Controlling underactuated mechanical systems using direct Lyapunov approach by solving for the potential function

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B.S., Bangladesh University of Professionals, 2016

A REPORT

submitted in partial fulfillment of the requirements for the degree

MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

2021

Approved by:

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Abstract

Underactuated Mechanical System (UMS) is the system with n degrees of freedom out of which only $m < n$ are actuated. Among various methods for controlling UMS, the Direct Lyapunov Approach (DLA) is one. In this method, the time derivative of the candidate Lyapunov function produces a relation that is solved via a matching method. The equation of motion corresponding to a system is applied to the matching conditions of the DLA method. The potential of the system is calculated by solving these matching conditions. The Conditions of stability are analyzed to find bounds on the parameters. Control Law is formulated and applied to the system to see the result of the system under the DLA method. Two underactuated systems being controlled with the DLA method have been studied which are, Inertia Wheel System and Inverted Pendulum Cart system.

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Chapter 1 - Introduction

Underactuated mechanical systems (UMS) are the systems where the degree of freedom *n* is more than the number of control inputs *m*. So, $m < n$. Control of these UMS is one of the most active fields of control system because it has very diverse engineering applications. This type of system is very commonly used in space and undersea robots, mobile robots, flexible robots, walking, branchiating, and gymnastic robots.

Various methods to control UMS have been studied in the past. The Partial feedback linearization (PFL) method provides a natural global change of coordinates which transforms the system into a strict feedback form. Then the control method is applied to the new form of the system [1]. Passivity based control (PBC) is the control method based on the passivity of UMS. The characteristic that ensures the derivative of total energy to be negative semidefinite is called the passivity of a system. In the PBC method, the total energy of the system is regulated to the equivalent value of a desired equilibrium state [2]. The Interconnection and Damping Assignment –Passivity Based Control (IDA-PBC) method shapes the Hamiltonian/Lagrangian to a desired form with suitable equilibrium states and structural features by using control inputs. In the IDA-PBC method, damping is injected into the system to ensure the passivity of the system [3]. For the systems in which PBC is difficult to implement, the Backstepping method can be used. In Backstepping the system is transformed into a new recursive form in which PBC can be easily applied to [4]. In Sliding Mode Control (SMC) the dynamics of the system is altered by applying a discontinuous control input that forces the system to 'slide' along a predefined state surface [5]. There are also Fuzzy Control (FC) methods for the systems that cannot be well defined or modeled. The FC is considered as a non-mathematical approach that models the qualitative aspects of human knowledge and reasoning processes without employing precise quantitative analysis [6]. There are also some Optimal Control methods in which the goal is to find a control law that minimizes (or maximizes) the cost function of the system [7]. These are some of the popular control methods that have been studied in recent years. The interested reader is referred to a survey [8].

Warren N. White, Mikil Foss, and Xin Guo presented the Direct Lyapunov Approach (DLA) for controller design for underactuated, nonlinear mechanical systems [8]. The time derivative of the Lyapunov function of the UMS produces a relation that is solved using the matching method. The relation is divided into three equations with respective matching components. These three equations are called the three Matching Conditions. These matching conditions consist of linear and partial differential equations. These equations can be analytically solved to find the control law that will ensure the stability of the system. In this report, the Direct Lyapunov approach is being implemented to control two UMS, The Inertia Wheel system, and the Inverted Pendulum Cart system.

Chapter 2 - Review of the DLA Method and Matching Conditions

Lyapunov functions are usually used as a tool to verify the stability of a given control method. In the DLA method, a Lyapunov candidate function is directly applied to the design of a control law for an underactuated, non-linear mechanical system. The time derivative of the Lyapunov function and the describing dynamic equation of the system gives a relationship which can be used to formulate the stabilizing control law of the system. To solve this equation and find the control law, the equation is divided into three matching conditions. These three matching conditions are chosen such that the 1st matching condition consists of products of three generalized velocities, the 2nd matching condition consists of products of two generalized velocities, and the 3rd matching condition consists of products of single generalized velocity and the functions of generalized coordinates. These matching conditions are differential or partial differential equations of the generalized coordinates. These matching conditions are solved to find out the control law to stabilize the system.

Many UMSs have their first-order or second-order constraints. The first-order constraint is the constraint consisting of generalized coordinate and its first derivative, and the second-order constraint is the constraint consisting of generalized coordinate, its first derivative, and its second derivative. The differential constraint is holonomic if it is integrable and is non-holonomic if it is non-integrable. A UMS is holonomic if its constraint is completely integrable and is non-holonomic if its constraint is non-integrable or partially integrable. The dynamic equations of UMS with holonomic constraints are determined from the Euler-Lagrange equations, which is

$$\frac{\partial}{\partial t}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right)-\frac{\partial L(q,\dot{q})}{\partial q}=M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G_g(q)=\tau \tag{1}$$

where $L(q,\dot{q})\in\mathbb{R}^{n\times n}$ is defined as the Lagrangian of a system which is the difference between the kinetic energy and the potential energy of the system. The vector q and its time derivative constitute a set of state variables for the mechanical system with n generalized coordinates. $M(q)\in\mathbb{R}^{n\times n}$ is the positive definite mass and/or inertia matrix, $C(q,\dot{q})\in\mathbb{R}^{n\times n}$ is the coefficient matrix for the centripetal and Coriolis forces and torques, $\tau\in\mathbb{R}^{n}$ is the actuation vector, and $G_g(q)$ contains forcing terms stemming from the gradient of the potential energy corresponding to the original system. Note that τ consists of m actuators where $n>m$.

