Cell decompositions of Hilbert schemes

by

Jacob Bucciarelli

B.A. California State University, Sacramento, 2017

M.A California State University, Sacramento, 2020

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2026

Abstract

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Using Bialynicki-Birula cell decomposition, one can decompose the Hilbert scheme of points in $\mathbb{C}^2$ into affine cells, which can be studied combinatorially using Young diagrams. Given a monomial ideal I corresponding to a partition $\lambda \vdash n$, we consider the subvariety of the Hilbert scheme of points in $\mathbb{C}^2$ consisting of ideals that contain I. Intersecting this subvariety with the Bialynicki-Birula cells yields subdivision. We wish to know when these subdivisions form a cell decomposition (i.e. each piece is homeomorphic to affine complex space). We study this question for a series of concave partitions.

This work then leads to a conjecture for a combinatorial statistic which can recover the polynomial: the dimension of those affine complex spaces. We then provide computational evidence which supports a specialized case of the conjecture given by Neguț.

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Major Professor
Dr. Mikhail Mazin

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Acknowledgments

I would first like to thank my advisor, Dr. Mikhail Mazin. Throughout graduate school, I faced a number of health-related challenges, and I am deeply grateful for his patience, understanding, and support during those difficult times. His willingness to answer my questions, no matter how many times I needed something re-explained, and his encouragement throughout my studies made it possible for me to navigate those challenges.

I would also like to thank Dr. Gabe Kerr, Dr. Pietro Poggi-Corradini, and Dr. Bala Natarajan for serving on my dissertation committee and for their valuable insights and feedback. I would especially like to thank Dr. Gabe Kerr and Dr. Pietro Poggi-Corradini for their leadership as Undergraduate Director and Department Head, respectively. Along with Dr. Virginia Naibo, Graduate Director, and the department staff, Shaunie Groundwater and Courtney Albin, they helped cultivate a more open, welcoming, and supportive environment for both current and future students.

I am especially grateful to my fellow graduate students, who became some of my closest friends throughout this journey. I would not be graduating without their constant encouragement and countless late nights we spent studying together in the library. In particular, I would like to thank Caleb Fernelius, Angel Fuerte-Perez, Huy Truong, Elizabeth Hale, Austin Crabtree, and Jesse Osnes.

Finally, I would like to thank Dr. Kim Logan and Dr. Craig Spencer for their invaluable advice, mentorship, and encouragement as I navigated the challenges of the academic job market.

Chapter 1

Definitions and Background

1.1 Preliminaries

Throughout this thesis we will be working with various Young diagrams. For the author's own preference, we will be using French notation when referring to such diagrams. Much of these preliminary definitions and combinatorial statistics are referenced from Haglund's book on q, t-Catalan numbers [Hag08].

Definition 1.1.1. *A partition is a non-increasing finite sequence of positive integers* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$*. Let* $l(\lambda)$ *denote the number of rows (height) of the diagram and* λ_1 *denote the number of columns (width). A Young diagram of* λ *is an array of, left-justified unit squares, called cells. For each* $1 \leq i \leq s$*,* λ_i *denotes the* i^{th} *row. In French notation,* λ_1 *refers to the bottom row. For a cell* $c = (i, j)$*, or* c_{ij}*, in a Young diagram* λ*, we define the* $arm(c)$ *as the number of cells, in the diagram of* λ*, to the right of* c *and the* $leg(c)$ *as the number of cells, in the diagram of* λ*, above* c*. The coordinates of cells start from zero, so that cell* (i, j) *is on the* $(i+1)st$ *row in the* $(j+1)st$ *column. See Example 1.1.1.*

Remark 1.1.1. Throughout this thesis, we identify a partition with its Young diagram, as well as a Dyck path (1.1.4) with the partition it defines. The exact interpretation of these objects will be clear from the context.

Example 1.1.1.

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
c_{03} & \multicolumn{3}{c}{} \\
\cline{1-2}
c_{02} & c_{12} & \multicolumn{2}{c}{} \\
\cline{1-2}
c_{01} & c_{11} & \multicolumn{2}{c}{} \\
\hline
c_{00} & c_{10} & c_{20} & c_{30} \\
\hline
\end{array}$$

Here, $\lambda = (4, 2, 2, 1)$.

For example, $arm(c_{01}) = 1$, $leg(c_{01}) = 2$ and $arm(c_{10}) = leg(c_{10}) = 2$.

Definition 1.1.2. *Let* $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$ *denote the* n*th Catalan number, so*

$$(C_0, C_1, C_2, \dots) = (1, 1, 2, 5, 14, 42, \dots).$$

Definition 1.1.3. *A lattice path is a sequence of South and East steps in the first quadrant of the* $\mathbb{Z}^2$*-lattice, starting at* $(0, n)$ *and ending at* $(m, 0)$ *where* $m, n \in \mathbb{Z}_{\geq 0}$*. Let* $L_{(m,n)}$ *denote the set of all such paths, and* $L^+_{(m,n)} \subset L_{(m,n)}$ *consist of paths which remain weakly below the line* $y = -\frac{n}{m}x + n$*.*

Definition 1.1.4. *Throughout, a Dyck path will refer to an element of* $L^+_{(n,n)}$*. More generally, an* (m, n)*-Dyck path is an element of* $L^+_{(m,n)}$*. When necessary, we will assume that* m *and* n *are coprime. Additionally, we note that* $C_n = \#L^+_{(n,n)}$*.*

The following definition for triangular partitions comes from Bergeron and Mazin [BM23b].

Definition 1.1.5. *A partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ *is said to be triangular if there exists positive real numbers* r *and* s *such that*

$$\lambda_j = \lfloor r - jr/s \rfloor,$$

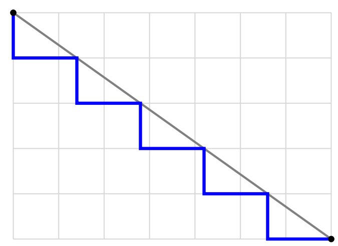
with j *running over integers that are less or equal to* s*. In other words, the cells* $c = (i, j) \in \mathbb{N}^2$ *of the triangular partition* λ_{rs} *are those that lie below the line joining* $(0, s)$ *to* $(r, 0)$*, i.e.*

$$(i, j) \in \lambda_{rs} \iff \frac{i+1}{r} + \frac{j+1}{s} \leq 1.$$

Triangular partitions have been considered under the name “plane corner cuts” in [OS99], and enumerated in [CRST99].

Remark 1.1.2. Young diagrams defined by Dyck paths in $L^+_{(r,s)}$ are sub-partitions of the triangular partition λ_{rs}. Under our convention, any Dyck path $\pi \in L^+_{(r,s)}$ can be viewed as a partition satisfying $\pi \subseteq \lambda_{rs}$.

Example 1.1.2.



Here we have the triangular partition $\lambda_{7,4}$, and all other Dyck paths $\pi \in L^+_{7,4}$ can be viewed as sub-partitions of $\lambda_{7,4}$.

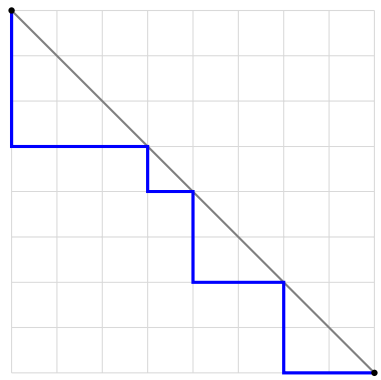
There are various combinatorial statistics, which are maps from $\{lattice\ paths\} \to \mathbb{Z}_{\geq 0}$. In particular, the area and dinv statistic are defined as follows.

Definition 1.1.6. *Let π be an (m,n)-Dyck path. Let $w_i(\pi)$ be the number of cells which lay to the right of the $((n+1)-i)$th row of π and weakly below the line $y = -\frac{n}{m}x + n$. Then the area sequence of π is $(w_1(\pi), w_2(\pi), \dots w_n(\pi))$, where w_1 refers to the top row, and the area of π is $area(\pi) = \sum_{i=1}^{n} w_i(\pi)$.*

Remark 1.1.3. The dinv statistic depends on the shape of the Young diagram. For clarity, we will start with the square case $m = n$ (Definition 1.1.7) before defining the statistic in full generality (Definition 1.1.10).

Definition 1.1.7. *Let $\pi \in L^+_{(n,n)}$ and let $(w_i(\pi))$ be its area sequence. For each i, define d_i to be the sum of all $j > i$ where $w_j = w_i$ or $w_j = w_i - 1$. Then $dinv(\pi) = \sum_{i=1}^{n} d_i$.*

Example 1.1.3. Let π be the Dyck path in the following picture.



The area sequence of π is $(0, 1, 2, 0, 0, 1, 0, 1)$ so that $area(\pi) = 5$
and $dinv(\pi) = 3 + 5 + 2 + 2 + 1 + 2 + 0 + 0 = 15$.

Definition 1.1.8. *The q, t-Catalan numbers are defined as* $C_n(q,t) = \sum\limits_{D} q^{area(D)} t^{dinv(D)}$*, where the sum is over all Dyck paths D in the $n \times n$ square.*

A more general construction exists where we replace the $n \times n$ square with a coprime $m \times n$ rectangle. The combinatorial statistics and Dyck paths are adjusted accordingly. This leads us to the rational Catalan numbers.

Definition 1.1.9. *Let m, n be coprime integers. The rational Catalan numbers are defined as* $C_{m,n} = \frac{1}{m+n}\binom{m+n}{m}$.

Remark 1.1.4. We note that this is a generalization of Definition 1.1.2 as setting $(m, n) = (n, n + 1)$ will recover the Catalan numbers.

Lemma 1.1.1. When m, n are co-prime integers, the rational Catalan numbers count the number of (m, n)-Dyck paths, $L^+_{(m,n)}$.

Proof. (*Sketch*) Let $\gcd(n, m) = 1$ and consider the set of lattice paths $L_{(m,n)}$, starting at $(0, n)$ and ending at $(m, 0)$ consisting of n South steps and m East steps. There are $\binom{m+n}{n}$ of these lattice paths by Defintion 1.1.4. Consider a $\mathbb{Z}_{n+m}$ action on $L_{(m,n)}$ where the action will shift the lattice path one position forward in its sequence of steps.

As $\gcd(n, m) = 1$, the $\mathbb{Z}_{n+m}$ action is free. That means for any path in $L_{(m,n)}$, all of the $m + n$ shifts are distinct. Note that each orbit has a unique path which stays under the diagonal. Then, there is a bijection

$$\{\text{orbits } of \ L_{(m,n)}\} \longleftrightarrow \{(m, n)\text{-Dyck Paths}\}.$$

Therefore, $|L^+_{(m,n)}| = |L_{(m,n)}/\mathbb{Z}_{n+m}|$. Thus, $|L^+_{(m,n)}| = \frac{1}{m+n}\binom{m+n}{n} = C_{m,n}$. □

Definition 1.1.10. *Let π be an (m, n)-Dyck path. Then*

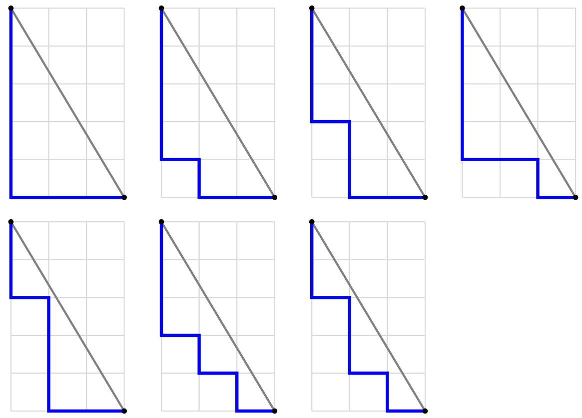
$$dinv(\pi) = \left|\left\{c \text{ below } \pi : \frac{arm(c)}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)}\right\}\right|.$$

Remark 1.1.5. There may be cases when $leg(c) = 0$ in which we will interpret $\frac{arm(c)+1}{leg(c)}$ as ∞. From now on, we will refer to this definition of dinv. Setting $m = n$ recovers the $n \times n$ case from Definition 1.1.7. This is a non-trivial result; see Lemma 3.3 in [GMV17].

The following theorem, which we will take as a definition, for rational q, t- Catalan numbers is due to Anton and Mellit [Mel21].

Definition 1.1.11. *Let m, n be coprime integers. The rational q, t-Catalan numbers are defined as $C_{m,n}(q, t) = \sum_{\pi \in L^+_{(m,n)}} q^{area(\pi)} t^{dinv(\pi)}$, where the sum is over all rational (m, n)-Dyck paths π in the $m \times n$ rectangle.*

Example 1.1.4. Consider $L^+_{(3,5)}$. There are seven elements here as follows:



Area statistic	dinv
$(0,0,1,1,2)$	0
$(0,0,1,1,1)$	1
$(0,0,1,0,1)$	2
$(0,0,1,1,0)$	1
$(0,0,0,0,1)$	2
$(0,0,1,0,0)$	3
$(0,0,0,0,0)$	4

Therefore, $C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + q^2t + qt^2 + qt^3 + t^4$.

Definition 1.1.12. *A partition λ is called concave if it consists of all boxes that fit under a concave down curve.*

The following two theorems provide a combinatorial characterization of concavity in terms of triangular partitions and horizontal step sequences. These theorems are adapted from Snellman and Paulsen. See [SP04] and [BM23a] for more background on enumerating concave partitions.

Theorem 1.1.1. A partition is concave if and only if it is a union of finitely many triangular partitions.

Definition 1.1.13. *A sequence $(a_1, a_2, \ldots, a_n)$ is called an almost non-increasing sequence if for any $i < j$ one has $a_j - a_i \leq 1$.*

Definition 1.1.14. [1] *For a partition λ, its horizontal step sequence $a = (a_1, a_2, \ldots, a_l)$ records the lengths of the maximal horizontal runs along the outer boundary of λ's Young diagram.*

Theorem 1.1.2. A partition λ with a horizontal step sequence $(a_1, a_2, \ldots, a_l)$ is concave if and only if for any $k \geq 1$ the sequence of length k partial sums of $(a_1, a_2, \ldots, a_l)$ is almost non-increasing.

Example 1.1.5. Consider the partition $\lambda = (9, 4, 3, 2, 1, 1, 1, 1, 1)$ and its horizontal step sequence $a = (5, 1, 1, 1, 0, 0, 0, 0, 1)$. Now for $k \geq 1$, we write out the sequence of length k partial sums of a, which we will label as a_{k_i}.

- $k = 1$, $a_{k_1} = (5, 1, 1, 1, 0, 0, 0, 0, 1)$
- $k = 2$, $a_{k_2} = (6, 2, 2, 1, 0, 0, 0, 1)$
- $k = 3$, $a_{k_3} = (7, 3, 2, 1, 0, 0, 1)$
- $k = 4$, $a_{k_4} = (8, 3, 2, 1, 0, 1)$
- $k = 5$, $a_{k_5} = (8, 3, 2, 1, 1)$
- $k = 6$, $a_{k_6} = (8, 3, 2, 2)$
- $k = 7$, $a_{k_7} = (8, 3, 3)$
- $k = 8$, $a_{k_8} = (8, 4)$
- $k = 9$, $a_{k_9} = (9)$

For all nine sequences, the largest increase from an earlier term to a later term is 1. So, all of these sequences are almost non-increasing. Thus λ is concave.

