

APPLICATIONS OF MODERN OPTIMAL CONTROL THEORY  
TO ENVIRONMENTAL CONTROL OF CONFINED SPACES

by 1264

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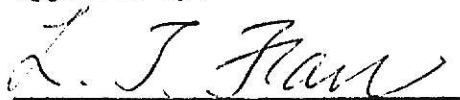
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## CHAPTER 1

### INTRODUCTION

This study contains results of the original investigation on the control of life support systems or more specifically the temperature control of life support systems by means of the modern control theory. A life support system is a system for creating, maintaining, and controlling an environment so as to permit personnel to function efficiently. The control of temperature is probably the most important role of the life support system.

The need for providing an automatic control system to an air-conditioning system has long been recognized [44, 45]. It is also a well known fact that use of the automatic control is necessary for the life support system of a space cabin or submarine or underground shelter [46, 47]. It appears that analysis and synthesis of the control systems for the air-conditioning and life support systems have so far been carried out by the classical approach [44, 45, 46, 47].

The classical approach to the analysis and synthesis of an automatic control system is essentially a trial-and-error procedure or a disturbance-response (or input-output) approach. Extensive use is made of the transform methods such as the Laplace transform (s-domain), Fourier transform ( $\omega$ -domain), and z transform (discrete time-domain). Even though mathematics is extensively used, the classical approach is essentially an empirical one [48].

In recent years, an approach to the analysis and synthesis of a control system, which is distinctly different from the classical one, has been developed. This modern approach is generally called the modern (optimal)

control theory [1, 2, 7, 8, 9, 10, 11, 31, 32, 40, 43, 48]. It is based on the state-space characterization of a system. The state-space is the abstract space whose coordinates are the state properties of the system or the variables which define the characteristics of the system [48]. This approach involves mainly maximization or minimization of an objective function (functional) which is a function of state (plant) and control variables which are in turn functions of time and/or distance coordinate. The objective function is specified, constraints are imposed on the state and decision variables, and an optimal control policy is determined by extremizing the objective function by means of mathematical techniques such as the calculus of variations, maximum principle, and dynamic programming [40, 41, 48]. This modern approach is entirely theoretical in the sense that no trial-and-error is involved in "adjusting the controller."

There are reasons to believe that the classical approach suffices in the analyses and syntheses of the control systems for a majority of air-conditioning and life support systems because usually the requirements are not extremely critical and specifications are not very tight. It is, therefore, justifiable that most of the control and dynamic investigations of air-conditioning and life support systems, which have appeared in the open literature, are based on the classical approach [27, 28, 49, 50, 51, 52, 53]. There is, however, a certain incentive in applying the modern approach to analysis and synthesis of automatic environmental control systems in space crafts, submarines, underground civil defense shelters and certain medical facilities. In these systems, very stringent requirements in the response time, control effort, and others are imposed. For example, the control system of a space craft must have an extremely small response time and

furthermore, the amount of energy required for the control effort must be very small because of the weight limitation imposed on the space craft.

In the present work, the emphases are on the use of the maximum principle and related variational techniques [1, 2, 8, 28, 31, 40]. Their applications will be illustrated by means of concrete numerical examples. It is said that use of the maximum principle and the calculus of variations gives rise to a control policy of an open loop nature [48, 54] which is not desirable for control of a space heating system in which room temperature variations are to be reduced and penalized [33]. In ref. [33], the dynamic programming technique is employed. It will be shown, however, that the maximum principle and related techniques can be advantageously employed for the types of systems and objective functions considered in this work.

In this study, only the modeling and control of deterministic systems are considered. However, when human and physiological factors are taken into account as part of a total life support system, use of the stochastic modeling and control may be more appropriate than use of the deterministic modeling and control because the system tends to be more stochastic than deterministic.

It is hoped that this work will stimulate the applications of and research on the modern optimal control theory to the environmental control of life support systems in general, including controls of humidity, purity and noise.

The computational algorithm of Pontryagin's maximum principle is introduced in Chapters 2, 3 and 4. Chapter 2 states the basic computational algorithm of the maximum principle and some immediate extensions of the basic algorithm. Chapter 3 discusses the necessary conditions for an optimum for processes in which equality constraints are imposed on the final state

variables. These problems can also be solved by employing the basic algorithm if the equality constraints imposed on the final state variables are considered as the final conditions of the state variables. Chapter 4 is the further extension of the basic algorithm, the necessary conditions for an optimum for systems with a bounded state variable space.

The reference system used in this study is one of a typical office for a senior executive, manager or department head located in a multi-story building. We wish to maintain the air temperature inside the office in a certain range for some physical or biological reason by using certain mechanical equipment, depending on the purposes of control.

Application of the computational algorithm of Pontryagin's maximum principle presented in Chapters 2, 3 and 4 to the environmental control whose mathematical model is derived in Chapter 5 is given in Chapters 6 through 8. Chapter 6 consists of three examples whose solutions are obtained by making use of the basic algorithm presented in Chapter 2. The first example is intended to find an optimal control policy for the minimum time system when the time constant of the heat exchanger is assumed to be negligibly small and when the dynamic behavior of the system element, room or cabin, is represented by one CST model. The statement of the second example is the same as that of the first one except that the response of the heat exchanger is assumed to be of the first order instead of the zeroth order. And the statement of the last example in this chapter is again the same as that of the first one except that the two tanks in series model replaces one CST model. Chapter 7 has two examples whose solutions are obtained by making use of the computational algorithm presented in Chapter 3. The first one is the reconsideration of the first example of Chapter 6 from a different point of view and the second problem is an example of three primary state variables with equality

constraints imposed on each state variable at the control terminal time. The statement of this problem is the same as that of the third one except that the time constant of the heat exchanger is not negligibly small. Chapter 8 is an example of two primary state variables with inequality constraints imposed on the state variables. The statement of this problem is the same as that in the third example of Chapter 6 with additional conditions, that is, inequality constraints.

Comparison of the results presented in Chapters 5 through 8 naturally leads us to consider the deviation of a system from its nominal behavior caused by deviation of system components and parameters from their nominal performance characteristics. This is the essence of the sensitivity analysis and is discussed in Chapter 9. The results enable us to demonstrate the sensitivity to (1) parameter variations; (2) changes in dimensions of mathematical models; and (3) change in constraints.

## CHAPTER 2

BASIC COMPUTATIONAL ALGORITHM OF  
PONTYAGIN'S MAXIMUM PRINCIPLE

In this chapter and the two to follow, we shall introduce the computational algorithm of Pontryagin's maximum principle. Here we shall state the basic algorithm of Pontryagin's maximum principle. Numerical examples will be provided in Chapter 6.

Consider that the dynamic behavior of a controlled system can be represented by a set of differential equations:

$$\frac{dx_i}{dt} = f_i [x_1(t), x_2(t), \dots, x_s(t); \theta_1(t), \theta_2(t), \dots, \theta_r(t)],$$

$$i = 1, 2, \dots, s \quad (2.1)$$

$$t_0 \leq t \leq T$$

or in vector form

$$\frac{dx}{dt} = f [x(t), \theta(t)], \quad t_0 \leq t \leq T \quad (2.1a)$$

where  $x(t)$  is an  $s$ -dimensional vector function representing the state of the process at time  $t$  and  $\theta(t)$  is an  $r$ -dimensional vector function representing the decision at time  $t$  [1]. The functions  $f_i$ ,  $i = 1, \dots, s$ , are single valued, bounded, differentiable with respect to the  $x$ 's with bounded first partial derivatives, and are continuous in the  $\theta$ 's on a product region  $x\theta$ , where  $x$  and  $\theta$  are closed regions in the  $s$ -dimensional  $x$ -space and  $r$ -dimensional  $\theta$ -space respectively [6]. Note that we are dealing with the



autonomous systems in which the right-hand side of the performance equation, equation (2.1), depends implicitly on time  $t$ , while the non-autonomous systems are those in which the right-hand side of the performance equation, equation (2.1), depends explicitly on time  $t$ .

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function,  $\theta(t)$ , subject to the  $p$ -dimensional constraints

$$h_i[\bar{\theta}(t)] \leq 0, \quad i = 1, 2, \dots, p \quad (2.2)$$

such that the performance index

$$S = \sum_{i=1}^s c_i x_i(T), \quad c_i = \text{constant} \quad (2.3)$$

is minimum (or maximum) when the initial conditions

$$x_i(t_0) = x_{i0}, \quad i = 1, 2, \dots, s \quad (2.4)$$

are given. The duration of control,  $T$ , is specified and the final conditions of state variables are unfixed. This type of problem is often called the free right-end problem (with fixed  $T$ ). The decision vector (or a collection of control variables) so chosen is called an optimal decision vector (or optimal control variables) and is denoted by  $\bar{\theta}(t)$ .

The procedure for solving the problem is to introduce an  $s$ -dimensional adjoint vector  $z(t)$  and a Hamiltonian function which satisfy the following relations

$$H[\bar{x}(t), \theta(t), z(t)] = \sum_{i=1}^s z_i(t) f_i[\bar{x}(t), \theta(t)] \quad (2.5)$$

$$\frac{dz_i}{dt} = - \frac{\partial H}{\partial x_i} = - \sum_{j=1}^s z_j \frac{f_j}{x_i}, \quad i = 1, 2, \dots, s \quad (2.6)$$

$$z_i(T) = c_i, \quad i = 1, 2, \dots, s \quad (2.7)$$

The set of equations, equations (2.1) and (2.6), constitutes a two-point split boundary value problem, whose solution depends on  $\theta(t)$ . The optimal decision vector  $\bar{\theta}(t)$  which makes  $S$  an extremum also makes the Hamiltonian an extremum for all  $t$ , i.e.,  $t_0 \leq t \leq T$  [1, 2, 3, 5, 7].

A necessary condition for  $S$  to be an extremum with respect to  $\theta(t)$  is

$$\frac{\partial H}{\partial \theta_i} = 0, \quad i = 1, 2, \dots, r \quad (2.8)$$

if the optimal decision vector is interior to the set of admissible decisions  $\theta(t)$  [the set given by equation (2.2)]. If  $\theta(t)$  is constrained, the optimal decision vector  $\bar{\theta}(t)$  is determined either by solving equation (2.8) for  $\theta(t)$  or by searching the boundary of the set. More specifically, the extremum value of Hamiltonian is maximum (or minimum) when the control variables are on the constraint boundary. Furthermore, the extremum value of the Hamiltonian is constant at every point of time under the optimal condition. It is worth noting that the final conditions of the adjoint variables,  $z_i(T)$ , are often given to be  $-c_i$  instead of  $c_i$  as shown in equation (2.7), in employing the maximum principle of Pontryagin. The use of such final conditions of  $z_i(t)$  gives rise to the condition that the Hamiltonian is maximum when the objective function is minimized, and minimum when the objective function is maximized as stated in the original version of the maximum principle of Pontryagin [1, 2, 8].

If both the initial and final conditions of state variables are given, the problem is said to be a boundary value problem. The basic algorithm presented except the condition given by equation (2.7) is still applicable. If optimization (usually minimization) of time  $t$  is involved in the objective function in a problem with an unfixed duration of control,  $T$ , the problem is then called a time optimal problem. In this case, the basic algorithm presented is still applicable with an additional condition that the extremal value of the Hamiltonian is not only a constant but also identical to zero. The simplest example of the time optimal control problem is one in which the performance index is of the form

$$S = \int_0^T dt$$

Such a problem is often called a minimum time problem.

## CHAPTER 3

NECESSARY CONDITIONS FOR OPTIMUM FOR PROCESSES  
IN WHICH STATE VARIABLES HAVE EQUALITY  
CONSTRAINT AT THE CONTROL  
TERMINAL TIME

Introduction

The basic computational algorithm of Pontryagin's maximum principle has been introduced in the preceding chapter. Here we shall extend the basic form to cover the optimal time problem with equality constraints imposed on the final state variables.

Kopp adjoined the equality constraints to the objective function via Lagrange multipliers and then solved the problem by a trial and error procedure [1, 7, 13]. Dean and Aris treated the problem by the Green function approach [14]. We shall first obtain the necessary conditions for optimum by adjoining the equality constraints to the objective function via Lagrange multipliers and taking weak variations of the resulting expression. The necessary conditions thus obtained will be applied to two concrete examples in Chapter 7.

Necessary Conditions for an Optimum in Problems with EqualityConstraints Imposed on the Final State Variables

Again let us consider the differential equations of the following form

$$\frac{dx_1}{dt} = f_1[x_1(t), x_2(t), \dots, x_s(t); \theta_1(t), \theta_2(t), \dots, \theta_r(t)] ,$$

$$i = 1, 2, \dots, s \quad (3.1)$$

with the initial conditions given by

$$x_i(t_0) = x_{i0}, \quad t = t_0 \quad (3.2)$$

Suppose that we wish to determine the control vector  $\theta(t)$  so as to minimize (or maximize)

$$S = \int_{t_0}^T F[\underline{x}(t), \theta(t)] dt \quad (3.3)$$

subject to the  $q$ -dimensional constraint on state variables at the unspecified terminal time,  $T$ , as shown below

$$g_i[\underline{x}(T)] = 0, \quad i = 1, 2, \dots, q \quad (3.4)$$

where initial time  $t_0$  is fixed and  $T$  is the unspecified control terminal time. Here the objective function, equation (3.3) is different from the form used in the preceding chapter. However, we can transform equation (3.3) into the form used in the preceding chapter by introducing an additional state variable  $x_{s+1}$  such that

$$x_{s+1}(t) = \int_{t_0}^t F[\underline{x}(t), \theta(t)] dt \quad (3.5)$$

It follows that

$$\begin{aligned} \frac{dx_{s+1}(t)}{dt} &= F[\underline{x}(t), \theta(t)] \\ &= f_{s+1}[\underline{x}(t), \theta(t)] \end{aligned} \quad (3.6)$$

$$x_{s+1}(t_0) = 0 \quad (3.7)$$

and hence the objective function now becomes

$$S = x_{s+1}(T) \quad (3.8)$$

or

$$S = \sum_{i=1}^{s+1} c_i x_i(T) \quad (3.9)$$

which is in the form used in the preceding chapter with

$$c_i = 0, \quad i = 1, 2, \dots, s$$

$$c_{s+1} = 1$$

Suppose that the equality constraints,  $g_i[\underline{x}(T)]$ , and the performance equations, equations (3.1) and (3.6), are adjoined to the objective function via Lagrange multipliers,  $v_i$  and  $z_i$ .

$$S' = \sum_{i=1}^{s+1} c_i x_i(T) + \sum_{i=1}^q v_i g_i[\underline{x}(T)] + \int_0^T \left\{ \sum_{i=1}^{s+1} z_i f_i[\underline{x}(t), \theta(t)] - \frac{dx_i}{dt} \right\} dt \quad (3.10)$$

We can then define the Hamiltonian

$$H[\underline{x}(t), \theta(t), z(t)] = \sum_{i=1}^{s+1} z_i f_i[\underline{x}(t), \theta(t)] \quad (3.11)$$

and substitute this relation to equation (3.10) to obtain

$$S' = \sum_{i=1}^{s+1} c_i x_i(T) + \sum_{i=1}^q v_i g_i[\underline{x}(T)] + \int_0^T \left\{ H[\underline{x}(t), \theta(t), z(t)] - \sum_{i=1}^{s+1} z_i(t) \frac{dx_i}{dt} \right\} dt \quad (3.12)$$

The first variation of  $S'$ ,  $\delta S'$ , which is defined as

$$\delta S' = S'[\bar{x}, \bar{\theta}, \bar{T}] - S'[\bar{x}, \bar{\theta}, \bar{T}]$$

may be obtained by letting

$$\begin{aligned} x_i(t) &= \bar{x}_i(t) + \delta x_i(t), & i &= 1, 2, \dots, s+1 \\ \theta_i(t) &= \bar{\theta}_i(t) + \delta \theta_i(t), & i &= 1, 2, \dots, r \\ T &= \bar{T} + \delta T \end{aligned} \quad (3.13)$$

$$z_i(t) = \bar{z}_i(t) + \delta z_i(t), \quad i = 1, 2, \dots, s+1$$

and then inserting these relations into equation (3.12) and carrying out the Taylor series expansion about  $\bar{x}$ ,  $\bar{\theta}$ ,  $\bar{z}$  and  $\bar{T}$ . Retaining only the linear terms of the resulting equation and then dropping the bar notation gives

$$\begin{aligned} \delta S' &= \delta T \left\{ H[\bar{x}(T), \bar{\theta}(T), \bar{z}(T)] + \sum_{i=1}^q v_i \frac{\partial g_i}{\partial T} + \sum_{i=1}^{s+1} c_i \frac{\partial x_i}{\partial T} - \sum_{i=1}^{s+1} z_i(T) \frac{dx_i(T)}{dT} \right\} \\ &+ \sum_{i=1}^{s+1} \delta x_i(T) \left\{ \sum_{j=1}^q v_j \frac{\partial g_j}{\partial x_i} - z_i(T) + c_i \right\} \\ &+ \int_0^T \left\{ \sum_{i=1}^{s+1} \delta x_i \left[ \frac{\partial H}{\partial x_i} + \frac{dz_i}{dt} \right] + \sum_{i=1}^r \delta \theta_i \frac{\partial H}{\partial \theta_i} + \sum_{i=1}^{s+1} \delta z_i \left[ \frac{\partial H}{\partial z_i} - \frac{dx_i}{dt} \right] \right\} dt \end{aligned} \quad (3.14)^*$$

We must set this first variation equation at zero to obtain the necessary conditions for a minimum. The resulting equations which determine the optimal

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\* Derivation of equation (3.14) is given in Appendix B.

control and state vectors are as follows:

$$H[\bar{x}(t), \theta(t), z(t)] = \sum_{i=1}^{s+1} z_i(t) f_i[\bar{x}(t), \theta(t)] \quad (3.15)$$

$$\frac{\partial H}{\partial z_i} = \frac{dx_i}{dt} = f_i[\bar{x}(t), \theta(t)], \quad i = 1, 2, \dots, s+1 \quad (3.16)$$

$$\frac{\partial H}{\partial x_i} = - \frac{dz_i}{dt} = \sum_{j=1}^{s+1} \frac{\partial f_j}{\partial x_i} z_j, \quad i = 1, 2, \dots, s+1 \quad (3.17)$$

$$\frac{\partial H}{\partial \theta_i} = 0 = \sum_{j=1}^s \frac{\partial f_j}{\partial \theta_i} z_j, \quad i = 1, 2, \dots, r \quad (3.18)$$

These represent the  $(2s+2)$  differential equations for the two-point split boundary value problems. The conditions at the initial time are given in equations (3.2) and (3.7), whereas those at the final time are

$$z_i(T) = \sum_{j=1}^q v_j \frac{\partial g_j}{\partial x_i(T)} + c_i, \quad i = 1, 2, 3, \dots, s+1 \quad (3.19)$$

and

$$H[\bar{x}(T), \theta(T), z(T)] + \sum_{i=1}^q \frac{\partial g_i}{\partial T} v_i + \sum_{i=1}^{s+1} c_i \frac{\partial x_i}{\partial T} - \sum_{i=1}^{s+1} z_i(T) \frac{dx_i(T)}{dT} = 0 \quad (3.20)$$

Equation (3.19) provides  $(s+1)$  conditions with  $q$  Lagrange multipliers to be determined. Equation (3.4) provides  $q$  equations which can be used for elimination of the Lagrange multipliers, and equation (3.20) provides one additional equation which can be used for determination of the unspecified terminal time.



It is worth noting that when the constraints  $g_i[\underline{x}(T)] = 0$  are not imposed on the final state variables, the necessary conditions derived here reduce to those derived for the basic problems presented in the preceding chapter. In other words, equations (3.19) and (3.20) reduce to

$$z_i(T) = c_i \quad (3.19a)$$

$$H[\underline{x}(T), \theta(T), z(T)] = 0 \quad (3.20a)$$

The set formed by these equations and equations (3.15) through (3.18) is identical to that formed by equations (2.5) through (2.8) in the preceding chapter. Equation (3.20a) is correct, because the minimum (or maximum) value of the Hamiltonian is identical to zero at every point of time  $t$  for the time optimal problem. The final condition,  $\underline{x}(T)$ , which is fixed in the fixed right-end problem can be considered as the simplest case of  $g_i[\underline{x}(T)] = 0$ .

## CHAPTER 4

NECESSARY CONDITIONS FOR AN OPTIMUM FOR PROCESSES WITH  
INEQUALITY CONSTRAINTS IMPOSED ON THE STATE VARIABLESIntroduction

The basic computational algorithm of Pontryagin's maximum principle has been introduced in Chapter 2 and the optimal control of the systems in which equality constraints are imposed on the state variables at the end of control action has been introduced in Chapter 3. In this chapter we shall consider the necessary conditions for optimum for a dynamic system whose state variables are constrained by a certain inequality condition or conditions.

Chang [6], Berkovitz [16] and Gamkveildze [2] have dealt with the fundamental aspects of the problem, and both theoretical and computational aspects are considered in papers by Dreyfus [17], Denham [18] and Denham Bryson [19]. Despite these and other efforts [1, 11, 15, 20], the optimal control problem with state variable constraints does not appear to be well understood. Here we shall first state the problem and then show the necessary conditions for optimum. Numerical examples will be given in Chapter 8.

A Dynamic System with State Variable Constraints

Again let us consider a continuous process whose dynamic behavior can be represented by the following set of differential equations

$$\frac{dx_i}{dt} = f_i[x_1(t), x_2(t), \dots, x_s(t); \theta_1(t), \theta_2(t), \dots, \theta_r(t)],$$

$$i = 1, 2, \dots, s \quad (4.1)$$

or in vector form

$$\frac{dx}{dt} = f[x(t), \theta(t)] \quad (4.1a)$$

where  $x(t)$  is an  $s$ -dimensional state vector and  $\theta(t)$  is an  $r$ -dimensional control vector. Now we wish to find a piecewise continuous control vector  $\theta(t)$  in the set such that the function of the final state

$$S = \sum_{i=1}^s c_i x_i(T), \quad c_i = \text{constant} \quad (4.2)$$

takes on its minimum (or maximum) value subject to the condition that  $x(t)$  stays within a specified region of the state space given by the inequalities

$$g_i[x(t), \theta(t)] \leq 0, \quad i = 1, 2, \dots, q \quad (4.3)$$

The duration of control,  $T$ , is specified. Functions  $f_i$ ,  $S$  and  $g_i$  are assumed to possess continuous derivatives to at least the second order [6, 20]. In addition to those, the state variables must satisfy certain initial or final conditions or both.

#### A Necessary Condition for Optimum

Chapter 2 stated the basic algorithm of Pontryagin's maximum principle for systems without state variable constraints. That is, the Hamiltonian  $H$  and the adjoint variables are defined by

$$H = \sum_{i=1}^s z_i(t) f_i[x(t), \theta(t)] \quad (4.4)$$

$$\frac{dz_i}{dt} = - \frac{\partial H}{\partial x_i} = - \sum_{i=1}^s z_i(t) \frac{f_i}{x_i}, \quad i = 1, 2, \dots, s \quad (4.5)$$

$$z_i(T) = c_i, \quad i = 1, 2, \dots, s \quad (4.6)$$

and the necessary condition for the objective function  $S$  to be an extremum with respect to  $\theta$  is

$$\frac{\partial H}{\partial \theta_i} = 0, \quad i = 1, 2, \dots, r \quad (4.7)$$

or  $H$  is the extremum on the boundary of the constraint and the extremum value of the Hamiltonian is constant at every point of time under the optimal condition.

For systems with  $q$  equality constraints,  $g_i$ ,  $i = 1, 2, \dots, q$ , imposed on the final state variables, the necessary condition for the objective function to be extremum remains the same except that equation (4.6) becomes

$$z_i(T) = \sum_{j=1}^q v_j \frac{\partial g_j}{\partial x_i(T)} + c_i, \quad i = 1, 2, \dots, s \quad (4.8)$$

and an additional condition at the control terminal time

$$\begin{aligned} H[x(T), \theta(T), z(T)] + \sum_{j=1}^q v_j \frac{\partial g_j}{\partial T} + \sum_{i=1}^s c_i \frac{\partial x_i}{\partial T} \\ - \sum_{i=1}^s z_i(T) \frac{dx_i(T)}{dT} = 0 \end{aligned} \quad (4.9)$$

Equations (4.8) and (4.9) were derived in the preceding chapter.

For the system with one primary state variable and with one inequality constraint imposed on the state variable, the condition that the Hamiltonian

is constant under the optimal condition remains the same. However, it has been explicitly indicated [9] that the condition given by equation (4.5) must be rewritten by employing the chain rule, as

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = - z_1 \frac{df_1}{dt} - z_1 \frac{df_1}{d\theta} \frac{d\theta}{dt} \quad (4.10)$$

because on the constraint boundary, the state variable and the control variable are related by the constraint. Here we wish to prove for systems with two primary state variables represented by

$$\frac{dx_1}{dt} = f_1[x_1, x_2, \theta_1, \theta_2] \quad (4.11)$$

$$\frac{dx_2}{dt} = f_2[x_1, x_2, \theta_1, \theta_2] \quad (4.12)$$

the necessary condition for the objective function of the form

$$S = \int_t^T F[x_1, x_2, \theta_1, \theta_2] dt \quad (4.13)$$

to be extremum subject to the inequality constraints

$$g_1[x_1, x_2, \theta_1, \theta_2] \geq 0 \quad (4.14)$$

$$g_2[x_1, x_2, \theta_1, \theta_2] \geq 0 \quad (4.15)$$

$$|\theta| \leq 1 \quad (4.16)$$

is that the Hamiltonian is also a constant along the constraint boundaries.

Introducing an additional state variable,  $x_3$ , such that

$$x_3(t) = \int_0^t F[\bar{x}_1, x_2, \theta_1, \theta_2] dt$$

It follows that

$$\frac{dx_3}{dt} = F[\bar{x}_1, x_2, \theta_1, \theta_2] \quad (4.17)$$

$$= f_3[\bar{x}_1, x_2, \theta_1, \theta_2]$$

$$x_3(0) = 0 \quad (4.18)$$

and

$$x_3(T) = S \quad (4.19)$$

or

$$S = \sum_{i=1}^3 c_i x_i(T)$$

$$c_1 = c_2 = 0, c_3 = 1 \quad (4.20)$$

Thus, the problem is transformed into that of minimizing  $x_3(T)$ .

The Hamiltonian is, according to equation (4.4),

$$H = z_1 f_1 + z_2 f_2 + z_3 f_3 \quad (4.21)$$

and the adjoint vector is defined by

$$\begin{aligned} \frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = & - z_1 \left[ \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_1} \right] \\ & - z_2 \left[ \frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_1} \right] \end{aligned}$$

$$- \left[ \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial F}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial F}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right] \quad (4.22)$$

$$\begin{aligned} \frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = & - z_1 \left[ \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right] \\ & - z_2 \left[ \frac{\partial f_2}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right] \\ & - \left[ \frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial F}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial F}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right] \end{aligned} \quad (4.23)$$

$$\frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = 0, \quad z_3(T) = 1 \quad (4.24)$$

Note that equations (4.22) and (4.23) are different from that defined in equation (4.5) because the state variables and the control variables are related through the equality constraints, equations (4.14) or (4.15). For this reason, differentiation of the Hamiltonian with respect to  $x_1$  must be carried out by employing the chain rule. Also note that equations (4.22) and (4.23) reduce to equation (4.5) when the state variable  $x(t)$  is interior to the set of constraints, equations (4.14) and (4.15). The solution of equation (4.24) is

$$z_3(t) = 1, \quad 0 \leq t \leq T \quad (4.25)$$

Thus equation (4.21) can be rewritten as

$$H = z_1 f_1 + z_2 f_2 + F \quad (4.26)$$

The derivative of the above equation with respect to  $t$  is

$$\begin{aligned}
\frac{dH}{dt} = \frac{dF}{dt} + f_1 \frac{dz_1}{dt} + f_2 \frac{dz_2}{dt} + z_1 \left[ \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 \right. \\
\left. + \frac{\partial f_1}{\partial x_2} f_2 \right] + z_2 \left[ \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial t} + \frac{\partial f_2}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_2} f_2 \right]
\end{aligned}
\tag{4.27}$$

Inserting equations (4.22) and (4.23) into equation (4.27) gives

$$\begin{aligned}
\frac{dH}{dt} = \frac{dF}{dt} + z_1 \left[ \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 \right] \\
+ z_2 \left[ \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial t} + \frac{\partial f_2}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_2} f_2 \right] \\
+ f_1 \left[ -z_1 \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_1} \right) \right. \\
- z_2 \left( \frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_1} \right) \\
- \left( \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial F}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_1} + \frac{\partial F}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_1} \right) \Big] \\
+ f_2 \left[ -z_1 \left( \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial f_1}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial f_1}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right) \right. \\
- z_2 \left( \frac{\partial f_2}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial f_2}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial f_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right) \\
- \left( \frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial F}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial F}{\partial \theta_2} \frac{\partial \theta_2}{\partial x_2} \right) \Big]
\end{aligned}
\tag{4.28}$$



Since  $x_1$ ,  $x_2$ ,  $\theta_1$  and  $\theta_2$  are functions only of  $t$ , we can write the following equations

$$\frac{\partial x_2}{\partial x_1} = \frac{dx_2}{dx_1} = \frac{dx_2}{dt} / \frac{dx_1}{dt} = f_2/f_1$$

$$\frac{\partial x_1}{\partial x_2} = \frac{dx_1}{dx_2} = \frac{dx_1}{dt} / \frac{dx_2}{dt} = f_1/f_2$$

$$\frac{\partial \theta_1}{\partial t} = \frac{d\theta_1}{dt}$$

$$\frac{\partial \theta_2}{\partial t} = \frac{d\theta_2}{dt}$$

$$\frac{\partial \theta_1}{\partial x_1} = \frac{d\theta_1}{dx_1} = \frac{d\theta_1}{dt} / \frac{dx_1}{dt} = \frac{d\theta_1}{dt} / f_1$$

$$\frac{\partial \theta_1}{\partial x_2} = \frac{d\theta_1}{dx_2} = \frac{d\theta_1}{dt} / \frac{dx_2}{dt} = \frac{d\theta_1}{dt} / f_2$$

$$\frac{\partial \theta_2}{\partial x_1} = \frac{d\theta_2}{dx_1} = \frac{d\theta_2}{dt} / \frac{dx_1}{dt} = \frac{d\theta_2}{dt} / f_1$$

$$\frac{\partial \theta_2}{\partial x_2} = \frac{d\theta_2}{dx_2} = \frac{d\theta_2}{dt} / \frac{dx_2}{dt} = \frac{d\theta_2}{dt} / f_2$$

Inserting these equations into equation (4.28) gives

$$\frac{dH}{dt} = -f_1 \frac{\partial F}{\partial x_1} - f_2 \frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial \theta_1} \frac{d\theta_1}{dt} - \frac{\partial F}{\partial \theta_2} \frac{d\theta_2}{dt} - f_2 z_2 \frac{\partial f_1}{\partial x_2} - z_2 f_2 \frac{\partial f_2}{\partial x_2}$$

$$\begin{aligned}
& - f_1 z_1 \frac{\partial f_1}{\partial x_1} - f_1 z_2 \frac{\partial f_2}{\partial x_1} - z_1 \frac{\partial f_1}{\partial \theta_1} \frac{d\theta_1}{dt} - z_1 \frac{\partial \theta_1}{\partial \theta_2} \frac{d\theta_2}{dt} - z_2 \frac{\partial f_2}{\partial \theta_1} \frac{d\theta_1}{dt} \\
& - z_2 \frac{\partial f_2}{\partial \theta_2} \frac{d\theta_2}{dt}
\end{aligned} \tag{4.29}$$

Note that we have used the relation

$$\frac{dF}{dt} = \frac{\partial F}{\partial x_1} f_1 + \frac{\partial F}{\partial x_2} f_2 + \frac{\partial F}{\partial \theta_1} \frac{d\theta_1}{dt} + \frac{\partial F}{\partial \theta_2} \frac{d\theta_2}{dt} \tag{4.30}$$

For those periods of an extremal solution on the constraint boundary,  $\theta(t)$  is determined in terms of the state variable and the independent variable  $t$  by the relation shown in equation (4.14). Thus the neighboring solutions for those periods must satisfy

$$\begin{aligned}
dg_1 &= \frac{\partial g_1}{\partial \theta_1} d\theta_1 + \frac{\partial g_1}{\partial \theta_2} d\theta_2 + \frac{\partial g_1}{\partial x_1} dx_1 + \frac{\partial g_1}{\partial x_2} dx_2 \\
&= 0
\end{aligned}$$

or solving for  $\frac{d\theta_1}{dt}$

$$\frac{d\theta_1}{dt} = - \left( \frac{\partial g_1}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial g_1}{\partial x_1} f_1 + \frac{\partial g_1}{\partial x_2} f_2 \right) \left( \frac{\partial g_1}{\partial \theta_1} \right)^{-1} \tag{4.31}$$

Inserting equation (4.26) into equation (4.24) leads us to

$$\frac{dH}{dt} = 0 \tag{4.32}$$

or  $H$  is also constant along constraint  $g_1 = 0$ . Similarly we can prove  $H$  is constant along constraint  $g_2 = 0$ .

