partial x} KD22(x,\theta)\right)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \end{bmatrix} \quad \textbf{(8.1)}$$

> $dKD_2 - dKD_3;$

$$-\frac{2\, m \sin(\theta) \left(16 \left(\frac{\partial}{\partial x} KD11(x,\theta)\right) + 3 \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \quad \textbf{(8.2)}$$

> $deq1 := dKD_1;$

$$deq1 := 0 \quad \textbf{(8.3)}$$

> $deq2 := dKD_2;$

$$deq2 := 0 \quad \textbf{(8.4)}$$

> $deq3 := dKD_3;$

$$deq3 := \frac{2\, m \sin(\theta) \left(16 \left(\frac{\partial}{\partial x} KD11(x,\theta)\right) + 3 \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \quad \textbf{(8.5)}$$

> $deq4 := dKD_4;$

$$deq4 := \frac{2\, m \sin(\theta) \left(16 \left(\frac{\partial}{\partial x} KD12(x,\theta)\right) + 3 \cos(\theta) \left(\frac{\partial}{\partial x} KD22(x,\theta)\right)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \quad \textbf{(8.6)}$$

Temporary variables

> $Temp := \left[\frac{\partial}{\partial x} KD11(x,\theta), \frac{\partial}{\partial x} KD12(x,\theta), \frac{\partial}{\partial x} KD21(x,\theta), \frac{\partial}{\partial x} KD22(x,\theta), \frac{\partial}{\partial \theta} KD11(x,\theta), \frac{\partial}{\partial \theta} KD12(x,\theta), \frac{\partial}{\partial \theta} KD21(x,\theta), \frac{\partial}{\partial \theta} KD22(x,\theta), KD11(x,\theta), KD12(x,\theta), KD21(x,\theta), KD22(x,\theta)\right]:$

> $deq1 := collect(deq1, Temp);$

$$deq1 := 0 \tag{10.1}$$

> $deq2 := collect(deq2, Temp);$

$$deq2 := 0 \tag{10.2}$$

> $deq3 := collect(deq3, Temp);$

$$deq3 := \frac{32\, m \sin(\theta) \left(\frac{\partial}{\partial x} KD11(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2} + \frac{6\, m \sin(\theta) \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \tag{10.3}$$

> $deq4 := collect(deq4, Temp);$

$$deq4 := \frac{32\, m \sin(\theta) \left(\frac{\partial}{\partial x} KD12(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2} + \frac{6\, m \sin(\theta) \cos(\theta) \left(\frac{\partial}{\partial x} KD22(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2} \tag{10.4}$$

PDE Solution

> $sys4 := [deq1, deq2, deq3, deq4];$

$$sys4 := \left[0, 0, \frac{32\, m \sin(\theta) \left(\frac{\partial}{\partial x} KD11(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2} + \frac{6\, m \sin(\theta) \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2}, \frac{32\, m \sin(\theta) \left(\frac{\partial}{\partial x} KD12(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2}\right. \tag{11.1}$$

$$+\frac{6\, m \sin(\theta) \cos(\theta) \left(\frac{\partial}{\partial x} KD22(x,\theta)\right)}{-4\, mb + 3\, m \cos(\theta)^2}$$

> sol4 := pdsolve(sys4, [KD11(x, θ), KD12(x, θ), KD21(x, θ), KD22(x, θ)]);

$$sol4 := \left\{KD11(x,\theta) = -\frac{3}{16}\cos(\theta)\, KD21(x,\theta) + \_F2(\theta),\ KD12(x,\theta) = -\frac{3}{16}\cos(\theta)\, KD22(x,\theta) + \_F1(\theta),\ KD21(x,\theta) = KD21(x,\theta),\ KD22(x,\theta) = KD22(x,\theta)\right\} \quad \textbf{(11.2)}$$

> pdetest(sol4, sys4);

$$[0, 0, 0, 0] \quad \textbf{(11.3)}$$

> $KDF := \begin{bmatrix} rhs(sol4_1) & rhs(sol4_2) \\ rhs(sol4_3) & rhs(sol4_4) \end{bmatrix};$

$$KDF := \begin{bmatrix} -\frac{3}{16}\cos(\theta)\, KD21(x,\theta) + \_F2(\theta) & -\frac{3}{16}\cos(\theta)\, KD22(x,\theta) + \_F1(\theta) \\ KD21(x,\theta) & KD22(x,\theta) \end{bmatrix} \quad \textbf{(11.4)}$$

> KDFdot := Matrix(map(diff, KDF, x))·xdot + Matrix(map(diff, KDF, θ))·θdot;

$$KDFdot := \left[\left[-\frac{3}{16}\, xdot \cos(\theta) \left(\frac{\partial}{\partial x} KD21(x,\theta)\right) + \theta dot \left(\frac{3}{16}\sin(\theta)\, KD21(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta} KD21(x,\theta)\right) + \frac{d}{d\theta}\_F2(\theta)\right),\right.$$
$$\left. -\frac{3}{16}\, xdot \cos(\theta) \left(\frac{\partial}{\partial x} KD22(x,\theta)\right) + \theta dot \left(\frac{3}{16}\sin(\theta)\, KD22(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta} KD22(x,\theta)\right) + \frac{d}{d\theta}\_F1(\theta)\right)\right],$$
$$\left.\left[\left(\frac{\partial}{\partial x} KD21(x,\theta)\right) xdot + \theta dot \left(\frac{\partial}{\partial \theta} KD21(x,\theta)\right),\ xdot \left(\frac{\partial}{\partial x} KD22(x,\theta)\right) + \left(\frac{\partial}{\partial \theta} KD22(x,\theta)\right)\theta dot\right]\right] \quad \textbf{(12.1)}$$

> QKD := Transpose(Multiply(KDF, qdot));

$$QKD := \left[\left(-\frac{3}{16}\cos(\theta)\, KD21(x,\theta) + \_F2(\theta)\right) xdot + \left(-\frac{3}{16}\cos(\theta)\, KD22(x,\theta)\right. \quad \textbf{(12.2)}$$

$$+\_F1(\theta)\Big)\,\theta dot,\ KD21(x,\theta)\,xdot + KD22(x,\theta)\,\theta dot\Big]$$

The differentiation of $\mathbf{q}^T\mathbf{K}_D$ **with respect to q is going to be called** ***dKDIq***

> $dKDIq := Matrix\left(2, 2, \left[\mathrm{diff}\left(QKD_1, \mathrm{x}\right), \mathrm{diff}\left(QKD_2, \mathrm{x}\right), \mathrm{diff}\left(QKD_1, \theta\right), diff\left(QKD_2, \theta\right)\right]\right);$

$$dKDIq := \left[\left[-\frac{3}{16}\,xdot\cos(\theta)\left(\frac{\partial}{\partial x}KD21(x,\theta)\right) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial x}KD22(x,\theta)\right)\theta dot, \left(\frac{\partial}{\partial x}KD21(x,\theta)\right)xdot + \left(\frac{\partial}{\partial x}KD22(x,\theta)\right)\theta dot\right],\right.$$
$$\left[\left(\frac{3}{16}\sin(\theta)\,KD21(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta}KD21(x,\theta)\right) + \frac{d}{d\theta}\_F2(\theta)\right)xdot + \theta dot\left(\frac{3}{16}\sin(\theta)\,KD22(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta}KD22(x,\theta)\right) + \frac{d}{d\theta}\_F1(\theta)\right), \left(\frac{\partial}{\partial \theta}KD21(x,\theta)\right)xdot$$
$$\left.\left. + \left(\frac{\partial}{\partial \theta}KD22(x,\theta)\right)\theta dot\right]\right] \tag{12.3}$$

>

> $Ltest := KDFdot - \frac{1}{2}\cdot Transpose(dKDIq) - \frac{1}{2}\cdot dKDIq;$

$$Ltest := \left[\left[\theta dot\left(\frac{3}{16}\sin(\theta)\,KD21(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta}KD21(x,\theta)\right) + \frac{d}{d\theta}\_F2(\theta)\right) + \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial x}KD22(x,\theta)\right)\theta dot,\right.\right. \tag{12.4}$$
$$-\frac{3}{16}\,xdot\cos(\theta)\left(\frac{\partial}{\partial x}KD22(x,\theta)\right) + \frac{1}{2}\,\theta dot\left(\frac{3}{16}\sin(\theta)\,KD22(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta}KD22(x,\theta)\right) + \frac{d}{d\theta}\_F1(\theta)\right)$$
$$-\frac{1}{2}\left(\frac{3}{16}\sin(\theta)\,KD21(x,\theta) - \frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial \theta}KD21(x,\theta)\right) + \frac{d}{d\theta}\_F2(\theta)\right)xdot - \frac{1}{2}\left(\frac{\partial}{\partial x}KD21(x,\theta)\right)xdot - \frac{1}{2}\left(\frac{\partial}{\partial x}KD22(x,\theta)\right)\theta dot\Big],$$
$$\Big[\frac{1}{2}\left(\frac{\partial}{\partial x}KD21(x,\theta)\right)xdot + \theta dot\left(\frac{\partial}{\partial \theta}KD21(x,\theta)\right) - \frac{1}{2}\left(\frac{\partial}{\partial x}KD22(x,\right.$$

$$\theta)\Big)\,\theta dot-\frac{1}{2}\left(\frac{3}{16}\sin(\theta)\,KD21(x,\theta)-\frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)+\frac{d}{d\theta}\_F2(\theta)\right)xdot-\frac{1}{2}\,\theta dot\left(\frac{3}{16}\sin(\theta)\,KD22(x,\theta)-\frac{3}{16}\cos(\theta)\left(\frac{\partial}{\partial\theta}KD22(x,\theta)\right)+\frac{d}{d\theta}\_F1(\theta)\right),xdot\left(\frac{\partial}{\partial x}KD22(x,\theta)\right)-\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)xdot\Big]\Big]$$

> $kd22sol := pdsolve(Ltest_{2,\,2}, KD22(x,\theta))$

$$kd22sol := KD22(x,\theta)=\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,dx+\_F1(\theta) \qquad \textbf{(12.5)}$$

> $Ltest := simplify(eval(Ltest, \text{kd22sol}));$

$$Ltest := \left[\left[\frac{1}{16}\,\theta dot\left(3\sin(\theta)\,KD21(x,\theta)+16\left(\frac{d}{d\theta}\_F2(\theta)\right)\right),\right.\right. \qquad \textbf{(12.6)}$$

$$-\frac{3}{32}\,xdot\cos(\theta)\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)+\frac{3}{32}\,\theta dot\sin(\theta)\left(\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,dx\right)+\frac{3}{32}\,\theta dot\sin(\theta)\,\_F1(\theta)-\frac{3}{32}\,\theta dot\cos(\theta)\left(\int\frac{\partial^2}{\partial\theta^2}KD21(x,\theta)\,dx\right)-\frac{3}{32}\,\theta dot\cos(\theta)\left(\frac{d}{d\theta}\_F1(\theta)\right)+\frac{1}{2}\,\theta dot\left(\frac{d}{d\theta}\_F1(\theta)\right)-\frac{3}{32}\,xdot\sin(\theta)\,KD21(x,\theta)-\frac{1}{2}\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}KD21(x,\theta)\right)xdot-\frac{1}{2}\,\theta dot\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)\Big],$$

$$\Big[\frac{1}{2}\left(\frac{\partial}{\partial x}KD21(x,\theta)\right)xdot+\frac{1}{2}\,\theta dot\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)-\frac{3}{32}\,xdot\sin(\theta)\,KD21(x,\theta)+\frac{3}{32}\,xdot\cos(\theta)\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)-\frac{1}{2}\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)-\frac{3}{32}\,\theta dot\sin(\theta)\left(\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,dx\right)-\frac{3}{32}\,\theta dot\sin(\theta)\,\_F1(\theta)+\frac{3}{32}\,\theta dot\cos(\theta)\left(\int\frac{\partial^2}{\partial\theta^2}KD21(x,\theta)\,dx\right)+\frac{3}{32}\,\theta dot\cos(\theta)\left(\frac{d}{d\theta}\_F1(\theta)\right)-\frac{1}{2}\,\theta dot\left(\frac{d}{d\theta}\_F1(\theta)\right),0\Big]\Big]$$

> $kd21sol := solve\left(Ltest_{1,\,1}, KD21(x,\theta)\right)$

$$kd21sol := -\frac{16}{3}\,\frac{\frac{d}{d\theta}\_F2(\theta)}{\sin(\theta)} \tag{12.7}$$

> $Ltest := simplify\left(eval\left(Ltest, KD21(x,\theta) = kd21sol\right)\right);$

$$Ltest := \left[\left[0, \frac{1}{96}\,\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\left(-9\,\theta dot\,\_F1(\theta)\cos(\theta)^4 \right.\right.\right. \tag{12.8}$$

$$+ 48\,xdot\cos(\theta)^3\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right) - 9\sin(\theta)\,\theta dot\cos(\theta)^3\left(\frac{d}{d\theta}\_F1(\theta)\right)$$

$$+ 48\,\theta dot\cos(\theta)^3\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$

$$+ 256\,\theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2$$

$$+ 48\sin(\theta)\,\theta dot\left(\frac{d}{d\theta}\_F1(\theta)\right)\cos(\theta)^2$$

$$+ 48\sin(\theta)\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)\cos(\theta)^2 + 48\sin(\theta)\,\theta dot\left(\int 1\right.$$

$$\left.dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2 + 18\,\theta dot\,\_F1(\theta)\cos(\theta)^2$$

$$- 48\,xdot\cos(\theta)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right) + 9\sin(\theta)\,\theta dot\cos(\theta)\left(\frac{d}{d\theta}\_F1(\theta)\right)$$

$$- 48\,\theta dot\cos(\theta)\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right) - 96\,\theta dot\cos(\theta)\left(\int 1\right.$$

$$\left.dx\right)\left(\frac{d}{d\theta}\_F2(\theta)\right) + 256\,\theta dot\cos(\theta)\sin(\theta)\left(\frac{d}{d\theta}\_F2(\theta)\right)$$

$$+ 48\sin(\theta)\,\theta dot\left(\int 1\,dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right) - 9\,\theta dot\,\_F1(\theta)$$

$$\left.\left. - 48\sin(\theta)\,\theta dot\left(\frac{d}{d\theta}\_F1(\theta)\right) - 256\,\theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\right)\right],$$

$$\left[-\frac{1}{96}\,\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\left(-9\,\theta dot\,\_F1(\theta)\cos(\theta)^4\right.\right.$$

$$+ 48\,xdot\cos(\theta)^3\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right) - 9\sin(\theta)\,\theta dot\cos(\theta)^3\left(\frac{d}{d\theta}\_F1(\theta)\right)$$

$$+48\,\theta dot\cos(\theta)^3\left(\int 1\,\mathrm{d}x\right)\left(\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\_F2(\theta)\right)$$
$$+256\,\theta dot\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\cos(\theta)^2$$
$$+48\sin(\theta)\,\theta dot\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F1(\theta)\right)\cos(\theta)^2$$
$$+48\sin(\theta)\,xdot\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)\right)\cos(\theta)^2+48\sin(\theta)\,\theta dot\left(\int 1\right.$$
$$\left.\mathrm{d}x\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\cos(\theta)^2+18\,\theta dot\,\_F1(\theta)\cos(\theta)^2$$
$$-48\,xdot\cos(\theta)\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)+9\sin(\theta)\,\theta dot\cos(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F1(\theta)\right)$$
$$-48\,\theta dot\cos(\theta)\left(\int 1\,\mathrm{d}x\right)\left(\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\_F2(\theta)\right)-96\,\theta dot\cos(\theta)\left(\int 1\right.$$
$$\left.\mathrm{d}x\right)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)\right)+256\,\theta dot\cos(\theta)\sin(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)\right)$$
$$+48\sin(\theta)\,\theta dot\left(\int 1\,\mathrm{d}x\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)-9\,\theta dot\,\_F1(\theta)$$
$$-48\sin(\theta)\,\theta dot\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F1(\theta)\right)-256\,\theta dot\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\Bigg),0\Bigg]\Bigg]$$

\>

\> $f1sol := dsolve\left(Ltest_{2,1}, \_F1(\theta)\right);$

$$f1sol := \_F1(\theta)=\frac{1}{3\cos(\theta)-16}\Bigg(\qquad\textbf{(12.9)}$$
$$\int\frac{16}{3}\,\frac{1}{\theta dot\,(3\sin(\theta)-\sin(3\,\theta))}\left(32\,\theta dot\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\right.$$
$$-3\,xdot\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\cos(3\,\theta)+3\,xdot\cos(\theta)\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)$$
$$-3\,\theta dot\,x\left(\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\_F2(\theta)\right)\cos(3\,\theta)+3\,\theta dot\cos(\theta)\,x\left(\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\_F2(\theta)\right)$$
$$-32\,\theta dot\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\_F2(\theta)\right)\cos(2\,\theta)-3\,xdot\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)\right)\sin(3\,\theta)$$

$$-3\sin(\theta)\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)-3\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\sin(3\,\theta)$$
$$-15\sin(\theta)\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)-32\,\theta dot\left(\frac{d}{d\theta}\_F2(\theta)\right)\sin(2\,\theta)$$
$$\left.\left.+24\,\theta dot\cos(\theta)\,x\left(\frac{d}{d\theta}\_F2(\theta)\right)\right)d\theta+\_C1\right)$$

