

Algebra of the infrared and Fukaya–Seidel categories

by

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B.S., Beijing Normal University, 2017

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Abstract

The algebra of the infrared was introduced by Gaiotto–Moore–Witten as an algebraic structure associated with massive two-dimensional $\mathcal{N} = (2, 2)$ theories. In the mathematical formulation developed by Kapranov, Kontsevich, and Soibelman, the combinatorics of finite point configurations in the plane, together with the geometry of secondary polytopes, gives rise to natural L_∞ - and A_∞ -algebras. These algebras encode the higher operations coming from instanton counting and provide an algebraic model for the algebra of the infrared.

This dissertation studies an extension of this construction to curve-valued potentials, with particular emphasis on the case where the base curve is an elliptic curve. Starting from a finite configuration of points on an elliptic curve, we construct associated L_∞ -algebras by passing to suitable lifts of the configuration and applying the combinatorics of secondary polytopes. After choosing an additional point, or stop, together with a lift to the universal cover, we construct corresponding A_∞ -algebras and an L_∞ -morphism to the derived derivation space, which is the deformation complex of the A_∞ -algebra.

A new feature of the curve-valued setting is that the resulting A_∞ -structure depends essentially on the choice of lift of the stop. This produces a chamber structure controlled by the fundamental group of the base curve. We formulate the resulting chamberwise algebraic picture and discuss its expected relation to Fukaya–Seidel categories over curves. In this way, the dissertation extends the algebra of the infrared beyond the classical planar setting and reveals new structures arising from the topology of the base curve.

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A new feature of the curve-valued setting is that the resulting A_∞ -structure depends essentially on the choice of lift of the stop. This produces a chamber structure controlled by the fundamental group of the base curve. We formulate the resulting chamberwise algebraic picture and discuss its expected relation to Fukaya–Seidel categories over curves. In this way, the dissertation extends the algebra of the infrared beyond the classical planar setting and reveals new structures arising from the topology of the base curve.

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Chapter 1

Introduction

1.1 Background

The algebra of the infrared was introduced by Gaiotto–Moore–Witten [9] as an algebraic framework for massive two-dimensional $\mathcal{N} = (2, 2)$ quantum field theories. In such a theory, the set of vacua is finite. For each ordered pair of vacua one has a space of solitons interpolating between them, and configurations of solitons and instantons give rise to higher algebraic operations.

One of the main structures in this formalism is an L_∞ -algebra associated to a generic finite configuration $A \subset \mathbb{R}^2$. The points of A represent the vacua of the theory. The operations are encoded by planar webs, namely plane graphs whose faces are labelled by elements of A , and whose edge directions are determined by the corresponding differences of vacua. The compactification of instanton moduli spaces has boundary strata described by broken web configurations, and the resulting boundary identities give precisely the L_∞ -relations.

After choosing a half-plane, or equivalently a direction at infinity, the same configuration also gives rise to a directed A_∞ -algebra. More generally, after adding coefficient data, one obtains a directed A_∞ -category. This directed structure is upper triangular, or semi-orthogonal, reflecting the linear order on the vacua determined by the chosen direction. In the physical picture, counts of instantons define Maurer–Cartan elements which deform this

directed A_∞ -category.

For Landau–Ginzburg models, the deformed A_∞ -categories are expected to recover the corresponding categories of D -branes. Mathematically, these categories are modeled by Fukaya–Seidel categories of Landau–Ginzburg potentials; see [16, 32].

1.2 The mathematical model

Kapranov–Kontsevich–Soibelman gave a mathematical interpretation of this Gaiotto–Moore–Witten construction using secondary polytopes and deformation theory [19]. In their approach, a finite configuration of points in an affine space determines a collection of secondary polytopes. The factorization properties of the faces of these polytopes encode the possible decompositions of polygons, and this combinatorics produces an L_∞ -algebra. In the two-dimensional case, after choosing a half-plane, one obtains a relative version of the construction which produces a directed A_∞ -algebra. The main universality theorem of [19] relates these two structures: the L_∞ -algebra associated to the point configuration maps to the deformation complex of the directed A_∞ -algebra, and in fact controls its deformations.

This mathematical model gives an algebraic counterpart of the physical picture in [9]. The points of the configuration represent vacua, the edges between them correspond to soliton sectors, and the polygonal decompositions model the possible degenerations of instanton moduli spaces. In this way, the algebra of the infrared provides a bridge between three types of data: the combinatorics of point configurations and secondary polytopes, the deformation theory of A_∞ -algebras, and the symplectic geometry of Fukaya–Seidel categories.

A further development of this point of view appears in the work of Kapranov–Soibelman–Soukhanov on perverse schobers and the algebra of the infrared [20], building on the theory of perverse schobers introduced by Kapranov–Schechtman [17]. There the algebra of the infrared is related to a categorified wall-crossing picture. Roughly speaking, instead of considering only vector spaces of solitons and algebraic operations among them, one organizes categories, functors, and wall-crossing data into a schober-like structure. This perspective is especially natural from the point of view of Fukaya–Seidel, where vanishing cycles, thimbles,

and monodromy functors are inherently categorical objects. For a recent survey of the relation between the algebra of the infrared, secondary polytopes, and perverse schobers, see Kapranov–Soibelman [18].

1.3 Algebra of the infrared with curve-valued potential

The purpose of the present paper is to study an analogue of this story for curve-valued potentials.

The first natural idea is to pass to the universal cover. In the elliptic curve case this obvious idea works particularly well. Indeed, if $E = \mathbb{C}/\Lambda$ is an elliptic curve, then its universal cover is the affine plane $\mathbb{C}$, and the deck transformations are translations. Therefore, after choosing lifts of the points of a finite configuration $A \subset E$, one obtains an ordinary finite point configuration $\tilde{A} \subset \mathbb{C}$. The usual affine notions of convex hull, straight line segment, polygonal subdivision, Euclidean area, and secondary polytope can then be applied upstairs. In this way the elliptic curve case reduces locally to the planar construction of [19], while the new global feature is the dependence on the choice of lifts, or equivalently on the sheet data with respect to the deck group Λ .

This also explains why the present construction is restricted to genus one. By the uniformization theorem, the universal cover of a compact Riemann surface of genus $g > 1$ is the disk, or equivalently the upper half-plane $\mathbb{H}$; see, for example, [7]. Thus such a curve can be written as

$$C \simeq \mathbb{H}/\Gamma,$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a cocompact Fuchsian group. Although one can still pass to the universal cover, the cover is no longer an affine vector space, and the deck transformations are no longer translations. Consequently there is no canonical affine notion of convex hull, polygonal subdivision, or secondary polytope for a lifted configuration of points. One could instead try to formulate a different theory using hyperbolic geodesic polygons or other additional geometric structures, but this would no longer be a formal repetition of the secondary-polytope

construction used here.

One can still expect local versions of the construction. On a sufficiently small coordinate chart of the base curve, the potential looks like a complex-valued holomorphic function, and the local Fukaya–Seidel and complex Morse models should resemble the usual affine case. The difficulty is global: as one moves between charts and around nontrivial loops in the base curve, the local data are transformed by monodromy. Therefore, instead of attaching a single secondary-polytope algebra to one affine lifted configuration, one should expect a structure which organizes local Fukaya–Seidel data together with their monodromy and gluing.

This suggests that a possible higher genus generalization would be categorified, or schober-theoretic, closer in spirit to the theory of perverse schobers and to its relation with the algebra of the infrared studied in [20]. In such a theory, the affine secondary-polytope combinatorics would be replaced by a more intrinsic categorical object encoding local categories, monodromy functors, and gluing data over the base curve. We do not develop this higher genus generalization here. The present paper focuses on the elliptic curve case, where the universal cover is the affine plane $\mathbb{C}$, so the algebra of the infrared construction of [19] can be applied directly to lifted configurations, while the monodromy information is still retained through the dependence on relative lifts.

1.4 The elliptic curve case

The elliptic curve case is the main curve-valued example studied in this thesis. It is developed in detail in the author’s paper [27]. We consider a holomorphic map

$$W : X \longrightarrow E,$$

where E is an elliptic curve. The critical values of W form a finite point configuration $A \subset E$. Since E is not affine, this configuration does not directly determine an ordinary secondary polytope. We therefore pass to the universal cover $\pi : \mathbb{C} \longrightarrow E$ and work with lifted point configurations in $\mathbb{C}$. After choosing suitable lifts of the critical values, one obtains

a finite configuration $\tilde{A} \subset \mathbb{C}$. Applying the secondary-polytope construction to $\tilde{A}$ gives an L_∞ -algebra, which we denote by $\mathfrak{g}_{\tilde{A}}$.

To construct the relative A_∞ -algebra, we choose an additional point $p \in E \setminus A$, called a stop, together with a lift $\tilde{p} \in \mathbb{C}$. In the classical complex-valued setting, the choice of a half-plane is used to break the cyclic symmetry of the vacua and to produce a directed, or upper triangular, A_∞ -algebra. In the elliptic curve setting there is no global affine half-plane with this property. Instead, the lifted stop $\tilde{p}$ provides a cut in the universal cover. This cut turns the cyclic order of the lifted configuration into a linear order, and hence determines the directed structure of the associated A_∞ -algebra.

Using this ordered lifted configuration, we construct a directed A_∞ -algebra $R_{\tilde{A}, \tilde{p}}$. We then construct an L_∞ -morphism

$$\Phi_{\tilde{A}, \tilde{p}}: \mathfrak{g}_{\tilde{A}} \longrightarrow R\mathrm{Der}(R_{\tilde{A}, \tilde{p}}),$$

where $R\mathrm{Der}(R_{\tilde{A}, \tilde{p}})$ denotes the derived derivation complex, or deformation complex, of $R_{\tilde{A}, \tilde{p}}$, as defined in Section 3.3. Then we prove the following theorem.

Theorem 1.1 (Theorem 8.2). *The L_∞ -morphism*

$$\Phi_{\tilde{A}, \tilde{p}}: \mathfrak{g}_{\tilde{A}} \longrightarrow R\mathrm{Der}(R_{\tilde{A}, \tilde{p}})$$

factors through an L_∞ -morphism

$$\Psi_{\tilde{A}, \tilde{p}}: \mathfrak{g}_{\tilde{A}} \longrightarrow \overrightarrow{C}^{\geq 1}(R_{\tilde{A}, \tilde{p}}, R_{\tilde{A}, \tilde{p}})[1].$$

Moreover, $\Psi_{\tilde{A}, \tilde{p}}$ is a quasi-isomorphism.

The eventual goal is to relate these algebraic structures to Fukaya–Seidel categories. For a curve-valued potential $W : X \rightarrow E$, one can define a Fukaya–Seidel category using admissible thimbles with a chosen basepoint on the curve, as in the framework of Kerr–Soibelman [22]. In contrast with the case of a potential valued in $\mathbb{C}$, admissible paths on E may wind around nontrivial cycles before reaching critical values. Therefore the set of

admissible thimbles is typically infinite. A finite distinguished collection of thimbles gives a finite directed Fukaya–Seidel subcategory and hence a total algebra R_S . We expect the algebra R_S to be obtained by deforming the algebra $R_{\tilde{A}_S}$ associated to the corresponding lifted critical value configuration.

Conjecture 1.2 (Conjecture 10.4). *The lifted complex Morse model should produce a Maurer–Cartan element*

$$\gamma_S \in \mathfrak{g}_{\tilde{A}_S}^1,$$

and the deformation of $R_{\tilde{A}_S}$ by γ_S should recover the total algebra of the finite Fukaya–Seidel subcategory generated by the chosen thimbles:

$$R_{\tilde{A}_S, \gamma_S} \simeq R_S.$$

This should be viewed as the curve-valued analogue of the conjectural comparison between the algebra of the infrared and the Fukaya–Seidel category in the complex-valued case. The dependence on the choice of lifts is not an artifact, but reflects the topology of the base curve. Indeed, changing lifts corresponds to acting by deck transformations, or equivalently to wrapping paths around nontrivial cycles of the elliptic curve. Thus the algebraic structures obtained from lifted point configurations carry a monodromy flavor. Different choices of lifts may give different L_∞ -algebras, different A_∞ -algebras, and different directed presentations of the Fukaya–Seidel category.

1.5 Relation to previous work and new features

Most of the algebraic constructions used in this paper are based on the framework of Kapranov–Kontsevich–Soibelman. In particular, the construction of the L_∞ -algebra from secondary polytopes, the factorization of faces of secondary polytopes, the relative directed A_∞ -algebra obtained after choosing an ordering datum, and the universality morphism to the deformation complex of this A_∞ -algebra are elliptic-curve analogues of the corresponding

constructions in [19]. Whenever an argument is a direct adaptation of the complex-valued case, we indicate this by referring to the relevant construction or result.

The main new feature of the present work is the dependence on lift data. In the elliptic-curve-valued setting, the critical values form a configuration on E , and the algebraic construction is performed after choosing lifts of these points to the universal cover. A simultaneous deck transformation of all lifted points gives an equivalent configuration, but changing the lift of individual points changes the relative sheet data and may change the associated L_∞ - and A_∞ -algebras. Similarly, the role of the half-plane in the complex-valued setting is replaced by the choice of a stop $p \in E \setminus A$ together with a lift $\tilde{p} \in \mathbb{C}$. The lifted stop determines a cut in the universal cover and hence a chamber-dependent linear ordering of the lifted configuration. This lift dependence and chamber dependence are the main new features of the elliptic-curve-valued theory.

On the Fukaya–Seidel side, the same phenomenon appears as the monodromy dependence of finite directed collections of admissible thimbles. Wrapping an admissible path around a nontrivial cycle of E changes the lift of the corresponding critical value by a deck transformation. Thus the algebraic dependence on lifted configurations reflects the geometric dependence on relative monodromy data. In this sense, the present work should be viewed as an elliptic-curve-valued version of the algebra of the infrared, together with an analysis of the additional chamber data which do not appear in the ordinary complex-valued setting.

1.6 Outline of this paper

The paper is organized as follows.

In Chapter 2, we recall the theory of secondary polytopes. We review the factorization properties of their faces and explain how these properties give rise to an algebra differential on a symmetric algebra. This is the basic combinatorial input for the construction of the L_∞ -algebra.

In Chapter 3, we collect the algebraic background needed later in the paper. We recall L_∞ -algebras, A_∞ -algebras, Maurer–Cartan elements, deformation complexes, and the Koszul

duality formalism used in the construction of the universality morphism.

In Chapter 4, we construct the L_∞ -algebra associated to a point configuration on an elliptic curve. The construction is carried out by choosing a lift of the point configuration to the universal cover $\mathbb{C}$, taking the convex hull of the lifted configuration, and applying the secondary-polytope construction to this lifted data.

In Chapter 5, we study the relative setting. We add an extra point on the elliptic curve, viewed as a basepoint or stop, and choose a lift of it. This breaks the cyclic symmetry and produces a directed A_∞ -algebra. We then construct an L_∞ -morphism from the L_∞ -algebra of the lifted configuration to the deformation complex of this A_∞ -algebra.

In Chapter 6, we introduce coefficient systems for the previous algebraic construction. These coefficient systems allow the L_∞ - and A_∞ -structures to incorporate soliton spaces or Floer-theoretic coefficient data, as in the algebra of the infrared.

In Chapter 7, we analyze the resulting A_∞ -algebra and the L_∞ -morphism in more detail. In particular, we study the dependence on the lift of the basepoint, the induced ordering of the configuration, and the effect of changing chambers.

In Chapter 8, we state and prove the main universality theorem. The theorem identifies the L_∞ -algebra constructed from the lifted configuration with the deformation complex controlling deformations of the associated directed A_∞ -algebra, in the chamber determined by the chosen lift and ordering.

In Chapter 9, we review Fukaya–Seidel categories from two viewpoints: the thimble model and the complex Morse model. We explain how vanishing paths give directed A_∞ -categories, and how thimbles can be described as stable manifolds of $\text{Re}(\zeta^{-1}W)$. This provides geometric motivation for the stops, lifts, and orderings used later in the elliptic-curve-valued setting.

Finally, in Chapter 10, we discuss the Fukaya–Seidel category of a curve-valued potential, finite distinguished collections of admissible thimbles, and the total algebra of the corresponding directed subcategory. By lifting the potential to the universal cover of the elliptic curve, we formulate a curve-valued version of the complex Morse model. This leads to conjectural comparisons between Maurer–Cartan deformations of the algebraic A_∞ -algebras and total algebras of Fukaya–Seidel subcategories.

Chapter 2

Secondary polytopes

We briefly recall the definition and basic properties of the secondary polytope associated to a planar point configuration. For the general theory of secondary polytopes, we refer to [11] for the original construction and to [5] for further developments and applications.

2.1 Triangulations and secondary polytopes

Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$ be a finite subset of points in general position and let $Q := \text{Conv}(A)$ be the convex hull of A .

Definition 2.1. A marked polytope is a pair (Q, A) , where $Q \subset \mathbb{R}^2$ is a convex polytope and $A \subset \mathbb{R}^2$ is a finite subset such that $Q = \text{Conv}(A)$. Equivalently, A contains all vertices of Q , but may also contain additional points on the boundary or in the interior of Q .

A marked subpolytope of (Q, A) is a marked polytope (Q', A') such that $A' \subset A$ and $Q' = \text{Conv}(A')$. We write $(Q', A') \subset (Q, A)$ to indicate that (Q', A') is a marked subpolytope of (Q, A) .

Definition 2.2. A polyhedral subdivision $\mathcal{S}$ of (Q, A) is a finite collection of pairs $\{(Q_i, A_i)\}_{i \in I}$ such that:

1. $Q = \bigcup_{i \in I} Q_i$, with $Q_i = \text{Conv}(A_i)$,

2. for $i \neq j$, the intersection $Q_i \cap Q_j$ is either empty or a common face of both Q_i and Q_j .

A subdivision is called a *triangulation* if each Q_i is a simplex.

Definition 2.3. A marked subpolytope $(Q', A') \subset (Q, A)$ is called *geometric* if $A' = A \cap Q'$. A polyhedral subdivision $\mathcal{P} = \{(Q'_\nu, A'_\nu)\}$ of (Q, A) is called *geometric* if each (Q'_ν, A'_ν) is a geometric marked subpolytope.

A subdivision $\mathcal{S}$ is said to be *regular* if there exists a height function $\omega : A \rightarrow \mathbb{R}$ such that $\mathcal{S}$ is induced by the lower faces of the convex hull of the lifted points

$$\{(a, \omega(a)) \mid a \in A\} \subset \mathbb{R}^2 \times \mathbb{R}.$$

Equivalently, $\mathcal{S}$ is regular if it arises as the projection of the lower envelope of a convex, piecewise-linear function on Q whose domains of linearity have vertices in A .

Regular subdivisions of (Q, A) form a finite poset under refinement: for two regular subdivisions $\mathcal{S}$ and $\mathcal{S}'$, we write

$$\mathcal{S}' \preceq \mathcal{S}$$

if $\mathcal{S}'$ is a refinement of $\mathcal{S}$.

A nontrivial regular subdivision $\mathcal{S}$ is called *coarse* if it has no regular coarsening other than the trivial subdivision. Equivalently, $\mathcal{S}$ is minimal among nontrivial regular subdivisions with respect to the refinement order.

The *secondary polytope* $\Sigma(A)$ is a convex polytope whose face poset is isomorphic to the poset of regular subdivisions of (Q, A) . More precisely:

- vertices of $\Sigma(A)$ correspond to regular triangulations of (Q, A) ;
- faces of $\Sigma(A)$ corresponds to regular subdivisions of (Q, A) ;
- inclusion of faces corresponds to refinement of subdivisions in the opposite direction;
- In particular, coarse regular subdivisions correspond to facets of $\Sigma(A)$.

Definition 2.4. For a triangulation $\mathcal{T}$, its GKZ-vector $\phi_{\mathcal{T}} \in \mathbb{R}^A$ is defined by

$$\phi_{\mathcal{T}}(a) = \sum_{\substack{\Delta \in \mathcal{T} \\ a \in \Delta}} \text{Area}(\Delta), \quad a \in A,$$

where the sum runs over all triangles Δ of $\mathcal{T}$ containing a , and $\text{Area}(\Delta)$ denotes the Euclidean area of Δ .

The secondary polytope $\Sigma(A)$ can be realized as the convex hull of the GKZ-vectors of all regular triangulations of (Q, A) .

Remark 2.5 (Fiber-polytope description). There is another equivalent description of the secondary polytope, due to Billera–Sturmfels, in terms of fiber polytopes; see [4]. Let $A \subset \mathbb{R}^d$ be a finite point configuration and let

$$Q = \text{Conv}(A).$$

Consider the standard simplex

$$\Delta_A = \left\{ (p_{\omega})_{\omega \in A} \in \mathbb{R}^A \mid p_{\omega} \geq 0, \quad \sum_{\omega \in A} p_{\omega} = 1 \right\}.$$

There is a natural affine projection

$$\pi : \Delta_A \longrightarrow Q, \quad (p_{\omega})_{\omega \in A} \longmapsto \sum_{\omega \in A} p_{\omega} \omega.$$

The secondary polytope can be described, up to the standard normalization, as the set of vector integrals

$$\int_Q s(q) d\text{Vol}(q) \in \mathbb{R}^A,$$

where $s : Q \rightarrow \Delta_A$ ranges over continuous sections of π .

More precisely, if $\mathcal{T}$ is a triangulation of A , then $\mathcal{T}$ determines a piecewise-linear section

$$s_{\mathcal{T}} : Q \longrightarrow \Delta_A$$

by writing each point $q \in Q$ in barycentric coordinates with respect to the simplex of $\mathcal{T}$ containing q . For such a section one has

$$\int_Q s_{\mathcal{T}}(q) d\text{Vol}(q) = \frac{1}{d+1} \phi_{\mathcal{T}}.$$

Indeed, the integral of each barycentric coordinate over a d -simplex Δ is $\text{Vol}(\Delta)/(d+1)$. Thus the Billera–Sturmfels fiber-polytope construction gives the same polytope as the GKZ construction by the vectors $\phi_{\mathcal{T}}$, up to the harmless overall scalar factor $1/(d+1)$, and possibly the additional normalization by $\text{Vol}(Q)$ depending on convention.

Proposition 2.6. *Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ be a finite point configuration which affinely spans $\mathbb{R}^d$. Then the secondary polytope $\Sigma(A)$ has dimension*

$$\dim \Sigma(A) = n - d - 1.$$

Proof. This is a standard property of secondary polytopes; see [11]. The secondary polytope lies in an affine subspace of $\mathbb{R}^A$ of codimension $d+1$, determined by the affine relations satisfied by all GKZ vectors. Since A affinely spans $\mathbb{R}^d$, it is full-dimensional in this affine subspace. Hence

$$\dim \Sigma(A) = |A| - (d+1) = n - d - 1.$$

□

2.2 Factorization property

Faces of $\Sigma(A)$ corresponding to a regular subdivision $\mathcal{S}$ are canonically identified with products of secondary polytopes of its maximal cells. This factorization property plays a central

role in the construction of algebra of the infrared.

Given a regular subdivision $\mathcal{S}$ of (Q, A) , let

$$\mathcal{S} = \{(Q_i, A_i)\}_{i \in I}$$

be its collection of maximal cells. Each (Q_i, A_i) is a subpolytope of (Q, A) in the above sense, and $\mathcal{S}$ may be viewed as a decomposition of (Q, A) into subpolytopes.

Recall that regular subdivisions of (Q, A) are in order-reversing correspondence with faces of the secondary polytope $\Sigma(A)$. Let $F_{\mathcal{S}} \subset \Sigma(A)$ denote the face corresponding to the regular subdivision $\mathcal{S}$. We then have the following factorization property.

Proposition 2.7. *There is a canonical identification*

$$F_{\mathcal{S}} \cong \prod_{i \in I} \Sigma(A_i),$$

where $\Sigma(A_i)$ denotes the secondary polytope of the subconfiguration (Q_i, A_i) . Under this identification, faces of $F_{\mathcal{S}}$ correspond to independent refinements of the subdivisions of the cells (Q_i, A_i) .

Equivalently, a regular refinement of $\mathcal{S}$ is uniquely determined by the choice, for each $i \in I$, of a regular subdivision of (Q_i, A_i) . This implies that the face poset of $F_{\mathcal{S}}$ is the product of the face posets of the secondary polytopes $\Sigma(A_i)$.

Remark 2.8. *The factorization property in Proposition 2.7 is one of the basic structural properties of secondary polytopes. In the form used here, it says that once a regular subdivision $\mathcal{S}$ is fixed, any further refinement of $\mathcal{S}$ is obtained independently by refining each of its maximal cells. Thus the corresponding face of the secondary polytope splits as the product of the secondary polytopes of the cells.*

A detailed proof of this factorization statement, in the language used in the algebra of the infrared, can be found in [20]. There the point configuration is often assumed to be in exceptional general position, which ensures that the relevant secondary polytopes and

their face stratifications behave generically. Under this genericity assumption, the faces corresponding to regular subdivisions have the expected dimensions, and the above product decomposition is compatible with the combinatorics of refinements.

This factorization property is the key structural input in the construction of the L_∞ algebra associated to (Q, A) : it allows one to express boundary components of faces of $\Sigma(A)$ in terms of products of lower-dimensional secondary polytopes, leading to higher multilinear operations.

2.3 The cellular chain complex of a polytope

In this paper, we work over a fixed field $\mathbb{k}$ of characteristic 0. All graded vector spaces are $\mathbb{Z}$ -graded, and we use cohomological grading conventions (differentials have degree +1). For a graded vector space V , the shift $V[1]$ is defined by $V[1]^i := V^{i+1}$. If $v \in V$ is homogeneous, its degree is denoted $|v|$.