This proof can probably be extended to the general cases. For a system whose dynamic behavior can be represented by equation (4.1), the necessary condition for the objective function, equation (4.2), to be an extremum on the boundary subject to the constraints, equation (4.3), is that the Hamiltonian remains constant on the constraint boundary or,

$$\frac{dH}{dt} = 0 \quad (4.33)$$

and the adjoint vector is of the form

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = \sum_{j=1}^s z_j \left[ \sum_{k=1}^s \frac{\partial f_j}{\partial x^k} \frac{\partial x^k}{\partial x_i} + \sum_{k=1}^r \frac{\partial f_j}{\partial \theta_k} \frac{\partial \theta_k}{\partial x_i} \right],$$

$$i = 1, 2, \dots, s \quad (4.34)$$

The condition  $H(T) = 0$  must hold when the control terminal time  $T$  is left free.

## CHAPTER 5

### MODELING AND SIMULATION

#### Introduction

In this chapter the reference system is dealt with at length. Initially, the performance equations of the system are derived, then the simulation is carried out, and finally the general performance equations terminate the chapter.

The major function of the environmental engineer is two-fold. He wants to establish and to maintain conditions under which human beings feel thermally comfortable. Thermal comfort is defined in ASHRAE standard [29] as that condition of mind which expresses satisfaction with the thermal environment. Since the condition for thermal comfort is influenced by various factors in a complex manner, a general expression for thermal comfort is very difficult to establish. Several attempts have been made to specify conditions for thermal comfort in the last forty years. Fanger [21], Lee et al. [22], Chatonnet and Cabanac [23], Gagge et al. [4] are some of the principal investigators in the field. They have used mainly the method of allowing two or more parameters to vary and keep constant all other factors which influence thermal comfort. Some valid solutions have been found.

On the other hand, the environmental engineer wishes to maintain the thermally comfort conditions. Some recent articles in ASHRAE Journal [25, 26] have demonstrated an increased interest in control theory and its application to the life support systems. Zermuehlen and Harrison [27] illustrated how the principles of control engineering can be applied in a study of room

temperature response to a sudden heat disturbance input. Harrison et al. [28] developed a model transfer function for an air-conditioned room which might be suitable for the investigation of short-term transient response. Eltimsahy [33] formulated the perturbation model of the domestic heating system and used the proposed model for further study of design and control of the heating systems by means of dynamic programming. This work considers the control of life support systems whose transient behavior may be described by systems of ordinary differential equations. A recently developed mathematical tool known as the maximum principle of Pontryagin [1, 2, 7, 8] has been used in order to determine control actions which are, in some sense, optimal.

### Modeling

A control system usually consists of three elements: the feedback element, the control element, and the system proper [30]. The feedback element in a life support control system or an environmental control system may be composed of a thermostat, humidistat and pressure regulator, or any combination of these, depending on the purpose of control. The control element may include a heat exchanger, humidifier, dehumidifier, blower, portable air-conditioner, or any combination of these, depending on the objective of control. For instance, both the thermostat and heat exchanger are often used to control the air temperature inside a building. The system proper may be a confined space, e.g., an underground shelter, a space vehicle, a space suit, a submarine or a building.

The system considered here is shown schematically in Figure 5.1. The confined space may be a typical office located in a multi-story building or

the cabin of a spaceship. Air or oxygen or a mixture of oxygen and nitrogen is circulated through the room or confined space via an air duct by mechanical means, e.g., a blower or fan. Control of air temperature in the system is accomplished with a duct system. The thermostat in the system adjusts the position of the control valve of the heat exchanger in order to provide the desired temperature.

The performance equations of the system, which represent the dynamic characteristics of the system and system components will be derived.

A. The System Proper. The following three main assumptions are made concerning the system proper:

1. Room or cabin air is well mixed, or stated in another way, air temperature within the system is uniform throughout at any instant in time.
2. The thermal capacitance of room walls, floor, ceiling and windows is neglected, as well as that of any furniture within the system.
3. Heat loss through the walls and windows is negligible.

The performance equation of the system proper can be obtained by using the continuity law or heat balance. For a room, the law states that the flow rate of heat into the system must either be absorbed inside the room or leave the room. Referring to Figure 5.2 we have

$$\underline{\text{heat in}} - \underline{\text{heat out}} = \underline{\text{heat accumulation}} \quad (5.1)$$

$$\underline{\text{heat in}} = q_{i1} + q_{i2} + q_{di} \quad (5.2)$$

$$\underline{\text{heat out}} = q_{o1} + q_{o2} \quad (5.3)$$

$$\underline{\text{heat accumulation}} = q_s \quad (5.4)$$

where

$q_{di}$  = the impulse heat disturbance rate in Kcal/sec.

$q_{i1}$  = the heat flow rate into the system by circulation air  
in Kcal/sec.

$q_{i2}$  = the heat flow rate into the system by fresh air in  
Kcal/sec.

$q_{o1}$  = the heat flow rate out of the system by circulation air in  
Kcal/sec.

$q_{o2}$  = the heat flow rate out of the system by exhausted air in  
Kcal/sec.

$q_s$  = the rate of heat stored inside the room in Kcal/sec.

The unit of  $q$ 's is in Kcal/sec. and the mks unit system is used in this study. Inserting equations (5.2), (5.3) and (5.4) into equation (5.1) gives

$$[q_{i1} + q_{i2} + q_{di}] - [q_{o1} + q_{o2}] = q_s \quad (5.5)$$

Based on the assumption of perfect mixing, the expressions for  $q_{i1}$ ,  $q_{i2}$ ,  $q_{o1}$ ,  $q_{o2}$ ,  $q_{di}$  are

$$\begin{aligned} q_{i1} &= Q_1 \int C_p (t_i - t_a) \\ &= Q_1 \int C_p T_i \end{aligned} \quad (5.6)$$

$$\begin{aligned} q_{i2} &= Q_2 \int C_p (t_2 - t_a) \\ &= Q_2 \int C_p T_2 \end{aligned} \quad (5.7)$$

$$\begin{aligned} q_{o1} &= Q_1 \int C_p (t_o - t_a) \\ &= Q_1 \int C_p T_o \end{aligned} \quad (5.8)$$

$$\begin{aligned} q_{o2} &= Q_2 \int C_p (t_c - t_a) \\ &= Q_2 \int C_p T_c \end{aligned} \quad (5.9)$$

$$\begin{aligned}
 q_{di} &= V_1 \int C_p (t_d - t_a) \delta(\alpha) \\
 &= V_1 \int C_p T_d \delta(\alpha)
 \end{aligned}
 \tag{5.10}$$

Note that here the disturbance is considered to be an impulse form. This disturbance term will appear as a forcing function which can be generally designated as  $\phi(\alpha)$  in the resulting differential equation.  $\phi(\alpha)$  can be written as

$$\phi(\alpha) = M_1 \delta(\alpha)$$

where

$$M_1 = V_1 \int C_p T_d$$

Also note that  $\delta(\alpha)$  has a unit of  $\text{sec}^{-1}$ . The rate at which heat energy is stored in the system proper can be expressed as

$$\begin{aligned}
 q_s &= V_1 \int C_p \frac{dt_c}{d\alpha} \\
 &= V_1 \int C_p \frac{dT_c}{d\alpha}
 \end{aligned}
 \tag{5.11}$$

where

$C_p$  = specific heat of air in Kcal/Kg °C.

$Q_1$  = air flow rate by circulation in  $\text{m}^3/\text{sec}$ .

$Q_2$  = flow rate of fresh air in  $\text{m}^3/\text{sec}$ .

$V_1$  = room volume in  $\text{m}^3$ .

$t_a$  = the reference temperature in °C.

$t_c$  = the room temperature in °C.

$t_d$  = the disturbance temperature in °C.



$t_1$  = the temperature of the circulation air into the system in  $^{\circ}\text{C}$ .

$t_2$  = outside air temperature in  $^{\circ}\text{C}$ .

$T_c$  =  $(t_c - t_a)$  in  $^{\circ}\text{C}$ .

$T_i$  =  $(t_i - t_a)$  in  $^{\circ}\text{C}$ .

$T_2$  =  $(t_2 - t_a)$  in  $^{\circ}\text{C}$ .

$T_d$  =  $(t_d - t_a)$  in  $^{\circ}\text{C}$ .

$\alpha$  = time in sec.

$\rho$  = air density in  $\text{Kg/m}^3$ .

The insertion of equation (5.6) through (5.11) into equation (5.5) yields

$$V_1 \rho C_p \frac{dT_c}{d\alpha} + [Q_1 \rho C_p + Q_2 \rho C_p] T_c = Q_1 \rho C_p T_i + Q_2 \rho C_p T_2 + V_1 \rho C_p T_d \delta(\alpha)$$

$Q_1$ ,  $Q_2$ ,  $\rho$ ,  $C_p$ ,  $V_1$ , and  $T_d$  are considered as constants here. Note that if the third assumption is not valid, the rate of heat loss through walls and windows can be included in the  $q_{di}$  term. The above equation can be simplified by dividing both sides of the equation by  $(Q_1 \rho C_p + Q_2 \rho C_p) = (Q_1 + Q_2) \rho C_p$

$$\left. \begin{aligned} Z_1 \frac{dT_c}{d\alpha} + T_c &= r_1 T_i + r_2 T_2 + Z_1 T_d \delta(\alpha) \\ T_c &= 0 \quad \text{at} \quad \alpha = 0 \end{aligned} \right\} \quad (5.12)$$

or in dimensionless form

$$\left. \begin{aligned} \frac{dx_1}{dt} + x_1 &= \frac{r_1 K_1 x_2}{K_4} + r_2 K_1 + K_1 \delta(t) \\ x_1 &= 0 \quad \text{at} \quad t = 0 \end{aligned} \right\} \quad (5.12a)$$

where

$$r_1 = \frac{Q_1}{Q_1 + Q_2}$$

$$r_2 = \frac{Q_2}{Q_1 + Q_2}$$

$$x_1 = \frac{K_1 T_c}{T_2}$$

$$x_2 = \frac{K_4 T_i}{T_2}$$

$$\phi = \frac{T_d}{T_2}$$

$$t = \alpha / \zeta_1 = \text{dimensionless time}$$

$$K_1 = \frac{T_2}{T_{c0}}$$

$$K_4 = \frac{T_2}{T_{i0}}$$

$\zeta_1$  = time constant of the system proper in sec.

$$= \frac{V_1}{Q_1 + Q_2}$$

$T_{c0}$  = room temperature at  $\alpha = 0^+$

$T_{i0}$  = temperature of the circulation air into the system at  $\alpha = 0^+$

Equation (5.12a) is the performance equation of the system proper. This performance equation can appear in another form in which the effect of the

disturbance is taking into account in the initial condition immediately after the onset of the process as shown below:

$$\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1 x_2}{K_4} + r_2 K_1$$

$$x_1 = 1 \quad \text{at} \quad t = 0^+$$

Instead of the impulse form, the heat disturbance may appear in other forms, such as the step, ramp or cyclic functions. The disturbance arises from various sources, such as sun load, turning on lights, opening windows or doors, temperature change in the incoming air, and heat generated by the people or animals.

If the temperature of the incoming recycle air,  $T_i$ , is kept constant, changing air flow rate may also accomplish the purpose of control. The performance equation for such a case can also be derived similarly from equation (5.5).

Note that if  $Q_2 = 0$  or  $r_2 = 0$ , equation (5.12) becomes

$$Z_1 \frac{dT_c}{d\alpha} + T_c = T_i + T_d \quad Z_1 \delta(\alpha) \quad (5.13)$$

This equation is applicable to underground shelters and spacecrafts and submarines under conditions where no fresh air enters the systems.

B. The Control Element. The heat exchanger, which is the control element of the system under consideration, can perform its control function in various ways, for example, changing the temperature or flow rate of the heat transfer medium, or changing both. The performance equation of the control element can be obtained again by employing the continuity law or heat balance, which can be expressed in equation form as follows:

$$\underline{\text{heat in}} - \underline{\text{heat out}} = \underline{\text{heat accumulation}} \quad (5.14)$$

$$\underline{\text{heat in}} = q_{mil} + q_{mi2}$$

$$\underline{\text{heat out}} = q_{m01} + q_{m02}$$

$$\underline{\text{heat accumulation}} = q_{ms}$$

where

$q_{mil}$  = heat flow rate brought into the machine unit by circulation air in Kcal/sec.

$q_{mi2}$  = heat flow rate brought into the machine unit by cooling water in Kcal/sec.

$q_{m01}$  = heat flow rate out of the machine unit by circulation air in Kcal/sec.

$q_{m02}$  = heat flow rate out of the machine unit by cooling water in Kcal/sec.

$q_{ms}$  = the heat flow rate stored in the machine unit in Kcal/sec.

Inserting these definitions into equation (5.14) gives

$$(q_{mil} + q_{mi2}) - (q_{m01} + q_{m02}) = q_{ms} \quad (5.15)$$

By assuming perfect mixing of both air and the heat transfer medium in the heat exchanger, ignoring the heat loss through the shell and neglecting the thermal capacitance of the heat exchanger, the expressions for  $q_{mil}$ ,  $q_{mi2}$ ,  $q_{m01}$ , and  $q_{m02}$  are as follows:

$$\begin{aligned} q_{mil} &= Q_1 \int C_p (t_c - t_a) \\ &= Q_1 \int C_p T_c \end{aligned} \quad (5.16)$$

$$\begin{aligned} q_{mi2} &= Q_w \int C_{pw} (t_{wc} - t_a) \\ &= Q_w \int C_{rw} T_{wc} \end{aligned} \quad (5.17)$$

$$\begin{aligned}
 q_{m01} &= Q_1 \int C_p (t_i - t_a) \\
 &= Q_1 \int C_p T_i
 \end{aligned} \tag{5.18}$$

$$\begin{aligned}
 q_{m02} &= Q_w \int_w C_{pw} (t_{wh} - t_a) \\
 &= Q_w \int_w C_{pw} T_{wh}
 \end{aligned} \tag{5.19}$$

The rate at which heat energy is stored in the heat exchanger can be expressed as

$$\begin{aligned}
 q_{ms} &= V_2 \int C_p \frac{dt_i}{d\alpha} \\
 &= V_2 \int C_p \frac{d(t_i - t_a)}{d\alpha} \\
 &= V_2 \int C_p \frac{dT_i}{d\alpha}
 \end{aligned} \tag{5.20}$$

where

$C_{pw}$  = specific heat of coolant in Kcal/Kg °C.

$Q_w$  = flow rate of coolant in m<sup>3</sup>/sec.

$t_{wc}$  = inlet temperature of coolant in °C.

$t_{wh}$  = outlet temperature of coolant in °C.

$V_2$  = volume of heat exchanger of machine unit in m<sup>3</sup>.

$\int_w$  = density of coolant in Kg/m<sup>3</sup>.

Insertion of equations (5.26) through (5.20) into equation (5.15) gives

$$\begin{aligned}
 (Q_1 \int C_p T_c + Q_w \int_w C_{pw} T_{wc}) - (Q_1 \int C_p T_i + Q_w \int_w C_{pw} T_{wh}) \\
 = V_2 \int C_p \frac{dT_i}{d\alpha}
 \end{aligned}$$

or dividing by  $Q_1 \int C_p$ ,

$$\tau_2 \frac{dT_1}{dt} + T_1 = T_c - \frac{Q_w \int_w C_{pw} (T_{wh} - T_{wc})}{Q_1 \int C_p} \quad (5.21)$$

where  $\tau_2$  is the time constant of the heat exchanger of the machine unit in sec. and is defined by

$$\tau_2 = V_2 / Q_1$$

Note that  $Q_w \int_w C_{pw} (T_{wh} - T_{wc})$  is the amount of heat removed from or added to the system which can be controlled by adjusting either  $Q_w$  when  $\int_w$ ,  $C_{pw}$ , and  $(T_{wh} - T_{wc})$  are constant, or  $(T_{wh} - T_{wc})$  when  $Q_w$ ,  $\int_w$ , and  $C_{pw}$  are kept constants, or both  $Q_w$  and  $(T_{wh} - T_{wc})$  when  $\int_w$  and  $C_{pw}$  are constants. In order to have a mathematically neat form, a hypothetical temperature  $T_r$  is defined.

$$T_r = Q_w \int_w C_{pw} (T_{wh} - T_{wc}) / Q_1 \int C_p$$

Inserting this definition into equation (5.21) yields

$$\tau_2 \frac{dT_1}{dt} + T_1 = T_c - T_r \quad (5.22)$$

or in dimensionless form

$$\frac{\tau_2}{\tau_1} \frac{dx_2}{dt} + x_2 = \frac{x_1 K_4}{K_1} - K_4 (K_2 \theta + K_3) \quad (5.22a)$$

where

$$K_2 = \frac{1}{2T_2} (T_{r \max} - T_{r \min})$$

$$K_3 = \frac{1}{2T_2} (T_{r \max} + T_{r \min})$$

$$\theta = \frac{T_r - \frac{1}{2}(T_{r \max} + T_{r \min})}{T_{r \max} - \frac{1}{2}(T_{r \max} + T_{r \min})}$$

= control variable

$$\frac{T_r}{T_2} = K_2 \theta + K_3$$

Equation (5.22) is the performance equation of the heat exchanger which is shown schematically in Figure 5.3. Note that  $\theta = 1$  when  $T_r = T_{r \max}$  and  $\theta = -1$  when  $T_r = T_{r \min}$ .

C. The Feedback Element - Thermostat. Here we simply assume that the sensing element measures the room temperature instantaneously and that there is no accumulation of heat in the element, or for simplicity, it will be assumed that the sensing element is the zeroth order element with its time constant,  $\tau_3$ , equal to zero. Reference 30 gives a detailed explanation of the response of the thermostat.

### Simulation

With the model in hand, a simulation should be carried out extensively by means of either a digital or analog computer. The results of simulation should then be compared to the known characteristics of the system or experimentally obtained data. The comparison enables us to determine the goodness of the model as an approximate representation of the system.

For illustration, let us consider a simple system in which the time constant of the heat exchanger is negligibly small, i.e.,  $\tau_2 \rightarrow 0$ . For this system we have from equation (5.22)

$$T_1 = T_c - T_r \quad (5.23)$$

This relation can also be obtained by simple (steady-state) heat balance around the heat exchanger. Note that  $T_r$  is positive whenever heat is removed from the system and negative when heat is added. Inserting equation (5.23) into equation (5.12) gives

$$Z_1 \frac{dT_c}{d\alpha} + r_2 T_c = r_2 T_2 + r_1 T_d \delta(\alpha) - r_1 T_r$$

$$T_c = 0 \quad \text{at} \quad \alpha = 0 \quad (5.24)$$

As mentioned previously this set of equations can be rewritten

$$Z_1 \frac{dT_c}{d\alpha} + r_2 T_c = r_2 T_2 - r_1 T_r \quad (5.24a)$$

$$T_c = T_{c0} \quad \text{at} \quad \alpha = 0^+$$

Steady State Value of  $T_r$  before Disturbance,  $T_{r0}$

The steady state value of  $T_r$  before disturbance,  $T_{r0}$ , can be evaluated by inserting

$$T_c = 0, \quad T_d = 0, \quad \text{and} \quad \frac{dT_c}{d\alpha} = 0$$

into equation (5.24). This gives rise to

$$T_{r0} = \frac{r_2 T_2}{r_1}, \quad r_1 \neq 0 \quad (5.25)$$

Note that the steady state value of  $T_r$ , which is denoted by  $T_{r0}$ , is zero when the outside air temperature,  $T_2$ , is zero, or when the ratio of the fresh air to the total air,  $r_2$ , is zero. This solution can also be obtained by either



over-all heat balance around the system or heat balances around the room and the heat exchanger.

1. Over-all heat balance around the system (Figure 5.4).

$$Q_2 C_p T_2 - (Q_2 C_p T_c + Q_1 C_p T_r) = 0$$

Therefore,

$$T_{r0} = \frac{Q_2 T_2}{Q_1} = \frac{r_2 T_2}{r_1} \quad (5.25a)$$

Recall that  $T_c$  equals zero.

2. Heat balances around the room and the heat exchanger.

$$(Q_1) \int C_p T_i + Q_2 \int C_p T_2 = 0$$

and

$$Q_1 \int C_p T_i + Q_1 \int C_p T_r = 0$$

Solving for  $T_{r0}$  from these equations, we obtain

$$T_{r0} = -T_r = \frac{r_2 T_2}{r_1}$$

#### Final Steady State Value of $T_r$ , $T_{rf}$

The final steady state value of  $T_r$  which is denoted by  $T_{rf}$ , can be obtained by letting

$$T_c = 0 \quad \text{and} \quad \frac{dT_c}{d\alpha} = 0$$

in equation (5.24). Hence

$$T_{rf} = \frac{1}{r_1} \int \mathcal{U}_1 T_d \delta(\alpha) + r_2 T_2 d\alpha$$

### Initial Value of $T_c$

The initial value of  $T_c$  at  $t = 0^+$  can be calculated by the following relation representing the heat balance between the condition before and that after the disturbance.

$$V_1 \rho C_p T_{c0} = V_1 \rho C_p T_d$$

or

$$T_{c0} = T_d \quad (5.26)$$

The desired final value of  $T_c$  is zero. Meanwhile, the lower bound of  $T_r$ ,  $T_{r \min}$ , is set at  $0^\circ\text{C}$ . Various cases with different upper bounds of  $T_r$ ,  $T_{r \max}$ , will be simulated.

### The Solution of $T_c$

Simulation of the desired model can be carried out when the form of  $T_r$  and the numerical values of the parameters are known. In case  $T_r$  is the step function, i.e.,  $T_r$  remains constant after  $t = 0$ , equation (5.24a) can be integrated as

$$T_c(\alpha) = e^{-\frac{r_2 \alpha}{Z_1}} \left[ A_1 + T_2 e^{\frac{r_2 \alpha}{Z_1}} - \frac{r_1 T_r}{r_2} \frac{r_2 \alpha}{Z_1} \right]$$

where  $A_1$  is an integration constant. The value of  $A_1$  can be determined by employing the initial condition at  $\alpha = 0^+$ , that is,

$$T_c = T_{c0} \quad \text{at} \quad \alpha = 0^+$$

Therefore,

$$A_1 = T_{c0} - T_2 + \frac{r_1 T_r}{r_2}$$

and

$$T_c(\alpha) = T_{c0} e^{-\frac{r_2 \alpha}{Z_1}} + T_2 \left[ 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right] - \frac{r_1 T_r}{r_2} \left[ 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right], \quad (5.27)$$

$$r_2 \neq 0$$

Note that we can also solve equation (5.24) by means of Laplace transform.

Laplace transformation of equation (5.26) gives,

$$T_c(s) = \frac{r_2 T_2}{s(Z_1 s + r_2)} - \frac{r_1 T_r}{s(Z_1 s + r_2)} + \frac{Z_1 T_d}{Z_1 s + r_2}$$

Inversion by partial fractions of the above equation gives

$$T_c(\alpha) = T_d e^{-\frac{r_2 \alpha}{Z_1}} + T_2 \left( 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right) - \frac{r_1 T_r}{r_2} \left( 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right)$$

which is identical to equation (5.27) because  $T_{c0} = T_d$  as given by equation (5.26).  $\alpha_f$  can be found from equation (5.27) by setting  $T_c = 0$ .

$$\alpha_f = -\frac{1}{r_2} \ln \left( \frac{r_1 T_r - r_2 T_2}{T_{c0} r_2 - T_2 r_2 + r_1 T_r} \right), \quad r_2 \neq 0 \quad (5.28)$$

For  $r_2 = 0$  or equivalently,  $r_1 = 1$ , equation (5.24) becomes

$$\frac{dT_c}{d\alpha} = T_d \delta(\alpha) - \frac{T_r \max}{Z_1}, \quad r_2 = 0$$

Integrating this equation, we have

$$T_c = -\frac{T_r \max}{Z_1} \alpha + T_{c0} \quad (5.27a)$$

$\alpha_f$  can be obtained by setting  $T_c = 0$  as

$$\alpha_f = \frac{T_{c0} Z_1}{T_{r \max}}, \quad r_2 = 0 \quad (5.28a)$$

### Numerical Examples

It is assumed that the volume of the reference system,  $V_1$ , is

$$\begin{aligned} V_1 &= 3^m \times 4^m \times 5^m \\ &= 60 \text{ m}^3 \end{aligned}$$

The flow rate of air in the system,  $Q$ , is

$$\begin{aligned} Q &= (\text{cross-sectional area of the system}) \\ &\quad \times (\text{air velocity in the system}) \\ &= (3^m \times 4^m) (.1 \text{ m/sec}) \\ &= 1.2 \text{ m}^3/\text{sec} \end{aligned}$$

and flow rates of circulation air and fresh air are

$$\begin{aligned} Q_1 &= 0.8Q = 0.96 \text{ m}^3/\text{sec} \\ Q_2 &= 0.2Q = 0.24 \text{ m}^3/\text{sec} \end{aligned}$$

The time constant of the system element,  $Z_1$ , is

$$Z_1 = \frac{V_1}{Q} = \frac{V_1}{Q_1 + Q_2} = \frac{60}{1.2} = 50 \text{ sec.}$$

Other numerical values employed are

$$T_2 = 10^\circ \text{C}$$

$$T_d = 20^\circ \text{C}$$

$$T_{r \min} = 0^\circ \text{C}$$

$$T_{r0} = \frac{r_2 T_2}{r_1} = \frac{0.2 \times 10}{0.8} = 2.5^\circ \text{C}$$

Here two examples with different heat removing (or control) capacities of the heat exchanger will be considered. The first example is for the case in which the maximum load (heat removing capacity) of the heat exchanger,  $T_{r \max}$  is set equal to  $2.5^\circ \text{C}$ , that is  $T_r = T_{r0} = 2.5^\circ \text{C}$  after  $t = 0$ . The second example is for the case in which the maximum load of the heat exchanger,  $T_{r \max}$ , is set to be  $30^\circ \text{C}$ .

$$\text{Case 1: } T_{r \max} = 2.5^\circ \text{C}$$

For this case, we have, from equations (5.27) and (5.28),

$$T_c(\alpha) = 20 e^{-\frac{r_2 \alpha}{50}} + 10 \left[ 1 - e^{-\frac{r_2 \alpha}{50}} \right] - \frac{2.5 r_1}{r_2} \left[ 1 - e^{-\frac{r_2 \alpha}{50}} \right]$$

$$\alpha_f = -\frac{50}{r_2} \ln \left[ \frac{2.5 r_1 - 10 r_2}{2.5 r_1 + 10 r_2} \right]$$

if  $r_2 \neq 0$ , and

$$T_c(\alpha) = 20 - \alpha/20$$

$$\alpha_f = 400$$

if  $r_2 = 0$ .

The results of simulation are shown schematically in Figure 5.5.

$$\text{Case 2: } T_{r \max} = 30^\circ \text{C}.$$

For this case we have equations (5.27) and (5.28)

$$T_c = 20 e^{-\frac{r_2 \alpha}{Z_1}} + 10 \left( 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right) - \frac{r_1}{r_2} \times 30 \left( 1 - e^{-\frac{r_2 \alpha}{Z_1}} \right)$$

$$\alpha_f = \frac{-Z_1}{r_2} \ln\left(\frac{3r_1 - r_2}{r_2 + 3r_1}\right) = -\frac{50}{r_2} \ln\left(\frac{3r_1 - r_2}{r_2 + 3r_1}\right)$$

if  $r_2 \neq 0$ , and

$$T_c = 20 - 0.6$$

$$\alpha_f = 33.3 \text{ sec}$$

if  $r_2 = 0$ .