In the DLA method the Hamiltonian of the system is taken as the candidate Lyapunov function. The Hamiltonian of a system is the total energy of the system which is the summation of the kinetic energy and the potential energy. In the DLA [9], the Hamiltonian of the stabilized system is written as

$$H_d=\frac{1}{2}\dot{q}^T K_D\dot{q}+\phi(q) \tag{2}$$

where the potential $\phi(q)\in\mathbb{R}$ is a positive definite function of q. $K_D\in\mathbb{R}^{n\times n}$ can be defined by

$$K_D=P(q)M(q) \tag{3}$$

where $P\in\mathbb{R}^{n\times n}$ is a nonsingular matrix satisfying the condition that K_D is symmetric and positive definite.

For a system to be stable it must dissipate energy. So, the time derivative of total energy must be at least negative semidefinite. Which indicates the time derivative of equation (2) must be at least negative semidefinite. Thus, the derivative is defined as the following format

$$\dot{H}_d = \frac{1}{2}\dot{q}^T \dot{K}_D \dot{q} + \dot{q}^T K_D \ddot{q} + \dot{q}^T \nabla_q \phi(q) = -\frac{1}{2}\dot{q}^T \overline{K_v} q \tag{4}$$

where $\overline{K_v} \in \mathbb{R}^{n \times n}$ is symmetric and must be at least positive semidefinite. Plugging in the K_D from equation (3) and the $\ddot{q}$ from equation (1) in equation (4) the following relationship is found.

$$\frac{1}{2}\dot{q}^T \dot{K}_D \dot{q} + \dot{q}^T K_D M^{-1} P^{-1} \nabla_q \phi(q) + \dot{q}^T K_D M^{-1}\left[\tau - C\dot{q} - G_g\right] = -\frac{1}{2}\dot{q}^T \overline{K_v} \dot{q}. \tag{5}$$

The equation (5) is then divided into three equations. These three equations are called the Matching Conditions. The τ_i are the inputs applied to each of the matching conditions. The 1st matching condition is taken such that it consists of the terms with the products of three generalized velocities. The 2nd matching condition is taken such that it consists of the terms with the products of two generalized velocities. And the 3rd matching condition is taken such that it involves terms that are the product of a single generalized velocity and a function of generalized coordinates. The three matching conditions are

$$\dot{q}^T K_D M^{-1}(-C\dot{q} + \tau_1) + \frac{1}{2}\dot{q}^T \dot{K}_D \dot{q} = 0, \tag{6}$$

$$\dot{q}^T K_D M^{-1} \tau_2 = -\dot{q}^T \overline{K}_v \dot{q}, \tag{7}$$

and

$$\dot{q}^T K_D M^{-1}(-G_g + \tau_3) + \dot{q}^T \nabla_q \phi(q) = 0 \tag{8}$$

where

$$\tau_1 + \tau_2 + \tau_3 = \tau. \tag{9}$$

From the 1st matching condition (6) the control input τ_1 consists of terms that are quadratic in the generalized velocities. This is undesirable because it can create an actuation of the same sign regardless of the sign of the generalized velocity. So, the elements of the 1st matching condition (6) are selected such that it sets the τ_1 as zero. Here, the 1st two matching conditions handle the damping of the system and the 3rd matching condition handles the potential energy portion of the system.

The 1st matching condition (6) produces a set of differential equations which can be used to determine the elements of K_D which comprise of functions of generalized coordinates and constants of integration. The step by step calculation of this process is shown in [9]. From the K_D matrix, the P matrix and its elements are calculated. The 2nd matching condition (7) is used to calculate the $\overline{K_v}$ in terms of the elements of the P matrix. It is also used to determine the τ_2 for the control law. The 3rd matching condition (8) gives a set of partial differential equations. Solving one of the equations the potential of the system can be found as a function of generalized coordinates and the elements of the P matrix. The other equation is used to determine the τ_3 for the control law. The summation of τ_2 and τ_3 are used as the controller input to the system to stabilize the nonlinear system under consideration.

Chapter 3 - Controlling Inertia Wheel Pendulum

3.1 The DLA Method Formulation

This method can be applied to control underactuated nonlinear systems. An example being Inertia Wheel Pendulum.

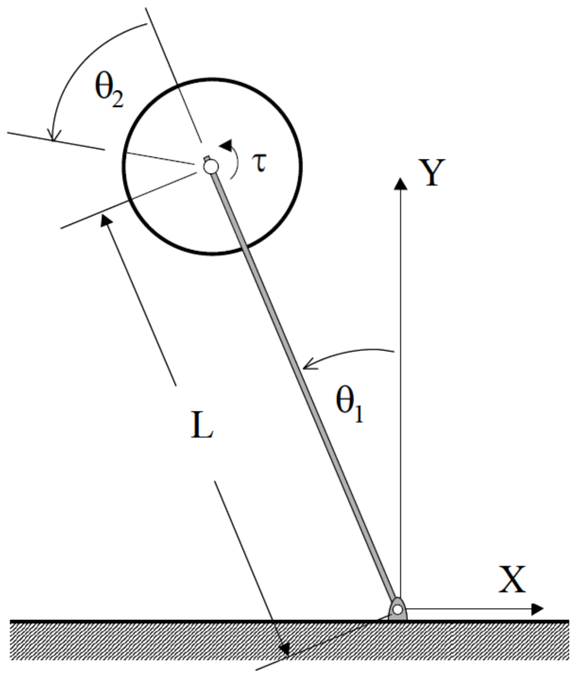

Figure 1: Inertia Wheel Pendulum

The equation of motion of this system is

$$\begin{bmatrix} I_2 & I_2 \\ I_2 & I_1 + I_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -mgL\sin(\theta_1) \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{10}$$

where m is the mass of the wheel, I_2 is the inertia of wheel, and I_1 is the inertia of the rod. Comparing equation (10) with equation (1)

$$q = \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}, \tag{11}$$

$$M = \begin{bmatrix} I_1 & I_2 \\ I_2 & I_1 + I_2 \end{bmatrix}, \tag{12}$$

and

$$G_g = \begin{bmatrix} 0 \\ -mgL\sin(\theta_1) \end{bmatrix}. \tag{13}$$

As the system in (10) consists of two generalized coordinates, let P, P^{-1}, τ_2 and τ_3 matrices be