> *Ltest* := *simplify*(*eval*(*Ltest*, *f1sol*));

$$Ltest:=\left[\left[0,\ \frac{1}{2}\,\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\left(\left(\cos(\theta)^3\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\right.\right.\right.\right.\tag{12.10}$$
$$-x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\cos(\theta)^3+\sin(\theta)\left(\int 1\,dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2$$
$$-\sin(\theta)\cos(\theta)^2x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+\cos(\theta)\,x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$
$$-2\cos(\theta)\left(\int 1\,dx\right)\left(\frac{d}{d\theta}\_F2(\theta)\right)-\cos(\theta)\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$
$$+2\cos(\theta)\,x\left(\frac{d}{d\theta}\_F2(\theta)\right)+\sin(\theta)\left(\int 1\,dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)$$
$$\left.\left.\left.\left.-\sin(\theta)\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\right)\theta dot\right)\right],$$
$$\left[-\frac{1}{2}\,\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\left(\left(\cos(\theta)^3\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\right.\right.\right.$$
$$-x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\cos(\theta)^3+\sin(\theta)\left(\int 1\,dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2$$
$$-\sin(\theta)\cos(\theta)^2x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+\cos(\theta)\,x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$
$$-2\cos(\theta)\left(\int 1\,dx\right)\left(\frac{d}{d\theta}\_F2(\theta)\right)-\cos(\theta)\left(\int 1\,dx\right)\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$
$$+2\cos(\theta)\,x\left(\frac{d}{d\theta}\_F2(\theta)\right)+\sin(\theta)\left(\int 1\,dx\right)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)$$
$$\left.\left.\left.\left.-\sin(\theta)\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\right)\theta dot\right),0\right]\right]$$

>

> $f2sol := dsolve\left(Ltest_{2,\,1}, \_F2(\theta)\right);$

$$f2sol := \_F2(\theta) = \_C1 + \_C2\cos(\theta) + \_C3\left(\int \frac{1}{2}\,\frac{\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\left(-1+\cos(2\,\theta)\right)}{\sin(\theta)}\,\mathrm{d}\theta\right) \tag{12.11}$$

> $Ltest := simplify(eval(\,Ltest, f2sol)\,);$

$$Ltest := \left[\left[0, -\frac{1}{2}\,\frac{1}{\sin(\theta)^6}\left(\theta dot\,\_C3\left(2\cos(\theta)^3\left(\int 1\,\mathrm{d}x\right)\sin(\theta) - 2\cos(\theta)^2\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta)\right.\right.\right.\right. \tag{12.12}$$

$$- 3\,x\cos(\theta)^4\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta) - \cos(\theta)^3\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right) + 3\cos(\theta)^4\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta)$$

$$+ 2\cos(\theta)^2\,x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta)$$

$$+ \cos(\theta)^3\,x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right) - 2\cos(\theta)^3\,x\sin(\theta) - \cos(\theta)^5\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right) + x\cos(\theta)^5\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)$$

$$- 3\,x\cos(\theta)^5\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\,\theta) - 3\cos(\theta)^3\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\,\theta) - 2\cos(\theta)^3\left(\int 1\,\mathrm{d}x\right)\sin(\theta)\cos(2\,\theta)$$

$$+ 3\cos(\theta)^3\,x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\,\theta) - 2\cos(\theta)\,x\sin(\theta)\cos(2\,\theta) - \left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta)$$

$$+ x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\,\theta)\sin(\theta) + 3\cos(\theta)^5\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\,\theta) + 2\cos(\theta)^3\,x\sin(\theta)\cos(2\,\theta)$$

$$+2\cos(\theta)\,x\sin(\theta)-4\cos(\theta)^2\left(\int 1\,dx\right)\sin(2\theta)-2x\cos(\theta)^4\sin(2\theta)$$

$$+4\cos(\theta)^2 x\sin(2\theta)+2\cos(\theta)\left(\int 1\,dx\right)\sin(\theta)\cos(2\theta)+2\cos(\theta)^4\left(\int 1\,dx\right)\sin(2\theta)+2\cos(\theta)\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)-2\cos(\theta)\left(\int 1\,dx\right)\sin(\theta)-2\cos(\theta)\,x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)+2\left(\int 1\,dx\right)\sin(2\theta)$$

$$-2x\sin(2\theta)\Big)\Big)\Big],$$

$$\Bigg[\frac{1}{2}\,\frac{1}{\sin(\theta)^6}\Bigg(\theta dot\_C3\,\Bigg(2\cos(\theta)^3\left(\int 1\,dx\right)\sin(\theta)-2\cos(\theta)^2\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)$$

$$-3x\cos(\theta)^4\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)-\cos(\theta)^3\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)+3\cos(\theta)^4\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)$$

$$+2\cos(\theta)^2 x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)$$

$$+\cos(\theta)^3 x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)-2\cos(\theta)^3 x\sin(\theta)-\cos(\theta)^5\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)+x\cos(\theta)^5\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)$$

$$-3x\cos(\theta)^5\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\theta)-3\cos(\theta)^3\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\theta)-2\cos(\theta)^3\left(\int 1\,dx\right)\sin(\theta)\cos(2\theta)$$

$$+3\cos(\theta)^3 x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\theta)-2\cos(\theta)\,x\sin(\theta)\cos(2\theta)-\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)$$

$$+x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\sin(2\theta)\sin(\theta)+3\cos(\theta)^5\left(\int 1\,dx\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)\cos(2\theta)+2\cos(\theta)^3 x\sin(\theta)\cos(2\theta)$$

$$+2\cos(\theta)\,x\sin(\theta)-4\cos(\theta)^2\left(\int 1\,\mathrm{d}x\right)\sin(2\,\theta)-2\,x\cos(\theta)^4\sin(2\,\theta)$$
$$+4\cos(\theta)^2 x\sin(2\,\theta)+2\cos(\theta)\left(\int 1\,\mathrm{d}x\right)\sin(\theta)\cos(2\,\theta)+2\cos(\theta)^4\left(\int 1\,\mathrm{d}x\right)\sin(2\,\theta)+2\cos(\theta)\left(\int 1\,\mathrm{d}x\right)\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)-2\cos(\theta)\left(\int 1\,\mathrm{d}x\right)\sin(\theta)-2\cos(\theta)\,x\operatorname{arctanh}\left(\frac{1}{\sin(\theta)}\right)+2\left(\int 1\,\mathrm{d}x\right)\sin(2\,\theta)$$
$$-2\,x\sin(2\,\theta)\Bigg)\Bigg),0\Bigg]\Bigg]$$

> *Ltest* := *simplify*(*eval*(*Ltest*, $\left(\int 1\,\mathrm{d}x\right)=x$));

$$Ltest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \textbf{(12.13)}$$

> $KDtesti := \left[\left[-\frac{3}{16}\cos(\theta)\,KD21(x,\theta)+\_F2(\theta),-\frac{3}{16}\cos(\theta)\,KD22(x,\theta)+\_F1(\theta)\right],\left[KD21(x,\theta),KD22(x,\theta)\right]\right]$:

> $KDtest := simplify\left(eval\left(KDtesti,\left[KD22(x,\theta)=\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,\mathrm{d}x+\_F1(\theta)\right]\right)\right)$;

$$KDtest := \left[\left[-\frac{3}{16}\cos(\theta)\,KD21(x,\theta)+\_F2(\theta),-\frac{3}{16}\cos(\theta)\left(\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,\mathrm{d}x\right)-\frac{3}{16}\cos(\theta)\,\_F1(\theta)+\_F1(\theta)\right],\left[KD21(x,\theta),\int\frac{\partial}{\partial\theta}KD21(x,\theta)\,\mathrm{d}x+\_F1(\theta)\right]\right] \qquad \textbf{(12.14)}$$

> $KDtest := simplify\left(eval\left(KDtest,\left[KD21(x,\theta)=-\frac{16}{3}\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)}{\sin(\theta)}\right]\right)\right)$;

$$KDtest := \left[\left[\frac{\cos(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\_F2(\theta)\right)+\_F2(\theta)\sin(\theta)}{\sin(\theta)},\right.\right. \qquad \textbf{(12.15)}$$

$$- \frac{1}{16} \frac{1}{-1+\cos(\theta)^2} \left(16 \cos(\theta) \left(\int 1 \, dx\right) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) \sin(\theta) - 16 \cos(\theta)^2 \left(\int 1 \, dx\right) \left(\frac{d}{d\theta} \_F2(\theta)\right) - 3 \cos(\theta) \_F1(\theta) + 3 \cos(\theta)^3 \_F1(\theta) + 16 \_F1(\theta) - 16 \_F1(\theta) \cos(\theta)^2\right)\Bigg],$$

$$\left[-\frac{16}{3} \frac{\frac{d}{d\theta} \_F2(\theta)}{\sin(\theta)}, \frac{1}{3} \frac{1}{-1+\cos(\theta)^2} \left(16 \sin(\theta) \left(\int 1 \, dx\right) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) - 16 \cos(\theta) \left(\int 1 \, dx\right) \left(\frac{d}{d\theta} \_F2(\theta)\right) - 3 \_F1(\theta) + 3 \_F1(\theta) \cos(\theta)^2\right)\right]\Bigg]$$

\>

\> $KDtest := simplify(eval(\ KDtest, [\ f1sol]));$

$$KDtest := \left[\left[\frac{\cos(\theta) \left(\frac{d}{d\theta} \_F2(\theta)\right) + \_F2(\theta) \sin(\theta)}{\sin(\theta)}, \right.\right.$$ **(12.16)**

$$- \frac{1}{48} \frac{1}{\sin(\theta)^2 \, \theta dot} \left(48 \cos(\theta)^2 \left(\int 1 \, dx\right) \left(\frac{d}{d\theta} \_F2(\theta)\right) \theta dot - 16 \cos(\theta)^2 \left(\int \frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)} \left(-16 \, \theta dot \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) + 3 \, xdot \cos(\theta)^3 \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) - 3 \, xdot \cos(\theta) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) + 3 \, \theta dot \, x \left(\frac{d^3}{d\theta^3} \_F2(\theta)\right) \cos(\theta)^3 - 3 \, \theta dot \cos(\theta) \, x \left(\frac{d^3}{d\theta^3} \_F2(\theta)\right) \right.\right.\right.$$

$$+16\,\theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2+3\sin(\theta)\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)\cos(\theta)^2$$

$$+3\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\sin(\theta)\cos(\theta)^2$$

$$+3\sin(\theta)\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+16\,\theta dot\cos(\theta)\sin(\theta)\left(\frac{d}{d\theta}\_F2(\theta)\right)$$

$$-6\,\theta dot\cos(\theta)\,x\left(\frac{d}{d\theta}\_F2(\theta)\right)\Bigg)\,d\theta\Bigg)-3\cos(\theta)^2\_C1\,\theta dot-48\sin(\theta)\Bigg($$

$$\int 1\,dx\Bigg)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\theta dot\cos(\theta)+16\left(\int\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\Bigg(\right.$$

$$-16\,\theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+3\,xdot\cos(\theta)^3\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)$$

$$-3\,xdot\cos(\theta)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+3\,\theta dot\,x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\cos(\theta)^3$$

$$-3\,\theta dot\cos(\theta)\,x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)+16\,\theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2$$

$$+3\sin(\theta)\,xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)\cos(\theta)^2$$

$$+3\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\sin(\theta)\cos(\theta)^2$$

$$+3\sin(\theta)\,\theta dot\,x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+16\,\theta dot\cos(\theta)\sin(\theta)\left(\frac{d}{d\theta}\_F2(\theta)\right)$$

$$\left.-6\,\theta dot\cos(\theta)\,x\left(\frac{d}{d\theta}\_F2(\theta)\right)\Bigg)\,d\theta\right)+3\_C1\,\theta dot\Bigg)\Bigg],$$

$$\left[-\frac{16}{3} \frac{\frac{d}{d\theta} \_F2(\theta)}{\sin(\theta)}, \frac{1}{3} \frac{1}{\sin(\theta)^2 \, \theta dot \, (3\cos(\theta) - 16)} \left(-48 \sin(\theta) \left(\right.\right.\right.$$

$$\left(\int 1 \, dx\right) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) \theta dot \cos(\theta) + 256 \sin(\theta) \, \theta dot \left(\int 1\right.$$

$$\left. dx\right) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) + 48 \cos(\theta)^2 \left(\int 1 \, dx\right) \left(\frac{d}{d\theta} \_F2(\theta)\right) \theta dot$$

$$- 256 \, \theta dot \cos(\theta) \left(\int 1 \, dx\right) \left(\frac{d}{d\theta} \_F2(\theta)\right) + 16 \left(\right.$$

$$\int \frac{1}{\sin(\theta) \left(-1 + \cos(\theta)^2\right)} \left(-16 \, \theta dot \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right)\right.$$

$$+ 3 \, xdot \cos(\theta)^3 \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) - 3 \, xdot \cos(\theta) \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right)$$

$$+ 3 \, \theta dot \, x \left(\frac{d^3}{d\theta^3} \_F2(\theta)\right) \cos(\theta)^3 - 3 \, \theta dot \cos(\theta) \, x \left(\frac{d^3}{d\theta^3} \_F2(\theta)\right)$$

$$+ 16 \, \theta dot \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) \cos(\theta)^2 + 3 \sin(\theta) \, xdot \left(\frac{d}{d\theta} \_F2(\theta)\right) \cos(\theta)^2$$

$$+ 3 \, \theta dot \, x \left(\frac{d^2}{d\theta^2} \_F2(\theta)\right) \sin(\theta) \cos(\theta)^2$$

$$+3\sin(\theta)\ \theta dot\ x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+16\ \theta dot\cos(\theta)\sin(\theta)\left(\frac{d}{d\theta}\_F2(\theta)\right)$$

$$-6\ \theta dot\cos(\theta)\ x\left(\frac{d}{d\theta}\_F2(\theta)\right)\Bigg)\,d\theta\Bigg)+3\_C1\ \theta dot-16\cos(\theta)^2\Bigg($$

$$\int\frac{1}{\sin(\theta)\left(-1+\cos(\theta)^2\right)}\Bigg(-16\ \theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)$$

$$+3\ xdot\cos(\theta)^3\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)-3\ xdot\cos(\theta)\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)$$

$$+3\ \theta dot\ x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)\cos(\theta)^3-3\ \theta dot\cos(\theta)\ x\left(\frac{d^3}{d\theta^3}\_F2(\theta)\right)$$

$$+16\ \theta dot\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\cos(\theta)^2+3\sin(\theta)\ xdot\left(\frac{d}{d\theta}\_F2(\theta)\right)\cos(\theta)^2$$

$$+3\ \theta dot\ x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)\sin(\theta)\cos(\theta)^2$$

$$+3\sin(\theta)\ \theta dot\ x\left(\frac{d^2}{d\theta^2}\_F2(\theta)\right)+16\ \theta dot\cos(\theta)\sin(\theta)\left(\frac{d}{d\theta}\_F2(\theta)\right)$$

$$-6\ \theta dot\cos(\theta)\ x\left(\frac{d}{d\theta}\_F2(\theta)\right)\Bigg)\,d\theta\Bigg)-3\cos(\theta)^2\_C1\ \theta dot\Bigg)\Bigg]\Bigg]$$

> $KDtest := simplify(eval(\ KDtest, [\ f2sol\ ])):$

> $KDtest := simplify\left(eval\left(\ KDtest, \left[\left(\int 1\,dx\right)=x\right]\right)\right):$

> $KDtest := simplify(eval(\ KDtest, [\_C3=0\ ]));$

$$KDtest := \begin{bmatrix} \_C1 & -\frac{1}{16}\_C1 \\ \frac{16}{3}\_C2 & \frac{\_C1}{3\cos(\theta)-16} \end{bmatrix} \quad \textbf{(12.17)}$$

>

KD should be symmetric

> $solve\left(KDtest_{2,1}-KDtest_{1,2}, \_C2\right)$

$$-\frac{3}{256}\_C1 \quad \textbf{(12.18)}$$

> $KDtest := simplify\left(eval\left(\ KDtest, \left[\_C2=-\frac{3}{256}\_C1\right]\right)\right);$

$$KDtest := \begin{bmatrix} \_C1 & -\frac{1}{16}\,\_C1 \\ -\frac{1}{16}\,\_C1 & \frac{\_C1}{3\cos(\theta) - 16} \end{bmatrix} \qquad \textbf{(12.19)}$$

>

As a conclusion KD is Lagrangian* for *the inverted pendulum cart system.