[1] Depending on the context, Definition 1.1.14 may or may not ignore the initial vertical segment from the southwest corner to the first row. For our purposes, we ignore it for the definition of concave partitions. Otherwise, we include it to match the notation in Equation 1.2.4.

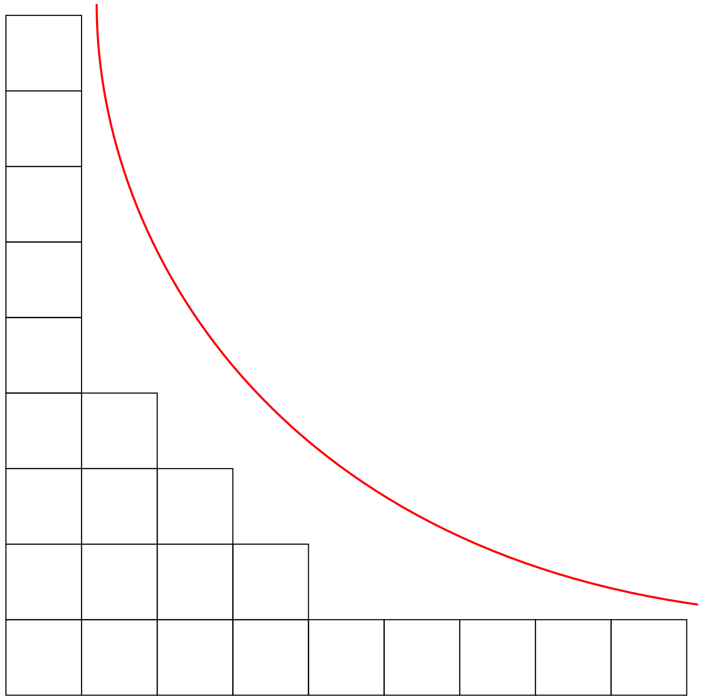


Figure 1.1: Concave partition $\lambda = (9, 4, 3, 2, 1, 1, 1, 1, 1)$

and its corresponding Young diagram

1.2 Motivation

Definition 1.2.1. *Let D be a Young diagram and let c_{ij} be a cell in D. The (q,t)-content of the cell c_{ij} is given by $q^i t^j$.*

Example 1.2.1. Consider the following Young diagram with (q,t)-content labeled by cell.

t^4			
t^3			
t^2			
t	qt	q^2t	
1	q	q^2	q^3

Definition 1.2.2. *A Young tableau is a Young diagram T where we label the cells with natural numbers. A standard Young tableau has labeling such that the entries in each row and column are increasing.*

Definition 1.2.3. *Given a standard tableau T of size n, we define a vector $z(T) = (z_k)_{1\leq k\leq n}$, where z_k is the (q,t)-content of the cell c_{ij} in T labeled by $k \in \mathbb{N}$.*

Example 1.2.2. Consider the tableau $T = (4, 2, 1, 1, 1)$

$$\begin{array}{|c|c|c|c}
\hline
9 \\
\cline{1-1}
8 \\
\cline{1-1}
7 \\
\hline
3 & 6 & 10 & \\
\hline
1 & 2 & 4 & \multicolumn{1}{c|}{5} \\
\hline
\end{array}$$

Then $z(T) = (1, q, t, q^2, q^3, qt, t^2, t^3, t^4, q^2t)$.

In 2020 Gorsky, Hawkes, Schilling, and Rainbolt [GHSR20] provide a purely combinatorial treatment of the following polynomial and show that in many cases the coefficients are nonnegative integers.

Definition 1.2.4.

$$f(a_1,\ldots,a_n) = \sum_T z_1^{a_1}\ldots z_n^{a_n} \prod_{i=2}^{n} \frac{1}{(1-z_i^{-1})(1-qtz_{i-1}/z_i)} \prod_{i<j} \omega(z_i/z_j), \tag{1.1}$$

where $a_i \geq 0$, the sum is over standard tableaux T with n boxes, z_k is the (q,t)-content q^it^j of the cell c_{ij} in T labeled by $k \in \mathbb{N}$, and $\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}$.

Definition 1.2.4 is motivated by the geometry of the flag Hilbert scheme of points on the plane. We briefly recall the relevant definitions and their connection with this polynomial, following [GHSR20].

Remark 1.2.1. As seen in Example 1.2.4, z_1 will always contribute a multiplication by 1, so the value of a_1, while necessary, is redundant. The authors of [GHSR20] re-index $f(a_1,\ldots,a_n)$ to $F(a_2,\ldots,a_n)$.

Definition 1.2.5. *The flag Hilbert scheme $FHilb^n(\mathbb{C}^2)$ is defined as the moduli space of flags*

$$FHilb^n(\mathbb{C}^2) = \{\mathbb{C}\,[x, y] = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n\},$$

where all I_k are ideals in $\mathbb{C}\,[x, y]$ of codimension k.

In chapter 3, we will discuss the torus action $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$, which lifts to an action on $FHilb^n(\mathbb{C}^2)$ and $Hilb^n(\mathbb{C}^2)$. The fixed points of this action correspond to the flags of monomial ideals which are in bijection with standard Young tableaux of size n.

Example 1.2.3. Consider the $n = 3$ case. The following diagram shows a monomial chain of ideals in $FHilb^3(\mathbb{C}^2)$ and its bijection to a chain of standard Young tableaux.

$$\begin{array}{ccccccc}
\mathbb{C}[x,y] & \supset & (x,y) & \supset & (x^2,y) & \supset & (x^2,xy,y^2) \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\varnothing & \supset & \begin{ytableau} 1 \end{ytableau} & \supset & \begin{ytableau} 1 & x \end{ytableau} & \supset & \begin{ytableau} y \\ 1 & x \end{ytableau}
\end{array}$$

The flag Hilbert scheme carries natural line bundles $\mathcal{L}_k = I_{k-1}/I_k$ which are equivariant with respect to the torus action. The (q, t)-content from the z_i's are the weights of a line bundle at a fixed point corresponding to a standard young tableaux T.

This polynomial gives a combinatorial model for approximating invariants of Coxeter links. Specifically, the polynomial computes the equivariant Euler characteristic of certain line bundles on the Hilbert scheme of points on the plane. It has connections to the elliptic Hall algebra action on symmetric functions which has a geometric realization via correspondence operators on the equivariant K-theory of the Hilbert schemes of $\mathbb{C}^2$ along with other connections to homology and shuffle theorems [SV13]. Using Tesler matrices, [GHSR20] shows Equation 1.1 is always a polynomial in q and t with integer coefficients.

Example 1.2.4. For $n = 2$, we compute $f(a_1, a_2)$.

Note that $z(T_{\ydiagram{1,1}}) = (1, t), z(T_{\ydiagram{2}}) = (1, q)$.

Then $T_{\square\square}$ contributes

$$q^{a_2} \cdot \frac{1}{(1-1/q)(1-qt/q)} \cdot \frac{(1-1/q)(1-qt/q)}{(1-q/q)(1-t/q)} = \frac{q^{a_2}}{1-t/q}.$$

Similarly, $T_{\substack{\square\\\square}}$ contributes $\frac{t^{a_2}}{(1-q/t)}$.

Then using geometric series,

$$f(a_1,a_2) = \frac{q^{a_2+1}-t^{a_2+1}}{q-t} = q^{a_2} + q^{a_2-1}t + \cdots + qt^{a_2-1} + t^{a_2}.$$

Remark 1.2.2. Note that in Example 1.2.4, that some of the individual factors in this product (both in the numerator and denominator) could vanish, and the convention is that we simply ignore these factors.

Remark 1.2.3. In the special case where $(a_1, a_2, \ldots, a_n)$ is the horizontal step sequence of a triangular partition (as defined in 1.1.5), Equation 1.1 agrees with the Rational q,t-Catalan numbers $C_{m,n}(q,t)$. This is a result Gorsky and Neguț [GN15].

Lemma 1.2.1 ([GHSR20])**.** Suppose that $a_i \geq 0$. Then $f(a_1,\ldots,a_n)$ is an integer polynomial in q and t. At $t=1$, this polynomial specializes to

$$f(a_1,\ldots,a_n)\Big|_{t=1} = \sum_{\mu\subseteq\lambda(a)} q^{|\lambda(a)|-|\mu|},$$

where $\lambda(a) = (a_2+\cdots+a_n, a_3+\cdots+a_n,\ldots,a_n)$ counts the number of horizontal steps.

Example 1.2.5. Recall Example 1.2.4 above. Note that $f(a_1,a_2) = C_{m,2} = \sum\limits_{D\subset\lambda(a_1,a_2)} t^{dinv(D)}q^{area(D)}$, where $\lambda(a_1,a_2) = \underbrace{\square\square\square\cdots\,\square}_{a_2}\underbrace{}_{a_1}$.

Note that $f(a_1,a_2)\Big|_{t=1} = 1+q+\cdots+q^{a_2}$ (satisfying Lemma 1.2.1).

In [GHSR20], the following conjecture was communicated to the authors by Andrei Neguț:

Conjecture 1.2.1. [Neguț] If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then $f(a_1, a_2, \ldots, a_n)$ is a polynomial in q and t with nonnegative coefficients.

It is expected that the higher cohomology of these line bundles vanish. This would imply the non-negativity of the coefficients of f. Even though the opposite implication is not known, Neguț's conjecture would still provide evidence supporting vanishing of the higher cohomology. For general $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, it is still an open problem to find an explicit statistic on partitions μ such that

$$f(a_1, \ldots, a_n) = \sum_{\mu \subseteq \lambda(a)} q^{|\lambda(a)| - |\mu|} t^{stat(\mu)}. \tag{1.2}$$

Currently, [GHSR20] proves this conjecture for $n \leq 3$ and $n = 4$, under more general conditions. In addition, they provide an explicit combinatorial statistic for $n = 3$.

For a certain class of concave partitions, this thesis seeks to provide a combinatorial statistic for the open problem given in Equation 1.2. We then provide computational evidence to verify the equality. This motivates the following specialization of Neguț's conjecture.

Conjecture 1.2.2. If $\lambda(a)$ is concave, then the coefficients of Equation 1.1 are nonnegative.

The underlying idea for the proof of this conjecture is to construct a variety $X_{\lambda,n}$ and intersect it with cell decompositions $C^{st}_{I_\mu}$, where the cells are enumerated by $\mu \subseteq \lambda$. Then, define $stat(\mu) = \dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$. We wish to show that these intersections are homeomorphic to affine spaces. This would then provide a candidate for the statistic in Equation 1.2. To this end, we study the Hilbert scheme of n points on the plane of $\mathbb{C}^2$, $Hilb^n(\mathbb{C}^2)$. Using Haiman charts [Hai98] and Bialynicki-Birula cell decomposition [BB73], we break down $Hilb^n(\mathbb{C}^2)$ into a disjoint union of simpler pieces (cells) which are known to be homeomorphic to affine spaces. The following describes the general construction of the methods used.

Chapter 2

Hilbert Schemes and Haiman Charts

2.1 Haiman Charts

Definition 2.1.1. *The Hilbert scheme of n points on the plane*

$$Hilb^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \ : \ \dim(\mathbb{C}[x,y]/I) = n\}$$

is defined as the moduli space of ideals of codimension n.

Definition 2.1.2. *Let $\lambda \subset (\mathbb{Z}_{\geq 0})^2$ be a Young diagram and $\overline{\lambda} := (\mathbb{Z}_{\geq 0})^2 \setminus \lambda$ be its complement. Then, define $I_\lambda = \langle x^k y^m | (k,m) \in \overline{\lambda} \rangle$ as the corresponding monomial ideal in $\mathbb{C}[x,y]$.*

Example 2.1.1. For $n = 3$, consider the ideals $I(a) = \langle x^2 + ay, xy, y^2 \rangle$ and $I_{\square\square\square} = \langle x^3, y \rangle$ where $I(a), I_{\square\square\square} \subset \mathbb{C}[x,y]$. Since $\mathbb{C}[x,y]/I(a) \cong \langle 1, x, y \rangle$ and $\mathbb{C}[x,y]/I_{\square\square\square} \cong \langle 1, x, x^2 \rangle$, both have dimension 3 and therefore are elements of $Hilb^3(\mathbb{C}^2)$.

Given a Young diagram λ of size n, Haiman introduced the open affine subvariety, called a Haiman chart, U_λ. This gives us a nice way to visualize a subset of $Hilb^n(\mathbb{C}^2)$ based on a fixed Young diagram.

Definition 2.1.3. *Let $U_\lambda \subset Hilb^n(\mathbb{C}^2)$ such that $U_\lambda := \{I \subset \mathbb{C}[x,y] : \{x^k y^m : (k,m) \in \lambda\}$ spans $\mathbb{C}[x,y]/I\}$. The set $\{x^k y^m : (k,m) \in \lambda\}$ provides a basis for $\mathbb{C}[x,y]/I$ for all*

$I \in U_\lambda$. Therefore, for every $I \in U_\lambda$ and any pair $(r,s)_{\mathbb{Z}_{\geq 0}}$, one has

$$x^r y^s \equiv \sum_{(k,m)\in\lambda} c_{rs}^{km}(I) x^k y^m \ (mod\ I).$$

Furthermore, we define

$$p_{r,s}^I := x^r y^s - \sum_{(k,m)\in\lambda} c_{rs}^{km}(I) x^k y^m \in I. \tag{2.1}$$

For a fixed $\lambda, (r,s), (k,m)$, one has a function $c_{rs}^{km} : U_\lambda \to \mathbb{C}$ such that $I \mapsto c_{rs}^{km}(I)$, the coefficient of $x^k y^m$ in the reduction of $x^r y^s$ *mod* I. The coefficients $c_{rs}^{km}(I)$ depend on the ideal I and form a collection of regular functions on U_λ.

Theorem 2.1.1. [Hai98] Let λ be a Young diagram of size n and let μ be a partition of n. The U_μ are open affine subvarieties which cover $Hilb^n(\mathbb{C}^2)$. The functions

$$\{c_{r,s}^{k,m}\}_{(k,m)\in\lambda}$$

generate the coordinate ring $\mathcal{O}_\lambda$ of U_λ.

There are relations on the c_{rs}^{km} coefficients as follows.