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{14}$$

$$P^{-1} = \begin{bmatrix} P^{-1}{}_{11} & P^{-1}{}_{12} \\ P^{-1}{}_{21} & P^{-1}{}_{22} \end{bmatrix} = \frac{1}{|P|}\begin{bmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{bmatrix}, \tag{15}$$

$$\tau_2 = Gu_2 = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}, \tag{16}$$

and

$$\tau_3 = Gu_3 = \begin{bmatrix} u_3 \\ 0 \end{bmatrix} \tag{17}$$

where

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{18}$$

where only the first axis is actuated. Here u_2 and u_3 are input torque applied to the wheel for 2nd matching condition (7) and 3rd matching condition (8). From matching condition (7) the following equation is found.

$$u_2 = -G^T P^{-1} \overline{K_v} \dot{q} \tag{19}$$

where

$$\overline{K_v} = \alpha \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} \end{bmatrix}. \tag{20}$$

From matching condition (8) the following two equations are found.

$$u_3 = -G^T P^{-1} \nabla_q \phi(q) = -P^{-1}{}_{11} \frac{\partial \phi(\theta_1, \theta_2)}{\partial \theta_2} - P^{-1}{}_{12} \frac{\partial \phi(\theta_1, \theta_2)}{\partial \theta_1}, \tag{21}$$

and

$$mgL \sin(\theta_1) + P^{-1}{}_{21} \frac{\partial \phi(\theta_1, \theta_2)}{\partial \theta_2} + P^{-1}{}_{22} \frac{\partial \phi(\theta_1, \theta_2)}{\partial \theta_1} = 0. \tag{22}$$

The solution of equation (22) gives the Potential of the system $\phi(\theta_1, \theta_2)$

$$\phi(\theta_1, \theta_2) = \phi_p(\theta_1) + \phi_h(\theta_1, \theta_2) \tag{23}$$

which consists of a particular solution $\phi_p(\theta_1)$ and a homogeneous solution $\phi_h(\theta_1, \theta_2)$. From the solution of the equation (22), these are

$$\phi_p(\theta_1) = \frac{mgL \cos(\theta_1)}{P^{-1}{}_{22}} \tag{24}$$

and

$$\phi_h(\theta_1, \theta_2) = N \left(\frac{-P^{-1}{}_{21} \theta_1 + \theta_2 P^{-1}{}_{22}}{P^{-1}{}_{22}} \right)^2 \tag{25}$$

where N is a constant multiplier.

3.2 Stability Constraints

The Hessian matrix of the potential can be formed as

$$H\left(\theta_1,\theta_2\right)=\begin{bmatrix}\dfrac{\partial^2\phi\left(\theta_1,\theta_2\right)}{\partial\theta_1^{\,2}} & \dfrac{\partial^2\phi\left(\theta_1,\theta_2\right)}{\partial\theta_1\partial\theta_2}\\ \dfrac{\partial^2\phi\left(\theta_1,\theta_2\right)}{\partial\theta_2\partial\theta_1} & \dfrac{\partial^2\phi\left(\theta_1,\theta_2\right)}{\partial\theta_2^{\,2}}\end{bmatrix}=\begin{bmatrix}2N & -\dfrac{2NP^{-1}{}_{21}}{P^{-1}{}_{22}}\\ -\dfrac{2NP^{-1}{}_{21}}{P^{-1}{}_{22}} & -\dfrac{mgL\cos\left(\theta_1\right)}{P^{-1}{}_{22}}+\dfrac{2NP^{-1}{}_{21}{}^{2}}{P^{-1}{}_{22}{}^{2}}\end{bmatrix}. \tag{26}$$

For a stable system, the Hessian must be positive definite. So, in (26), both the principle minors must be positive. So, $N>0$ and the following inequality must be true.

$$\left|H\left(\theta_1,\theta_2\right)\right|=-\frac{2NmgL\cos\left(\theta_1\right)}{P^{-1}{}_{22}}=-\frac{2NmgL\cos\left(\theta_1\right)}{\dfrac{P_{11}}{|P|}}>0 \tag{27}$$

where from equation (15) $P^{-1}{}_{22}=\dfrac{P_{11}}{|P|}$.

As K_D must be symmetric and positive definite, some constraints on the elements of the P matrix can be established. The K_D can be calculated as

$$K_D=\begin{bmatrix}I_2P_{11}+I_2P_{12} & P_{11}I_2+P_{12}\left(I_1+I_2\right)\\ I_2P_{21}+I_2P_{22} & P_{21}I_2+P_{22}\left(I_1+I_2\right)\end{bmatrix}. \tag{28}$$

For K_D to be positive definite, $K_{D_{1,1}}$ must be positive, which gives

$$I_2\left(P_{11}+P_{12}\right)>0 \tag{29}$$

For K_D to be positive definite, the determinant of K_D must be positive, which gives

$$\left|K_D\right|=I_1I_2\left(P_{11}P_{22}-P_{12}P_{21}\right)=|P||M|>0 \tag{30}$$

Here, I_1 and I_2 are positive constants and $|M|=I_1I_2>0$. So, equation (30) becomes

$$|P|=\left(P_{11}P_{22}-P_{12}P_{21}\right)>0 \tag{31}$$

where the left-hand side of the inequality is the determinant of P. So, the determinant of P must be positive. Then from equation (27) the P_{11} must be negative.

$$P_{11} < 0. \tag{32}$$

Simplifying equation (29) and (31) the following inequalities are found.

$$P_{12} > -P_{11}, \tag{33}$$

and

$$P_{21} < \frac{P_{11}P_{22}}{P_{12}}. \tag{34}$$

For the K_D to be symmetric the $K_{D_{12}}$ and $K_{D_{21}}$ must be equal, which gives the equation

$$P_{22} = P_{11} + P_{12}\left(\frac{I_1 + I_2}{I_2}\right) - P_{21}. \tag{35}$$

The equations (32) to (35) are the sets of constraints that must be met for a stable response of the system.