C.3 Direct Lyapunov Aproach for the IPC System

IPC_F1_DLA.mw

```
> restart :
> with(LinearAlgebra) :
```

Definitions

q is a vector of generalized coordinates

$$> q := \begin{bmatrix} x \\ \theta \end{bmatrix} :$$

qdot is a vector of generalized velocites

$$> qdot := \begin{bmatrix} xdot \\ \theta dot \end{bmatrix} :$$

Cmatrix is the matrix of Centripetal* and *coriolis coefficients

$$> C := \begin{bmatrix} 0 & \frac{\mathrm{m} \cdot \mathrm{l} \cdot \sin(\theta) \cdot \omega}{2} \\ 0 & 0 \end{bmatrix} :$$

Define the mass matrix* and *its inverse, where mb is the mass of the cart* + *pendulum, m is the mass of the pendulum

$$> mass := Matrix\left(2, 2, \left[mb, -\frac{m \cdot l \cdot \cos(\theta)}{2}, -\frac{m \cdot l \cdot \cos(\theta)}{2}, \frac{m \cdot l^2}{3}\right]\right);$$

$$mass := \begin{bmatrix} mb & -\frac{1}{2}\, m\, l \cos(\theta) \\ -\frac{1}{2}\, m\, l \cos(\theta) & \frac{1}{3}\, m\, l^2 \end{bmatrix} \quad \textbf{(1.1)}$$

$$> Imass := MatrixInverse(mass);$$

$$Imass := \begin{bmatrix} -\frac{4}{-4\, mb + 3\, m \cos(\theta)^2} & -\frac{6 \cos(\theta)}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \\ -\frac{6 \cos(\theta)}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} & -\frac{12\, mb}{m\, l^2\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \end{bmatrix} \quad \textbf{(1.2)}$$

G is the vector with the Gravity terms

$$> G := \begin{bmatrix} 0 \\ -\frac{m \cdot g \cdot l \cdot \sin(\theta)}{2} \end{bmatrix} :$$

Fm1 Control law matrix for FMC,* Eq. 3.13. Because more parameters are needed in order to place the poles in desired location ν and σ are added to *Fm1

$$> \ Fm1 := \begin{bmatrix} -F11(x,\theta)\cdot\theta dot + \nu & F11(x,\theta)\cdot xdot + \sigma \\ -F22(x,\theta)\cdot\theta dot & F22(x,\theta)\cdot xdot \end{bmatrix}:$$

Fmc1 control law matrix for FMC, Eq. 5.10. This matrix is added in order to *satisfy the FMC*

$$> \ Fmc1 := \begin{bmatrix} F33(qf, qdotf) & F44(qf, qdotf) \\ F44(qf, qdotf) & F55(qf, qdotf) \end{bmatrix}:$$

First Matching Condition

In this section the linear partial differential equations that determine the elements of the KD matrix will be found.

KD is not a constant

$$> \ KDT := \begin{bmatrix} KD11(x,\theta) & KD21(x,\theta) \\ KD21(x,\theta) & KD22(x,\theta) \end{bmatrix}:$$

The time derivative of KD is

$$> \ KDdotT := simplify(map(diff, xdot\cdot KDT, x) + map(diff, \theta dot\cdot KDT, \theta));$$

$$KDdotT := \left[\left[xdot\left(\frac{\partial}{\partial x}KD11(x,\theta)\right) + \theta dot\left(\frac{\partial}{\partial\theta}KD11(x,\theta)\right), xdot\left(\frac{\partial}{\partial x}KD21(x,\theta)\right) + \theta dot\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right)\right], \left[xdot\left(\frac{\partial}{\partial x}KD21(x,\theta)\right) + \theta dot\left(\frac{\partial}{\partial\theta}KD21(x,\theta)\right), xdot\left(\frac{\partial}{\partial x}KD22(x,\theta)\right) + \theta dot\left(\frac{\partial}{\partial\theta}KD22(x,\theta)\right)\right]\right] \tag{2.1}$$

First Matching Condition with inputsFm1 and matrix Fmc1

$$> \ FMCsim := simplify(KDdotT + Multiply(KDT, Multiply(MatrixInverse(mass), (Fm1 - C))) + Transpose(Multiply(KDT, Multiply(MatrixInverse(mass), (Fm1 - C)))) + Fmc1):$$

SimTest represents the symmetry proof

$$> \ SymTest := FMCsim_{1,2} - FMCsim_{2,1}$$

$$SymTest := 0 \tag{2.2}$$

$$> \ mx := simplify(Matrix(2, 2, [coeff(FMCsim_{1,1}, xdot), coeff(FMCsim_{1,2}, xdot), coeff(FMCsim_{2,1}, xdot), coeff(FMCsim_{2,2}, xdot)])):$$

$$> \ mth := simplify(Matrix(2, 2, [coeff(FMCsim_{1,1}, \theta dot), coeff(FMCsim_{1,2}, \theta dot), coeff(FMCsim_{2,1}, \theta dot), coeff(FMCsim_{2,2}, \theta dot)])):$$

> $sys := simplify([mth_{1,1}, mth_{2,1}, mth_{2,2}, mx_{1,1}, mx_{2,1}, mx_{2,2}]):$

Solving for the forces on sys the results will be called Forsol

> $Forsol := pdsolve(sys, [F11(x,\theta), F22(x,\theta), KD22(x,\theta), KD21(x,\theta), KD11(x,\theta)]);$

$$
\begin{aligned}
Forsol := &\{F11(x,\theta) = F11(x,\theta), F22(x,\theta) = F22(x,\theta), KD11(x,\theta) = 0, \\
&KD21(x,\theta) = 0, KD22(x,\theta) = 0\}, \left\{F11(x,\theta) = -\frac{2\,F22(x,\theta)\,mb}{m\,l\cos(\theta)}, F22(x,\theta) \right. \\
&\left. = F22(x,\theta), KD11(x,\theta) = 0, KD21(x,\theta) = 0, KD22(x,\theta) = \_C1\right\}, \left\{F11(x,\right. \\
&\theta) = -\frac{3}{2}\,\frac{\cos(\theta)\,F22(x,\theta)}{l}, F22(x,\theta) = F22(x,\theta), KD11(x,\theta) = \_C1, \\
&\left. KD21(x,\theta) = 0, KD22(x,\theta) = 0\right\}, \left\{F11(x,\theta) = \frac{1}{4}\left((-6\,m\,l\,(\_C1\,\theta\right.\right. \\
&+\_C2)\cos(\theta) + 12\,mb\,(\_C1\,x - 4\,\_C3))\,F22(x,\theta) - 3\,l^2\,\_C1\,m\left(-\frac{4}{3}\,mb\right. \\
&\left.\left.+m\cos(\theta)^2\right)\right)\Big/\left(l\left(\left(-\frac{3}{2}\,\_C1\,x + 6\,\_C3\right)\cos(\theta) + l\,(\_C1\,\theta\right.\right. \\
&\left.\left.+\_C2)\right)m\Big), F22(x,\theta) = F22(x,\theta), KD11(x,\theta) = \frac{1}{4}\,(\_C1\,\theta + \_C2)^2, \\
&KD21(x,\theta) = -\frac{1}{4}\,(\_C1\,\theta + \_C2)\,(\_C1\,x - 4\,\_C3), KD22(x,\theta) \\
&\left.= \frac{1}{4}\,(\_C1\,x - 4\,\_C3)^2\right\}, \left\{F11(x,\theta) = \left(2\,l\,m\left((\_C1\,\_C5 - \_C2\,\_C4)\,\theta\right.\right.\right. \\
&\left.+\left(-\frac{1}{2}\,\_C2^2 + \_C1\,\_C3\right)x + \_C2\,\_C5 - 2\,\_C4\,\_C3\right)\cos(\theta) \\
&+4\,((\_C1\,\_C6 - 2\,\_C4^2)\,\theta + (\_C1\,\_C5 - \_C2\,\_C4)\,x + \_C2\,\_C6 \\
&-2\,\_C4\,\_C5)\,mb)\Big/((-8\,\_C4^2 + 4\,\_C1\,\_C6)\,\theta^2 + ((8\,\_C1\,\_C5
\end{aligned}
\tag{3.1}
$$

$$- 8\,\_C2\,\_C4)\,x + 8\,\_C2\,\_C6 - 16\,\_C4\,\_C5)\,\theta + (-2\,\_C2^2 + 4\,\_C1\,\_C3)\,x^2$$

$$+ (-16\,\_C4\,\_C3 + 8\,\_C2\,\_C5)\,x - 8\,\_C5^2 + 8\,\_C3\,\_C6), F22(x,\theta) =$$

$$-\Big(2\,l\,\Big(\Big(\Big(-3\,\_C4^2 + \frac{3}{2}\,\_C1\,\_C6\Big)\,\theta + \Big(\frac{3}{2}\,\_C1\,\_C5 - \frac{3}{2}\,\_C2\,\_C4\Big)\,x$$

$$+ \frac{3}{2}\,\_C2\,\_C6 - 3\,\_C4\,\_C5\Big)\cos(\theta) + l\,\Big((\_C1\,\_C5 - \_C2\,\_C4)\,\theta + \Big($$

$$-\frac{1}{2}\,\_C2^2 + \_C1\,\_C3\Big)\,x + \_C2\,\_C5 - 2\,\_C4\,\_C3\Big)\Big)\,m\Big)\Big/\Big((-12\,\_C4^2$$

$$+ 6\,\_C1\,\_C6)\,\theta^2 + ((12\,\_C1\,\_C5 - 12\,\_C2\,\_C4)\,x + 12\,\_C2\,\_C6$$

$$- 24\,\_C4\,\_C5)\,\theta + (-3\,\_C2^2 + 6\,\_C1\,\_C3)\,x^2 + (-24\,\_C4\,\_C3$$

$$+ 12\,\_C2\,\_C5)\,x - 12\,\_C5^2 + 12\,\_C3\,\_C6\Big), KD11(x,\theta) = \frac{1}{2}\,\_C1\,\theta^2 + \_C2\,\theta$$

$$+ \_C3, KD21(x,\theta) = \frac{1}{2}\,(-\_C1\,x + 2\,\_C4)\,\theta - \frac{1}{2}\,x\,\_C2 + \_C5, KD22(x,\theta)$$

$$= \frac{1}{2}\,\_C1\,x^2 - 2\,\_C4\,x + \_C6\Big\}$$

Solution 5 is selected for being simpler
Testing the forces and KD values on the FMC, the rsult will be called FMCtest

> $FMCTest := simplify(eval(FMCsim, [Forsol_{5,1}, Forsol_{5,2}, Forsol_{5,3}, Forsol_{5,4}, Forsol_{5,5}]));$

$$FMCTest := \Big[\Big[\frac{1}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\,(6\,\_C1\cos(\theta)\,\nu\,\theta\,x - 4\,\_C1\,l\,\nu\,\theta^2 \tag{3.2}$$

$$- 8\,l\,\nu\,\_C3 + 6\cos(\theta)\,\nu\,\_C2\,x - 12\cos(\theta)\,\nu\,\theta\,\_C4 - 12\cos(\theta)\,\nu\,\_C5$$

$$- 4\,F33(qf, qdotf)\,l\,mb + 3\,F33(qf, qdotf)\,m\,l\cos(\theta)^2 - 8\,l\,\nu\,\_C2\,\theta),$$

$$\frac{1}{2}\,\frac{1}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\,(-6\,\_C1\cos(\theta)\,\nu\,x^2$$

$$- 3\,\_C1\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta + 4\,\_C1\,x\,l\,\nu\,\theta + 6\,\_C1\,x\cos(\theta)\,\sigma\,\theta$$

$$- 4\,\_C1\,l\,\sigma\,\theta^2 + 2\,\_C1\,m\,l^2\sin(\theta)\,\omega\,\theta^2 + 4\,\_C3\,m\,l^2\sin(\theta)\,\omega - 8\,\_C3\,l\,\sigma$$

$$+24\,x\cos(\theta)\,\nu\,\_C4+6\,x\cos(\theta)\,\sigma\,\_C2-3\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C2$$
$$+4\,x\,l\,\nu\,\_C2-8\,l\,\nu\,\theta\,\_C4-8\,F44(qf,qdotf)\,l\,mb+6\,F44(qf,$$
$$qdotf)\,m\,l\cos(\theta)^2+4\,m\,l^2\sin(\theta)\,\omega\,\_C2\,\theta+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C5$$
$$+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta\,\_C4-8\,l\,\sigma\,\_C2\,\theta-12\cos(\theta)\,\sigma\,\theta\,\_C4$$
$$-12\cos(\theta)\,\sigma\,\_C5-8\,l\,\nu\,\_C5-12\cos(\theta)\,\nu\,\_C6)\Big],$$
$$\Big[\frac{1}{2}\,\frac{1}{l\left(-4\,mb+3\,m\cos(\theta)^2\right)}\Big(-6\,\_C1\cos(\theta)\,\nu\,x^2$$
$$-3\,\_C1\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta+4\,\_C1\,x\,l\,\nu\,\theta+6\,\_C1\,x\cos(\theta)\,\sigma\,\theta$$
$$-4\,\_C1\,l\,\sigma\,\theta^2+2\,\_C1\,m\,l^2\sin(\theta)\,\omega\,\theta^2+4\,\_C3\,m\,l^2\sin(\theta)\,\omega-8\,\_C3\,l\,\sigma$$
$$+24\,x\cos(\theta)\,\nu\,\_C4+6\,x\cos(\theta)\,\sigma\,\_C2-3\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C2$$
$$+4\,x\,l\,\nu\,\_C2-8\,l\,\nu\,\theta\,\_C4-8\,F44(qf,qdotf)\,l\,mb+6\,F44(qf,$$
$$qdotf)\,m\,l\cos(\theta)^2+4\,m\,l^2\sin(\theta)\,\omega\,\_C2\,\theta+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C5$$
$$+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta\,\_C4-8\,l\,\sigma\,\_C2\,\theta-12\cos(\theta)\,\sigma\,\theta\,\_C4$$
$$-12\cos(\theta)\,\sigma\,\_C5-8\,l\,\nu\,\_C5-12\cos(\theta)\,\nu\,\_C6\Big),$$
$$\frac{1}{l\left(-4\,mb+3\,m\cos(\theta)^2\right)}\Big(-6\,\_C1\,x^2\cos(\theta)\,\sigma$$
$$+3\,\_C1\,x^2\,m\,l\cos(\theta)\sin(\theta)\,\omega+4\,\_C1\,x\,l\,\sigma\,\theta-2\,\_C1\,x\,m\,l^2\sin(\theta)\,\omega\,\theta$$
$$+24\,x\cos(\theta)\,\sigma\,\_C4-12\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C4-2\,x\,m\,l^2\sin(\theta)\,\omega\,\_C2$$
$$+4\,x\,l\,\sigma\,\_C2-8\,l\,\sigma\,\theta\,\_C4+4\,m\,l^2\sin(\theta)\,\omega\,\_C5-12\cos(\theta)\,\sigma\,\_C6$$
$$-8\,l\,\sigma\,\_C5-4\,F55(qf,qdotf)\,l\,mb+3\,F55(qf,qdotf)\,m\,l\cos(\theta)^2$$
$$+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C6+4\,m\,l^2\sin(\theta)\,\omega\,\theta\,\_C4\Big)\Big]\Big]$$

Testing the inputs on Eq. 5.9, the rsult will be called FTiTest

$$> Fm1 := simplify\left(eval\left(Fm1, \left[Forsol_{5,1}, Forsol_{5,2}, Forsol_{5,3}, Forsol_{5,4}, Forsol_{5,5}\right]\right)\right):$$

> $F1 := simplify(Multiply(Fm1, qdot));$

$$F1 := \begin{bmatrix} \theta dot\, \sigma + xdot\, \nu \\ 0 \end{bmatrix} \quad (3.3)$$

> $Fmc1 := simplify\left(eval\left(Fmc1, \left[Forsol_{5,\,1}, Forsol_{5,\,2}, Forsol_{5,\,3}, Forsol_{5,\,4}, Forsol_{5,\,5}\right]\right)\right):$

Solving for Forsol in system of equations from the FMC, the result will be called var3