Let P be a convex polytope. We define its cellular chain complex with coefficients in orientation lines by

$$C_\bullet(P) := \bigoplus_{\emptyset \neq F \subset P} \text{or}(F)[\dim F],$$

where the sum ranges over all (nonempty) faces F of P , $\text{or}(F)$ denotes the one-dimensional orientation line of F , and $[\dim F]$ denotes the degree shift. The differential is the usual cellular boundary map

$$d : \text{or}(F) \longrightarrow \bigoplus_{F' \prec F} \text{or}(F')$$

summed over codimension-one faces $F' \prec F$ with the induced orientation signs. One has $d^2 = 0$.

Let (Q, A) be a fixed marked polytope and let $(Q', A') \subset (Q, A)$ range over all marked subpolytopes. Each such (Q', A') has its own secondary polytope $\Sigma(A')$.

For every (Q', A') we consider the *top cell* of $\Sigma(A')$, i.e. the oriented fundamental class

$$v_{A'} \in \text{or}(\Sigma(A'))[\dim \Sigma(A')].$$

We now define the graded vector space

$$V := \bigoplus_{(Q', A') \subset (Q, A)} V_{A'}, \quad V_{A'} := \text{or}(\Sigma(A'))[\dim \Sigma(A')]. \quad (2.1)$$

Thus V has a *single generator* $v_{A'}$ *per subconfiguration* $A' \subset A$. Only top-dimensional faces of the various $\Sigma(A')$ appear in V .

We form the free graded commutative algebra

$$S^\bullet(V) = \bigoplus_{n \geq 0} \text{Sym}^n(V).$$

If $x, y \in V$ are homogeneous, their symmetric product satisfies

$$x \odot y = (-1)^{|x||y|} y \odot x.$$

Every element of $S^\bullet(V)$ is therefore a finite linear combination of monomials of the form $v_{A'_1} \odot \cdots \odot v_{A'_n}$.

Fix (Q', A') . Inside the chain complex $C_\bullet(\Sigma(A'))$ the fundamental class $v_{A'}$ has boundary

$$\partial v_{A'} = \sum_{\substack{F \subset \Sigma(A') \\ \text{codim } F = 1}} \varepsilon(F) [F].$$

By the theory of secondary polytopes, each codimension-1 face F associated to a coarse regular subdivision $P'' = \{(Q''_\nu, A''_\nu)\}$ factorizes as a product of smaller secondary polytopes:

$$F \cong \prod_{\nu} \Sigma(A''_\nu).$$

Thus, we define the differential on a generator by

$$d(v_{A'}) := \sum_{P''} \varepsilon(P'') \bigodot_{\nu} v_{A''}, \quad (2.2)$$

where the sum runs over all coarse regular subdivisions of (Q', A') . This defines a linear map

$$d : V \longrightarrow S^{\bullet}(V),$$

of degree +1.

We extend d to all of $S^{\bullet}(V)$ as a graded derivation:

$$d(x \odot y) := d(x) \odot y + (-1)^{|x|} x \odot d(y), \quad x, y \in S^{\bullet}(V).$$

This uniquely determines a degree+1 endomorphism

$$d : S^{\bullet}(V) \rightarrow S^{\bullet}(V).$$

Let us compute $d^2(v_{A'})$. By definition (2.2) and the Leibniz rule, $d^2(v_{A'})$ is a signed sum over *two-step subdivisions*

$$(Q', A') \longrightarrow P'' = \{A''_{\nu}\} \longrightarrow P''' = \{A'''_{\mu}\},$$

i.e. refinements of the coarse subdivision P'' in exactly one component.

Geometrically this corresponds to taking the boundary of the codimension-1 face $F_{P''} \subset \Sigma(A')$: its boundary consists of codimension-2 faces. Every codimension-2 face $G \subset \Sigma(A')$ arises *twice*, with opposite signs: once via P'' then a refinement of some (Q''_{ν}, A''_{ν}) , and once via a distinct intermediate subdivision. This is exactly the combinatorial identity $\partial^2 = 0$ in the cellular chain complex $C_{\bullet}(\Sigma(A'))$.

Because factorization identifies each G with a product

$$G \cong \Sigma(A_1''') \times \cdots \times \Sigma(A_k'''),$$

the two algebraic contributions to $v_{A_1'''} \odot \cdots \odot v_{A_k'''}$ coming from the two subdivision paths have opposite signs. Thus they cancel.

Therefore

$$d^2(v_{A'}) = 0 \quad \text{for all } A'.$$

Since d is a derivation, it follows that

$$d^2 = 0 \quad \text{on all of } S^\bullet(V).$$

In summary, the pair $(S^\bullet(V), d)$ is a graded commutative dg-algebra whose differential is geometrically induced from the cellular boundary maps of all secondary polytopes $\Sigma(A')$.

Chapter 3

A_∞ -algebras, L_∞ -algebras and Koszul duality

This chapter collects the algebraic background used throughout the paper. We use standard conventions for A_∞ - and L_∞ -algebras in terms of square-zero coderivations on the bar coalgebra. General background on A_∞ -algebras can be found in [21]; background on L_∞ -algebras and their relation to deformation theory can be found in [25, 28]; and the coderivation formalism for deformation theory is discussed in [8, 24]. General background on operads and homotopy algebra may also be found in [29].

3.1 A_∞ -algebras

In this section we recall the basic definitions of A_∞ -algebras and A_∞ -morphisms. The notion of an A_∞ -algebra is a homotopy invariant weakening of the notion of a differential graded algebra. Instead of requiring the product to be strictly associative on the chain level, one allows associativity to hold up to a coherent system of higher homotopies.

Let V be a graded vector space over a field $\mathbb{k}$.

Definition 3.1. An A_∞ -algebra structure on V is given by a collection of multilinear maps

$$m_n : V^{\otimes n} \longrightarrow V[2 - n], \quad n \geq 1,$$

satisfying the Stasheff identities. Explicitly, for each $n \geq 1$, one has

$$\sum_{\substack{r+s+t=n \\ s \geq 1}} (-1)^{r+st} m_{r+1+t}(\text{id}^{\otimes r} \otimes m_s \otimes \text{id}^{\otimes t}) = 0. \quad (3.2)$$

Here m_n has cohomological degree $2 - n$, and the signs are determined by the Koszul sign rule.

The first few Stasheff identities explain the meaning of the definition. For $n = 1$, equation (3.2) gives

$$m_1^2 = 0.$$

Thus m_1 is a differential on V . For $n = 2$, the identity says that m_1 is compatible with the binary product m_2 . In other words, m_2 is a chain map up to the usual Koszul signs. For $n = 3$, the identity says that m_2 is associative up to a homotopy controlled by m_3 . The higher maps

$$m_n, \quad n \geq 3,$$

then encode higher homotopies among these associativity relations.

Thus an A_∞ -algebra may be viewed as an associative algebra up to a coherent system of higher homotopies. In particular, the binary operation m_2 induces an associative product on the cohomology

$$H^\bullet(V, m_1).$$

Example 1. Every differential graded algebra is an A_∞ -algebra. Indeed, let (V, d, μ) be a differential graded algebra. Then one sets

$$m_1 = d, \quad m_2 = \mu, \quad m_n = 0 \quad \text{for } n \geq 3.$$

The Stasheff identities reduce to the conditions that $d^2 = 0$, that d satisfies the Leibniz rule with respect to μ , and that μ is strictly associative.

The correct notion of morphism between A_∞ -algebras is also homotopy-theoretic. An A_∞ -morphism is not simply a chain map compatible with products. Instead, it consists of a collection of maps whose higher components encode the failure of strict compatibility with the A_∞ -operations.

Let $(V, \{m_n^V\})$ and $(W, \{m_n^W\})$ be A_∞ -algebras.

Definition 3.3. An A_∞ -morphism

$$f : V \longrightarrow W$$

is a collection of multilinear maps

$$f_n : V^{\otimes n} \longrightarrow W[1 - n], \quad n \geq 1,$$

satisfying the identities

$$\begin{aligned} & \sum_{\substack{r+s+t=n \\ s \geq 1}} (-1)^{r+st} f_{r+1+t}(\text{id}^{\otimes r} \otimes m_s^V \otimes \text{id}^{\otimes t}) \\ &= \sum_{\substack{k \geq 1 \\ i_1 + \dots + i_k = n}} (-1)^\epsilon m_k^W(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_k}), \end{aligned} \tag{3.4}$$

where

$$\epsilon = \sum_{j=1}^k (k-j)(i_j-1).$$

The first component

$$f_1 : V \longrightarrow W$$

has degree 0. The identity (3.4) for $n = 1$ says that f_1 is a chain map:

$$f_1 m_1^V = m_1^W f_1.$$

For $n = 2$, the identity says that f_1 preserves the product m_2 up to a homotopy controlled by f_2 . The higher maps f_n , for $n \geq 3$, give the higher coherence data.

Definition 3.5. *An A_∞ -morphism $f : V \rightarrow W$ is called strict if*

$$f_n = 0 \quad \text{for all } n \geq 2.$$

In this case f_1 is a chain map which is strictly compatible with all A_∞ -operations.

Definition 3.6. *An A_∞ -morphism $f : V \rightarrow W$ is called an A_∞ -quasi-isomorphism if its first component*

$$f_1 : (V, m_1^V) \longrightarrow (W, m_1^W)$$

is a quasi-isomorphism of cochain complexes.

Thus, for A_∞ -algebras, quasi-isomorphism is detected by the linear term f_1 . The higher components f_n are nevertheless essential: they record the compatibility of f_1 with the higher algebraic structures.

There is an equivalent and often more conceptual way to package the above definitions. After applying the standard suspension, the operations $\{m_n\}_{n \geq 1}$ can be assembled into a degree-one coderivation

$$b : T^c(V[1]) \longrightarrow T^c(V[1])$$

on the reduced tensor coalgebra

$$T^c(V[1]) = \bigoplus_{n \geq 1} V[1]^{\otimes n}.$$

The Stasheff identities are equivalent to the single equation

$$b^2 = 0.$$

In this language, an A_∞ -morphism $f : V \rightarrow W$ is equivalently a coalgebra morphism

$$F : T^c(V[1]) \longrightarrow T^c(W[1])$$

compatible with the corresponding coderivations:

$$F \circ b_V = b_W \circ F.$$

This point of view is useful because it shows that A_∞ -algebras are controlled by differential graded Lie algebras of coderivations. In later sections, we will use this philosophy to relate A_∞ -structures to deformation complexes and to L_∞ -morphisms into derived derivation algebras.

For further background on A_∞ -algebras and bar constructions, see [13]. For a more systematic treatment of A_∞ -categories, modules, and twisted complexes, see [26].

3.2 L_∞ -algebras

In this section we recall the basic definitions of L_∞ -algebras and L_∞ -morphisms. An L_∞ -algebra is a homotopy-theoretic generalization of a differential graded Lie algebra. Instead of requiring the Jacobi identity to hold strictly, one allows it to hold up to a coherent system of higher homotopies.

Let $\mathfrak{g}$ be a graded vector space over a field $\mathbb{k}$.

Definition 3.7. *An L_∞ -algebra structure on $\mathfrak{g}$ is a collection of graded skew-symmetric multilinear maps*

$$\ell_n : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}[2 - n], \quad n \geq 1,$$

satisfying the higher Jacobi identities. Explicitly, for every $n \geq 1$ and homogeneous elements

$x_1, \dots, x_n \in \mathfrak{g}$, one has

$$\sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \sum_{\sigma \in \text{Sh}(i, n-i)} \chi(\sigma; x) (-1)^{i(j-1)} \ell_j \left(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)} \right) = 0. \quad (3.8)$$

Here $\text{Sh}(i, n-i)$ denotes the set of $(i, n-i)$ -shuffles, and $\chi(\sigma; x)$ is the Koszul sign obtained by permuting the homogeneous elements $x_1, \dots, x_n$ according to σ .

The first few identities explain the meaning of the definition. For $n = 1$, equation (3.8) gives

$$\ell_1^2 = 0.$$

Thus ℓ_1 is a differential on $\mathfrak{g}$. For $n = 2$, the identity says that ℓ_1 is compatible with the bracket ℓ_2 . In other words, ℓ_1 acts as a derivation of ℓ_2 , up to the usual Koszul signs. For $n = 3$, the identity says that the bracket ℓ_2 satisfies the graded Jacobi identity up to a homotopy controlled by ℓ_3 . The higher operations

$$\ell_n, \quad n \geq 3,$$

then encode higher coherence relations among these homotopies.

Thus an L_∞ -algebra may be viewed as a Lie algebra up to a coherent system of higher homotopies. The bracket ℓ_2 induces a graded Lie bracket on the cohomology

$$H^\bullet(\mathfrak{g}, \ell_1).$$

Example 2. Every differential graded Lie algebra is an L_∞ -algebra. Let $(\mathfrak{g}, d, [\cdot, \cdot])$ be a differential graded Lie algebra. Then one sets

$$\ell_1 = d, \quad \ell_2 = [\cdot, \cdot], \quad \ell_n = 0 \quad \text{for } n \geq 3.$$

The L_∞ -identities reduce to the conditions that $d^2 = 0$, that d is compatible with the bracket,

and that the bracket satisfies the graded Jacobi identity.

One of the main reasons L_∞ -algebras appear in deformation theory is that they have a natural Maurer–Cartan equation. This equation generalizes the Maurer–Cartan equation in a differential graded Lie algebra.

Definition 3.9. Let $(\mathfrak{g}, \{\ell_n\}_{n \geq 1})$ be an L_∞ -algebra. A degree 1 element

$$\alpha \in \mathfrak{g}^1$$

is called a Maurer–Cartan element if it satisfies

$$\sum_{n \geq 1} \frac{1}{n!} \ell_n(\alpha, \dots, \alpha) = 0.$$

For a differential graded Lie algebra, this equation becomes

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

In general, the higher brackets contribute higher-order correction terms.

In many applications, the sum in the Maurer–Cartan equation is infinite. Therefore one usually assumes that $\mathfrak{g}$ is nilpotent, filtered, or completed, so that the above series is well-defined. In the finite-dimensional or nilpotent situations considered in many algebraic constructions, this convergence issue is harmless.

Given a Maurer–Cartan element, one can twist the L_∞ -structure to obtain a new L_∞ -algebra.

Let $\alpha \in \mathfrak{g}^1$ be a Maurer–Cartan element. The twisted operations are defined by

$$\ell_n^\alpha(x_1, \dots, x_n) = \sum_{k \geq 0} \frac{1}{k!} \ell_{n+k}(\underbrace{\alpha, \dots, \alpha}_{k \text{ times}}, x_1, \dots, x_n).$$

The Maurer–Cartan equation for α ensures that the operations $\{\ell_n^\alpha\}_{n \geq 1}$ again satisfy the L_∞ -identities.

The twisted differential is

$$\ell_1^\alpha(x) = \ell_1(x) + \ell_2(\alpha, x) + \frac{1}{2!}\ell_3(\alpha, \alpha, x) + \cdots .$$

This is the differential controlling deformations around the point α . Thus Maurer–Cartan elements may be viewed as deformation parameters, and twisting describes the deformation complex at such a point.

The natural notion of morphism between L_∞ -algebras is also homotopy-theoretic. An L_∞ -morphism is not simply a linear map preserving all brackets. Instead, it consists of a collection of maps whose higher components encode the failure of strict compatibility.

Let

$$(\mathfrak{g}, \{\ell_n^{\mathfrak{g}}\}) \quad \text{and} \quad (\mathfrak{h}, \{\ell_n^{\mathfrak{h}}\})$$

be L_∞ -algebras.

Definition 3.10. *An L_∞ -morphism*

$$F : \mathfrak{g} \longrightarrow \mathfrak{h}$$

is a collection of graded skew-symmetric multilinear maps

$$F_n : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{h}[1 - n], \quad n \geq 1,$$

satisfying the usual compatibility identities with the higher brackets on $\mathfrak{g}$ and $\mathfrak{h}$.

The first component

$$F_1 : \mathfrak{g} \longrightarrow \mathfrak{h}$$

has degree 0. The first L_∞ -morphism identity says that F_1 is a chain map:

$$F_1 \ell_1^{\mathfrak{g}} = \ell_1^{\mathfrak{h}} F_1.$$

The next identity says that F_1 preserves the bracket ℓ_2 up to a homotopy controlled by F_2 . The higher maps F_n , for $n \geq 3$, give higher coherence data.

Definition 3.11. *An L_∞ -morphism $F : \mathfrak{g} \rightarrow \mathfrak{h}$ is called strict if*

$$F_n = 0 \quad \text{for all } n \geq 2.$$

In this case F_1 is strictly compatible with all L_∞ -operations.

Definition 3.12. *An L_∞ -morphism $F : \mathfrak{g} \rightarrow \mathfrak{h}$ is called an L_∞ -quasi-isomorphism if its first component*

$$F_1 : (\mathfrak{g}, \ell_1^{\mathfrak{g}}) \longrightarrow (\mathfrak{h}, \ell_1^{\mathfrak{h}})$$

is a quasi-isomorphism of cochain complexes.

An L_∞ -morphism sends Maurer–Cartan elements to Maurer–Cartan elements. More precisely, if $\alpha \in \mathfrak{g}^1$ is a Maurer–Cartan element, then, under suitable nilpotence or convergence assumptions,

$$F_*(\alpha) = \sum_{n \geq 1} \frac{1}{n!} F_n(\alpha, \dots, \alpha)$$

is a Maurer–Cartan element of $\mathfrak{h}$. This property is one of the main reasons L_∞ -morphisms are useful in deformation theory.

There is a compact way to package the definition of an L_∞ -algebra. After applying the suspension, the higher brackets ℓ_n can be assembled into a degree-one coderivation

$$Q : S^c(\mathfrak{g}[1]) \longrightarrow S^c(\mathfrak{g}[1])$$

on the cofree cocommutative coalgebra

$$S^c(\mathfrak{g}[1]) = \bigoplus_{n \geq 1} S^n(\mathfrak{g}[1]).$$

The higher Jacobi identities are equivalent to the single equation

$$Q^2 = 0.$$

Thus an L_∞ -algebra can be described equivalently as a codifferential on the cofree cocommutative coalgebra generated by $\mathfrak{g}[1]$.

In this language, an L_∞ -morphism

$$F : \mathfrak{g} \rightarrow \mathfrak{h}$$

is a coalgebra morphism

$$S^c(\mathfrak{g}[1]) \longrightarrow S^c(\mathfrak{h}[1])$$

compatible with the corresponding codifferentials. This coalgebra description is often the cleanest way to state and prove functorial properties of L_∞ -algebras, since all higher compatibility identities are contained in one equation.

In later sections, we will use L_∞ -algebras as deformation-theoretic objects. In particular, the L_∞ -algebras constructed from secondary polytopes will act on A_∞ -algebras through L_∞ -morphisms to derived derivation complexes.

3.3 Derived derivation spaces, Hochschild complexes, and Koszul duality

We briefly recall the deformation-theoretic background used below, following [19, Section 7] and the general framework of [23, 24].

Let $\mathcal{P}$ be a dg operad over a field $\mathbb{k}$, and let A be a $\mathcal{P}$ -algebra in the category of dg vector spaces over $\mathbb{k}$. We denote by

$$\alpha : \mathcal{P} \longrightarrow \mathrm{End}(A)$$

the corresponding morphism of dg operads.

Definition 3.13. A homogeneous linear map $\theta : A \rightarrow A$ is called a $\mathcal{P}$ -derivation if it is compatible with all operations coming from $\mathcal{P}$. Equivalently, for every homogeneous $p \in \mathcal{P}(n)$, one has

$$[\theta, \alpha(p)] = 0,$$

where the bracket is the operadic commutator in the endomorphism operad. We write $\text{Der}_{\mathcal{P}}(A)$ for the graded vector space of $\mathcal{P}$ -derivations of all degrees.

Lemma 3.14. The graded vector space $\text{Der}_{\mathcal{P}}(A)$ is a dg Lie algebra. The bracket is the graded commutator

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|} \eta \circ \theta,$$

and the differential is

$$d_{\text{Der}}(\theta) = d_A \circ \theta - (-1)^{|\theta|} \theta \circ d_A.$$

Proof. We first check that $\text{Der}_{\mathcal{P}}(A)$ is closed under the graded commutator. Let $\theta, \eta \in \text{Der}_{\mathcal{P}}(A)$. Then for every $p \in \mathcal{P}$, we have

$$[\theta, \alpha(p)] = 0, \quad [\eta, \alpha(p)] = 0.$$

By the graded Jacobi identity in the endomorphism operad,

$$[[\theta, \eta], \alpha(p)] = [\theta, [\eta, \alpha(p)]] - (-1)^{|\theta||\eta|} [\eta, [\theta, \alpha(p)]] = 0.$$

Hence $[\theta, \eta]$ is again a $\mathcal{P}$ -derivation.

Next we check that the differential preserves derivations. Since $\alpha : \mathcal{P} \rightarrow \text{End}_A$ is a morphism of dg operads, we have

$$d_{\text{End}}(\alpha(p)) = \alpha(d_{\mathcal{P}}p).$$

If $\theta \in \text{Der}_{\mathcal{P}}(A)$, then

$$[\theta, \alpha(p)] = 0$$

for all $p \in \mathcal{P}$. Applying the differential in the endomorphism operad gives

$$0 = d_{\text{End}}[\theta, \alpha(p)] = [d_{\text{End}}\theta, \alpha(p)] + (-1)^{|\theta|}[\theta, d_{\text{End}}\alpha(p)].$$

Using $d_{\text{End}}\alpha(p) = \alpha(d_{\mathcal{P}}p)$, the second term vanishes because θ is a $\mathcal{P}$ -derivation. Therefore

$$[d_{\text{End}}\theta, \alpha(p)] = 0.$$

Thus $d_{\text{Der}}\theta = d_{\text{End}}\theta$ is again a $\mathcal{P}$ -derivation. Hence $\text{Der}_{\mathcal{P}}(A)$ is closed under both the graded commutator and the differential.

The graded commutator satisfies the graded antisymmetry and graded Jacobi identity because it is the commutator bracket in the endomorphism dg algebra. Therefore $\text{Der}_{\mathcal{P}}(A)$ is a dg Lie algebra. $\square$

The ordinary derivation complex is not homotopy invariant in general: a quasi-isomorphism of $\mathcal{P}$ -algebras $A \rightarrow A'$ need not induce a quasi-isomorphism

$$\text{Der}_{\mathcal{P}}(A) \longrightarrow \text{Der}_{\mathcal{P}}(A').$$

Thus $\text{Der}_{\mathcal{P}}(A)$ is not, by itself, the correct object for describing the deformation theory of A up to homotopy. In order to obtain a homotopy-invariant deformation complex, one derives the assignment

$$A \longmapsto \text{Der}_{\mathcal{P}}(A).$$

Concretely, one chooses a cofibrant replacement

$$\tilde{A} \longrightarrow A$$

in the category of $\mathcal{P}$ -algebras and defines

$$R\text{Der}_{\mathcal{P}}(A) := \text{Der}_{\mathcal{P}}(\tilde{A}).$$

Here $\tilde{A} \rightarrow A$ is a quasi-isomorphism of $\mathcal{P}$ -algebras, with $\tilde{A}$ chosen sufficiently free so that derivations out of $\tilde{A}$ detect deformations of A in a homotopy-invariant way. Different choices of cofibrant replacement give quasi-isomorphic dg Lie algebras, so the quasi-isomorphism type of $R\mathrm{Der}_{\mathcal{P}}(A)$ depends only on A .

We call $R\mathrm{Der}_{\mathcal{P}}(A)$ the derived derivation complex of A . Since

$$R\mathrm{Der}_{\mathcal{P}}(A) = \mathrm{Der}_{\mathcal{P}}(\tilde{A})$$

is a dg Lie algebra, it can also be regarded as an L_{∞} -algebra with only two nonzero structure maps: the unary bracket is the differential d_{Der} , and the binary bracket is the graded commutator of derivations. This is the sense in which the derived derivation complex carries a natural L_{∞} -structure.

In characteristic zero, this dg Lie algebra governs the formal deformation theory of A as a $\mathcal{P}$ -algebra. More precisely, for a local Artin dg algebra B with maximal ideal $\mathfrak{m}_B$, deformations of the $\mathcal{P}$ -algebra structure on A over B are described by Maurer–Cartan elements of

$$R\mathrm{Der}_{\mathcal{P}}(A) \otimes \mathfrak{m}_B.$$

If we write $L = R\mathrm{Der}_{\mathcal{P}}(A)$, then such an element $\gamma \in L^1 \otimes \mathfrak{m}_B$ satisfies the Maurer–Cartan equation

$$d_L \gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

Gauge equivalence of Maurer–Cartan elements corresponds to equivalence of deformations. Thus the formal moduli problem of deformations of A as a $\mathcal{P}$ -algebra is encoded by the dg Lie algebra $R\mathrm{Der}_{\mathcal{P}}(A)$, or equivalently by its associated L_{∞} -algebra.

In the case where $\mathcal{P} = \mathrm{Ass}$ and A is an associative algebra, the derived derivation complex is identified with the shifted truncated Hochschild cochain complex

$$C^{\geq 1}(A, A)[1] = \left\{ \mathrm{Hom}_k(A, A) \xrightarrow{\delta_0} \mathrm{Hom}_k(A^{\otimes 2}, A) \xrightarrow{\delta_1} \mathrm{Hom}_k(A^{\otimes 3}, A) \longrightarrow \cdots \right\}.$$

The dg Lie bracket is the shifted Gerstenhaber bracket, equivalently the bracket induced by the brace operations on Hochschild cochains.

Similarly, if $\mathcal{P} = \text{Lie}$ and L is a Lie algebra, then the derived derivation complex is identified with the shifted truncated Chevalley–Eilenberg cochain complex with coefficients in the adjoint representation:

$$C_{\text{Lie}}^{\geq 1}(L, L)[1] = \{L^* \otimes L \longrightarrow \Lambda^2 L^* \otimes L \longrightarrow \Lambda^3 L^* \otimes L \longrightarrow \cdots\}.$$

This complex controls deformations of the Lie bracket on L .