3.3 Results of the DLA Method on Inertia Wheel

Choosing appropriate P so that K_D is symmetric and positive definite and the α so that $\overline{K_v}$ is at least, positive semidefinite, the potential of the system looks like the following figure.

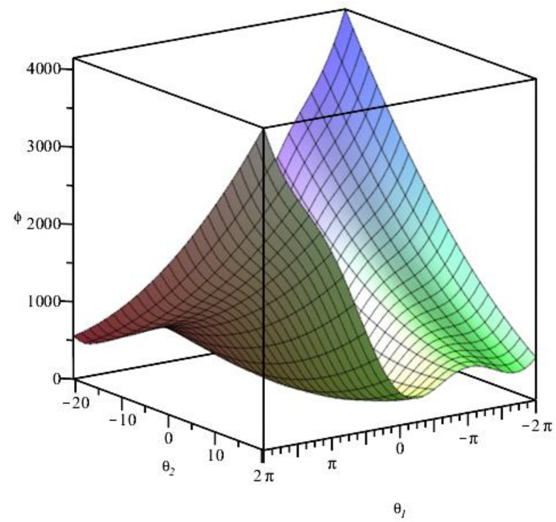

Figure 2: Inertia wheel pendulum Potential vs. States

The gradient of the potential can also be examined to understand the system.

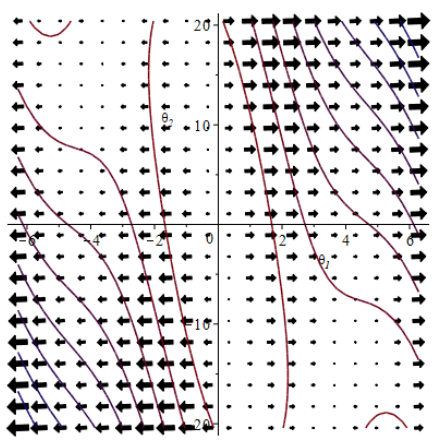

Figure 3: Inertia wheel Gradient of potential with states

From the above two figures is can be seen that, the minimum point of the potential is $(\theta_1,\theta_2)=(0,0)$. Also, the gradients are directed outward from $(\theta_1,\theta_2)=(0,0)$. So, the system will stabilize at $(\theta_1,\theta_2)=(0,0)$.

The control law obtained from (19) and (21) gives the following result.