> $var3 := solve\left(\left\{FMCTest_{1,\,1}, FMCTest_{2,\,1}, FMCTest_{2,\,2}\right\}, [\mathrm{F33}(\mathrm{qf}, \mathrm{qdotf}), \mathrm{F44}(\mathrm{qf}, \mathrm{qdotf}), F55(\mathrm{qf}, \mathrm{qdotf})]\right);$

$$var3 := \left[\left[F33(qf, qdotf) = -\frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)}\left(2\, \nu\, \left(3\, \_C1 \cos(\theta)\, \theta\, x - 2\, \_C1\, l\, \theta^2 - 4\, l\, \_C3 + 3 \cos(\theta)\, \_C2\, x - 6 \cos(\theta)\, \theta\, \_C4 - 6 \cos(\theta)\, \_C5 - 4\, l\, \_C2\, \theta\right)\right), F44(qf, qdotf) = \frac{1}{2}\, \frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)}\left(6\, \_C1 \cos(\theta)\, \nu\, x^2 + 3\, \_C1\, x\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \theta - 4\, \_C1\, x\, l\, \nu\, \theta - 6\, \_C1\, x \cos(\theta)\, \sigma\, \theta + 4\, \_C1\, l\, \sigma\, \theta^2 - 2\, \_C1\, m\, l^2 \sin(\theta)\, \omega\, \theta^2 - 4\, \_C3\, m\, l^2 \sin(\theta)\, \omega + 8\, \_C3\, l\, \sigma - 24\, x \cos(\theta)\, \nu\, \_C4 - 6\, x \cos(\theta)\, \sigma\, \_C2 + 3\, x\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C2 - 4\, x\, l\, \nu\, \_C2 + 8\, l\, \nu\, \theta\, \_C4 - 4\, m\, l^2 \sin(\theta)\, \omega\, \_C2\, \theta - 6\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C5 - 6\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \theta\, \_C4 + 8\, l\, \sigma\, \_C2\, \theta + 12 \cos(\theta)\, \sigma\, \theta\, \_C4 + 12 \cos(\theta)\, \sigma\, \_C5 + 8\, l\, \nu\, \_C5 + 12 \cos(\theta)\, \nu\, \_C6\right), F55(qf, qdotf) = -\frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)}\left(-6\, \_C1\, x^2 \cos(\theta)\, \sigma + 3\, \_C1\, x^2\, m\, l \cos(\theta) \sin(\theta)\, \omega + 4\, \_C1\, x\, l\, \sigma\, \theta - 2\, \_C1\, x\, m\, l^2 \sin(\theta)\, \omega\, \theta + 24\, x \cos(\theta)\, \sigma\, \_C4 - 12\, x\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C4 - 2\, x\, m\, l^2 \sin(\theta)\, \omega\, \_C2\right. \quad (3.4)$$

$$+4\,x\,l\,\sigma\,\_C2-8\,l\,\sigma\,\theta\,\_C4+4\,m\,l^2\sin(\theta)\,\omega\,\_C5-12\cos(\theta)\,\sigma\,\_C6$$
$$-8\,l\,\sigma\,\_C5+6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C6+4\,m\,l^2\sin(\theta)\,\omega\,\theta\,\_C4)\,]]$$

Testing the final control inputs on the FMC, the result will be called FMCtest

> $FMCTest := simplify(eval(FMCTest, var3_1))$

$$FMCTest := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \textbf{(3.5)}$$

The FMC is satisfied !

Substituing the solution *Forsol* in *Fmc1*

> $Fmc1i := simplify(eval(\mathrm{Fmc1}, var3_1));$

$$Fmc1i := \Bigg[\Bigg[-\frac{1}{l\,(-4\,mb+3\,m\cos(\theta)^2)}\,(2\,\nu\,(3\,\_C1\cos(\theta)\,\theta\,x-2\,\_C1\,l\,\theta^2 \qquad \textbf{(3.6)}$$
$$-4\,l\,\_C3+3\cos(\theta)\,\_C2\,x-6\cos(\theta)\,\theta\,\_C4-6\cos(\theta)\,\_C5-4\,l\,\_C2\,\theta)),$$
$$\frac{1}{2}\,\frac{1}{l\,(-4\,mb+3\,m\cos(\theta)^2)}\,(6\,\_C1\cos(\theta)\,\nu\,x^2$$
$$+3\,\_C1\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta-4\,\_C1\,x\,l\,\nu\,\theta-6\,\_C1\,x\cos(\theta)\,\sigma\,\theta$$
$$+4\,\_C1\,l\,\sigma\,\theta^2-2\,\_C1\,m\,l^2\sin(\theta)\,\omega\,\theta^2-4\,\_C3\,m\,l^2\sin(\theta)\,\omega+8\,\_C3\,l\,\sigma$$
$$-24\,x\cos(\theta)\,\nu\,\_C4-6\,x\cos(\theta)\,\sigma\,\_C2+3\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C2$$
$$-4\,x\,l\,\nu\,\_C2+8\,l\,\nu\,\theta\,\_C4-4\,m\,l^2\sin(\theta)\,\omega\,\_C2\,\theta$$
$$-6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C5-6\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta\,\_C4+8\,l\,\sigma\,\_C2\,\theta$$
$$+12\cos(\theta)\,\sigma\,\theta\,\_C4+12\cos(\theta)\,\sigma\,\_C5+8\,l\,\nu\,\_C5+12\cos(\theta)\,\nu\,\_C6)\Bigg],$$
$$\Bigg[\frac{1}{2}\,\frac{1}{l\,(-4\,mb+3\,m\cos(\theta)^2)}\,(6\,\_C1\cos(\theta)\,\nu\,x^2$$
$$+3\,\_C1\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\theta-4\,\_C1\,x\,l\,\nu\,\theta-6\,\_C1\,x\cos(\theta)\,\sigma\,\theta$$
$$+4\,\_C1\,l\,\sigma\,\theta^2-2\,\_C1\,m\,l^2\sin(\theta)\,\omega\,\theta^2-4\,\_C3\,m\,l^2\sin(\theta)\,\omega+8\,\_C3\,l\,\sigma$$
$$-24\,x\cos(\theta)\,\nu\,\_C4-6\,x\cos(\theta)\,\sigma\,\_C2+3\,x\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C2$$
$$-4\,x\,l\,\nu\,\_C2+8\,l\,\nu\,\theta\,\_C4-4\,m\,l^2\sin(\theta)\,\omega\,\_C2\,\theta$$

$$-6\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C5 - 6\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \theta\, \_C4 + 8\, l\, \sigma\, \_C2\, \theta$$

$$+ 12 \cos(\theta)\, \sigma\, \theta\, \_C4 + 12 \cos(\theta)\, \sigma\, \_C5 + 8\, l\, \nu\, \_C5 + 12 \cos(\theta)\, \nu\, \_C6 \Big)$$

$$, -\frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \Big(\big(4\, l\, \theta\, \_C4 + 6 \cos(\theta)\, \_C6 - 2\, \_C1\, x\, l\, \theta$$

$$+ 3 \cos(\theta)\, \_C1\, x^2 - 12 \cos(\theta)\, x\, \_C4 - 2\, x\, l\, \_C2 + 4\, l\, \_C5 \big) \big(m\, l \sin(\theta)\, \omega$$

$$- 2\, \sigma \big) \Big) \Big] \Big]$$

In order to avoid odd functions $\_C1 = 0, \_C2 = 0, \_C4 = 0$

> $Fmc1i := simplify(eval(Fmc1i, [\_C1 = 0, \_C2 = 0, \_C4 = 0]));$

$$Fmc1i := \Bigg[\Bigg[\frac{4\, \nu\, (2\, l\, \_C3 + 3 \cos(\theta)\, \_C5)}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)},$$

$$-\frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \big(2\, \_C3\, m\, l^2 \sin(\theta)\, \omega - 4\, \_C3\, l\, \sigma$$

$$+ 3\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C5 - 6 \cos(\theta)\, \sigma\, \_C5 - 4\, l\, \nu\, \_C5$$

$$- 6 \cos(\theta)\, \nu\, \_C6 \big) \Bigg],$$

$$\Bigg[-\frac{1}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \big(2\, \_C3\, m\, l^2 \sin(\theta)\, \omega - 4\, \_C3\, l\, \sigma$$

$$+ 3\, m\, l \cos(\theta) \sin(\theta)\, \omega\, \_C5 - 6 \cos(\theta)\, \sigma\, \_C5 - 4\, l\, \nu\, \_C5$$

$$- 6 \cos(\theta)\, \nu\, \_C6 \big), -\frac{2\, (3 \cos(\theta)\, \_C6 + 2\, l\, \_C5)\, (m\, l \sin(\theta)\, \omega - 2\, \sigma)}{l\left(-4\, mb + 3\, m \cos(\theta)^2\right)} \Bigg] \Bigg] \tag{3.7}$$

Substituing the solution *Forsol* into the KD matrix

> $KDT := simplify\big(eval\big(KDT, \big[Forsol_{5,\,1}, Forsol_{5,\,2}, Forsol_{5,\,3}, Forsol_{5,\,4}, Forsol_{5,\,5}\big]\big)\big);$

$$KDT := \left[\left[\frac{1}{2}\, \_C1\, \theta^2 + \_C2\, \theta + \_C3, -\frac{1}{2}\, \_C1\, x\, \theta + \theta\, \_C4 - \frac{1}{2}\, x\, \_C2 + \_C5 \right], \left[-\frac{1}{2}\, \_C1\, x\, \theta + \theta\, \_C4 - \frac{1}{2}\, x\, \_C2 + \_C5, \frac{1}{2}\, \_C1\, x^2 - 2\, \_C4\, x + \_C6 \right] \right] \tag{3.8}$$

In order to avoid odd function $\_C1 = 0, \_C2 = 0, \_C4 = 0$

> $KDT := simplify(eval(KDT, [\_C1 = 0, \_C2 = 0, \_C4 = 0]));$

$$KDT := \begin{bmatrix} \_C3 & \_C5 \\ \_C5 & \_C6 \end{bmatrix} \tag{3.9}$$

The determinant of the KD matrix is

> simplify(Determinant(KDT));

$$\_C3\ \_C6 - \_C5^2 \tag{3.10}$$

The P matrix and its determinant are

> P := simplify(Multiply(KDT, MatrixInverse(mass)));

$$P := \begin{bmatrix} -\frac{2\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)}{l\,(-4\,mb + 3\,m\cos(\theta)^2)} & -\frac{6\,(\_C3\cos(\theta)\,m\,l + 2\,\_C5\,mb)}{m\,l^2\,(-4\,mb + 3\,m\cos(\theta)^2)} \\ -\frac{2\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)}{l\,(-4\,mb + 3\,m\cos(\theta)^2)} & -\frac{6\,(\_C5\cos(\theta)\,m\,l + 2\,\_C6\,mb)}{m\,l^2\,(-4\,mb + 3\,m\cos(\theta)^2)} \end{bmatrix} \tag{3.11}$$

> Determinant(P);

$$-\frac{12\,(\_C3\,\_C6 - \_C5^2)}{l^2\,m\,(-4\,mb + 3\,m\cos(\theta)^2)} \tag{3.12}$$

Second Matching Condition

Define the matrix Kv

The Kv matrix is obtain by the product of the first column of the P matrix and its transpose

> Kv := α·Multiply(Column(P, [1]), Transpose(Column(P, [1])));

$$Kv := \left[\left[\frac{4\,\alpha\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)^2}{l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2}, \frac{4\,\alpha\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)}{l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2}\right], \left[\frac{4\,\alpha\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)}{l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2}, \frac{4\,\alpha\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)^2}{l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2}\right]\right] \tag{4.1}$$

> convert(Kv, string)

"Matrix(2, 2, [[4*alpha*(2*l*_C3+3*cos(theta)*_C5)^2/l^2/(-4*mb+3*m*cos(theta)^2)^2,4*alpha*(2*l*_C3+3*cos(theta)*_C5)/l^2/(-4*mb+3*m*cos(theta)^2)^2*(3*cos(theta)*_C6+2*l*_C5)],[4*alpha*(2*l*_C3+3*cos(theta)*_C5)/l^2/(-4*mb+3*m*cos(theta)^2)^2*(3*cos(theta)*_C6+2*l*_C5),4*alpha*(3*cos(theta)*_C6+2*l*_C5)^2/l^2/(-4*mb+3*m*cos(theta)^2)^2]])" **(4.2)**

The vector F2 is obtain by the product of the P matrix and the Kv matrix.

This result is then multiply by the vector of generalized velocities

> F2f := -Multiply(MatrixInverse(P), Kv);

$$F2f := \left[\left[-\frac{2\,(\_C5\cos(\theta)\,m\,l + 2\,\_C6\,mb)\,\alpha\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)^2}{(\_C3\,\_C6 - \_C5^2)\,l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2} \right.\right. \quad \textbf{(4.3)}$$

$$\ldots\,\_C5)\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5))\Big/\left((\_C3\,\_C6 - \_C5^2)\,l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2\right),$$

$$-(2\,(\_C5\cos(\theta)\,m\,l + 2\,\_C6\,mb)\,\alpha\,(2\,l\,\_C3$$

$$+ 3\cos(\theta)\,\_C5)\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5))\Big/\left((\_C3\,\_C6 - \_C5^2)\,l^2\,(\right.$$

$$\left.-4\,mb + 3\,m\cos(\theta)^2)^2\right)$$

$$\left. + \frac{2\,(\_C3\cos(\theta)\,m\,l + 2\,\_C5\,mb)\,\alpha\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)^2}{(\_C3\,\_C6 - \_C5^2)\,l^2\,(-4\,mb + 3\,m\cos(\theta)^2)^2}\right],$$

$$\left.\left[0, 0\right]\right]$$

> simplify(Eigenvalues(Kv));

$$\left[\left[0\right],\right. \quad \textbf{(4.4)}$$

$$\left[(4\,\alpha\,(4\,l^2\,\_C5^2 + 12\,l\,\_C5\,\_C6\cos(\theta) + 9\,\_C6^2\cos(\theta)^2 + 4\,l^2\,\_C3^2\right.$$

$$+ 12\,l\,\_C3\,\_C5\cos(\theta) + 9\,\_C5^2\cos(\theta)^2))\Big/(l^2\,(16\,mb^2 - 24\,mb\,m\cos(\theta)^2$$

$$\left.\left. + 9\,m^2\cos(\theta)^4))\right]\right]$$

> Fm2 := simplify(Multiply(F2f, qdot));

$$Fm2 := \quad \textbf{(4.5)}$$

$$\left[\left[\frac{1}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\,(2\,\alpha\,(2\,xdot\,l\,\_C3\right.\right.$$

$$\left. + 3\,xdot\,\_C5\cos(\theta) + 2\,\theta dot\,l\,\_C5 + 3\,\theta dot\,\_C6\cos(\theta)))\right],$$

$$\left.\left[0\right]\right]$$

Third Matching Condition

Define the gradient of the $\Phi(x, \theta)$

> $PHM := Matrix\left(2, 1, \left[\frac{\partial}{\partial x}\Phi(x,\theta), \frac{\partial}{\partial \theta}\Phi(x,\theta)\right]\right);$

$$PHM := \begin{bmatrix} \frac{\partial}{\partial x}\Phi(x,\theta) \\ \frac{\partial}{\partial \theta}\Phi(x,\theta) \end{bmatrix} \tag{5.1}$$

Using the third matching condition two equations can be obtained

> $TMC := convert\left(\begin{bmatrix} F3 \\ 0 \end{bmatrix}, Matrix\right) - convert(G, Matrix) + Multiply(MatrixInverse(P), PHM);$

$$TMC := \left[\left[F3 + \frac{1}{2}\frac{(\_C5\cos(\theta)\, m\, l + 2\,\_C6\, mb)\left(\frac{\partial}{\partial x}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2} - \frac{1}{2}\frac{(\_C3\cos(\theta)\, m\, l + 2\,\_C5\, mb)\left(\frac{\partial}{\partial \theta}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2}\right], \left[\frac{1}{2}m\, g\, l\sin(\theta) - \frac{1}{6}\frac{(3\cos(\theta)\,\_C6 + 2\, l\,\_C5)\, l\, m\left(\frac{\partial}{\partial x}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2} + \frac{1}{6}\frac{(2\, l\,\_C3 + 3\cos(\theta)\,\_C5)\, l\, m\left(\frac{\partial}{\partial \theta}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2}\right]\right] \tag{5.2}$$

From the second row of TMC $\Phi(\theta, r)$ is found

> $TMCphi := TMC_{2,1};$

$$TMCphi := \frac{1}{2}m\, g\, l\sin(\theta) - \frac{1}{6}\frac{(3\cos(\theta)\,\_C6 + 2\, l\,\_C5)\, l\, m\left(\frac{\partial}{\partial x}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2} + \frac{1}{6}\frac{(2\, l\,\_C3 + 3\cos(\theta)\,\_C5)\, l\, m\left(\frac{\partial}{\partial \theta}\Phi(x,\theta)\right)}{\_C3\,\_C6 - \_C5^2} \tag{5.3}$$