The results of simulation are shown schematically in Figure 5.5.

Similarly, we can carry out the simulation by employing the dimensionless form of the performance equation. The performance equation in dimensionless form can be obtained by combining equations (5.12a) and (5.22a) and setting  $Z_2 = 0$  as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_1 + K_1 \sigma \delta(t) - r_1 K_1 K_2 \theta - r_1 K_1 K_3 \quad (5.29)$$

Boundary conditions are

$$\begin{array}{lll} x_1(0) = 0 & \text{at} & t = 0 \\ x_1(T) = 0 & \text{at} & t = T \end{array}$$

As mentioned previously, this set of equations can also be written as

$$\begin{array}{lll} \frac{dx_1}{dt} + r_2 x_1 = r_2 K_1 - r_1 K_1 K_2 \theta - r_1 K_1 K_3 \\ x_1(0^+) = 1 & \text{at} & t = 0^+ \\ x_1(T) = 0 & \text{at} & t = T \end{array} \quad (5.29a)$$

First of all, let us assume  $\theta$  is a given control action and is equal to  $\theta_{\max} = 1$ . Then equation (5.29a) can be integrated as follows

$$x_1(t) = e^{-r_2 t} \left[ A_2 + K_1 e^{r_2 t} - \frac{r_1 K_1 K_2}{r_2} e^{r_2 t} - \frac{r_1 K_1 K_3}{r_2} e^{r_2 t} \right],$$

$$r_2 \neq 0$$

Application of the boundary condition,  $x_1(0^+) = 1$  at  $t = 0^+$ , yields

$$A_2 = 1 - K_1 + \frac{r_1 K_1 K_2}{r_2} + \frac{r_1 K_1 K_3}{r_2}$$

and

$$\begin{aligned} x_1(t) &= e^{-r_2 t} + K_1 (1 - e^{-r_2 t}) - \frac{r_1 K_1 K_2}{r_2} (1 - e^{-r_2 t}) - \frac{r_1 K_1 K_3}{r_2} (1 - e^{-r_2 t}), \\ r_2 &\neq 0 \end{aligned} \quad (5.30)$$

Final time,  $T$ , corresponding to the end of control, can be obtained by using the condition

$$x_1(T) = 0 \quad \text{at} \quad t = T$$

This yields

$$T = -\frac{1}{r_2} \ln \left( \frac{r_1 K_1 K_2 + r_1 K_1 K_3 - r_2 K_1}{r_2 - r_2 K_1 + r_1 K_1 K_2 + r_1 K_1 K_3} \right) \quad r_2 \neq 0 \quad (5.31)$$

For  $r_2 = 0$ , equation (5.29) becomes

$$\frac{dx_1}{dt} = K_1 \delta(t) - K_1 K_2 - K_1 K_3$$

or in integrated form

$$x_1(t) = -K_1(K_2 + K_3)t + 1, \quad r_2 = 0 \quad (5.30a)$$

T can be obtained by employing

$$x_1 = 0 \quad \text{at} \quad t = T$$

This gives

$$T = \frac{1}{K_1(K_2 + K_3)}, \quad r_2 = 0 \quad (5.31a)$$

Two examples which correspond to the problems solved in dimensional form will be considered here.

$$\text{Case 1: } T_{r \text{ max}} = 2.5^\circ\text{C}$$

For this case, we have, from the definitions of  $K_2$  and  $K_3$  and equations (5.30) and (5.31),

$$K_2 = \frac{1}{2T_2} (T_{r \text{ max}} - T_{r \text{ min}}) = \frac{1}{20} (2.5 - 0) = 0.125$$

$$K_3 = \frac{1}{2T_2} (T_{r \text{ max}} + T_{r \text{ min}}) = \frac{1}{20} (2.5 + 0) = 0.125$$

$$x_1(t) = e^{-r_2 t} + 0.5(1 - e^{-r_2 t}) - \frac{0.0625 r_1}{r_2} (1 - e^{-r_2 t}),$$

$$r_2 \neq 0$$

$$T = -\frac{1}{r_2} \ln \frac{0.135 r_1 - 0.5 r_2}{0.135 r_1 + 0.5 r_2}, \quad r_2 \neq 0$$

$$x_1(t) = -0.135 t + 1, \quad r_2 = 0$$

$$T = \frac{1}{0.125}, \quad r_2 = 0$$



The results of simulation are shown in Figure 5.6.

Case 2:  $T_{r \text{ max}} = 30^\circ\text{C}$

For this case, we have

$$K_2 = \frac{1}{2T_2} (T_{r \text{ max}} - T_{r \text{ min}}) = \frac{1}{20} (30 - 0) = 1.5$$

$$K_3 = \frac{1}{2T_2} (T_{r \text{ max}} + T_{r \text{ min}}) = \frac{1}{20} (30 + 0) = 1.5$$

$$x_1(t) = e^{-r_2 t} + 0.5(1 - e^{-r_2 t}) - 1.5 \frac{r_1}{r_2} (1 - e^{-r_2 t}), \quad r_2 \neq 0$$

$$T = -\frac{1}{r_2} \ln \frac{1.5r_1 - 0.5r_2}{0.5r_2 + 1.5r_1}, \quad r_2 \neq 0$$

$$x_1(t) = -1.5t + 1, \quad r_2 = 0$$

$$T = \frac{1}{1.5}, \quad r_2 = 0$$

The results of simulation are shown in Figure 5.6 and tabulated in Table 5.1.

Note that the numerical examples are restricted to the cooling problems for simplicity. However, the performance equations developed here can be applied to the case in which air is heated in the heat exchanger, i.e.,  $T_r$  is negative. In other words, the performance equations can take into account both the heating and cooling actions of the heat exchanger.

#### General System Equations

In the preceding section, the performance equations have been derived for the cases in which the disturbance is of the form of the impulse function and air in the cabin or room is completely mixed. However, the

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procedure for deriving the performance equations is fairly general and can be extended to cases in which the disturbances are of the form other than the impulse function and air in the room or cabin is far from being completely mixed.

First, let us consider the case resulting from step heat disturbance while other conditions remain unchanged from those considered in the preceding section.  $q_{ds}$  has the form

$$q_{ds} = (Q_1 + Q_2) \int C_p T_d U_0(\alpha) \quad (5.32)$$

The performance equation, for the system element can be obtained as

$$Z_1 \frac{dT_c}{d\alpha} + T_c = r_1 T_1 + r_2 T_2 + T_d U_0(\alpha) \quad (5.33)$$

or in dimensionless form

$$\frac{dx_1}{dt} + x_1 = \frac{K_1 r_1 x_2}{K_4} + r_2 K_1 + K_1 \sigma U_0(t) \quad (5.33a)$$

Similarly, for the system with ramp heat disturbance, we have

$$q_{dr} = \frac{(Q_1 + Q_2)^2}{V_1} \int C_p T_d R(\alpha) \quad (5.34)$$

$$Z_1 \frac{dT_c}{d\alpha} + T_c = r_1 T_1 + r_2 T_2 + \frac{T_d R(\alpha)}{Z_1} \quad (5.35)$$

and

$$\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1 x_2}{K_4} + r_2 K_1 + K_1 \sigma R(t) \quad (5.35a)$$

In general, the dimensionless performance equation for the system element can be written as

$$\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1 x_2}{K_4} + r_2 K_1 + K_1 \delta F(t) \quad (5.36)$$

where  $F(t)$  stands for the functional form of the heat disturbance which can be impulse function  $\delta(t)$ , unit step function  $U(t)$ , ramp function  $R(t)$  [or  $t U(t)$ ], cyclic disturbance or any other disturbance. This equation together with equation (5.22a) form the complete dimensionless system equations, that is,

$$\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1 x_2}{K_4} + r_2 K_1 + K_1 \delta F(t) \quad (5.36)$$

$$\frac{z_2}{z_1} \frac{dx_2}{dt} + x_2 = \frac{K_4 x_1}{K_1} - K_4 (K_2 \theta + K_3) \quad (5.22a)$$

It is worth noting that the initial condition for  $T_c$  is

$$T_c = T_{c0} = 0 \quad \text{at} \quad t = 0$$

The performance equations for various types of heat disturbance are tabulated in Table 5.2.

#### Performance Equations for Two Compartment Model

Next, let us consider the case in which air in the room or cabin is no longer in the state of complete mixing. Specifically, we shall consider the case in which flow of air in the room can be characterized by two completely stirred tanks (or pools or compartments) (2 CST's) in series.

The following assumptions must be added to those already made for the system proper in the preceding section:

1. The room is divided into two well-mixed compartments in series.

Volume of each pool is denoted by  $V_{11}$  and  $V_{12}$ , and the temperature in

each pool is denoted by  $T_{11}$  and  $T_{12}$ .

2. Backflow of air from the second compartment to the first compartment is negligible.
3. Disturbances are equally distributed over the system.
4. Fresh air comes into the first compartment at a constant flow rate, while exhaust air is released from the second compartment at a constant flow rate.

The schematic diagram of the system is shown in Figure 5.7. The performance equation for each pool can be obtained by using the transient heat balance around each compartment. Thus, for pool 1

$$[\text{heat in}] - [\text{heat out}] = [\text{heat accumulation}] \quad (5.37)$$

or

$$\begin{aligned} & [Q_1 T_1] c_p + Q_2 T_2 c_p + \frac{V_{11}}{V_1} T_d \delta(\alpha) V_{11} c_p - (Q_1 + Q_2) T_{c1} c_p \\ & = V_{11} c_p \frac{dT_{c1}}{d\alpha} \end{aligned} \quad (5.38)$$

Dividing this equation by  $c_p(Q_1 + Q_2)$  yields

$$Z_{11} \frac{dT_{c1}}{d\alpha} + T_{c1} = r_1 T_1 + r_2 T_2 + \frac{Z_{11}}{Z_1} Z_{11} T_d \delta(\alpha) \quad (5.39)$$

$$T_{c1} = 0 \quad \text{at} \quad \alpha = 0$$

or

$$Z_{11} \frac{dT_{c1}}{d\alpha} + T_{c1} = r_1 T_1 + r_2 T_2$$

$$T_{c1} = T_{c10} \quad \text{at} \quad \alpha = 0^+$$

where  $\tau_{11}$  is the time constant of pool 1 and is defined by

$$\tau_{11} = \frac{V_{11}}{Q_1 + Q_2}$$

Note that

$$\frac{V_{11}}{V_1} = \frac{V_{11}/(Q_1 + Q_2)}{V_1/(Q_1 + Q_2)} = \frac{\tau_{11}}{\tau_1}$$

Similarly, for pool 2 we have

$$\begin{aligned} & \frac{[(Q_1 + Q_2)T_{c1} \int C_p + \frac{V_{12}}{V_1} T_d \delta(\alpha) V_{12} \int C_p] - (Q_1 + Q_2) \int C_p T_{c2}}{=} \\ & = V_{12} \int C_p \frac{dT_{c2}}{d\alpha} \end{aligned} \quad (5.40)$$

Again dividing this equation by  $\int C_p (Q_1 + Q_2)$  yields

$$\tau_{12} \frac{dT_{c2}}{d\alpha} + T_{c2} = T_{c1} + \frac{\tau_{12}}{\tau_1} \tau_{12} T_d \delta(\alpha) \quad (5.41)$$

$$T_{c2} = 0 \quad \text{at} \quad \alpha = 0$$

or

$$\tau_{12} \frac{dT_{12}}{d\alpha} + T_{c2} = T_{c1}$$

$$T_{c2} = T_{c20} \quad \text{at} \quad \alpha = 0^+$$

where  $\tau_{12}$  is the time constant of pool 2 and is defined by

$$\tau_{12} = \frac{V_{12}}{Q_1 + Q_2}$$

For the heat exchanger, we have from equation (5.22)

$$\mathcal{Z}_2 \frac{dT_i}{d\mathcal{Z}} + T_i = T_{c2} - T_r$$

Equations (5.39), (5.41) and (5.42) are the performance equations of the system. We may rewrite these in dimensionless form by defining

$$x_{11} = \frac{K_{11}T_{c1}}{T_2}$$

$$x_{12} = \frac{K_{12}T_{c2}}{T_2}$$

$$x_2 = \frac{K_4 T_i}{T_2}$$

$$t = \frac{\mathcal{L}}{\mathcal{Z}_1}$$

Introducing these definitions into equations (5.39), (5.41) and (5.42), we have

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_2 + a_{12} + a_{13}\delta(t) \quad (5.43)$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = a_{21}x_{11} + a_{23}\delta(t) \quad (5.44)$$

$$\frac{dx_2}{dt} + rx_2 = a_{42}x_{12} - a_5\theta - a_6$$

where

$$a_{11} = r_1 K_{11} r_{11}/K_4$$

$$a_{12} = r_{11} r_2 K_{11}$$

$$a_{13} = T_d K_{11}/T_2 r_{11}$$

$$a_{21} = r_{12} K_{12}/K_{11}$$

$$a_{23} = T_d K_{12}/T_2 r_{12}$$

$$a_{42} = r K_4/K_{12}$$

$$a_5 = r K_2 K_4$$

$$a_6 = r K_3 K_4$$

$$r_{11} = \frac{z_1}{z_{11}}$$

$$r_{12} = \frac{z_1}{z_{12}}$$

$$r = \frac{z_1}{z_2}$$

The performance equations derived in this section can also be used for simulation either on a digital or on an analog computer by following the procedure presented in the preceding section.



TABLE 5.1  
SIMULATION RESULTS FOR IMPULSE HEAT DISTURBANCE

| $r_1$ | $r_2$ | Dimensional                       |                                  | Dimensionless   |               |
|-------|-------|-----------------------------------|----------------------------------|-----------------|---------------|
|       |       | $T_r \text{ max} = 2.5 \text{ C}$ | $T_r \text{ max} = 30 \text{ C}$ | $K_2=K_3=0.125$ | $K_2=K_3=1.5$ |
|       |       | $\phi_I(\text{sec})$              | $\phi_f(\text{sec})$             | T               | T             |
| 1     | 0     | 400                               | 33.3                             | 8               | .66           |
| .8    | .2    | $\infty$                          | 41.8                             | $\infty$        | .84           |
| 0     | 1     | $\infty$                          | $\infty$                         | $\infty$        | $\infty$      |

TABLE 5.2  
THE PERFORMANCE EQUATION FOR SYSTEM ELEMENT

| Type of Heat Disturbances | The Dimensionless Performance Equations                                                                                                           | Initial Condition of $T_c$ |
|---------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|
| Impulse $\delta(t)$       | $\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1}{K_4} x_2 + r_2 K_1 + K_1 \delta(t)$ <p>or</p> $\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1}{K_4} x_2 + r_2 K_1$ | 0<br>or<br>1               |
| Step $u(t)$               | $\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1}{K_4} x_2 + r_2 K_1 + K_1 \delta u_0(t)$                                                                   | 0                          |
| Ramp $R(t)$               | $\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1}{K_4} x_2 + r_2 K_1 + K_1 \delta R(t)$                                                                     | 0                          |
| General $F(t)$            | $\frac{dx_1}{dt} + x_1 = \frac{r_1 K_1}{K_4} x_2 + r_2 K_1 + K_1 \delta F(t)$                                                                     | 0                          |

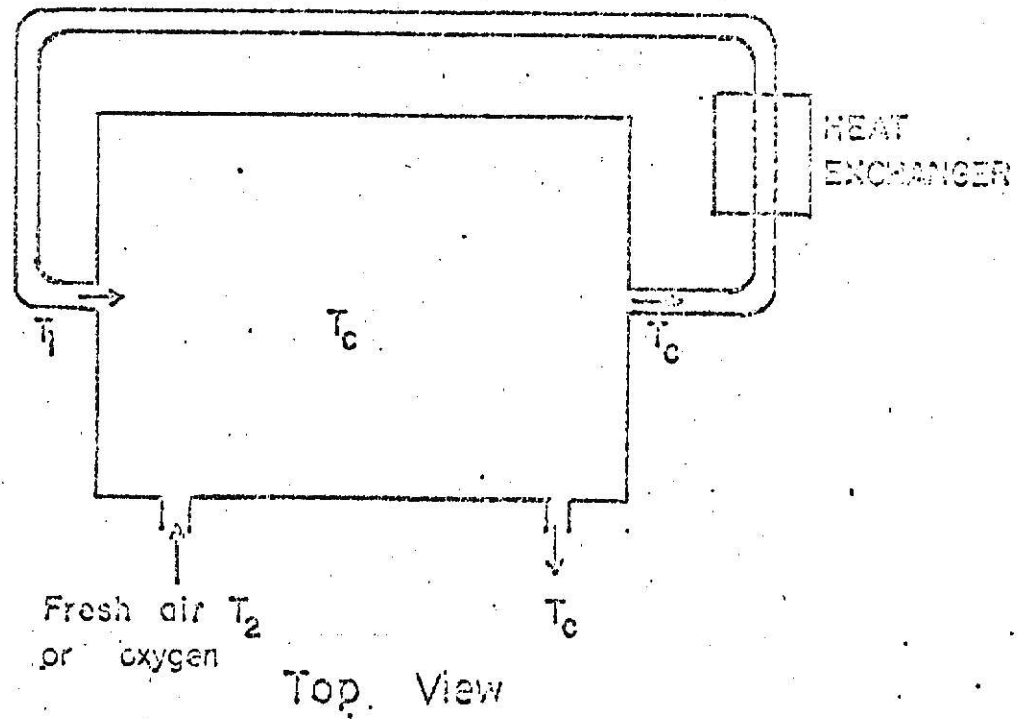
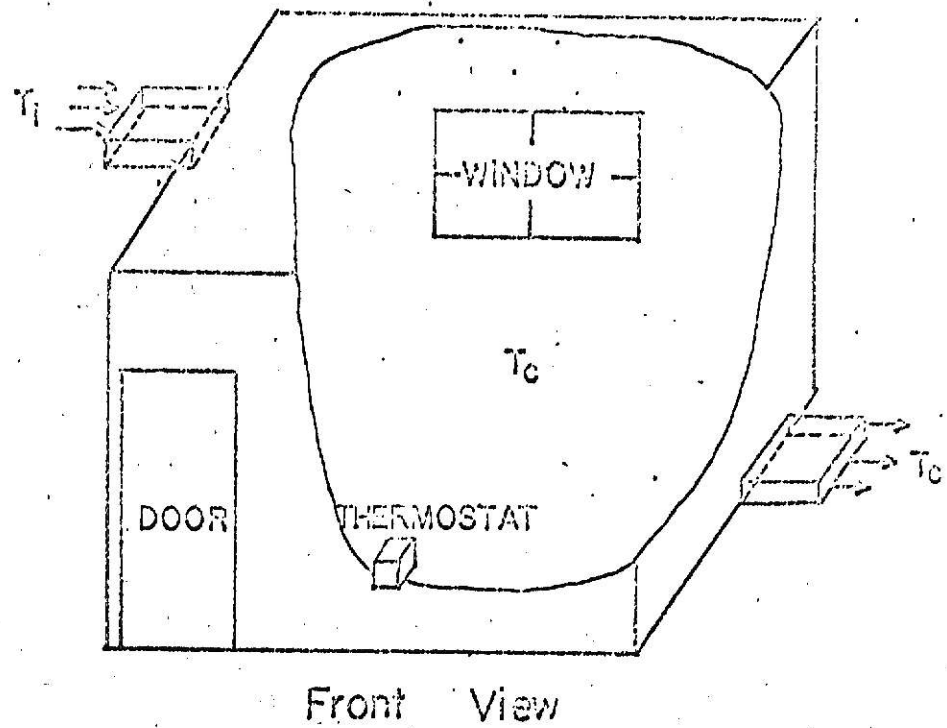


Figure 5.1. AIR-CONDITIONED ROOM

**THIS BOOK CONTAINS  
NUMEROUS PAGE  
NUMBERS THAT ARE  
ILLEGIBLE**

**THIS IS AS RECEIVED  
FROM THE  
CUSTOMER**

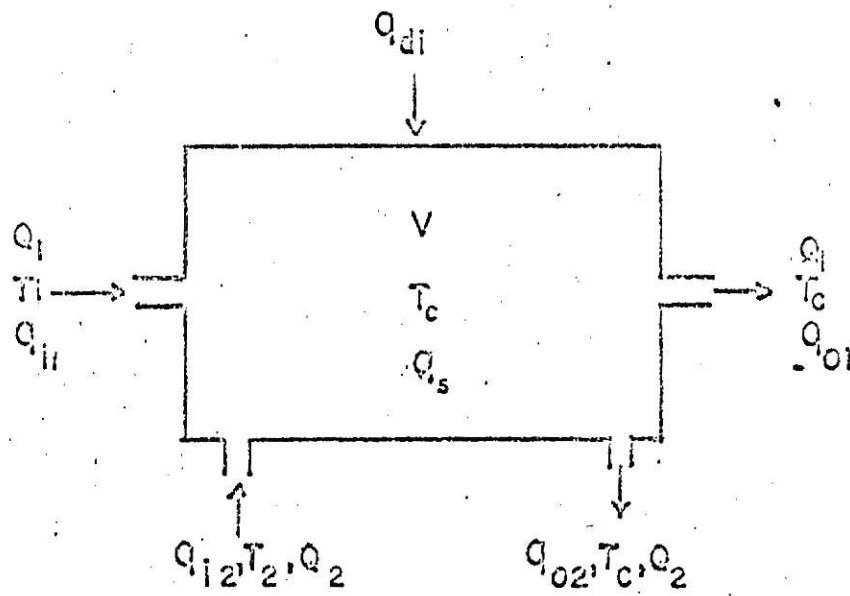


Figure 5.2. ROOM HEAT FLOW RATES

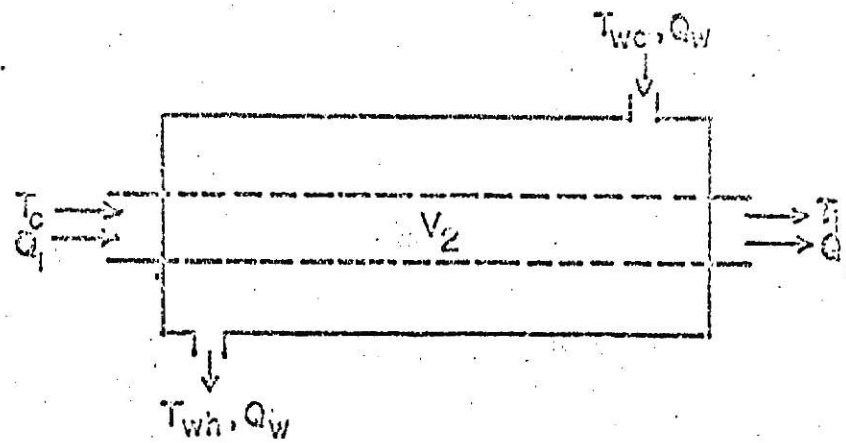


Figure 5.3. SCHEMATIC DIAGRAM OF HEAT EXCHANGER

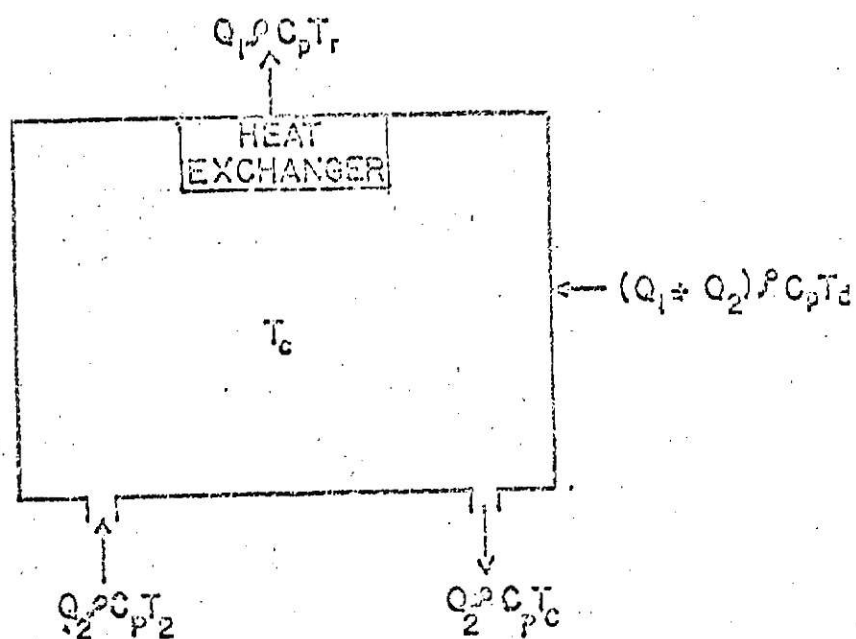


Figure 5.4. OVER-ALL HEAT BALANCE OF THE SYSTEM WITH  $Z_2 = 0$

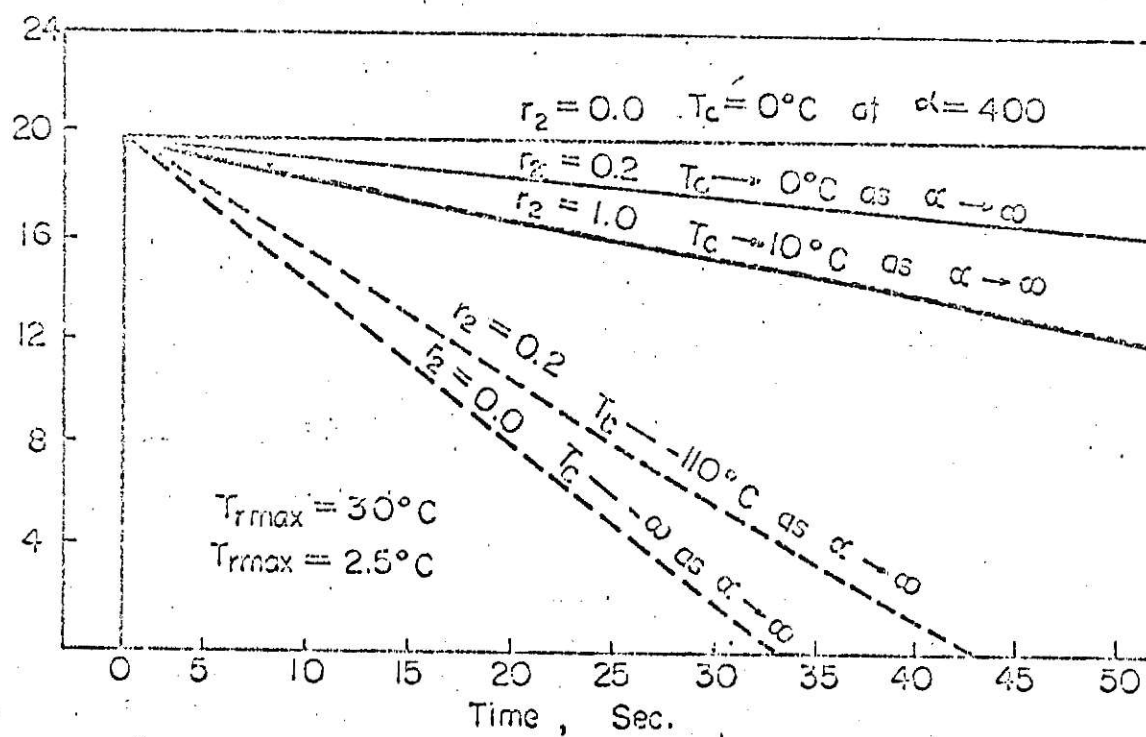


Figure 5.5. RESULT OF SIMULATION OF EQUATION (5.27) WITH  $T_2 = 10^\circ\text{C}$  AND  $T_d = 20^\circ\text{C}$ .



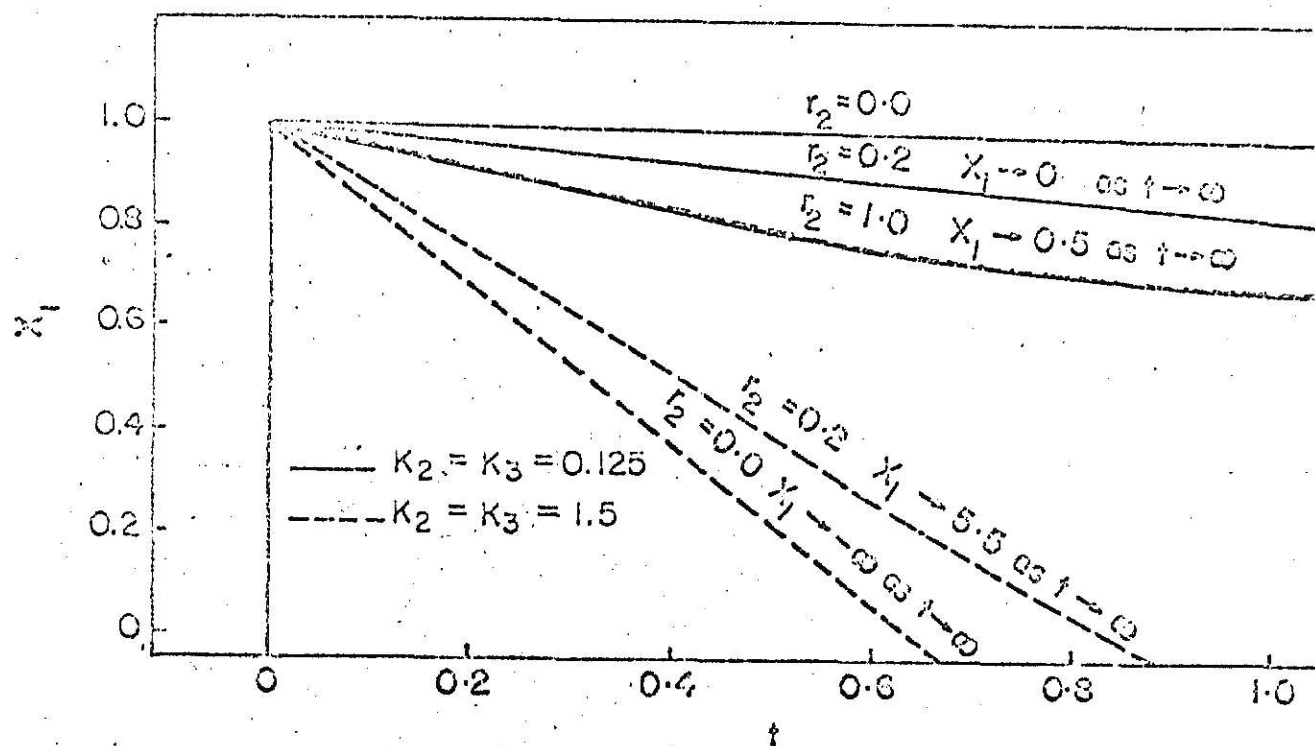


Figure 5.6. RESULTS OF SIMULATION OF EQUATION (5.29) WITH  $T_2 = 10^\circ \text{C}$  AND  $T_d = 20^\circ \text{C}$ .