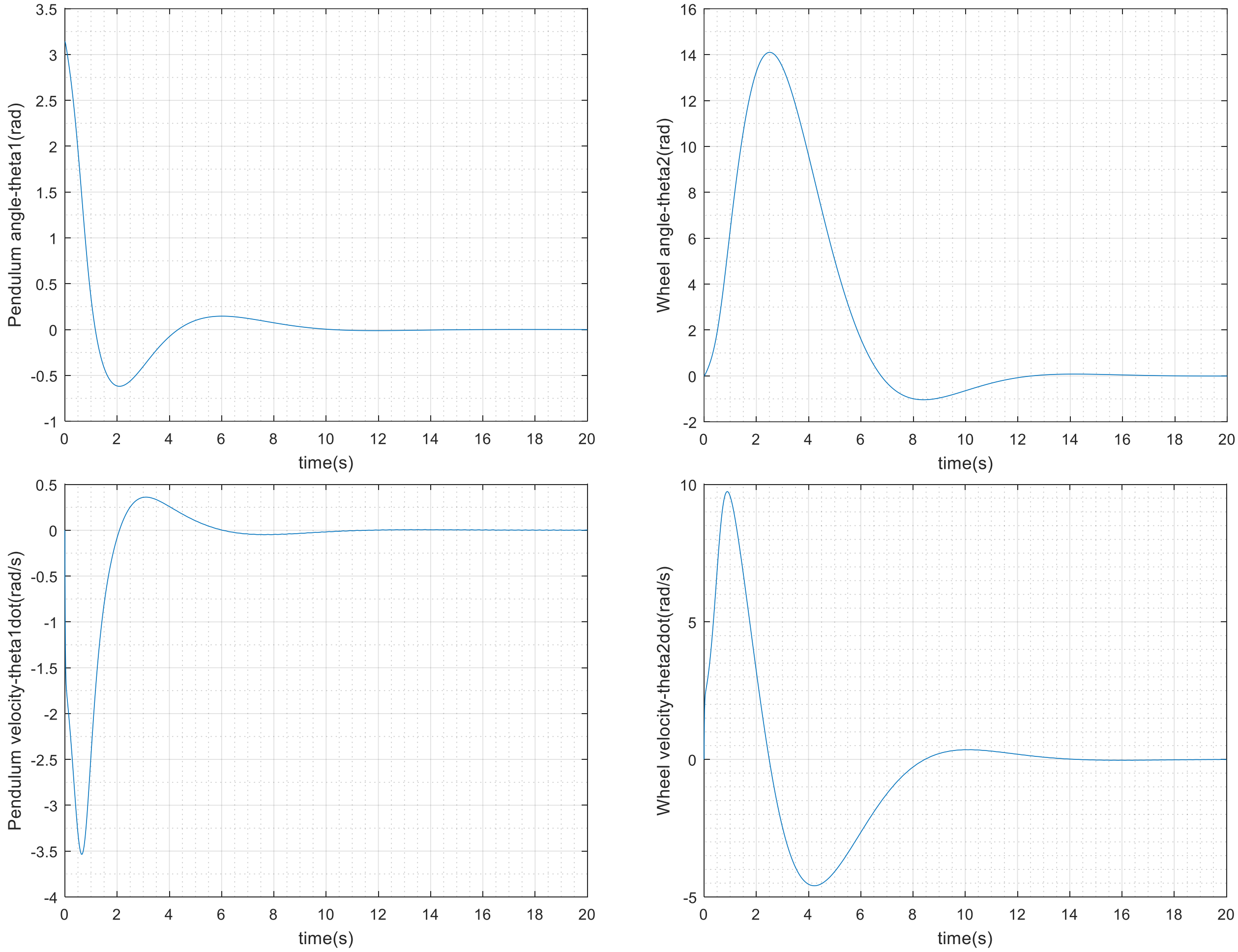

Figure 4: Inertia Wheel Pendulum performance

These results indicate that the method is successful to stabilize the Inertia wheel pendulum system. Choosing different constants can give an even better performance. Here the value of α improves the settling time of the system.

Chapter 4 - Inverted Pendulum Cart

4.1 The DLA Method Formulation

This method was also applied to control Inverted Pendulum Cart. This is also an underactuated nonlinear system.

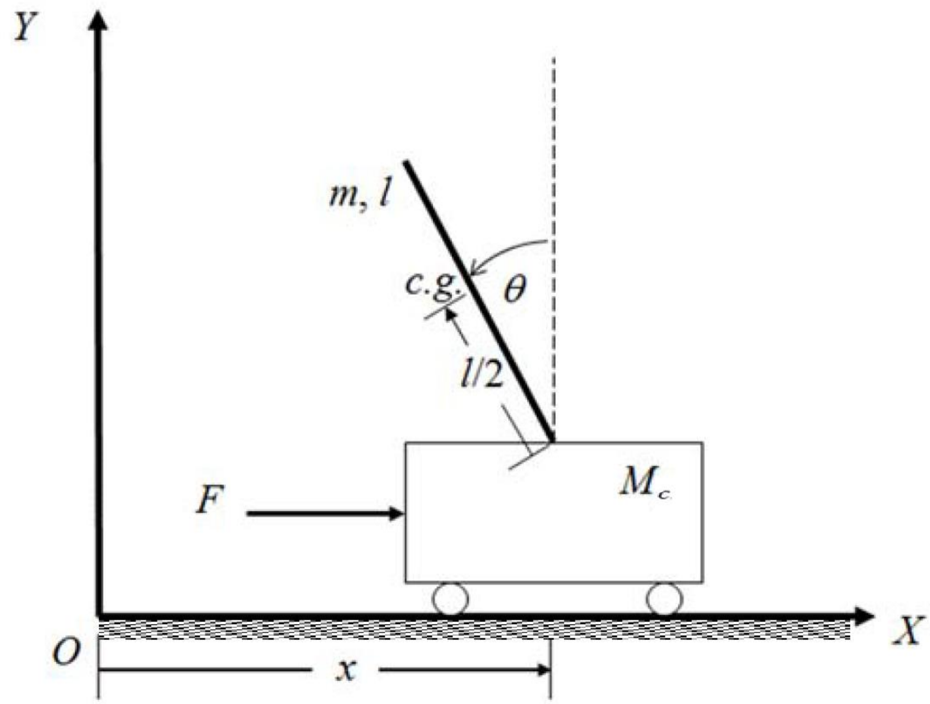

Figure 5: Inverted Pendulum Cart

The equation of motion of the inverted pendulum cart system is

$$\begin{bmatrix} M_T & -\frac{1}{2}ml\cos(\theta) \\ -\frac{1}{2}ml\cos(\theta) & I_P \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} 0 & \frac{1}{2}ml\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}+\begin{bmatrix} 0 \\ -\frac{1}{2}mgl\sin(\theta) \end{bmatrix}=\begin{bmatrix} F \\ 0 \end{bmatrix}. \quad (36)$$

Comparing equation (36) to equation (1),

$$q=\begin{bmatrix} x \\ \theta \end{bmatrix}, \quad (37)$$

$$M = \begin{bmatrix} M_T & -\frac{1}{2}ml\cos(\theta) \\ -\frac{1}{2}ml\cos(\theta) & I_P \end{bmatrix}, \tag{38}$$

$$C = \begin{bmatrix} 0 & \frac{1}{2}ml\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}, \tag{39}$$

$$G_g = \begin{bmatrix} 0 \\ -\frac{1}{2}mgl\sin(\theta) \end{bmatrix}, \tag{40}$$

and

$$\tau = \begin{bmatrix} F \\ 0 \end{bmatrix}. \tag{41}$$

Here total mass $M_T = M_c + m$ and I_P is the inertia of the rod. As the system has two generalized coordinates, similar P, P^{-1}, τ_2, τ_3, G and $\overline{K_v}$ can be assumed as equation (14), (15), (16), (17), (18) and (20). Then, from matching conditions (7) and (8) the following equations are found.

$$u_2 = -G^T P^{-1} \overline{K_v} \dot{q}, \tag{42}$$

$$\overline{K_v} = \alpha \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} \end{bmatrix}, \tag{43}$$

$$u_3 = -{P^{-1}}_{11} \frac{\partial \phi(x,\theta)}{\partial x} - -{P^{-1}}_{12} \frac{\partial \phi(x,\theta)}{\partial \theta}, \tag{44}$$

and

$$\frac{1}{2}mgl\sin(\theta) + {P^{-1}}_{21} \frac{\partial \phi(x,\theta)}{\partial x} + {P^{-1}}_{22} \frac{\partial \phi(x,\theta)}{\partial \theta} = 0. \tag{45}$$