Lemma 2.1.1.

$$c_{r+1,s}^{km} = \sum_{(a,b)\in\lambda} c_{rs}^{ab} c_{a+1,b}^{km}$$

and

$$c_{r,s+1}^{km} = \sum_{(a,b)\in\lambda} c_{rs}^{ab} c_{a,b+1}^{km}$$

Proof. Multiplying 2.1 by x, we note that

$$I \ni xP_{r,s}^I = x^{r+1} y^s - \sum_{(a,b)\in\lambda} c_{rs}^{ab} x^{a+1} y^b$$

$$= x^{r+1}y^s - \sum_{(a,b)\in\lambda} c_{rs}^{ab}\big(P_{a+1,b}^I + \sum_{(k,m)\in\lambda} c_{a+1,b}^{km}x^k y^m\big)$$

$$\Rightarrow x^{r+1}y^s - \sum_{(a,b)\in\lambda}\sum_{(k,m)\in\lambda} c_{rs}^{ab}c_{a+1,b}^{km}x^k y^m \in I.$$

Since $P_{r+1,s}^I$ is unique, we can equate the above coefficients with the coefficients of $P_{r+1,s}^I$ to get

$$c_{r+1,s}^{km} = \sum_{(a,b)\in\lambda} c_{rs}^{ab}c_{a+1,b}^{km}.$$

In a similar fashion, multiplying by y instead will lead to

$$c_{r,s+1}^{km} = \sum_{(a,b)\in\lambda} c_{rs}^{ab}c_{a,b+1}^{km}.$$

$\square$

Example 2.1.2. Let $\lambda = (2)$ and consider $I = \left\langle p_{0,1}^I, p_{2,0}^I \right\rangle = \langle y - c_{01}^{00} - c_{01}^{10}x, x^2 - c_{20}^{00} - c_{20}^{10}x\rangle \in U_\lambda$. Note that $xp_{0,1}^I + c_{01}^{10}(I)p_{2,0}^I = xy - (c_{01}^{00}(I) + c_{01}^{10}(I)c_{20}^{10}(I))x - c_{01}^{10}(I)c_{20}^{00}(I))$ and since these $p_{r,s}^I$ are unique, this must be $p_{1,1}^I$. Thus, $c_{01}^{00}(I) + c_{01}^{10}(I)c_{20}^{10}(I) = c_{11}^{10}(I)$ and $c_{01}^{10}(I)c_{20}^{00}(I) = c_{11}^{00}(I)$.

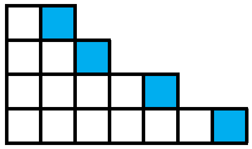
Definition 2.1.4. *We will define the east and north boundary of D as follows*

$$EB(\lambda) = \{(k,m) \notin \lambda \ : \ (k-1,m) \in \lambda\}$$
$$NB(\lambda) = \{(k,m) \notin \lambda \ : \ (k,m-1) \in \lambda\}.$$

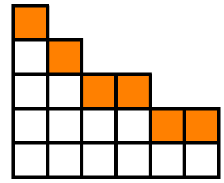
For $(r,s) \in \lambda$ we note that

$$c_{rs}^{km} = \begin{cases} 1, & (r,s) = (k,m); \\ 0, & otherwise \end{cases}$$

Example 2.1.3.



$EB(\lambda) = \{(1,3),(2,2),(4,1),(6,0)\}$



$NB(\lambda) =$

$\{(0,4),(1,3),(2,2),(3,2),(4,1),(5,1)\}$

Using Lemma 2.1.1 and Definition 2.1.4, we can further extend the relations for the coefficients c^{km}_{rs}. Suppose that $(r-1,s) \in \overline{\lambda}$, $(k,m) \in \lambda$, and $k > 0$. Then we get

$$c^{k,m}_{r,s} \underset{2.1.1}{=} \sum_{(a,b)\in\lambda} c^{a,b}_{r-1,s} c^{k,m}_{a+1,b} = c^{k-1,m}_{r-1,s} + \sum_{(a,b)\in EB(\lambda)} c^{a-1,b}_{r-1,s} c^{k,m}_{a,b}. \tag{2.2}$$

Similarly, if $(r,s-1) \in \overline{\lambda}$, $(k,m) \in \lambda$, and $m > 0$, we get

$$c^{k,m}_{r,s} = c^{k,m-1}_{r,s-1} + \sum_{(a,b)\in NB(\lambda)} c^{a,b-1}_{r,s-1} c^{k,m}_{a,b}. \tag{2.3}$$

Furthermore, if $(r-1,s) \in \overline{\lambda}$ and $(0,m) \in \lambda$, then

$$c^{0,m}_{r,s} = \sum_{(a,b)\in EB(\lambda)} c^{a-1,b}_{r-1,s} c^{0,m}_{a,b}. \tag{2.4}$$

Similarly, if $(r,s-1) \in \overline{\lambda}$ and $(k,0) \in \lambda$, then

$$c^{k,0}_{r,s} = \sum_{(a,b)\in NB(\lambda)} c^{a,b-1}_{r,s-1} c^{k,0}_{a,b}. \tag{2.5}$$

We will extend the notations by setting $c^{km}_{rs} := 0$ for any $(k,m) \notin \lambda$, in particular for any negative k or m. Now relations 2.4 and 2.5 are special cases of relations 2.2 and 2.3, respectively. However, it is often important to emphasize the distinction.

The following relations on the basic polynomials $p^I_{r,s}$ follow from the relations 2.2 and 2.3

Corollary 2.1.1.

$$p^I_{r+1,s} = xp^I_{r,s} + \sum_{(k,m)\in EB(\lambda)} c^{k-1,m}_{r,s} p^I_{k,m}, \tag{2.6}$$

and

$$p^I_{r,s+1} = yp^I_{r,s} + \sum_{(k,m)\in NB(\lambda)} c^{k,m-1}_{r,s} p^I_{k,m}. \tag{2.7}$$

Given an ideal $I \in U_\lambda$, one gets a unique basis $\{p^I_{r,s}\}_{(r,s)\in\bar{\lambda}}$ with polynomials $p^I_{r,s}$ satisfying the above relations.

Example 2.1.4. This example highlights the equality from equation 2.2

c_{03}			
c_{02}	c_{12}		
c_{01}	c_{11}		
c_{00}	c_{10}	c_{20}	c_{30}

Lets consider $(k,m) = (2,0) = c_{20} \in \lambda$, $r = 3, s = 3$ with Young diagram $\lambda = (4,2,2,1)$. Then,

$$c^{20}_{33} = c^{00}_{23}c^{20}_{10} + c^{01}_{23}c^{20}_{11} + c^{02}_{23}c^{20}_{12} + c^{03}_{23}c^{20}_{13} + c^{10}_{23}c^{20}_{20} + c^{11}_{23}c^{20}_{21} + c^{12}_{23}c^{20}_{22} + c^{20}_{23}c^{20}_{30} + c^{30}_{23}c^{20}_{40}$$

$$\underset{2.1.4}{=} \quad c^{10}_{23} + c^{03}_{23}c^{20}_{13} + c^{11}_{23}c^{20}_{21} + c^{12}_{23}c^{20}_{22} + c^{20}_{23}c^{20}_{30} + c^{30}_{23}c^{20}_{40},$$

noting that the red coefficients are zero and the cyan coefficient is 1.

2.1.1 Algebro-Geometric Background and Connection to c^{km}_{rs}

Let $X := \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \cdots = f_k(x) = 0\} \subset \mathbb{C}^n$ where $\{f_1, f_2, \ldots, f_k\}$ is a set of polynomial functions over $\mathbb{C}$ such that $f_i(X) = 0$. Then consider the ideal $I = \langle f_1, f_2, \ldots, f_k\rangle$ and the coordinate ring on X, $\mathcal{O}_X := \mathbb{C}[x_1, x_2, \ldots, x_n]/I$.

Lemma 2.1.2. For any $x \in X$, the set $M_x := \{[f] \in \mathcal{O}_X : f(x) = 0\}$, is a maximal ideal of $\mathcal{O}_X$. Additionally, over $\mathbb{C}$, all maximal ideals of $\mathcal{O}_X$ are of the form M_x.

Proof. Fix $x \in X$. Consider the natural evaluation map $ev_x : \mathcal{O}_X \to \mathbb{C}$ where $ev_x([f]) = f(x)$. Note that ev_x is a well-defined surjective ring homomorphism with $\ker(ev_x) = M_x$. By the first isomorphism theorem,

$$\mathcal{O}_X/M_x \cong \mathbb{C}.$$

Thus, $\mathcal{O}_X/M_x$ is a field and therefore M_x is a maximal ideal of $\mathcal{O}_X$.

To show the reverse direction, consider any maximal ideal M of $\mathcal{O}_X$. By the correspondence theorem for rings, there is a bijection

$$\{\textit{Ideals of}\ \mathbb{C}[x_1, x_2, \ldots, x_n]\ \textit{that contain}\ I\} \longleftrightarrow \{\textit{Ideals of}\ \mathcal{O}_X\}$$

given by the natural projection map $\pi : \mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathcal{O}_X$ where $\pi(f) = [f]$. Then, there exists a maximal ideal $\widetilde{M} = \pi^{-1}(M)$ such that $I \subset \widetilde{M} \subset \mathbb{C}[x_1, x_2, \ldots x_n]$. By Hilbert Nullstellensatz, every maximal ideal of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is of the form $\widetilde{M} = \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n\rangle$, for some $x_0 = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. Consider a generator $f_i \in I \subset \widetilde{M}$. Then, $f_i = \sum_{j=1}^{n} h_j(x_j - a_j)$, where $h_j \in \mathbb{C}[x_1, \ldots, x_n]$ and we note that $f_i(x_0) = 0\ \forall i \in \{1, 2, \ldots, k\}$. Thus, $x_0 \in X$ which implies that M_{x_0} is a maximal ideal of $\mathcal{O}_X$. Now note that $\pi^{-1}(M_{x_0}) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(x_0) = 0\}$, which is maximal in $\mathbb{C}[x_1, \ldots, x_n]$ as π is surjective. By Nullstellensatz, $\pi^{-1}(M_{x_0}) = \langle x_1 - b_1, x_2 - b_2, \ldots, x_n - b_n\rangle$ for some $b \in \mathbb{C}^n$. Since each generator $x_i - b_i$ vanishes at x_0, we have $a_i - b_i = 0$, hence $b_i = a_i$. Then $\pi^{-1}(M_{x_0}) = \widetilde{M} = \pi^{-1}(M)$. Since π is surjective, we get that $M_{x_0} = M$. $\square$

Now we would like to apply this to our situation as follows. Let λ be a Young diagram of size n. By 2.1.1, we have an open affine subvariety U_λ of $Hilb^n(\mathbb{C}^2)$ and coordinate ring $\mathcal{O}_{U_\lambda}$ on U_λ generated by $\{c_{rs}^{km}\}_{(k,m)\in\lambda}$. Then, for $I_\lambda = \langle x^k y^m : (k, m) \in \overline{\lambda}\rangle \in U_\lambda$, we consider the maximal ideal $M_{I_\lambda} = \{f \in \mathcal{O}_{U_\lambda} : f(I_\lambda) = 0\}$ of $\mathcal{O}_{U_\lambda}$. Note that for any $(r, s) \in \overline{\lambda}$ and $(k, m) \in \lambda$ one has $c_{rs}^{km}(I_\lambda) = 0$. It then follows that the maximal ideal M_{I_λ} is generated by the functions $\{c_{rs}^{km} : (r, s) \in \overline{\lambda}, (k, m) \in \lambda\}$.

Definition 2.1.5. *The cotangent space of U_λ at a point I_λ is defined as $T^*_{I_\lambda} U_\lambda = M_{I_\lambda}/M^2_{I_\lambda}$.*

2.2 Classes of Essential Arrows

A natural way of visualizing a function $c_{r,s}^{k,m} : U_\lambda \to \mathbb{C}$ is via the arrow $[(r, s) \to (k, m)]$ pointing from a box (r, s) outside the diagram λ to a box (k, m) inside of λ. See Figure 2.1

for an illustration of arrows and the constructions below.

Definition 2.2.1. *Two arrows are called equivalent if they can be translated one into another so that the tail of the arrow remains in $\overline{\lambda}$, while the head of the arrow remains in λ. In other words, the equivalence relation is generated by the following two conditions:*

1. *Arrows $[(r,s) \to (k,m)]$ and $[(r-1,s) \to (k-1,m)]$ are equivalent whenever $(r,s),(r-1,s) \in \overline{\lambda}$ and $(k,m),(k-1,m) \in \lambda$.*

2. *Arrows $[(r,s) \to (k,m)]$ and $[(r,s-1) \to (k,m-1)]$ are equivalent whenever $(r,s),(r,s-1) \in \overline{\lambda}$ and $(k,m),(k,m-1) \in \lambda$.*

Rearranging relations 2.2 and 2.3 we have

$$c_{r,s}^{k,m} - c_{r-1,s}^{k-1,m} = \sum_{(a,b)\in EB(\lambda)} c_{r-1,s}^{a-1,b} c_{a,b}^{k,m} \tag{2.8}$$

and

$$c_{r,s}^{k,m} - c_{r,s-1}^{k,m-1} = \sum_{(a,b)\in NB(\lambda)} c_{r,s-1}^{a,b-1} c_{a,b}^{k,m}, \tag{2.9}$$

which implies that the functions corresponding to equivalent arrows are equal modulo $(M_{I_\lambda})^2$.

Definition 2.2.2. *An arrow is called escaping if it can translated so that the tail remains in $\overline{\lambda}$ and the head crosses one of the coordinate axes. In other words, the arrow is escaping if it is equivalent to $[(r,s) \to (k,m)]$ such that either $k = 0$ and $(r-1,s) \in \overline{\lambda}$ or $m = 0$ and $(r,s-1) \in \overline{\lambda}$. An arrow from $\overline{\lambda}$ to λ that is not escaping is called essential.*

See Figure 2.1 on the left and in the center for equivalence classes of essential and escaping arrows respectively. Relations 2.4 and 2.5 imply that if $[(r,s) \to (k,m)]$ is escaping, then $c_{r,s}^{k,m} \in (M_{I_\lambda})^2$. It follows that the equivalence classes of essential arrows correspond to a basis in the cotangent space $M_{I_\lambda}/(M_{I_\lambda})^2$. Note that every strictly South-West arrow, i.e. $[(r,s) \to (k,m)]$ where $k < r$ and $m < s$, is escaping.

Definition 2.2.3. *An arrow* $[(r,s) \to (k,m)]$ *is called a NW-arrow if* $m \geq s$ *and* $k < r$. *A NW-arrow* $[(r,s) \to (k,m)]$ *is called extreme if it cannot be moved North or West, i.e.* $(r-1,s) \in \lambda$ *and* $(k,m+1) \in \overline{\lambda}$.

Similarly, an arrow $[(r,s) \to (k,m)]$ *is called a SE-arrow if* $m < s$ *and* $k \geq r$. *A SE-arrow* $[(r,s) \to (k,m)]$ *is called extreme if it cannot be moved South or East, i.e.* $(r,s-1) \in \lambda$ *and* $(k+1,m) \in \overline{\lambda}$.

It follows that every equivalence class of essential arrows contains exactly one extreme arrow (see Figure 2.1 on the left for an example of an extreme NW representative). Furthermore, there is a bijection between the set of boxes of λ and the set of extreme NW-arrows.

The bijection between the boxes of λ and the extreme NW-arrows is given by

$$(a,b) \mapsto [(a + arm(a,b) + 1, b) \to (a, b + leg(a,b))].$$

Similarly, the correspondence

$$(a,b) \mapsto [(a, b + leg(a,b) + 1) \to (a + arm(a,b), b)]$$

gives a bijection between the set of boxes of λ and the set of extreme SE-arrows. See Figure 2.1 on the right for an illustration of the *leg* and *arm* statistics, and the extreme arrows corresponding to a box.