The operadic description becomes especially transparent when $\mathcal{P}$ is a quadratic Koszul operad. Let $\mathcal{P}^!$ denote its Koszul dual operad. Then a weak $\mathcal{P}$ -algebra structure, equivalently a $\mathcal{P}_\infty$ -structure, on a graded vector space A can be encoded by a square-zero derivation $d^2 = 0$ on the free $\mathcal{P}^!$ -algebra $F_{\mathcal{P}^!}(A^*[-1])$, see [14].

This is the form of Koszul duality used in [19]. Two basic cases are particularly important:

- an L_∞ -structure on a graded vector space L is equivalent to a square-zero derivation on the completed symmetric algebra $S_+^\bullet(L^*[-1])$;
- an A_∞ -structure on a graded vector space R is equivalent to a square-zero derivation on the completed tensor algebra $T_+^\bullet(R^*[-1])$.

For the applications in this paper, the most important general principle is the following consequence of this formalism, and we give a detailed proof.

Proposition 3.15 (cf. [19]). *Assume that the graded vector spaces involved are of finite type, or otherwise work with completed algebras and continuous duals. The following two types of data are equivalent.*

(i) *A datum consisting of:*

- (a) *an A_∞ -algebra structure on R ;*
- (b) *an L_∞ -algebra structure on L ;*

(c) an L_∞ -morphism

$$\alpha: L \longrightarrow \mathrm{RDer}(R) \simeq C^{\geq 1}(R, R)[1].$$

(ii) A continuous degree-one derivation

$$d: \mathcal{A} \longrightarrow \mathcal{A}, \quad d^2 = 0,$$

on the completed graded algebra

$$\mathcal{A} := \widehat{S}^\bullet(L^\vee[-1]) \widehat{\otimes} \widehat{T}^\bullet(R^\vee[-1]),$$

such that

$$d(\widehat{S}^\bullet(L^\vee[-1]) \otimes 1) \subset \widehat{S}^\bullet(L^\vee[-1]) \otimes 1,$$

and such that d preserves the two augmentation ideals

$$I_L := \widehat{S}_+^\bullet(L^\vee[-1]) \widehat{\otimes} \widehat{T}^\bullet(R^\vee[-1])$$

and

$$I_R := \widehat{S}^\bullet(L^\vee[-1]) \widehat{\otimes} \widehat{T}_+^\bullet(R^\vee[-1]).$$

Proof. Write

$$\mathcal{O}_L := \widehat{S}^\bullet(L^\vee[-1]), \quad \mathcal{O}_R := \widehat{T}^\bullet(R^\vee[-1]),$$

so that

$$\mathcal{A} = \mathcal{O}_L \widehat{\otimes} \mathcal{O}_R.$$

Suppose first that a differential d as in (ii) is given. Since d preserves the subalgebra $\mathcal{O}_L \otimes 1$, its restriction defines a square-zero derivation

$$d_L: \mathcal{O}_L \longrightarrow \mathcal{O}_L.$$

Moreover, the preservation of I_L implies that d_L preserves the augmentation ideal

$$\mathfrak{m}_L = \widehat{S}_+^\bullet(L^\vee[-1]).$$

Thus d_L determines an L_∞ -algebra structure on L .

Let

$$\varepsilon_L: \mathcal{O}_L \longrightarrow \mathbb{k}$$

be the augmentation. Since d preserves I_L , it induces a square-zero derivation on the quotient

$$\mathcal{A}/I_L \simeq \mathcal{O}_R.$$

Equivalently, this differential is given by

$$d_R := (\varepsilon_L \otimes \text{id}) \circ d|_{1 \otimes \mathcal{O}_R}.$$

Because d also preserves I_R , the derivation d_R preserves $\widehat{T}_+^\bullet(R^\vee[-1])$. Therefore d_R determines an uncurved A_∞ -algebra structure on R .

It remains to extract the L_∞ -morphism. For a generator $r^\vee \in R^\vee[-1]$, decompose

$$d(1 \otimes r^\vee) = \sum_{n \geq 0, m \geq 1} d_{n,m}(r^\vee),$$

where

$$d_{n,m}: R^\vee[-1] \longrightarrow S^n(L^\vee[-1]) \otimes T^m(R^\vee[-1]).$$

The component with $n = 0$ is precisely the A_∞ differential d_R . The components with $n \geq 1$ are the mixed terms.

Dualizing and undoing the shifts, the maps $d_{n,m}$ with $n \geq 1$ give maps

$$\alpha_{n,m}: S^n(L[-1]) \longrightarrow \text{Hom}_{\mathbb{k}}(R^{\otimes m}, R)[1 - m].$$

For each $n \geq 1$, summing over m gives

$$\alpha_n: S^n(L[-1]) \longrightarrow C^{\geq 1}(R, R)[1].$$

The identity $d^2 = 0$ on the generators $R^\vee[-1]$ is exactly the collection of compatibility relations saying that $\{\alpha_n\}_{n \geq 1}$ defines an L_∞ -morphism

$$\alpha: L \longrightarrow C^{\geq 1}(R, R)[1].$$

Conversely, suppose that an A_∞ -structure on R , an L_∞ -structure on L , and an L_∞ -morphism

$$\alpha: L \longrightarrow C^{\geq 1}(R, R)[1]$$

are given. Let

$$d_L: \mathcal{O}_L \longrightarrow \mathcal{O}_L, \quad d_R: \mathcal{O}_R \longrightarrow \mathcal{O}_R$$

be the corresponding square-zero derivations.

Write the components of α as

$$\alpha_{n,m}: S^n(L[-1]) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(R^{\otimes m}, R)[1-m], \quad n, m \geq 1.$$

After dualizing, these give maps

$$d_{n,m}: R^\vee[-1] \longrightarrow S^n(L^\vee[-1]) \otimes T^m(R^\vee[-1]).$$

Define

$$d_{\mathrm{mix}} := \sum_{n,m \geq 1} d_{n,m}.$$

Define d on the generators by

$$d|_{L^\vee[-1]} := d_L, \quad d|_{R^\vee[-1]} := d_R + d_{\mathrm{mix}},$$

and extend it uniquely to $\mathcal{A}$ as a graded derivation. In particular, for homogeneous $a \in \mathcal{O}_L$ and $x \in \mathcal{O}_R$,

$$d(a \otimes x) = d_L(a) \otimes x + (-1)^{|a|} a d(1 \otimes x).$$

Since every component of d_{mix} has positive symmetric degree and positive tensor degree, the derivation d preserves both I_L and I_R . The equation $d^2 = 0$ on $L^\vee[-1]$ is equivalent to the L_∞ -identities on L , while the equation $d^2 = 0$ on $R^\vee[-1]$ is equivalent to the A_∞ -identities on R together with the L_∞ -morphism identities for α . Therefore $d^2 = 0$.

The two constructions are inverse to one another. □

Chapter 4

Construction of the L_∞ -algebra

Let $A \subset E$ be a finite point configuration on the elliptic curve E . The purpose of this section is to explain the geometric structure from which our L_∞ -algebra will arise. In the classical planar situation, the relevant combinatorial object is the secondary polytope associated to a finite point configuration. Our goal is to formulate an analogous picture in the elliptic-curve setting.

4.1 Fixed-domain construction

Let E be an elliptic curve, and let $\pi : \mathbb{C} \rightarrow E$ be the universal covering map. We use the affine structure on $\mathbb{C}$ to define polygonal regions on E .

Definition 4.1. *Let $A \subset E$ be a finite point configuration. Choose lifts of the points of A to $\mathbb{C}$, and denote the resulting lifted configuration by $\tilde{A} \subset \mathbb{C}$. Suppose that the convex hull*

$$\tilde{P} := \text{Conv}(\tilde{A})$$

is contained in a domain on which π is injective. Equivalently,

$$\pi|_{\tilde{P}} : \tilde{P} \rightarrow \pi(\tilde{P})$$

is a homeomorphism onto its image.

The subset

$$P := \pi(\tilde{P}) \subset E$$

is called the polygonal region associated to the lifted configuration $\tilde{A}$. The pair (P, A) is called a marked polygonal region. Here $A \subset P$ is regarded as the set of marked points of the region. We do not require all points of A to lie on the boundary of P ; some points may lie in the interior of P .

If $\tilde{P}' \subset \tilde{P}$ is a convex subpolygon, and if

$$\tilde{A}' := \tilde{A} \cap \tilde{P}', \quad A' := \pi(\tilde{A}') \subset A,$$

then

$$P' := \pi(\tilde{P}')$$

is called a polygonal subregion of P , and the pair (P', A') is called a marked subpolygon of (P, A) .

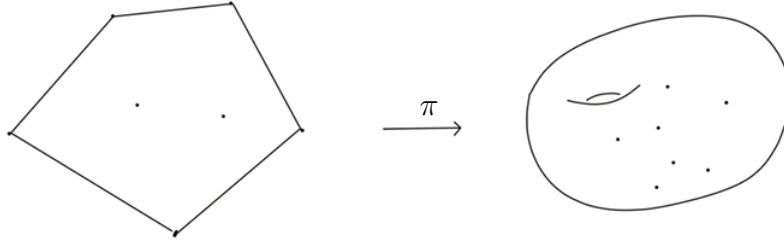


Figure 4.1: Lifted configuration in $\mathbb{C}$.

Remark 4.2. This definition uses a special feature of elliptic curves: the universal cover of E is the affine plane $\mathbb{C}$, and the deck transformations are translations. Hence notions such as convex hull, straight line segment, Euclidean polygon, affine triangulation, and secondary polytope are inherited directly from $\mathbb{C}$.

For a curve of genus $g > 1$, the universal cover is the disk rather than an affine vector space. Although one can still pass to the universal cover, there is no canonical affine structure

compatible with the deck group. Consequently there is no canonical analogue of the convex hull of a lifted point configuration, nor a direct secondary polytope associated to it. One could introduce extra geometric data, for example a hyperbolic metric, and study geodesic polygons or related objects. However, such choices are additional noncanonical structures and do not give a formal repetition of the construction used here. For this reason, the present construction is restricted to the elliptic curve case.

Let E be an elliptic curve, written as a complex torus $E \cong \mathbb{C}/\Lambda$. We use the standard viewpoint that a holomorphic quadratic differential determines a flat surface structure away from its zeroes; see [36] for background on flat surfaces and translation structures. The translation-invariant differential dz on $\mathbb{C}$ descends to a nowhere-vanishing holomorphic 1-form on E . Hence $q = dz^{\otimes 2}$ defines a holomorphic quadratic differential on E without zeroes. The quadratic differential q determines a flat metric on E , given in a local coordinate $z = x + iy$ by

$$ds_q^2 = |q| = |dz|^2 = dx^2 + dy^2.$$

Accordingly, the associated area form is

$$dA_q = \frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy.$$

Therefore, for any polygonal region $P \subset E$, its area with respect to the flat structure defined by q is

$$\text{Area}_q(P) = \int_P dA_q.$$

Equivalently, after choosing a lift $\tilde{P} \subset \mathbb{C}$, one may compute

$$\text{Area}_q(P) = \int_{\tilde{P}} dx dy,$$

which is independent of the choice of lift since deck transformations act on $\mathbb{C}$ by translations and preserve the Euclidean area form.

Once the lift $(\tilde{P}, \tilde{A}) \subset \mathbb{C}$ is fixed, it is an ordinary finite point configuration in a convex

polygon in the affine plane. Therefore the usual theory of regular subdivisions, triangulations, and secondary polytopes applies to $(\tilde{P}, \tilde{A})$. Whenever we speak about subdivisions, triangulations, or secondary polytopes of the marked polygonal region (P, A) , we mean the corresponding objects associated to the chosen lift $(\tilde{P}, \tilde{A})$.

Definition 4.3. *Let (P, A) be a marked polygonal region on E , obtained from a lifted configuration*

$$(\tilde{P}, \tilde{A}) \subset \mathbb{C}$$

as above. A triangulation of (P, A) is the projection to E of an ordinary affine triangulation $\tilde{\mathcal{T}}$ of $(\tilde{P}, \tilde{A})$. Equivalently, it is a decomposition of P into geodesic triangles whose vertices belong to A , such that distinct triangles meet only along common faces and the union of all triangles is P . We always require that this decomposition lifts to a genuine affine triangulation of $(\tilde{P}, \tilde{A})$.

Let $\mathcal{T}$ be such a triangulation. For each triangle $\Delta \in \mathcal{T}$, choose its lift $\tilde{\Delta} \in \tilde{\mathcal{T}}$. Since $\pi|_{\tilde{P}}$ is a homeomorphism onto P , this lift is unique. We define the area of Δ by

$$\text{Area}_q(\Delta) := \int_{\Delta} dA_q = \int_{\tilde{\Delta}} dx \wedge dy,$$

where $q = dz^{\otimes 2}$ and $dA_q = \frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$ is the associated flat area form.

For a triangulation $\mathcal{T}$ of (P, A) , define its GKZ vector

$$\phi_{\mathcal{T}} \in \mathbb{R}^A$$

by

$$\phi_{\mathcal{T}} = \sum_{\Delta \in \mathcal{T}} \text{Area}_q(\Delta) \sum_{a_i \in \text{Vert}(\Delta)} e_i,$$

where $\{e_i\}_{a_i \in A}$ is the standard basis of $\mathbb{R}^A$. In other words, the a_i -coordinate of $\phi_{\mathcal{T}}$ is

$$(\phi_{\mathcal{T}})_i = \sum_{\Delta \in \mathcal{T}, a_i \in \text{Vert}(\Delta)} \text{Area}_q(\Delta).$$

Definition 4.4. *The secondary polytope associated to the marked polygonal region (P, A) is*

$$\Sigma(P, A) := \text{Conv} \{ \phi_{\mathcal{T}} \mid \mathcal{T} \text{ is a triangulation of } (P, A) \} \subset \mathbb{R}^A.$$

When the marked polygonal region (P, A) is clear from context, we write

$$\Sigma(A) := \Sigma(P, A).$$

Similarly, if $(P', A') \subset (P, A)$ is a marked subpolygon, we write $\Sigma(A')$ for the secondary polytope associated to (P', A') .

The definition is exactly the usual GKZ construction applied to the lifted point configuration $(\tilde{P}, \tilde{A}) \subset \mathbb{C}$. The projection to E introduces no additional ambiguity, because $\pi|_{\tilde{P}}$ is injective and the area form dA_q pulls back to the standard Euclidean area form $dx \wedge dy$ on $\mathbb{C}$. Therefore the secondary polytope $\Sigma(P, A)$ has the same combinatorial properties as the classical secondary polytope of a planar point configuration.

In particular, it satisfies the usual factorization property. Namely, if S is a regular polygonal subdivision of (P, A) , obtained from a regular subdivision $\tilde{S}$ of $(\tilde{P}, \tilde{A})$, then the face of $\Sigma(P, A)$ corresponding to S is naturally identified with the product of the secondary polytopes of the cells of S :

$$F_S \simeq \prod_{C \in S} \Sigma(C, A_C).$$

Here C runs over the two-dimensional cells of the subdivision, and $A_C := A \cap C$ denotes the set of marked points lying in C . This is the standard GKZ factorization theorem applied upstairs to $(\tilde{P}, \tilde{A})$.

The following construction is the elliptic-curve analogue of the L_∞ -algebra construction of [19, Section 3]. Define the graded vector space

$$V := \bigoplus_{(P', A') \subset (P, A)} V_{A'}, \quad V_{A'} := \text{or}(\Sigma(A'))[\dim \Sigma(A')]. \quad (4.5)$$

Let $v_{A'} \in V_{A'}$ denote the generator given by the oriented fundamental class of $\Sigma(A')$. Using the factorization property (Proposition 2.7), we define a degree +1 derivation

$$d : S^\bullet(V) \rightarrow S^\bullet(V)$$

by prescribing it on generators:

$$d(v_{A'}) := \sum_{\mathcal{S}} \varepsilon(\mathcal{S}) \bigodot_{i \in I(\mathcal{S})} v_{A_i}, \quad (4.6)$$

where the sum runs over all *coarse subdivisions* $\mathcal{S}$ of (P', A') with cells $\{(P_i, A_i)\}_{i \in I(\mathcal{S})}$, and where $\varepsilon(\mathcal{S}) \in \{\pm 1\}$ is the sign determined by the induced orientation on the corresponding codimension-1 face of $\Sigma(A')$. Then we can extend d to $S^\bullet(V)$ by the graded Leibniz rule.

Proposition 4.7. *The derivation d satisfies $d^2 = 0$. Hence $(S^\bullet(V), d)$ is a commutative dg algebra.*

Proof. This is identical to the planar case: $d(v_{A'})$ is the cellular boundary of the top cell of $\Sigma(A')$ expressed via factorization, and $d^2 = 0$ follows from $\partial^2 = 0$ in the cellular chain complex of $\Sigma(A')$. $\square$

By Section 3, the differential d on the algebra $S^\bullet(V)$ determines an L_∞ -algebra structure on the shifted dual

$$\mathring{\mathfrak{g}} := V^*[-1] = \bigoplus_{(P', A') \subset (P, A)} E_{A'}, \quad E_{A'} := V_{A'}^*[-1] = \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1]. \quad (4.8)$$

However, the L_∞ -algebra $\mathring{\mathfrak{g}}$ is typically far too large for our purposes: the direct sum in (4.8) runs over *all* marked subpolygons (P', A') , including many which are not relevant to the geometry on the elliptic curve. To obtain the correct algebra, we restrict to the *geometric summands* in the sense of Definition 2.3.

We define a subspace of $\mathfrak{g}$ using the geometric summands:

$$\mathfrak{g} := \bigoplus_{\substack{(P', A') \subset (P, A) \\ \text{geometric}}} E_{A'} \subset \dot{\mathfrak{g}}.$$

Proposition 4.9. *The subspace $\mathfrak{g} \subset \dot{\mathfrak{g}}$ is closed under all L_∞ -operations ℓ_n . Hence $\mathfrak{g}$ is an L_∞ -subalgebra of $\dot{\mathfrak{g}}$.*

Proof. Recall that the L_∞ -structure on $\dot{\mathfrak{g}} = V^*[-1]$ is obtained from the derivation

$$d : S^\bullet(V) \rightarrow S^\bullet(V), \quad d^2 = 0,$$

by Koszul duality (Section 3). Concretely, write

$$d|_V = \sum_{k \geq 1} d_k, \quad d_k : V \rightarrow S^k(V),$$

and let ℓ_k be the dual k -ary bracket on $\dot{\mathfrak{g}}$. Thus it suffices to show that the restriction of d to the geometric part of V lands in the symmetric algebra generated by geometric generators.

Let (Q', A') be geometric and consider the generator

$$v_{A'} \in V_{A'} = \text{or}(\Sigma(A'))[\dim \Sigma(A')] \subset V.$$

By definition of d (via cellular boundaries and factorization), every monomial appearing in $d(v_{A'})$ is indexed by a coarse regular subdivision

$$\mathcal{S} = \{(Q_i, A_i)\}_{i \in I}$$

of the marked polytope (Q', A') , and has the form

$$\pm v_{A_1} \odot \cdots \odot v_{A_{|I|}}. \tag{4.10}$$

We claim that each cell (Q_i, A_i) is geometric. Since S is a subdivision of (Q', A') , we have

$$A_i = A' \cap Q_i.$$

Since (Q', A') is geometric, $A' = \tilde{A} \cap Q'$. Hence

$$A_i = A' \cap Q_i = (\tilde{A} \cap Q') \cap Q_i = \tilde{A} \cap Q_i,$$

because $Q_i \subset Q'$. Therefore each (Q_i, A_i) is geometric.

Consequently, every factor v_{A_i} in (4.10) is a geometric generator. Hence

$$d(V_{\text{geom}}) \subset S^\bullet(V_{\text{geom}}),$$

where $V_{\text{geom}} \subset V$ is the direct sum of $V_{A'}$ over geometric (Q', A') .

Dualizing and shifting, this implies that the coderivation defining the L_∞ -structure preserves the cofree coalgebra generated by $\mathfrak{g}[1] \subset \dot{\mathfrak{g}}[1]$, equivalently

$$\ell_n(\mathfrak{g}, \dots, \mathfrak{g}) \subset \mathfrak{g} \quad \text{for all } n \geq 1.$$

Thus $\mathfrak{g}$ is an L_∞ -subalgebra of $\dot{\mathfrak{g}}$. □

4.2 Construction for arbitrary lifted configurations

In the construction above, we chose the lifts of the points of A inside a fixed fundamental domain of the universal cover $\pi : \mathbb{C} \rightarrow E$. This choice is a convenient normalization, but it is not essential for the algebraic construction. More generally, one may choose an arbitrary lift

$$\tilde{A} = \{\tilde{a}_i\}_{i \in A} \subset \mathbb{C}$$

of the configuration A , allowing the points $\tilde{a}_i$ to lie in different fundamental domains.

For such a lifted configuration, the construction is performed upstairs in $\mathbb{C}$. Namely, we form the convex hull

$$\tilde{P} := \text{Conv}(\tilde{A}) \subset \mathbb{C}$$

and apply the usual construction of triangulations, secondary polytopes, and their factorization properties to the finite point configuration $(\tilde{P}, \tilde{A})$. This produces an L_∞ -algebra, which we denote by $\mathfrak{g}_{\tilde{A}}$. Unlike the case where $\tilde{P}$ projects injectively to E , the projection $\pi(\tilde{P})$ may now wrap around the elliptic curve or self-overlap. Thus one should not necessarily regard $\pi(\tilde{P}) \subset E$ as a polygonal region in the sense defined above. The point is rather that the secondary-polytopal construction only requires the affine point configuration $\tilde{A} \subset \mathbb{C}$.

The resulting L_∞ -algebra depends on the chosen lift $\tilde{A}$. If all points of $\tilde{A}$ are translated by the same deck transformation $\lambda \in \Lambda$, then

$$\tilde{A} + \lambda = \{\tilde{a}_i + \lambda\}_{i \in A}$$

is affinely isomorphic to $\tilde{A}$. Since the GKZ construction is invariant under affine translations, this gives a canonical identification

$$\mathfrak{g}_{\tilde{A}+\lambda} \simeq \mathfrak{g}_{\tilde{A}}.$$

On the other hand, changing the lift of only some of the points changes the relative positions of the lifted configuration. It may change the convex hull, the collection of triangulations, the associated secondary polytopes, and hence the L_∞ -algebra. Therefore the algebra $\mathfrak{g}_{\tilde{A}}$ should be regarded as attached not only to the configuration $A \subset E$, but to the additional choice of sheet data $\tilde{A}$.

Remark 4.11. *This dependence on the lift is important for the relation with Fukaya–Seidel categories. After choosing a lift of the regular value, a choice of lifts of the critical values determines a corresponding collection of lifted thimbles in the universal cover. Different choices of sheets may lead, after projection to the elliptic curve, to different directed collections. Thus the family of L_∞ -algebras*

$$\{\mathfrak{g}_{\tilde{A}}\}_{\tilde{A}}$$

as $\tilde{A}$ varies over possible lifts of A , should be viewed as encoding the algebraic models associated with these different lifted presentations of the Fukaya–Seidel category.

4.3 Examples in lower dimensions

We now give several low-dimensional examples of the L_∞ -algebras constructed above.

Example 3 (Three points). *Let*

$$B = \{a, b, c\} \subset E$$

be three distinct points, and choose lifts

$$\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$$

which form a nondegenerate triangle. Assume that there are no additional lifted points of $\tilde{A}$ inside

$$\text{Conv}(\tilde{a}, \tilde{b}, \tilde{c}).$$

Then the secondary polytope $\Sigma(B)$ is a point. Hence the corresponding generator

$$e_{abc} \in E_B \subset \mathfrak{g}$$

does not support any nontrivial higher operation.

Example 4 (Four points in convex position). *Let*

$$B = \{a, b, c, d\} \subset E$$

and suppose that we can choose lifts

$$\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}$$

which are in convex position. Then

$$\text{Conv}(\tilde{B})$$

is a quadrilateral. A diagonal gives a coarse regular subdivision into two triangles. For example, the diagonal $(\tilde{a}, \tilde{c})$ gives the two triangular cells

$$B_1 = \{a, b, c\}, \quad B_2 = \{a, c, d\}.$$

The corresponding codimension-one face of $\Sigma(B)$ contributes a term

$$\pm v_{abc} \odot v_{acd}$$

to the differential of v_{abcd} . After dualizing, this gives a binary operation

$$\ell_2(e_{abc}, e_{acd}) = \pm e_{abcd}.$$

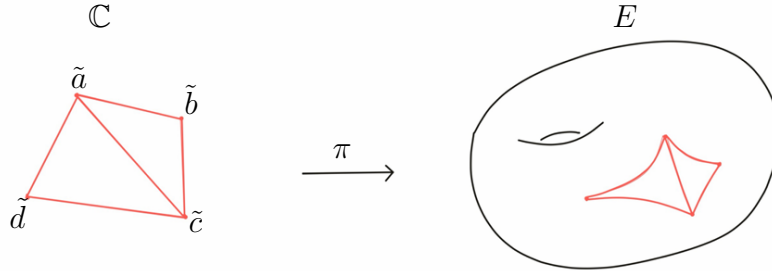


Figure 4.2: Four points in convex position.