The K_D and P are same as calculated in [8]. K_D is defines as

$$K_D = \begin{bmatrix} K_{D11} & K_{D12}(\theta) \\ K_{D12}(\theta) & K_{D22}(\theta) \end{bmatrix}, \tag{46}$$

where

$$K_{D12}(\theta) = \frac{mlK_{D11}\cos(\theta)}{2M_T} + C_1 f_1(\theta), \tag{47}$$

$$K_{D22}(\theta) = \frac{K_{D11}I_P}{M_T} - \frac{C_1 ml\cos(\theta) f_1(\theta)}{M_T} - \frac{C_2}{2} f_1(\theta)^2 \tag{48}$$

where

$$f_1(\theta) = \sqrt{8M_T I_P - 2m^2 l^2 \cos(\theta)^2}. \tag{49}$$

And P is defined as

$$P = \begin{bmatrix} \frac{K_{D11}}{M_T} + f_2(\theta) & \frac{2M_T}{ml\cos(\theta)} f_2(\theta) \\ f_3(\theta) - 2mlC_2\cos(\theta) & \frac{K_{D11}}{M_T} - f_2(\theta) - 4C_2M_T \end{bmatrix} \tag{50}$$

where

$$f_2(\theta) = \frac{4mlC_1\cos(\theta)}{f_1(\theta)}, \tag{51}$$

$$f_3(\theta) = \frac{4C_1(2M_T I_P - m^2 l^2 \cos^2(\theta))}{M_T f_1(\theta)}. \tag{52}$$

Here K_{D11}, C_1, and C_2 are constants.

Solving equation (45) the potential $\phi(x,\theta)$ is found which consists of a particular solution $\phi_p(\theta)$ and a homogeneous solution $\phi_h(x,\theta)$.

$$\phi(x,\theta) = \phi_p(\theta) + \phi_h(x,\theta). \tag{53}$$

The characteristic function for homogeneous solution is found to be a combination of an algebraic function and some elliptic integral functions. For better understanding, some new constants are defined to simplify the solution. The particular solution of the potential $\phi_p(\theta)$ is as follows.

$$\phi_p(\theta) = \int \frac{mlg\sin(\theta)\, f_1(\theta) C3}{2M_T\left(4C1\cos(\theta) lmM_T + KD11 f_1(\theta)\right)} d\theta \tag{54}$$

where

$$C3 = 8C1^2 M_T{}^2 + 4C2KD11M_T{}^2 - KD11^2. \tag{55}$$

The Characteristic function is defined as

$$char(x,\theta) = (A_T(\theta) + E_T(\theta) + x). \tag{56}$$

Here $A_T(\theta)$ and $E_T(\theta)$ are consecutively the first part and the second part of the characteristic function. New constants $C4$ and $C5$ are defined for better understanding and simpler presentation. $A_T(\theta)$ is found out to be

$$A_T(\theta) = 2M_T lm\left(-\frac{\left(4C1^2 + C2KD11\right)\sin(\theta)}{C4} + \frac{8C1^2 I_P M_T C3 \operatorname{arctanh}\left(\frac{\left(8C1^2 l^2 m^2 M_T{}^2 + KD11^2 l^2 m^2\right)\sin(\theta)}{ml\sqrt{C5C4}}\right)}{C4ml\sqrt{C5C4}} \right) \tag{57}$$

where

$$C4 = 8C1^2 M_T{}^2 + KD11^2, \tag{58}$$

and

$$C5 = 8C1^2 l^2 m^2 M_T{}^2 + KD11^2 l^2 m^2 - 4I_P KD11^2 M_T. \tag{59}$$

The $E_T(\theta)$ is the term in characteristic function consisting of all the elliptic integrals. $E_T(\theta)$ is found to be

$$E_T(\theta) = \frac{\sqrt{M_T I_P} C1}{\sqrt{2} KD11 C4^2} (32C1^2 M_T{}^2 C3 E_{pi}(\theta) + 8C4KD11\left(-2C2M_T{}^2 + KD11\right) E_e(\theta) + 4KD11^2 C3 E_f(\theta)) \tag{60}$$

where the elliptic integrals $E_{pi}(\theta)$, $E_e(\theta)$, and $E_f(\theta)$ can be defined as

$$E_{pi}(\theta) = E_{pi}(z,v,k) = \int_0^z \frac{1}{\left(1-vt^2\right)\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt, \tag{61}$$

$$E_e(\theta) = E_e(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt, \tag{62}$$

and

$$E_f(\theta) = E_f(z,k) = \int_0^z \frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt \tag{63}$$

where

$$z = \cos(\theta), \tag{64}$$

$$v = \frac{\left(8C1^2 M_T{}^2 + KD11^2\right) m^2 l^2}{4 M_T I_P KD11^2}, \tag{65}$$

and

$$k = \frac{\sqrt{\dfrac{m^2 l^2}{M_T I_P}}}{2}. \tag{66}$$

The homogeneous solution of the potential $\phi_h(x,\theta)$ should be defined in a way so that the potential will be positive definite. It is defined as

$$\phi_h(x,\theta) = N\left(A_T(\theta) + E_T(\theta) + x\right)^2 \tag{67}$$

where N is a constant multiplier.