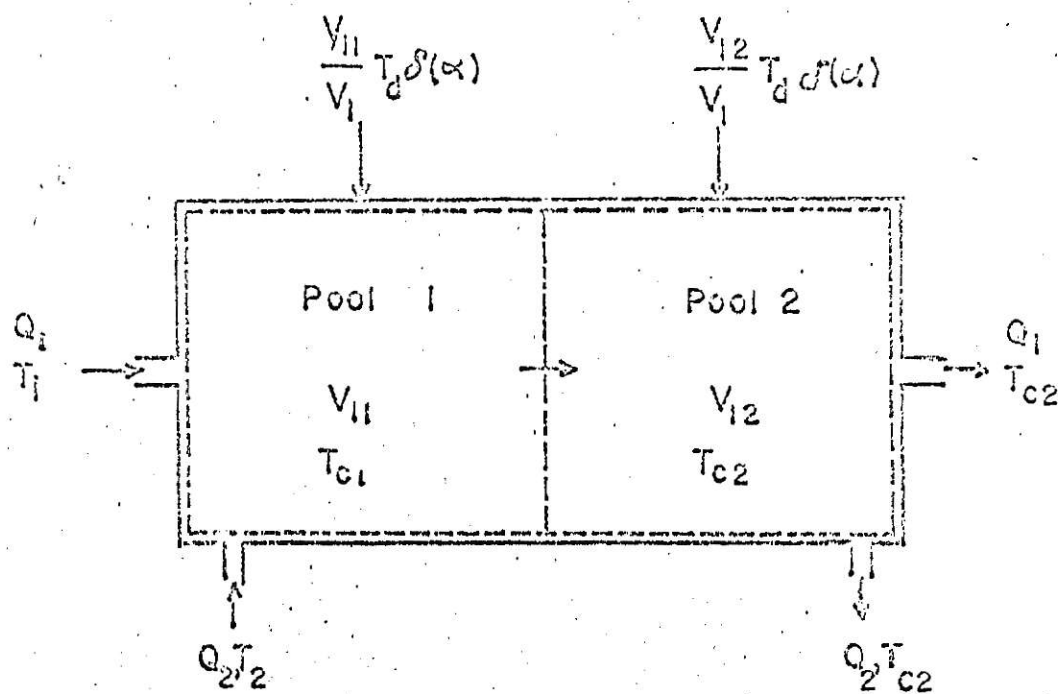


Figure 5.7. SCHEMATIC EXPRESSION OF THE MODIFIED  
SYSTEM ELEMENT

## CHAPTER 6

### NUMERICAL EXAMPLES I

In this chapter and the two to follow, the application of Pontryagin's maximum principle to control the life support system is presented, and several examples show how the maximum principle may be employed in the analysis and design of optimal control systems.

The basic form of the maximum principle presented in Chapter 2 will be applied to concrete examples. Procedures and computational approaches employed will be given in detail.

Example 6.1. Suppose that the dynamic behavior of a life support system consisting of an air-conditioned room or cabin subject to the impulse heat disturbance and a heat exchanger with negligibly small time constant ( $\tau_2 \rightarrow 0$ ), can be represented by the following equation [equation (5.29a)]:

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_1 - r_1 K_1 K_2 \theta - r_1 K_1 K_3 \quad (6.1)$$

with

$$\begin{array}{lll} x_1(0) = 1 & \text{at} & t = 0^+ \\ x_1(T) = 0 & \text{at} & t = T \end{array}$$

where  $T$  is the unspecified final control time. We wish to determine  $\theta$  so that the response of the system can return to its desired state in a minimum period of time, that is, to minimize

$$S_1 = \int_0^T dt \quad (6.2)$$

If an additional state variable  $x_2$  is introduced as

$$x_2(T) = \int_0^T dt$$

it follows that

$$\frac{dx_2}{dt} = 1, \quad x_2(0) = 0 \quad (6.3)$$

The problem is thus transformed into that of minimizing  $x_2(T)$ .

According to equation (2.5), the Hamiltonian is

$$\begin{aligned} H[\underline{z}(t), \underline{x}(t), \underline{\theta}(t)] \\ = z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} \\ = z_1 [-r_2 x_1 + r_2 K_1 - r_1 K_1 K_2 \theta - r_1 K_1 K_3] + z_2 \end{aligned} \quad (6.4)$$

The components of the adjoint vector, according to equation (2.6), are defined by

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = r_2 z_1 \quad (6.5)$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0, \quad z_2(T) = 1 \quad (6.6)$$

Solutions of equations (6.5) and (6.6) are

$$z_1(t) = A e^{r_2 t} \quad (6.7)$$

$$z_2(t) = 1, \quad 0 \leq t \leq T \quad (6.8)$$

where  $A$  is the integration constant to be determined later. Inserting

equation (6.8) into equation (6.4) yields

$$H = -r_2 K_1 K_2 z_1 \theta - r_2 z_1 x_1 + r_2 K_1 z_1 - r_1 K_1 K_3 z_1 + 1 \quad (6.9)$$

Therefore the switching function,  $H^*$ , the portion of  $H$  which depends on  $\theta$ , is

$$H^* = - r_1 K_1 K_2 z_1 \theta \quad (6.10)$$

Recall that minimization of the Hamiltonian with respect to  $\theta$  corresponds to that of the objective function. Equation (6.10), however, indicates that the minimization of the Hamiltonian with respect to  $\theta$  is equivalent to that of the switching function. Thus, minimization of the switching function corresponds to that of the objective function. Equation (6.10) also indicates that for the switching function to assume the minimum value,  $\theta$  must assume its minimum allowable or its maximum allowable value depending on the sign of the coefficient of  $\theta$ .

$$\theta = \theta_{\max} = 1 \quad \text{if} \quad r_1 K_1 K_2 z_1 > 0 \quad (6.11)$$

$$\theta = \theta_{\min} = -1 \quad \text{if} \quad r_1 K_1 K_3 z_1 < 0$$

Time optimal control policy of this type is of bang-bang type  $[1, 3, 8]$ .

In the case where the coefficient of  $\theta$  in equation (6.10) vanishes, we have the possibility of singular control  $[12]$ . For singular control, the control variable takes on values which are intermediate to  $\theta_{\max}$  and  $\theta_{\min}$ ; hence the name intermediate control is also used in place of the singular control  $[12]$ . Also inertialess control will be considered. An inertialess controller has the ability to shift from  $\theta_{\max}$  to  $\theta_{\min}$  instantaneously and vice versa.

The maximum principle now requires that the system equations, equations (6.1) and (6.3), be integrated simultaneously with the adjoint equation (6.5) so that the two-point boundary conditions

$$\begin{aligned} x_1(0) &= 1, & x_1(T) &= 0 \\ x_2(0) &= 0, & x_2(T) &= \text{undetermined} \\ z_1(0) &= \text{undetermined}, & z_1(T) &= \text{undetermined} \end{aligned}$$

are satisfied. For this minimum time problem the extremum of the Hamiltonian must vanish at every point of its response.

In order to bring the initial deviated state  $x_1(0^+) = 1$  to the final desired operating state  $x_1(T) = 0$ , we intuitively reject the control  $\theta = \theta_{\min} = -1$  (which corresponds to the minimum cooling action). Equation (6.1) can be integrated with the conditions

$$\theta = \theta_{\max} = 1 \tag{6.12}$$

and

$$x_1(0) = 1 \quad \text{at} \quad t = 0^+ \tag{6.13}$$

as

$$\begin{aligned} x_1(t) &= e^{-r_2 t} + K_1(1 - e^{-r_2 t}) - \frac{r_1 K_1 K_2}{r_2} (1 - e^{-r_2 t}) \\ &\quad - \frac{r_1 K_1 K_3}{r_2} (1 - e^{-r_2 t}) \end{aligned} \tag{6.14}$$

The integration constant  $A$  in equation (6.7) can be determined by using the condition that minimum  $H$  is zero for all the process times in time optimal control. At  $t = 0^+$ , we have from equations (6.7), (6.9), (6.12) and (6.13)

$$A = z_1(0^+) = \frac{1}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}$$

and

$$z_1(t) = \frac{-1}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} e^{r_2 t} \quad (6.15)$$

Equation (6.15) implies that  $z(t)$  will not change sign except when  $t$  approaches negative infinity when  $z_1(t) \rightarrow 0$ , or stated in another way, control will not shift from  $\theta_{\max}$  to  $\theta_{\min}$  (or from  $\theta_{\min}$  to  $\theta_{\max}$ ). Therefore, this problem is a particular case of bang-bang control which has the bang part only. The optimal control policy starts with  $T_r \max$  and then keeps operating at the upper bound of  $T_r$  until the final desired state is reached. The final control time can be obtained from equations (6.9) and (6.12) together with the final condition

$$x_1(T) = 0 \quad \text{at} \quad t = T$$

as follows

$$H = z_1(T) [-r_2 x_1(T) + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3] + 1 = 0$$

or solving for  $z_1(T)$

$$z_1(T) = \frac{-1}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} \quad (6.16)$$

Also we have, from equation (6.15), at  $t = T$

$$z_1(T) = \frac{-1}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} e^{r_2 T} \quad (6.17)$$

Solving for  $T$  from equations (6.16) and (6.17) gives

$$T = \frac{1}{r_2} \ln \frac{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} \quad (6.18)$$

This solution may be verified by inserting it into equation (6.14) as

$$\begin{aligned}
 x_1(T) &= e^{-r_2 T} + K_1(1 - e^{-r_2 T}) - \frac{r_1 K_1 K_2}{r_2} (1 - e^{-r_2 T}) - \frac{r_1 K_1 K_3}{r_2} (1 - e^{-r_2 T}) \\
 x_1(T) &= e^{-r_2 \sqrt{\frac{1}{r_2}} \ln \frac{-r_2 + K_1 r_2 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}} \\
 &\quad + K_1 \left\{ 1 - e^{-r_2 \sqrt{\frac{1}{r_2}} \ln \frac{-r_2 + K_1 r_2 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}} \right\} \\
 &\quad - \frac{r_1 K_1 K_2}{r_2} \left\{ 1 - e^{-r_2 \sqrt{\frac{1}{r_2}} \ln \frac{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}} \right\} \\
 &\quad - \frac{r_1 K_1 K_3}{r_2} \left\{ 1 - e^{-r_2 \sqrt{\frac{1}{r_2}} \ln \frac{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}} \right\}
 \end{aligned}$$

$$x_1(T) = 0$$

This indicates that the Hamiltonian is kept at zero at every point of its response in this minimum time problem. For

$$r_1 = 0.8 ,$$

$$r_2 = 0.2$$

$$K_1 = 0.5 ,$$

$$K_2 = 1.5$$

$$K_3 = 1.5 ,$$

$$\delta = 2 ,$$

we have from equations (6.14), (6.15) and (6.18)

$$z_1(t) = \frac{1}{1.3} e^{0.2t} = 0.769 e^{0.2t} \quad (6.19)$$



$$x_1(t) = 6.5e^{-0.2t} - 5.5 \quad (6.20)$$

$$T = 0.8353$$

and from equation (6.19)

$$z_1 = \frac{1}{1.3} = 0.769 \quad \text{at} \quad t = 0$$

$$z_1 = \frac{1}{1.1} = 0.909 \quad \text{at} \quad t = T$$

Equations (6.19) and (6.20) are graphically shown in Figure 6.1. The state variable  $x_1$  approaches asymptotically to the final state, the control variable remains at unity until the final state is reached and the adjoint vector increases asymptotically. The optimal control can be verified by computing  $H$  at an arbitrary point, say 0.5, of the time coordinate as follows:

$$t = 0.5$$

$$z_1(t) = e^{-0.1}/1.3$$

$$x_1(t) = 6.5e^{-0.1} - 5.5$$

and

$$\begin{aligned} H &= z_1(t) \cdot [-r_2 x_1 + r_2 K_1 + K_1 f(t) - r_1 K_1 K_2 - r_1 K_1 K_3] + 1 \\ &= \frac{e^{-0.1}}{1.3} [-0.2(6.5e^{-0.1} - 5.5) + 0.2 \times 0.5 + 0 - 1.2] + 1 \\ &= 0 \end{aligned}$$

This computation shows that the minimum value of  $H$  is zero at every point of this continuous process.

Four cases with different cooling capacities of the heat exchangers are considered here.  $T_r \text{ max}$  and  $T_r \text{ min}$  take the following values for these

four cases:

$$\text{Case 1: } T_{r \max} = 30^{\circ}\text{C} , \quad T_{r \min} = 0^{\circ}\text{C}$$

$$\text{Case 2: } T_{r \max} = 20^{\circ}\text{C} , \quad T_{r \min} = 0^{\circ}\text{C}$$

$$\text{Case 3: } T_{r \max} = 10^{\circ}\text{C} , \quad T_{r \min} = 0^{\circ}\text{C}$$

$$\text{Case 4: } T_{r \max} = 5^{\circ}\text{C} , \quad T_{r \min} = 0^{\circ}\text{C}$$

The numerical solutions for these cases are obtained from equations (6.14), (6.15) and (6.18) and are tabulated in Table 6.1.

Example 6.2. Generally, responses of the heat exchanger as well as the cabin are not always instantaneous. Suppose that for the system considered in the preceding example that the time constant of the heat exchanger,  $\tau_2$ , is not so small as to be ignored. The performance equations for such a system have been derived in Chapter five [equations (5.12a) and (5.22a)]

$$\frac{dx_1}{dt} + x_1 = a_1 x_2 + a_2 \quad (6.21)$$

$$\frac{dx_2}{dt} + r x_2 = a_4 x_1 - a_5 \theta - a_6 \quad (6.22)$$

with the initial conditions

$$x_1(0^+) = 1, \quad x_2(0^+) = 1 \quad \text{at} \quad t = 0^+ \quad (6.23)$$

The decision variable  $\theta$  is constrained as

$$|\theta| \leq 1$$

where

$$r = \frac{\tau_1}{\tau_2}, \quad (\tau_2 \neq 0)$$

$$a_1 = r_1 K_1 / K_4$$

$$a_2 = r_2 K_1$$

$$a_3 = K_1$$

$$a_4 = \frac{r K_4}{K_1}$$

$$a_5 = r K_2 K_4$$

$$a_6 = r K_3 K_4$$

We wish to determine the control variable  $\theta$  so that the state variables may be brought from the initial deviated state

$$x_1 = 1, \quad x_2 = 1 \quad \text{at} \quad t = 0^+$$

to the final desired state

$$x_1 = 0, \quad x_2 = 1 \quad \text{at} \quad t = T$$

in a minimum time. In other words

$$S_1 = \int_0^T dt$$

is to be minimized.

If an additional state variable  $x_3$  is introduced as

$$x_3(t) = \int_0^t dt, \quad (6.24)$$

it follows that

$$\frac{dx_3}{dt} = 1, \quad x_3(0^+) = 0 \quad (6.25)$$

and

$$x_3(T) = \int_0^T dt = S_1 \quad (6.26)$$

The problem is now transformed into that of minimizing  $x_3(T)$  because  $x_3(T)$  and  $S_1$  are identical.

According to equation (2.5), the Hamiltonian is,

$$\begin{aligned}
 H(z, x, \theta) &= z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt} \\
 &= z_1[-x_1 + a_1x_2 + a_2] + z_2[-rx_2 + a_4x_1 - a_5\theta - a_6] + z_3 \quad (6.27)
 \end{aligned}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = z_1 - a_4z_2 \quad (6.28)$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = rz_2 - a_1z_1 \quad (6.29)$$

$$\frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = 0, \quad z_3(T) = 1 \quad (6.30)$$

From equation (6.30), the solution of  $z_3$  is

$$z_3(t) = 1, \quad 0^+ \leq t \leq T \quad (6.31)$$

Hence, the Hamiltonian can be rewritten as

$$H = z_1(-x_1 + a_1x_2 + a_2) + z_2(-rx_2 + a_4x_1 - a_5\theta - a_6) + 1 \quad (6.32)$$

and the switching function  $H^*$ , the portion of  $H$  which depends on  $\theta$ , is

$$H^* = -a_5z_2\theta \quad (6.33)$$

Inspection of  $H^*$  shows the basic structure of the time optimal control policy is of the bang-bang type as in the first example. The conditions for which

the Hamiltonian be minimum are

$$\begin{aligned} \theta &= \theta_{\max} = 1 & \text{if } a_5 z_2 > 0 \\ \theta &= \theta_{\min} = -1 & \text{if } a_5 z_2 < 0 \end{aligned} \quad (6.34)$$

These conditions also imply that if switching occurs, it will be at

$$a_5 z_2 = 0 \quad (6.35)$$

provided that the controller shifts from  $\theta_{\max}$  to  $\theta_{\min}$  instantaneously and inertialessly, or vice versa.

Now the maximum principle requires that the system equations and the adjoint vectors, equations (6.21), (6.22), (6.25), (6.28) and (6.29), be integrated simultaneously in such a way that the two-point boundary conditions

$$\begin{aligned} x_1(0^+) &= 1, & x_1(T) &= 0 \\ x_2(0^+) &= 1, & x_2(T) &= 1 \\ x_3(0^+) &= 0, & x_3(T) &\text{ undetermined} \\ z_1(0^+) &\text{ undetermined}, & z_1(T) &\text{ undetermined} \\ z_2(0^+) &\text{ undetermined}, & z_2(T) &\text{ undetermined} \end{aligned}$$

be satisfied. Meanwhile, the Hamiltonian must remain at zero at every point of its response under the optimal condition.

In order to bring the initial deviated state

$$x_1(0^+) = 1, \quad x_2(0^+) = 1$$

to the final desired state

$$x_1(T) = 0, \quad x_2(T) = 1,$$

we intuitively start the control from  $\theta = \theta_{\max} = 1$ . (This corresponds to the minimum cooling action). Substituting this condition into equations (6.21) and (6.22), and eliminating  $x_2$  gives

$$\frac{d^2 x_1}{dt^2} + (1+r) \frac{dx_1}{dt} + (r - a_1 a_4) x_1 + (a_1 a_5 + a_1 a_6 - r a_2) = 0$$

The solution of this equation is

$$x_1 = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + K, \quad 0 \leq t \leq t_s \quad (6.36)$$

where  $A_1$  and  $A_2$  are constant and their values will be determined later.

$\lambda_1$  and  $\lambda_2$  are roots of the characteristic equation

$$\lambda^2 + (1+r)\lambda + (r - a_1 a_4) = 0$$

and  $K$  is the particular solution and its value is

$$K = \frac{r a_2 - a_1 a_5 - a_1 a_6}{r - a_1 a_4}$$

From equation (6.30), the solution of  $x_2$  is,

$$x_2 = \frac{1}{a_1} [(\lambda_1 + 1) A_1 e^{\lambda_1 t} + (\lambda_2 + 1) A_2 e^{\lambda_2 t} + K - a_2] \quad (6.37)$$

The initial conditions applied to equations (6.36) and (6.37) give

$$A_1 = \frac{1}{\lambda_1 - \lambda_2} (a_1 + a_2 - \lambda_2 - 1 + K\lambda_2) \quad (6.38)$$

$$A_2 = 1 - A_1 - K \quad (6.39)$$

Suppose that the control switches from  $\theta_{\max} = 1$  to  $\theta_{\min} = -1$  at a certain time. Switching time  $t_s$ ,  $x_1(t)$  and  $x_2(t)$  are solved from equations (6.21) and (6.22). The results are

$$x_1(t) = D_1 e^{\lambda_1 t} + D_2 e^{\lambda_2 t} + K' \quad t_s \leq t \leq T \quad (6.40)$$

$$x_2(t) = \frac{1}{a_1} [(1 + \lambda_1) D_1 e^{\lambda_1 t} + (1 + \lambda_2) D_2 e^{\lambda_2 t} + K - a_2] \quad t_s \leq t \leq T \quad (6.41)$$

where

$$K' = \frac{ra_2 + a_1 a_5 - a_1 a_5}{r - a_1 a_4}$$

and  $D_1$  and  $D_2$  are constant, and their values can be determined by using the continuity of  $x$  with respect to  $t$ . We have from equations (6.36), (6.37), (6.40) and (6.41) at  $t = t_s$

$$\begin{aligned} x_1(t_s) &= A_1 e^{\lambda_1 t_s} + A_2 e^{\lambda_2 t_s} + K \\ &= D_1 e^{\lambda_1 t_s} + D_2 e^{\lambda_2 t_s} + K' \end{aligned} \quad (6.42)$$

and

$$x_2(t_s) = \frac{1}{a_1} [(1 + \lambda_1) A_1 e^{\lambda_1 t_s} + (1 + \lambda_2) A_2 e^{\lambda_2 t_s} + K - a_2]$$

$$= \frac{1}{a_1} [(1 + \lambda_1) D_1 e^{\lambda_1 t_s} + (1 + \lambda_2) D_2 e^{\lambda_2 t_s} + K' - a_2] \quad (6.43)$$

Solving for  $D_1$  and  $D_2$  in terms of  $A_1$ ,  $A_2$  and  $t_s$  from equations (6.42) and (6.43),

$$D_1 = A_1 - E_1 e^{\lambda_1 t_s} \quad (6.44)$$

and

$$D_2 = A_2 - E_2 e^{\lambda_2 t_s} \quad (6.45)$$

where

$$E_1 = \frac{\lambda_2 (K' - K)}{\lambda_1 + \lambda_2}$$

$$E_2 = \frac{\lambda_1 (K' - K)}{\lambda_1 + \lambda_2}$$

The value of  $t_s$  and that of  $T$  can be determined by employing the final conditions at  $t = T$ . Thus, equations (6.40) and (6.41) become

$$D_1 e^{\lambda_1 T} + D_2 e^{\lambda_2 T} + K' = 0 \quad (6.46)$$

$$(1 + \lambda_1) D_1 e^{\lambda_1 T} + (1 + \lambda_2) D_2 e^{\lambda_2 T} + K' - a_2 = a_1 \quad (6.47)$$

Subtracting equation (6.46) from equation (6.47) yields

$$\lambda_1 D_1 e^{\lambda_1 T} + \lambda_2 D_2 e^{\lambda_2 T} = a_1 + a_2 \quad (6.48)$$

Solving for  $D_2 e^{\lambda_2 T}$  from the above equation and equation (6.46) gives



$$D_2 e^{\lambda_2 T} = E_4 \quad (6.49)$$

Subtracting equation (6.49) from equation (6.46) gives

$$D_1 e^{\lambda_1 T} = E_3 \quad (6.50)$$

where

$$E_3 = \frac{a_1 + a_2 + K'\lambda_2}{\lambda_1 - \lambda_2}$$

$$E_4 = \frac{a_1 + a_2 + \lambda_1 K'}{\lambda_2 - \lambda_1}$$

We have four unknowns,  $D_1$ ,  $D_2$ ,  $t_s$  and  $T$  in four equations (6.44), (6.45), (6.49) and (6.50). Solving for  $t_s$ , we have

$$\left( \frac{E_3}{A_1 - E_1 e^{-\lambda_1 t_s}} \right)^{\lambda_2} = \left( \frac{E_4}{A_2 - E_2 e^{-\lambda_2 t_s}} \right)^{\lambda_1} \quad (6.51)$$

$t_s$  can be solved from this equation by a trial and error procedure, and  $D_1$ ,  $D_2$  and  $T$  can be obtained from equations (6.44), (6.45) and (6.49). The solutions for four cases are shown in Table 6.2 and Figures 6.2, 6.3 and 6.4.

Figure 6.2 is the phase plane plot,  $x_2$  vs  $x_1$ , for different cases with fixed  $r$ , while Figure 6.3 shows that of Case 1 for different values of  $r$ . Both figures show a common feature that all the trajectories of  $x_2$  vs  $x_1$  show one switching point. However, the speed of response can be observed in Figure 6.4.  $x_1$  decreases asymptotically from  $t = 0$  to  $t = 0.847$ , and then approaches linearly to the final desired state. Response of  $x_2$  concaves downward from the initial stage to  $t = 0.1$ , and then increases linearly to

$t = 0.847$ , and finally decreases linearly to the end. This figure also shows the optimal control policy; it operates at  $\theta = 1$  from  $t = 0$  to  $t = 0.847$ , and then switches to  $\theta = -1$  and keeps operating at  $\theta = -1$  until the final desired conditions are obtained. Additional results are tabulated in Table 6.2.

A comparison of Figures 6.1 and 6.4 shows that the time lag of the heat exchanger is not too important.

Example 6.3. Suppose that a life support system consists of an air-conditioned room and a heat exchanger as in the preceding two examples. However, the flow of air in the room can be characterized by the two CST's-in-series model. The performance equations of such a system have been derived in the section entitled, "General Performance Equations" in the preceding chapter. They are

$$\frac{dx_{11}}{dt} + r_{11} x_{11} = a_{11} a_{42}' x_{12} - a_{11} a_5' \theta - a_{11} a_6' + a_{12} (t) \quad (6.52)$$

$$\frac{dx_{12}}{dt} + r_{12} x_{12} = a_{21} x_{11} \quad (6.53)$$

where

$$a_{42}' = K_4/K_{12}$$

$$a_5' = K_2 K_4$$

$$a_6' = K_3 K_4$$

The initial and the final conditions are

$$\begin{array}{lll} x_{11}(0^+) = x_{12}(0^+) = 1 & \text{at} & t = 0^+ \\ x_{11}(T) = x_{12}(T) = 0 & \text{at} & t = T \end{array} \quad (6.54)$$

where  $T$  is unspecified. We are to minimize

$$S = \int_0^T dt \quad (6.55)$$

Introducing an additional state variable

$$x_3(t) = \int_0^t dt,$$

we have

$$\frac{dx_3}{dt} = 1, \quad x_3(0) = 0 \quad (6.56)$$

The problem is thus transformed into that of minimizing  $x_3(T)$ .