In particular, one gets that $\dim(M_{I_\lambda}/M_{I_\lambda}^2) = 2|\lambda| = \dim(Hilb^n(\mathbb{C}^2))$, which is consistent with $Hilb^n(\mathbb{C}^2)$ being smooth. One also concludes that functions $c_{r,s}^{k,m}$ corresponding to the extreme arrows $[(r,s) \to (k,m)]$ form a local coordinate system near $I_\lambda \in U_\lambda$.

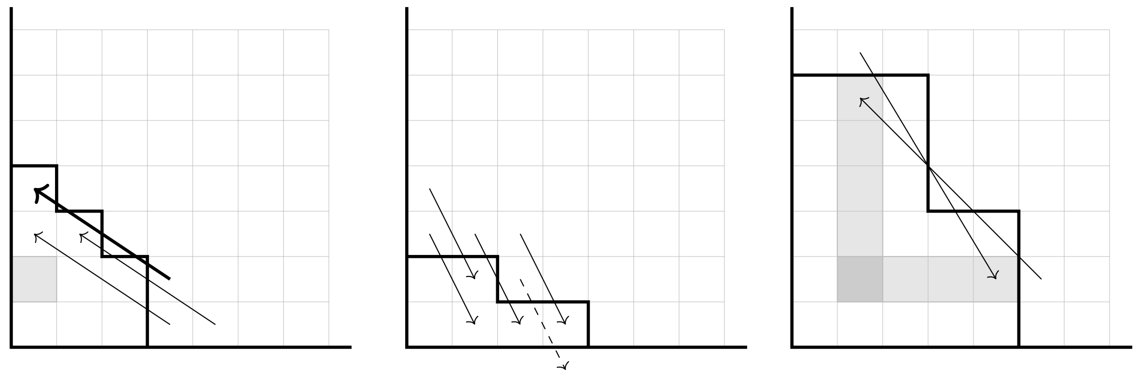


Figure 2.1: On the left: an equivalence class of essential arrows. The extreme representative $[(3,1) \to (0,3)]$ is in bold. In the center: an equivalence class of escaping arrows. On the right: the box $(1,1)$ and the two corresponding extreme arrows. Note that $arm(1,1) = 3$ and $leg(1,1) = 4$.

2.3 Torus Action

Definition 2.3.1. *The Torus action* $(\mathbb{C}^*)^2$ *acts on* $\mathbb{C}^2$ *as follows*

$$(t,s) \cdot (x,y) \mapsto (t^{-1}x, s^{-1}y).$$

It also acts on $\mathbb{C}[x,y]$ *in a similar manner:*

$$(t,s) \cdot p(x,y) \mapsto p(t^{-1}x, s^{-1}y).$$

Together this implies that $(\mathbb{C}^*)^2$ acts on $Hilb^n(\mathbb{C}^2)$.

Remark 2.3.1. The fixed points of $Hilb^n(\mathbb{C}^2)$ are in bijection with the monomial ideals enumerated by partitions of n. We also note that this action maps ideals to ideals and preserves dimension. That is to say

$$\dim_{\mathbb{C}} (\mathbb{C}[x,y]/I) = \dim_{\mathbb{C}} (\mathbb{C}[x,y]/(s,t) \cdot I).$$

Example 2.3.1. Let I be fixed by $(\mathbb{C}^*)^2$ and consider $I \ni p(x,y) = x^2 + xy + y^3$. Note that $(1,t) \cdot p(x,y) = x^2 + \frac{xy}{t} + \frac{y^3}{t^3}$ and as $t \to \infty$, we see that $x^2 \in I$ as $Hilb^n(\mathbb{C}^2)$ is closed. Using $(t,1)$ we can similarly show that $y^3 \in I$ and lastly, $p(x,y) - (x^2 + y^3) = xy \in I$.

Lemma 2.3.1. The c_{rs}^{km} functions are eigenvectors of the torus action $(\mathbb{C}^*)^2$.

Proof. Let $I \in U_\lambda \subset Hilb^n(\mathbb{C}^2)$ and let $p_{r,s}^I \in I$ be a basis polynomial. Now we apply the action:

$$(t_1,t_2)\cdot p_{r,s}^I(x,y) = (t_1,t_2)\cdot(x^r y^s - \sum_{(k,m)\in\lambda} c_{rs}^{km}(I)x^k y^m) = \left(\frac{1}{t_1^r t_2^s}\right)x^r y^s - \sum_{(k,m)\in\lambda} t_1^{-k}t_2^{-m}c_{rs}^{km}(I) \in (t_1,t_2)\cdot I.$$

Multiplying through by $t_1^r t_2^s$ gives

$$x^r y^s - \sum_{(k,m)\in\lambda} t_1^{r-k}t_2^{s-m}c_{rs}^{km}(I) \in (t_1,t_2)\cdot I.$$

Note that a polynomial of this form is unique in U_λ, thus we must have that

$$p_{r,s}^{(t_1,t_2)\cdot I} = x^r y^s - \sum_{(k,m)\in\lambda} t_1^{r-k}t_2^{s-m}c_{rs}^{km}(I) \in (t_1,t_2)\cdot I.$$

Therefore, $c_{km}^{rs}((t_1,t_2)\cdot I) = t_1^{r-k}t_2^{s-m}c_{rs}^{km}(I)$. Hence, we see that $(t_1,t_2)\cdot c_{rs}^{km} = t_1^{r-k}t_2^{s-m}c_{rs}^{km}$ $\square$

Definition 2.3.2. *Fix positive coprime integers (p,q) and let $T_{p,q} := \{(t^p,t^q) \in (\mathbb{C}^*)^2\}$ be a one-parameter subgroup of $(\mathbb{C}^*)^2$. More specifically, $T_{p,q}$ is the image of an injective homomorphism $f : \mathbb{C}^* \to (\mathbb{C}^*)^2$ where $t \mapsto (t^p,t^q)$.*

Definition 2.3.3. *A polynomial $f(x,y) \in \mathbb{C}[x,y]$ is (p,q)-quasihomogeneous of degree d if for every $t \in \mathbb{C}^*$, $f(t^p x, t^q y) = t^d f(x,y)$. Equivalently, f is (p,q)-quasihomogeneous of degree d if $ap + bq = d$ for all monomial terms $x^a y^b$ of f.*

Example 2.3.2. Consider $f(x,y) = x^4 y + y^3 \in \mathbb{C}[x,y]$. Then f is $(1,2)$-quasihomogenous of degree 6 since $f(tx,t^2y) = t^6 f(x,y)$. In addition, we can verify by checking each term, $4(1) + 1(2) = 6$ and $0(1) + 3(2) = 6$.

Definition 2.3.4. *An ideal $I \in Hilb^n(\mathbb{C}^2)$ is called invariant (or fixed) under the torus action $(\mathbb{C}^*)^2$ if $(t_1,t_2)\cdot I = I$ for all $(t_1,t_2) \in (\mathbb{C}^*)^2$.*

Remark 2.3.2. We note that for $I \in Hilb^n(\mathbb{C}^2)$, the generators of I are (p,q)-quasihomogenous functions if and only if I is fixed by the $T_{p,q}$-action.

The following theorem was communicated to us by Mikhail Mazin (unpublished). The result also follows from the work of Buryak and Feigin [BF14]. For completeness of the exposition, we include a proof here.

Theorem 2.3.1. The $T_{p,q}$-action on the Hilbert scheme $Hilb^n(\mathbb{C}^2)$ has isolated fixed points only if and only if the following inequality holds:

$$p + q > n,$$

in which case the fixed points coincide with the fixed points of the $(\mathbb{C}^*)^2$-action
(i.e. the monomial ideals).

Proof. First, let us prove that if $n \geq p+q$ then there exist non-isolated fixed points. Without loss of generality one can assume that $q \geq p$. Consider the following ideals:

$$I_{p+q+k}(\lambda) := \langle xy^{1+k}, x^2y, x^q + \lambda y^p \rangle, \quad k \in \{0, 1, \ldots, p-1\},$$
$$I_{2p+q}(\lambda) := \langle x^2y, x^q + \lambda y^p \rangle,$$

where $\lambda \neq 0$. We note that $I_n(\lambda) \in Hilb^n(\mathbb{C}^2)$ for $n \in \{p+q, \ldots, 2p+q-1\}$ (see Figure 2.2 for an illustration). Furthermore, since the generators are (p,q)-quasihomogeneous, it follows that these ideals are fixed by $T_{p,q}$. One can extend this construction by setting

$$I_{n+(p+1)}(\lambda) := x \cdot I_n(\lambda) + \langle y^{p+1} \rangle$$

for all $n \geq p+q$. This provides non-isolated fixed points for the $T_{p,q}$ action on $Hilb^n(\mathbb{C}^2)$ for $n \geq p+q$.

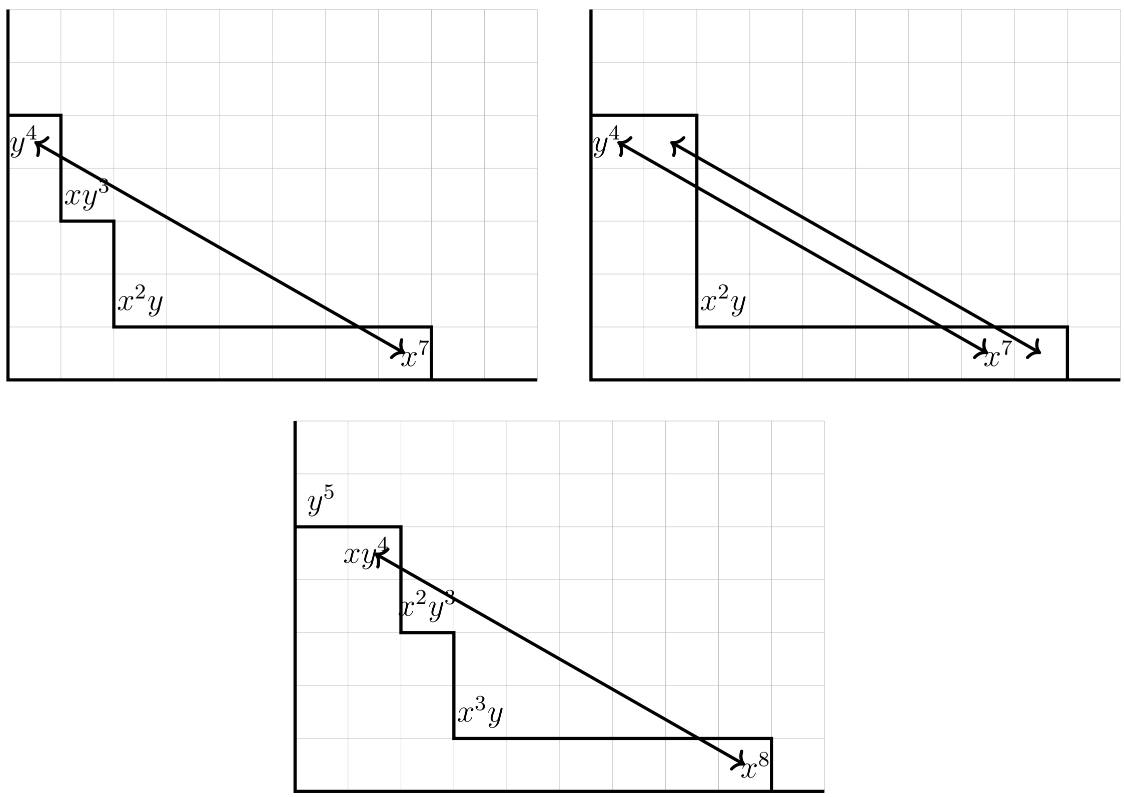


Figure 2.2: In this example we have $(p, q) = (4, 7)$. On the left we schematically illustrate the ideal $I_{13} = I_{4+7+2} = \langle xy^3, x^2y, x^7 + y^4 \rangle \in Hilb^{13}(\mathbb{C}^2)$, and on the right we illustrate $I_{15} = I_{2\cdot 4+7} = \langle x^2y, x^7 + y^4 \rangle \in Hilb^{15}(\mathbb{C}^2)$.

Now, suppose that an ideal $I \in Hilb^n(\mathbb{C}^2)$ is fixed by $T_{p,q}$. It then follows that I is generated by (p,q)-quasihomogeneous polynomials. Suppose that I is not monomial. Let d be the minimal integer such that the dth quasihomogeneous component of I does not have a monomial basis. Then there exists a polynomial

$$p(x,y) = x^l y^{m+kp} + a_1 x^{l+q} y^{m+(k-1)p} + \ldots + a_k x^{l+kq} y^m \in I,$$

such that

1. $pl + qm + kpq = d$, i.e. $p(x,y)$ is of quasi-degree d,
2. monomials
$$\{x^l y^{m+kp}, x^{l+q} y^{m+(k-1)p}, \ldots, x^{l+(k-1)q} y^{m+p}\}$$
are linearly independent modulo the ideal I. (It then follows that $a_k \neq 0$ and the monomials
$$\{x^{l+q} y^{m+(k-1)p}, \ldots, x^{l+(k-1)q} y^{m+p}, x^{l+kq} y^m\}$$
are also linearly independent modulo I.)

Existence of such a polynomial is easy to verify (with $k > 0$). But then none of the monomials

$$x^l y^m, x^l y^{m+1}, \ldots, x^l y^{m+kp-1}, x^{l+1} y^m, \ldots, x^{l+kq-1} y^m$$

can be in I and, since the quasihomogeneous components of I of degrees less than d are generated by monomials, they are also linearly independent modulo I. Furthermore, one can add $x^l y^{m+kp}$ or $x^{l+kq} y^m$ to the list, and they are still going to be linearly independent modulo I. We conclude that

$$n = \dim \mathbb{C}[x,y]/I \geq kp + kq \geq p + q.$$

We illustrate the construction in Figure 2.3. $\square$

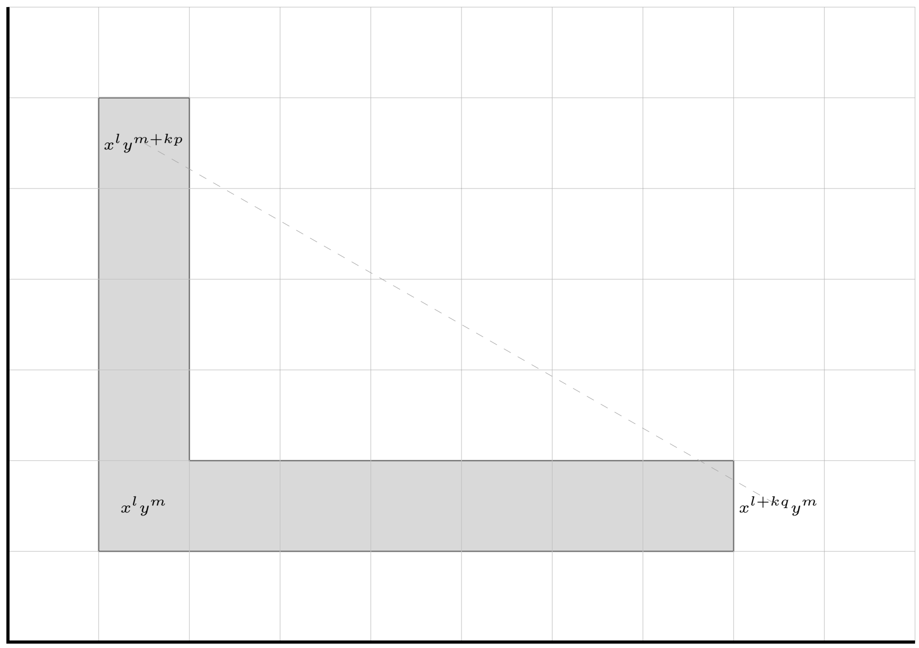
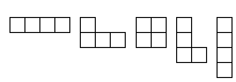


Figure 2.3: Illustration for the second part of the proof of Theorem 2.3.1: the monomials in the highlighted area are linearly independent modulo I.