> $solphi := pdsolve(TMCphi);$

$$solphi := \Phi(x,\theta) = \frac{1}{\_C5}\Bigg(-g\,\_C5^2 \ln(2\,l\,\_C3 + 3\cos(\theta)\,\_C5) + g\ln(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)\,\_C6\,\_C3 + \_F1\Bigg(\frac{1}{\_C5\sqrt{4\,l^2\,\_C3^2 - 9\,\_C5^2}}\Bigg(x\,\_C5\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2} + 4\,l\,\_C5^2 \arctan\left(\frac{(2\,l\,\_C3 - 3\,\_C5)\tan\left(\frac{1}{2}\theta\right)}{\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2}}\right) - 4\,l\arctan\left(\frac{(2\,l\,\_C3 - 3\,\_C5)\tan\left(\frac{1}{2}\theta\right)}{\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2}}\right)\_C6\,\_C3 + 2\,\_C6\arctan\left(\tan\left(\frac{1}{2}\theta\right)\right)\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2}\Bigg)\Bigg)\_C5\Bigg) \quad \mathbf{(5.4)}$$

From the first row of TMC F3 is found

> $f3 := simplify(F3 - (TMC_{1,1}));$

$$f3 := -\frac{1}{2}\frac{1}{\_C3\,\_C6 - \_C5^2}\left(\left(\frac{\partial}{\partial x}\Phi(x,\theta)\right) l\,m\cos(\theta)\,\_C5 + 2\left(\frac{\partial}{\partial x}\Phi(x,\theta)\right)\_C6\,mb - \left(\frac{\partial}{\partial \theta}\Phi(x,\theta)\right)\_C3\cos(\theta)\,m\,l - 2\left(\frac{\partial}{\partial \theta}\Phi(x,\theta)\right)\_C5\,mb\right) \quad \mathbf{(5.5)}$$

> $PhiIifii := simplify(rhs(solphi));$

$$PhiIifii := \frac{1}{\_C5}\Bigg(-g\,\_C5^2\ln(2\,l\,\_C3 + 3\cos(\theta)\,\_C5) + g\ln(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)\,\_C6\,\_C3 + \_F1\Bigg(-\frac{1}{\_C5\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2}}\Bigg(-x\,\_C5\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2} + 4\,l\,\_C5^2\arctan\left(\frac{(2\,l\,\_C3 - 3\,\_C5)(-1+\cos(\theta))}{\sqrt{4\,l^2\,\_C3^2-9\,\_C5^2}\,\sin(\theta)}\right) \quad \mathbf{(5.6)}$$

$$- 4\, l \arctan\left(\frac{(2\, l\, \_C3 - 3\, \_C5)\,(-1+\cos(\theta))}{\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}\,\sin(\theta)}\right) \_C6\, \_C3$$

$$+ 2\, \_C6 \arctan\left(\frac{-1+\cos(\theta)}{\sin(\theta)}\right)\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}\Bigg)\Bigg)\, \_C5\Bigg)$$

> $$PhiIifi := simplify\left(\frac{1}{\_C5}\left(-g\, \_C5^2 \ln(2\, l\, \_C3 + 3\cos(\theta)\, \_C5) + g \ln(2\, l\, \_C3\right.\right.$$

$$+ 3\cos(\theta)\, \_C5)\, \_C6\, \_C3 + F7\cdot\left(-\frac{1}{\_C5\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}}\right($$

$$-x\, \_C5\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}$$

$$+ 2\, \_C6 \arctan\left(\frac{-1+\cos(\theta)}{\sin(\theta)}\right)\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}$$

$$+ 4\, l\, \_C5^2 \arctan\left(\frac{(2\, l\, \_C3 - 3\, \_C5)\,(-1+\cos(\theta))}{\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}\,\sin(\theta)}\right)$$

$$\left.\left.\left.- 4\, l \arctan\left(\frac{(2\, l\, \_C3 - 3\, \_C5)\,(-1+\cos(\theta))}{\sqrt{4\, l^2\, \_C3^2 - 9\, \_C5^2}\,\sin(\theta)}\right) \_C6\, \_C3\right)\right)^2 \_C5\right)\right):$$

Input contribution from FMC, F_1

> $F1i := simplify(F1_1);$

$$F1i := \theta dot\, \sigma + xdot\, \nu \quad \textbf{(5.7)}$$

Input contribution from SMC, F_2

> $F2i := simplify(Fm2_1);$

$$F2i := \frac{2\,\alpha\,(2\, xdot\, l\, \_C3 + 3\, xdot\, \_C5\cos(\theta) + 2\,\theta dot\, l\, \_C5 + 3\,\theta dot\, \_C6\cos(\theta))}{l\,(-4\, mb + 3\, m\cos(\theta)^2)} \quad \textbf{(5.8)}$$

Input contribution from TMC, F_3

> $F3i := simplify(eval(f3, \Phi(x, \theta) = PhiIifi))$:

Contribution from FMC to Lyapunov, Fmc_1

> $Fmc1i := Fmc1i;$

$$Fmc1i := \left[\left[\frac{4\,\nu\,(2\,l\,\_C3 + 3\cos(\theta)\,\_C5)}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}, -\frac{1}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\,(2\,\_C3\,m\,l^2\sin(\theta)\,\omega - 4\,\_C3\,l\,\sigma + 3\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C5 - 6\cos(\theta)\,\sigma\,\_C5 - 4\,l\,\nu\,\_C5 - 6\cos(\theta)\,\nu\,\_C6)\right], \left[-\frac{1}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\,(2\,\_C3\,m\,l^2\sin(\theta)\,\omega - 4\,\_C3\,l\,\sigma + 3\,m\,l\cos(\theta)\sin(\theta)\,\omega\,\_C5 - 6\cos(\theta)\,\sigma\,\_C5 - 4\,l\,\nu\,\_C5 - 6\cos(\theta)\,\nu\,\_C6), -\frac{2\,(3\cos(\theta)\,\_C6 + 2\,l\,\_C5)\,(m\,l\sin(\theta)\,\omega - 2\,\sigma)}{l\,(-4\,mb + 3\,m\cos(\theta)^2)}\right]\right] \tag{5.9}$$

The control input of the system is

Define the control law

> $Fc := simplify(eval(F1i + F2i + F3i))$:

> $T := \left[\, diff(Fc, x) \;\; diff(Fc, \theta) \;\; diff(Fc, xdot) \;\; diff(Fc, \theta dot) \,\right]$:

> $T1 := simplify\left(eval\left(T, \left[\arctan\left(\frac{2\cos(\theta)\,\_C3\,l - 3\,\_C5\cos(\theta) - 2\,\_C3\,l + 3\,\_C5}{\sin(\theta)\sqrt{4\,\_C3^2\,l^2 - 9\,\_C5^2}}\right) = 0\right]\right)\right)$:

> $T1 := simplify\left(eval\left(T1, \left[\arctan\left(\frac{-1 + \cos(\theta)}{\sin(\theta)}\right) = 0\right]\right)\right)$:

> $T1 := simplify(eval(T1, [\,\theta dot = 0\,]))$:

> $T1 := simplify(eval(T1, [\cos(\theta) = 1, xdot = 0]));$

$$T1 := \left[\frac{l\,(-4\,mb + 3\,m)\,F7}{2\,l\,\_C3 + 3\,\_C5}, -\frac{1}{2}\,\frac{1}{4\,l^2\,\_C3^2 + 12\,l\,\_C5\,\_C3 + 9\,\_C5^2}\,(6\,g\,m\,l^2\,\_C3^2 \right. \tag{6.1}$$

$$+24\,m\,\_C3\,l^2\,F7\,x\sin(\theta)+9\,\_C3\,l\,g\,\_C5\,m+12\,\_C3\,l\,\_C5\,g\,mb$$

$$+16\,l^2\,\_C5\,F7\,mb-12\,l^2\,\_C5\,F7\,m+18\,l\,m\,\_C5\,F7\,x\sin(\theta)$$

$$+24\,l\,\_C6\,F7\,mb+24\,l\,\_C5\,F7\,x\sin(\theta)\,mb-18\,l\,m\,\_C6\,F7$$

$$+18\,\_C5^2\,mb\,g\Big),\ \frac{3\,\nu\,l\,m+6\,\alpha\,\_C5+4\,\alpha\,l\,\_C3-4\,\nu\,l\,mb}{l\,(-4\,mb+3\,m)},$$

$$\frac{3\,\sigma\,l\,m+6\,\alpha\,\_C6+4\,\alpha\,l\,\_C5-4\,\sigma\,l\,mb}{l\,(-4\,mb+3\,m)}\Bigg]$$

T2 is the linearized control law

> $T2 := simplify(eval(T1, [\,\theta=0\,, x=0\,]))\,;$

$$T2 := \left[\frac{l\,(-4\,mb+3\,m)\,F7}{2\,l\,\_C3+3\,\_C5},\ \frac{1}{2}\,\frac{1}{4\,l^2\,\_C3^2+12\,l\,\_C5\,\_C3+9\,\_C5^2}\Big(\right. \tag{6.2}$$

$$-6\,g\,m\,l^2\,\_C3^2-16\,l^2\,\_C5\,F7\,mb-9\,\_C3\,l\,g\,\_C5\,m-12\,\_C3\,l\,\_C5\,g\,mb$$

$$+12\,l^2\,\_C5\,F7\,m+18\,l\,m\,\_C6\,F7-24\,l\,\_C6\,F7\,mb-18\,\_C5^2\,mb\,g\Big),$$

$$\frac{3\,\nu\,l\,m+6\,\alpha\,\_C5+4\,\alpha\,l\,\_C3-4\,\nu\,l\,mb}{l\,(-4\,mb+3\,m)},$$

$$\left.\frac{3\,\sigma\,l\,m+6\,\alpha\,\_C6+4\,\alpha\,l\,\_C5-4\,\sigma\,l\,mb}{l\,(-4\,mb+3\,m)}\right]$$

Linearization of the control law and system

> $sysid := [mb=5, m=1, l=.7, g=9.81]:$

> $LCond := [x=0, \mathrm{xdot}=0, \theta=0, \theta dot=0]:$

The equation of motion of the inverted pendulum cart system is

> $Eqs := simplify\left(Multiply\left(MatrixInverse(mass), \left(\begin{bmatrix}\tau\\0\end{bmatrix} - Multiply(C, qdot) - \mathrm{G}\right)\right)\right);$

$$Eqs := \begin{bmatrix} -\dfrac{4\,\tau-2\,m\,l\sin(\theta)\,\omega\,\theta dot+3\cos(\theta)\,m\,g\sin(\theta)}{-4\,mb+3\,m\cos(\theta)^2} \\ \dfrac{3\,(-2\cos(\theta)\,\tau+\cos(\theta)\,m\,l\sin(\theta)\,\omega\,\theta dot-2\,mb\,g\sin(\theta))}{l\,(-4\,mb+3\,m\cos(\theta)^2)} \end{bmatrix} \tag{7.1}$$

> $A := [[0, 0, 1, 0], [0, 0, 0, 1], [\mathrm{diff}(Eqs[1], x), \mathrm{diff}(Eqs[1], \theta), \mathrm{diff}(Eqs[1], xdot), \mathrm{diff}(Eqs[1], \theta dot)], [\mathrm{diff}(Eqs[2], x), \mathrm{diff}(Eqs[2], \theta), \mathrm{diff}(Eqs[2], xdot), \mathrm{diff}(Eqs[2], \theta dot)]];$

$$A := \left[\left[0, 0, 1, 0\right], \left[0, 0, 0, 1\right], \left[0, -\frac{-2\, m\, l \cos(\theta)\, \omega\, \theta dot - 3 \sin(\theta)^2 m\, g + 3 \cos(\theta)^2 m\, g}{-4\, mb + 3\, m \cos(\theta)^2} - \frac{6\, (4\, \tau - 2\, m\, l \sin(\theta)\, \omega\, \theta dot + 3 \cos(\theta)\, m\, g \sin(\theta))\, m \cos(\theta) \sin(\theta)}{\left(-4\, mb + 3\, m \cos(\theta)^2\right)^2}, 0, \frac{2\, m\, l \sin(\theta)\, \omega}{-4\, mb + 3\, m \cos(\theta)^2}\right], \left[0, \frac{3\, \left(2 \sin(\theta)\, \tau - \sin(\theta)^2 m\, l\, \omega\, \theta dot + \cos(\theta)^2 m\, l\, \omega\, \theta dot - 2\, mb\, g \cos(\theta)\right)}{l\, \left(-4\, mb + 3\, m \cos(\theta)^2\right)} + \frac{1}{l\, \left(-4\, mb + 3\, m \cos(\theta)^2\right)^2} (18\, (-2 \cos(\theta)\, \tau + \cos(\theta)\, m\, l \sin(\theta)\, \omega\, \theta dot - 2\, mb\, g \sin(\theta))\, m \cos(\theta) \sin(\theta)), 0, \frac{3 \cos(\theta)\, m \sin(\theta)\, \omega}{-4\, mb + 3\, m \cos(\theta)^2}\right]\right] \quad (7.2)$$

> $A := map(eval, A, LCond) :$

> $A := eval(A, sysid);$

(7.3)

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.731176470 & 0 & 0 \\ 0 & 24.73109243 & 0 & 0 \end{bmatrix} \quad (7.3)$$

> $B := \begin{bmatrix} 0 \\ 0 \\ \mathrm{diff}(Eqs[1], \tau) \\ \mathrm{diff}(Eqs[2], \tau) \end{bmatrix}$:

> $B := simplify(map(eval, B, LCond))$:

> $B := simplify(map(eval, B, sysid))$;

$$B := \begin{bmatrix} 0 \\ 0 \\ \frac{4}{17} \\ 0.5042016808 \end{bmatrix} \quad (7.4)$$

> $LinCi := T2$:

> $LinC3i := map(eval, LinCi, sysid)$;

$$LinC3i := \left[-\frac{11.9\,F7}{1.4\,\_C3 + 3\,\_C5}, \frac{1}{2} \frac{1}{1.96\,\_C3^2 + 8.4\,\_C3\,\_C5 + 9\,\_C5^2} \left(-28.8414\,\_C3^2 - 33.32\,\_C5\,F7 - 473.823\,\_C3\,\_C5 - 71.4\,\_C6\,F7 - 882.90\,\_C5^2\right), 1.000000000\,\nu - 0.5042016808\,\alpha\,\_C5 - 0.2352941177\,\alpha\,\_C3, 1.000000000\,\sigma - 0.5042016808\,\alpha\,\_C6 - 0.2352941177\,\alpha\,\_C5 \right] \quad (7.5)$$

The gain matrix is

> k := [-24.2610, 169.4551, -25.0697 , 35.4992];

$$k := [-24.2610, 169.4551, -25.0697, 35.4992] \quad (7.6)$$

F_1=0

>

> $LinC3 := -map(eval, LinC3i, [\nu = 0, \sigma = 0])$;

$$LinC3 := \left[\frac{11.9\,F7}{1.4\,\_C3 + 3\,\_C5}, \right. \quad (8.1)$$

$$-\frac{1}{2}\frac{1}{1.96\,\_C3^2+8.4\,\_C3\,\_C5+9\,\_C5^2}\left(-28.8414\,\_C3^2-33.32\,\_C5\,F7-473.823\,\_C3\,\_C5-71.4\,\_C6\,F7-882.90\,\_C5^2\right),\ 0.5042016808\,\alpha\,\_C5+0.2352941177\,\alpha\,\_C3,\ 0.5042016808\,\alpha\,\_C6+0.2352941177\,\alpha\,\_C5\Big]$$

> sol := solve({LinC3[1] - k[1], LinC3[2] - k[2], LinC3[3] - k[3], LinC3[4] - k[4]});

$$sol := \left\{F7=\frac{304.1079958}{\alpha},\ \_C3=\frac{219.9055702}{\alpha},\ \_C5=-\frac{152.3441711}{\alpha},\ \_C6=\frac{141.5006931}{\alpha},\ \alpha=\alpha\right\} \quad \textbf{(8.2)}$$

> K := map(eval, LinC3, [F7 = 304.1079958/α, _C3 = 219.9055702/α, _C5 = -152.3441711/α, _C6 = 141.5006931/α, α = α]);

$$K := \begin{bmatrix} -24.26099999 & 169.4551000 & -25.06970001 & 35.49919997 \end{bmatrix} \quad \textbf{(8.3)}$$

> Acl := eval(A-Multiply(B, K));

Acl := [[0., 0., 1., 0.],
[0., 0., 0., 1.],
[5.70847058588235257, -38.1406117652941106, 5.89875294352941104, -8.35275293411764608],
[12.2324369728467843, -60.7084538101320845, 12.6401848821937772, -17.8987562919293097]] **(8.4)**