According to equation (2.5), the Hamiltonian is,

$$\begin{aligned} H(z, x, \theta) = & z_{11}(-r_{11}x_{11} + a_{11}a_{42}'x_{12} - a_{11}a_5'\theta - a_{11}a_6' + a_{12}) \\ & + z_{12}(-r_{12}x_{12} + a_{21}x_{11}) + z_2 \end{aligned} \quad (6.57)$$

According to the definition of the adjoint variables, we have

$$\frac{dz_{11}}{dt} = -\frac{\partial H}{\partial x_{11}} = r_{11}z_{11} - r_{12}z_{12} \quad (6.58)$$

$$\frac{dz_{12}}{dt} = -\frac{\partial H}{\partial x_{12}} = -a_{11}a_{42}'z_{11} + r_{12}z_{12} \quad (6.59)$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \quad z_2(T) = 1$$

The solution of  $z_2$  can be obtained from this equation as

$$z_2(t) = 1, \quad 0 \leq t \leq T \quad (6.60)$$

Equation (6.57) can be rewritten as

$$\begin{aligned}
H(z, x, \theta) = & z_{11}(-r_{11}x_{11} + a_{11}a_{42}'x_{12} - a_{11}a_5'\theta - a_{11}a_6' + a_{12}) \\
& + z_{12}(-r_{12}x_{12} + a_{21}x_{11}) + 1 \quad (6.61)
\end{aligned}$$

Therefore, the switching function  $H^*$  is

$$H^* = -a_{11}a_5'z_{11}\theta \quad (6.62)$$

Inspection of  $H^*$  shows that the optimal controller should be of a bang-bang type. The control action for this problem, however, is constrained in such a manner that

$$|\theta| \leq 1 \quad (6.63)$$

The conditions for which the Hamiltonian is to be minimum are

$$\begin{aligned}
\theta = \theta_{\max} = 1 & \quad \text{if} \quad -a_{11}a_5'z_{11} < 0 \\
\theta = \theta_{\min} = -1 & \quad \text{if} \quad -a_{11}a_5'z_{11} > 0
\end{aligned} \quad (6.64)$$

In order to bring the initial deviated state,

$$x_{11}(0^+) = x_{12}(0^+) = 1 \quad \text{at} \quad t = 0^+,$$

to the final desired operating state,

$$x_{11}(T) = x_{12}(T) = 0, \quad \text{at} \quad t = T,$$

we intuitively employ the control action of  $\theta = \theta_{\max} = 1$  (maximum cooling action). Substituting this value of  $\theta$  into equations (6.52) and (6.53) and then eliminating  $x_{11}$ , we have

$$\begin{aligned} \frac{d^2 x_{12}}{dt^2} + (r_{11} + r_{12}) \frac{dx_{12}}{dt} + (r_{11}r_{12} - a_{11}a_{42}'a_{21})x_{12} \\ + a_{11}a_5'a_{21} + a_{11}a_{21}a_6' - a_{12}a_{21} = 0 \end{aligned} \quad (6.65)$$

Solution of  $x_{12}$  can be written in the form

$$x_{12} = Ae^{\lambda_{11}t} + be^{\lambda_{12}t} + K, \quad 0 \leq t \leq t_s \quad (6.66)$$

where  $\lambda_{11}$  and  $\lambda_{12}$  are roots of the characteristic equation

$$\lambda^2 + (r_{11} + r_{12})\lambda + (r_{11}r_{12} - a_{11}a_{21}a_{42}') = 0$$

and

$$K = \frac{a_{11}a_5'a_{21} + a_{11}a_{21}a_6' - a_{12}a_{21}}{a_{11}a_{42}'a_{21} - r_{11}r_{12}}$$

Inserting equation (6.66) and its derivative to equation (6.53) and solving for  $x_{11}$  yield

$$\begin{aligned} x_{11} = \frac{1}{a_{21}} [(\lambda_{11} + r_{12})Ae^{\lambda_{11}t} + (\lambda_{12} + r_{12})Be^{\lambda_{12}t} + r_{12}K], \\ 0 \leq t \leq t_s \end{aligned} \quad (6.67)$$

Constants A and B in equations (6.66) and (6.67) can be determined by employing the initial condition, equation (6.54) and Cramer's rule as follows:

$$A = \frac{\begin{vmatrix} a_{21} - r_{12}K & r_{12} + \lambda_{12} \\ 1 - K & 1 \end{vmatrix}}{\begin{vmatrix} r_{12} + \lambda_{11} & r_{12} + \lambda_{12} \\ 1 & 1 \end{vmatrix}} = \frac{a_{21} - r_{12} - \lambda_{12} + \lambda_{12}K}{\lambda_{11} - \lambda_{12}}$$

and

$$B = \frac{\begin{vmatrix} r_{12} + \gamma_{11} & a_{21} - r_{12}^K \\ 1 & 1 - K \end{vmatrix}}{\lambda_{11} - \lambda_{12}} = \frac{r_{12} + \lambda_{11} - \lambda_{11}^K - a_{21}}{\lambda_{11} - \lambda_{12}}$$

For  $\theta = -1$ ,  $x_{11}(t)$  and  $x_{12}(t)$  are solved by using equations (6.52) and (6.53).

$$x_{11}(t) = \frac{1}{a_{21}} \left[ (\lambda_{11} + r_{12}) D_1 e^{\lambda_{11} t} + (\lambda_{12} + r_{12}) D_2 e^{\lambda_{12} t} + r_{12}^K \right],$$

$$t_s \leq t \leq T \quad (6.68)$$

and

$$x_{12}(t) = D_1 e^{\lambda_{11} t} + D_2 e^{\lambda_{12} t} + K', \quad t_s \leq t \leq T \quad (6.69)$$

where

$$K' = \frac{a_{11} a_{21} a_{6'} - a_{11} a_{5'} a_{21} - a_{12} a_{21}}{a_{11} a_{21} a_{42'} - r_{11} r_{12}}$$

Constants  $D_1$  and  $D_2$  can be specified by noting that  $x_{11}$  and  $x_{12}$  are continuous with respect to  $t$ . We obtain from equations (6.67) through (6.69) at  $t = t_s$

$$x_{12}(t_s) = D_1 e^{\lambda_{11} t_s} + D_2 e^{\lambda_{12} t_s} + K'$$

$$= A e^{\lambda_{11} t_s} + B e^{\lambda_{12} t_s} + K \quad (6.70)$$

and

$$x_{12}(t_s)$$

$$= \frac{1}{a_{21}} \left[ (\lambda_{11} + r_{12}) D_1 e^{\lambda_{11} t_s} + (\lambda_{12} + r_{12}) D_2 e^{\lambda_{12} t_s} + r_{12}^K \right]$$

$$= \frac{1}{c_{21}} \left[ (\lambda_{11} + r_{12}) A e^{\lambda_{11} t_s} + (\lambda_{12} + r_{12}) B e^{\lambda_{12} t_s} + r_{12}^K \right] \quad (6.71)$$

Solving for  $D_1$  and  $D_2$  from these equations leads to

$$D_1 = A - E_1 e^{-\lambda_{11} t_s} \quad (6.72)$$

$$D_2 = B - E_2 e^{-\lambda_{12} t_s} \quad (6.73)$$

where

$$E_1 = \frac{\lambda_{11}(K' - K)}{\lambda_{11} + \lambda_{12}}$$

$$E_2 = \frac{\lambda_{12}(K' - K)}{\lambda_{11} + \lambda_{12}}$$

We see that  $D_1$  and  $D_2$  are functions of  $t_s$ . The value of  $t_s$  and that of  $T$  can be obtained by using the final conditions

$$x_{11}(T) = x_{12}(T) = 0 \quad \text{at } t = T$$

Equations (6.68) and (6.69) thus become

$$D_1 e^{\lambda_{11} T} + D_2 e^{\lambda_{12} T} + K' = 0 \quad (6.74)$$

$$\frac{1}{a_{21}} [(\lambda_{11} + r_{12}) D_1 e^{\lambda_{11} T} + (\lambda_{12} + r_{12}) D_2 e^{\lambda_{12} T} + r_{12} K'] = 0 \quad (6.75)$$

Eliminating  $T$  from these equations and letting

$$E_3 = \frac{\lambda_{11} K'}{\lambda_{12} - \lambda_{11}}$$

$$E_4 = \frac{K' \lambda_{12}}{\lambda_{11} - \lambda_{12}},$$

we obtain

$$\left( \frac{E_4}{A_1 - E_1 e^{-\lambda_{11} t_s}} \right)^{\lambda_{12}} = \left( \frac{E_3}{A_2 - E_2 e^{-\lambda_{12} t_s}} \right)^{\lambda_{11}} \quad (6.76)$$

$t_s$  can be solved from this equation by a trial and error procedure. Then  $D_1$ ,  $D_2$  and  $T$  can be calculated directly from equations (6.72) through (6.73).

The solutions of this problem are shown schematically in Figures 6.5, 6.6 and 6.7 and are tabulated in Table 6.3. The solutions are very similar to those of the preceding example. However, one distinct difference between the response of the dimensionless room temperature in this problem and that in the preceding one is that the dimensionless room temperature can become negative in this problem while it cannot be below zero in the preceding one.



TABLE 6.1

OPTIMAL SOLUTIONS OF ONE CST MODEL  
TOGETHER WITH  $t_2 = 0$  [EXAMPLE 6.1]

| Case Number | Two Bounds of<br>Control Variable                                 | $K_2$ | $K_3$ | Final Time T |
|-------------|-------------------------------------------------------------------|-------|-------|--------------|
| 1           | $T_{r \max} = 30^\circ\text{C}$<br>$T_{r \min} = 0^\circ\text{C}$ | 1.50  | 1.5   | 0.8353       |
| 2           | $T_{r \max} = 20^\circ\text{C}$<br>$T_{r \min} = 0^\circ\text{C}$ | 1.00  | 1.00  | 1.2566       |
| 3           | $T_{r \max} = 10^\circ\text{C}$<br>$T_{r \min} = 0^\circ\text{C}$ | 0.50  | 0.50  | 2.5541       |
| 4           | $T_{r \max} = 5^\circ\text{C}$<br>$T_{r \min} = 0^\circ\text{C}$  | 0.25  | 0.25  | 5.493        |

TABLE 6.2

OPTIMAL SOLUTIONS OF ONE CST MODEL  
TOGETHER WITH  $t_2 \neq 0$  [EXAMPLE 6.2]

| Case | $T_{r \max}$ | $T_{r \min}$ | $K_2$ | $K_3$ | $r$ | $t_s$ | $x_{1s}$ | $x_{2s}$ | $T$     |
|------|--------------|--------------|-------|-------|-----|-------|----------|----------|---------|
| 1    | 30           | 0            | 1.5   | 1.5   | 10  | 0.847 | 0.074    | 10.57    | 1.0770  |
|      |              |              |       |       | 50  | 0.838 | 0.014    | 11.59    | 0.8875  |
|      |              |              |       |       | 100 | 0.837 | 0.008    | 11.80    | 0.8615  |
|      |              |              |       |       | 200 | 0.836 | 0.004    | 11.92    | 0.8485  |
|      |              |              |       |       | 500 | 0.835 | 0.002    | 11.97    | 0.8397  |
| 2    | 20           | 0            | 1.0   | 1.0   | 10  | 1.266 | 0.048    | 7.10     | 1.4584  |
|      |              |              |       |       | 50  | 1.256 | 0.012    | 7.80     | 1.2990  |
|      |              |              |       |       | 100 | 1.248 | 0.009    | 7.88     | 1.2767  |
|      |              |              |       |       | 200 | 1.247 | 0.005    | 7.95     | 1.2664  |
|      |              |              |       |       | 500 | 1.246 | 0.0002   | 7.98     | 1.25613 |
| 3    | 10           | 0            | 0.5   | 0.5   | 10  | 2.600 | 0.024    | 3.78     | 2.6331  |
|      |              |              |       |       | 50  | 2.560 | 0.025    | 3.90     | 2.5860  |
|      |              |              |       |       | 100 | 2.556 | 0.010    | 3.94     | 2.5654  |
|      |              |              |       |       | 200 | 2.555 | 0.008    | 3.96     | 2.5598  |
|      |              |              |       |       | 500 | 2.554 | 0.003    | 3.99     | 2.5545  |
| 4    | 5            | 0            | 0.25  | 0.25  | 10  | 5.620 | 0.013    | 1.89     | 5.7103  |
|      |              |              |       |       | 50  | 5.501 | 0.008    | 1.93     | 5.6211  |
|      |              |              |       |       | 100 | 5.498 | 0.005    | 1.98     | 5.5010  |
|      |              |              |       |       | 200 | 5.494 | 0.001    | 1.985    | 5.4971  |
|      |              |              |       |       | 500 | 5.493 | 0.0009   | 1.99     | 5.4935  |

TABLE 6.3  
OPTIMAL SOLUTIONS OF TWO CST'S-IN-SERIES  
MODEL WITH  $t_2 = 0$  /EXAMPLE 6.3/

| Case | $T_{r \max}$ | $T_{r \min}$ | $K_2$ | $K_3$ | $r_{11}$ | $x_{11s}$ | $x_{12s}$ | $t_s$ | T     |
|------|--------------|--------------|-------|-------|----------|-----------|-----------|-------|-------|
| 1    | 30           | 0            | 1.5   |       | 1.2      | -0.0760   | 0.7205    | 0.645 | 0.992 |
|      |              |              |       |       | 1.5      | -0.2298   | 0.4874    | 0.518 | 1.047 |
|      |              |              |       |       | 2.0      | -0.6031   | 0.3862    | 0.471 | 1.078 |
|      |              |              |       |       | 5.0      | -3.8727   | 0.1852    | 0.598 | 1.007 |
|      |              |              |       |       | 10.0     | -9.4477   | 0.0896    | 0.715 | 0.956 |
| 2    | 20           | 0            | 1.0   |       | 1.2      | -0.0596   | 0.4538    | 0.975 | 1.275 |
|      |              |              |       |       | 1.5      | -0.1703   | 0.2984    | 0.775 | 1.225 |
|      |              |              |       |       | 2.0      | -0.4526   | 0.2552    | 0.715 | 1.245 |
|      |              |              |       |       | 5.0      | -2.5520   | 0.1024    | 0.910 | 1.25  |
|      |              |              |       |       | 10.0     | -6.0425   | 0.05105   | 1.060 | 1.25  |
| 3    | 10           | 0            | 0.5   |       | 1.2      | -0.0343   | 0.1476    | 2.015 | 2.215 |
|      |              |              |       |       | 1.5      | -0.0998   | 0.1177    | 1.635 | 1.955 |
|      |              |              |       |       | 2.0      | -0.2285   | 0.0882    | 1.500 | 1.860 |
|      |              |              |       |       | 5.0      | -1.1295   | 0.0353    | 1.875 | 2.115 |
|      |              |              |       |       | 10.0     | -2.6253   | 0.0171    | 2.180 | 2.310 |
| 4    | 5            | 0            | 0.25  |       | 1.2      | -0.0169   | 0.0192    | 4.520 | 4.625 |
|      |              |              |       |       | 1.5      | -0.0414   | 0.0223    | 3.730 | 3.940 |
|      |              |              |       |       | 2.0      | -0.8831   | 0.0167    | 3.460 | 3.640 |
|      |              |              |       |       | 5.0      | -0.3865   | 0.0082    | 4.250 | 4.360 |
|      |              |              |       |       | 10.0     | -0.8734   | 0.0059    | 4.780 | 4.841 |

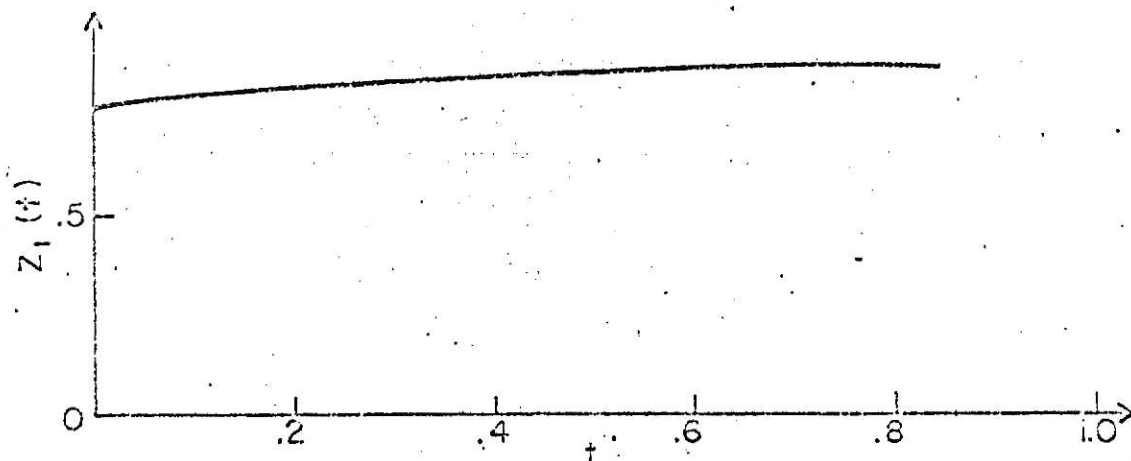
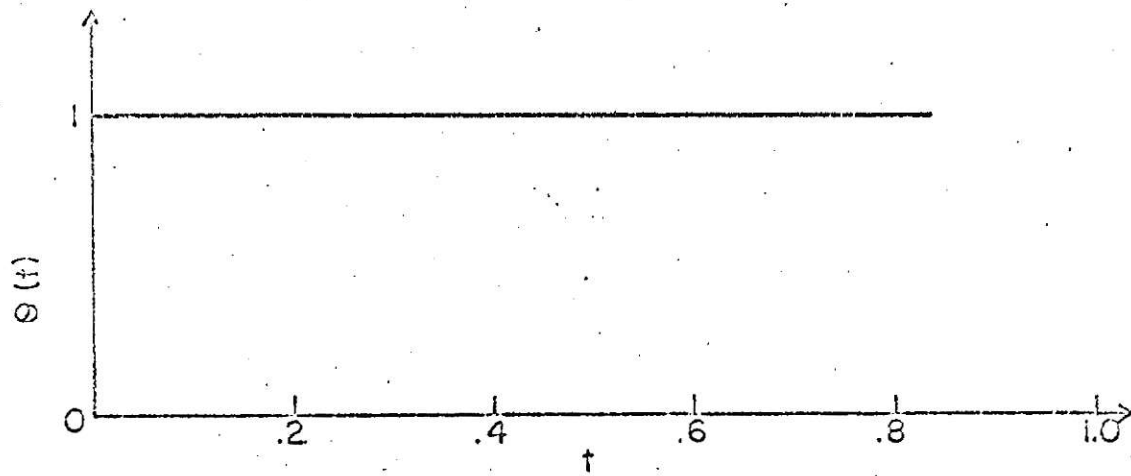
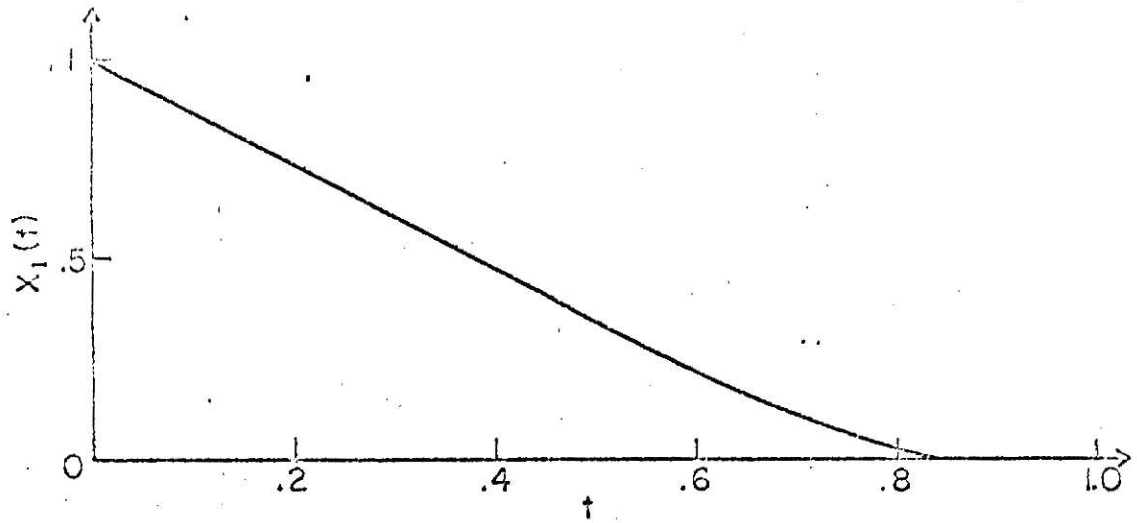


Figure 6.1. SYSTEM RESPONSE OF ONE CST MODEL WITH  $z_2 = 0$   
EXAMPLE 6.1

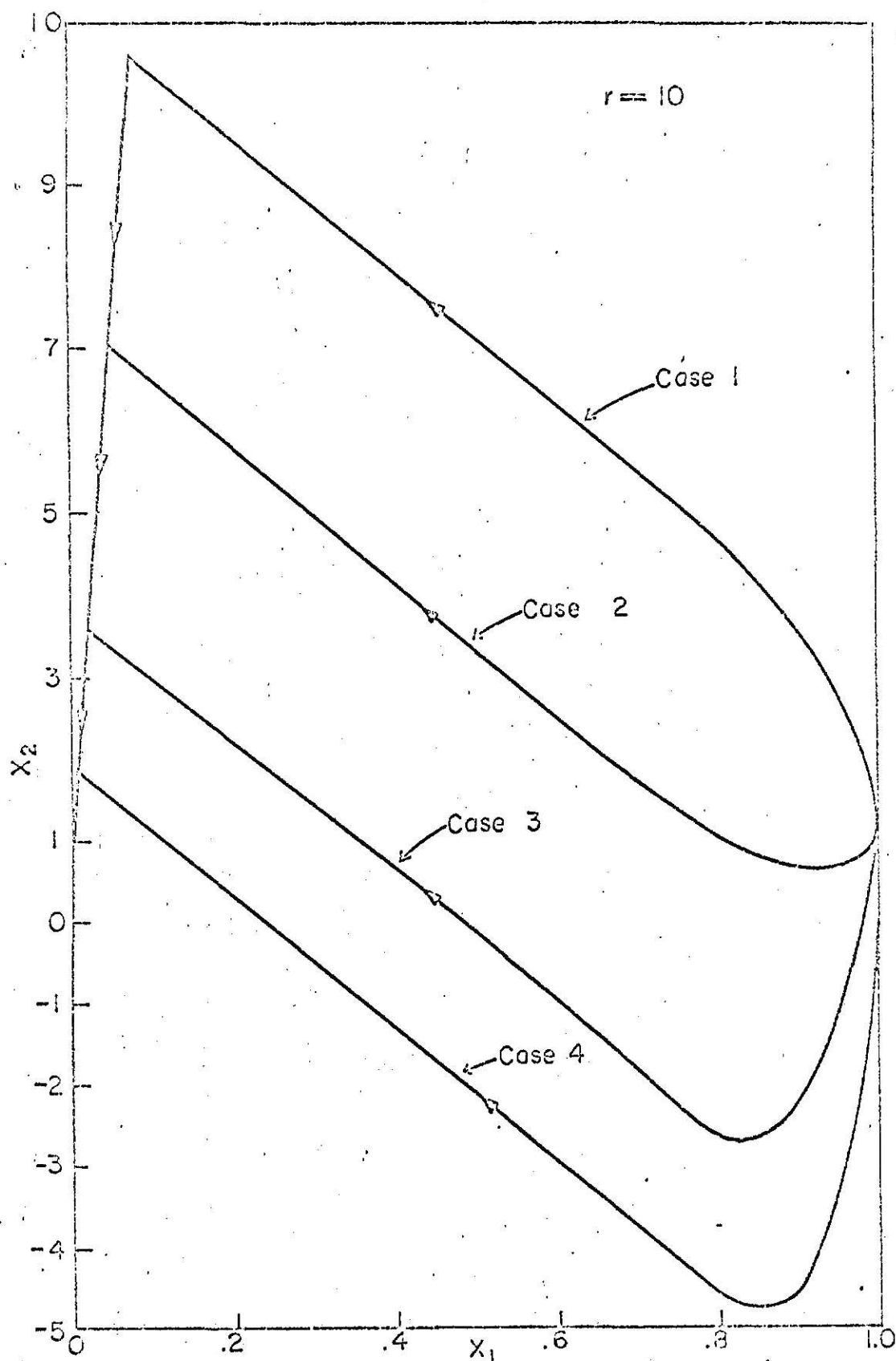


Figure 6.2. PHASE PLANE PLOT FOR DIFFERENT CASES WITH  $z_2 \neq 0$  AND  $r = 10$  [EXAMPLE 6.2]

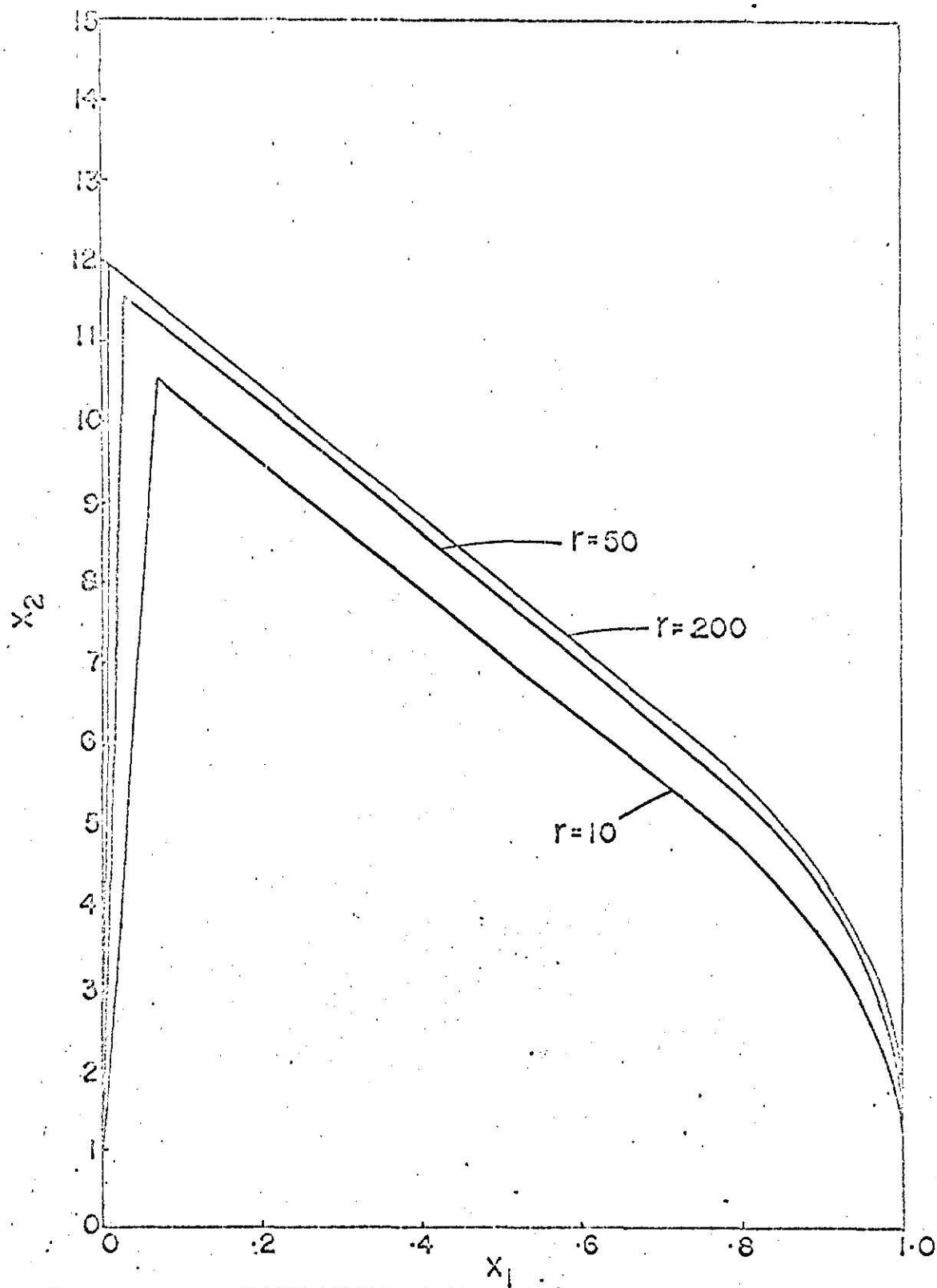


Figure 6.3. PHASE PLANE PLOT FOR CASE 1 WITH  $z_2 = 0$  AND DIFFERENT  $r$  VALUES [EXAMPLE 6.2]

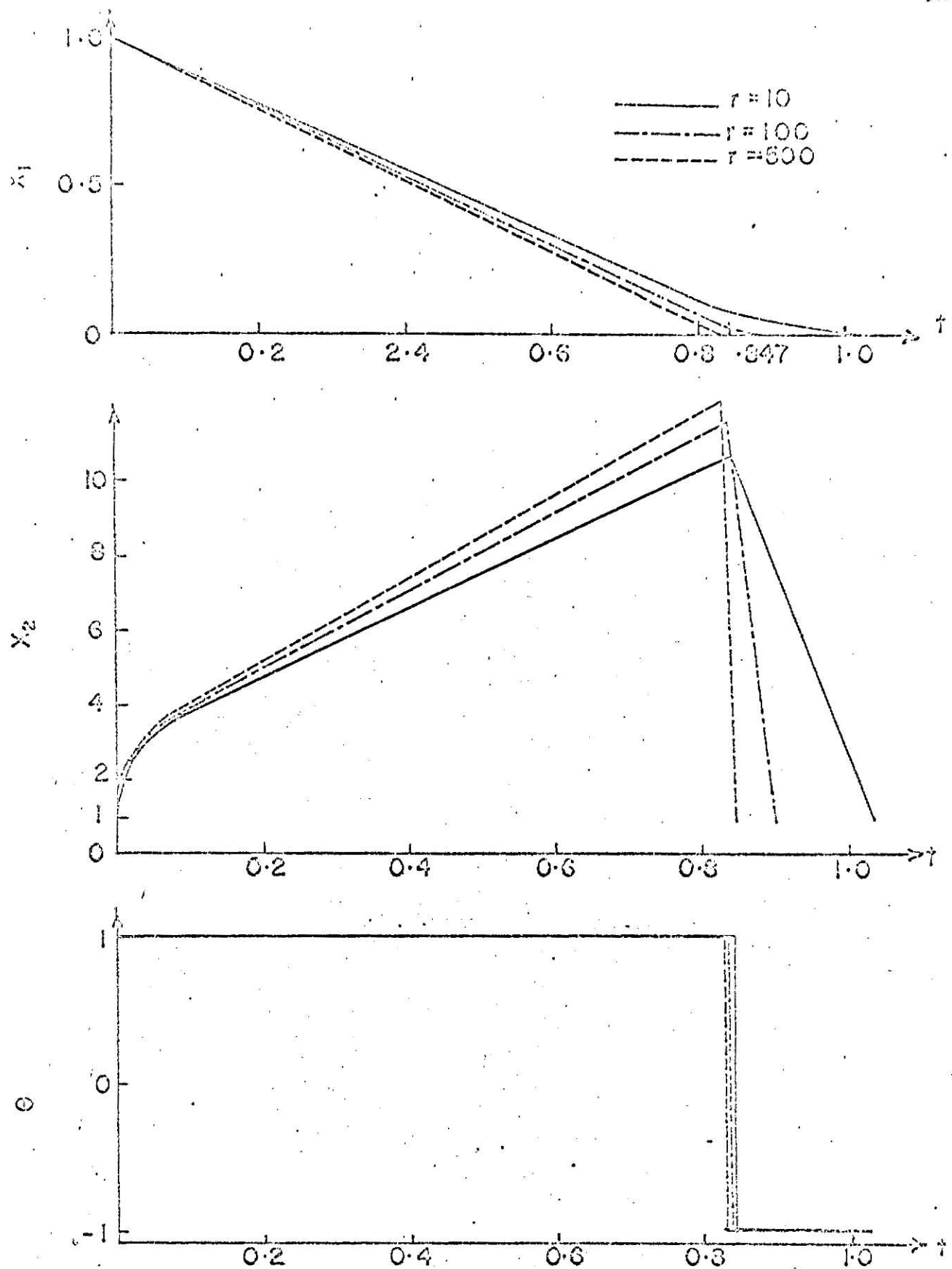


Figure 6.4. SYSTEM RESPONSE OF CASE 1 OF ONE CST MODEL WITH  $u_2 \neq 0$  [EXAMPLE 6.2]