Similarly, the other diagonal $(\tilde{b}, \tilde{d})$ gives

$$\ell_2(e_{abd}, e_{bcd}) = \pm e_{abcd}.$$

Example 5 (Triangle with one interior point). Let

$$B = \{a, b, c, d\} \subset E$$

and suppose that the chosen lifts satisfy

$$\tilde{d} \in \text{Int Conv}(\tilde{a}, \tilde{b}, \tilde{c}).$$

Thus $\tilde{a}, \tilde{b}, \tilde{c}$ are the vertices of a triangle, and $\tilde{d}$ is an interior point. The coarse regular subdivision into three triangles is

$$\{\tilde{a}, \tilde{b}, \tilde{d}\}, \quad \{\tilde{b}, \tilde{c}, \tilde{d}\}, \quad \{\tilde{c}, \tilde{a}, \tilde{d}\}.$$

Equivalently, writing

$$B_1 = \{a, b, d\}, \quad B_2 = \{b, c, d\}, \quad B_3 = \{c, a, d\},$$

the corresponding face of the secondary polytope contributes a term

$$\pm v_{abd} \odot v_{bcd} \odot v_{cad}$$

to the differential of v_{abcd} . Dualizing, we obtain a ternary operation

$$\ell_3(e_{abd}, e_{bcd}, e_{cad}) = \pm e_{abcd}.$$

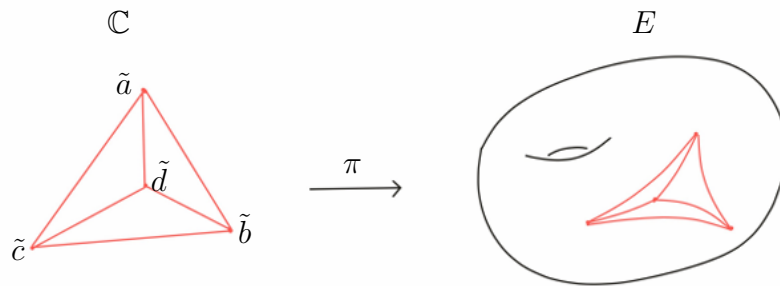


Figure 4.3: Triangle with one interior point.

Example 6 (General pattern). *Let*

$$B = \{a_1, \dots, a_m\} \subset E$$

and choose lifts

$$\tilde{B} = \{\tilde{a}_1, \dots, \tilde{a}_m\} \subset \mathbb{C}.$$

Suppose that $\text{Conv}(\tilde{B})$ admits a coarse regular subdivision into k maximal cells

$$B_1, \dots, B_k.$$

Then the corresponding codimension-one face of the secondary polytope $\Sigma(B)$ factors as

$$\prod_{i=1}^k \Sigma(B_i).$$

This contributes a term

$$\pm v_{B_1} \odot \dots \odot v_{B_k}$$

to the differential of v_B . After dualizing, this gives a potentially nonzero k -ary bracket

$$\ell_k(e_{B_1}, \dots, e_{B_k}) = \pm e_B.$$

Recall [12] that an L_∞ -algebra L is called nilpotent if there exists $r_0 > 0$ such that all r -ary iterated superpositions of the higher brackets vanish identically as maps

$$L^{\otimes r} \longrightarrow L$$

for all $r > r_0$.

Proposition 4.12. *The L_∞ -algebra $\mathfrak{g}_{\tilde{A}}$ is nilpotent.*

Proof. By construction, the matrix coefficients of the operation λ_n are indexed by coarse sub-

divisions of marked subpolytopes $(\tilde{Q}', \tilde{A}') \subset (\tilde{Q}, \tilde{A})$ into n marked subpolytopes. Therefore the matrix coefficients of an r -ary iterated superposition of the operations λ_n are indexed by subdivisions of marked subpolytopes of $(\tilde{Q}, \tilde{A})$ into r marked subpolytopes, not necessarily coarse.

Since $\tilde{A}$ is finite, there is a uniform bound on the number of nonempty marked subpolytopes which can occur in such a subdivision. In particular, no such subdivision exists for $r \geq |\tilde{A}|$. Hence all r -ary iterated superpositions vanish for $r \geq |\tilde{A}|$, and $\mathfrak{g}_{\tilde{A}}$ is nilpotent. $\square$

Chapter 5

Relative setting: A_∞ -algebra and L_∞ -morphism

In this chapter, we introduce an additional marked point $p \in E$, called the stop.¹ This extra datum allows us to refine the L_∞ -algebra to an A_∞ -algebra. The underlying graded vector space remains unchanged, while the operations are modified to reflect the presence of the stop.

Moreover, to the L_∞ -algebra and the A_∞ -algebra, we associate an L_∞ -morphism from the former to the derived derivation space of the latter. In this way, the L_∞ -structure acts by infinitesimal deformations of the relative A_∞ -algebra.

5.1 Construction of the A_∞ -algebra

We fix the following data. Choose a stop $p \in E$. As before, we fix lifts of the points of A to a single fundamental domain of the universal covering $\pi : \mathbb{C} \rightarrow E = \mathbb{C}/\Lambda$. We also choose a lift $\tilde{p} \in \mathbb{C}$ of the stop p . Thus, upstairs, we have the lifted configuration contained in one fundamental domain, together with the point $\tilde{p}$, which may lie in an arbitrary translate of that domain, see Figure 5.1.

¹The terminology is reminiscent of the role of stops in partially wrapped Fukaya categories; see [10, 33]. In the present paper, the stop is used as an ordering datum in the relative secondary-polytope construction.

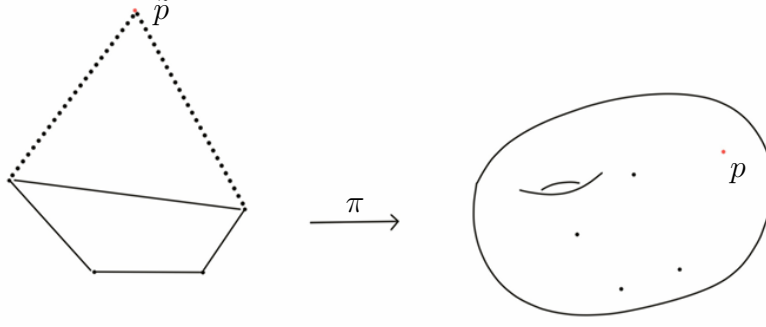


Figure 5.1: *Lifted configuration with $\tilde{p}$.*

Notation. Let $A \subset E$ be a finite point configuration, let $p \in E$ be the stop. We fix a lift $\tilde{p} \in \mathbb{C}$ of p . We denote by $\tilde{A}^\circ \subset \mathbb{C}$ the chosen set of lifts of the points of A , and we set $\tilde{A} := \tilde{A}^\circ \cup \{\tilde{p}\}$. Thus $\tilde{A}^\circ$ consists of the lifts of the original marked points, while $\tilde{A}$ denotes the full lifted configuration, including the lift of the stop.

We now introduce the basic combinatorial objects used in the A_∞ construction. Set $\tilde{Q} = \text{Conv}(\tilde{A}) \subset \mathbb{C}$. Let (Q', A') be a subpolygon of $(\tilde{Q}, \tilde{A})$. We say that (Q', A') is *rooted* if $\tilde{p} \in \text{Vert}(Q')$, and *unrooted* otherwise. The relative A_∞ -construction below is modeled on the relative two-dimensional construction of [19, Section 8].

Let $\tilde{\mathfrak{g}}$ be the L_∞ -algebra associated to the marked polygon $(\tilde{Q}, \tilde{A})$. By construction, $\tilde{\mathfrak{g}}$ admits a decomposition as a graded vector space. More precisely, we have

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}_{\text{root}},$$

where $\mathfrak{g}$ is spanned by the summands corresponding to unrooted subpolygons, and $\mathfrak{g}_{\text{root}}$ is spanned by those corresponding to rooted subpolygons.

Proposition 5.1. *With notation as above, both $\mathfrak{g}$ and $\mathfrak{g}_{\text{root}}$ are L_∞ -subalgebras of $\tilde{\mathfrak{g}}$. Moreover, $\mathfrak{g}_{\text{root}}$ is an L_∞ -ideal in $\tilde{\mathfrak{g}}$.*

Proof. The L_∞ -brackets on $\tilde{\mathfrak{g}}$ are defined by combining subpolygons $Q'_1, \dots, Q'_n$ into a coarser subdivision whose underlying marked polygon is denoted by Q' . The key point is that rootedness is preserved under this operation in the evident way.

If each Q'_i is unrooted, then none of them has $\tilde{p}$ as a vertex. Consequently, the resulting

polygon Q' is again unrooted. It follows that the higher brackets of elements of $\mathfrak{g}$ remain in $\mathfrak{g}$. Hence $\mathfrak{g}$ is an L_∞ -subalgebra of $\tilde{\mathfrak{g}}$.

On the other hand, if at least one of the Q'_i is $\tilde{p}$ -rooted, then the polygon obtained by combining them still has $\tilde{p}$ as a vertex, so Q' is rooted. Therefore, any higher bracket with at least one input in $\mathfrak{g}_{\text{root}}$ takes values in $\mathfrak{g}_{\text{root}}$. This shows that $\mathfrak{g}_{\text{root}}$ is an L_∞ -ideal in $\tilde{\mathfrak{g}}$. In particular, $\mathfrak{g}_{\text{root}}$ is itself an L_∞ -subalgebra. $\square$

We now explain how the rooted part $\mathfrak{g}_{\text{root}}$ can be refined from an L_∞ -algebra to an A_∞ -algebra. The essential additional input is the choice of the lift $\tilde{p}$ of the stop. We shall always choose the lift $\tilde{p}$ so that it is sufficiently far from the fundamental domain containing $\tilde{A}^\circ$. This separation ensures that the stop is distinguished from the original configuration and allows us to impose an ordering on the points as seen from $\tilde{p}$, for instance in the counterclockwise direction around $\tilde{p}$, see Figure 5.2. It is precisely this additional ordering data that enters the definition of the A_∞ -structure.

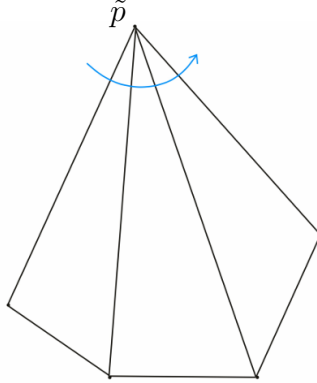


Figure 5.2: *The choice of $\tilde{p}$ breaks the cyclic symmetry and determines a linear order.*

For each subpolytope (Q', A') we set $V_{A'} := \text{or}(\Sigma(Q', A')) [\dim \Sigma(Q', A')]$, and define the graded vector space

$$V_r := \bigoplus_{(Q', A') \text{ rooted}} V_{A'}. \quad (5.2)$$

A *rooted subdivision* of the rooted polygon (Q', A') is a collection of rooted subpolygons $(Q'_1, A'_1), \dots, (Q'_k, A'_k)$ such that $Q' = Q'_1 \cup \dots \cup Q'_k$, the intersection $Q'_i \cap Q'_j$ is a common face (possibly empty), and each Q'_i contains the same root vertex $\tilde{p}$.

The ordering of points in $\tilde{A}^\circ$ extends to rooted subpolygons: each rooted subpolygon comes equipped with an ordering of its non-root vertices. For every rooted subdivision

$$(Q', A') \rightsquigarrow ((Q'_1, A'_1), \dots, (Q'_k, A'_k)),$$

we produce an ordered list of rooted subpolygons.

This ordering allows us to lift the differential d from the free commutative algebra $S^\bullet(V_r)$ generated by V_r to a differential on the tensor algebra $T^\bullet(V_r)$, viewed as the free associative algebra generated by V_r . More precisely, let $P'' = \{(Q''_\nu, A''_\nu)\}_\nu$ be a coarse rooted subdivision of a rooted marked polygon (Q', A') . Using the ordered list of rooted subpolygons determined above, we define

$$V_{P''} := \bigotimes_{\nu} V_{A''_\nu} \subset T^\bullet(V_r),$$

where the embedding is given by tensor multiplication in the prescribed order.

We then define the action of d on the generators of $T^\bullet(V_r)$ as follows. For a summand $V_{A'} \subset V_r$, we let d be given by the top-degree part of the cellular chain differential in $C_\bullet(\Sigma(A'))$, now regarded as a map

$$d = \sum_{P''} d_{P''}, \quad d_{P''}: V_{A'} \longrightarrow V_{P''} \subset T^\bullet(V_r),$$

where the sum ranges over all coarse rooted subdivisions P'' of (Q', A') , and where each map $d_{P''}$ is induced by the corresponding component of the chain differential in $C_\bullet(\Sigma(A'))$. Finally, we extend d to the whole tensor algebra $T^\bullet(V_r)$ by the Leibniz rule and get the following proposition.

Proposition 5.3. *The differential d defined above satisfies $d^2 = 0$. In particular, $(T(V_r), d)$ is a differential graded algebra.*

Define

$$\dot{R}_{\tilde{p}} := V_r^*[-1] = \bigoplus_{(Q', A') \text{ rooted}} E_{A'}, \quad (5.4)$$

where

$$E_{A'} := \text{or}(\Sigma(Q', A'))^* [-1 - \dim \Sigma(Q', A')].$$

This determines an A_∞ -algebra structure on $\dot{R}_{\tilde{p}}$. Then we restrict to the *geometric* summands. Define the graded subspace

$$R_{\tilde{p}} := \bigoplus_{(Q', A') \text{ geom}} E_{A'} \subset \dot{R}_{\tilde{p}}. \quad (5.5)$$

We claim that $R_{\tilde{p}}$ is an A_∞ -subalgebra of $\dot{R}_{\tilde{p}}$.

Proposition 5.6. *The graded vector space $R_{\tilde{p}}$ defined in (5.5) carries a natural A_∞ -algebra structure, obtained by restricting the A_∞ structure on $\dot{R}_{\tilde{p}}$.*

Proof. Recall that the A_∞ structure maps

$$m_k : (\dot{R}_{\tilde{p}})^{\otimes k} \longrightarrow \dot{R}_{\tilde{p}}$$

are obtained by dualizing the tensor differential d on $T(V_r)$. A nonzero contribution to

$$m_k(E_{A'_1}, \dots, E_{A'_k})$$

arises precisely from a coarse rooted regular subdivision

$$(Q', A') \rightsquigarrow ((Q'_1, A'_1), \dots, (Q'_k, A'_k)),$$

where each (Q'_i, A'_i) appears as an input cell and (Q', A') is the output cell.

If all input cells (Q'_i, A'_i) are geometric, then the output cell (Q', A') is also geometric. Indeed, every marked point of $\tilde{A} \cap Q'$ lies in at least one cell Q'_i . Since that cell is geometric, the point belongs to A'_i , hence to A' . Therefore

$$A' = \tilde{A} \cap Q',$$

so the output cell is geometric. Consequently, the output summand $E_{A'}$ lies in $R_{\tilde{p}}$. This shows that the collection $\{E_{A'}\}_{(Q', A') \in \mathcal{R}_{\text{geom}}}$ is closed under all A_∞ operations.

By the discussion above, the A_∞ structure maps m_k preserve the subspace $R_{\tilde{p}}$. Hence they restrict to maps

$$m_k : R_{\tilde{p}}^{\otimes k} \longrightarrow R_{\tilde{p}},$$

satisfying the A_∞ relations inherited from $\dot{R}_{\tilde{p}}$. This endows $R_{\tilde{p}}$ with the structure of an A_∞ -subalgebra. $\square$

Proposition 5.7. *The A_∞ -algebra $R_{\tilde{p}}$ is a graded associative algebra. More precisely,*

$$m_k|_{R_{\tilde{p}}^{\otimes k}} = 0 \quad \text{for all } k \neq 2,$$

and the only nonzero operation is the binary product

$$m_2 : R_{\tilde{p}} \otimes R_{\tilde{p}} \longrightarrow R_{\tilde{p}}.$$

Proof. The A_∞ -operations on $\dot{R}_{\tilde{p}}$ are obtained by dualizing the tensor differential on $T^\bullet(V_r)$. A nonzero contribution to m_k is indexed by a coarse rooted subdivision of a rooted marked polygon into k rooted marked subpolygons.

In dimension two, a coarse rooted subdivision into rooted subpolygons has only two possible types. This is the same dichotomy of coarse subdivisions used in the relative two-dimensional construction of [19, Section 8]. Such a subdivision may have two maximal rooted pieces; these subdivisions give the binary product m_2 . Alternatively, it may have one piece, corresponding to the operation of removing an internal marked point from the marking; these subdivisions give the unary operation m_1 in the larger algebra $\dot{R}_{\tilde{p}}$.

However, the one-piece subdivisions do not preserve the geometric subspace. Indeed, if (Q', A') is geometric, so that $A' = \tilde{A} \cap Q'$, then removing an internal marked point produces a marked polygon $(Q', A' \setminus \{a\})$, which is no longer geometric. Hence the unary operation m_1 vanishes after restricting to the geometric subalgebra $R_{\tilde{p}}$.

Therefore, on $R_{\tilde{p}}$, all operations m_k with $k \neq 2$ vanish. The A_∞ -relations then reduce to the associativity of m_2 . Hence $R_{\tilde{p}}$ is a graded associative algebra. $\square$

We can therefore describe the product on $R_{\tilde{p}}$ explicitly, see Figure 5.3. Let (Q_1, A_1) and (Q_2, A_2) be geometric rooted marked subpolygons, and let

$$E_{A_1}, E_{A_2} \subset R_{\tilde{p}}$$

be the corresponding summands. The product

$$m_2 : E_{A_1} \otimes E_{A_2} \longrightarrow R_{\tilde{p}}$$

is zero unless the following conditions hold:

1. $Q_1 \cup Q_2$ is again a geometric marked subpolygon;
2. $Q_1 \cap Q_2$ is a common boundary edge;
3. with respect to the orientation determined by the stop $\tilde{p}$, the polygon Q_1 lies on the left of Q_2 .

If these conditions hold, then $Q := Q_1 \cup Q_2$ is a convex geometric marked subpolygon, with marked set

$$A_Q = A_1 \cup A_2,$$

and the product is the canonical gluing map

$$m_2 : E_{A_1} \otimes E_{A_2} \longrightarrow E_{A_Q}$$

associated to the coarse subdivision

$$(Q, A_Q) \rightsquigarrow ((Q_1, A_1), (Q_2, A_2)).$$

Equivalently, for basis elements we may write

$$E_{A_1} \cdot E_{A_2} = \begin{cases} \pm E_{A_1 \cup A_2}, & \text{if } Q_1 \cup Q_2 \text{ is geometric and } Q_1 \text{ lies on the left of } Q_2, \\ 0, & \text{otherwise.} \end{cases}$$

The sign is the one induced by the orientation convention for the corresponding coarse subdivision.

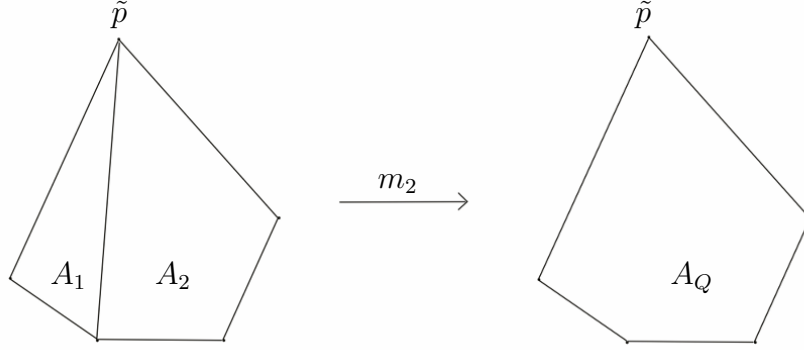


Figure 5.3: m_2 in $R_{\tilde{p}}$.

5.2 L_∞ -morphism

Now we explain how the subdivision calculus produces a natural L_∞ -morphism from the L_∞ -algebra of unrooted polygons to the dg Lie algebra of derived derivations of the A_∞ -algebra $R_{\tilde{p}}$ constructed above.

Consider all coarse subdivisions of all marked subpolygons of $(\tilde{Q}, \tilde{A})$, both unrooted and rooted. Collecting all coarse-subdivision contributions gives a square-zero derivation

$$d: S^\bullet(V) \otimes T^\bullet(V_r) \longrightarrow S^\bullet(V) \otimes T^\bullet(V_r).$$

By construction, d preserves the subalgebra $S^\bullet(V) \otimes 1$ and the two augmentation ideals

$$S_+^\bullet(V) \otimes T^\bullet(V_r), \quad S^\bullet(V) \otimes T_+^\bullet(V_r).$$

Therefore Proposition ?? produces an L_∞ -morphism

$$\phi: \mathfrak{g} \longrightarrow C^{\geq 1}(R_{\tilde{p}}, R_{\tilde{p}})[1].$$

5.3 Dependence on the lift of the stop and chamber structure

The A_∞ -algebra constructed above, as well as the induced L_∞ -morphism to its derived derivation space, depends on the choice of a lift $\tilde{p} \in \mathbb{C}$ of the stop $p \in E$. Let $R_{\tilde{p}}$ denote the corresponding A_∞ -algebra. We now explain how this dependence is organized by the deck group of the universal covering and how it leads naturally to a chamber structure.

Fix once and for all a lift $\tilde{A}^\circ \subset \mathbb{C}$ of the configuration A , contained in a chosen fundamental domain. The set of lifts of the stop is then

$$\pi^{-1}(p) = \tilde{p} + \Lambda, \quad \Lambda \cong \pi_1(E),$$

so that the deck group acts freely and transitively on the possible choices of $\tilde{p}$.

Although the set of lifts is discrete, when $\tilde{p}$ is taken far away from the fixed domain containing $\tilde{A}^\circ$, the relevant ordering data is determined only by the asymptotic direction of $\tilde{p}$. More precisely, after choosing a base point c in the fixed domain, one associates to $\tilde{p}$ the unit vector

$$u(\tilde{p}) := \frac{\tilde{p} - c}{\|\tilde{p} - c\|} \in S^1.$$

For $\tilde{p}$ sufficiently distant, the induced order on $\tilde{A}$ depends only on $u(\tilde{p})$, and is locally constant as a function of this direction.

The wall set is

$$\mathcal{W} := \left\{ \pm \frac{\tilde{a}_i - \tilde{a}_j}{\|\tilde{a}_i - \tilde{a}_j\|} \mid \tilde{a}_i, \tilde{a}_j \in \tilde{A}^\circ, i \neq j \right\} \subset S^1,$$

and a chamber is a connected component of $S^1 \setminus \mathcal{W}$.

The order changes only when $u(\tilde{p})$ crosses one of finitely many walls. These walls are determined by pairs of points of $\tilde{A}$: they are the directions for which two points of $\tilde{A}$ become aligned as seen from infinity, equivalently the directions parallel to differences $\tilde{a}_i - \tilde{a}_j$. Thus S^1 is decomposed into finitely many open chambers, and the induced ordering on $\tilde{A}$ is constant on each chamber.

It follows that the rooted A_∞ -algebra and the associated L_∞ -morphism are constant within a fixed chamber. More precisely, if $\tilde{p}$ and $\tilde{p}'$ determine the same chamber, then they induce the same ordering on all rooted subpolygons, hence the same ordered tensor embeddings used in the definition of the algebra differential on

$$S^\bullet(V) \otimes T^\bullet(V_r).$$

Therefore the corresponding A_∞ -algebras are naturally identified after identifying the ordered combinatorial data, and under this identification the associated L_∞ -morphisms agree.

When $\tilde{p}$ crosses a wall, the induced order changes, typically by an adjacent transposition. As a result, the corresponding ordered tensor decompositions of rooted subpolygons change as well. Hence the resulting A_∞ -algebra need not remain strictly isomorphic to the one on the other side of the wall, and the associated L_∞ -morphism also changes. In this way, the family of rooted A_∞ -algebras obtained from different lifts of p is naturally organized by a wall-crossing structure on the circle of directions at infinity.

From this point of view, the deck action of $\pi_1(E)$ on the set of lifts of the stop does not simply act by automorphisms of a single fixed A_∞ -algebra. Rather, it produces a collection of A_∞ -algebras, together with corresponding L_∞ -morphisms, indexed by chambers of asymptotic directions. The passage from one chamber to another should be understood as a wall-crossing transformation.

Proposition 5.8. *Let $C \subset S^1 \setminus \mathcal{W}$ be a chamber. For any two sufficiently distant lifts $\tilde{p}$ and $\tilde{p}'$ of the stop p , let*

$$(\tilde{Q}, \tilde{A}), \quad (\tilde{Q}', \tilde{A}')$$

be the corresponding marked polygons, where $\tilde{A}$ and $\tilde{A}'$ include $\tilde{p}$ and $\tilde{p}'$, respectively. If

$$u(\tilde{p}), u(\tilde{p}') \in C,$$

then the induced counterclockwise orderings on $\tilde{A}^\circ$ and $(\tilde{A}')^\circ$, and more generally on the non-root vertices of every rooted subpolygon of $(\tilde{Q}, \tilde{A})$ and $(\tilde{Q}', \tilde{A}')$, coincide, see Figure 5.4. Consequently, the corresponding rooted A_∞ -algebras $R_{\tilde{p}}$ and $R_{\tilde{p}'}$ are naturally isomorphic after identifying the ordered combinatorial data, and the associated L_∞ -morphisms

$$\Phi_{\tilde{p}}: \mathfrak{g} \longrightarrow C^{\geq 1}(R_{\tilde{p}}, R_{\tilde{p}})[1]$$

and

$$\Phi_{\tilde{p}'}: \mathfrak{g} \longrightarrow C^{\geq 1}(R_{\tilde{p}'}, R_{\tilde{p}'})[1]$$

also agree.

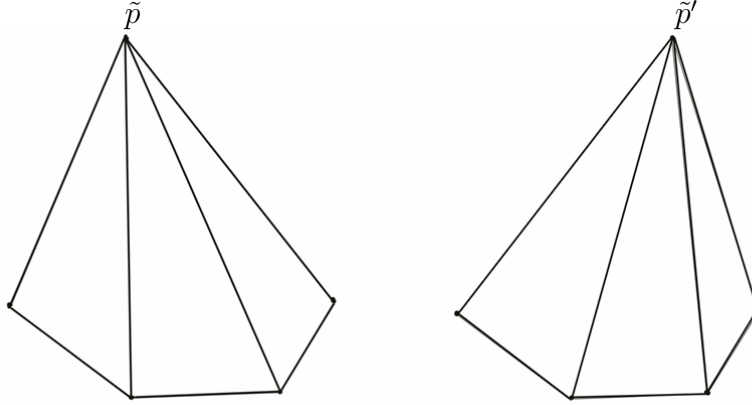


Figure 5.4: Different lifts of p .