> EACLi := Eigenvalues(Acl);

$$EACLi := \begin{bmatrix} -5.00006536005109581+0.\,I \\ -2.99993244076262888+0.\,I \\ -2.00000277379308944+2.00001956630095235\,I \\ -2.00000277379308944-2.00001956630095235\,I \end{bmatrix} \quad \textbf{(8.5)}$$

Evaluating the matrices with the parameters values

> KDf := simplify(eval(KDT, [F7 = 304.1079958/α, _C3 = 219.9055702/α, _C5 = -152.3441711/α, _C6 = 141.5006931/α, α = α]));

$$KDf := \begin{bmatrix} \frac{219.9055702}{\alpha} & -\frac{152.3441711}{\alpha} \\ -\frac{152.3441711}{\alpha} & \frac{141.5006931}{\alpha} \end{bmatrix} \quad \textbf{(8.6)}$$

> $KDf := simplify(eval(KDf, [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02, \alpha = 3]));$

$$KDf := \begin{bmatrix} 73.30185673 & -50.78139037 \\ -50.78139037 & 47.16689770 \end{bmatrix} \quad \textbf{(8.7)}$$

> $Determinant(KDf);$

$$878.671570 \quad \textbf{(8.8)}$$

> $Kvf := simplify\left(eval\left(Kv, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$

> $Kvf := simplify(eval(Kvf, [\alpha = 3])):$

> $Kvf := simplify(eval(Kvf, sysid));$

$$Kvf := \Bigg[\quad \textbf{(8.9)}$$

$$\left[\frac{1}{400. - 120. \cos(\theta)^2 + 9. \cos(\theta)^4}\left(0.0001000000000\left(5.683774646\ 10^9 \cos(\theta)^2 - 7.657447275\ 10^9 \cos(\theta) + 2.579117856\ 10^9\right)\right),\right.$$

$$\left. -\frac{1}{400. - 120. \cos(\theta)^2 + 9. \cos(\theta)^4}\left(0.0002000000000\left(-3.104316405\ 10^9 \cos(\theta) + 8.93368848\ 10^8 + 2.639608874\ 10^9 \cos(\theta)^2\right)\right)\right],$$

$$\left[-\frac{1}{400. - 120. \cos(\theta)^2 + 9. \cos(\theta)^4}\left(0.0002000000000\left(-3.104316405\ 10^9 \cos(\theta) + 8.93368848\ 10^8 + 2.639608874\ 10^9 \cos(\theta)^2\right)\right)\right.$$

,

$$\left.\frac{1}{400. - 120. \cos(\theta)^2 + 9. \cos(\theta)^4}\left(0.0001000000000\left(4.903456199\ 10^9 \cos(\theta)^2 - 4.927269897\ 10^9 \cos(\theta) + 1.237799811\ 10^9\right)\right)\right]\Bigg]$$

> $P := simplify(Multiply(KDf, MatrixInverse(mass))):$

> $P := simplify(eval(P, sysid));$

$$P := \left[\left[\frac{1.000000000\ 10^{-7}\left(-2.932074269\ 10^9 + 4.352690604\ 10^9 \cos(\theta)\right)}{-20. + 3. \cos(\theta)^2},\right.\right. \quad \textbf{(8.10)}$$

$$-\frac{1.000000000\ 10^{-7}\ (6.283016293\ 10^{9}\cos(\theta)-6.218129434\ 10^{10})}{-20.+3.\cos(\theta)^{2}}\Bigg],$$

$$\Bigg[-\frac{1.000000000\ 10^{-7}\ (-2.031255616\ 10^{9}+4.042876947\ 10^{9}\cos(\theta))}{-20.+3.\cos(\theta)^{2}},$$

$$\frac{5.000000000\ 10^{-7}\ (8.70538121\ 10^{8}\cos(\theta)-1.155107699\ 10^{10})}{-20.+3.\cos(\theta)^{2}}\Bigg]\Bigg]$$

$> PhiIif := simplify\left(eval\left(PhiIifi, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$

$> PhiIif := simplify(eval(PhiIif, sysid)):$

$> PhiIif := simplify(eval(PhiIif, [\alpha = 3]));$

$$PhiIif := 2153.034264 - 169.7426569\ln(3.310406433\ 10^{7} - 4.9143281\ 10^{7}\cos(\theta)) + 101.3693320\,x^{2} + 87.23733370\,x\,\mathrm{arctanh}\left(\frac{2.264484578\,(-1.+\cos(\theta))}{\sin(\theta)}\right) + 376.6164630\,x\,\arctan\left(\frac{-1.+\cos(\theta)}{\sin(\theta)}\right) + 18.76887280\,\mathrm{arctanh}\left(\frac{2.264484578\,(-1.+\cos(\theta))}{\sin(\theta)}\right)^{2} + 162.0559960\,\mathrm{arctanh}\left(\frac{2.264484578\,(-1.+\cos(\theta))}{\sin(\theta)}\right)\arctan(1/(\sin(\theta))(-1.+\cos(\theta))) + 349.8098430\arctan\left(\frac{-1.+\cos(\theta)}{\sin(\theta)}\right)^{2} \quad \textbf{(8.11)}$$

$> F1 := simplify\left(eval\left(F1i, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$

$> F1 := simplify(eval(F1, [\alpha = 3, \nu = 0])):$

$> Fmc1 := simplify\left(eval\left(Fmc1i, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$

$> Fmc1 := simplify(eval(Fmc1, [\alpha = 3])):$

$> Fmc1Matlab := simplify(eval(Fmc1, sysid)):$

>

> $F2 := simplify\left(eval\left(F2i, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right);$

$$F2 := -\frac{1}{l\left(-4.\,mb + 3.\,m\cos(\theta)^2\right)}\left(2.000000000\ 10^{-7}\left(-4.398111404\ 10^{9}\ xdot\ l + 4.570325133\ 10^{9}\ xdot\cos(\theta) + 3.046883422\ 10^{9}\ \theta dot\ l - 4.245020793\ 10^{9}\ \theta dot\cos(\theta)\right)\right) \quad \textbf{(8.12)}$$

> $F2 := simplify(eval(F2, sysid)):$

> $F2 := simplify(eval(F2, [\alpha = 3, \nu = 0, \sigma = 0])):$

> $F3 := simplify\left(eval\left(F3i, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$

> $F3 := simplify(eval(F3, sysid)):$

> $F3 := simplify(eval(F3, [\alpha = 3, \nu = 0, \sigma = 0])):$

> $convert(F1, string)$

"`θdot`*sigma" **(8.13)**

> $Fmc1Matlab := simplify(eval(Fmc1Matlab, [\alpha = 3, \nu = 0, \sigma = 0])):$

> $convert(Fmc1Matlab, string);$

"Matrix(2, 2, [[0.,.1000000000e-6*sin(theta)*omega*(-1026225995.+1523441712.*cos(theta))/(-20.+3.*cos(theta)^2)],[.1000000000e-6*sin(theta)*omega*(-1026225995.+1523441712.*cos(theta))/(-20.+3.*cos(theta)^2), -.1000000000e-6*sin(theta)*omega*(2830013863.*cos(theta)-1421878931.)/(-20.+3.*cos(theta)^2)]])" **(8.14)**

> $convert(F2, string)$

"-.1000000000e-6*(-8796222811.*xdot+.1305807181e11*xdot*cos(theta)+6093766845.*`θdot`-.1212863084e11*`θdot`*cos(theta))/(-20.+3.*cos(theta)^2)" **(8.15)**

> $convert(F3, string)$

"-.9716473972e-1*(-6082068288.*arctanh(2.264484578*(-1.+cos(theta))/sin(theta))+912310243.*cos(theta)^2*arctanh(2.264484578*(-1.+cos(theta))/sin(theta))-7520109265.*cos(theta)*sin(theta)-.1413466400e11*x-.2625718776e11*arctan((-1.+cos(theta))/sin(theta))+3938578165.*cos(theta)^2*arctan((-1.+cos(theta))/sin(theta))+2120199600.*cos(theta)^2*x+.7442446524e11*sin(theta))/(-99312193.+147429843.*cos(theta))" **(8.16)**

> $convert(\mathrm{PhiIif}, string)$

"2153.034264-169.7426569*ln(33104064.33-49143281.*cos(theta))+101.3693320* **(8.17)**

x^2+87.23733370*x*arctanh(2.264484578*(-1.+cos(theta))/sin(theta))
+376.6164630*x*arctan((-1.+cos(theta))/sin(theta))+18.76887280*arctanh
(2.264484578*(-1.+cos(theta))/sin(theta))^2+162.0559960*arctanh
(2.264484578*(-1.+cos(theta))/sin(theta))*arctan((-1.+cos(theta))/sin(theta))
+349.8098430*arctan((-1.+cos(theta))/sin(theta))^2"

The Potential Φ

> $PhiGraphi := simplify(eval(\mathrm{PhiIifi}, \mathrm{sysid})):$

> $$PhiGraph := simplify\left(eval\left(PhiGraphi, \left[\_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right):$$

> $PhiGraph := simplify(eval(PhiGraph, [\alpha = 3]));$

$$PhiGraph := 2153.034270 - 169.7426571 \ln\left(3.310406433\ 10^{7} - 4.9143281\ 10^{7} \cos(\theta)\right) + 0.9999999990\ F7\ x^2 + 0.8605890160\ F7\ x \operatorname{arctanh}\left(\frac{2.264484578\ (-1. + \cos(\theta))}{\sin(\theta)}\right) + 3.715289977\ F7\ x \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right) + 0.1851533644\ F7 \operatorname{arctanh}\left(\frac{2.264484578\ (-1. + \cos(\theta))}{\sin(\theta)}\right)^2 + 1.598668875\ F7 \operatorname{arctanh}\left(\frac{2.264484578\ (-1. + \cos(\theta))}{\sin(\theta)}\right) \arctan\left(1 / (\sin(\theta))\,(-1. + \cos(\theta))\right) + 3.450844910\ F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^2 \tag{9.1}$$

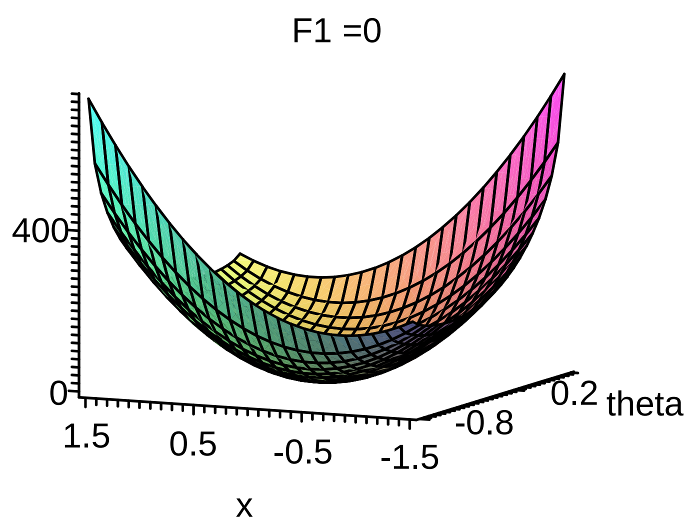
> $$PhiGraph := 0.9999999990\ F7\ x^2 + 0.8605890160\ F7\ x \operatorname{arctanh}\left(\frac{2.264484578\ (-1. + \cos(\theta))}{\sin(\theta)}\right) + 3.450844910\ F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^2 + 1.598668875\ F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right) \operatorname{arctanh}\left(1 / (\sin(\theta))\,(2.264484578\ (-1. + \cos(\theta)))\right) + 0.1851533644\ F7 \operatorname{arctanh}\left(\frac{2.264484578\ (-1. + \cos(\theta))}{\sin(\theta)}\right)^2 :$$

> $PhiGraph := simplify\left(eval\left(PhiGraph, \left[F7 = \frac{304.1079958}{\alpha}\right]\right)\right):$

> $PhiGraph := simplify(eval(PhiGraph, [\alpha = 3])):$

> $f := (x, \theta) \rightarrow PhiGraph:$

> $plot3d(f(x, \theta), \theta = -0.8 .. 0.8, x = -1.5 .. 1.5, axes = FRAME, orientation = [205, 80], style = PATCH, title = \text{" F1 =0"});$



The Hessian

> $Hessiani := simplify((Matrix(2, 2, [diff(diff(\text{PhiIifi}, x), x), diff(diff(\text{PhiIifi}, x), \theta), diff(diff(PhiIifi, \theta), x), diff(diff(PhiIifi, \theta), \theta)]))):$

> $Hessiani := simplify(eval(Hessiani, [\cos(\theta) = 1, x = 0]));$

$$Hessiani := \left[\left[2\,F7, \frac{2\,F7\,(2\,l\,\_C5 + 3\,\_C6)}{2\,l\,\_C3 + 3\,\_C5}\right], \left[\frac{2\,F7\,(2\,l\,\_C5 + 3\,\_C6)}{2\,l\,\_C3 + 3\,\_C5}, \frac{1}{4\,l^2\,\_C3^2 + 12\,l\,\_C5\,\_C3 + 9\,\_C5^2}\left(-6\,g\,\_C6\,l\,\_C3^2 + 6\,\_C3\,\_C5^2\,g\,l - 9\,\_C3\,\_C5\,g\,\_C6 + 8\,F7\,\_C5^2\,l^2 + 24\,F7\,\_C6\,\_C5\,l + 9\,\_C5^3\,g + 18\,F7\,\_C6^2\right)\right]\right] \tag{10.1}$$

> $Hessian := simplify\left(eval\left(Hessiani, \left[F7 = \frac{304.1079958}{\alpha}, \_C3 = \frac{219.9055702}{\alpha}, \_C5 = -\frac{152.3441711}{\alpha}, \_C6 = \frac{141.5006931}{\alpha}, \alpha = \alpha\right]\right)\right);$

$$Hessian := \left[\left[\frac{608.2159916}{\alpha}, \right.\right. \tag{10.2}$$

$$-\frac{608.2159916\,(3.046883422\,10^9\,l - 4.245020793\,10^9)}{\alpha\,(4.398111404\,10^9\,l - 4.570325133\,10^9)}\Bigg],$$

$$\Bigg[-\frac{608.2159916\,(3.046883422\,10^9\,l - 4.245020793\,10^9)}{\alpha\,(4.398111404\,10^9\,l - 4.570325133\,10^9)},$$

$$-(100.\,(1.043413771\,10^9\,g\,l - 1.084269985\,10^9\,g - 5.646372299\,10^9\,l^2 + 1.573343282\,10^{10}\,l - 1.096017475\,10^{10}))/(\alpha\,(1.934338392\,10^9\,l^2 - 4.020159817\,10^9\,l + 2.088787182\,10^9))\Bigg]\Bigg]$$

> *Hessian* := *simplify*(*eval*(*Hessian*, sysid));

$$Hessian := \begin{bmatrix} \frac{608.2159916}{\alpha} & -\frac{861.2460905}{\alpha} \\ -\frac{861.2460905}{\alpha} & \frac{2779.788406}{\alpha} \end{bmatrix} \quad \textbf{(10.3)}$$

> *Hessian* := *simplify*(*eval*(*Hessian*, [α = 3]));

$$Hessian := \begin{bmatrix} 202.7386639 & -287.0820302 \\ -287.0820302 & 926.5961353 \end{bmatrix} \quad \textbf{(10.4)}$$

> *Determinant*(*Hessian*);

$$1.054407703\,10^5 \quad \textbf{(10.5)}$$

> *Eigenvalues*(*Hessian*);

$$\begin{bmatrix} 102.705819174577528 + 0.\,I \\ 1026.62898002542261 + 0.\,I \end{bmatrix} \quad \textbf{(10.6)}$$

F_1 is non-zero

> *LinC3* := -*simplify*(*eval*(*LinC3i*, σ = 0));

$$LinC3 := \Bigg[\frac{59.50000000\,F7}{7.\,\_C3 + 15.\,\_C5}, \frac{1}{49.\,\_C3^2 + 210.\,\_C3\,\_C5 + 225.\,\_C5^2}(0.002500000000\,(1.44207\,10^5\,\_C3^2 + 1.66600\,10^5\,\_C5\,F7 + 2.369115\,10^6\,\_C3\,\_C5 + 3.57000\,10^5\,\_C6\,F7 + 4.414500\,10^6\,\_C5^2)), -\nu + 0.5042016808\,\alpha\,\_C5 + 0.2352941177\,\alpha\,\_C3, 0.5042016808\,\alpha\,\_C6 + 0.2352941177\,\alpha\,\_C5\Bigg] \quad \textbf{(11.1)}$$