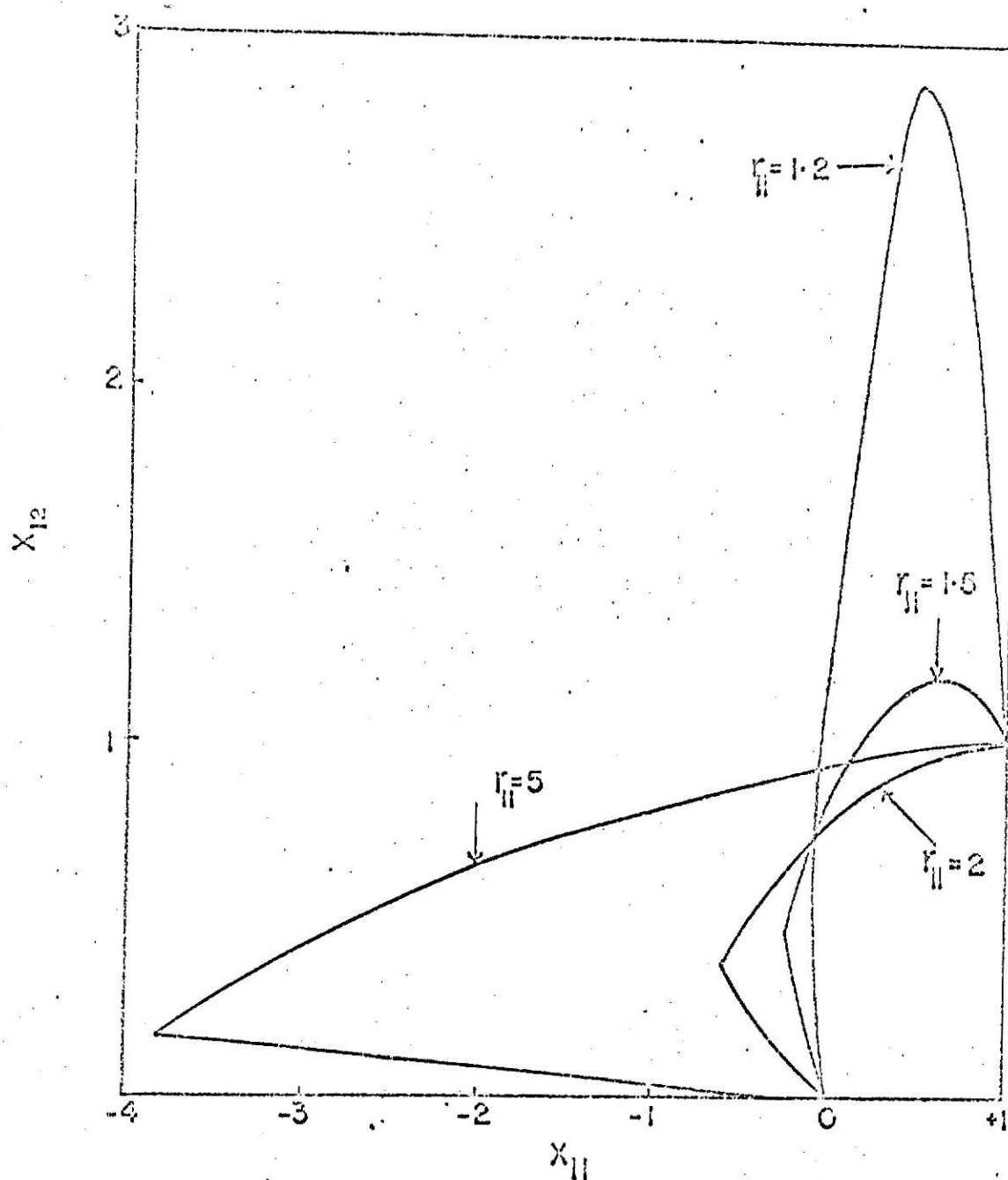


Figure 6.5. PHASE PLANE PLOT FOR CASE 1 OF TWO CST MODEL  
WITH  $\zeta_2 = 0$  AND DIFFERENT  $r_{II}$  VALUES [EXAMPLE 6.3]



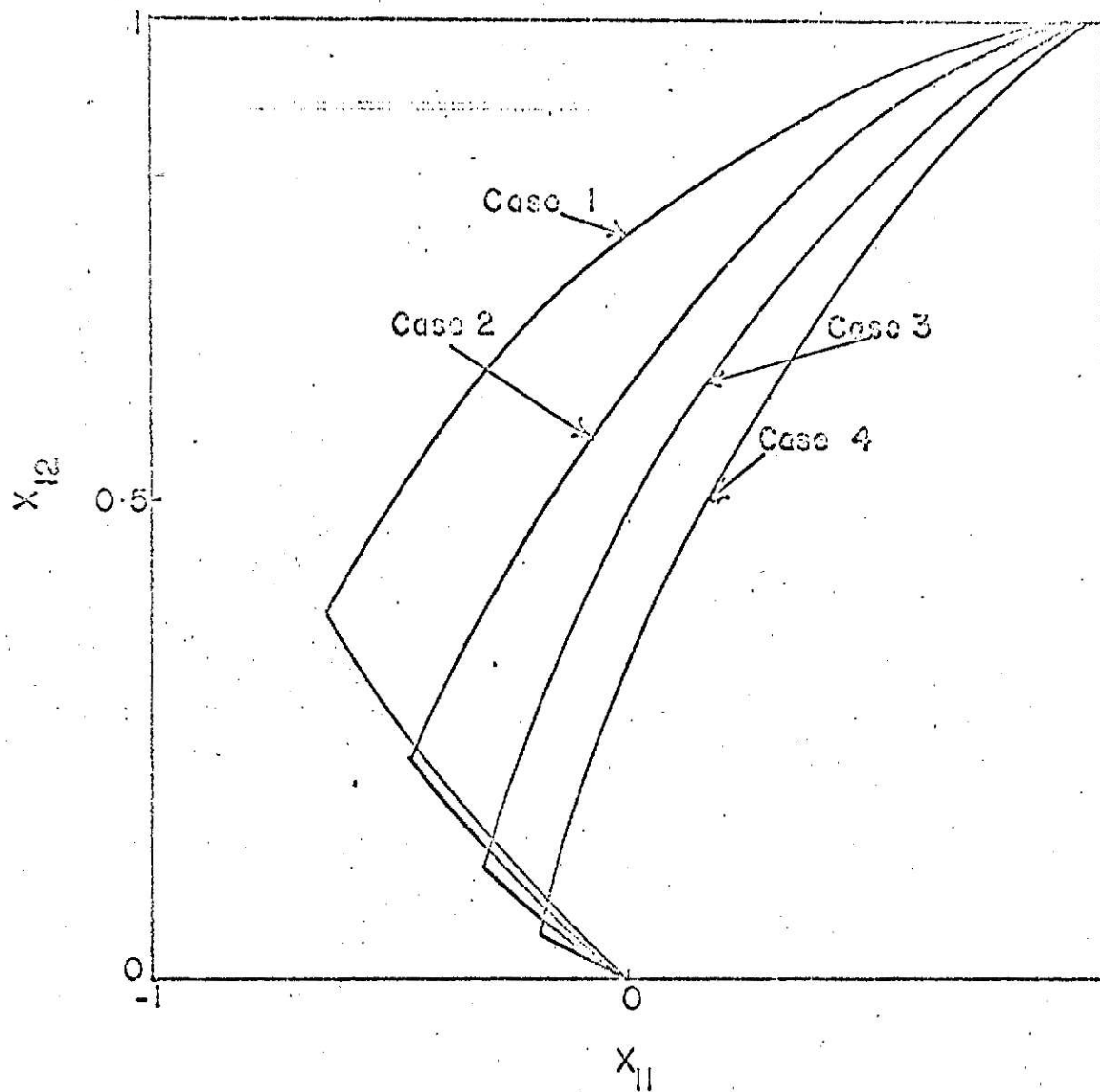


Figure 6.6. PHASE PLANE PLOT FOR THE DIFFERENT CASES OF TWO  
CST'S-IN-SERIES MODEL WITH  $\zeta_2 = 0$  AND  $r = 2$   
[EXAMPLE 6.3]

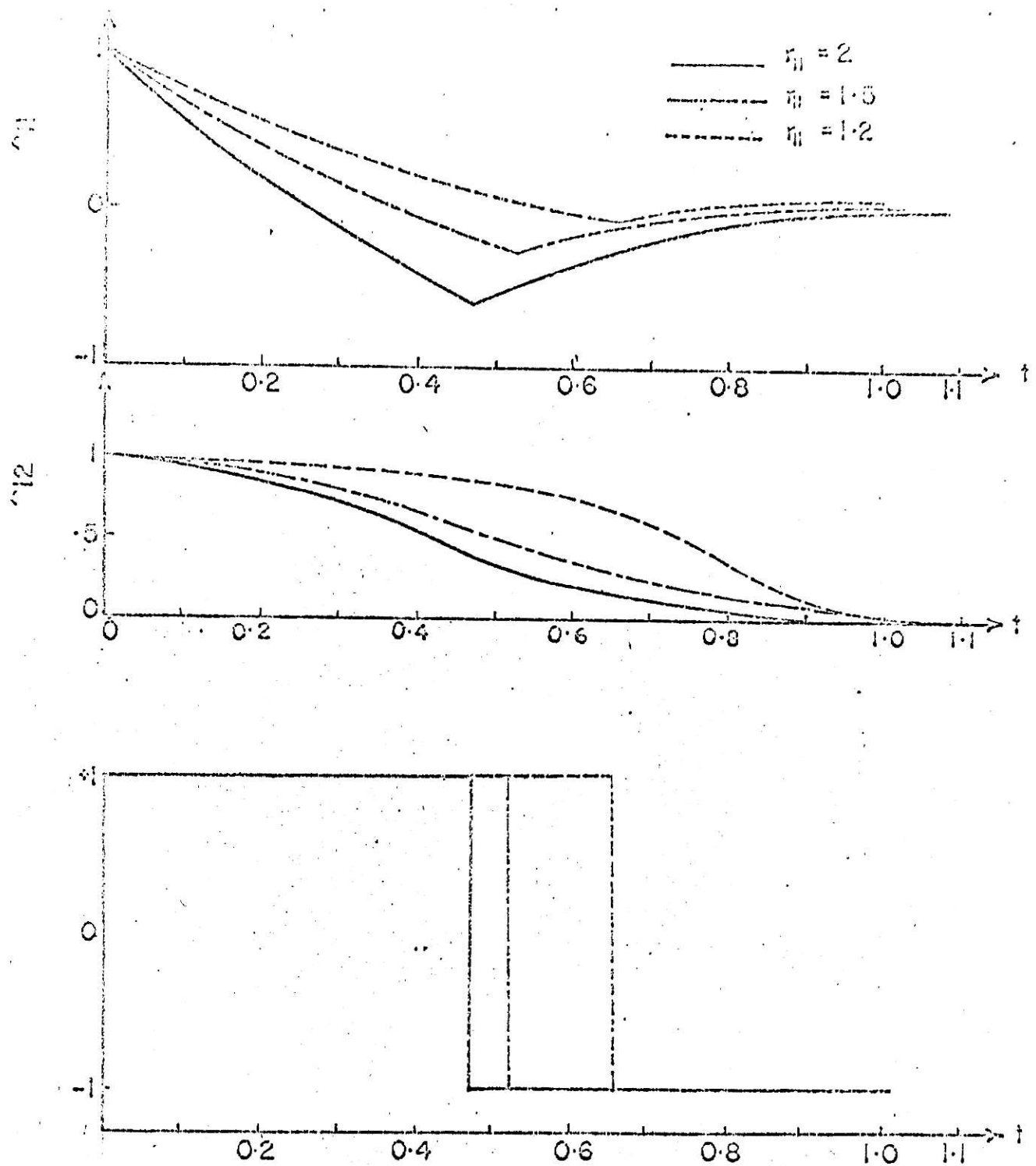


Figure 6.7. SYSTEM RESPONSE OF CASE 1 OF TWO CST'S-IN-SERIES MODEL WITH  $\zeta_2 = 0$  [EXAMPLE 6.3]

# CHAPTER 7

## NUMERICAL EXAMPLES II

Here the necessary conditions developed in Chapter 3 will be applied to the following examples.

Example 7.1. Let us reconsider Example 6.1. The statement of the problem remains the same. Now the square form of the final condition of the state variable can be considered as an equality constraint on the state variable at the control terminal time, i.e.,

$$g_1[x(T)] = \frac{1}{2} [x_1(T)]^2 = 0 \quad (7.1)$$

We now wish to show that at the optimal condition the two necessary conditions at the control terminal time, equations (3.19) and (3.20), are satisfied. Employing equations (3.19) and (3.20), we have

$$z_1(T) = \frac{1}{2} v_1 \frac{\partial [x_1(T)]^2}{\partial x_1(T)} + c_1 = v_1 x_1(T) + c_1 \quad (7.2)$$

$$z_2(T) = \frac{1}{2} v_2 \frac{\partial [x_1(T)]^2}{\partial x_2(T)} + c_2 = c_2 \quad (7.3)$$

$$\begin{aligned} & H[x(T), \theta(T), z(T)] + v_1 x_1(T) \frac{dx_1(T)}{dT} + [c_1 \frac{dx_1(T)}{dT} + c_2 \frac{dx_2(T)}{dT}] \\ & - [z_1(T) \frac{dx_1(T)}{dT} + z_2(T) \frac{dx_2(T)}{dT}] \\ & = H[x(T), \theta(T), z(T)] \\ & = 0 \end{aligned} \quad (7.4)$$

Since

$$c_1 = 0$$

$$c_2 = 1,$$

equations (7.2) and (7.3) become

$$z_1(T) = v_1 x_1(T) \quad (7.2a)$$

$$z_2(T) = 1 \quad (7.3a)$$

Equations (7.4), (7.2a) and (7.3a) assure us that this type of problem can be solved by making use of the necessary conditions presented in the preceding section as well as those presented in Chapter 2.

Example 7.2. As mentioned in Example 6.2, the response of the heat exchanger as well as that of the cabin or room is not always instantaneous. In Example 6.3 we considered the system consisting of a room or cabin with the flow of air characterized by the two CST's-in-series model and a heat exchanger whose time constant is negligibly small. Here we consider the slightly different system in which the response of the heat exchanger is not instantaneous. The performance equations are [see equations (5.43) through (5.45)]:

$$\frac{dx_{11}}{dt} + r_{11} x_{11} = a_{11} x_2 + a_{12} \quad (7.5)$$

$$\frac{dx_{12}}{dt} + r_{12} x_{12} = a_{21} x_{11} \quad (7.6)$$

$$\frac{dx_2}{dt} + rx_2 = a_{42} x_{12} - a_5 \theta - a_6 \quad (7.7)$$

with the initial and final conditions

$$\begin{aligned} x_{11}(0^+) = x_{12}(0^+) = x_2(0^+) = 1 & \quad \text{at } t = 0^+ \\ x_{11}(T) = x_{12}(T) = 0, x_2(T) = 1 & \quad \text{at } t = T \end{aligned} \quad (7.8)$$

where  $T$  is unspecified. The control variable,  $\theta$ , is constrained as

$$|\theta| \leq 1 \quad (7.9)$$

We wish to determine a piecewise continuous control variable  $\theta$  so that the response of the system can return to its desired state in a minimum period of time, that is

$$S = \int_0^T dt \quad (7.10)$$

is minimized.

We shall first make use of the basic computational algorithm of the maximum principle presented in Chapter 2 to solve the problem. Let

$$x_3(t) = \int_0^t dt$$

or

$$\frac{dx_3}{dt} = 1, \quad x_3(0) = 0 \quad (7.11)$$

Hence we have

$$S' = x_3(T)$$

$$\begin{aligned} H = & z_{11}[-r_{11}x_{11} + a_{11}x_2 + a_{12}] \\ & + z_{12}[-r_{12}x_{12} + a_{21}x_{11}] \\ & + z_2[-rx_2 + a_{42}x_{12} - a_5\theta - a_6] + z_3 \end{aligned} \quad (7.12)$$

$$\frac{dz_{11}}{dt} = r_{11}z_{11} - a_{21}z_{12} \quad (7.13)$$

$$\frac{dz_{12}}{dt} = r_{12}z_{12} - a_{42}z_2 \quad (7.14)$$

$$\frac{dz_2}{dt} = rz_2 - a_{11}z_{11} \quad (7.15)$$

$$\frac{dz_3}{dt} = 0, \quad z_3(T) = 1 \quad (7.16)$$

It follows from equation (7.16) that

$$z_3(t) = 1, \quad 0 \leq t \leq T \quad (7.17)$$

Equation (7.12) can then be rewritten as

$$\begin{aligned} H = & z_{11}[-r_{11}x_{11} + a_{11}x_2 + a_{12}] + z_{12}[-r_{12}x_{12} + a_{21}x_{11}] \\ & + z_2[-rx_2 + a_{42}x_{12} - a_5\theta - a_6] + 1 \end{aligned} \quad (7.18)$$

Therefore,

$$H^* = -a_5z_2\theta \quad (7.19)$$

An optimal control corresponding to this case should be of the bang-bang type.

Thus the conditions for optimal control (minimum  $H^*$ ) are

$$\theta = -1, \quad \text{if} \quad -a_5z_2 > 0$$

$$\theta = +1, \quad \text{if} \quad -a_5z_2 < 0$$

In order to bring the initial deviated state,

$$x_{11}(0) = x_{12}(0) = x_2(0) = 1 \quad \text{at} \quad t = 0$$

to the final desired state,

$$x_{11}(T) = x_{12}(T) = 0, \quad x_2(T) = 1 \quad \text{at} \quad t = T,$$

we shall first apply the control  $\theta = 1$ . In other words, we have  $\theta = 1$  in the interval  $0 \leq t \leq t_{s1}$ . Substitution of  $\theta = 1$  into equations (7.7) through (7.11) and subsequent elimination of  $x_{11}$  and  $x_2$  from the resulting expressions give rise to

$$\begin{aligned} & \frac{d^3 x_{12}}{dt^3} + (r + r_{11} + r_{12}) \frac{d^2 x_{12}}{dt^2} + (r_{11} r_{12} + rr_{11} + rr_{12}) \frac{dx_{12}}{dt} \\ & + (rr_{11} r_{12} - a_{11} a_{21} a_{42}) x_{12} \\ & = ra_{12} a_{21} - a_5 a_{11} a_{21} \theta - a_{11} a_{21} a_6 \end{aligned} \quad (7.20)$$

The solution of  $x_{12}$  has the form

$$x_{11} = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{\lambda_3 t} + K, \quad 0 \leq t \leq t_{s1} \quad (7.21)$$

where A, B and C are constants and  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  are roots of the characteristic equation

$$\begin{aligned} & \lambda^3 + (r + r_{11} + r_{12})\lambda^2 + (r_{11} r_{12} + rr_{11} + rr_{12})\lambda \\ & + (rr_{11} r_{12} - a_{11} a_{21} a_{42}) = 0 \end{aligned} \quad (7.22)$$

and K is a constant and its value equals to

$$K = \frac{ra_{12} a_{21} - a_5 a_{11} a_{21} - a_{11} a_{21} a_6}{rr_{11} r_{12} - a_{11} a_{21} a_{42}} \quad (7.23)$$

Inserting equation (7.21) and its first and second derivatives into equations (7.5) through (7.7) and then solving for  $x_{11}$  and  $x_2$  lead to

$$x_{11} = \frac{1}{c_{21}} [(\lambda_1 + r_{12})Ae^{\lambda_1 t} + (\lambda_2 + r_{12})Be^{\lambda_2 t} + (\lambda_3 + r_{12})Ce^{\lambda_3 t} + r_{12}K],$$

$$0 \leq t \leq t_{s1} \quad (7.24)$$

and

$$x_2 = \frac{1}{a_{11}a_{21}} \left\{ A\lambda_1^2 + (r_{11} + r_{12})\lambda_1 + r_{11}r_{12} \right\} e^{\lambda_1 t} + B\lambda_2^2$$

$$+ (r_{11} + r_{12})\lambda_2 + r_{11}r_{12} \left\{ e^{\lambda_2 t} + C\lambda_3^2 + (r_{11} + r_{12})\lambda_3 \right.$$

$$\left. + r_{11}r_{12} \right\} e^{\lambda_3 t} + r_{11}r_{12}K - a_{12}a_{21} \Big\} ,$$

$$0 \leq t \leq t_{s1} \quad (7.25)$$

Let

$$A_1 = \lambda_1 + r_{12}$$

$$A_2 = \lambda_2 + r_{12}$$

$$A_3 = \lambda_3 + r_{12}$$

$$A_4 = \lambda_1^2 + (r_{11} + r_{12})\lambda_1 + r_{11}r_{12}$$

$$A_5 = \lambda_2^2 + (r_{11} + r_{12})\lambda_2 + r_{11}r_{12}$$

$$A_6 = \lambda_3^2 + (r_{11} + r_{12})\lambda_3 + r_{11}r_{12}$$

$$A_7 = a_{21} - r_{12}K$$



$$A_8 = 1 - K$$

$$A_9 = a_{11}a_{21} + a_{12}a_{21} - r_{11}r_{12}K$$

Then equations (7.21), (7.24) and (7.25) can be rewritten as

$$x_{11} = \frac{1}{a_{21}} [A_1 Ae^{\lambda_1 t} + A_2 Be^{\lambda_2 t} + A_3 Ce^{\lambda_3 t} + r_{12}K], \quad 0 \leq t \leq t_{s1} \quad (7.26)$$

$$x_{12} = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{\lambda_3 t} + K, \quad 0 \leq t \leq t_{s1} \quad (7.27)$$

$$x_2 = \frac{1}{a_{11}a_{21}} [A_4 Ae^{\lambda_1 t} + A_5 Be^{\lambda_2 t} + A_6 Ce^{\lambda_3 t} + r_{11}r_{12}K - a_{12}a_{21}], \quad 0 \leq t \leq t_{s1} \quad (7.28)$$

Values of A, B and C can be determined by employing the initial conditions

$$x_{11}(0^+) = x_{12}(0^+) = x_2(0^+) = 1 \quad \text{at} \quad t = 0^+$$

Hence, equations (7.26) through (7.28) become

$$A_1 A + A_2 B + A_3 C = A_7$$

$$A + B + C = A_8$$

$$A_4 A + A_5 B + A_6 C = A_9$$

Solving for A, B and C by using Crammer's rule, we have

$$A = A_{11}/A_{10}$$

$$B = A_{12}/A_{10}$$

$$C = A_{13}/A_{10}$$

where

$$A_{10} = A_1 A_6 + A_2 A_4 + A_3 A_5 - A_1 A_5 - A_2 A_6 - A_3 A_4$$

$$A_{11} = A_6 A_7 + A_2 A_9 + A_3 A_5 A_8 - A_5 A_7 - A_2 A_6 A_8 - A_3 A_9$$

$$A_{12} = A_1 A_6 A_8 + A_4 A_7 + A_3 A_0 - A_1 A_9 - A_6 A_7 - A_3 A_4 A_8$$

$$A_{13} = A_1 A_9 + A_2 A_8 A_4 + A_7 A_5 - A_1 A_5 A_8 - A_2 A_9 - A_4 A_7$$

Similarly, for  $t_{s1} \leq t \leq t_{s2}$ , we have

$$\theta = -1$$

$$x_{11} = \frac{1}{a_{21}} [A_1 A' e^{\lambda_1 t} + A_2 B' e^{\lambda_2 t} + A_3 C' e^{\lambda_3 t} + r_{12} K'],$$

$$t_{s1} \leq t \leq t_{s2} \quad (7.29)$$

$$x_{12} = A' e^{\lambda_1 t} + B' e^{\lambda_2 t} + C' e^{\lambda_3 t} + K',$$

$$t_{s1} \leq t \leq t_{s2} \quad (7.30)$$

and

$$x_{13} = \frac{1}{a_{11} a_{21}} [A_4 A' e^{\lambda_1 t} + A_5 B' e^{\lambda_2 t} + A_6 C' e^{\lambda_3 t} + r_{11} r_{12} K' - a_{12} a_{21}],$$

$$t_{s1} \leq t \leq t_{s2} \quad (7.31)$$

where

$$K' = \frac{r a_{12} a_{21} + a_5 a_{11} a_{21} - a_{11} a_{21} a_6}{r r_{11} r_{12} - a_{11} a_{21} a_{42}}$$

and  $A'$ ,  $B'$  and  $C'$  are unknown constants. For  $t_{s2} \leq t \leq T$ , we have

$$\theta = 1$$

$$x_{11} = \frac{1}{a_{21}} [\bar{A}_1 A'' e^{\lambda_1 t} + A_2 B'' e^{\lambda_2 t} + A_3 C'' e^{\lambda_3 t} + r_{12} K],$$

$$t_{s2} \leq t \leq T \quad (7.32)$$

$$x_{12} = A'' e^{\lambda_1 t} + B'' e^{\lambda_2 t} + C'' e^{\lambda_3 t} + K,$$

$$t_{s2} \leq t \leq T \quad (7.33)$$

$$x_2 = \frac{1}{a_{11} a_{21}} [\bar{A}_4 A'' e^{\lambda_1 t} + A_5 B'' e^{\lambda_2 t} + A_6 C'' e^{\lambda_3 t} + r_{11} r_{12} K - a_{12} a_{21}],$$

$$t_{s2} \leq t \leq T \quad (7.34)$$

where  $A''$ ,  $B''$  and  $C''$  are unknown constants.

We know that  $x_{11}$ ,  $x_{12}$  and  $x_2$  are continuous functions of  $t$ . Therefore,  $A'$ ,  $B'$ ,  $C'$  and  $A''$ ,  $B''$ ,  $C''$  can be determined by using the continuity of  $x_{11}$ ,  $x_{12}$  and  $x_2$  with respect to  $t$  at  $t = t_{s1}$  and  $t = t_{s2}$ . Thus

$$x_{11}(t_{s1}) = \frac{1}{a_{21}} [\bar{A}_1 A e^{\lambda_1 t_{s1}} + A_2 B e^{\lambda_2 t_{s1}} + A_3 C e^{\lambda_3 t_{s1}} + r_{12} K]$$

$$= \frac{1}{a_{21}} [\bar{A}_1 A' e^{\lambda_1 t_{s1}} + A_2 B' e^{\lambda_2 t_{s1}} + A_3 C' e^{\lambda_3 t_{s1}} + r_{12} K'] \quad (7.35)$$

$$x_{12}(t_{s1}) = A e^{\lambda_1 t_{s1}} + B e^{\lambda_2 t_{s1}} + C e^{\lambda_3 t_{s1}} + K$$

$$= A' e^{\lambda_1 t_{s1}} + B' e^{\lambda_2 t_{s1}} + C' e^{\lambda_3 t_{s1}} + K \quad (7.36)$$

and

$$\begin{aligned}
 x_{13}(t_{s1}) &= \frac{1}{a_{11}a_{21}} \left[ A_4 A e^{\lambda_1 t_{s1}} + A_5 B e^{\lambda_2 t_{s1}} + A_6 C e^{\lambda_3 t_{s1}} \right. \\
 &\quad \left. + r_{11} r_{12}^K - a_{12} a_{21} \right] \\
 &= \frac{1}{a_{11}a_{21}} \left[ A_4 A' e^{\lambda_1 t_{s1}} + A_5 B' e^{\lambda_2 t_{s1}} + A_6 C' e^{\lambda_3 t_{s1}} \right. \\
 &\quad \left. + r_{11} r_{12}^K - a_{12} a_{21} \right] \tag{7.37}
 \end{aligned}$$

Solving for A', B' and C' in terms of  $t_{s1}$  from equations (7.35) through (7.37) gives

$$A' = \frac{A_{18}}{A_{17}} \tag{7.38}$$

$$B' = \frac{A_{19}}{A_{17}} \tag{7.39}$$

$$C' = \frac{A_{20}}{A_{17}} \tag{7.40}$$

where

$$A_{17} = (A_1 A_6 + A_2 A_4 + A_3 A_5 - A_1 A_5 - A_2 A_6 - A_3 A_4) e^{\lambda_1 t_{s1}} e^{\lambda_2 t_{s1}} e^{\lambda_3 t_{s1}}$$

$$A_{18} = (A_6 A_{14} + A_2 A_{16} + A_3 A_5 A_{15} - A_5 A_{14} - A_2 A_6 A_{15} - A_3 A_{16}) e^{\lambda_2 t_{s1}} e^{\lambda_3 t_{s1}}$$

$$A_{19} = (A_1 A_6 A_{15} + A_4 A_{14} + A_3 A_{16} - A_1 A_{16} - A_6 A_{14} - A_3 A_4 A_{15}) e^{\lambda_1 t_{s1}} e^{\lambda_3 t_{s1}}$$

$$A_{20} = (A_1 A_6 + A_2 A_4 A_{15} + A_{14} A_5 - A_1 A_5 A_{15} - A_2 A_{16} - A_4 A_{14}) e^{\lambda_1 t_{s1}} e^{\lambda_2 t_{s1}}$$

$$A_{14} = a_{21} x_{11}(t_{s1}) - r_{12} K'$$

$$A_{15} = x_{12}(t_{s1}) - K'$$

$$A_{16} = A_{11} a_{21} x_2(t_{s1}) + a_{12} a_{21} - r_{11} r_{12} K'$$

Similarly,  $A''$ ,  $B''$  and  $C''$  can be determined by

$$A'' = \frac{A_{25}}{A_{24}} \quad (7.41)$$

$$B'' = \frac{A_{26}}{A_{24}} \quad (7.42)$$

$$C'' = \frac{A_{27}}{A_{24}} \quad (7.43)$$

where

$$A_{24} = (A_1 A_6 + A_2 A_4 + A_3 A_5 - A_1 A_5 - A_2 A_6 - A_3 A_4) e^{\lambda_1 t_{s2}} e^{\lambda_2 t_{s2}} e^{\lambda_3 t_{s2}}$$

$$A_{25} = (A_6 A_{21} + A_2 A_{23} + A_3 A_5 A_{22} - A_5 A_{21} - A_2 A_6 A_{22} - A_3 A_{23}) e^{\lambda_2 t_{s2}} e^{\lambda_2 t_{s2}}$$

$$A_{26} = (A_1 A_6 A_{22} + A_4 A_{21} + A_3 A_{23} - A_1 A_{23} - A_6 A_{21} - A_3 A_4 A_{22}) e^{\lambda_1 t_{s2}} e^{\lambda_2 t_{s2}}$$

$$A_{27} = (A_1 A_{23} + A_2 A_4 A_{22} + A_5 A_{21} - A_1 A_5 A_{22} - A_2 A_{23} - A_4 A_{21}) e^{\lambda_1 t_{s2}} e^{\lambda_2 t_{s2}}$$

$$A_{21} = a_{21}x_{11}(t_{s2}) - r_{12}K$$

$$A_{22} = x_{12}(t_{s2}) - K$$

$$A_{23} = a_{11}a_{21}x_{22}(t_{s2}) + a_{12}a_{21} - r_{11}r_{12}K$$

We can find  $t_{s1}$ ,  $t_{s2}$  and  $T$  from equations (7.32) through (7.34) and the final conditions

$$x_{11}(T) = x_{12}(T) = 0, \quad x_2(T) = 1, \quad \text{at} \quad t = T$$

Thus

$$A_1 A'' e^{\lambda_1 T} + A_2 B'' e^{\lambda_2 T} + A_3 C'' e^{\lambda_3 T} + r_{12}K = 0 \quad (7.44)$$

$$A'' e^{\lambda_1 T} + B'' e^{\lambda_2 T} + C'' e^{\lambda_3 T} + K = 0 \quad (7.45)$$

$$A_4 A'' e^{\lambda_1 T} + A_5 B'' e^{\lambda_2 T} + A_6 C'' e^{\lambda_3 T} + r_{11}r_{12}K - a_{12}a_{21} - a_{11}a_{21} = 0 \quad (7.46)$$