Proof. For $\tilde{p}$ sufficiently far from the fixed domain containing the lifts of the original points A , the counterclockwise order on $\tilde{A}^\circ$ as seen from $\tilde{p}$ depends only on the direction $u(\tilde{p})$. This order changes exactly when two points of $\tilde{A}^\circ$ become aligned as viewed from infinity, that is, when $u(\tilde{p})$ is parallel to $\tilde{a}_i - \tilde{a}_j$ for some $i \neq j$. These exceptional directions are precisely the walls in $\mathcal{W}$. Hence the induced order is constant on each chamber C .

The same holds for every rooted subpolygon, since its ordering is induced from the order

on the non-root vertices. Therefore, whenever $u(\tilde{p})$ and $u(\tilde{p}')$ belong to the same chamber, all ordered tensor embeddings used in the definition of the algebra differential on

$$S^\bullet(V) \otimes T^\bullet(V_r)$$

are the same for $(\tilde{Q}, \tilde{A})$ and $(\tilde{Q}', \tilde{A}')$. It follows that the resulting rooted A_∞ -algebras are naturally identified after identifying the ordered combinatorial data. Under this identification, the associated L_∞ -morphisms agree. $\square$

For later use, we introduce notation for the chamber determined by a fixed ordering of the lifted configuration. Let $S = (\tilde{a}_1, \dots, \tilde{a}_n)$ be an ordered collection of points in the fixed lifted configuration $\tilde{A}^\circ$. We say that a chamber

$$\mathfrak{C} \subset S^1 \setminus \mathcal{W}$$

is *compatible with S* if, for any sufficiently distant lift $\tilde{p}$ with $u(\tilde{p}) \in \mathfrak{C}$, the order on $\tilde{A}^\circ$ induced by viewing the points from $\tilde{p}$ agrees with the fixed order $\tilde{a}_1 < \tilde{a}_2 < \dots < \tilde{a}_n$. When such a compatible chamber is fixed, we denote it by $\mathfrak{C}_S$.

The chamber $\mathfrak{C}_S$ determines the rooted directed A_∞ -algebra constructed above. To emphasize the dependence on this choice of chamber, we write

$$R_{\mathfrak{C}_S}$$

for the corresponding chamberwise A_∞ -algebra. Equivalently, if $\tilde{p}$ is any sufficiently distant lift with $u(\tilde{p}) \in \mathfrak{C}_S$, then

$$R_{C_S} := R_{\tilde{p}}.$$

By Proposition 5.8, this notation is independent of the choice of such $\tilde{p}$ inside the chamber.

Similarly, we denote the associated chamberwise L_∞ -morphism by

$$\Phi_{\mathfrak{C}_S} : \mathfrak{g} \longrightarrow R\mathrm{Der}(R_{\mathfrak{C}_S}).$$

Chapter 6

Algebras with coefficients

We introduce a coefficient system on the lifted marked polygon and use it to refine the L_∞ -algebra and A_∞ -algebra constructed above.

6.1 Coefficient systems and factorization sheaves

Fix a stop $p \in E$, choose a sufficiently distant lift $\tilde{p}$, and let $(\tilde{Q}, \tilde{A})$ denote the corresponding lifted marked polygon, where $\tilde{A}$ now includes $\tilde{p}$. We assume throughout that $\tilde{Q}$ is endowed with the orientation induced from the standard orientation of $\mathbb{C} \cong \mathbb{R}^2$.

A *system of coefficients* on $(\tilde{Q}, \tilde{A})$ consists of the following data: for each oriented geodesic edge $\sigma = [\tilde{a}_i, \tilde{a}_j]$ with endpoints in $\tilde{A}$, a cochain complex N_σ , together with an identification $N_{\bar{\sigma}} \simeq N_\sigma^*$, where $\bar{\sigma}$ denotes the same edge with the opposite orientation.

For every unrooted marked subpolygon $(\tilde{Q}', \tilde{A}') \subset (\tilde{Q}, \tilde{A})$, we define

$$N_{\tilde{A}'} := \bigotimes_{\sigma \subset \partial \tilde{Q}'} N_\sigma,$$

where the tensor product runs over the oriented boundary edges of $\tilde{Q}'$, taken in the induced boundary orientation.

More generally, if $\mathcal{P} = \{(\tilde{Q}_\nu, \tilde{A}_\nu)\}$ is a coarse subdivision of $(\tilde{Q}, \tilde{A})$ into marked sub-

polygons, we set

$$N_{\mathcal{P}} := \bigotimes_{\nu} N_{\tilde{A}_{\nu}}.$$

If $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then every internal edge of $\mathcal{P}'$ appears twice with opposite orientations. Using the duality $N_{\tilde{\sigma}} \simeq N_{\tilde{\sigma}}^*$ and the evaluation pairing $N_{\tilde{\sigma}} \otimes N_{\tilde{\sigma}} \longrightarrow k$, we obtain a morphism of cochain complexes

$$\gamma_{\mathcal{P}'\mathcal{P}}: N_{\mathcal{P}'} \longrightarrow N_{\mathcal{P}}.$$

These maps are transitive with respect to chains of refinements. Consequently, the complexes $N_{\mathcal{P}}$, together with the maps $\gamma_{\mathcal{P}'\mathcal{P}}$, define a constructible complex of sheaves on the secondary polytope $\Sigma(\tilde{A}')$, constant on each open face corresponding to a coarse subdivision. We denote this sheaf by $\mathcal{N}_{\tilde{A}'}$.

The sheaves $\mathcal{N}_{\tilde{A}'}$ satisfy the same factorization property as in the classical situation. Namely, if $\mathcal{P} = \{(\tilde{Q}_{\nu}, \tilde{A}_{\nu})\}$ is a coarse subdivision of $(\tilde{Q}', \tilde{A}')$, then the corresponding face of $\Sigma(\tilde{A}')$ is identified with the product

$$F_{\mathcal{P}} \simeq \prod_{\nu} \Sigma(\tilde{A}_{\nu}),$$

and the restriction of $\mathcal{N}_{\tilde{A}'}$ to this face is identified with the exterior tensor product

$$\mathcal{N}_{\tilde{A}'}|_{F_{\mathcal{P}}} \simeq \boxtimes_{\nu} \mathcal{N}_{\tilde{A}_{\nu}}.$$

Thus the collection of sheaves $\{\mathcal{N}_{\tilde{A}'}\}$ forms a factorizing system on the family of secondary polytopes attached to marked subpolygons of $(\tilde{Q}, \tilde{A})$.

We now form the cellular cochain complexes of the factorizing sheaves $\mathcal{N}_{\tilde{A}'}$. For each unrooted marked subpolygon $(\tilde{Q}', \tilde{A}') \subset (\tilde{Q}, \tilde{A})$, let

$$E_{\tilde{A}'} := N_{\tilde{A}'} \otimes \text{or}(\Sigma(\tilde{A}'))[-\dim \Sigma(\tilde{A}') - 1].$$

Define

$$\mathfrak{g}_{\mathcal{N}}^{\bullet} := \bigoplus_{(\tilde{Q}', \tilde{A}') \subset (\tilde{Q}, \tilde{A})} E_{\tilde{A}'}.$$

The cellular cochain differentials of the sheaves $\mathcal{N}_{\tilde{A}'}$, together with the factorization property above, assemble into a coderivation

$$d: S^{\bullet}(\mathfrak{g}_{\mathcal{N}}^{\bullet}[1]) \longrightarrow S^{\bullet}(\mathfrak{g}_{\mathcal{N}}^{\bullet}[1]), \quad d^2 = 0.$$

Equivalently, $\mathfrak{g}_{\mathcal{N}}^{\bullet}$ carries a natural L_{∞} -algebra structure.

As in the coefficient-free case, one may restrict to those summands indexed by geometric unrooted subpolygons of $(\tilde{Q}, \tilde{A})$. We denote the resulting L_{∞} -subalgebra by

$$\mathfrak{g}_{\mathcal{N}} \subset \mathfrak{g}_{\mathcal{N}}^{\bullet}.$$

The degree-one component of $\mathfrak{g}_{\mathcal{N}}$ is therefore

$$\mathfrak{g}_{\mathcal{N}}^1 = \bigoplus_{(\tilde{Q}', \tilde{A}') \subset (\tilde{Q}, \tilde{A})} N_{\tilde{A}'}^{-\dim \Sigma(\tilde{A}')} \otimes \text{or}(\Sigma(\tilde{A}')).$$

In particular, a Maurer–Cartan element in $\mathfrak{g}_{\mathcal{N}}$ assigns to each marked triangle an element of $N_{\tilde{A}'}^0$, to each circuit an element of $N_{\tilde{A}'}^{-1}$, and so on.

We now pass to the relative setting. Let $\mathfrak{g}_{\tilde{\mathcal{N}}}$ be the L_{∞} -algebra associated to all rooted and unrooted subpolygons of $(\tilde{Q}, \tilde{A})$. The decomposition into rooted and unrooted marked subpolygons induces a decomposition

$$\mathfrak{g}_{\tilde{\mathcal{N}}} = \mathfrak{g}_{\mathcal{N}} \ltimes \mathfrak{g}_{\mathcal{N}, \text{root}},$$

where $\mathfrak{g}_{\mathcal{N}}$ is spanned by the summands corresponding to unrooted subpolygons and $\mathfrak{g}_{\mathcal{N}, \text{root}}$ is spanned by the summands corresponding to rooted subpolygons. Exactly as in the coefficient-free case, $\mathfrak{g}_{\mathcal{N}, \text{root}}$ is an L_{∞} -ideal.

The choice of a sufficiently distant lift $\tilde{p}$ induces a counterclockwise ordering on the non-root vertices of every rooted subpolygon. This allows one to lift the rooted part from an L_∞ -algebra to an A_∞ -algebra. We denote the resulting A_∞ -algebra by $R_{N,\text{root}}$.

Looking at all coarse subdivisions of all marked subpolygons of $(\tilde{Q}, \tilde{A})$, both unrooted and rooted, we obtain a square-zero algebra derivation

$$d: S^\bullet(V_N) \otimes T^\bullet(V_{N,\text{root}}) \longrightarrow S^\bullet(V_N) \otimes T^\bullet(V_{N,\text{root}}).$$

By construction, d preserves the subalgebra

$$S^\bullet(V_N) \otimes 1$$

and the two augmentation ideals

$$S_+^\bullet(V_N) \otimes T^\bullet(V_{N,\text{root}})$$

and

$$S^\bullet(V_N) \otimes T_+^\bullet(V_{N,\text{root}}).$$

Its restriction to $S^\bullet(V_N) \otimes 1$ is precisely the differential defining $\mathfrak{g}_N$, while its component of symmetric degree zero determines the A_∞ -structure on $R_{N,\text{root}}$. Therefore, by Proposition ??, d determines an L_∞ -morphism

$$\Phi_N: \mathfrak{g}_N \longrightarrow \text{RDer}_{\text{Ass}}(R_{N,\text{root}}).$$

This is the coefficient-enhanced version of the L_∞ -morphism constructed in the previous chapter.

6.2 Bimodule Coefficients

Now we enrich the coefficient-free constructions of the previous chapters by allowing dg-algebras at vertices and bimodules along edges.

Let us denote the elements of $\tilde{A}$ by $i, j, k, \dots$

Definition 6.1. *An extended system of coefficients on the configuration $A \subset E$ consists of the following data:*

- (1) *For each $i \in A$, an associative dg-algebra S_i .*
- (2) *For each ordered pair (i, j) with $i, j \in A$, a differential graded (S_i, S_j) -bimodule N_{ij} , assumed projective of finite rank over the graded algebra underlying $S_i \otimes_{\mathbb{k}} S_j^{op}$.*
- (3) *For each pair (i, j) , a pairing*

$$\beta_{ij} : N_{ij} \otimes_{\mathbb{k}} N_{ji} \longrightarrow S_i \otimes_{\mathbb{k}} S_j$$

which is a morphism of $(S_i \otimes_{\mathbb{k}} S_j, S_i \otimes_{\mathbb{k}} S_j)$ -bimodules.

- (4) (Non-degeneracy) *The induced morphism of (S_i, S_j) -bimodules*

$$\beta_{ij}^t : N_{ij} \longrightarrow \text{Hom}_{S_j \otimes_{\mathbb{k}} S_i^{op}}(N_{ji}, S_j \otimes_{\mathbb{k}} S_i^{op})$$

is an isomorphism.

- (5) (Symmetry) *The diagram*

$$\begin{array}{ccc} N_{ij} \otimes_{\mathbb{k}} N_{ji} & \xrightarrow{\beta_{ij}} & S_i \otimes_{\mathbb{k}} S_j \\ \text{perm} \downarrow & & \downarrow \text{perm} \\ N_{ji} \otimes_{\mathbb{k}} N_{ij} & \xrightarrow{\beta_{ji}} & S_j \otimes_{\mathbb{k}} S_i \end{array}$$

is commutative, where perm denotes the permutation of tensor factors.

Remark 6.2. If all $S_i = \mathbb{k}$ and all N_{ij} are one-dimensional with β_{ij} the standard evaluation pairing, then we recover the coefficient-free setting.

Let $(Q', A') \subset (\tilde{Q}, \tilde{A})$ be an unrooted marked subpolygon. Write its boundary vertices in counterclockwise order as $i_0, i_1, \dots, i_m$. Define first the *linear tensor product*

$$L_{A'} := N_{i_0 i_1} \otimes_{S_{i_1}} N_{i_1 i_2} \otimes_{S_{i_2}} \cdots \otimes_{S_{i_m}} N_{i_{m-1} i_m}, \quad (6.3)$$

which is an (S_{i_0}, S_{i_m}) -bimodule.

Then define the *cyclic tensor product*

$$N_{A'} := L_{A'} \otimes_{S_{i_0}^{op} \otimes_{\mathbb{k}} S_{i_m}} N_{i_m i_0}. \quad (6.4)$$

Let $\sigma = (\dots, h, i, j, k, \dots)$ and $\tau = (\dots, p, j, i, q, \dots)$ be two closed oriented edge paths (cyclic words in vertices of A') which share the opposite oriented edges (i, j) in σ and (j, i) in τ . Concatenating along $[i, j]$ means erasing these two edges and forming a new closed path $\sigma *_{[i, j]} \tau$.

The pairing β_{ij} produces a *concatenation map*

$$\gamma_{\sigma, \tau}^{[i, j]} : N_{\sigma} \otimes_{\mathbb{k}} N_{\tau} \longrightarrow N_{\sigma *_{[i, j]} \tau}, \quad (6.5)$$

defined on decomposable tensors as follows, see Figure 6.1.

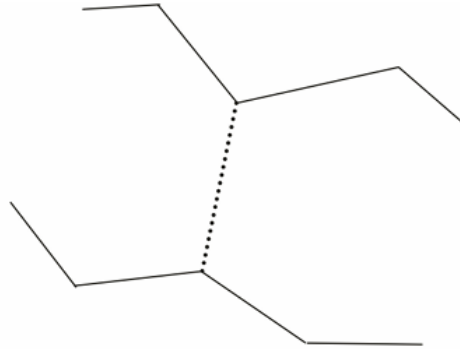


Figure 6.1: The concatenation map.

Write

$$n_\sigma = \cdots \otimes n_{hi} \otimes n_{ij} \otimes n_{jk} \otimes \cdots \in N_\sigma, \quad n_\tau = \cdots \otimes n_{pj} \otimes n_{ji} \otimes n_{iq} \otimes \cdots \in N_\tau,$$

and expand

$$\beta_{ij}(n_{ij} \otimes n_{ji}) = \sum_{\nu} s'_\nu \otimes s''_\nu, \quad s'_\nu \in S_i, \quad s''_\nu \in S_j.$$

Then

$$\gamma_{\sigma,\tau}^{[i,j]}(n_\sigma \otimes n_\tau) = \sum_{\nu} (\cdots \otimes (n_{pj} \cdot s''_\nu) \otimes n_{jk} \otimes \cdots \otimes (n_{hi} \cdot s'_\nu) \otimes n_{iq} \otimes \cdots), \quad (6.6)$$

where the dots indicate the unchanged tensor factors, and the multiplications are the right S_j -action on N_{pj} and the right S_i -action on N_{hi} (respectively; equivalently one may rewrite using the left actions depending on conventions). The symmetry axiom in Definition 6.1 ensures this is compatible with swapping the roles of (i, j) and (j, i) .

Now let

$$\mathcal{P} = \{(Q''_\nu, A''_\nu)\}$$

be a subdivision of a marked subpolygon (Q', A') . We define

$$N_{\mathcal{P}} := \bigotimes_{\nu} N_{A''_\nu}.$$

Each internal edge of the subdivision gives rise to a concatenation map of the form (6.5), and these maps commute with each other (so the order of applying them does not matter). Applying all of them yields a canonical *composition map*

$$\gamma_{\mathcal{P}} : N_{\mathcal{P}} \longrightarrow N_{A'}. \quad (6.7)$$

Whenever $\mathcal{P}'$ refines $\mathcal{P}$, the pairings β_{ij} along the intermediate edges define contraction maps

$$\gamma_{\mathcal{P}'\mathcal{P}} : N_{\mathcal{P}'} \longrightarrow N_{\mathcal{P}}.$$

These maps are compatible with iterated refinements and therefore define a constructible complex of sheaves $\mathcal{N}_{A'}$ on the secondary polytope $\Sigma(A')$, whose stalk at the face corresponding to a subdivision $\mathcal{P}$ is $N_{\mathcal{P}}$ and whose restriction maps are the maps $\gamma_{\mathcal{P}'\mathcal{P}}$ constructed from the pairings β_{ij} .

For each unrooted marked subpolygon (Q', A') , let

$$E_{A'} := N_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1].$$

In relative setting, for each rooted marked subpolygon (Q', A') , with vertices $\tilde{p}, i_0, i_1, \dots, i_m$ in counterclockwise order, we define similarly

$$F_{A'} := L_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1].$$

We then set

$$\mathfrak{g}_{\mathcal{N}} := \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{unrooted}}} E_{A'}, \quad R_{\mathcal{N}, \text{root}} := \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{rooted}}} F_{A'}.$$

The factorization maps $\gamma_{P'P}$, together with the cellular differentials of the secondary polytopes, assemble into a square-zero algebra derivation

$$d: S^\bullet(\mathfrak{g}_{\mathcal{N}}[1]) \otimes T^\bullet(R_{\mathcal{N}, \text{root}}[1]) \longrightarrow S^\bullet(\mathfrak{g}_{\mathcal{N}}[1]) \otimes T^\bullet(R_{\mathcal{N}, \text{root}}[1]).$$

By construction, d preserves the subalgebra

$$S^\bullet(\mathfrak{g}_{\mathcal{N}}[1]) \otimes 1$$

and the two augmentation ideals

$$S_+^\bullet(\mathfrak{g}_{\mathcal{N}}[1]) \otimes T^\bullet(R_{\mathcal{N}, \text{root}}[1])$$

and

$$S^\bullet(\mathfrak{g}_N[1]) \otimes T_+^\bullet(R_{N,\text{root}}[1]).$$

Its restriction to the symmetric subalgebra defines the L_∞ -structure on $\mathfrak{g}_N$, while its component of symmetric degree zero defines the A_∞ -structure on $R_{N,\text{root}}$. The remaining positive-symmetric-degree components determine, by Proposition ??, an L_∞ -morphism

$$\Phi_N : \mathfrak{g}_N \longrightarrow \text{RDer}_{\text{Ass}}(R_{N,\text{root}}).$$

Proposition 6.8. *The extended system of coefficients on $(\tilde{Q}, \tilde{A})$ gives rise to:*

1. *an L_∞ -algebra*

$$\mathfrak{g}_{\mathcal{N}} = \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{unrooted}}} E_{A'};$$

2. *an A_∞ -algebra*

$$R_{\mathcal{N},\text{root}} = \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{rooted}}} F_{A'};$$

3. *an L_∞ -morphism*

$$\Phi_{\mathcal{N}} : \mathfrak{g}_{\mathcal{N}} \longrightarrow \text{RDer}_{\text{Ass}}(R_{\mathcal{N},\text{root}}).$$

Proof. The factorization maps associated to refinements of coarse subdivisions define a differential on

$$S^\bullet(\mathfrak{g}_{\mathcal{N}}[1]) \otimes T^\bullet(R_{\mathcal{N},\text{root}}[1]),$$

whose restriction to the symmetric factor gives the L_∞ -structure on $\mathfrak{g}_{\mathcal{N}}$, while the rooted tensor factor yields the A_∞ -structure on $R_{\mathcal{N},\text{root}}$. The resulting mixed differential then determines the L_∞ -morphism to the derived derivation space by the associative version of the formalism recalled in Proposition ??. $\square$

When we introduce the coefficient system, the dependence of the A_∞ -algebra on the choice of lift of the stop is also naturally organized by the fundamental group of E . For $\tilde{p}$

chosen sufficiently distant, the A_∞ -algebra and the associated L_∞ -morphism depend only on the asymptotic direction of $\tilde{p}$. The $\pi_1(E)$ -action on the set of lifts therefore induces a variation of these structures through the corresponding directions at infinity. This variation is piecewise constant, with jumps occurring precisely when the direction crosses a wall.

Proposition 6.9. *The action of $\pi_1(E)$ on the set of lifts of the stop p induces a chamberwise constant family of rooted A_∞ -algebras and associated L_∞ -morphisms. More precisely, if two sufficiently distant lifts $\tilde{p}$ and $\tilde{p}'$ lie in the same chamber of asymptotic directions, then they determine the same rooted A_∞ -algebra and the same L_∞ -morphism.*

In this sense, the $\pi_1(E)$ -action does not in general preserve a single fixed A_∞ -algebra. Rather, it produces a family of such algebras, indexed chamberwise by asymptotic directions of lifts of the stop, and crossing a wall may change the resulting A_∞ -structure.

Chapter 7

Analysis of the A_∞ -algebra and the L_∞ -morphism

Fix a chamber $\mathfrak{C}_S$ for the asymptotic direction of the chosen lift $\tilde{p}$ of the stop. This determines a total order on the non-root vertices $\tilde{A}^\circ := \tilde{A} \setminus \{\tilde{p}\}$, which we write as

$$i_1 < i_2 < \cdots < i_r.$$

All constructions in this section are understood with respect to this fixed order. Once the order is fixed, the analysis of the directed A_∞ -algebra and of the corresponding L_∞ -morphism follows [19, Section 11].

7.1 A_∞ -algebra

Recall that the rooted A_∞ -algebra is

$$R = \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{rooted}}} F_{A'}.$$

For $i < j$ in $\widetilde{A}^\circ$, let $R_{ij} \subset R$ be the direct sum of the summands $F_{A'}$ corresponding to rooted subpolygons (Q', A') such that the two edges of Q' incident to the root $\tilde{p}$ meet the vertices i and j . Thus

$$R = \bigoplus_{i < j} R_{ij}.$$

Proposition 7.1. *The higher products m_n , $n \geq 3$, on R vanish. The only nontrivial binary products are the maps*

$$\mu_{ijk} : R_{ij} \otimes R_{jk} \longrightarrow R_{ik}, \quad i < j < k.$$

Hence R is a strictly upper-triangular dg-algebra without unit.

If one works with an extended coefficient system and sets $R_{ii} := S_i$, then

$$R := \bigoplus_{i \leq j} R_{ij}$$

becomes a triangular associative dg-algebra with unit.

Proof. By construction, the A_∞ -operations on R are defined by coarse subdivisions of rooted polygons into rooted subpolygons. In dimension two, a coarse subdivision of a rooted polygon into rooted pieces has either one part, corresponding to the differential m_1 , or two parts, corresponding to the binary product m_2 . There are no coarse subdivisions contributing to m_n for $n \geq 3$. This proves the vanishing of the higher products.

Moreover, the only way to compose two rooted polygons is to glue them along a common rooted edge, so that the output again has root edges at the two outer vertices. Thus the only nonzero binary compositions are of the form

$$R_{ij} \otimes R_{jk} \rightarrow R_{ik}, \quad i < j < k.$$

This shows that R is strictly upper-triangular. The final statement is immediate from the bimodule structures on the R_{ij} . □

7.2 L_∞ -morphism

We now analyze the L_∞ -morphism

$$\Phi: \mathfrak{g} \longrightarrow C^{\geq 1}(R, R)[1]$$

constructed in the previous section. Since

$$\mathfrak{g} = \bigoplus_{\substack{(Q', A') \subset (\tilde{Q}, \tilde{A}) \\ \text{unrooted}}} E_{A'}, \quad R = \bigoplus_{\substack{(P, B) \subset (\tilde{Q}, \tilde{A}) \\ \text{rooted}}} F_B,$$

the morphism Φ is determined by its matrix elements with respect to these direct sum decompositions. Equivalently, for each finite marked subpolygon (Q', A') and rooted marked subpolygons $(P_1, B_1), \dots, (P_m, B_m), (P, C)$, we have a map

$$\Phi_{A'}^{(B_1, \dots, B_m | C)}: E_{A'} \longrightarrow \text{Hom}(F_{B_1} \otimes \dots \otimes F_{B_m}, F_C)[1 - m].$$

It is often more convenient to write this in transposed form as

$$\Phi_{A'; B_1, \dots, B_m}^C: E_{A'} \otimes F_{B_1} \otimes \dots \otimes F_{B_m} \longrightarrow F_C.$$

Let $(Q', A') \subset (\tilde{Q}, \tilde{A})$ be a finite marked subpolygon. Consider the rooted convex hull

$$\widehat{Q}' := \text{Conv}(A' \cup \{\tilde{p}\}), \quad \widehat{A}' := \tilde{A} \cap \widehat{Q}'.$$

Since the chamber is fixed, the root $\tilde{p}$ determines two distinguished boundary arcs on $\partial Q'$ (See Figure 7.1):

- the *positive boundary* $\partial^+ Q'$ (drawn in red), consisting of those sides of Q' facing the root $\tilde{p}$;
- the *negative boundary* $\partial^- Q'$ (drawn in blue), consisting of the complementary sides of

$\partial Q'$.