> *sol* := *solve*({$LinC3_1 - k_1$, $LinC3_2 - k_2$, $LinC3_3 - k_3$, $LinC3_4 - k_4$});

$$sol := \Bigg\{F7 = \frac{3.207194748\,10^{-15}\,(1.051071429\,10^{15}\,\alpha\,\_C6 - 5.390677767\,10^{16})}{\alpha}, \quad \textbf{(11.2)}$$

$$\_C3 = \frac{1.123661653\ 10^{-15}\left(3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17}\right)}{\alpha},$$

$$\_C5 = -\frac{3.399999999\ 10^{-9}\left(6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10}\right)}{\alpha}, \_C6$$

$$= \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563\}$$

> $K := map\left(eval, LinC3, \left[F7 = \frac{3.207194748\ 10^{-15}\left(1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16}\right)}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\left(3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17}\right)}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\left(6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10}\right)}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563\right]\right):$

Notice that only the variuables affected by the input F1 and $\_C6$ are going to be modified. Now $\_C6 = 50$

> $K := map(eval, K, [\alpha = 3, \_C6 = 28]);$

$$K := \left[\ -24.26100010\quad 169.4551009\quad -25.06970000\quad 35.49920001\ \right] \tag{11.3}$$

> $Acl := eval(A - Multiply(B, K));$

$Acl := [[0., 0., 1., 0.],$ **(11.4)**

$[0., 0., 0., 1.],$

$[5.70847061176470572, -38.1406119770588177, 5.89875294117647098,$

$-8.35275294352941167],$

$[12.2324370283089703, -60.7084542639135947, 12.6401848771517624,$

$-17.8987563120973796]]$

> $EACLi := Eigenvalues(Acl);$

$$EACLi := \begin{bmatrix} -5.00006502647233653 + 0.\ I \\ -2.00000275508543801 + 2.00001946515687745\ I \\ -2.00000275508543801 - 2.00001946515687745\ I \\ -2.99993283427765700 + 0.\ I \end{bmatrix} \tag{11.5}$$

Evaluating the matrices with the parameters values from actuation case, the results are:

> $KDf := simplify\left(eval\left(KDT, \left[F7\right.\right.\right.$

$$= \frac{3.207194748\ 10^{-15}\ (1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3$$

$$= \frac{1.123661653\ 10^{-15}\ (3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 =$$

$$-\frac{3.399999999\ 10^{-9}\ (6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6$$

$$= \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563 \Big]\Big)\Big) :$$

> $KDf := simplify(eval(KDf, [Ib = 0.4, m = 1.5, g = 9.81, Ro = 0.02, C6 = 50])) :$

> $KDf := simplify(eval(KDf, [\alpha = 3, \_C6 = 28]));$

$$KDf := \begin{bmatrix} 7.927630577 & -9.709466673 \\ -9.709466673 & 28 \end{bmatrix} \quad \textbf{(11.6)}$$

> $Determinant(KDf)$

$$127.6999131 \quad \textbf{(11.7)}$$

> $Kvf := simplify\Big(eval\Big(Kvf, \Big[F7$

$$= \frac{3.207194748\ 10^{-15}\ (1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3$$

$$= \frac{1.123661653\ 10^{-15}\ (3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 =$$

$$-\frac{3.399999999\ 10^{-9}\ (6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6$$

$$= \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563 \Big]\Big)\Big) :$$

> $Kvf := simplify(eval(Kvf, sysid)) :$

> $Kvf := simplify(eval(Kvf, [\alpha = 3, \_C6 = 28]));$

$$Kvf := \Big[\quad \textbf{(11.8)}$$

$$\Big[\frac{1}{400. - 120.\ \cos(\theta)^2 + 9.\ \cos(\theta)^4} (0.0001000000000\ (5.683774646\ 10^{9}$$

$$\cos(\theta)^2 - 7.657447275\ 10^{9} \cos(\theta) + 2.579117856\ 10^{9})),$$

$$-\frac{1}{400. - 120.\ \cos(\theta)^2 + 9.\ \cos(\theta)^4} (0.0002000000000\ ($$

$$-3.104316405\ 10^{9}\cos(\theta)+8.93368848\ 10^{8}+2.639608874\ 10^{9}\cos(\theta)^{2})\big)\Big],$$

$$\Big[-\frac{1}{400.-120.\cos(\theta)^{2}+9.\cos(\theta)^{4}}\big(0.0002000000000\,(-3.104316405\ 10^{9}\cos(\theta)+8.93368848\ 10^{8}+2.639608874\ 10^{9}\cos(\theta)^{2})\big),$$

$$\frac{1}{400.-120.\cos(\theta)^{2}+9.\cos(\theta)^{4}}\big(0.0001000000000\,(4.903456199\ 10^{9}\cos(\theta)^{2}-4.927269897\ 10^{9}\cos(\theta)+1.237799811\ 10^{9})\big)\Big]\Big]$$

> $P := simplify(Multiply(KDf, MatrixInverse(mass)))$:

> $P := simplify(eval(P, sysid))$;

$$P := \left[\left[\frac{8.000000000\ 10^{-8}\,(-3.96381529\ 10^{8}+1.040300001\ 10^{9}\cos(\theta))}{-20.+3.\cos(\theta)^{2}}, -\frac{1.200000000\ 10^{-7}\,(5.66259327\ 10^{8}\cos(\theta)-9.907619060\ 10^{9})}{-20.+3.\cos(\theta)^{2}}\right], \left[-\frac{1.000000000\ 10^{-8}\,(-3.883786669\ 10^{9}+2.400000000\ 10^{10}\cos(\theta))}{-20.+3.\cos(\theta)^{2}}, \frac{1.000000000\ 10^{-7}\,(8.32240001\ 10^{8}\cos(\theta)-3.428571429\ 10^{10})}{-20.+3.\cos(\theta)^{2}}\right]\right] \quad \textbf{(11.9)}$$

> $PhiIif := simplify\Big(eval\Big(PhiIifi, \Big[F7 = \frac{3.207194748\ 10^{-15}\,(1.051071429\ 10^{15}\,\alpha\,\_C6-5.390677767\ 10^{16})}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\,(3.035422690\ 10^{15}\,\alpha\,\_C6-2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\,(6.30252101\ 10^{8}\,\alpha\,\_C6-4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha=\alpha, \nu=-0.2778938021\,\alpha\,\_C6+39.32216563\Big]\Big)\Big)$:

> $PhiIif := simplify(eval(PhiIif, [\alpha=3, \_C6=28]))$:

> $PhiIif := simplify(eval(PhiIif, sysid))$;

$$PhiIif := 68.73005780 \operatorname{arctanh}\left(\frac{1.493704932\ (-1.+\cos(\theta))}{\sin(\theta)}\right)^2 + 1222.745278 \arctan\left(\frac{-1.+\cos(\theta)}{\sin(\theta)}\right)^2 + 2466.035604 + 36.75761872\, x^2 + 100.5264485\, x \operatorname{arctanh}\left(\frac{1.493704932\ (-1.+\cos(\theta))}{\sin(\theta)}\right) + 579.7928446 \operatorname{arctanh}\left(\frac{1.493704932\ (-1.+\cos(\theta))}{\sin(\theta)}\right) \arctan(1/ (\sin(\theta))(-1.+\cos(\theta))) - 129.0210281 \ln(2.219736563\ 10^9 - 5.825680000\ 10^9 \cos(\theta)) + 424.0090818\, x \arctan\left(\frac{-1.+\cos(\theta)}{\sin(\theta)}\right) \tag{11.10}$$

>

> $F1 := simplify\left(eval\left(F1i, \left[F7 = \frac{3.207194748\ 10^{-15}\ (1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\ (3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\ (6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563\right]\right)\right):$

> $F1 := simplify(eval(F1, [\alpha = 3, \_C6 = 28, \sigma = 0]));$

$$F1 := 15.97908625\ xdot \tag{11.11}$$

> $Fmc1 := simplify\left(eval\left(Fmc1i, \left[F7 = \frac{3.207194748\ 10^{-15}\ (1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\ (3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\ (6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563\right]\right)\right):$

> $Fmc1 := simplify(eval(Fmc1, sysid)):$

> $Fmc1Matlab := simplify(eval(Fmc1, [\alpha = 3, \_C6 = 28, \sigma = 0])):$

>

> $F2 := simplify\left(eval\left(F2i, \left[F7 = \frac{3.207194748\ 10^{-15}\,(1.051071429\ 10^{15}\,\alpha\,\_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\,(3.035422690\ 10^{15}\,\alpha\,\_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\,(6.30252101\ 10^{8}\,\alpha\,\_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\,\alpha\,\_C6 + 39.32216563\right]\right)\right):$

> $F2 := simplify(eval(F2, sysid)):$

> $F2 := simplify(eval(F2, [\alpha = 3, \_C6 = 28, \sigma = 0]));$

$$F2 := -\frac{1}{-20. + 3.\cos(\theta)^2}\,\big(2.000000000\ 10^{-7}\,(-4.75657830\ 10^{8}\,xdot + 1.248360004\ 10^{9}\,xdot\cos(\theta) + 5.82568000\ 10^{8}\,\theta dot - 3.600000001\ 10^{9}\,\theta dot\cos(\theta))\big) \qquad \textbf{(11.12)}$$

> $F3 := simplify\left(eval\left(F3i, \left[F7 = \frac{3.207194748\ 10^{-15}\,(1.051071429\ 10^{15}\,\alpha\,\_C6 - 5.390677767\ 10^{16})}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\,(3.035422690\ 10^{15}\,\alpha\,\_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\,(6.30252101\ 10^{8}\,\alpha\,\_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\,\alpha\,\_C6 + 39.32216563\right]\right)\right):$

> $F3 := simplify(eval(F3, [\alpha = 3, \_C6 = 28, \sigma = 0])):$

> $F3 := simplify(eval(F3, sysid)):$

> $convert(\mathrm{F1}, string);$

"15.97908625*xdot" **(11.13)**

>

> $convert(Fmc1Matlab, string);$

"Matrix(2, 2, [[-.3000000000e-5*(-337803445.+886562318.*cos(theta))/(-20.+3.*cos(theta)^2),.1000000000e-7*(-1109868277.*sin(theta)*omega+2912840006.*cos(theta)*sin(theta)*omega+.3834980700e12*cos(theta)-.6205936166e11)/(-20.+3.*cos(theta)^2)],[.1000000000e-7*(-1109868277.*sin(theta)* **(11.14)**

omega+2912840006.*cos(theta)*sin(theta)*omega+.3834980700e12*cos(theta)
-.6205936166e11)/(-20.+3.*cos(theta)^2),-.1000000000e-7*sin(theta)*omega*
(.1680000001e11*cos(theta)-2718650673.)/(-20.+3.*cos(theta)^2)]])"

> *convert*(*F2*, *string*)

"-.2000000000e-6*(-475657830.*xdot+1248360004.*xdot*cos(theta)+582568000. **(11.15)**
`θdot`-3600000001.`θdot`*cos(theta))/(-20.+3.*cos(theta)^2)"

> *convert*(*F3*, *string*)

"10.*(.1055527349e11*arctanh(1.493704932*(-1.+cos(theta))/sin(theta)) **(11.16)**
+1224878332.*cos(theta)*sin(theta)-1583291101.*cos(theta)^2*arctanh
(1.493704932*(-1.+cos(theta))/sin(theta))+7719158394.*x-.2143122036e11*
sin(theta)+.4452076310e11*arctan((-1.+cos(theta))/sin(theta))-6678114463.*
cos(theta)^2*arctan((-1.+cos(theta))/sin(theta))-1157873750.*cos(theta)^2*x)/
(-1664802419.+4369260010.*cos(theta))"

> *convert*(PhiIif, *string*)

"68.73005780*arctanh(1.493704932*(-1.+cos(theta))/sin(theta))^2+1222.745278* **(11.17)**
arctan((-1.+cos(theta))/sin(theta))^2+2466.035604+36.75761872*
x^2+100.5264485*x*arctanh(1.493704932*(-1.+cos(theta))/sin(theta))
+579.7928446*arctanh(1.493704932*(-1.+cos(theta))/sin(theta))*arctan((-1.+
cos(theta))/sin(theta))-129.0210281*ln(2219736563.-5825680000.*cos(theta))
+424.0090818*x*arctan((-1.+cos(theta))/sin(theta))"

The Potential Φ

> $PhiGraphi := simplify(eval(\mathrm{PhiIifi}, \mathrm{sysid}))$:

> $$PhiGraph := simplify\Bigg(eval\Bigg(PhiGraphi, \Bigg[\_C3 = \frac{1.123661653\ 10^{-15}\,(3.035422690\ 10^{15}\,\alpha\,\_C6 - 2.338099784\ 10^{17})}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\,(6.30252101\ 10^{8}\,\alpha\,\_C6 - 4.437400000\ 10^{10})}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\,\alpha\,\_C6 + 39.32216563\Bigg]\Bigg)\Bigg):$$

> $PhiGraph := simplify(eval(PhiGraph, [\alpha = 3, \_C6 = 28, \sigma = 0]));$

$$PhiGraph := 2.734825490\,F7\,x\,\mathrm{arctanh}\left(\frac{1.493704925\,(-1. + \cos(\theta))}{\sin(\theta)}\right) + 2584.311340 + 15.77328620\,F7\,\mathrm{arctanh}\left(\frac{1.493704925\,(-1. + \cos(\theta))}{\sin(\theta)}\right)\mathrm{arctan}(1 \tag{12.1}$$

$$/ (\sin(\theta)) (-1. + \cos(\theta))) + 11.53513745\, F7\, x \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)$$

$$- 129.0204786 \ln(5.549341367\ 10^{9} - 1.456420003\ 10^{10} \cos(\theta))$$

$$+ 1.000001801\, F7\, x^{2}$$

$$+ 1.869815542\, F7\, \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right)^{2}$$

$$+ 33.26482968\, F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^{2}$$

$> PhiGraph := 2.734825380\, F7\, x\, \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right)$
$+ 1.000001801\, F7\, x^{2} + 33.26482968\, F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^{2}$
$+ 1.869815375\, F7\, \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right)^{2}$
$+ 15.77328620\, F7 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right) \operatorname{arctanh}(1/$
$(\sin(\theta)) (1.493704925\ (-1. + \cos(\theta)))) :$

$> PhiGraph := simplify\left(eval\left(PhiGraph, \left[F7 = \frac{3.207194748\ 10^{-15}\ (1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16})}{\alpha}\right]\right)\right) :$

$> PhiGraph := simplify(eval(PhiGraph, [\alpha = 3, \_C6 = 28, \sigma = 0]));$

$$PhiGraph := 100.5264289\, x\, \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right) \tag{12.2}$$

$$+ 36.75796293\, x^{2} + 1222.745175 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^{2}$$

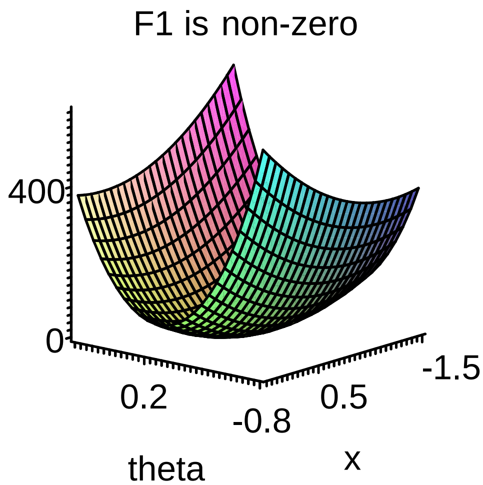
$$+ 68.73048047 \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right)^{2}$$

$$+ 579.7928253 \operatorname{arctanh}\left(\frac{1.493704925\ (-1. + \cos(\theta))}{\sin(\theta)}\right) \arctan(1/$$

$$(\sin(\theta)) (-1. + \cos(\theta)))$$

$> f := (x, \theta) \rightarrow PhiGraph :$

$> plot3d(f(x, \theta), \theta = -0.8 .. 0.8, x = -1.5 ... 1.5, axes = FRAME, orientation = [130, 75], style = PATCH, title = \text{"F1 is non-zero"});$



The Hessian

$> Hessiani := simplify\left(\left(Matrix\left(2, 2, \left[diff\left(diff\left(\mathrm{PhiIifi}, \mathrm{x}\right), \mathrm{x}\right), diff\left(diff\left(\mathrm{PhiIifi}, \mathrm{x}\right), \theta\right), diff\left(diff\left(PhiIifi, \theta\right), x\right), diff\left(diff\left(PhiIifi, \theta\right), \theta\right)\right]\right)\right)\right) :$