Example 2.3.3. Let $a \in \mathbb{C}$, $n = 4$. Consider the one-parameter family of ideals, $I_a = \langle xy + ay^2, x^2, y^3 \rangle \in Hilb^4(\mathbb{C}^2)$. Note $T_{1,1} \curvearrowright I_a = \left\langle \frac{xy+ay^2}{t^2}, \frac{x^2}{t^2}, \frac{y^3}{t^3} \right\rangle = \langle xy + ay^2, x^2, y^3 \rangle = I_a$. Thus, $T_{1,1} \curvearrowright I_a$ is fixed by the action. Now consider $T_{3,5} \curvearrowright I_a = \left\langle \frac{xy}{t^8} + a\frac{y^2}{t^{10}}, \frac{x^2}{t^6}, \frac{y^3}{t^{15}} \right\rangle = \left\langle xy + \frac{ay^2}{t^2}, x^2, y^3 \right\rangle$. Note that $T_{3,5} \curvearrowright I_a = I_a \iff a = 0$. More generally, for $p + q > 4$, the only fixed ideals are the monomial ones. That is $\langle x^4, y \rangle$, $\langle x^3, xy, y^2 \rangle$, $\langle x^2, y^2 \rangle$, $\langle x^2, xy, y^3 \rangle$, and $\langle x, y^4 \rangle$. These correspond to partitions , respectively.

2.4 Bialynicki-Birula Cells

Throughout this section we will assume that $p, q > 0$ with $\gcd(p, q) = 1$ and $p + q > n = |\lambda|$, where λ is a Young diagram.

Definition 2.4.1. *According to Bialynicki-Birula [BB73],[BB76] this $T_{p,q}$-action defines two cell decompositions, stable and unstable as follows*

$$C^{st}_{I_\lambda} = \{ I \in Hilb^n(\mathbb{C}^2) \ : \ \lim_{t \to \infty} (t^p, t^q) \cdot I = I_\lambda \}$$

$$C_{I_\lambda}^{unst} = \{I \in Hilb^n(\mathbb{C}^2) \ : \ \lim_{t\to 0}(t^p,t^q)\cdot I = I_\lambda\},$$

where $C_{I_\lambda}^{st}$ is a decomposition of the zero fiber $Hilb_0^n(\mathbb{C}^2)$ and $C_{I_\lambda}^{unst}$ is a decomposition of $Hilb^n(\mathbb{C}^2)$. These cells are known to be homeomorphic to affine spaces.

Let $I \in C_{I_\lambda}^{st}$. Then, for any $(r,s) \in \overline{\lambda}$. By definition,

$$\lim_{t\to\infty}(t^p,t^q)\cdot P_{r,s}^I = x^r y^s.$$

Thus, it must be the case that $(t^p,t^q)\cdot c_{rs}^{ks}(I) = 0$. By Lemma 2.3.1, we know that $(t^p,t^q)\cdot c_{rs}^{km}(I) = (t^p)^{r-k}(t^q)^{s-m}c_{rs}^{km}(I)$, which then implies that $c_{rs}^{km}(I) = 0$ or $p(r-k)+q(s-m) < 0$. Assuming this set-up, we give the following definition.

Definition 2.4.2. *The $c_{rs}^{km}(I)$ coefficients of $P_{r,s}^I$ are called positive with respect to $T_{p,q}$ if $p(r-k)+q(s-m) < 0$.*

Lemma 2.4.1. For $I \in C_{I_\lambda}^{st}$, the $c_{rs}^{km}(I)$ which are not positive with respect to $T_{p,q}$ are zero. Furthermore, without loss of generality, for $p \ll q$, only the $c_{rs}^{km}(I)$ corresponding to northwest arrows are non-zero.

Proof. Using formulas (2.2-2.5), we can express any positive $c_{rs}^{km}(I)$ as a polynomial of extreme positives, where the extreme positives are the coordinates on $C_{I_\lambda}^{st}$. Therefore, all non-positive $c_{rs}^{km}(I)$ are zero. For the other part, note that since $(r,s) \in \overline{\lambda}$ and $(k,m) \in \lambda$, at least one of the $r-k$ or $s-m$ is positive. As $p \ll q$, we get $r-k < 0$ or $s-m > 0$. These cases only hold for northwest arrows. $\square$

The following example will provide a blueprint for the general methods used to prove Theorem 3.1.1 in Chapter 4.

Example 2.4.1. Let $\mu = (2,1,1)$, $I_\mu = \langle y^3, xy, x^2\rangle$, and let $p \ll q$. For $I \in C_{I_\mu}^{st}$, we know only the northwest arrows will be nonzero by Lemma 2.4.1. So, I must be of the form $I = \langle y^3, xy - c_{11}^{02}(I)y^2, x^2 - c_{20}^{01}(I)y - c_{20}^{02}(I)y^2\rangle$. Note that $c_{20}^{01}(I)$ is an escaping arrow.

Using formulas 2.6 and 2.7, we can express it in terms of extremes. First, we shift north by multiplying P^I_{02} by y to get

$$P^I_{21} = yP^I_{20} + c^{02}_{20}(I)P^I_{03}.$$

Then equating coefficients gives $c^{01}_{20}(I) = c^{02}_{21}(I)$. Now, to shift west we multiply P^I_{11} by x to get

$$P^I_{21} = xP^I_{11} + c^{02}_{11}(I)P^I_{12}.$$

Equating again gives $c^{02}_{21}(I) = 0$. Thus, $c^{01}_{20}(I) = 0$. Therefore,

$$C^{st}_{I_\mu} = \{\langle y^3, xy - c^{02}_{11}(I)y^2, x^2 - c^{02}_{20}(I)y^2\rangle : I \in Hilb^4(\mathbb{C}^2)\}.$$

We can verify by checking the following

$$\lim_{t\to\infty}(t^p, t^q) \cdot \langle y^3, xy - c^{02}_{11}(I)y^2, x^2 - c^{02}_{20}(I)y^2\rangle = \left\langle \frac{y^3}{t^{3q}}, \frac{xy}{t^{p+q}} - \frac{c^{02}_{11}(I)y^2}{t^{2q}}, \frac{x^2}{t^{2p}} + \frac{c^{02}_{20}(I)y^2}{t^{2q}} \right\rangle = I_\mu$$

(the terms with $c^{02}_{11}(I), c^{02}_{20}(I)$ go to zero in the limit). Thus, $C^{st}_{I_\mu} \cong \mathbb{C}^2$.

Theorem 2.4.1. The Haiman chart U_λ is invariant under the torus action $(\mathbb{C}^*)^2$.

Proof. Recall the definition of U_λ given in Defintion 2.1.3. Note that for any $I \notin U_\lambda$, we have that $\{x^k y^m : (k,m) \in \lambda\}$ does not provide a basis for $\mathbb{C}[x,y]/I$.

Then there exists some relation

$$\sum_{(k,m)\in\lambda} d^{k,m}(I)x^k y^m \equiv 0 \ (mod\ I)$$

where not all $d^{k,m}$ are zero. So we have that $\sum_{(k,m)\in\lambda} d^{k,m}x^k y^m \in I$.

We will now act on this element by the torus.

$$(t_1, t_2) \cdot \sum_{(k,m)\in\lambda} d^{k,m}x^k y^m = \sum_{(k,m)\in\lambda} d^{k,m}t_1^{-k}t_2^{-m}x^k y^m \in (t_1, t_2) \cdot I.$$

This implies that $(t_1, t_2) \cdot I \notin U_\lambda$ as the action just rescales the coefficients. Hence the complement of U_λ is invariant under $(\mathbb{C}^*)^2$. As the torus action is invertible, we then have that U_λ is also invariant under $(\mathbb{C}^*)^2$. $\square$

Corollary 2.4.1. For all Young diagrams λ and all one-parameter subgroups $T_{p,q}$, we have that $C^{st}_{I_\lambda} \subset U_\lambda$ and $C^{unst}_{I_\lambda} \subset U_\lambda$.

Proof. First note that U_λ is open and contains I_λ. By definition, for $I \in C^{st}_{I_\lambda}$, we have that $U_\lambda \cap T_{p,q} \cdot I \neq \emptyset$. Thus, there exists some ideal $I' \in U_\lambda \cap T_{p,q} \cdot I$. By theorem 2.4.1, we must have $T_{p,q} \cdot I' \subset U_\lambda$. Then, As both I' and I lie in the same $T_{p,q}$-orbit, we have $T_{p,q} \cdot I \subset U_\lambda$. Therefore, $I \in U_\lambda$. The same process can be applied to show $C^{unst}_{I_\lambda} \subset U_\lambda$. $\square$

Let $\lambda \vdash n$, $\mu \subseteq \lambda$, and $\gcd(p,q)_{\mathbb{Z}_{\geq 0}} = 1$. Let $X_{\lambda,n} \subset Hilb^n(\mathbb{C}^2)$ be the subvariety consisting of ideals that contain I_λ. Then one has the following diagram:

$$\begin{array}{ccccccc} X_{\lambda,n} & \subset & Hilb^n_0(\mathbb{C}^2) & \subset & Hilb_n(\mathbb{C}^2) & = & \bigcup_{\mu \vdash n} U_\mu \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \\ & & \bigsqcup_{\mu \vdash n} C^{st}_{I_\mu} & & \bigsqcup_{\mu \vdash n} C^{unst}_{I_\mu} & & \end{array}$$

where $C^{st}_{I_\mu} \cong \mathbb{C}^k$ and $C^{unst}_{I_\mu} \cong \mathbb{C}^{2n-k}$.

Consider

$$X_{\lambda,n} = \bigsqcup_{\mu \vdash n} C^{st}_{I_\mu} \cap X_{\lambda,n}.$$

Question 2.4.1. For which $\lambda, T_{p,q}$ are all of these intersections $C^{st}_{I_\mu} \cap X_{\lambda,n}$ homeomorphic to $\mathbb{C}^k$ for some k?

Question 2.4.2. For the cases in which the above is true, we set $stat(\mu) = \dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$. Does this statistic satisfy Equation 1.2?

In chapter 4, we define an (r, s) polynomial using $stat(\mu)$. We then show that Equation 1.2 is equal to our (r, s) polynomial, up to a change of variable. We provide computational evidence to support this.

The connection between the generalized q,t-Catalan numbers and combinatorics is already established in the cases for when λ is triangular [GN15] and for $\ell(\lambda) \leq 4$ [GHSR20]. In addition, hook shaped partitions are straight-forward. In Chapter 3, we study another family of partitions $\lambda = (\ell, 2, \underbrace{1, 1, \ldots, 1}_{(k-2)-times})$ and apply the geometric approach described in this chapter.

Chapter 3

Results and Future Directions

3.1 Notation and Set-Up

Throughout this chapter, we assume that $p, q > 0$ with $\gcd(p, q) = 1$, $p \ll q$, and $p + q > n = |\lambda|$, where λ is a Young diagram. Let $u \leq k \in \mathbb{N}$, $v \leq \ell \in \mathbb{N}$. The following figure provides an arbitrary sub-partition $\mu \subseteq \lambda$ where $\lambda = (\ell, 2, \underbrace{1, 1, \ldots, 1}_{(k-2)-times})$ with height $l(\lambda) = k$, width $\lambda_1 = \ell$, $|\lambda| = n$. Given, $\mu \subseteq \lambda$, we will be considering extreme arrows of $\mathcal{O}_{U_\mu}$. Figure 3.1 shows an arbitrary case.

Theorem 3.1.1. For the four general cases and the seven special cases, $C^{st}_{I_\mu} \cap X_{\lambda,n} \cong \mathbb{C}^k$, where $k \in \{0, 1, 2\}$.

Since this proof requires us checking each $\mu \subseteq \lambda$, we break it into general and special cases. For each case, we list the northwest extreme arrows and then it suffices to check that the northwest arrows emanating from the intersection of the north and east boundary union $\{c_{0,k}, c_{\ell,0}\}$ do not contribute any non-trivial relations on these extreme arrows. See figure 3.2 for diagrams of the general cases and figure 3.3 for diagrams of the special cases. Before we start the process of handling each case, there is a useful lemma that we will prove first.

Lemma 3.1.1. For $(r, s) \in \overline{\lambda}$ and $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$, the coefficients $c^{km}_{rs}(I)$ of $P^I_{r,s}$ are all zero.

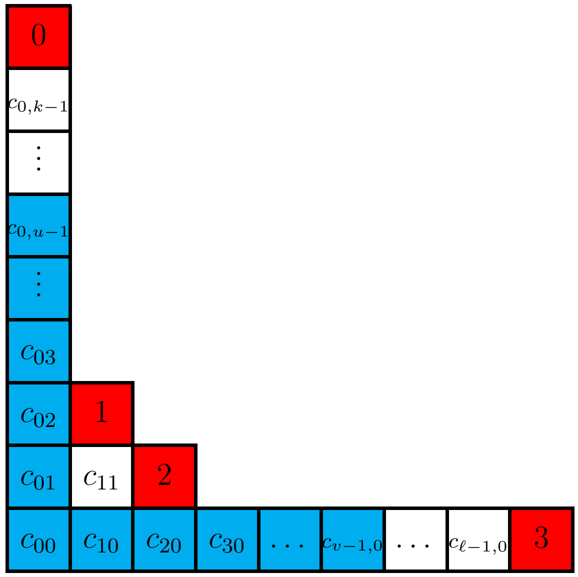
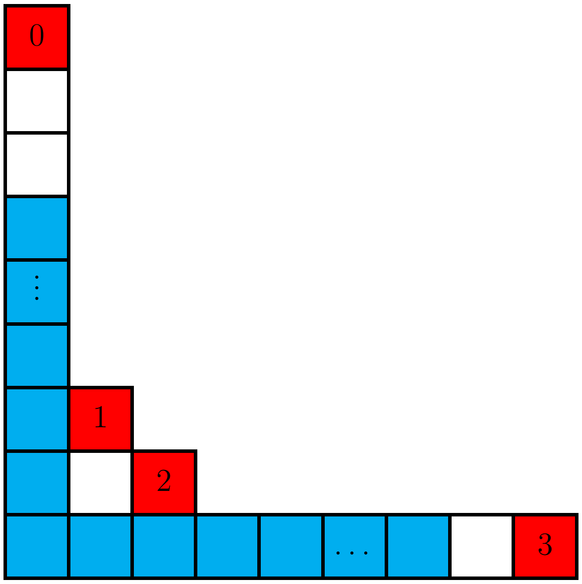


Figure 3.1: Pictorially, the cyan boxes denote an arbitrary sub-partition $\mu = (v, \underbrace{1,1,1,1}_{(u-1)-times}) \subset \lambda$ while the red boxes denote the intersection of the north and east boundary union $\{c_{0,k}, c_{\ell,0}\}$.