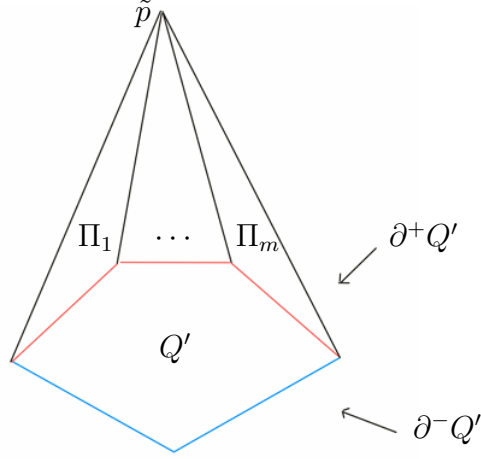


Figure 7.1: *The positive and negative boundary components associated to Q' .*

Write the edges of $\partial^+ Q'$ as

$$\eta_1, \dots, \eta_m$$

in counterclockwise order around $\tilde{p}$. For each ν , let Π_ν be the rooted triangle having root $\tilde{p}$ and opposite side η_ν , and put

$$D_\nu := \tilde{A} \cap \Pi_\nu.$$

Then we obtain a subdivision

$$(\hat{Q}', \hat{A}') = (Q', A') \cup (\Pi_1, D_1) \cup \dots \cup (\Pi_m, D_m).$$

More generally, let $(P, C) \subset (\tilde{Q}, \tilde{A})$ be a rooted marked subpolygon, and let $(Q', A') \subset (P, C)$ be a finite marked subpolygon such that $\partial^- Q'$ is contained in the finite part of ∂P . The two remaining finite parts of ∂P will be called the left handle (green) and right handle (purple) and denoted λ, ρ . Such a subdivision is called a 1-finite subdivision, see Figure 7.2.

Let $\eta_1, \dots, \eta_m$ be the edges of $\partial^+ Q'$ in counterclockwise order around $\tilde{p}$, and let $P_1, \dots, P_m$ be the rooted subpolygons between $\tilde{p}$ and these edges. Put

$$B_\nu := \tilde{A} \cap P_\nu.$$

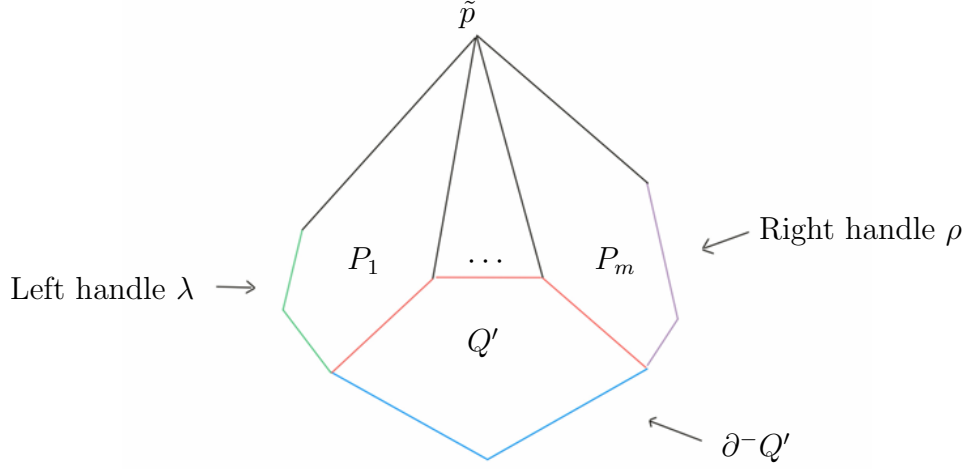


Figure 7.2: *General 1-finite subdivision and handles.*

If one works with coefficients, let N_λ, N_ρ denote the linear tensor products along the two handles. Then

$$F_{B_1} = N_\lambda \otimes_S E_{\eta_1}, \quad F_{B_\nu} = E_{\eta_\nu} \quad (2 \leq \nu \leq m-1), \quad F_{B_m} = E_{\eta_m} \otimes_S N_\rho,$$

and

$$F_C = N_\lambda \otimes_S E_{A'} \otimes_S N_\rho.$$

In the coefficient-free case, one simply omits the factors N_λ and N_ρ .

Let

$$\gamma_{Q'}: E_{A'} \otimes F_{D_1} \otimes \cdots \otimes F_{D_m} \longrightarrow F_{\widehat{A}'}$$

be the composition map obtained by contracting along the intermediate edges of the basic rooted subdivision of $(\widehat{Q'}, \widehat{A}')$. We then define

$$\Phi_{A'; B_1, \dots, B_m}^C := \text{Id}_{N_\lambda} \otimes_S \gamma_{Q'} \otimes_S \text{Id}_{N_\rho}.$$

This gives a map

$$E_{A'} \otimes F_{B_1} \otimes \cdots \otimes F_{B_m} \longrightarrow F_C.$$

Proposition 7.2. *The maps*

$$\Phi_{A';B_1,\dots,B_m}^C$$

obtained from all pairs (P, Q') consisting of a rooted subpolygon $P \subset (\tilde{Q}, \tilde{A})$ and a finite subpolygon $Q' \subset P$ with $\partial^- Q'$ contained in the finite part of ∂P , are precisely the matrix elements of the L_∞ -morphism

$$\Phi: \mathfrak{g} \longrightarrow C^{\geq 1}(R, R)[1].$$

Proof. The mixed differential on

$$S^\bullet(\mathfrak{g}[1]) \otimes T^\bullet(R[1])$$

is determined by coarse subdivisions containing exactly one finite piece and any number of rooted pieces. Such a subdivision is uniquely encoded by a pair (P, Q') as above. The corresponding component of the mixed differential is exactly the map

$$\Phi_{A';B_1,\dots,B_m}^C,$$

and these components exhaust all matrix elements of Φ . □

Chapter 8

Universality Theorem

Fix an ordered collection S of points in the lifted configuration $\tilde{A}^\circ$, let $\mathfrak{g}_{\tilde{A}^\circ}$ be the L_∞ -algebra constructed from $\tilde{A}^\circ$, and let $\mathfrak{C}_S \subset S^1 \setminus \mathcal{W}$ be a chamber compatible with this ordering in the sense of Section 5.3. Let $R_{\mathfrak{C}_S}$ be the triangular dg-algebra associated to the rooted construction in this chamber. We write

$$R_{\mathfrak{C}_S} = \bigoplus_{i \leq j} R_{ij}, \quad R_{ii} = \mathcal{S}_i,$$

where the triangular condition is understood with respect to the order determined by $\mathfrak{C}_S$. We are interested in deformations of $R_{\mathfrak{C}_S}$ which preserve this triangular structure and do not deform the diagonal algebras $\mathcal{S}_i$. Such deformations are governed by the directed Hochschild complex.

Definition 8.1. *The directed Hochschild complex of $R_{\mathfrak{C}_S}$ is the subcomplex*

$$\vec{C}^\bullet(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S}) \subset C^\bullet(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})$$

whose degree- n term is

$$\vec{C}^n(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S}) := \bigoplus_{i_0 < i_1 < \dots < i_n} \mathrm{Hom}_{\mathcal{S}_{i_0} \otimes \mathcal{S}_{i_n}^{\mathrm{op}}} \left(R_{i_0 i_1} \otimes_{\mathcal{S}_{i_1}} R_{i_1 i_2} \otimes_{\mathcal{S}_{i_2}} \dots \otimes_{\mathcal{S}_{i_{n-1}}} R_{i_{n-1} i_n}, R_{i_0 i_n} \right).$$

By construction, $\overrightarrow{C}^\bullet(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})$ is a subcomplex of the ordinary Hochschild complex and is closed under the Hochschild bracket. Hence $\overrightarrow{C}^\bullet(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1]$ is a dg-Lie subalgebra of $C^\bullet(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1]$.

The chamberwise L_∞ -morphism constructed above is denoted

$$\Phi_{\mathfrak{C}_S} : \mathfrak{g}_{\tilde{A}^\circ} \longrightarrow C^{\geq 1}(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1].$$

Since the construction preserves the directed order determined by $\mathfrak{C}_S$, this morphism is expected to factor through the directed Hochschild complex. The following theorem is the elliptic-curve version of the universality theorem of [19, Section 12]. The proof follows the same deformation-theoretic strategy, while keeping track of the additional lift and chamber data which appear for elliptic-curve-valued potentials.

Theorem 8.2 (Universality Theorem). *Let $\mathfrak{C}_S$ be a chamber compatible with the ordered lifted configuration S . Then the L_∞ -morphism*

$$\Phi_{\mathfrak{C}_S} : \mathfrak{g}_{\tilde{A}^\circ} \longrightarrow C^{\geq 1}(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1]$$

factors through an L_∞ -morphism

$$\Psi_{\mathfrak{C}_S} : \mathfrak{g}_{\tilde{A}^\circ} \longrightarrow \overrightarrow{C}^{\geq 1}(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1].$$

Moreover, $\Psi_{\mathfrak{C}_S}$ is a quasi-isomorphism.

Thus the L_∞ -algebra $\mathfrak{g}_{\tilde{A}^\circ}$ governs the same deformation problem as the directed Hochschild complex of the triangular dg-algebra $R_{\mathfrak{C}_S}$: namely, deformations preserving the triangular structure determined by the chamber $\mathfrak{C}_S$ and fixing the diagonal algebras $\mathcal{S}_i$.

Remark 8.3. *Within a fixed chamber $\mathfrak{C}_S$, the induced order on every rooted subpolygon is constant, so the triangular algebra $R_{\mathfrak{C}_S}$, the directed Hochschild complex, and the universality morphism $\Psi_{\mathfrak{C}_S}$ are fixed. When the asymptotic direction of the lift of the stop crosses a wall,*

the induced order on rooted subpolygons changes. Consequently, the triangular algebra may jump, and one obtains a different directed Hochschild complex and a different universality morphism. Hence the elliptic construction gives a family of universality statements indexed by chambers.

Proof of the Universality Theorem. For brevity, we fix a chamber $\mathfrak{C}_S$ and write

$$\vec{\mathcal{C}}^\bullet := \vec{\mathcal{C}}^{\geq 1}(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1].$$

We first prove the factorization statement. By Proposition 7.2, the matrix components of ψ are the maps

$$\psi_{A'}^{(B_1, \dots, B_m | C)}$$

coming from 1-finite subdivisions. Such a component can be nonzero only when

$$F_{B_1} \subset R_{i_0 i_1}, \quad F_{B_2} \subset R_{i_1 i_2}, \quad \dots, \quad F_{B_m} \subset R_{i_{m-1} i_m},$$

and

$$F_C \subset R_{i_0 i_m}$$

for some strictly increasing sequence $i_0 < \dots < i_m$. Moreover, these maps are multilinear over the intermediate algebras $S_{i_1}, \dots, S_{i_{m-1}}$. Therefore ψ takes values in the ordered Hochschild complex, so it factors through an L_∞ -morphism

$$\Psi: \mathfrak{g}_{\tilde{A}^\circ} \longrightarrow \vec{\mathcal{C}}^\bullet.$$

It remains to prove that Ψ is a quasi-isomorphism. We organize the argument in four steps.

Step 1: Interpretation of $\vec{\mathcal{C}}^\bullet$ via closed paths. Recall that

$$R_{ij} = \bigoplus_{Q'} F_{A \cap Q'},$$

where Q' runs over rooted marked polygons with rooted edges $[\tilde{p}, i]$ and $[j, \tilde{p}]$.

A sequence $P_0, P_1, \dots, P_n$ of marked rooted polygons is called *admissible* if there exist indices

$$i_0 < i_1 < \dots < i_n$$

such that P_0 has rooted edges $[\tilde{p}, i_0]$ and $[i_n, \tilde{p}]$, while for each $\nu = 1, \dots, n$, the polygon P_ν has infinite edges $[\tilde{p}, i_{\nu-1}]$ and $[i_\nu, \tilde{p}]$.

For such an admissible sequence, write $B_\nu = A \cap P_\nu$, and set

$$F_{P_1, \dots, P_n}^{P_0} := \text{Hom}_{S_{i_0} \otimes S_{i_n}^{\text{op}}} \left(F_{B_1} \otimes_{S_{i_1}} \dots \otimes_{S_{i_{n-1}}} F_{B_n}, F_{B_0} \right).$$

Then

$$\vec{\mathcal{C}}^n = \bigoplus_{(P_0, \dots, P_n) \text{ admissible}} F_{P_1, \dots, P_n}^{P_0}.$$

Geometrically, each summand may be viewed as the cyclic tensor product attached to the closed edge path obtained by traversing the negative boundaries

$$\partial^- P_1, \partial^- P_2, \dots, \partial^- P_n$$

and then returning along $\partial^- P_0$ with the opposite orientation.

Step 2: Filtration by handle length. Such a closed path may retrace itself on the left or on the right; these retraced parts are called the *left* and *right handles*. Let $G^l \vec{\mathcal{C}}^\bullet$ be the direct sum of those summands for which the sum of the lengths of the two handles is at least l . This gives a decreasing filtration

$$\vec{\mathcal{C}}^\bullet = G^0 \vec{\mathcal{C}}^\bullet \supset G^1 \vec{\mathcal{C}}^\bullet \supset G^2 \vec{\mathcal{C}}^\bullet \supset \dots$$

We claim that each $G^l \vec{\mathcal{C}}^\bullet$ is a subcomplex. The differential on $\vec{\mathcal{C}}^\bullet$ is the sum

$$d + \delta,$$

where d is induced by the internal differential of R , while δ is the Hochschild differential. The term d does not change the combinatorial shape of the path, hence preserves the filtration.

For δ , the first and last Hochschild terms amount to attaching an additional infinite polygon on the left or on the right; this increases the length of one of the handles.

The intermediate Hochschild terms correspond to splitting one of the polygons P_ν along an edge of the form $[\tilde{p}, s]$, where s is an intermediate vertex; these operations do not change the total handle length. Hence δ preserves the filtration, and so does $d + \delta$.

Step 3: The degree-zero graded piece. Consider the induced morphism

$$\Psi: \mathfrak{g} \longrightarrow \mathrm{gr}_G^0 \vec{C}^\bullet = \vec{C}^\bullet / G^1 \vec{C}^\bullet.$$

The complex $\mathrm{gr}_G^0 \vec{C}^\bullet$ is the direct sum of cyclic tensor products corresponding to closed paths without handles.

On the other hand, $\mathfrak{g}$ is the direct sum of the cyclic tensor products associated with the boundaries of convex unrooted marked polygons. These boundary cycles are particular handle-free paths. Moreover, by Proposition 7.2, the only matrix component of Ψ on a summand $E_{A'} \subset \mathfrak{g}$ which does not land in $G^1 \vec{C}^\bullet$ is the transpose of the basic contraction map

$$\gamma_{Q'}: N_{\partial Q'} \longrightarrow E_{A'},$$

and this transpose is the identity

$$\gamma_{Q'^t}: E_{A'} \xrightarrow{\sim} N_{\partial Q'} = E_{A'}.$$

Therefore Ψ is injective on $\mathfrak{g}$, and its image consists exactly of those handle-free summands for which the negative boundaries

$$\partial^- P_1, \dots, \partial^- P_n$$

are segments and, together with $\partial^- P_0$, bound a convex unrooted polygon.

So it suffices to prove that the cokernel of this embedding is exact. As a graded vector space, $\text{Coker}(\Psi)$ is the direct sum of the remaining handle-free summands, i.e. those not coming from convex unrooted polygons.

In gr_G^0 , the first and last Hochschild terms disappear, because they create handles and therefore land in G^1 . The only remaining part of the differential comes from splitting the polygons P_ν along intermediate vertices. Such a splitting does not change the underlying coefficient factor; it only records a choice of split. Hence each summand of $\text{Coker}(\Psi)$ decomposes as

$$F_{P_1, \dots, P_n}^{P_0} \otimes C^\bullet(\Delta^{r-1}),$$

where r is the number of available intermediate vertices at which such splittings can occur, and $C^\bullet(\Delta^{r-1})$ is the augmented simplicial cochain complex of the $(r-1)$ -simplex.

This complex is exact whenever $r \geq 1$. Therefore only those admissible sequences survive for which there are no possible splittings and which cannot be obtained by gluing two adjacent polygons into a larger convex polygon. The first condition says that each P_ν for $\nu \geq 1$ is a triangle, so each $\partial^- P_\nu$ is a segment. The second says that these segments form an upwardly convex broken line from i_0 to i_n , which together with the downwardly convex broken line $\partial^- P_0$ bounds a convex unrooted polygon. But these are precisely the summands already lying in the image of Ψ . Hence $\text{Coker}(\Psi)$ is exact, and Ψ induces a quasi-isomorphism

$$\mathfrak{g} \xrightarrow{\sim} \text{gr}_G^0 \vec{C}^\bullet.$$

Step 4: Higher graded pieces. It remains to show that, for each $l \geq 1$,

$$\text{gr}_G^l \vec{C}^\bullet$$

is exact. The argument is the same as in Step 3. In the associated graded, the first and last Hochschild terms again disappear, because they strictly increase handle length. Thus the induced differential is given only by splittings of the polygons P_ν , together with the internal

differential d coming from R .

Fix one connected shape of picture with total handle length l . The corresponding summands form a subcomplex. Since $l \geq 1$, at least one of the two handles is nontrivial. Suppose, for instance, that the left handle is nontrivial. Then the union of that handle with the first adjacent segment admits a nontrivial family of splittings. Exactly as above, this produces a tensor factor of the form

$$C^\bullet(\Delta^{r-1})$$

with $r \geq 1$, hence an exact factor. The same applies if the nontrivial handle is the right one. Therefore every summand in $\mathrm{gr}_G^l \vec{C}^\bullet$ is exact, and so the whole graded piece is exact.

Since Ψ induces a quasi-isomorphism on gr_G^0 and all higher graded pieces are exact, Ψ itself is a quasi-isomorphism. This completes the proof. $\square$

Chapter 9

A Survey on Fukaya–Seidel Categories

The purpose of this chapter is to recall the basic construction of the Fukaya–Seidel category associated to a Lefschetz fibration over the complex plane or over a disc. The presentation here is intended as background for the later chapters, where we consider curve-valued potentials and compare the resulting algebraic structures with directed Fukaya–Seidel type categories.

The Fukaya–Seidel category was introduced by Seidel as a categorical invariant of a symplectic Lefschetz fibration

$$W: X \longrightarrow \mathbb{C}$$

or, more generally, of a Lefschetz fibration over a bordered Riemann surface. It can be viewed as a directed version of the Fukaya category, where the direction is determined by a choice of distinguished vanishing paths in the base. We refer to Seidel’s book [32] and his earlier work [30] for the foundational construction, and to [2] for an accessible survey. For more recent developments relating Fukaya–Seidel categories and partially wrapped Fukaya categories, see also [10].

9.1 Fukaya–Seidel categories from Lefschetz thimbles

Let $(X, \omega = d\theta)$ be an exact symplectic manifold. A smooth map

$$W: X \longrightarrow \mathbb{C}$$

is called an exact Lefschetz fibration if it has finitely many critical points, and near each critical point it is locally modeled by a complex Morse singularity. More precisely, for each critical point $x \in X$, there are local complex coordinates $z_1, \dots, z_n$ centered at x and a local coordinate on the base centered at $W(x)$ such that

$$W(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2.$$

The critical values are assumed to be distinct. We denote the set of critical values by

$$\text{Critv}(W) \subset \mathbb{C}.$$

Fix a regular value $b_0 \in \mathbb{C} \setminus \text{Critv}(W)$. The smooth fiber

$$M = W^{-1}(b_0)$$

is an exact symplectic manifold. The Fukaya–Seidel category is built from Lagrangian submanifolds in M called vanishing cycles, or equivalently from Lagrangian thimbles in the total space X .

A *vanishing path* is an embedded path

$$\gamma: [0, 1] \longrightarrow \mathbb{C}$$

such that $\gamma(0) = b_0$, $\gamma(1)$ is a critical value of W , and $\gamma(t)$ is a regular value for $0 \leq t < 1$.

Parallel transport along γ determines a Lagrangian sphere

$$V_\gamma \subset M,$$

called the *vanishing cycle*. Geometrically, V_γ consists of those points in the reference fiber which collapse to the corresponding critical point under symplectic parallel transport along γ .

The associated *Lefschetz thimble* is a Lagrangian disc

$$\Delta_\gamma \subset X$$

whose boundary is the vanishing cycle:

$$\partial\Delta_\gamma = V_\gamma \subset M.$$

Thus the thimble lives in the total space X , while the vanishing cycle lives in the reference fiber M .

The directed nature of the Fukaya–Seidel category comes from an ordering of vanishing paths. Suppose that the critical values are

$$c_1, \dots, c_m \in \mathbb{C}.$$

A distinguished basis of vanishing paths is a collection

$$\Gamma = (\gamma_1, \dots, \gamma_m)$$

such that each γ_i starts at the fixed regular value b_0 , ends at c_i , and the paths are otherwise disjoint. Near b_0 , the tangent directions of the paths are ordered, usually clockwise or counterclockwise.

This ordering produces an ordered collection of vanishing cycles

$$(V_1, \dots, V_m)$$

in the reference fiber M , where $V_i = V_{\gamma_i}$. The corresponding Fukaya–Seidel category is a directed A_∞ -category whose objects are these vanishing cycles, or equivalently the corresponding thimbles.

Let $\mathbb{k}$ be a ground field. For simplicity, assume that all Lagrangians are graded, spin, and exact, so that the Floer complexes and higher products are well-defined without obstruction.

Given a distinguished basis of vanishing paths, the directed Fukaya–Seidel category

$$\mathcal{FS}(W; \Gamma)$$

has objects

$$L_1, \dots, L_m,$$

where L_i denotes the vanishing cycle V_i or the corresponding thimble Δ_i .

The morphism spaces are defined by

$$\mathrm{hom}_{\mathcal{FS}}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j), & i < j, \\ \mathbb{k} \cdot e_i, & i = j, \\ 0, & i > j. \end{cases}$$

Here $CF^*(L_i, L_j)$ is the Floer cochain complex between the two vanishing cycles, and e_i is the strict identity morphism of L_i .

The higher products

$$\mu^d: \mathrm{hom}(L_{i_{d-1}}, L_{i_d}) \otimes \cdots \otimes \mathrm{hom}(L_{i_0}, L_{i_1}) \longrightarrow \mathrm{hom}(L_{i_0}, L_{i_d})[2-d]$$

are defined by counting pseudo-holomorphic polygons with boundary on the ordered sequence

of Lagrangians

$$L_{i_0}, L_{i_1}, \dots, L_{i_d}.$$

Because the category is directed, the morphism spaces vanish unless the indices are weakly increasing. This gives the Fukaya–Seidel category an upper triangular form.

Given the ordered collection of objects $L_1, \dots, L_m$, one can form the total endomorphism algebra

$$R = \bigoplus_{1 \leq i, j \leq m} \text{hom}_{\mathcal{FS}}(L_i, L_j).$$

Because of directedness, this algebra has the form

$$R = \bigoplus_{i \leq j} \text{hom}_{\mathcal{FS}}(L_i, L_j).$$

Thus R is analogous to an upper triangular matrix algebra. The diagonal pieces are

$$\text{hom}_{\mathcal{FS}}(L_i, L_i) = \mathbb{k} \cdot e_i,$$

while the off-diagonal pieces are Floer complexes.

The A_∞ -operations on the category induce multilinear maps

$$\mu^d: R^{\otimes d} \longrightarrow R[2 - d],$$

making R into an A_∞ -algebra. In this sense, the Fukaya–Seidel category may be studied through its total directed algebra.

This upper triangular structure is important in deformation theory. If the diagonal algebras are fixed and only the off-diagonal morphism spaces and higher compositions are deformed, then the relevant deformation complex is a directed Hochschild complex. This viewpoint is closely related to the algebraic structures studied later in this thesis.

One can also define the Fukaya–Seidel category for a Lefschetz fibration

$$W: X \longrightarrow \mathbb{D}$$

over a disc $\mathbb{D} \subset \mathbb{C}$. In this case the basepoint b_0 is chosen on the boundary $\partial\mathbb{D}$, and the vanishing paths connect b_0 to the critical values in the interior of the disc.

The disc case is often the cleanest setting for the directed construction. The boundary point b_0 gives a distinguished place from which all vanishing paths start, and the cyclic order of the tangent directions at b_0 gives an ordering of the objects.

Thus the Fukaya–Seidel category over a disc is naturally attached to the data

$$(W, \mathbb{D}, b_0, \Gamma),$$

where Γ is a distinguished basis of vanishing paths.

Changing the distinguished basis changes the directed presentation of the category. However, different distinguished bases are related by Hurwitz moves. On the level of Fukaya–Seidel categories, these moves correspond to mutations of the ordered collection of vanishing cycles. Therefore the directed category depends on the choice of basis, but its associated derived category is expected to be invariant up to equivalence.

Now consider a Lefschetz fibration

$$W: X \longrightarrow \mathbb{C}.$$

Assume that all critical values lie in a sufficiently large disc

$$\mathbb{D}_R \subset \mathbb{C}.$$

Then one may choose a basepoint $b_0 \in \partial\mathbb{D}_R$ and define

$$\mathcal{FS}(W)$$

by applying the disc construction to the restricted fibration

$$W^{-1}(\mathbb{D}_R) \longrightarrow \mathbb{D}_R.$$

Equivalently, one may choose a direction at infinity in the complex plane. The choice of such a direction determines a reference point near infinity and an ordering of vanishing paths. Therefore, for fibrations over $\mathbb{C}$, the Fukaya–Seidel category is again directed, but the directed structure depends on a choice of cut or ray at infinity.