In principle, we can solve for  $T$ ,  $t_{s1}$  and  $t_{s2}$  from the above equations by a trial and error procedure. However,  $t_{s1}$  and  $t_{s2}$  appear implicitly in these equations, which increase the difficulty of solving the problem. To circumvent this difficulty, we shall continue to solve this problem by considering the square form of the final conditions of the state variables as equality constraints and by employing the additional necessary conditions developed in Chapter 3. The equality constraints on the final state variables are

$$g_1[x(T)] = \frac{1}{2} [x_{11}(T) - 0]^2 = \frac{1}{2} [x_{11}(T)]^2 = 0 \quad (7.47)$$

$$g_2 \int x(T) \int = \frac{1}{2} \int x_{12}(T) - 0 \int^2 = \frac{1}{2} \int x_{12}(T) \int^2 = 0 \quad (7.48)$$

$$g_3 \int x(T) \int = \frac{1}{2} \int x_2(T) - 1 \int^2 = 0 \quad (7.49)$$

Therefore, equations (3.19) and (3.20) for this problem are

$$z_{11}(T) = v_{11} x_{11}(T) + c_{11} \quad (7.50)$$

$$z_{12}(T) = v_{12} x_{12}(T) + c_2 \quad (7.51)$$

$$z_2(T) = v_2 \int x_2(T) - 1 \int + c_3 \quad (7.52)$$

$$z_3(T) = c_4 \quad (7.53)$$

$$\begin{aligned} H \int x(T), \theta(T), z(T) \int + \left\{ v_{11} x_{11}(T) \frac{dx_{11}}{dT} + v_{12} x_{12}(T) \frac{dx_{12}}{dT} \right. \\ \left. + v_2 \frac{dx_2(T)}{dT} + v_2 \int x_2(T) - 1 \int \frac{dx_2(T)}{dT} \right\} + \int c_1 \frac{dx_{11}(T)}{dT} \\ + c_2 \frac{dx_{12}(T)}{dT} + c_3 \frac{dx_2(T)}{dT} + c_4 \frac{dx_3(T)}{dT} \int - \int z_{11}(T) \frac{dx_{11}(T)}{dT} \\ + z_{12}(T) \frac{dx_{12}(T)}{dT} + z_2(T) \frac{dx_2(T)}{dT} + z_3(T) \frac{dx_3}{dT} \int = 0 \end{aligned}$$

By substituting equations (7.50) through (7.53) into the above equation, it can be shown that

$$H \int x(T), \theta(T), z(T) \int = 0 \quad (7.54)$$

Since the objective function, equation (7.10), has been transformed into the following form

$$S = c_1 x_{11}(T) + c_2 x_{12}(T) + c_3 x_2(T) + c_4 x_3(T), \quad (7.55)$$

$$c_1 = c_2 = c_3 = 0, \quad c_4 = 1,$$

equations (7.50) through (7.53) become

$$z_{11}(T) = v_{11} x_{11}(T) \quad (7.56)$$

$$z_{12}(T) = v_{12} x_{12}(T) \quad (7.57)$$

$$z_2(T) = v_2 [x_2(T) - 1] \quad (7.58)$$

$$z_3(T) = 1 \quad (7.59)$$

and from equations (7.12) and (7.14), we have

$$\begin{aligned} & H[x(T), \theta(T), z(T)] \\ &= 0 \\ &= z_{11}(T) [-r_{11} x_{11}(T) + a_{11} x_2(T) - a_{12}] \\ &\quad + z_{12}(T) [-r_{12} x_{12}(T) + a_{21} x_{11}(T)] \\ &\quad + z_2(T) [-r x_2(T) + a_{42} x_{12}(T) - a_5 \theta(T) - a_6] + 1 \end{aligned} \quad (7.60)$$

Determination of the control terminal time,  $T$ , from the above equation in conjunction with the performance equations, equation for the adjoint variables and constraints is very difficult, if not impossible. However,  $T$  may be obtained by using the gradient procedure with the penalty function approach. This penalty function can be written as



$$\begin{aligned}
S'' = & \int c_1 x_{11}(T) + c_2 x_{12}(T) + c_3 x_2(T) + c_4 x_3(T) \int \\
& + \frac{1}{2} \left\{ v_{11} \int x_{11}(T) \int^2 + v_{12} \int x_{12}(T) \int^2 + v_2 \int x_2(T) - 1 \int^2 \right\}
\end{aligned}
\tag{7.61}$$

and the control terminal time,  $T$ , can be determined by the condition

$$\begin{aligned}
\frac{dS''}{dT} \\
= & 1 + v_{11} x_{11}(T) \frac{dx_{11}(T)}{dT} + v_{12} x_{12}(T) \frac{dx_{12}(T)}{dT} + v_2 \int x_2(T) - 1 \int \\
= & 0
\end{aligned}
\tag{7.62}$$

In other words, the control terminal time is chosen so that the penalty function, equation (7.61) is at a minimum with respect to this terminal time. It is possible, however, that the terminal time determined by equation (7.62) may not be the time which minimizes the penalty function. If there is any question concerning this assumption, we may examine the sign of the second derivative of  $S''$  with respect to  $T$  when the first derivative of  $S''$  with respect to  $T$  is zero, or we may carry out an exhaustive search or random search around this point to assure that it is indeed a minimum point.

Since the control policy is of the bang-bang type shown in equation (7.19) and the performance equations are linear, the number of the switching points is one less than the dimension of the system as mentioned previously. Also from Example 6.2, the time lag of the heat exchanger is almost negligible. Thus the initial trial control pattern is assumed to be

$$\begin{aligned}
\theta = 1 & \quad 0 \leq t \leq 0.468 \\
\theta = -1 & \quad 0.468 < t \leq 1.100 \\
\theta = 1 & \quad t > 1.100
\end{aligned}
\tag{7.63}$$

Then we integrate the performance equations, equations (25) through (27), with the assumed  $\theta(t)$  until the derivative of the penalty function is zero and the sign of the second derivative of the penalty function is positive. We finally carry out the direct search around switching points to assure that  $S''$  is indeed a minimum. By use of this final time and value of  $x(t)$ , we can solve the adjoint equations, equations (33) through (35), backward from  $T$  to  $0$  with the final conditions given by equations (76) through (78) and then check the condition  $H = 0$ .

Let

$$t_b = T - t$$

Then

$$dt_b = -dt$$

Equations (7.13) through (7.15) become

$$\frac{dz_{11}}{dt_b} = a_{21}z_{12} - r_{11}z_{11} \quad (7.64)$$

$$\frac{dz_{12}}{dt_b} = a_{42}z_2 - r_{12}z_{12} \quad (7.65)$$

$$\frac{dz_2}{dt_b} = a_{11}z_{11} - rz_2 \quad (7.66)$$

Eliminating  $z_{11}$  and  $z_{12}$  from these equations, we have

$$\begin{aligned} \frac{d^3 z_2}{dt_b^3} + (r + r_{11} + r_{12}) \frac{d^2 z_2}{dt_b^2} + (rr_{11} + r_{11}r_{12} + rr_{12}) \frac{dz_2}{dt_b} \\ + (rr_{11}r_{12} - a_{11}a_{21}a_{42})z_2 = 0 \end{aligned}$$

Solution of  $x_2$  has the form

$$z_2 = D e^{\lambda_1 t_b} + E e^{\lambda_2 t_b} + F e^{\lambda_3 t_b} \quad (7.67)$$

where D, E and F are unknown constants. Inserting equation (7.67) into equations (7.64) through (7.66) and solving for  $z_{11}$  and  $z_{12}$ , we have

$$z_{11} = \frac{1}{a_{11}} \left[ D(\lambda_1 + r) e^{\lambda_1 t_b} + E(\lambda_2 + r) e^{\lambda_2 t_b} + F(\lambda_3 + r) e^{\lambda_3 t_b} \right] \quad (7.68)$$

and

$$z_{12} = \frac{1}{a_{11} a_{21}} \left[ D \left[ \lambda_1^2 + (r + r_{12}) \lambda_1 + r r_{11} \right] e^{\lambda_1 t_b} + E \left[ \lambda_2^2 + (r + r_{11}) \lambda_2 + r r_{11} \right] e^{\lambda_2 t_b} + F \left[ \lambda_3^2 + (r + r_{11}) \lambda_3 + r r_{11} \right] e^{\lambda_3 t_b} \right] \quad (7.69)$$

Let

$$B_1 = \lambda_1 + r$$

$$B_2 = \lambda_2 + r$$

$$B_3 = \lambda_3 + r$$

$$B_4 = \lambda_1^2 + (r + r_{11}) \lambda_1 + r r_{11}$$

$$B_5 = \lambda_2^2 + (r + r_{11}) \lambda_2 + r r_{11}$$

$$B_6 = \lambda_3^2 + (r + r_{11}) \lambda_3 + r r_{11}$$

Equations (7.67) through (7.69) become

$$z_{11} = \frac{1}{a_{11}} \left[ D B_1 e^{\lambda_1 t_b} + E B_2 e^{\lambda_2 t_b} + F B_3 e^{\lambda_3 t_b} \right] \quad (7.70)$$

$$z_{12} = \frac{1}{a_{11}a_{21}} [DB_4 e^{\lambda_1 t_b} + EB_5 e^{\lambda_2 t_b} + FB_6 e^{\lambda_3 t_b}] \quad (7.71)$$

$$z_2 = D e^{\lambda_1 t_b} + E e^{\lambda_2 t_b} + F e^{\lambda_3 t_b} \quad (7.72)$$

At  $t = T$ ,  $t_b = 0$ , equations (7.56) through (7.58) become

$$z_{11}(0) = v_{11}x_{11}(0) \quad (7.73)$$

$$z_{12}(0) = v_{12}x_{12}(0) \quad (7.74)$$

$$z_2(0) = v_2 x_2(0) - 1 \quad (7.75)$$

Also at  $t_b = 0$  equations (7.70) through (7.72) become

$$DB_1 + EB_2 + FB_3 = a_{11}z_{11}(0) \quad (7.76)$$

$$DB_4 + EB_5 + FB_6 = a_{11}a_{21}z_{12}(0) \quad (7.77)$$

$$D + E + F = z_2(0) \quad (7.78)$$

Solving for D, E and F from these equations, we have

$$D = \frac{B_{11}}{B_{10}}$$

$$E = \frac{B_{12}}{B_{10}}$$

$$F = \frac{B_{13}}{B_{10}}$$

where

$$B_{10} = B_1B_5 + B_2B_6 + B_3B_4 - B_1B_6 - B_2B_4 - B_3B_5$$

$$B_{11} = B_5 B_7 + B_2 B_6 B_9 + B_3 B_8 - B_6 B_7 - B_2 B_8 - B_3 B_5 B_9$$

$$B_{12} = B_1 B_8 + B_6 B_7 + B_3 B_4 B_9 - B_1 B_9 B_6 - B_4 B_7 - B_3 B_8$$

$$B_{13} = B_1 B_5 B_9 + B_2 B_8 + B_4 B_7 - B_1 B_8 - B_2 B_4 B_9 - B_5 B_7$$

$$B_7 = a_{11} v_{11} x_{12}(0)$$

$$B_8 = a_{11} a_{21} v_{12} x_{12}(0)$$

$$B_9 = v_2 \sqrt{x_2}(0) - 17$$

The control pattern is shown in the figure by the dotted line. The value of the penalty function is 1.0960 which is 0.018 longer than that of the case in which the response of the heat exchanger is negligibly small, that is,  $z_2 \rightarrow 0$  (example 6.3). Practically, the difference can be neglected. In general, the response of the heat exchanger is almost instantaneous, especially when the time constant of the heat exchanger is much smaller than that of the system element.

The optimal response of  $x_{11}$  is also given in Figure 7.1a and the corresponding responses of  $x_{12}$  and  $x_2$  are given in Figure 7.1b. Values of the system parameters employed in obtaining the numerical results correspond to those of Case 1 in the previous chapters. The condition  $H = 0$  holds.

This approach is fairly useful for the multi-dimensional systems with nonlinear control variables. Since the extrem of the Hamiltonian corresponds to that of the objective function [1, 8], minimization of the objective function can be achieved by minimizing the Hamiltonian with respect to the control variable. And the steepest change of the objective function  $S''$  can be obtained by calculating the gradient  $\frac{\partial H}{\partial \theta_1}$  and then making  $\Delta \theta$  directed opposite to the

gradient, that is

$$\Delta \theta = -K \left[ \frac{\partial H}{\partial \theta} \right]$$

Thus the new trial control  $\theta'(t)$  is

$$\theta'(t) = \theta(t) + \Delta \theta$$

The process is repeated until convergence of the penalty function is obtained.

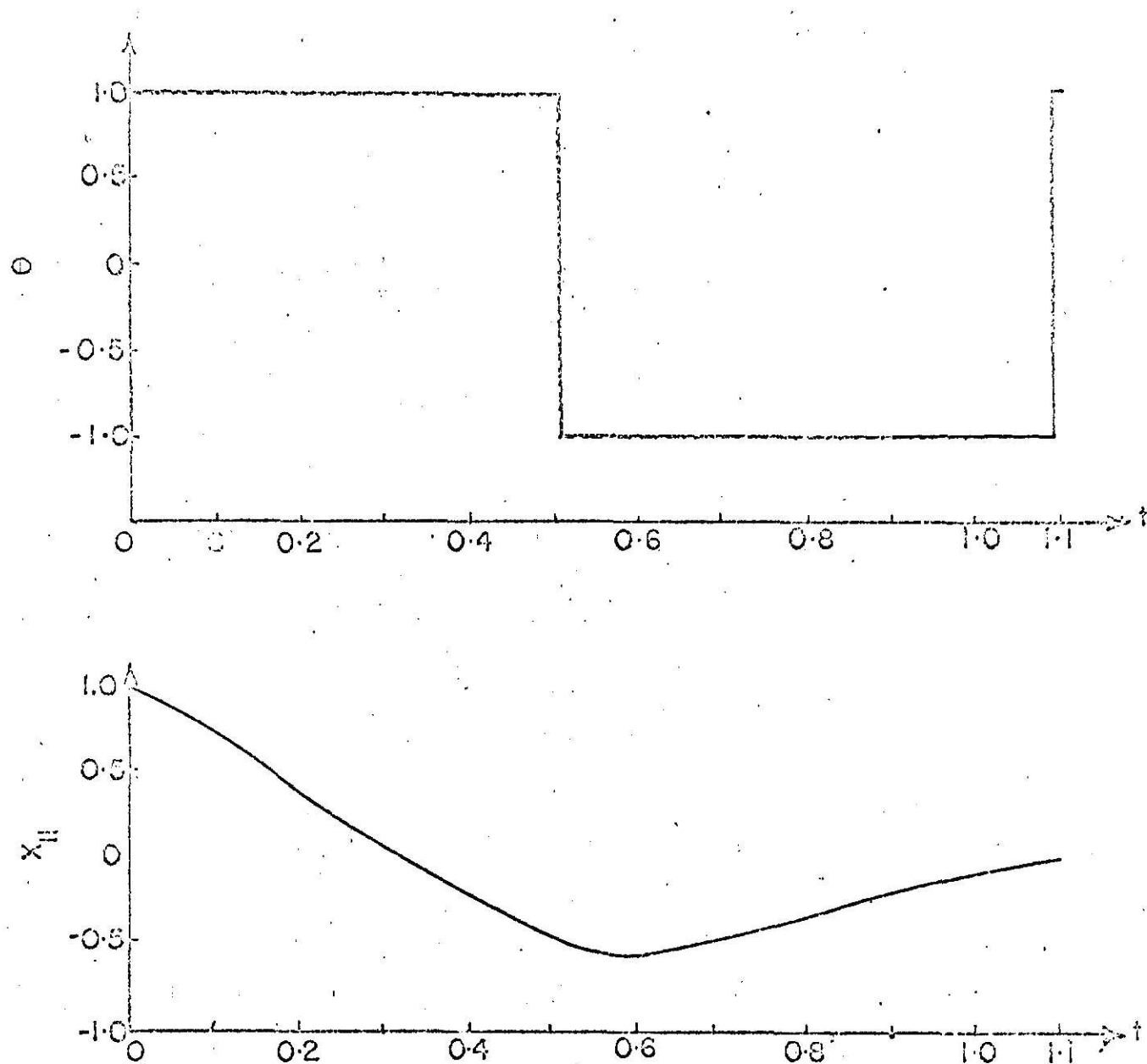


Figure 7.1a. SYSTEM RESPONSE FOR CASE 1 OF TWO CST'S-IN-SERIES  
MODEL WITH  $\zeta_2 = 0$ ,  $r_{11} = 2$  AND  $r = 10$

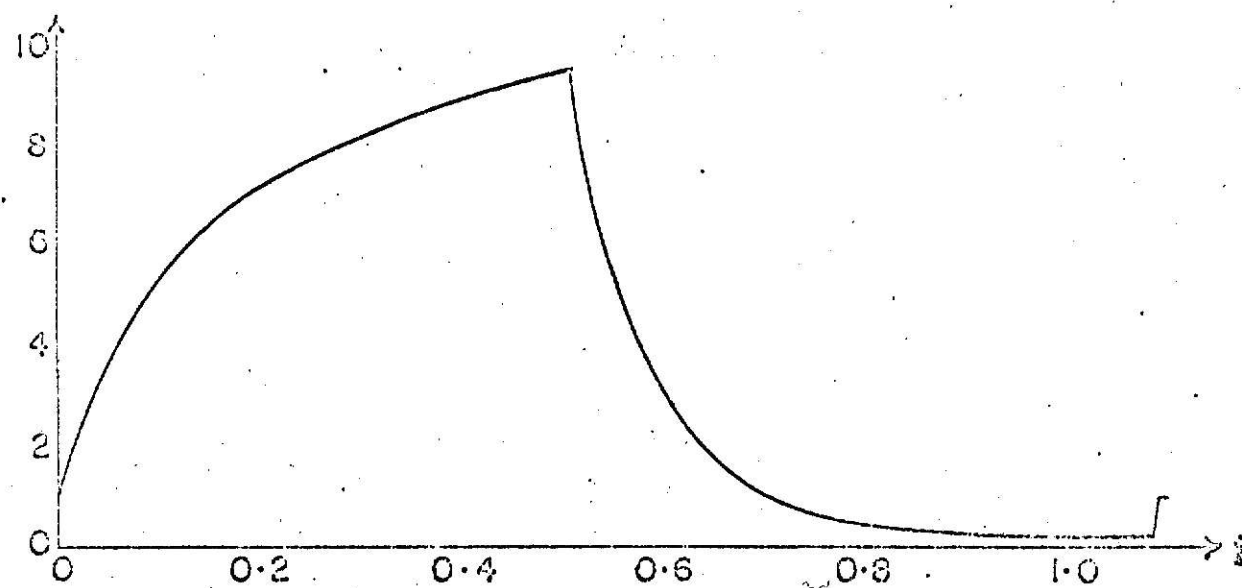
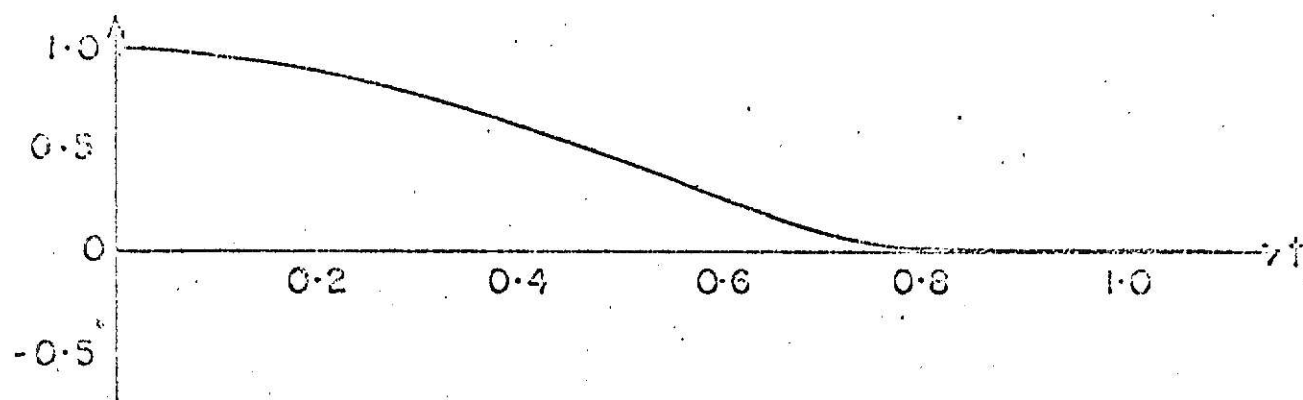


Figure 7.1b. SYSTEM RESPONSE FOR CASE 1 OF TWO CST'S-IN-SERIES MODEL  
WITH  $\zeta_2 \neq 0$  AND  $r_{11} = 2$  AND  $r = 10$



CHAPTER 8  
NUMERICAL EXAMPLES III

Example 6.3 is reconsidered here. In this example, the dimensionless room temperature assumes negative values during part of the control period. Very often, however, the room temperature has to be higher than a certain value in certain range for some physical or biological reasons. This requirement becomes the constraint of the problem. The performance equations are

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}a_{42}'x_{12} - a_{11}a_5'\theta - a_{11}a_6' + a_{12} \quad (8.1)$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = a_{21}x_{11} \quad (8.2)$$

$$x_{11}(0) = x_{12}(0) = 1 \quad \text{at} \quad t = 0 \quad (8.3)$$

$$x_{11}(T) = x_{12}(T) = 0 \quad \text{at} \quad t = T \quad (8.4)$$

where  $T$  is unspecified. The control variable and the state variable are constrained as

$$|\theta| \leq 1 \quad (8.5a)$$

$$x_{11} \geq m \quad (8.5b)$$

We wish to find a piecewise continuous control variable  $\theta$  such that the system can be brought back from the initial deviated state, equation (8.3), to the final desired state, equation (8.4), in a minimum period of time, that is,

$$S = \int_0^T dt \quad (8.6)$$

is minimized. Note that a constraint is not imposed on  $x_{12}$  because it is known from Example 6.3 that  $x_{12}$  does not cool down to the negative dimensionless temperature during operation.

When the inequality constraint,  $x_{11} > m$ , holds, the solution obtained in Example 6.3 is still applicable. They are [see equations (6.66) through (6.69)]:

$$x_{11} = \frac{1}{a_{21}} [(\lambda_{11} + r_{12})Ae^{\lambda_{11}t} + (\lambda_{12} + r_{12})Be^{\lambda_{12}t} + r_{12}K], \quad 0 \leq t \leq t_a \quad (8.7)$$

$$x_{12} = Ae^{\lambda_{11}t} + Be^{\lambda_{12}t} + K \quad 0 \leq t \leq t_a \quad (8.8)$$

$$x_{11} = \frac{1}{a_{21}} [(\lambda_{11} + r_{12})D_1e^{\lambda_{11}t} + (\lambda_{12} + r_{12})D_2e^{\lambda_{12}t} + r_{12}K'], \quad t_a \leq t \leq T \quad (8.9)$$

$$x_{12} = D_1e^{\lambda_{11}t} + D_2e^{\lambda_{12}t} + K', \quad t_a \leq t \leq T \quad (8.10)$$

where

$$K = \frac{a_{11}a_5'a_{21} + a_{11}a_{21}a_6' - a_{12}a_{21}}{a_{11}a_{42}'a_{21} - r_{11}r_{12}}$$

$$K' = \frac{a_{11}a_{21}a_6' - a_{11}a_5'a_{21} - a_{12}a_{21}}{a_{11}a_{42}'a_{21} - r_{11}r_{12}}$$

$$A = \frac{a_{21} - r_{12} - \lambda_{12} + \lambda_{12}K}{\lambda_{11} - \lambda_{12}}$$

$$B = \frac{r_{12} + \lambda_{11} - \lambda_{11} K}{\lambda_{11} - \lambda_{12}}$$

$$\lambda_{11} = \frac{1}{2} \left[ -(r_{11} + r_{12}) + \sqrt{(r_{11} + r_{12})^2 - r(r_{11}r_{12} - a_{11}a_{21}a_{42}')} \right]$$

$$\lambda_{12} = \frac{1}{2} \left[ -(r_{11} + r_{12}) - \sqrt{(r_{11} + r_{12})^2 - 4(r_{11}r_{12} - a_{11}a_{21}a_{42}')} \right]$$

and where  $D_1$  and  $D_2$  are unknowns, their value will be determined later.  $t_a$  is the arrival time when the state variable  $x_{11}$  reaches the boundary, i.e.,  $x_{11} = m$ .  $t_d$  is the departure time when  $x_{11}$  departs from the boundary.

On the boundary,  $x_{11} = m$ . Therefore, equations (8.1) and (8.2) become

$$mr_{11} = a_{11}a_{21}a_{42}'x_{12} - a_{11}a_5'\theta - a_{11}a_6' + a_{12} \quad (8.11)$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = ma_{21} \quad (8.12)$$

or solving for  $x_{12}$  and  $\theta$

$$x_{12} = \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t}, \quad t_a \leq t \leq t_d \quad (8.13)$$

$$\theta = \frac{1}{a_{11}a_5'} \left[ a_{11}a_{42}'x_{12} - a_{11}a_6' + a_{12} - mr_{11} \right], \quad t_a \leq t \leq t_d \quad (8.14)$$

$$x_{11} = m, \quad t_a \leq t \leq t_d \quad (8.15)$$

where  $C$  is an unknown. However, its value can be determined by inserting  $x_{11} = m$  into equation (8.7) and solving for  $t_a$ . Then  $t_a$  may be substituted into equation (8.8) to solve for  $C$ . Thus

$$x_{12} = Ae^{\lambda_{11}t_a} + Be^{\lambda_{12}t_a} + K, \quad t = t_a$$

and from equation (8.13),

$$x_{12} = \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t_a}, \quad t = t_a$$

or solving for C

$$C = e^{r_{12}t_a} \left[ Ae^{\lambda_{11}t_a} + Be^{\lambda_{12}t_a} - \frac{ma_{21}}{r_{12}} + K \right] \quad (8.16)$$

Because of continuity of  $x_{11}$  and  $x_{12}$  with respect to  $t$ , we have from equations (8.9) and (8.15), (8.10) and (8.13) at  $t = t_d$

$$\begin{aligned} x_{11}(t_d) &= m \\ &= \frac{1}{a_{21}} \left[ (\lambda_{11} + r_{12})D_1 e^{\lambda_{11}t_a} + (\lambda_{12} + r_{12})D_2 e^{\lambda_{12}t_a} \right. \\ &\quad \left. + r_{12}K \right] \end{aligned}$$

and

$$\begin{aligned} x_{12}(t_d) &= \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t_a} \\ &= D_1 e^{\lambda_{11}t_d} + D_2 e^{\lambda_{12}t_d} + K' \end{aligned}$$

Solving for  $D_1$  and  $D_2$  from these equations leads us to the following expressions

$$D_1 = \frac{\begin{vmatrix} -K + \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t_d} & e^{\lambda_{12}t_d} \\ -r_{12}K' + ma_{21} & (\lambda_{12} + r_{12})e^{\lambda_{11}t_d} \\ e^{\lambda_{11}t_d} & e^{\lambda_{12}t_d} \\ (\lambda_{11} + r_{12})e^{\lambda_{11}t_d} & (\lambda_{12} + r_{12})e^{\lambda_{12}t_d} \end{vmatrix}}{\begin{vmatrix} r_{12}K' - ma_{21} + (\lambda_{12} + r_{12}) \left( -K' + \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t_d} \right) \\ (\lambda_{12} - \lambda_{11})e^{\lambda_{11}t_d} \end{vmatrix}}$$

(8.17)

and

$$D_2 = \frac{-r_{12}K' + ma_{21} + (\lambda_{11} + r_{12}) \left( K' - \frac{ma_{21}}{r_{12}} - Ce^{-r_{12}t_d} \right)}{(\lambda_{12} - \lambda_{11})e^{\lambda_{12}t_d}} \quad (8.18)$$

We now see that  $D_1$  and  $D_2$  are functions of  $t_d$ . Its values and that of  $T$  can be obtained by making use of the final conditions of equations (8.9) and (8.10) at  $t = T$ . Thus

$$\frac{1}{a_{21}} [(\lambda_{11} + r_{11})D_1 e^{\lambda_{11}T} + (\lambda_{12} + r_{12})D_2 e^{\lambda_{12}T} + r_{12}K'] = 0 \quad (8.19)$$

$$D_1 e^{\lambda_{11}T} + D_2 e^{\lambda_{12}T} + K' = 0 \quad (8.20)$$

Solving for  $D_1 e^{\lambda_{11}T}$  from equation (8.20) and then inserting it into equation (8.19) yields

$$D_2 = \frac{\lambda_{11}K'}{(\lambda_{12} - \lambda_{11})e^{\lambda_{12}T}} \quad (8.21)$$

Inserting equation (8.21) into equation (8.20) and solving for  $D_1$  gives

$$D_1 = \frac{\lambda_{12}^{K'}}{(\lambda_{12} - \lambda_{11})e^{\lambda_{11}T}} \quad (8.22)$$

Equating equations (8.17) and (8.22), (8.18) and (8.21) gives

$$\begin{aligned} D_1 &= \frac{-r_{12}^{K'} - ma_{21} + (\lambda_{12} - r_{12}) \left(-K' + \frac{ma_{21}}{r_{12}} + Ce^{-r_{12}t_d}\right)}{(\lambda_{12} - \lambda_{11})e^{\lambda_{11}T}} \\ &= \frac{\lambda_{12}^{K'}}{(\lambda_{12} - \lambda_{11})e^{\lambda_{11}T}} \end{aligned}$$

and

$$\begin{aligned} D_2 &= \frac{-r_{12}^{K'} + ma_{21} + (\lambda_{11} + r_{12}) \left(K' - \frac{ma_{21}}{r_{12}} - Ce^{-r_{12}t_d}\right)}{(\lambda_{12} - \lambda_{11})e^{\lambda_{12}t_d}} \\ &= \frac{\lambda_{11}^{K'}}{(\lambda_{12} - \lambda_{11})e^{\lambda_{12}T}} \end{aligned}$$

or eliminating the common factor  $(\lambda_{12} - \lambda_{11})$  in these equations gives

$$\begin{aligned} \lambda_{12}^{K'} e^{\lambda_{11}t_d} &= e^{\lambda_{11}T} \left[ r_{12}^{K'} - ma_{21} + (\lambda_{12} + r_{12}) \left(-K' + \frac{ma_{21}}{r_{12}} \right. \right. \\ &\quad \left. \left. + Ce^{-r_{12}t_d} \right) \right] \quad (8.23) \end{aligned}$$

$$\lambda_{11} K' e^{\lambda_{12} t_d} = e^{\lambda_{12} T} \left[ -r_{12} K' + m a_{21} + (\lambda_{11} + r_{12}) \left( K' - \frac{m a_{21}}{r_{12}} \right) - c e^{-r_{12} t_d} \right] \quad (8.24)$$

$t_d$  and  $T$  are solved from equations (8.23) and (8.24) by a trial and error procedure. Then  $D_1$  and  $D_2$  can be obtained directly from equations (8.17) and (8.18) by substituting the value of  $t_d$  in them.