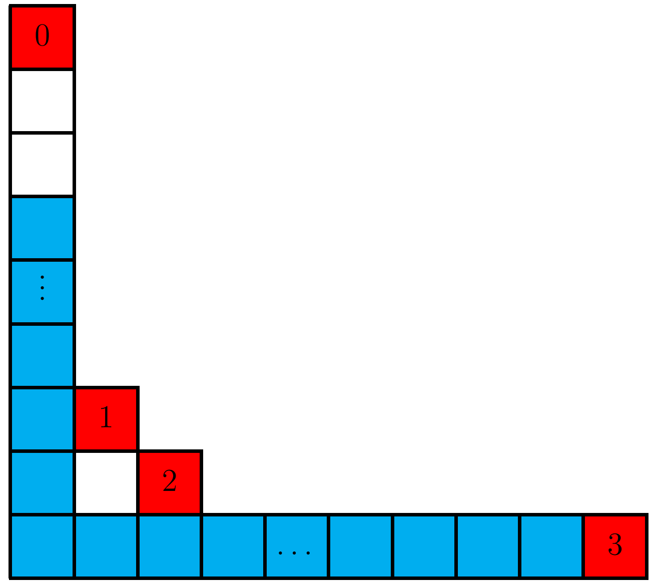
Proof. Let $(r,s) \in \overline{\lambda}$ and $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$. Consider $P^I_{r,s} = x^r y^s - \sum_{(k,m)\in\mu} c^{km}_{rs}(I) x^k y^m \in I$. As $(r,s) \in \overline{\lambda}$, we have that $x^r y^s \in I$ which implies that $\sum_{(k,m)\in\mu} c^{km}_{rs}(I) x^k y^m \in I$. By Corollary 2.4.1, we have that $C^{st}_{I_\mu} \subset U_\mu$, so that $I \in U_\mu$. Thus, the set $\{x^k y^m : (k,m) \in \mu\}$ provides a basis for $\mathbb{C}[x,y]/I$. Then, we must have that $c^{km}_{rs}(I) = 0$ for $(k,m) \in \mu$. Thus, $P^I_{r,s}$ is just the monomial $x^r y^s$. $\square$

Remark 3.1.1. Arrows whose tail emanate from $(r,s) \in \overline{\lambda}$ come in three types: escaping, essential, and extreme. Although Lemma 3.1.1 tells us that these coefficients are zero, we must still express these arrows in terms of extremes (if needed) to check that any possible relations on the non-zero extremes are trivial.

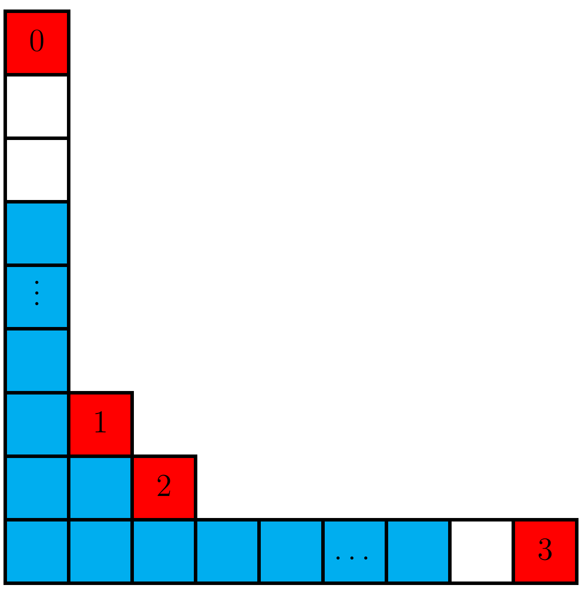
For each case, we will denote E as the set of northwest extreme arrows (possibly including zero) and T_1, T_2, T_3 as the northwest arrows whose tail emanates from box 1, 2, or 3, respectively.



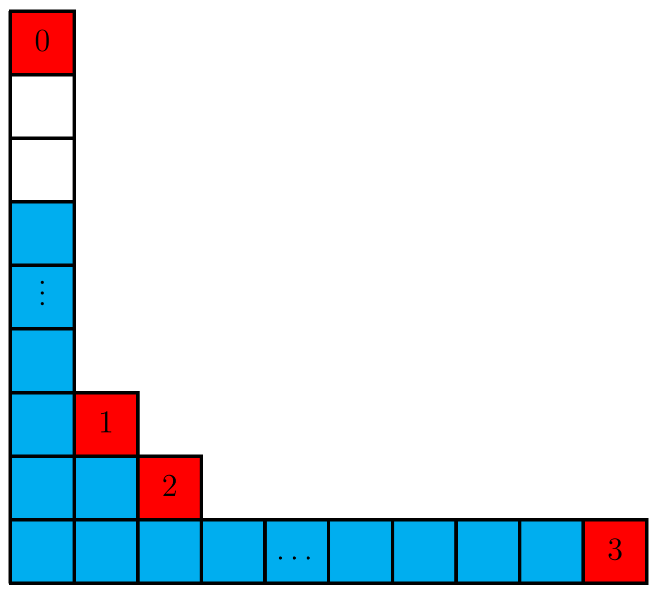
Case 1: $v \geq 2, u \geq 3$, with box $(1,1)$ not included and at least one box of the first row of λ not contained in μ
$\mathbb{C}^2$



Case 2: $v = \ell, u \geq 3$, with box $(1,1)$ not included
$\mathbb{C}$



Case 3: $v \geq 2, u \geq 3$, with box $(1,1)$ included and at least one box of the first row of λ not contained in μ
$\mathbb{C}^2$



Case 4: $v = \ell, u \geq 3$, with box $(1,1)$ included
$\mathbb{C}^0$

Figure 3.2: The four general cases

Proof. (Of Theorem 3.1.1) To simplify notation, we suppress the dependence on I throughout the proof and write c^{km}_{rs} and P_{rs} in place of $c^{km}_{rs}(I)$ and P^I_{rs}, respectively. We also note that, for each case, by Lemma 3.1.1, $T_1 = T_2 = T_3 = \{0\}$.

3.2 Case 1

Let $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$ and consider the collection of northwest extreme arrows
$E := \{c^{0,u-1}_{11}, c^{0,u-1}_{12}, c^{0,u-1}_{13}, \ldots, c^{0,u-1}_{1,u-2}, c^{0,u-1}_{v,0}\} \subseteq \mathcal{O}_{U_\mu}$.

3.2.1 Box 1

Note that the northwest arrows whose tail is in box 1 are $T_1 := \{c^{03}_{12}, c^{04}_{12}, \ldots, c^{0,u-1}_{12}\}$. Using Lemma 3.1.1, we can conclude that $c^{0,u-1}_{1,i} = 0$ for $2 \le i \le u-2$. Now we will express the arrows of T_1 in terms of extremes. We will shift them north as follows:

$$P_{12} = xy^2 - c^{03}_{12}y^3 - c^{04}_{12}y^4 - \cdots - c^{0,u-1}_{12}y^{u-1} \in I$$

$$yP_{12} = xy^3 - c^{03}_{12}y^4 - c^{04}_{12}y^5 - \cdots - c^{0,u-2}_{12}y^{u-1} - c^{0,u-1}_{12}y^u \in I$$

As $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$, we have that $y^u \in I$ so that $P_{13} = yP_{12} + c^{0,u-1}_{12}y^u$. Now, we equate coefficients:

$$\begin{aligned} c^{03}_{12} &= c^{04}_{13} \\ c^{04}_{12} &= c^{05}_{13} \\ &\vdots \\ c^{0,u-2}_{12} &= c^{0,u-1}_{13} \end{aligned} \tag{3.1}$$

Observe that $c^{0,u-1}_{13}$ is already zero, as noted above. Thus, $c^{0,u-2}_{12} = 0$. We can continue this process of shifting north until all remaining elements of T_1 have been expressed as an extreme arrow that is zero. Thus, T_1 gives no relations on E.

3.2.2 Box 2

Note that all arrows with tail in box 2 are escaping. Those arrows are $T_2 := \{c^{02}_{21}, c^{03}_{21}, \ldots, c^{0,u-1}_{21}\}$. Consider the following equations:

$P_{21} = x^2y - c_{21}^{02}y^2 - c_{21}^{03}y^3 - \cdots - c_{21}^{0,u-1}y^{u-1} \in I$
$P_{11} = xy - c_{11}^{02}y^2 - c_{11}^{03}y^3 - \cdots - c_{11}^{0,u-1}y^{u-1} \in I$
$xP_{11} = x^2y - c_{11}^{02}xy^2 - c_{11}^{03}xy^3 - \cdots - c_{11}^{0,u-1}xy^{u-1} \in I$. Note that

$$P_{21} = xP_{11} + \sum_{2 \leq j \leq u-1} c_{11}^{0,j}P_{1,j}.$$

Equating coefficients, we get the following:

$$\begin{aligned}
c_{21}^{02} &= 0 \\
c_{21}^{03} &= c_{11}^{02}c_{12}^{03} \\
c_{21}^{04} &= c_{11}^{02}c_{12}^{04} + c_{11}^{03}c_{13}^{04} \\
c_{21}^{05} &= c_{11}^{02}c_{12}^{05} + c_{11}^{03}c_{13}^{05} + c_{11}^{04}c_{14}^{05} \\
&\vdots \\
c_{21}^{0,u-1} &= c_{11}^{02}c_{12}^{0,u-1} + c_{11}^{03}c_{13}^{0,u-1} + \cdots + c_{11}^{0,u-2}c_{1,u-2}^{0,u-1}.
\end{aligned} \tag{3.2}$$

Note that each term of the equations given in Equation 3.2 has a factor of $c_{1,i}^{0,j}$ for some $3 \leq j \leq u-1$, $2 \leq i \leq u-2$. By the conditions imposed in Section 3.2.1, these $c_{1,i}^{0,j}$ are zero with no relations on E. Thus, T_2 gives no relations on E.

3.2.3 Box 3

Note that $T_3 := \{c_{\ell,0}^{01}, c_{\ell,0}^{02}, \ldots, c_{\ell,0}^{0,u-1}\}$. Consider these equations:
$P_{\ell-1,0} = x^{\ell-1} - c_{\ell-1,0}^{01}y - c_{\ell-1,0}^{02}y^2 + \cdots - c_{\ell-1,0}^{0,u-1}y^{u-1}$
$xP_{\ell-1,0} = x^{\ell} - c_{\ell-1,0}^{01}xy - c_{\ell-1,0}^{02}xy^2 - \cdots - c_{\ell-1,0}^{0,u-1}xy^{u-1}$
Then,

$$P_{\ell,0} = xP_{\ell-1,0} + \sum_{1 \leq j \leq u-1} c_{\ell-1,0}^{0,j}P_{1,j}.$$

Equating coefficients gives

$$
\begin{aligned}
c^{01}_{\ell,0} &= 0 \\
c^{02}_{\ell,0} &= c^{01}_{\ell-1,0}c^{02}_{11} \\
c^{03}_{\ell,0} &= c^{01}_{\ell-1,0}c^{03}_{11} + c^{02}_{\ell-1,0}c^{03}_{12} \\
c^{04}_{\ell,0} &= c^{01}_{\ell-1,0}c^{04}_{11} + c^{02}_{\ell-1,0}c^{04}_{12} + c^{03}_{\ell-1,0}c^{04}_{13} \\
&\vdots \\
c^{0,u-1}_{\ell,0} &= c^{01}_{\ell-1,0}c^{0,u-1}_{11} + c^{02}_{\ell-1,0}c^{0,u-1}_{12} + \cdots + c^{0,u-2}_{\ell-1,0}c^{0,u-1}_{1,u-2}.
\end{aligned}
\tag{3.3}
$$

For the equations given in Equation 3.3, By the conditions imposed in Section 3.2.1, all but the first terms are zero with no relations on E. For the first term in each equation, the non-escaping coefficients $c^{0,j}_{11}$, $2 \le j \le u-2$ can be shifted north. Similar to Section 3.2.1, note that $P_{12} = yP_{11} - c^{0,u-1}_{11}y^u$ and equating coefficients gives us the following:

$$
\begin{aligned}
c^{02}_{11} &= c^{03}_{12} \\
c^{03}_{11} &= c^{04}_{12} \\
&\vdots \\
c^{0,u-2}_{11} &= c^{0,u-1}_{12}
\end{aligned}
\tag{3.4}
$$

Again, as shown in Section 3.2.1, we know that the extreme arrow $c^{0,u-1}_{12} = 0$, thus $c^{0,u-2}_{11} = 0$. We can continue this process of shifting north until all remaining elements of T_3 have been expressed as an extreme arrow that is zero. Thus, T_3 gives no relations on E.

To conclude case 1, we have shown that $T_1 = T_2 = T_3 = \{0\}$ and are free of relations on $E' = \{c^{0,u-1}_{11}, c^{0,u-1}_{v,0}\}$. Therefore, $C^{st}_{I_\mu} \cap X_{\lambda,n} \cong \mathbb{C}^2$.

3.3 Case 2

Let $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$ and consider the collection of extreme arrows $E := \{c^{0,u-1}_{11}, c^{0,u-1}_{12}, c^{0,u-1}_{13}, \ldots, c^{0,u-1}_{1,u-2}, c^{0,u-1}_{\ell,0}\} \subseteq \mathcal{O}_{U_\mu}$.

Remark 3.3.1. Following the same line of reasoning given in Sections 3.2.1 and 3.2.2, we can conclude that $c_{1,i}^{0,u-1} = 0$ for $2 \leq i \leq u-2$ and that $T_1 = T_2 = \{0\}$ are free of relations on E. Using Lemma 3.1.1, we see that $c_{\ell,0}^{0,u-1} = 0$, so the only non-zero extreme arrow is $E' = \{c_{11}^{0,u-1}\}$. Note that this remark accounts for boxes 1 and 2, so only the case of box 3 needs to be checked.