This point is important. The affine base $\mathbb{C}$ has a preferred notion of infinity, but it does not have a unique preferred ordering of critical values. Choosing a ray at infinity gives a way to cut the plane and linearly order the vanishing paths. A different choice of ray may produce a different directed presentation. These presentations are related by mutations and monodromy.

The geometry of a Lefschetz fibration is encoded by its monodromy. Let c_i be a critical value and let V_i be the corresponding vanishing cycle. The local monodromy around c_i is given by a symplectic Dehn twist

$$\tau_{V_i}$$

along the vanishing cycle V_i .

If the critical values are ordered by a distinguished basis of vanishing paths, then the total monodromy around all critical values is expressed as a product of Dehn twists

$$\tau_{V_m} \circ \cdots \circ \tau_{V_1}.$$

Changing the distinguished basis changes this factorization by Hurwitz moves. Thus the Fukaya–Seidel category records not only the fiberwise Fukaya category, but also the way in which the fiber changes as one moves around the critical values in the base.

The directed Fukaya–Seidel category has a natural semiorthogonal form. Since

$$\mathrm{hom}(L_i, L_j) = 0 \quad \text{for } i > j,$$

the ordered collection

$$(L_1, \dots, L_m)$$

behaves like a semiorthogonal collection. Passing to the triangulated envelope, one obtains a category generated by these objects with morphisms only in the chosen direction.

From the algebraic point of view, the total algebra of the category is an upper triangular A_∞ -algebra. The diagonal pieces correspond to the individual objects, and the off-diagonal pieces correspond to Floer complexes between different objects. Higher products describe compositions among these morphism spaces.

This is the form that appears in the deformation-theoretic approach: one starts with a directed algebra and studies deformations preserving the triangular structure. Such deformations are governed by a directed Hochschild complex. This perspective will be used later when comparing Fukaya–Seidel categories with algebraic structures coming from configurations of points and secondary polytopes.

9.2 Fukaya–Seidel categories from complex Morse theory

We now recall a second viewpoint on Fukaya–Seidel categories, namely the complex Morse model. This viewpoint is especially useful for us because it describes Lefschetz thimbles directly in terms of gradient trajectories of a holomorphic function. It is also close to the soliton or instanton picture appearing in the algebra of the infrared.

Let

$$W: X \longrightarrow \mathbb{C}$$

be a holomorphic Morse function on a Kähler manifold (X, ω, J, g) . Here

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

is the associated Kähler metric. We assume, for simplicity, that W has finitely many non-degenerate critical points and distinct critical values.

For each phase

$$\zeta \in S^1,$$

we consider the real-valued function

$$h_\zeta = \operatorname{Re}(\zeta^{-1}W).$$

A ζ -gradient trajectory is a solution of

$$\frac{du}{dt} + \nabla \operatorname{Re}(\zeta^{-1}W)(u(t)) = 0. \tag{9.1}$$

Thus the trajectories are ordinary negative gradient trajectories, but applied to the real part of the rotated holomorphic function $\zeta^{-1}W$.

Since W is holomorphic, the imaginary part

$$\operatorname{Im}(\zeta^{-1}W)$$

is constant along solutions of (9.1). Indeed, if $u(t)$ is a solution, then

$$\frac{d}{dt} \operatorname{Im}(\zeta^{-1}W)(u(t)) = d \operatorname{Im}(\zeta^{-1}W) (-\nabla \operatorname{Re}(\zeta^{-1}W)) = 0,$$

because the gradients of the real and imaginary parts of a holomorphic function are orthogonal with respect to the Kähler metric. On the other hand,

$$\frac{d}{dt} \operatorname{Re}(\zeta^{-1}W)(u(t)) = -|\nabla \operatorname{Re}(\zeta^{-1}W)(u(t))|^2 \leq 0.$$

Therefore the image of a ζ -gradient trajectory under W lies on an affine line in the complex plane parallel to the direction ζ , and moves monotonically along that line.

Let p be a critical point of W , and let

$$c = W(p)$$

be the corresponding critical value. The stable manifold of p for the negative gradient flow of h_ζ is

$$\Delta_p^\zeta = \left\{ x \in X \mid \lim_{t \rightarrow +\infty} u_x(t) = p \right\},$$

where $u_x(t)$ denotes the solution of (9.1) with initial condition $u_x(0) = x$. This stable manifold is the Lefschetz thimble associated to the critical point p and the phase ζ .

Under W , the thimble Δ_p^ζ projects to the ray

$$c + \mathbb{R}_{\geq 0}\zeta \subset \mathbb{C}.$$

Thus the phase ζ determines the direction of the vanishing path in the base. The vanishing cycle is obtained by intersecting the thimble with a regular fiber:

$$V_p^\zeta = \Delta_p^\zeta \cap W^{-1}(b),$$

where b is a regular value on the ray $c + \mathbb{R}_{> 0}\zeta$.

Near a nondegenerate critical point, the complex Morse lemma gives holomorphic coordinates $z_1, \dots, z_n$ centered at p such that

$$W(z_1, \dots, z_n) = c + z_1^2 + \dots + z_n^2.$$

If we take $\zeta = 1$, then

$$h_1 = \operatorname{Re}(W) = \operatorname{Re}(c) + \sum_{j=1}^n (x_j^2 - y_j^2),$$

where $z_j = x_j + iy_j$. The negative gradient flow is

$$\frac{dx_j}{dt} = -2x_j, \quad \frac{dy_j}{dt} = 2y_j.$$

Hence the stable manifold of the critical point is

$$\{y_1 = \cdots = y_n = 0\} \cong \mathbb{R}^n.$$

On this stable manifold,

$$W - c = x_1^2 + \cdots + x_n^2 \in \mathbb{R}_{\geq 0}.$$

Therefore the thimble projects to the positive real ray starting at the critical value c .

For a general phase $\zeta \in S^1$, the corresponding stable thimble is obtained by rotating the local coordinates. Locally it is given by

$$z_j = \zeta^{1/2} x_j, \quad x_j \in \mathbb{R}.$$

Then

$$W - c = \zeta(x_1^2 + \cdots + x_n^2),$$

so the image of the thimble is the ray

$$c + \mathbb{R}_{\geq 0}\zeta.$$

This local computation explains why the complex Morse model produces exactly the same geometric objects as the Lefschetz fibration picture: stable manifolds of the rotated real part of W are Lefschetz thimbles, and their projections to the base are vanishing paths.

The vanishing cycle can also be seen in the local model. Intersecting the thimble with the fiber

$$W^{-1}(c + r\zeta), \quad r > 0,$$

gives

$$x_1^2 + \cdots + x_n^2 = r,$$

which is a sphere

$$S^{n-1}.$$

Thus the local vanishing cycle is the standard Lagrangian sphere in the smooth fiber.

The stable manifolds Δ_p^ζ are Lagrangian submanifolds of X . One way to see this is to use the compatibility between the holomorphic function W and the Kähler structure. The flow of

$$-\nabla \operatorname{Re}(\zeta^{-1}W)$$

preserves the condition that the imaginary part $\operatorname{Im}(\zeta^{-1}W)$ is constant, and the stable manifold is built from directions on which the symplectic form restricts trivially.

In the local model, the thimble is

$$\mathbb{R}^n \subset \mathbb{C}^n$$

after a unitary rotation. This is the standard exact Lagrangian subspace. Globally, the thimble is obtained by parallel transporting the local vanishing cycle along the ray

$$c + \mathbb{R}_{\geq 0}\zeta.$$

Therefore the complex Morse model and the symplectic parallel transport model give equivalent descriptions of Lefschetz thimbles.

The complex Morse model also gives a natural way to describe solitons between critical points. Let p_i and p_j be two critical points of W , with critical values

$$c_i = W(p_i), \quad c_j = W(p_j).$$

A ζ -soliton from p_i to p_j is a solution

$$u: \mathbb{R} \longrightarrow X$$

of the equation

$$\frac{du}{dt} + \nabla \operatorname{Re}(\zeta^{-1}W)(u(t)) = 0 \quad (9.2)$$

with asymptotic conditions

$$\lim_{t \rightarrow -\infty} u(t) = p_i, \quad \lim_{t \rightarrow +\infty} u(t) = p_j.$$

For such a trajectory to exist, the two critical values must lie on a line in the direction ζ . Indeed,

$$\operatorname{Im}(\zeta^{-1}W)$$

is constant along the trajectory, so we must have

$$\operatorname{Im}(\zeta^{-1}c_i) = \operatorname{Im}(\zeta^{-1}c_j).$$

Equivalently,

$$c_i - c_j \in \mathbb{R}\zeta.$$

Moreover, the real part decreases along the flow, so the orientation of the trajectory is determined by the inequality

$$\operatorname{Re}(\zeta^{-1}c_i) > \operatorname{Re}(\zeta^{-1}c_j).$$

Thus the phase ζ determines which critical values can be connected by solitons. This is the beginning of the wall-crossing picture: as ζ varies, the relative positions of the critical values with respect to the ζ -direction change. When two critical values become aligned, new soliton trajectories may appear or disappear.

The Fukaya–Seidel category can be described using either vanishing paths or complex Morse theory. In the vanishing path description, one chooses a distinguished basis of paths in the base and obtains an ordered collection of vanishing cycles

$$L_1, \dots, L_m.$$

In the complex Morse description, one chooses a phase ζ and considers the stable thimbles

$$\Delta_{p_1}^\zeta, \dots, \Delta_{p_m}^\zeta$$

associated to the critical points of W .

The ordering of the objects comes from the ordering of the corresponding critical values in the base. For example, after rotating by ζ^{-1} , one may order the critical values by the real coordinate

$$\operatorname{Re}(\zeta^{-1}c_i).$$

This produces a directed category whose morphisms go only in one direction:

$$\operatorname{hom}(L_i, L_j) = 0 \quad \text{for } i > j.$$

The nonzero morphism spaces are Floer complexes between vanishing cycles:

$$\operatorname{hom}(L_i, L_j) = CF^*(L_i, L_j), \quad i < j.$$

The higher products are defined by counting holomorphic polygons with boundary on the ordered collection of vanishing cycles.

From the complex Morse viewpoint, these Floer-theoretic operations should be thought of as higher-dimensional analogues of counting gradient trajectories. The unary operation counts strips, the binary operation counts triangles, and the higher operations count polygons. Thus the Fukaya–Seidel category is a categorification of the Morse complex associated to the holomorphic function W .

There is a particularly transparent model when the symplectic manifold is a cotangent bundle. Let X be a Kähler manifold and let

$$F: X \longrightarrow \mathbb{C}$$

be a holomorphic function. Consider the cotangent bundle

$$T^*X$$

with its canonical holomorphic symplectic form. The zero section and the graph of the differential of F define holomorphic Lagrangian submanifolds:

$$L_0 = X, \quad L_1 = \text{Graph}(dF).$$

This is the holomorphic analogue of the usual real Morse theory model, where one compares the zero section with the graph of df for a real-valued Morse function f .

In ordinary Floer theory, Floer's theorem identifies pseudo-holomorphic strips between the zero section and the graph of df with gradient trajectories of f . In the holomorphic setting, the corresponding statement involves the Fueter equation. Doan and Rezchikov prove that, in the cotangent model above, the relevant Fueter strips correspond to complex gradient trajectories of F [6]. This gives a complexified version of the classical relationship between Floer theory and Morse theory.

This result supports the idea that the Fukaya–Seidel category of a holomorphic function can be understood through complex Morse theory. The generators are critical points or vanishing cycles, the morphisms are controlled by trajectories between them, and the higher operations are controlled by appropriate moduli spaces of higher-dimensional solutions.

Remark 9.3. *The complex Morse model is especially relevant when one wants to replace the base $\mathbb{C}$ by a Riemann surface. Suppose that*

$$W: X \longrightarrow E$$

is a potential valued in an elliptic curve. Unlike $\mathbb{C}$, the elliptic curve has no distinguished point at infinity and no global linear coordinate. Therefore the usual Fukaya–Seidel construction cannot be applied directly by choosing rays going to infinity.

However, locally or after passing to the universal cover

$$\pi: \mathbb{C} \longrightarrow E,$$

one can recover a complex-valued model. If $\widetilde{W}$ denotes a lift of W to the universal cover, then one can again consider functions of the form

$$\operatorname{Re}(\zeta^{-1}\widetilde{W}).$$

The associated gradient trajectories project to trajectories in the elliptic curve. The choice of lift is therefore analogous to the choice of a branch of the complex-valued potential.

This explains the role of a stop or basepoint in the curve-valued setting. In the classical Fukaya–Seidel category over $\mathbb{C}$, the point at infinity or the boundary point of a disc gives a way to order the thimbles. For an elliptic curve, one must instead choose extra data, such as a point

$$p \in E$$

and a lift

$$\widetilde{p} \in \mathbb{C}.$$

This choice cuts the universal cover into a domain where the lifted critical values can be ordered. Different lifts of p can give different orderings, and hence different directed A_∞ -algebras.

From the complex Morse viewpoint, this dependence is natural. A directed Fukaya–Seidel category is not determined only by the set of critical values; it also depends on how those critical values are seen from a chosen direction, cut, or lift. In the affine case, this choice is often hidden in the choice of a ray at infinity. In the elliptic curve case, the choice becomes visible and leads to a wall-crossing phenomenon as the lift of the stop crosses walls in the universal cover.

Thus the complex Morse model provides a bridge between the classical Fukaya–Seidel

category and the algebraic construction studied in this thesis. The stable thimbles are replaced by lifted configurations, the ordering of critical values is controlled by the choice of a lift of the stop, and the higher operations are encoded by configurations of trajectories or polygons. This is the geometric motivation for the directed A_∞ -algebras constructed later.

9.3 Example: the A_2 -model

We illustrate the construction with a simple Lefschetz fibration whose Fukaya–Seidel category is the directed category of the quiver

$$1 \longrightarrow 2.$$

Consider the polynomial

$$W: \mathbb{C} \longrightarrow \mathbb{C}, \quad W(z) = z^3 - 3z.$$

Its critical points are determined by

$$W'(z) = 3z^2 - 3 = 0,$$

so

$$z = 1, \quad z = -1.$$

The corresponding critical values are

$$W(1) = -2, \quad W(-1) = 2.$$

Thus W has two nondegenerate critical points and two distinct critical values.

Choose a regular basepoint $b_0 \in \mathbb{C}$ far to the right of both critical values, and choose two distinguished vanishing paths from b_0 to the critical values 2 and -2 . These paths determine

two vanishing cycles in the reference fiber

$$M = W^{-1}(b_0).$$

Since W is a cubic polynomial, the fiber M consists of three points. The vanishing cycles are zero-dimensional spheres, i.e. each is a pair of points in M .

After choosing the standard distinguished basis of vanishing paths, we may represent the two vanishing cycles as

$$L_1 = \{q_1, q_2\}, \quad L_2 = \{q_2, q_3\},$$

where

$$M = \{q_1, q_2, q_3\}.$$

Thus L_1 and L_2 intersect in exactly one point:

$$L_1 \cap L_2 = \{q_2\}.$$

The directed Fukaya–Seidel category has two objects,

$$L_1, \quad L_2.$$

The morphism spaces are

$$\mathrm{hom}(L_i, L_j) = \begin{cases} \mathbb{k} \cdot e_i, & i = j, \\ CF^*(L_i, L_j), & i < j, \\ 0, & i > j. \end{cases}$$

In this example,

$$CF^*(L_1, L_2) = \mathbb{k} \cdot a,$$

where a is the generator corresponding to the intersection point q_2 . Therefore the only

nonzero morphism spaces are

$$\mathrm{hom}(L_1, L_1) = \mathbb{k} \cdot e_1, \quad \mathrm{hom}(L_2, L_2) = \mathbb{k} \cdot e_2,$$

and

$$\mathrm{hom}(L_1, L_2) = \mathbb{k} \cdot a.$$

The reverse morphism space vanishes:

$$\mathrm{hom}(L_2, L_1) = 0.$$

The A_∞ -operations are very simple in this case. There are no nontrivial differentials, so

$$\mu^1 = 0.$$

The only nontrivial binary products are the compositions with the identity morphisms:

$$\mu^2(e_2, a) = a, \quad \mu^2(a, e_1) = a,$$

together with

$$\mu^2(e_i, e_i) = e_i.$$

There are no nontrivial higher products, because there is only one non-identity morphism a .

Hence

$$\mu^d = 0 \quad \text{for } d \geq 3.$$

Thus the total algebra of this Fukaya–Seidel category is

$$R = \mathrm{hom}(L_1, L_1) \oplus \mathrm{hom}(L_1, L_2) \oplus \mathrm{hom}(L_2, L_2).$$

Equivalently,

$$R = \mathbb{k}e_1 \oplus \mathbb{k}a \oplus \mathbb{k}e_2.$$

It has the form of an upper triangular matrix algebra

$$R \cong \begin{pmatrix} \mathbb{k} & \mathbb{k}a \\ 0 & \mathbb{k} \end{pmatrix}.$$

Algebraically, this is the path algebra of the quiver

$$1 \longrightarrow 2.$$

This example shows the basic feature of the Fukaya–Seidel category: the geometry of vanishing paths produces an ordered collection of thimbles, and the resulting category is directed. In more complicated examples, the same upper triangular form remains, but the Floer complexes and higher products may be nontrivial.

9.4 Example: the mirror of $\mathbb{P}^2$

We now discuss a slightly more complicated example, namely the standard Landau–Ginzburg mirror of $\mathbb{P}^2$. Consider

$$W: (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}, \quad W(x, y) = x + y + \frac{1}{xy}.$$

This is the usual toric mirror superpotential for $\mathbb{P}^2$; see, for example, Auroux–Katzarkov–Orlov [3] and Abouzaid [1]. Its critical points are determined by

$$\frac{\partial W}{\partial x} = 1 - \frac{1}{x^2 y} = 0, \quad \frac{\partial W}{\partial y} = 1 - \frac{1}{xy^2} = 0.$$

Thus

$$x^2y = 1, \quad xy^2 = 1.$$

Dividing the two equations gives $x = y$. Hence

$$x^3 = 1.$$

Therefore the critical points are

$$(1, 1), \quad (\omega, \omega), \quad (\omega^2, \omega^2),$$

where

$$\omega = e^{2\pi i/3}.$$

The corresponding critical values are

$$W(1, 1) = 3, \quad W(\omega, \omega) = 3\omega, \quad W(\omega^2, \omega^2) = 3\omega^2.$$

Thus the three critical values form the vertices of an equilateral triangle in the complex plane.

Choose a basepoint $b_0 \in \mathbb{C}$ away from the critical values, together with a distinguished basis of vanishing paths. This gives three Lefschetz thimbles

$$\Delta_0, \Delta_1, \Delta_2$$

and three vanishing cycles

$$L_0, L_1, L_2$$

in a reference fiber of W . The directed Fukaya–Seidel category

$$\mathcal{FS}(W)$$

has objects

$$L_0, \quad L_1, \quad L_2.$$

As usual, the morphism spaces are upper triangular:

$$\mathrm{hom}(L_i, L_j) = 0 \quad \text{for } i > j,$$

and

$$\mathrm{hom}(L_i, L_i) = \mathbb{k} \cdot e_i.$$

Under the homological mirror symmetry identification, the directed Fukaya–Seidel category of this Landau–Ginzburg model is identified with the subcategory of $D^b \mathrm{Coh}(\mathbb{P}^2)$ generated by the exceptional collection

$$(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$$

on $\mathbb{P}^2$. This point of view is developed in Abouzaid’s work on toric mirror symmetry [1].

Therefore the morphism spaces can be modeled as

$$\mathrm{hom}(L_0, L_1) \cong H^0(\mathbb{P}^2, \mathcal{O}(1)),$$

$$\mathrm{hom}(L_1, L_2) \cong H^0(\mathbb{P}^2, \mathcal{O}(1)),$$

and

$$\mathrm{hom}(L_0, L_2) \cong H^0(\mathbb{P}^2, \mathcal{O}(2)).$$

Hence

$$\dim \mathrm{hom}(L_0, L_1) = 3, \quad \dim \mathrm{hom}(L_1, L_2) = 3,$$

and

$$\dim \mathrm{hom}(L_0, L_2) = 6.$$

Let

$$X_0, X_1, X_2$$

be homogeneous coordinates on $\mathbb{P}^2$. Then

$$H^0(\mathbb{P}^2, \mathcal{O}(1)) = \text{Span}_{\mathbb{k}}\{X_0, X_1, X_2\},$$

while

$$H^0(\mathbb{P}^2, \mathcal{O}(2)) = \text{Span}_{\mathbb{k}}\{X_0^2, X_0X_1, X_0X_2, X_1^2, X_1X_2, X_2^2\}.$$

The most important operation is the binary product

$$\mu^2: \text{hom}(L_1, L_2) \otimes \text{hom}(L_0, L_1) \longrightarrow \text{hom}(L_0, L_2).$$

Under the above identification, this is simply multiplication of homogeneous polynomials:

$$H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{O}(1)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(2)).$$

For example,

$$\mu^2(X_i, X_j) = X_iX_j.$$

The product is commutative on the polynomial side, so the nine formal products X_iX_j span a six-dimensional space, with relations

$$X_iX_j = X_jX_i.$$

Thus the total algebra of this directed Fukaya–Seidel category has the form

$$R = \begin{pmatrix} \mathbb{k} & H^0(\mathbb{P}^2, \mathcal{O}(1)) & H^0(\mathbb{P}^2, \mathcal{O}(2)) \\ 0 & \mathbb{k} & H^0(\mathbb{P}^2, \mathcal{O}(1)) \\ 0 & 0 & \mathbb{k} \end{pmatrix}.$$

Equivalently,

$$R = \mathbb{k}e_0 \oplus \mathbb{k}e_1 \oplus \mathbb{k}e_2 \oplus \text{hom}(L_0, L_1) \oplus \text{hom}(L_1, L_2) \oplus \text{hom}(L_0, L_2).$$

This example is more complicated than the A_2 -model because there are now three objects and nontrivial compositions between off-diagonal morphism spaces. The composition

$$\text{hom}(L_1, L_2) \otimes \text{hom}(L_0, L_1) \rightarrow \text{hom}(L_0, L_2)$$

is the algebraic shadow of counting holomorphic triangles with boundary on the three vanishing cycles

$$L_0, \quad L_1, \quad L_2.$$

Thus the example already displays the main feature of a Fukaya–Seidel category: the directed algebra is upper triangular, but its off-diagonal pieces interact through nontrivial A_∞ -operations.

For the purposes of this thesis, the lesson is the following. A choice of distinguished vanishing paths gives an ordering of the thimbles, and this ordering produces an upper triangular algebra. The diagonal pieces are fixed, while the off-diagonal pieces record Floer complexes between different thimbles. Higher products encode counts of holomorphic polygons. This is the same general algebraic shape that appears later in the directed A_∞ -algebras associated to lifted point configurations.

Chapter 10

Fukaya–Seidel categories over a base curve

In this chapter we sketch an application of the previous considerations to Fukaya–Seidel categories over a curve base. For the relevant general framework over an arbitrary base curve, we refer to [22]. Our goal is to formulate the expected deformation principle: the total A_∞ -algebra R_S of a Fukaya–Seidel category should be obtainable as a Maurer–Cartan deformation of a combinatorial A_∞ -algebra R .

10.1 Fukaya–Seidel category on a base curve

Let (X, ω) be an exact Kähler manifold, Y a complex curve, and

$$f : X \rightarrow Y$$

a holomorphic map with Morse critical points and pairwise distinct critical values. Fix a regular value (basepoint) $z \in Y$ and a nonzero real tangent vector $\partial_z \in T_z Y$.

Choose a small embedded disc $D_z \subset Y$ centered at z such that $-\partial_x$ at the center corresponds to ∂_z under an identification $D_z \simeq D = \{w \in \mathbb{C} : |w| \leq 1\}$. On the model disc D one considers a Hamiltonian flow which is the identity on ∂D and which, morally, pushes

Lagrangians slightly along the chosen direction at the basepoint. Transporting this by the identification $D_z \simeq D$ gives a symplectomorphism ϕ supported near z ; extend it by the identity on $Y \setminus D_z$. (We keep the notation ϕ for the resulting perturbation used in Floer theory.)

Write $F_z = f^{-1}(z)$ for the smooth fiber over z , and set

$$X_z := X \setminus F_z.$$

An *f-admissible path* is an immersed path $\delta : [0, 1] \rightarrow Y$ such that

$$\delta([0, 1)) \subset Y \setminus \text{Critv}(f), \quad \delta(1) \in \text{Critv}(f), \quad \delta(0) = z, \quad \delta'(0) \neq \partial_z.$$

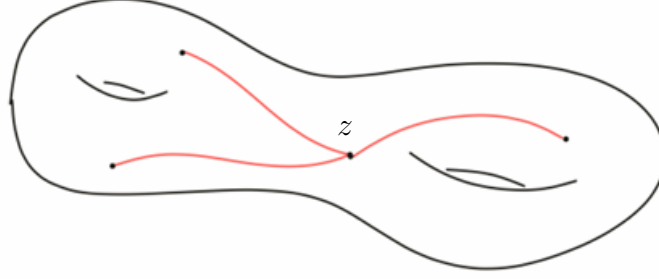


Figure 10.1: *Admissible paths.*

Let $T_\delta \subset X$ be the associated vanishing thimble over δ and let $T_\delta^\circ := T_\delta \cap X_z$ be the *open thimble*. An *admissible Lagrangian thimble* is such an open thimble $T_\delta^\circ \subset X_z$.