4.2 Stability Constraints

The Hessian of the potential with respect to states is

$$H(x,\theta)=\begin{bmatrix} 2N & 2\mathrm{N}\left(\dfrac{\partial A_T(\theta)}{\partial\theta}+\dfrac{\partial E_T(\theta)}{\partial\theta}\right) \\ 2\mathrm{N}\left(\dfrac{\partial A_T(\theta)}{\partial\theta}+\dfrac{\partial E_T(\theta)}{\partial\theta}\right) & \dfrac{\partial^2\phi_p(\theta)}{\partial\theta^2}+2\mathrm{N}\left(\dfrac{\partial A_T(\theta)}{\partial\theta}+\dfrac{\partial E_T(\theta)}{\partial\theta}\right)^2+2N\left(A_T(\theta)+E_T(\theta)+x\right)\left(\dfrac{\partial^2 A_T(\theta)}{\partial\theta^2}+\dfrac{\partial^2 E_T(\theta)}{\partial\theta^2}\right)\end{bmatrix}. \tag{68}$$

For a stable system, the Hessian must be positive definite. So, in (68), both the principle minors must be positive. So, $N>0$ and the following inequality must be true.

$$\left|H(x,\theta)\right|=2N\left(\frac{\partial^2\phi_p(\theta)}{\partial\theta^2}+2N\left(A_T(\theta)+E_T(\theta)+x\right)\left(\frac{\partial^2 A_T(\theta)}{\partial\theta^2}+\frac{\partial^2 E_T(\theta)}{\partial\theta^2}\right)\right)>0. \tag{69}$$

For $x=0$ and $\theta=0$ equation (69) becomes

$$|H|=-\frac{mlg\sqrt{-2m^2l^2+8M_TI_P}\left(8C1^2M_T{}^2+4C2KD11M_T{}^2-KD11^2\right)}{2M_T\left(4mlC1M_T+KD11\sqrt{-2m^2l^2+8M_TI_P}\right)}=-\frac{mgl}{2}\frac{|P|}{P_{11}}>0. \tag{70}$$

To meet the conditions of stability the constants $KD11$, $C1$ and $C2$ must satisfy some constraints. For K_D to be positive definite, $KD11$ must be positive. This condition gives the constraint

$$KD11>0. \tag{71}$$

And the determinant of K_D must be positive. The determinant of P must be positive because $|K_D|=|P||M|$.

$$|P|=-\frac{8C1^2M_T{}^2+4C2KD11M_T{}^2-KD11^2}{M_T{}^2}>0. \tag{72}$$

As the determinant of P is positive, equation (70) indicates that the P_{11} must be negative. Which gives the condition

$$\frac{KD11}{M_T}+\frac{4ml\cos(\theta)C1}{\sqrt{-2m^2l^2\cos(\theta)^2+8M_TI_P}}<0. \tag{73}$$

From equation (61), for the elliptic integral $E_{pi}(\theta)$ to be bounded, the following condition must be true [10].

$$vz^2=\frac{\left(8C1^2M_T{}^2+KD11^2\right)m^2l^2\cos^2(\theta)}{4M_TI_PKD11^2}<1. \tag{74}$$

For the system to be stable the inequalities (71), (72), (73) and (74) must be met.

4.3 Inequalities Simplification

Simplifying equation (72) the following inequality is found.

$$C2<-\frac{2C1^2}{KD11}+\frac{KD11}{4M_T{}^2} \tag{75}$$

Simplifying equation (73) the following inequality is found.

$$KD11< f_4(\theta)C1 \tag{76}$$

where

$$f_4(\theta)=-\frac{4ml\cos(\theta)M_T}{\sqrt{-2m^2l^2\cos(\theta)^2+8M_TI_P}}. \tag{77}$$

For the calculation of the $f_4(\theta)$ with varying θ, the following values of the various parameters of the Inverted Pendulum were considered.

$$m=1\,Kg, \tag{78}$$

$$M_T=m+5=6\,Kg, \tag{79}$$

$$l = 0.7\ m, \tag{80}$$

$$I_P = \frac{ml^2}{3} = 0.1633333333\ Kgm^2, \tag{81}$$

$$g = 9.81\ N/m^2, \tag{82}$$

and

$$\theta = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ rad. \tag{83}$$

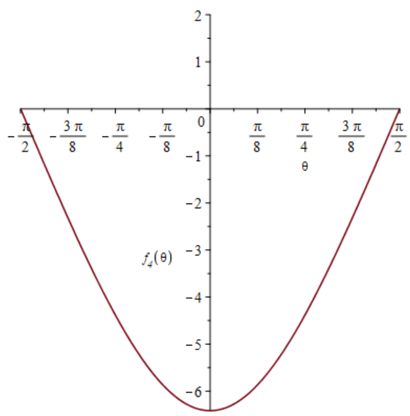

Figure 6: Varying f4 function with angle

As from equation (76) $KD11$ is positive, so $C1$ must be negative. From above figure of $f_4(\theta)$ with varying θ, it is seen that as the value of θ is increasing the $f_4(\theta)$ increases and goes towards zero. So, right-hand side of the inequality will go towards zero, and the bound on $KD11$ will become smaller. So,

$$C1 < 0. \tag{84}$$

Simplifying inequality (74) yields

$$KD11^2 > \left(\frac{16m^2l^2\cos^2(\theta)M_T{}^2}{-2m^2l^2\cos^2(\theta)+8M_TI_P}\right)C1^2 = f_4(\theta)^2\,C1^2. \tag{85}$$

For squared inequalities all the sign options must be taken into consideration. For better understanding this inequality (85) the absolute value is taken for the right-hand side of the inequality (85)

$$KD11^2 > \left| f_4(\theta) C1 \right|^2 . \tag{86}$$

Taking square root of this inequality (86), the following two inequalities are found.

$$KD11 < -\left| f_4(\theta) C1 \right| \tag{87}$$

and

$$KD11 > \left| f_4(\theta) C1 \right| . \tag{88}$$

As, $KD11$ is positive the first inequality (87) cannot be true. So, only the second inequality (88) must be true. This equation (88) contradicts with equation (76). This is the reason there is no such combination of $KD11$ and $C1$ that satisfies inequalities (76) and (88).