3.3.1 Box 3

Note that $T_3 := \{c_{\ell,0}^{01}, c_{\ell,0}^{02}, \ldots, c_{\ell,0}^{0,u-1}\}$. Similar to Section 3.2.3, except for $c_{\ell,0}^{0,u-1}$, all other arrows in T_3 are escaping and we use a similar argument to show they will not impose any new relations on the extremes. Instead of shifting west first, we need to shift north and then west. We first note that shifting north one unit will not add any extra relations on the coefficients. Equating $P_{\ell,0}$ to $yP_{\ell,0}$ leads to

$$\begin{aligned} c_{\ell,0}^{01} &= c_{\ell,1}^{02} \\ c_{\ell,0}^{02} &= c_{\ell,1}^{03} \\ &\vdots \\ c_{\ell,0}^{0,u-2} &= c_{\ell,1}^{0,u-1} \end{aligned} \tag{3.5}$$

Now, shifting west one unit gives

$$P_{\ell,1} = xP_{\ell-1,1} + \sum_{2 \leq j \leq u-1} c_{\ell-1,1}^{0,j} P_{1,j}.$$

Equating coefficients then gives us

$$
\begin{aligned}
c^{01}_{\ell,0} &= 0 \\
c^{02}_{\ell,0} &= c^{02}_{\ell-1,1}c^{03}_{12} \\
c^{03}_{\ell,0} &= c^{02}_{\ell-1,1}c^{04}_{12} + c^{03}_{\ell-1,1}c^{04}_{13} \\
c^{04}_{\ell,0} &= c^{02}_{\ell-1,1}c^{05}_{12} + c^{03}_{\ell-1,1}c^{05}_{13} + c^{04}_{\ell-1,1}c^{05}_{14} \\
&\vdots \\
c^{0,u-2}_{\ell,0} &= c^{02}_{\ell-1,1}c^{0,u-1}_{12} + c^{03}_{\ell-1,1}c^{0,u-1}_{13} + \cdots + c^{0,u-2}_{\ell-1,1}c^{0,u-1}_{1,u-2}.
\end{aligned} \tag{3.6}
$$

We now note that, besides the last equation, each term in the equations given in Equation 3.7 has a non-escaping factor which is zero by conditions imposed by Remark 3.3.1. For the last equation, we know again by Remark 3.3.1 that each term has an extreme arrow which is zero as a factor. Therefore $T_3 = \{0\}$.

To conclude case 2, we have shown that $T_1 = T_2 = T_3 = \{0\}$ and are free of relations on $E' = \{c^{0,u-1}_{11}\}$. Therefore, $C^{st}_{I_\mu} \cap X_{\lambda,n} \cong \mathbb{C}$.

3.4 Case 3

Let $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$ and consider the collection of extreme arrows
$E := \{c^{0,u-1}_{12}, c^{0,u-1}_{13}, \ldots, c^{0,u-1}_{1,u-2}, c^{0,u-1}_{21}, c^{11}_{v,0}, c^{0,u-1}_{v,0}\} \subseteq \mathcal{O}_{U_\mu}$.

3.4.1 Box 1

From Section 3.2.1, we can conclude that $c^{0,u-1}_{1,i} = 0$ for $2 \leq i \leq u-2$ and $T_1 = \{0\}$ is free of relations on E.

3.4.2 Box 2

We will shift the arrows of $T_2 = \{c_{21}^{02}, c_{21}^{03}, \ldots, c_{21}^{0,u-1}\}$ north and then west. This will follow the same steps as in Section 3.3.1. Equating P_{22} to yP_{21} gives

$$\begin{aligned} c_{21}^{02} &= c_{22}^{03} \\ c_{21}^{03} &= c_{22}^{04} \\ &\vdots \\ c_{21}^{0,u-2} &= c_{22}^{0,u-1}. \end{aligned} \tag{3.7}$$

Now, shifting west one unit gives

$$P_{22} = xP_{12} + \sum_{3 \leq j \leq u-1} c_{12}^{0,j} P_{1,j}.$$

Equating coefficients then gives us

$$\begin{aligned} c_{21}^{02} &= 0 \\ c_{21}^{03} &= c_{12}^{03} c_{13}^{04} \\ c_{21}^{04} &= c_{12}^{03} c_{13}^{05} + c_{12}^{04} c_{14}^{05} \\ c_{21}^{05} &= c_{12}^{03} c_{13}^{06} + c_{12}^{04} c_{14}^{06} + c_{12}^{05} c_{15}^{06} \\ &\vdots \\ c_{21}^{0,u-2} &= c_{12}^{03} c_{13}^{0,u-1} + c_{12}^{04} c_{14}^{0,u-1} + \cdots + c_{12}^{0,u-2} c_{1,u-2}^{0,u-1}. \end{aligned} \tag{3.8}$$

Note that each term in every equation of Equation 3.8 has a factor with tail emanating from c_{12}. These arrows are zero by conditions imposed by Section 3.4.1. Thus, they are all zero by Lemma 3.1.1. Lastly, $c_{21}^{0,u-1}$ is also zero from Lemma 3.1.1. So, T_2 is free of relations on E.

3.4.3 Box 3

Note $T_3 := \{c^{01}_{\ell,0}, c^{02}_{\ell,0}, \ldots, c^{0,u-1}_{\ell,0}, c^{11}_{\ell,0}\}$. Similar to Section 3.2.3, we will shift west one unit. This time however, there are some slight changes when equating the coefficients.

$$P_{\ell,0} = xP_{\ell-1,0} + \sum_{2\leq j\leq u-1} c^{0,j}_{\ell-1,0}P_{1,j} + c^{11}_{\ell-1,0}P_{21}.$$

Equating coefficients gives

$$\begin{aligned}
c^{11}_{\ell,0} &= c^{01}_{\ell-1,0}\\
c^{01}_{\ell,0} &= 0\\
c^{02}_{\ell,0} &= c^{11}_{\ell-1,0}c^{02}_{21}\\
c^{03}_{\ell,0} &= (c^{02}_{\ell-1,0}c^{03}_{12}) + c^{11}_{\ell-1,0}c^{03}_{21}\\
c^{04}_{\ell,0} &= (c^{02}_{\ell-1,0}c^{04}_{12} + c^{03}_{\ell-1,0}c^{04}_{13}) + c^{11}_{\ell-1,0}c^{04}_{21}\\
c^{05}_{\ell,0} &= (c^{02}_{\ell-1,0}c^{05}_{12} + c^{03}_{\ell-1,0}c^{05}_{13} + c^{04}_{\ell-1,0}c^{05}_{14}) + c^{11}_{\ell-1,0}c^{05}_{21}\\
&\vdots\\
c^{0,u-1}_{\ell,0} &= (c^{02}_{\ell-1,0}c^{0,u-1}_{12} + c^{03}_{\ell-1,0}c^{0,u-1}_{13} + \cdots + c^{0,u-2}_{\ell-1,0}c^{0,u-1}_{1,u-2}) + c^{11}_{\ell-1,0}c^{0,u-1}_{21}.
\end{aligned} \tag{3.9}$$

Besides the first equation, each term on the right hand side of Equation 3.9 has a factor which is zero as imposed by Sections 3.4.1 and 3.2.2. We now take a moment to deal with the first equation. We note that there exists some $m \in \mathbb{N}$ such that $v = \ell - m$. For case 3, we must have that $m \geq 2$ and we can show that $c^{01}_{\ell-1,0} = 0$ by shifting once more to the west. We follow the same logic as before and equate coefficients once again.

$$P_{\ell-1,0} = xP_{\ell-2,0} + \sum_{2\leq j\leq u-1} c^{0,j}_{\ell-2,0}P_{1,j} + c^{11}_{\ell-2,0}P_{21}.$$

Now we see that $c^{01}_{\ell-1,0} = 0$. Thus, $c^{11}_{\ell,0} = 0$.

To conclude case 3, we have shown that $T_1 = T_2 = T_3 = \{0\}$ and are free of relations on $E' = \{c^{11}_{v,0}, c^{0,u-1}_{v,0}\}$. Therefore, $C^{st}_{I_\mu} \cap X_{\lambda,n} \cong \mathbb{C}^2$.

3.5 Case 4

Let $I \in C^{st}_{I_\mu} \cap X_{\lambda,n}$ and consider the collection of extreme arrows $E := \{c^{0,u-1}_{12}, c^{0,u-1}_{13}, \ldots, c^{0,u-1}_{1,u-2}, c^{0,u-1}_{21}, c^{11}_{\ell,0}, c^{0,u-1}_{\ell,0}\} \subseteq \mathcal{O}_{U_\mu}$.

Remark 3.5.1. By Lemma 3.1.1, we see that $c^{0,u-1}_{21} = c^{11}_{\ell,0} = c^{0,u-1}_{\ell,0} = 0$ and the rest of the elements of E are also zero, which follows by Section 3.2.1. Since all extreme arrows are zero, we do not need to check the T_1, T_2 or T_3. Thus $E' = \{0\}$ and $C^{st}_{I_\mu} \cap X_{\lambda,n} \cong \mathbb{C}^0$.

General Cases

Case 1	Case 2	Case 3	Case 4
k=2	k=1	k=2	k=0

3.6 Special Cases

Now we handle the special cases. The methods used to show T_1, T_2, T_3 do not contribute any non-trivial relations are similar to those used in the general cases. Refer to the list of tableau in Section 3.6.1 below for the following cases.

Case 1: There are no extreme arrows.

Case 2: The only extreme arrows are $c^{01}_{\ell,0}$ and $c^{11}_{\ell,0}$, which are both zero by Lemma 3.1.1.

Case 3: The only extreme arrow is $c^{01}_{\ell,0}$, which is zero by Lemma 3.1.1

Case 4: There is one extreme arrow at $c^{01}_{\ell-1,0}$. The only arrow in T_3 is $c^{01}_{\ell,0}$. This arrow is escaping and shifting to west once and comparing coefficients will show that $c^{01}_{\ell,0} = 0$.

Case 5: Note $E = \{c^{01}_{\ell-1,0}, c^{11}_{\ell-1,0}\}$ and $T_3 = \{c^{01}_{\ell,0}, c^{11}_{\ell,0}\}$. Shifting west once gives $P_{\ell,0} = xP_{\ell-1,0} + c^{11}_{\ell-1,0}P_{21}$. Equating coefficients then gives $c^{01}_{\ell,0} = 0$ and $c^{11}_{\ell,0} = c^{01}_{\ell-1,0}$. However, Lemma 3.1.1 gives that $c^{11}_{\ell,0} = 0$, so $c^{01}_{\ell-1,0} = 0$ and $E' = \{c^{11}_{\ell-1,0}\}$.

Case 6: Note $E = \{c^{01}_{\ell-2,0}, c^{11}_{\ell-2,0}\}$ and $T_3 = \{c^{01}_{\ell,0}, c^{11}_{\ell,0}\}$. As in case 5, we shift once and equate to get $c^{01}_{\ell,0} = 0$ and $c^{11}_{\ell,0} = c^{01}_{\ell-1,0}$. Now we shift west once more to get $P_{\ell-2,0} = xP_{\ell-1,0} + c^{11}_{\ell-2,0}P_{21}$. Equating coefficients gives $c^{01}_{\ell-1,0} = 0$ and $c^{11}_{\ell-1,0} = c^{01}_{\ell-2,0}$ and we note that

$c^{11}_{\ell-1,0} \neq 0$ as its tail is inside λ. Thus, $E' = E$.

Case 7: Note that $E = \{c^{0,u-1}_{10}, c^{0,u-1}_{11}\}$. Arrows emanating from T_1, T_2, and T_3 can be handled exactly as in Section 3.2.

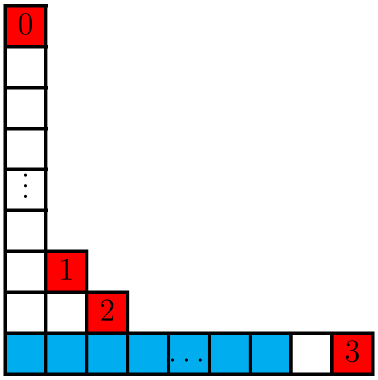
To conclude the section on special cases, we provide the table below.

Special Cases

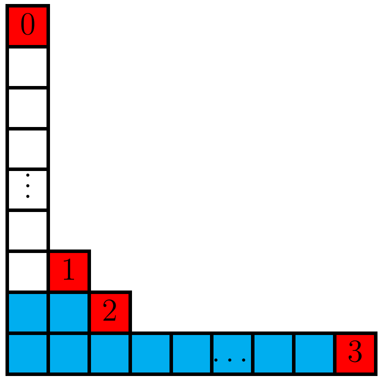
Case 1-3	**Case 4-5**	**Case 6-7**
k=0	k=1	k=2

$\square$

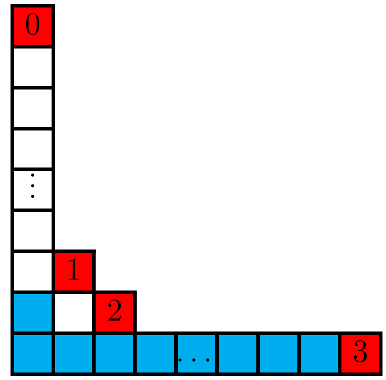
3.6.1 Special Case Tableau



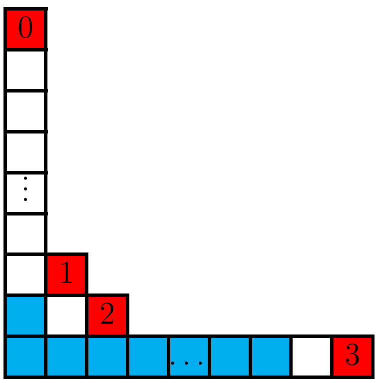
Case 1: $v \geq 0, u = 1$
$\mathbb{C}^0$



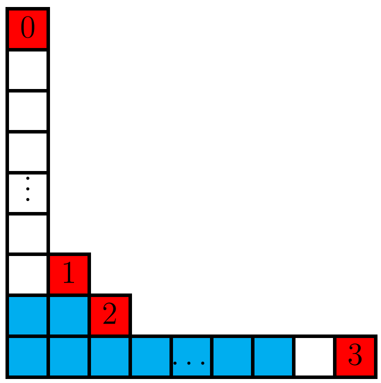
Case 2: $v = \ell, u = 2$, with box (1,1) included
$\mathbb{C}^0$



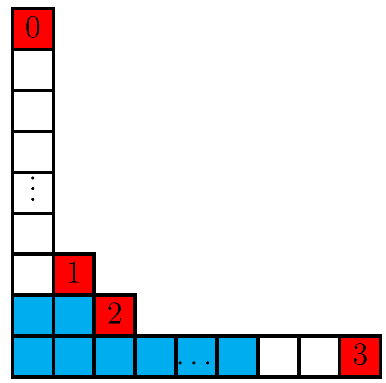
Case 3: $v = \ell, u = 2$, with box $(1,1)$ not included
$\mathbb{C}^0$



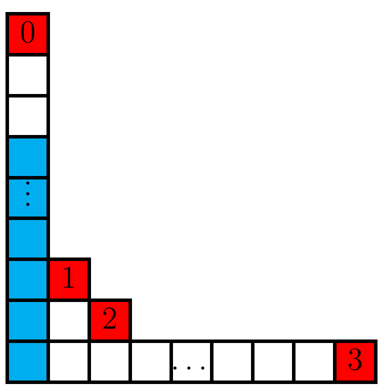
Case 4: $v \geq 2, u = 2$, with box $(1,1)$ not included and at least one box of the first row of λ not contained in μ
$\mathbb{C}$



Case 5: $v = \ell - 1, u = 2$, with box $(1,1)$ included
$\mathbb{C}$



Case 6: $v \geq 2, u = 2$, with box $(1,1)$ included and at least two boxes of the first row of λ not contained in μ
$\mathbb{C}^2$



Case 7: $v = 1, u \geq 3$
$\mathbb{C}^2$

Figure 3.3: The seven special cases

Chapter 4

Connection to Generalized Polynomial

Definition 4.0.1. *We define an (r, s)-polynomial as*

$$p_{k,\ell}(r, s) := \sum_{\mu \subseteq \lambda} r^{|\lambda|-|\mu|} s^{\dim(\mu)},$$

where $\dim(\mu) = \dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$.