$> Hessiani := simplify(eval(Hessiani, [\cos(\theta) = 1, x = 0])) :$

$$> Hessian := simplify\left(eval\left(Hessiani, \left[F7 = \frac{3.207194748\ 10^{-15}\left(1.051071429\ 10^{15}\ \alpha\ \_C6 - 5.390677767\ 10^{16}\right)}{\alpha}, \_C3 = \frac{1.123661653\ 10^{-15}\left(3.035422690\ 10^{15}\ \alpha\ \_C6 - 2.338099784\ 10^{17}\right)}{\alpha}, \_C5 = -\frac{3.399999999\ 10^{-9}\left(6.30252101\ 10^{8}\ \alpha\ \_C6 - 4.437400000\ 10^{10}\right)}{\alpha}, \_C6 = \_C6, \alpha = \alpha, \nu = -0.2778938021\ \alpha\ \_C6 + 39.32216563 \right]\right)\right) :$$

$> Hessian := simplify(eval(Hessian, \mathrm{sysid})) :$

$>$

$> Hessian := simplify(eval(Hessian, [\alpha = 3, \_C6 = 28, \sigma = 0]));$

$$Hessian := \begin{bmatrix} 73.51579346 & -287.0820303 \\ -287.0820303 & 1329.511648 \end{bmatrix} \quad \textbf{(13.1)}$$

$> Determinant(Hessian);$

$$15324.01160 \quad \textbf{(13.2)}$$

$> Eigenvalues(Hessian);$

$$\begin{bmatrix} 11.0084790579423952 + 0.\,I \\ 1392.01896240205724 + 0.\,I \end{bmatrix} \tag{13.3}$$

>

The control law numerical value is

> $tauf := simplify\big(eval\big(F1 + F2 + F3, \big[F6 = \big(34.53815001\,(5.285592337\;10^{29}\,p21 - 4.741744895\;10^{26}\,p22)\,p21^2\big) \big/ \big((1.200482133\;10^{15}\,p21 - 1.568000000\;10^{12}\,p22)\,(4.382731333\;10^{14}\,p21 - 3.919999999\;10^{11}\,p22)\big), \alpha = \alpha, \nu = -0.5861921286\,\alpha\,p21 + 0.001418721037\,\alpha\,p22 - 34.13190000, p21 = p21, p22 = p22, \sigma = -1.\,\alpha\,p21 + 118.4029000\big]\big)\big):$

> $tauf := simplify(eval(tauf, [p21 = 1.75 \cdot p11])):$

> $tauf := simplify(eval(tauf, [p11 = 0.05491210805\,p22])):$

> $tauf := simplify\big(eval\big(tauf, \big[p22 = 5\;10^{-9}, \alpha = 1.5\big]\big)\big):$

> $tauf := simplify\big(eval\big(tauf, \big[p22 = 5\;10^{-9}, \alpha = 1.5\big]\big)\big):$

> $tauf := simplify(eval(tauf, sysid));$

$$\begin{aligned} tauf := -\Big(10.\,\Big(&-3.736651676\;10^{10}\,xdot + 5.650265530\;10^{10}\,xdot\cos(\theta) \\ &+ 1.170687953\;10^{11}\,xdot\cos(\theta)^2 - 2.094503476\;10^{10}\,xdot\cos(\theta)^3 \\ &- 1.939721232\;10^{10}\,\theta dot + 1.707735955\;10^{11}\,\theta dot\cos(\theta) \\ &- 3.145867208\;10^{11}\,\theta dot\cos(\theta)^2 \\ &+ 2.111054698\;10^{11}\,\mathrm{arctanh}\left(\frac{1.493704932\,(-1. + \cos(\theta))}{\sin(\theta)}\right) \\ &- 6.333164250\;10^{10}\cos(\theta)^2\,\mathrm{arctanh}\left(\frac{1.493704932\,(-1. + \cos(\theta))}{\sin(\theta)}\right) \\ &+ 2.449756664\;10^{10}\cos(\theta)\sin(\theta) - 3.674634996\;10^{9}\cos(\theta)^3\sin(\theta) \\ &+ 4.749873303\;10^{9}\cos(\theta)^4\,\mathrm{arctanh}\left(\frac{1.493704932\,(-1. + \cos(\theta))}{\sin(\theta)}\right) \\ &+ 1.543831679\;10^{11}\,x - 4.631495018\;10^{10}\cos(\theta)^2\,x \\ &- 4.286244072\;10^{11}\sin(\theta) + 6.429366110\;10^{10}\sin(\theta)\cos(\theta)^2 \\ &+ 8.904152620\;10^{11}\,\mathrm{arctan}\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right) \end{aligned} \tag{13.4}$$

$$- 2.671245786\ 10^{11} \cos(\theta)^2 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)$$

$$+ 2.003434339\ 10^{10} \cos(\theta)^4 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)$$

$$+ 3.473621250\ 10^{9} \cos(\theta)^4 x\Bigg)\Bigg) \Bigg/ \Big(3.329604838\ 10^{10}$$

$$- 8.738520020\ 10^{10} \cos(\theta) - 4.994407257\ 10^{9} \cos(\theta)^2$$

$$+ 1.310778003\ 10^{10} \cos(\theta)^3\Big)$$

The potential numerical value is

> $phif := PhiGraph;$

$$phif := 100.5264289\, x \operatorname{arctanh}\left(\frac{1.493704925\,(-1. + \cos(\theta))}{\sin(\theta)}\right) + 36.75796293\, x^2 \quad \textbf{(13.5)}$$

$$+ 1222.745175 \arctan\left(\frac{-1. + \cos(\theta)}{\sin(\theta)}\right)^2$$

$$+ 68.73048047 \operatorname{arctanh}\left(\frac{1.493704925\,(-1. + \cos(\theta))}{\sin(\theta)}\right)^2$$

$$+ 579.7928253 \operatorname{arctanh}\left(\frac{1.493704925\,(-1. + \cos(\theta))}{\sin(\theta)}\right) \arctan\Big(1 \Big/$$

$$(\sin(\theta))(-1. + \cos(\theta))\Big)$$

```
%%%%%%%%%%%  C.4 IINVERTED PENDULUM CART SIM, F1 is zero  %%%%%%%%%%%
%%%%%%%%%%%             IPC_Nonlinear_Sys.m             %%%%%%%%%%%
function dxdt = IPC_NonLinear_Sys(u)

%% Main Vectors
x            = u(1);            % feedback array
theta        = u(2);
xdot         = u(3);
thetadot     = u(4);
%% Generalized quantities
q            = [x theta]';       % Generalized coordinates
qdot         = [xdot thetadot]'; % Generalized velocities

%% Physical parameter values
mb           = 5;               % kg       - mass of cart+pendulum
m            = 1;               % kg       - mass of pendulum
l            = .7;              % kg m^2   - inertia of beam
g            = 9.81;            % m/sec^2 - acceleration of gravity

%% Linear model parameters
alpha    =3;
C6       = 141.5006932/alpha;
F5       = 304.1079958/alpha;
C3       = 219.9055703/alpha;
C5       = -152.3441711/alpha;
%No actuation
nu       = 0;
sigma    = 0;
omega    = thetadot;

%% The G,M,C,  P and KD matrices
            %gravity terms
G            = [0; -(1/2)*m*g*l*sin(theta)];
            %mass matrix
mass         = [mb -(1/2)*m*l*cos(theta); -(1/2)*m*l*cos(theta) (1/3)*m*l^2];
            %Centripetal and coriolis matrix
C            =   [0, (1/2)*m*l*sin(theta)*thetadot; 0, 0];

%% FMC
            %KD matrix
KD           =  [C3  C5; C5  C6];
            % P matrix
P            =  KD*inv(mass);
            % Determinant of P matrix
Pdet         =  det(P);
%% SMC
            %Kv Matrix
Kv           = [4*alpha*(2*C3*l+3*C5*cos(theta))^2/l^2/(-4*mb+3*m*cos(theta)
^2)^2,4*alpha*(2*C3*l+...
            3*C5*cos(theta))/l^2/(-4*mb+3*m*cos(theta)^2)^2*(2*l*C5+3*cos
(theta)*C6);4*alpha*(2*C3*l+...
```

```
            3*C5*cos(theta))/l^2/(-4*mb+3*m*cos(theta)^2)^2*(2*l*C5+3*cos
(theta)*C6),4*alpha*(2*l*C5+...
            3*cos(theta)*C6)^2/l^2/(-4*mb+3*m*cos(theta)^2)^2];

%% Evaluate the control law
            % F1= F1m + F1m2
 F1         = 0;

 F2         = -.1000000000e-6*(-.1212863085e11*cos(theta)*thetadot+.
1305807181e11*cos(theta)*xdot-...
            8796222814.*xdot+6093766845.*thetadot)/(-20.+3.*cos(theta)^2);

 F3         = -1.943294797*(1414080884.*cos(theta)^2*atanh(2.264484581*(cos
(theta)-1.)/sin(theta))-...
            9427205895.*atanh(2.264484581*(cos(theta)-1.)/sin(theta))-.
1165616936e11*sin(theta)*cos(theta)-...
            .2190872918e11*x-.4069864100e11*atan((cos(theta)-1.)/sin(theta))
+6104796154.*cos(theta)^2*atan((cos(theta)-...
            1.)/sin(theta))+3286309375.*cos(theta)^2*x+.1153579211e12*sin
(theta))/(-3078677984.+4570325133.*cos(theta));

tau         = F1+F2+F3;

%% Lyapunov
Fmc1       =[0.,-.1000000000e-6*sin(theta)*omega*(-1026225995.+1523441712.
*cos(theta))/(-20.+3.*cos(theta)^2);.1000000000e-6*sin(theta)*omega*(-...
            1026225995.+1523441712.*cos(theta))/(-20.+3.*cos(theta)^2),-.
1000000000e-6*sin(theta)*omega*(2830013863.*cos(theta)-...
            1421878931.)/(-20.+3.*cos(theta)^2)];

 PHi        = 2735.928392-169.7426575*log(1026225995.-1523441711.*cos(theta))
+101.3693319*x^2+...
            376.6164626*x*atan((cos(theta)-1.)/sin(theta))+87.
23733425*x*atanh(2.264484581*(cos(theta)-...
            1.)/sin(theta))+349.8098434*atan((cos(theta)-1.)/sin(theta))
^2+162.0559964*atan((cos(theta)-...
            1.)/sin(theta))*atanh(2.264484581*(cos(theta)-1.)/sin(theta))+...
            18.76887275*atanh(2.264484581*(cos(theta)-1.)/sin(theta))^2;
 Vc         = 662;
 V          = 0.5*qdot'*KD*qdot+PHi+Vc;
 Vdot       = -qdot'*(Kv+Fmc1)*qdot;

%% Evaluate the Dynamic
qdotdot     = inv(mass)*([tau;0]-C*qdot-G);
ddx         = qdotdot(1);
ddtheta     = qdotdot(2);
%% M-File output
dxdt        = [xdot;thetadot;ddx;ddtheta;tau(1);P(1,1);P(1,2);P(2,1);P(2,2);
V;Vdot;PHi;Pdet];
%% End of the Function
```

```
%%%%%%%%%%%  C.5 IINVERTED PENDULUM CART SIM, F1 is non-zero %%%%%%%%%%%
%%%%%%%%%%%             IPC_Nonlinear_Sys.m                  %%%%%%%%%%%
function dxdt = IPC_NonLinear_Sys(u)

%% Main Vectors
x            = u(1);             % feedback array
theta        = u(2);
xdot         = u(3);
thetadot     = u(4);
%% Generalized quantities
q            = [x theta]';       % Generalized coordinates
qdot         = [xdot thetadot]'; % Generalized velocities

%% Physical parameter values
mb           = 5;                % kg       - mass of cart+pendulum
m            = 1;                % kg       - mass of pendulum
l            = .7;               % kg m^2   - inertia of beam
g            = 9.81;             % m/sec^2 - acceleration of gravity

%% Linear model parameters
alpha        =  3;
C6           =  28;
C3           =  5.618308264*10^(-16)*(6.070845378*10^15*alpha*C6-4.676199568↙
*10^17)/alpha;
C5           = -4.250000000*10^(-10)*(5.042016807*10^9*alpha*C6-3.549920000↙
*10^11)/alpha;
F5           = 4.008993435*10^(-14)*(8.408571429*10^13*alpha*C6-4.312542214↙
*10^15)/alpha;
nu           = -.2778938021*alpha*C6+39.32216563;
sigma        = -.2778938021*alpha*C6+39.32216563;
omega        = thetadot;
%% The G,M,C,  P and KD matrices
             %gravity terms
G            = [0; -(1/2)*m*g*l*sin(theta)];
             %mass matrix
mass         = [mb -(1/2)*m*l*cos(theta); -(1/2)*m*l*cos(theta) (1/3)*m*l^2];
             %Centripetal and coriolis matrix
C            =   [0, (1/2)*m*l*sin(theta)*thetadot; 0, 0];

%% FMC
             %KD matrix
KD           =  [C3  C5; C5  C6];
             % P matrix
P            =  KD*inv(mass);
             % Determinant of P matrix
Pdet         =  det(P);
%% SMC
             %Kv Matrix
Kv           = [4*alpha*(2*C3*l+3*C5*cos(theta))^2/l^2/(-4*mb+3*m*cos(theta)↙
^2)^2,4*alpha*(2*C3*l+...
             3*C5*cos(theta))/l^2/(-4*mb+3*m*cos(theta)^2)^2*(2*l*C5+3*cos↙
```

```
(theta)*C6);4*alpha*(2*C3*l+...
            3*C5*cos(theta))/l^2/(-4*mb+3*m*cos(theta)^2)^2*(2*l*C5+3*cos↙
(theta)*C6),4*alpha*(2*l*C5+...
            3*cos(theta)*C6)^2/l^2/(-4*mb+3*m*cos(theta)^2)^2];

%% Evaluate the control law
            % F1= F1m + F1m2
 F1         = nu*xdot ;

 F2         = -.2000000000e-6*(-3600000001.*cos(theta)*thetadot+1248360000.↙
*cos(theta)*xdot-475657830.*xdot+582568000.*thetadot)/(-20.+...
            3.*cos(theta)^2);

 F3         = -.1062302965*(5322973250.*cos(theta)^2*atanh(1.493704911*(-1.↙
+cos(theta))/sin(theta))-...
            4118001935.*sin(theta)*cos(theta)+.2245160710e11*cos(theta)↙
^2*atan((-1.+cos(theta))/sin(theta))-...
            .2595156342e11*x+3892734436.*cos(theta)^2*x-.1496773804e12*atan↙
((-1.+cos(theta))/sin(theta))-...
            .3548649126e11*atanh(1.493704911*(-1.+cos(theta))/sin(theta))+...
            .7205107440e11*sin(theta))/(-59457229.+156045000.*cos(theta));

tau         = F1+F2+F3;

%% Lyapunov
 Fmc1       =[-.3000000000e-5*(-337803445.+886562318.*cos(theta))/(-20.+3.↙
*cos(theta)^2),-.1000000000e-7*(-1109868277.*sin(theta)*omega+...
            2912840006.*cos(theta)*sin(theta)*omega-.6205936166e11+.↙
3834980700e12*cos(theta))/(-20.+...
            3.*cos(theta)^2);.1000000000e-7*(-1109868277.*sin(theta)↙
*omega+2912840006.*cos(theta)*sin(theta)*omega-.6205936166e11+...
            .3834980700e12*cos(theta))/(-20.+3.*cos(theta)^2),-.1000000000e-↙
7*sin(theta)*omega*(.1680000001e11*cos(theta)-2718650673.)/(-20.+...
            3.*cos(theta)^2)];

 PHi        = 2735.928392-169.7426575*log(1026225995.-1523441711.*cos(theta))↙
+101.3693319*x^2+87.23733425*x*atanh(2.264484581*(-1.+...
            cos(theta))/sin(theta))+376.6164626*x*atan((-1.+cos(theta))/sin↙
(theta))+18.76887275*atanh(2.264484581*(-1.+...
            cos(theta))/sin(theta))^2+162.0559964*atanh(2.264484581*(-1.+cos↙
(theta))/sin(theta))*atan((-1.+cos(theta))/sin(theta))+...
            349.8098434*atan((-1.+cos(theta))/sin(theta))^2;

 Vc         = 662;
 V          = 0.5*qdot'*KD*qdot+PHi+Vc;
 Vdot       = -qdot'*(Kv+Fmc1)*qdot;


%% Evaluate the Dynamic

qdotdot     = inv(mass)*([tau;0]-C*qdot-G);
```

```
ddx         = qdotdot(1);
ddtheta     = qdotdot(2);
%% M-File output
dxdt        = [xdot;thetadot;ddx;ddtheta;tau(1);P(1,1);P(1,2);P(2,1);P(2,2);↙
V;Vdot;PHi;Pdet];
%% End of the Function
```

C.6 Simulink file for the inverted pendulum cart system when F1 is zero and nonzero