Using the holomorphic volume form (as in the standard Fukaya–Seidel setup), equip T_δ° with:

- a grading α , and
- a Pin structure β ,

and write the resulting Lagrangian brane as

$$L_\delta := (T_\delta^\circ, \alpha, \beta).$$

Define an A_∞ -category $A_z(f)$ as follows:

- **Objects:** admissible Lagrangian branes L_δ .
- **Morphisms:** for objects L_0, L_1 ,

$$\mathrm{Hom}_{A_z(f)}(L_0, L_1) := CF^*(L_0, \phi(L_1)),$$

the Floer complex (for a universal choice of perturbation data ensuring transversality).

- **Higher compositions:** the A_∞ -operations μ^k are defined by counts of suitable pseudoholomorphic polygons with boundary on the branes, with signs determined by the Pin structures and degrees determined by the gradings.

Definition 10.1. *The Fukaya–Seidel category of f at the basepoint z is the category of twisted complexes over $A_z(f)$:*

$$\mathcal{FS}_z(f) := \mathrm{Tw}(A_z(f)).$$

Remark 10.2. *When $Y = \mathbb{C}$ there is an alternative description via a certain $\mathbb{Z}/2$ -cover construction, but for general base curves Y this route may not be available. In this paper we use the above definition over an arbitrary base curve.*

In the case where the base is a general curve, a finite directed model of the Fukaya–Seidel category depends on the choice of a distinguished collection of admissible paths

$$S = (\delta_1, \dots, \delta_r).$$

Indeed, unlike the classical case of a potential valued in the complex plane, there is in general no preferred half-plane at the basepoint which selects a canonical finite ordered system of thimbles. The admissible paths may leave the basepoint in different tangent directions, and may also wind around nontrivial cycles of the base curve before reaching a critical value.

Equivalently, if $\gamma \in \pi_1(Y, z)$ and δ is an admissible path from z to a critical value w_i , then the concatenation

$$\gamma \cdot \delta$$

(after a small smoothing near z) gives another admissible path with the same endpoint. Thus the fundamental group of the base curve acts on the set of admissible paths, and hence on the set of admissible thimbles. In particular, even when f has only finitely many critical values, the collection of admissible thimbles is typically infinite.

For this reason, the category $A_z(f)$ should be viewed as a large A_∞ -category. A finite distinguished collection S determines a finite directed subcategory

$$A_S \subset A_z(f),$$

generated by the branes

$$L_{\delta_1}, \dots, L_{\delta_r},$$

together with the chosen ordering of this collection. The corresponding total algebra is

$$R_S = \bigoplus_{i \leq j} \mathrm{Hom}_{A_z(f)}^\bullet(L_{\delta_i}, L_{\delta_j}),$$

with the A_∞ -operations inherited from $A_z(f)$ and restricted to this ordered collection. We use the directed convention that

$$\mathrm{Hom}_{A_S}^\bullet(L_{\delta_i}, L_{\delta_i}) = \mathbb{k}e_i, \quad \mathrm{Hom}_{A_S}^\bullet(L_{\delta_i}, L_{\delta_j}) = 0 \quad \text{for } i > j.$$

We next record the effect of monodromy on the directed subcategories introduced above. We will use the standard naturality of Floer complexes under symplectomorphisms. Let

$$\Psi : X_z \longrightarrow X_z$$

be a symplectomorphism preserving the class of admissible Lagrangian branes. For two branes L_0, L_1 , the Floer complexes

$$CF^\bullet(L_0, L_1) \quad \text{and} \quad CF^\bullet(\Psi(L_0), \Psi(L_1))$$

are naturally identified, provided the Floer data are transported by Ψ .

Applying the same monodromy element to the whole distinguished collection produces an equivalent directed subcategory, via the induced autoequivalence of the Fukaya category of the fiber.

However, this should be distinguished from applying monodromy to only one object. Replacing a single path δ_i by a wrapped path $\gamma \cdot \delta_i$, while keeping the other paths fixed, generally changes the relative position of this thimble with respect to the others. Therefore the Floer complexes need not be quasi-isomorphic, and the higher products involving this object may also change. This one-sided operation is the source of nontrivial monodromy-type transformations among the various directed presentations.

Different choices of S may therefore give different directed subcategories and different total algebras R_S . In the curve-valued setting, the total algebra is not a single object canonically attached only to f and z , but rather a presentation-dependent object attached to the additional choice of a finite distinguished collection of admissible thimbles.

This dependence is one of the new features of the curve-valued case. The fundamental group of the base curve produces different presentations of the Fukaya–Seidel category by acting on admissible paths. Thus one expects the family of total algebras $\{R_S\}_S$ to be related by monodromy-type transformations.

10.2 The lifted complex Morse model and Maurer–Cartan element

We now explain how the complex Morse model of the Fukaya–Seidel category is adapted to the curve–valued setting. In this subsection we specialize to the case where the base curve is an elliptic curve $E = \mathbb{C}/\Lambda$ and the potential is a holomorphic map

$$f : X \longrightarrow E.$$

We assume that f has finitely many Morse critical points $\text{Crit}(f) = \{x_1, \dots, x_r\}$, with pairwise distinct critical values $w_i := f(x_i) \in E$.

Let $\pi : \mathbb{C} \longrightarrow E$ be the universal covering map. We form the fiber product

$$\tilde{X} := X \times_E \mathbb{C}.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{C} \\ p \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & E, \end{array}$$

where $p : \tilde{X} \rightarrow X$ is the covering map and $\tilde{f} : \tilde{X} \rightarrow \mathbb{C}$ is the lifted potential. The Kähler form, holomorphic volume form, and brane structures on X are pulled back to $\tilde{X}$.

Fix a finite distinguished collection of admissible paths

$$S = (\delta_1, \dots, \delta_r)$$

from the basepoint $z \in E$ to the critical values w_i . Choose a lift $\tilde{z} \in \mathbb{C}$ of z . Each admissible path δ_i has a unique lift

$$\tilde{\delta}_i : [0, 1] \longrightarrow \mathbb{C}$$

starting at $\tilde{z}$. We write $\tilde{w}_i := \tilde{\delta}_i(1)$ for the corresponding lift of the critical value w_i . Thus the distinguished collection S determines a lifted point configuration

$$\tilde{A}_S = \{\tilde{w}_1, \dots, \tilde{w}_r\} \subset \mathbb{C}.$$

This is the configuration to which the algebraic construction of the previous sections is applied.

The admissible thimble associated to δ_i also lifts. Let $L_{\delta_i} \subset X_z$ be the admissible Lagrangian brane associated to δ_i . The lift $\tilde{\delta}_i$ determines a distinguished lift of this thimble to $\tilde{X}$, which we denote by $\tilde{L}_i \subset \tilde{X}_{\tilde{z}}$.

For a fixed admissible path δ_i , the lifted brane $\tilde{L}_i$ is not a new local object. Rather, it is the distinguished lift of the same brane determined by the chosen lift $\tilde{\delta}_i$. Since $\tilde{\delta}_i$ projects homeomorphically to δ_i , the covering map

$$p : \tilde{X} \longrightarrow X$$

identifies the lifted brane with the downstairs brane:

$$\tilde{L}_i \simeq L_{\delta_i}.$$

Thus, for one path at a time, passing to the universal cover only chooses a representative upstairs. The lift becomes essential only for a finite collection of paths, through the relative positions of the lifted endpoints

$$\tilde{w}_i = \tilde{\delta}_i(1).$$

The absolute choice of $\tilde{z}$ is not essential, see [30, 31]. Downstairs, the simultaneous deck transformation corresponds to applying the same global monodromy element to every admissible path in S . By the monodromy invariance discussed above, global monodromy does not change the directed Fukaya–Seidel subcategory up to A_∞ -quasi-equivalence, nor the corresponding total algebra up to quasi-isomorphism. Therefore, on the Fukaya–Seidel

side, the absolute lift of z is not part of the essential data. What matters is the relative lifted configuration of the chosen paths, modulo simultaneous deck transformations.

This distinction matches the phenomenon already observed in the algebraic construction of the A_∞ -algebra. There one chooses a lift $\tilde{p}$ of the stop or basepoint in the universal cover, and the resulting directed algebra depends on the relative position of $\tilde{p}$ with respect to the lifted point configuration. A simultaneous deck transformation of the whole lifted picture does not change the essential directed Fukaya–Seidel subcategory. However, changing the relative lift of one endpoint, or moving $\tilde{p}$ across a wall, changes the combinatorics of the lifted configuration. Thus the chamber structure in the algebraic A_∞ -model is reflected on the Fukaya–Seidel side by the monodromy dependence of finite directed collections of thimbles.

Assume that

$$\{\tilde{w}_1, \dots, \tilde{w}_r\} \subset \mathbb{C}$$

is generic with respect to the chosen direction; for example, no two points have the same imaginary part in the coordinate where ∂_z corresponds to $-\partial_x$ at $\tilde{z}$. For a pair of lifted critical values $\tilde{w}_i, \tilde{w}_j$, define

$$\zeta_{ij} = \left(\frac{\tilde{w}_i - \tilde{w}_j}{|\tilde{w}_i - \tilde{w}_j|} \right)^{-1}.$$

Then

$$\operatorname{Re}(\zeta_{ij}\tilde{f}) : \tilde{X} \longrightarrow \mathbb{R}$$

is the real Morse function used to define solitons between the corresponding lifted critical points.

Definition 10.3. *A ζ_{ij} -soliton from $\tilde{x}_i$ to $\tilde{x}_j$ is a downward gradient trajectory of*

$$\operatorname{Re}(\zeta_{ij}\tilde{f})$$

connecting these two critical points. Its image under $\tilde{f}$ lies over the straight segment

$$[\tilde{w}_i, \tilde{w}_j] \subset \mathbb{C}.$$

Let N_{ij} be the graded vector space generated by such ζ_{ij} -solitons from $\tilde{x}_i$ to $\tilde{x}_j$. The grading is given by the Maslov index, using the pulled-back holomorphic volume form on $\tilde{X}$. These spaces play the role of the coefficient spaces attached to the edges of the lifted point configuration.

More generally, let

$$\tilde{Q}' = \text{Conv}(\tilde{w}_{i_0}, \dots, \tilde{w}_{i_m}) \subset \mathbb{C}$$

be a polygon whose vertices are contained in $\tilde{A}_S$. A lifted gradient polygon over $\tilde{Q}'$ consists of a cyclic sequence of solitons

$$\phi = (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_m i_0}),$$

where each $\phi_{i_\nu i_{\nu+1}}$ is a $\zeta_{i_\nu i_{\nu+1}}$ -soliton and projects under $\tilde{f}$ to the edge

$$[\tilde{w}_{i_\nu}, \tilde{w}_{i_{\nu+1}}].$$

As in the complex-valued case, the lifted gradient polygons should be viewed as the asymptotic data for instantons. We now describe the expected moduli spaces more explicitly. We do not attempt here to give a complete analytic construction of these moduli spaces, including compactness, transversality, gluing, and orientations. Rather, we describe the complex Morse model expected to produce the Maurer–Cartan element associated to the lifted configuration.

Fix a lifted polygon

$$\tilde{Q}' = \text{Conv}(\tilde{w}_{i_0}, \dots, \tilde{w}_{i_m}) \subset \mathbb{C}$$

with vertices in $\tilde{A}_S$. Let

$$\Sigma_{\tilde{Q}'}$$

be a punctured disc with $m + 1$ strip-like ends, one end for each oriented edge

$$[\tilde{w}_{i_\nu}, \tilde{w}_{i_{\nu+1}}].$$

On the end corresponding to this edge, we use coordinates (s, t) and the phase

$$\zeta_{i_\nu i_{\nu+1}} = \left(\frac{\tilde{w}_{i_\nu} - \tilde{w}_{i_{\nu+1}}}{|\tilde{w}_{i_\nu} - \tilde{w}_{i_{\nu+1}}|} \right)^{-1}.$$

Choose a domain-dependent perturbation datum on $\Sigma_{\tilde{Q}'}$ which, on this end, becomes the fixed phase $\zeta_{i_\nu i_{\nu+1}}$. Then an instanton associated to $\tilde{Q}'$ is a map

$$u : \Sigma_{\tilde{Q}'} \longrightarrow \tilde{X}$$

satisfying the corresponding Witten equation, or complex gradient flow equation, see [35].

We use it in the sense of the complex Morse theory model for Fukaya–Seidel categories; see Gaiotto–Moore–Witten [9], Haydys [15] and Wang [34]. For a survey of this perspective, see Doan–Rezchikov [6, Section 2.2].

In local strip-like coordinates this equation has the form

$$\partial_s u + I(u) \left(\partial_t u + \nabla \operatorname{Re}(\zeta \tilde{f})(u) \right) = 0,$$

where I is the complex structure on $\tilde{X}$, and where ζ is the phase determined by the perturbation datum. Equivalently, one may write it as an inhomogeneous Cauchy–Riemann equation

$$(du - X_{\operatorname{Im}(\zeta \tilde{f})} \otimes dt)^{0,1} = 0,$$

up to the usual convention on the sign of the Hamiltonian vector field. With our sign convention, the s -independent solutions on the end are precisely downward gradient trajectories

of $\text{Re}(\zeta_{i_\nu i_{\nu+1}} \tilde{f})$.

Now choose a cyclic collection of solitons

$$\phi = (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_m i_0})$$

along the boundary of $\tilde{Q}'$. We define $\mathcal{M}_{\tilde{Q}'}(\phi)$ to be the moduli space of solutions u of the above Witten equation satisfying the following asymptotic conditions: on the strip-like end corresponding to the edge $[\tilde{w}_{i_\nu}, \tilde{w}_{i_{\nu+1}}]$, the map u converges, as $s \rightarrow +\infty$, to the prescribed soliton $\phi_{i_\nu i_{\nu+1}}$.

Thus the boundary and end behavior of u is encoded by the cyclic soliton data ϕ .

Informally, such an instanton fills the polygon $\tilde{Q}'$ in the total space $\tilde{X}$. Its projection under the lifted potential $\tilde{f} : \tilde{X} \rightarrow \mathbb{C}$ has asymptotic shape controlled by the polygon $\tilde{Q}'$, and along each edge it limits to the corresponding soliton. The moduli space $\mathcal{M}_{\tilde{Q}'}(\phi)$ should therefore be regarded as the space of Witten solutions with boundary asymptotics prescribed by the lifted gradient polygon ϕ .

The expected dimension of this moduli space is determined by the Maslov degrees of the solitons. We denote it by

$$\dim \mathcal{M}_{\tilde{Q}'}(\phi) = d(\phi) - 1.$$

In particular, when $d(\phi) = 1$, the moduli space is expected to be zero-dimensional. After choosing coherent orientations, the signed count of points in this zero-dimensional moduli space defines a coefficient

$$\#\mathcal{M}_{\tilde{Q}'}(\phi) \in \mathbb{k}.$$

Equivalently, the count gives a multilinear contribution associated to the cyclic tensor

$$\phi_{i_0 i_1} \otimes \phi_{i_1 i_2} \otimes \dots \otimes \phi_{i_m i_0}.$$

Summing these contributions over all lifted polygons $\tilde{Q}'$ with vertices in $\tilde{A}_S$, and over all

cyclic collections of solitons ϕ of degree $d(\phi) = 1$, one obtains an element

$$\gamma_S = \sum_{\tilde{Q}'} \sum_{d(\phi)=1} \# \mathcal{M}_{\tilde{Q}'}(\phi) \phi \in \mathfrak{g}_{\tilde{A}_S}^1.$$

Here $\mathfrak{g}_{\tilde{A}_S}$ is the L_∞ -algebra associated to the lifted configuration $\tilde{A}_S$, with coefficients given by the soliton spaces N_{ij} . Thus γ_S should be understood as the instanton-counting element attached to the lifted complex Morse model.

The Maurer–Cartan equation for γ_S is expected to follow from the compactification of one-dimensional moduli spaces. Indeed, when

$$\dim \mathcal{M}_{\tilde{Q}'}(\phi) = 1,$$

the compactified moduli space should have boundary strata of two types. The first type comes from breaking of instantons along intermediate soliton data. The second type comes from degenerations in which the polygon $\tilde{Q}'$ splits into smaller lifted polygons. Combinatorially, these degenerations are indexed by polygonal subdivisions of $\tilde{Q}'$.

These boundary decompositions are precisely the terms appearing in the L_∞ Maurer–Cartan equation

$$d\gamma_S + \frac{1}{2}[\gamma_S, \gamma_S] + \frac{1}{3!}\ell_3(\gamma_S, \gamma_S, \gamma_S) + \cdots = 0.$$

The vanishing of the signed boundary of each compactified one-dimensional moduli space therefore gives the Maurer–Cartan equation. In this sense, the L_∞ -operations coming from polygonal subdivisions encode the possible boundary strata of the instanton moduli spaces.

10.3 Comparison with the complex-valued Morse model

We now compare the lifted construction above with the usual complex-valued Morse model for Fukaya–Seidel categories, as used in the algebra of the infrared; see [9, 19]. In the classical

setting one starts with a holomorphic Morse function

$$W : X \longrightarrow \mathbb{C}.$$

The critical values of W form a finite point configuration in the affine plane $\mathbb{C}$. Thus the complex Morse model is organized directly by the affine geometry of the target: one can form straight line segments between critical values, define phases by taking differences of critical values, and consider gradient trajectories of the real functions

$$\operatorname{Re}(\zeta_{ij}W), \quad \zeta_{ij} = \left(\frac{W(x_i) - W(x_j)}{|W(x_i) - W(x_j)|} \right)^{-1}.$$

The corresponding solitons, gradient polygons, and instanton moduli spaces are therefore defined directly in the W -plane.

In the curve-valued setting considered here, the potential is a holomorphic map

$$f : X \longrightarrow E = \mathbb{C}/\Lambda.$$

The critical values lie in E , not in an affine space. Hence there is no globally defined difference $w_i - w_j$ between two critical values and no global affine plane in which to draw the soliton web. To recover the complex Morse model, we pass to the universal cover. The fiber product

$$\tilde{X} = X \times_E \mathbb{C}$$

carries a lifted holomorphic function

$$\tilde{f} : \tilde{X} \longrightarrow \mathbb{C}.$$

After choosing a lift of the basepoint and lifts of the admissible paths, the critical values acquire lifts

$$\tilde{A}_S = \{\tilde{w}_1, \dots, \tilde{w}_r\} \subset \mathbb{C}.$$

The complex Morse theory is then performed upstairs, using the ordinary affine geometry of $\mathbb{C}$. In this lifted model, straight segments, phases, solitons, lifted gradient polygons, and the expected instanton moduli spaces are defined in the same formal way as in the complex-valued model.

The difference between the two settings is therefore not local but global. A single lifted path does not produce a genuinely new local thimble. Indeed, let δ_i be an admissible path from z to w_i , and let $\tilde{\delta}_i$ be its lift starting at a chosen lift $\tilde{z}$. The lift $\tilde{\delta}_i$ determines a corresponding lifted brane $\tilde{L}_i$ in the fiber $\tilde{X}_{\tilde{z}}$. Since $\tilde{\delta}_i$ projects homeomorphically to δ_i , the covering map

$$p : \tilde{X} \longrightarrow X$$

restricts to an identification of fibers

$$p_{\tilde{z}} : \tilde{X}_{\tilde{z}} \xrightarrow{\sim} X_z.$$

Under this identification, the lifted brane $\tilde{L}_i$ corresponds to the original brane L_{δ_i} :

$$p_{\tilde{z}}(\tilde{L}_i) = L_{\delta_i}.$$

Equivalently, after identifying the two fibers by $p_{\tilde{z}}$, the brane $\tilde{L}_i$ may be regarded as the same local brane as L_{δ_i} .

However, for a finite distinguished collection

$$S = (\delta_1, \dots, \delta_r),$$

the relative positions of the lifted endpoints

$$\tilde{w}_i = \tilde{\delta}_i(1)$$

carry additional information. A simultaneous deck transformation of all lifts translates the

whole picture in $\mathbb{C}$ and does not change the essential directed Fukaya–Seidel subcategory, up to the corresponding global monodromy equivalence. By contrast, changing the lift of only one endpoint is equivalent downstairs to composing the corresponding path with a loop in E . This can change the monodromy applied to the associated thimble and can therefore change the directed collection.

This is the geometric origin of the sheet dependence in the algebraic model. The lifted point configuration $\tilde{A}_S$ determines the straight segments, phases, and polygonal combinatorics used in the construction of the L_∞ -algebra and the associated A_∞ -algebra. Different relative choices of lifts may give different configurations in the affine plane $\mathbb{C}$, and hence different algebraic models. Thus the elliptic curve case should be viewed as a lifted version of the complex-valued Morse model, together with additional monodromy data coming from the deck group Λ .

10.4 Expected results

We conclude this chapter with a conjecture relating the lifted complex Morse model to the Fukaya–Seidel category. This conjecture should be viewed as curve-valued analogues of the conjectural comparison proposed by Gaiotto–Moore–Witten [9] and formulated mathematically, in the complex-valued case, by Kapranov–Kontsevich–Soibelman [19].

Conjecture 10.4. *Let $S = (\delta_1, \dots, \delta_r)$ be a finite ordered distinguished collection of admissible paths in the curve-valued Fukaya–Seidel category. Let $L_{\delta_1}, \dots, L_{\delta_r}$ be the corresponding branes, and let R_S be the directed total algebra of the subcategory generated by this ordered collection.*

The choice of S , together with a choice of lift of the basepoint, determines a lifted critical value configuration

$$\tilde{A}_S = \{\tilde{w}_1, \dots, \tilde{w}_r\} \subset \mathbb{C}$$

and hence an L_∞ -algebra $\mathfrak{g}_{\tilde{A}_S}$ as in Section 4.2.

Choose a lift $\tilde{p}$ of the stop lying in a chamber $\mathfrak{C}_S$ such that the linear order on $\tilde{A}_S$ induced

by $\tilde{p}$ agrees with the ordering of the distinguished collection S . This chamber determines the directed algebra $R_{\tilde{A}_S, \mathfrak{C}_S}$, or equivalently $R_{\tilde{A}_S, \tilde{p}}$, appearing in the A_∞ -construction in Section 5.3.

The lifted complex Morse model associated to the thimbles in S defines a Maurer–Cartan element $\gamma_S \in \mathfrak{g}_{\tilde{A}_S}^1$. Under the chamberwise universality morphism in Section 8

$$\Psi_{\mathfrak{C}_S} : \mathfrak{g}_{\tilde{A}_S} \longrightarrow \overrightarrow{C}^{\geq 1}(R_{\mathfrak{C}_S}, R_{\mathfrak{C}_S})[1],$$

the Maurer–Cartan element γ_S determines a deformation of the directed A_∞ -algebra $R_{\mathfrak{C}_S}$. Denote the resulting deformed A_∞ -algebra by $R_{\mathfrak{C}_S, \gamma_S}$. Then there is an A_∞ -quasi-isomorphism

$$R_{\mathfrak{C}_S, \gamma_S} \simeq R_S.$$

In other words, after choosing the chamber whose induced order agrees with the ordered distinguished collection S , the algebra of the infrared recovers the directed total algebra of the Fukaya–Seidel subcategory generated by S .

Remark 10.5. The chamber $\mathfrak{C}_S$ is part of the comparison data. On the algebraic side, different chambers for the lift of the stop/basepoint may give different directed A_∞ -algebras. On the Fukaya–Seidel side, the ordered collection S fixes the directed presentation of the subcategory. Thus the comparison uses the chamber for which the induced order of the lifted critical values agrees with the order of S .

The conjecture should be read as depending on the chosen distinguished collection S . If the collection is changed, then the lifted critical value configuration may also change, and hence the corresponding L_∞ -algebra and Maurer–Cartan element may change. This is one of the main new features of the curve-valued case: wrapping admissible paths around nontrivial cycles of the base curve is reflected algebraically by replacing some lifted critical values by deck translates. The following example illustrates this dependence.

Let

$$S = (\delta_1, \dots, \delta_r)$$

be a finite distinguished collection, and let

$$\tilde{A}_S = \{\tilde{w}_1, \dots, \tilde{w}_r\}$$

be the corresponding lifted critical value configuration. Suppose that we wrap some of the admissible paths around nontrivial cycles of E . Equivalently, choose elements

$$\lambda_1, \dots, \lambda_r \in \Lambda,$$

where $\lambda_i = 0$ means that the path δ_i is not changed. We write

$$S^\lambda = (\lambda_1 \delta_1, \dots, \lambda_r \delta_r), \quad \lambda = (\lambda_1, \dots, \lambda_r).$$

Here $\lambda_i \delta_i$ denotes the admissible path obtained by first winding around the loop corresponding to λ_i , and then following δ_i , with a small smoothing near the basepoint.

The lift of $\lambda_i \delta_i$ starting at the fixed lift $\tilde{z}$ ends at

$$\tilde{w}_i + \lambda_i.$$

Therefore the partially wrapped collection S^λ determines the lifted configuration

$$\tilde{A}_{S^\lambda} = \{\tilde{w}_1 + \lambda_1, \dots, \tilde{w}_r + \lambda_r\} \subset \mathbb{C}.$$

Thus partial wrapping changes the algebraic input of the algebra construction. In general, the relevant L_∞ -algebra is no longer $\mathfrak{g}_{\tilde{A}_S}$, but rather $\mathfrak{g}_{\tilde{A}_{S^\lambda}}$.

The lifted complex Morse model for the new thimble collection produces a Maurer–Cartan element

$$\gamma_{S^\lambda} \in \mathfrak{g}_{\tilde{A}_{S^\lambda}}^1.$$

Under the chamberwise universality morphism, this Maurer–Cartan element gives a defor-

mation

$$R_{\tilde{A}_{S\lambda}, \gamma_{S\lambda}}.$$

By the conjectural comparison above, this deformed algebra is expected to recover the total algebra of the Fukaya–Seidel subcategory generated by the partially wrapped collection:

$$R_{\tilde{A}_{S\lambda}, \gamma_{S\lambda}} \simeq R_{S\lambda}.$$

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