Conjecture 4.0.1. Let λ be a concave partition of shape $\lambda = (\ell, 2, \underbrace{1, 1, \ldots, 1}_{(k-2)-times})$ with $p, q > 0$, $\gcd(p, q) = 1$, $p \ll q$ and $p + q > n = |\lambda|$. Then, up to a change of variable, $p_{k,\ell}(r, s)$ equals $f(a_1, a_2, \ldots, a_n)$ from the open problem given in Equation 1.2, where the sequence a_i is the horizontal step sequence of λ. More explicitly,

$$p_{k,\ell}(q, s) = [k+\ell+1]_{q,s} + qs[k+\ell-2]_{q,s} + qs[k+\ell-3]_{q,s} + q^2s^2[k-3]_{q,s}[\ell-1]_{q,s} + q^2s^2[k-1]_{q,s}[\ell-3]_{q,s},$$

where the $[\cdot]_{q,s}$ notation is given in Lemma 4.2.1.

The rest of the chapter deals with computing each cases contribution to $p_{k,\ell}(r, s)$ using the values of $\dim(\mu) = \dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$ given in the proof of Theorem 3.1.1. We conclude with providing computational evidence of Conjecture 4.0.1. Please refer to tables 3.5 and 3.6 for the following sections on the general and special contributions.

4.1 General Cases

For case 1, we get

$$s^2 \sum_{v\geq 2, u\geq 3}^{v=\ell-1, u=k} r^{(\ell+k)-(u+v-1)} = s^2 r^{\ell+k+1} \Big(\sum_{u\geq 3}^{k} r^{-u}\Big)\Big(\sum_{v\geq 2}^{\ell-1} r^{-v}\Big)$$

$$= s^2 r^{\ell+k+1} \Big(\frac{r^{k-2}-1}{r^k(r-1)}\Big)\Big(\frac{r^{\ell-2}-1}{r^{\ell-1}(r-1)}\Big)$$

$$= s^2 r^2 \frac{(r^{k-2}-1)(r^{\ell-2}-1)}{(r-1)^2}, \tag{4.1}$$

where

$$\sum_{u\geq 3}^{k} r^{-u} = \frac{1}{r^k}\sum_{u\geq 3}^{k} r^{k-u} = \frac{1}{r^k}\sum_{i=0}^{k-3} r^i = \frac{r^{k-2}-1}{r^k(r-1)}$$

and similarly for $\sum_{v\geq 2}^{\ell-1} r^{-v}$. For case 2, we get

$$s\sum_{u\geq 3}^{k} r^{(\ell+k)-(u+\ell-1)} = sr^{k+1}\sum_{u\geq 3}^{k} r^{-u} = sr\Big(\frac{r^{k-2}-1}{r-1}\Big). \tag{4.2}$$

For case 3, we get

$$s^2 \sum_{v\geq 2, u\geq 3}^{v=\ell-1, u=k} r^{(\ell+k)-(u+v)} = s^2 r^{\ell+k} \Big(\sum_{u\geq 3}^{k} r^{-u}\Big)\Big(\sum_{v\geq 2}^{\ell-1} r^{-v}\Big)$$

$$= s^2 r \frac{(r^{k-2}-1)(r^{\ell-2}-1)}{(r-1)^2}. \tag{4.3}$$

For case 4, we get

$$\sum_{u\geq 3}^{k} r^{(\ell+k)-(u+\ell)} = r^k \sum_{u\geq 3}^{k} r^{-u} = \frac{r^{k-2}-1}{r-1}. \tag{4.4}$$

4.1.1 Special Cases

In a similar fashion from Section 4.1, we list the contributions from the seven special cases in the following table

Case 1	Case 2	Case 3	Case 4
$r^k\left(\frac{r^{\ell+1}-1}{r-1}\right)$	r^{k-2}	r^{k-1}	$sr^k\left(\frac{r^{\ell-1}-1}{r-1}\right)$

Case 5	Case 6	Case 7
sr^{k-1}	$s^2r^k\left(\frac{r^{\ell-3}-1}{r-1}\right)$	$s^2r^\ell\left(\frac{r^{k-2}-1}{r-1}\right)$

Table 4.1: Contributions from the seven special cases

4.2 Applying the Transformation (General)

Map $s \mapsto s^{-1}$ and multiply the expression by $s^{|\lambda|}$. Then, set $q = rs$.

For case 1, Equation 4.1 becomes

$$q^2s^2\left(\frac{q^{k-2}-s^{k-2}}{q-s}\right)\left(\frac{q^{\ell-2}-s^{\ell-2}}{q-s}\right) = q^{k+\ell-4}s^2\left(\sum_{i=0}^{k-3} q^{-i}s^i\right)\left(\sum_{j=0}^{\ell-3} q^{-j}s^j\right). \tag{4.5}$$

For case 2, Equation 4.2 becomes

$$qs^{\ell+1}\left(\frac{q^{k-2}-s^{k-2}}{q-s}\right) = q^{k-2}s^{\ell+1}\sum_{i=0}^{k-3} q^{-i}s^i. \tag{4.6}$$

For case 3, Equation 4.3 becomes

$$qs^3\left(\frac{q^{k-2}-s^{k-2}}{q-s}\right)\left(\frac{q^{\ell-2}-s^{\ell-2}}{q-s}\right) = q^{k+\ell-5}s^3\left(\sum_{i=0}^{k-3} q^{-i}s^i\right)\left(\sum_{j=0}^{\ell-3} q^{-j}s^j\right). \tag{4.7}$$

For case 4, Equation 4.4 becomes

$$s^{\ell+3}\left(\frac{q^{k-2}-s^{k-2}}{q-s}\right) = q^{k-3}s^{\ell+3}\sum_{i=0}^{k-3} q^{-i}s^i. \tag{4.8}$$

4.2.1 Applying the Transformation (Special)

As we did above in Section 4.1.1, we will again provide a table computing all the special case transformation contributions in a similar manner to Section 4.2.

Case 1	**Case 2**	**Case 3**	**Case 4**
$q^{\ell+k}\sum_{i=0}^{\ell} q^{-i}s^i$	$q^{k-2}s^{\ell+2}$	$q^{k-1}s^{\ell+1}$	$q^{\ell+k-2}s\sum_{i=0}^{\ell-2} q^{-i}s^i$

Case 5	**Case 6**	**Case 7**
$q^{k-1}s^{\ell}$	$q^k s^{\ell-2}\sum_{i=0}^{\ell-4} q^i s^{-i}$	$q^{\ell+k-3}s\sum_{i=0}^{k-3} q^{-i}s^i$

Table 4.2: Contributions from the seven special cases after transformation

Define $S_n := \sum_{i=0}^{n} q^{-i}s^i$. We sum the entries from Equations 4.1-4.4 and Table 4.2 to get a polynomial in q and s.

$$\begin{aligned} p_{k,\ell}(q,s) &:= q^{k+\ell-5}s^2(q+s)S_{k-3}S_{\ell-3} + q^{k-3}s\big(qs^{\ell} + s^{\ell+2} + q^{\ell}\big)S_{k-3} \\ &+ q^{k+\ell}S_{\ell} + q^{k+\ell-2}sS_{\ell-2} + q^{k-2}s^{\ell+2} + q^{k-1}s^{\ell}(s+1) \\ &+ q^k s^{\ell-2}\sum_{i=0}^{\ell-4} q^i s^{-i}. \end{aligned} \tag{4.9}$$

We will now show that $p_{k,\ell}(q,s)$ is symmetric in both q,s and k,ℓ which will provide evidence of Conjecture 4.0.1.

Lemma 4.2.1. For $n \in \mathbb{Z}_{\geq 0}$, define $[n]_{q,s} := q^{n-1} + q^{n-2}s + q^{n-3}s^2 + \cdots + qs^{n-2} + s^{n-1}$. Then, for $m \leq n$,

1. $[m]_{q,s}[n]_{q,s} = [m+n-1]_{q,s} + qs[m+n-3]_{q,s} + \cdots + q^{m-1}s^{m-1}[n-m+1]_{q,s}$

2. $[m]_{q,s}[n]_{q,s} = [m+n-1]_{q,s} + qs[m-1]_{q,s}[n-1]_{q,s}$.

Proof. 1. Note

$$[m]_{q,s}[n]_{q,s} = (q^{m-1} + \cdots + s^{m-1})(q^{n-1} + \cdots + s^{n-1})$$

$$= q^{m+n-2} + 2q^{m+n-3}s + \cdots + mq^{n-1}s^{m-1} + \cdots + mq^{m-1}s^{n-1} + (m-1)q^{m-2}s^{n} + \cdots + s^{m+n-2}$$

$$= [m+n-1]_{q,s} + qs[m+n-3]_{q,s} + \cdots + q^{m-1}s^{m-1}[n-m+1]_{q,s}$$

2. By applying part 1, we see that

$$qs[m-1]_{q,s}[n-1]_{q,s} = qs[m+n-3]_{q,s} + \cdots + q^{m-1}s^{m-1}[n-m+1]_{q,s},$$

which gives the result.

□

Lemma 4.2.2. The polynomial $p_{k,\ell}(q,s)$ is symmetric in both q,s and k,ℓ.

Proof. Using the $[\cdot]_{q,s}$ notation and applying Lemma 4.2.1, we will re-express Equation 4.9 by handling terms of degree $k+\ell$, $k+\ell-1$ and $k+\ell-2$, respectively.

- Degree $k+\ell$

$$q^{k-3}s^{\ell+3}S_{k-3} + q^{k+\ell}S_{\ell} + q^{k-1}s^{\ell+1} = s^{\ell+3}[k-2]_{q,s} + q^{k}[\ell+1]_{q,s} + q^{k-1}s^{\ell+1} + q^{k-2}s^{\ell-2} = [k+\ell+1]_{q,s}.$$

- Degree $k+\ell-1$

$$q^{k-2}s^{\ell+1}S_{k-3} + q^{k-1}s^{\ell} + q^{k+\ell-2}sS_{\ell-2} = qs^{\ell+1}[k-2]_{q,s} + q^{k-1}s^{\ell} + q^{k}s[\ell-1]_{q,s} = qs[k+\ell-2]_{q,s}.$$

- Degree $k+\ell-2$

$$
\begin{aligned}
&q^{k+\ell-5}s^2(q+s)S_{k-3}S_{\ell-3}+q^{k+\ell-3}sS_{k-3}+q^k s^{\ell-2}\sum_{i=0}^{\ell-4}q^i s^{-i} \\
&\quad = qs^2(q+s)[k-2]_{q,s}[\ell-2]_{q,s}+q^\ell s[k-2]_{q,s}+q^k s^2[\ell-3]_{q,s} \\
&\quad = [k-2]_{q,s}\big(qs^2[\ell-1]_{q,s}+q^2s^3[\ell-3]_{q,s}+q^\ell s\big)+q^k s^2[\ell-3]_{q,s} \\
&\quad = [k-2]_{q,s}\big(qs[\ell]_{q,s}+q^2s^3[\ell-3]_{q,s}\big)+q^k s^2[\ell-3]_{q,s} \\
&\quad = qs[k-2]_{q,s}[\ell]_{q,s}+[\ell-3]_{q,s}\big(q^2s^3[k-2]_{q,s}+q^k s^2\big) \\
&\quad = qs[k-2]_{q,s}[\ell]_{q,s}+q^2s^2[k-1]_{q,s}[\ell-3]_{q,s} \\
&\quad = qs[k+\ell-3]_{q,s}+q^2s^2[k-3]_{q,s}[\ell-1]_{q,s}+q^2s^2[k-1]_{q,s}[\ell-3]_{q,s}.
\end{aligned}
\tag{4.10}
$$

Finally, putting it all together, we have

$$p_{k,\ell}(q,s)=[k+\ell+1]_{q,s}+qs[k+\ell-2]_{q,s}+qs[k+\ell-3]_{q,s}+q^2s^2[k-3]_{q,s}[\ell-1]_{q,s}+q^2s^2[k-1]_{q,s}[\ell-3]_{q,s}.$$

which clearly has both q,s and k,ℓ symmetry. □

Example 4.2.1. For $\lambda=(4,2,1,1)$, we will list all possible dimensions for $C^{st}_{I_\mu}\cap X_{\lambda,n}$ given any sub-partition $\mu\subseteq\lambda$ with $p\ll q$.

Size (n)	μ	$\dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$
0	$\emptyset$	0
1	(1)	0
2	(1,1)	1
	(2)	0
3	(1,1,1)	2
	(2,1)	1
	(3)	0
4	(1,1,1,1)	2
	(2,1,1)	2
	(2,2)	2
	(3,1)	1
	(4)	0

Size (n)	μ	$\dim(C^{st}_{I_\mu} \cap X_{\lambda,n})$
5	$(2,1,1,1)$	2
	$(2,2,1)$	2
	$(3,1,1)$	2
	$(3,2)$	1
	$(4,1)$	0
6	$(2,2,1,1)$	2
	$(3,1,1,1)$	2
	$(3,2,1)$	2
	$(4,1,1)$	1
	$(4,2)$	0
7	$(3,2,1,1)$	2
	$(4,1,1,1)$	1
	$(4,2,1)$	0
8	$(4,2,1,1)$	0

After applying the transformation on each table entry and then summing up the terms as in Equation 4.9, we get the following

$$p_{4,4}(q,s) = q^8 + q^7 s + q^6 s^2 + q^5 s^3 + q^4 s^4 + q^3 s^5 + q^2 s^6 + q s^7 + s^8 + q^6 s + q^5 s^2 + q^4 s^3$$

$$+q^3 s^4 + q^2 s^5 + q s^6 + q^5 s + 3q^4 s^2 + 3q^3 s^3 + 3q^2 s^4 + q s^5.$$

Applying code from Section 4.3, we verify that $p_{4,4}(q,s) = f(0,2,1,0,1)$, satisfying Conjecture 4.0.1.

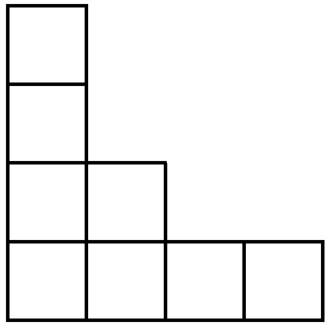


Figure 4.1: $\lambda = (4,2,1,1)$ with horizontal step sequence $a = (0,2,1,0,1)$.

4.3 Code and Computations

Using Sage code that computes the function of [GHSR20], we compare it with the polynomial given in Equation 4.9. As the computation involves iterating over all standard Young tableaux of size n, memory constraints prevent us from testing large values of both parameters simultaneously. Therefore, we restrict one of k or ℓ to a fixed value while allowing the other to increase.

Conjecture 4.0.1 was computationally verified for all pairs (k, ℓ) where $k \in [3, 6]$ and $\ell \leq 40$, as well as for the symmetric cases.

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