Chapter 5 - Conclusion

In this report, the DLA method and its formulation technique are discussed. The formulation of this method depends on the dynamic equations of motion governing the behavior of mechanical systems. The main challenge from a control engineer perspective is understanding the formulation and analyzing the matching conditions to find appropriate elements of the P matrix that will result in a stable system.

The DLA method is applied to the Inertia wheel system. It is seen that the inertia wheel system has a constant mass matrix. So, the elements of the P matrix are also constant which makes this method simple to implement. The DLA is very efficient in controlling the Inertia wheel system. It produces a good response with a small settling time and a low overshoot. The DLA method is also applied to the inverted pendulum cart system. This system has a mass matrix that is a function of a state. So, the elements of the P matrix also become functions of a state. For this system, two of the conditions that ensure the stability of the system are contradictory. So, the system could not be controlled using this method. Further research might be conducted to see the result of the DLA method with other UMSs such as Acrobot, Pendubot, Beam and ball, etc.

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Appendix A - Codes for Inertia Wheel Response Simulation

The code for calculating P matrix in matlab:

```
clc;
clear all;
R = 10;       %range in which the constants are calculated
dR = 1;       %step size in range the constants are tested

P= [];        %empty P matrix
n=0;
I1 = 0.4;     %kgm^2
I2 = 1;       %kgm^2
m = 1;        %kg
L = 1;        %m
g = 9.8;
for(p11 = -R:dR:-1) % as P11 must be negative

  for( p21 = -R:dR:R)
    for( p22 = -R:dR:R)
      p12 = (p21 + p22 - p11)/(1+(I1/I2));
      %initializing all conditions
      ans1= 0;
      ans2= 0;
      %condition-1 check
      if( p12 > -p11)
        ans1=1;
      end
      %condition-2 check
      if((p11*p22)>(p12*p21))
        ans2=1;
      end

      %if all conditions are met record the constants in P matrix
      if((ans1==1)&&(ans2==1))
        P = [P; p11 p12 p21 p22];
        n=n+1;
      end
    end

  end
end
```

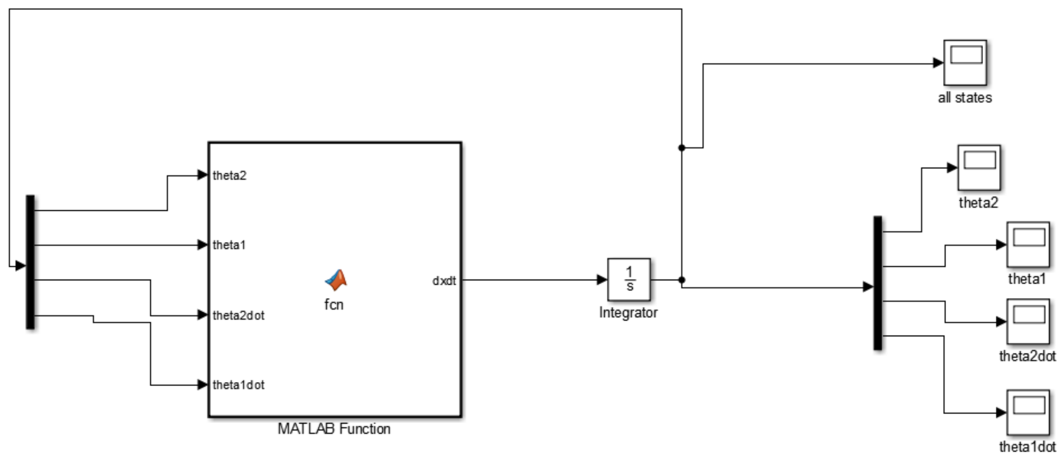

Figure 7: Simulation block

Code used for simulation:

```
function dxdt = fcn(theta2,theta1,theta2dot,theta1dot)

I1 = 0.4;      %kgm^2
I2 = 1;        %kgm^2
m = 1;         %kg
L = 1;         %m
g = 9.80099;   %m/s^2

Mass = [I2 I2; I2 I1+I2];   %M matrix
Cm = [0 0; 0 0];            %C matrix
G = [0; -m*L*g*sin(theta1)];%G__g matrix
%Elements of chosen P matrix
p11 = -1;
p12 = 6.428571428571429
p21 = -7;
p22 = 15;
p = [p11 p12; p21 p22];
invp = inv(p);

%calculations for control input T2
alpha = 5;
Kv = alpha*[p11^2 p11*p21; p11*p21 p21^2];

T2 = -inv(p)*Kv;
%calculation for control input T3

%coefficient of theta2
a = -((invp(1,1)*invp(2,2))-(invp(1,2)*invp(2,1)))*2/invp(2,2);
%coefficient of theta1
b =((invp(1,1)*invp(2,2))-(invp(1,2)*invp(2,1)))*2*invp(2,1)/(invp(2,2)^2);
%coefficient of sin(theta1)
c = m*g*L*invp(1,2)/invp(2,2);
```

```
%control law for chosen P matrix
T = a*theta2+b*theta1+c*sin(theta1)+T2(1,1)*theta2dot+T2(1,2)*theta1dot;

BI = [T; 0];             %Tau matrix
qd = [theta2dot theta1dot]';%derivative of state
qdd = inv(Mass)*(-Cm*qd-G+BI);%double derivative of state eqn(1)
dxdt = [theta2dot; theta1dot; qdd];


end
```