The solution of the problem is tabulated in Table 8.1 and is shown schematically in Figure 8.1. The optimal control policy is of the bang-bang type as shown in Example 6.3. However, because of the existence of inequality constraints on the state variable  $x_{11}$ , the optimal control policy  $\theta$  takes some intermediate value other than 1 or -1 during part of the operation.

A check of the Hamiltonian is very difficult, if not impossible. However, simulation of the problem with  $r_{11} = 2$  and  $x_{11} \geq -0.2$  has been carried out extensively by the phase plane approach and the results of simulation are tabulated in Table 8.2 and are also shown graphically in Figure 8.2. These results affirm that the solutions in Table 8.1 are truly optimal.

TABLE 8.1

OPTIMAL SOLUTIONS OF CASE 1 OF THE TWO CST'S-IN-SERIES MODEL  
 WITH  $z_2 = 0$  AND INEQUALITY CONSTRAINT  $x_{11} \geq m$

| $r_{11}$ | $m = -0.2$ | $m = -0.4$ | No constraint<br>on $x_{11}$ |
|----------|------------|------------|------------------------------|
| 1.2      | 0.992      | 0.992      | 0.992                        |
| 1.5      | 1.051      | 1.047      | 1.047                        |
| 2.0      | 1.233      | 1.103      | 1.078                        |
| 5.0      | 2.483      | 1.998      | 1.007                        |
| 10.0     | 3.457      | 2.862      | 0.956                        |



TABLE 8.2

SIMULATION OF CASE 1 OF THE TWO CST'S-IN-SERIES MODEL  
 WITH  $z_2 = 0$ ,  $r_{11} = 2$  AND  $x_{11} \geq -0.2$

| Control Variable $\theta$ |                 |                 |               | T           |
|---------------------------|-----------------|-----------------|---------------|-------------|
| $t < t_a$                 | $t_a < t < t_s$ | $t_s < t < t_d$ | $t_d < t < T$ |             |
| 1.0                       | -1.0            | 1.0             | -1.0          | No Solution |
| 1.0                       | -1.0            | 0.8             | -1.0          | No Solution |
| 1.0                       | -1.0            | 0.6             | -1.0          | No Solution |
| 1.0                       | -0.8            | 1.0             | -1.0          | 2.535       |
| 1.0                       | -0.8            | 0.8             | -1.0          | 2.363       |
| 1.0                       | -0.8            | 0.6             | -1.0          | 2.249       |
| 1.0                       | -0.6            | 1.0             | -1.0          | 1.435       |
| 1.0                       | -0.6            | 0.8             | -1.0          | 1.388       |
| 1.0                       | -0.6            | 0.6             | -1.0          | 1.336       |

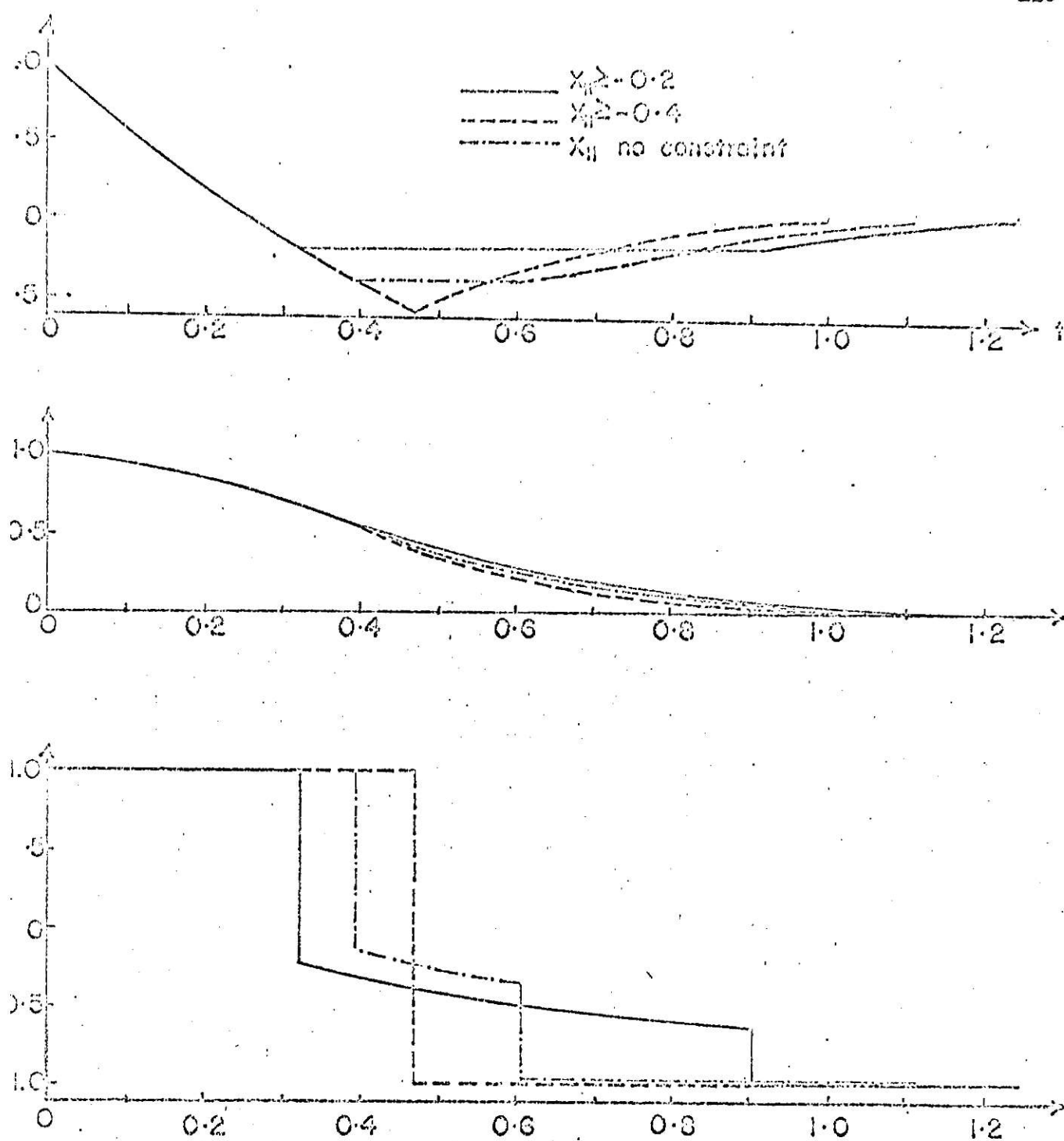


Figure 8.1. SYSTEM RESPONSES FOR CASE 1 OF TWO CST'S-IN-SERIES MODEL  
WITH  $z_2 = 0$ ,  $r_{11} = 2$  AND  $x_{11}$  IS CONSTRAINED

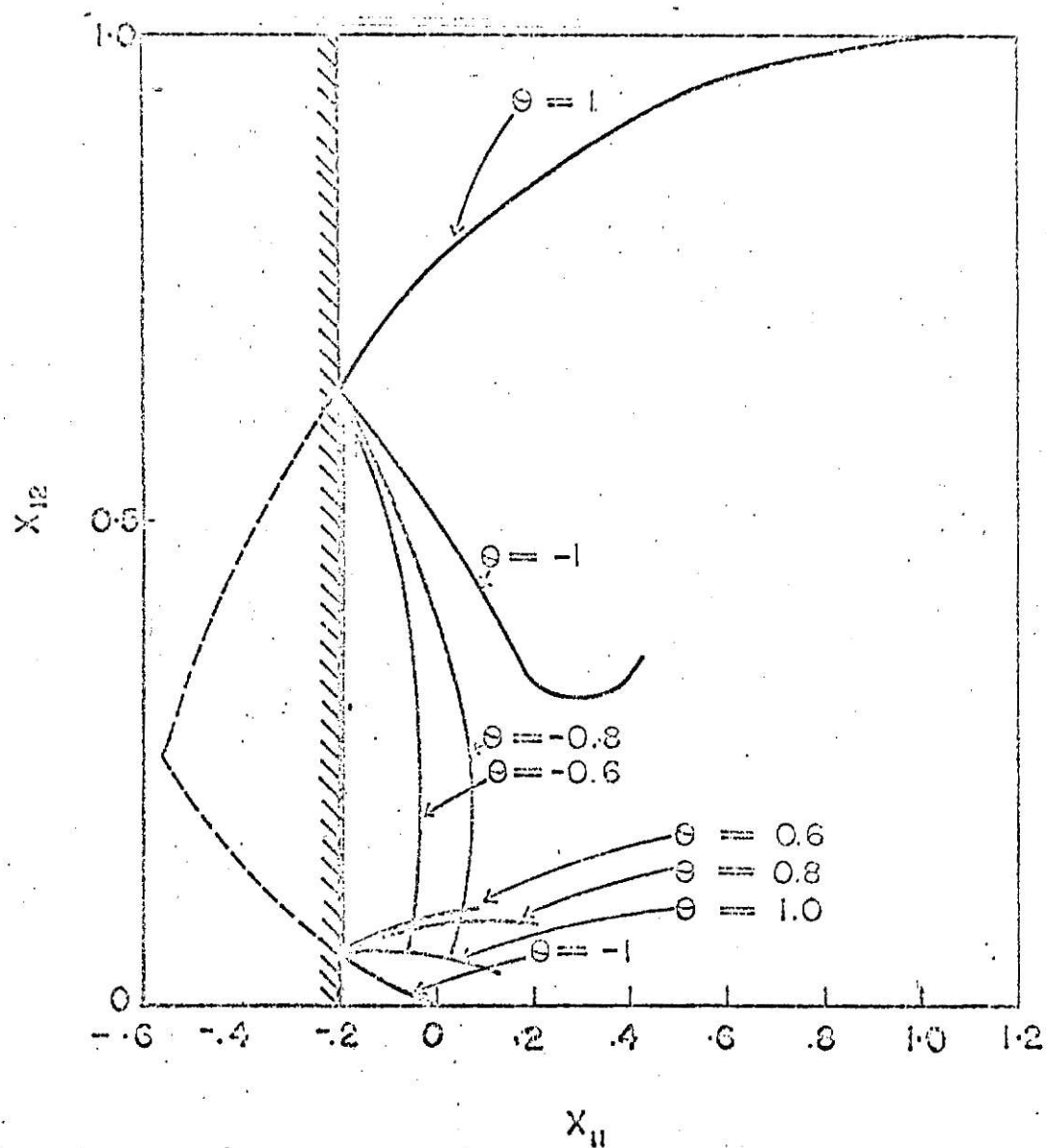


Figure 8.2. SIMULATION OF CASE 1 OF TWO CST'S-IN-SERIES MODEL WITH  $\zeta_2 = 0$ ,  $r_{11} = 2$  AND  $x_{11} \geq -0.2$

## CHAPTER 9

### OPTIMALITY AND SENSITIVITY ANALYSIS

The preceding chapters are concerned with the thermal modeling and simulation of confined spaces and life support systems and the optimal control of such systems. Examination of optimal results presented in the last four chapters naturally leads us to consider the deviation of a system from its nominal or optimal behavior caused by deviation of system components and parameters from their nominal performance characteristics. This is the essence of the sensitivity analysis.

Tomovic [34] and Takamatsu [35] discussed the role of sensitivity analysis in engineering problems. They indicated that there are several areas of sensitivity analysis. While knowledge of the sensitivity of the performance of a system as predicted by its model to parameter variation is important, there are other aspects of sensitivity analysis which are important for a particular problem or system. These are: (1) sensitivity to change in dimensions of the mathematical model representing the system; (2) sensitivity to transition from the continuous model and the discrete model in describing the system; (3) sensitivity to the influence of various functional blocks (system components) which comprise a system; and (4) sensitivity to change in the constraints. Tomovic [36] discussed the continuous sensitivity analysis to be used in analyzing the stability of a process. Demski [37] discussed the broad applications of sensitivity analysis in engineering and management sciences. Books by Pagurek [38] and Sage [8] are suggested references for sensitivity analysis of control system.

Herein we shall make use of the results presented in Chapters 5 through 8 to demonstrate the sensitivity to (1) parameter variation; (2) change in dimensions of mathematical models; and (3) change in constraints.

#### Sensitivities to Parameter Variation

In the literature pertaining to optimal control almost all the intensive studies in the field of sensitivity are related to the evaluation of the sensitivity of performance of a system with respect to small parameter variations [34, 39]. The specific values of the parameters used for design and control will differ to some extent from the actual values. Therefore, it is of practical importance to the process designer to know how sensitive is the process designed by him to these parameter uncertainties which may be due to, for example, the environmental and aging effects, the choice of a mathematical model both for the controlled system and the controller, and the measurements. Suppose that the value estimated for a parameter differs by 10 percent from the true value. Does deviation of this magnitude significantly affect the optimal design of the process? The sensitivity analysis used to attempt to obtain quantitative answers to this question is thus an integral part for the complete optimal design of a process.

Table 5.1 and Figures 5.6 and 5.7 show the effect of variation of the parameter  $\gamma_1$  (the recycle ratio of air) on the optimal conditions. It can be seen that the system is not sensitive to the variation of this parameter when the magnitude of the variation is small. The effect, however, is noticeable when the variation is large. Table 6.2 and Figures 6.3 and 6.4 indicate the effect of the change of the parameter  $\gamma$  (the ratio of time constants of the system element, cabin or room, to that of the heat exchanger). This effect is very small. Table 6.3 and Figures 6.5 and 6.7 show the effect of variation of

the parameter  $\gamma_{11}$  (the volume fraction of the first pool) on the optimal conditions. This effect is substantial.

#### The Sensitivity to Change in Dimensions of the Mathematical Models

This area of sensitivity analysis pertains to the sensitivity of the values of the variables to changes in the number of parameters or the order of the mathematical model. The designer is interested in the validity of his mathematical model when the dimensions of the parameter space of the system are reduced.

To illustrate this aspect of the sensitivity analysis, the dimensionless room temperature as a function of the dimensionless time under the optimal conditions presented in the previous chapters is summarized in Figure 9.1. Curves 1 and 2 represent the change of the dimensionless room temperature as a function of time for the system with one CST room or cabin. Curve 1 is for the system with the heat exchanger having a negligibly small time constant ( $Z_2 \rightarrow 0$ ,  $Z_1 = 50$  sec.). Curve 2 is for the system containing a heat exchanger with small but not a negligible time constant ( $Z_2 = 5$  sec.,  $Z_1 = 50$  sec.). Comparison of Curves 1 and 2 indicates that there is definitely a small but not negligible effect of the dimensional change of the system equation which is caused by neglecting the time constant of the heat exchanger. A similar conclusion can be obtained by comparing Curve 3 with Curve 4 which are for the systems represented by the two CST's-in-series model. Comparison of the group of curves, Curves 1 and 2, with the group, Curves 3 and 4, shows that the complexity of the model describing the system component (room or cabin) has a definite effect on the predicted performance of the system. It can be seen that such an effect is substantial for the particular models considered here, which are the one CST model and the two CST's-in-series model.

### The Sensitivity to Change in Constraints

The system sensitivity to changes in constraints imposed on the system, or more specifically, imposed on the state variable (temperature) of the system is discussed here.

For this purpose, results presented in Chapter 8 are summarized in Figure 9.2. Note that the systems considered are all represented by the two CST's-in-series.

Since the constraint is imposed on  $x_{11}$  (dimensionless temperature of the first compartment of the model), naturally the effect of the change of constraint on this state variable cannot be neglected. However, the effect on the dimensionless temperature of the second compartment, which is also the exit temperature of air from the room or cabin, is negligibly small. The effect of the change in the constraint on control policy, however, is very appreciable as indicated by the plot of  $\theta$  vs.  $t$  in the same figure. Observations made here are valid for the particular model considered and for the particular values of the model parameters employed.

### Concluding Remarks

It should be evident that thorough consideration must be given to numerous aspects of the system in order to achieve a meaningful sensitivity analysis. The following aspects must be considered in order to obtain the desired information:

1. Sensitivity to parameter variation.
2. Sensitivity to change in dimensions of the mathematical models.
3. Sensitivity to change in constraints.
4. Sensitivity to transition between continuous and discrete models.

5. Sensitivity to the influence of the functional blocks of a system. Items 1, 2 and 3 have been considered in this chapter and items 4 and 5 will be briefly discussed below.

In certain cases, an alternative may exist in representing the model either by a continuous model or a discrete model. If the predicted behavior of the system is very sensitive to the type of the model selected, the use of an appropriate model becomes important.

Complex systems generally consist of several functional blocks, stages or units. Complex chain reactions may follow alternative reaction paths and produce different intermediate reactants. The sensitivity of the entire system or process to variation in types of function blocks which are contained in the system and to their relative locations is called the structural sensitivity [31].



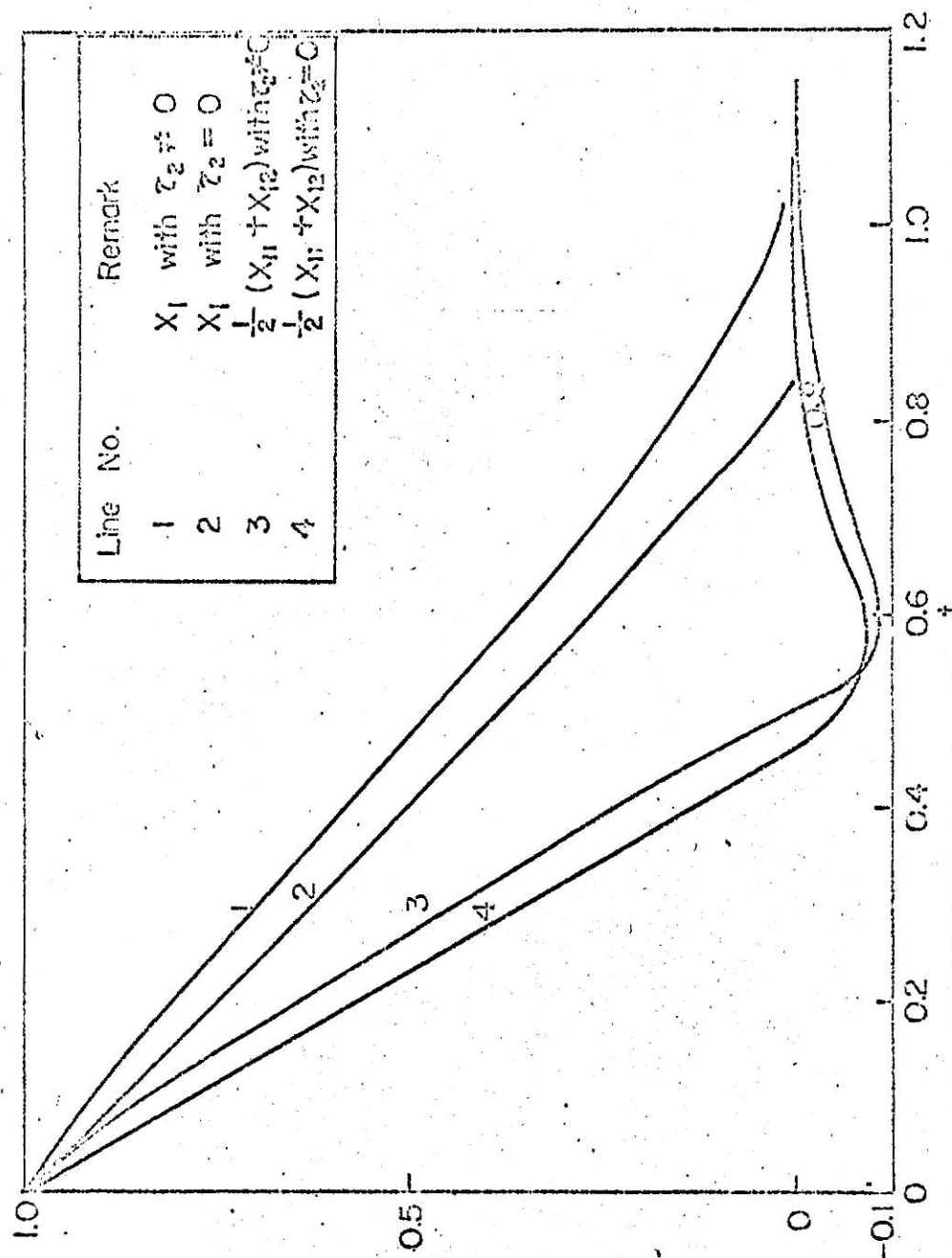


Figure 9.1. THE SENSITIVITY TO CHANGE IN DIMENSIONS OF THE MATHEMATICAL MODEL

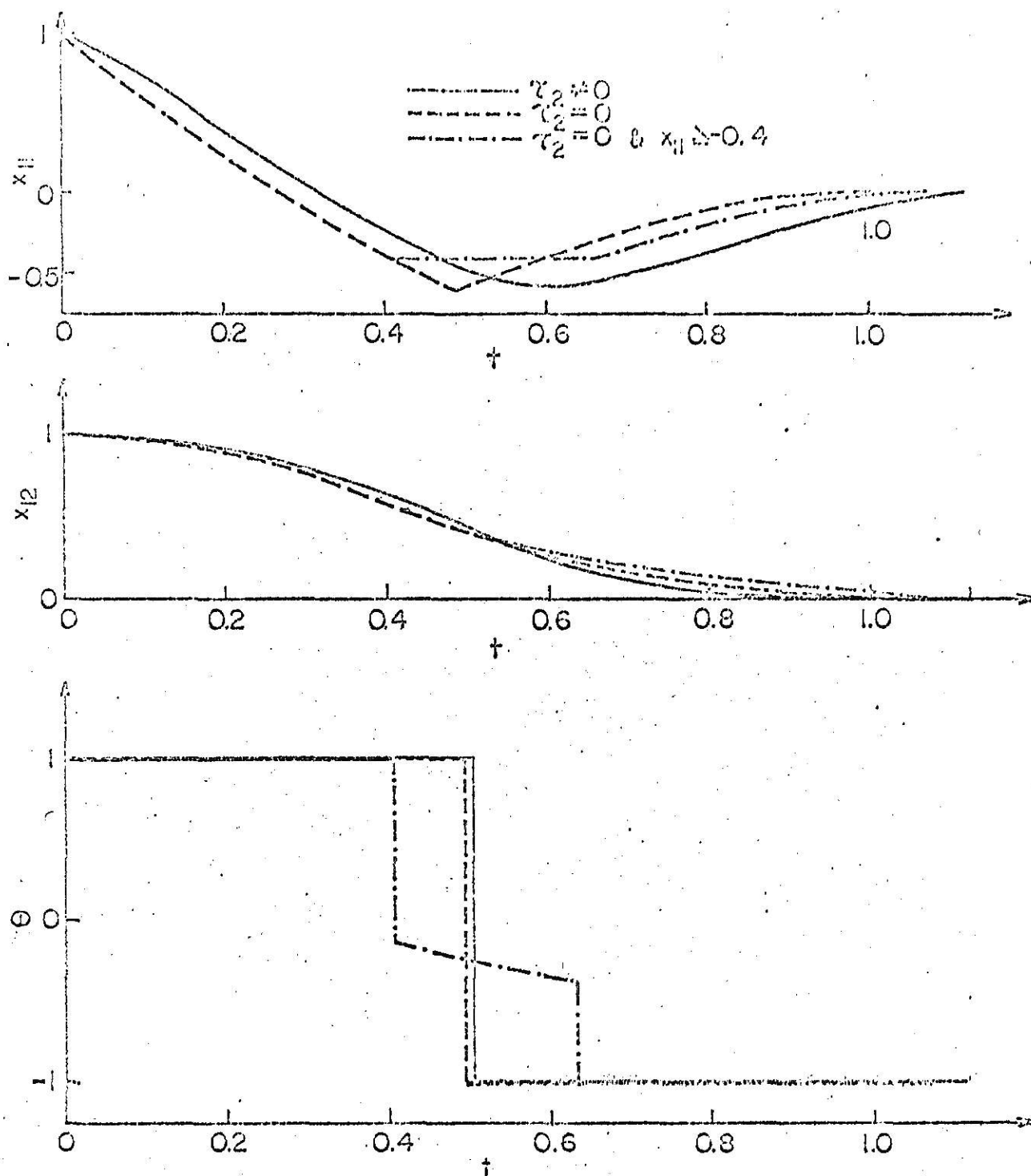


Figure 9.2. THE SENSITIVITY TO SHIFT OF THE CONSTRAINTS OF THE MATHEMATICAL MODEL

## CHAPTER 10

## SUMMARY AND RECOMMENDATION

The results presented in this study affirm that the modern optimal control theory can be successfully applied to the study of thermal comfort systems. The modeling of these systems as proposed in Chapter 5 is adequate to represent a large scale of dynamic behavior of a complex life support system.

The basic computational algorithm of Pontryagin's maximum principle was stated rather briefly but completely in Chapter 2. Then we showed that the fixed right-end problem can also be solved by considering the problem as one with equality constraints on the final state variables in Chapter 3. Finally, for the theoretical part, we proved in Chapter 4 that the necessary conditions for optimum of Pontryagin's maximum principle for processes with inequality constraints imposed on the state variables remain the same as those for processes without state variable constraints. One exception, however, is that the adjoint variables possess the form shown in equation (4.34) instead of equation (4.5) on the constraint boundary.

Pontryagin's maximum principle which handles constraints is a much better form for the calculus of variations for this problem than the Bolza form which cannot treat constraints [9]. The maximum principle can also be used to evaluate the number of the switching points of the bang-bang control policy via the switching function and adjoint vectors. Examples in Chapters 6 through 8 took advantage of this rule. Furthermore, the maximum principle can be applied not only to the system with linear performance equations but also to those with non-linear performance equations. Bellman [4] has proven

theoretically the number of the switching points is one less than the dimension of the problem for linear systems. But this theory cannot be applied to non-linear systems.

The response of the heat exchanger is almost instantaneous, especially when the time constant of the heat exchanger is much smaller than that of the system element.

Through Lagrange multipliers the optimization problems represented by either algebraic or differential equations can be handled in exactly the same approach.

Based on the discussions in the preceding chapter it is evident that thorough consideration must be given to numerous aspects of the system in order to obtain a meaningful sensitivity analysis.

Some possible immediate extensions are:

1. The proposed model can be made more realistic, though not perfect, by considering more and more pools in series. In this case, the time constants of the sensing element and the heat exchanger are no longer negligible. Computation of the optimal trajectory will be very difficult, if not impossible. However, the high-speed digital computer is believed to be able to compute the optimal control patterns. These theoretical solutions will finally be compared with the experimental data which will be measured in the environmental laboratory, School of Engineering, Kansas State University.

2. Other forms of the objective functions, for example those given below, can be considered,

$$S = \int_0^T \bar{L}x_1^2 dt$$

$$S = \int_0^T [\beta + \gamma_1(x_1)^2] dt$$

$$S = \int_0^T [\theta]^2 dt$$

$$S = \int_0^T [\beta + \mu(\theta)^2] dt$$

$$S = \int_0^T [\beta + r_1(x_1)^2 + \mu(\theta)^2] dt$$

$$S = \int_0^T |\theta| dt$$

$$S = \int_0^T [r_1(x_1)^2 + \mu(\theta)^2] dt$$

Those objective functions have different physical significance [1, 3, 8].

3. Solution of the performance equations which consists of combination of the ordinary differential equation representing the dynamic behavior of the system element, and the partial differential equation representing the dynamic behavior of the heat exchanger, will be more realistic, though the procedure of solving this problem is quite sophisticated.

4. Of equal importance to the optimal control of air temperature in the life support systems is the optimal control of humidity and air cleanliness in such systems, for unless a system can be kept at some desired operating point, it matters little whether or not that operating point is at an optimum one. The scheme used for the optimal temperature control is believed to be easily applied to the control of humidity and air cleanliness in the life support systems.

5. Some other forms of the inequality constraints, for example,  $ax_{11} + bx_{12} = \text{constant}$ , which may have some physical importance and should be associated with the problem solved in Chapter 8.

In conclusion this study is exploratory in nature and seeks to develop a mathematical model for an air-conditioned room which may be suitable for the dynamic control of air temperature. The forms of the mathematical model, introduction of the modern optimal control theory in the field of environmental engineering are the major issues.

The author wishes that this work will stimulate others to work toward the development of the mathematical model and better utilization of the modern control theory in the design and improvement of comfort control systems.

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## APPENDIX A

## A MINIMUM INTEGRAL ERROR AND MINIMUM TIME PROBLEM

We wish to choose control variable  $\theta$  subject to

$$|\theta| \leq 1 \quad (\text{A.1})$$

such that the performance index

$$S_2 = \int_0^T (x_1)^2 dt, \quad T \text{ is unspecified} \quad (\text{A.2})$$

is minimized, where  $x_1$  is governed by equation (5.29a) with the following given initial and final conditions

$$\begin{aligned} x_1(0^-) &= 0 & \text{at} & \quad t = 0^- \\ x_1(0^+) &= 1 & \text{at} & \quad t = 0^+ \\ x_1(T) &= 0 & \text{at} & \quad t = T \end{aligned} \quad (\text{A.3})$$

Initially, an additional state variable  $x_2$  is introduced as

$$x_2(t) = \int_0^t (x_1)^2 dt \quad (\text{A.4})$$

It follows that

$$\frac{dx_2}{dt} = x_1^2, \quad x_2(0) = 0 \quad (\text{A.5})$$

and

$$x_2(T) = \int_0^T (x_1)^2 dt = S_2 \quad (\text{A.6})$$

Thus the problem is transformed to that of minimizing  $x_2(T)$ .

The Hamiltonian function and adjoint variables are

$$H = z_1[r_2x_1 + r_2K_1 - r_1K_1K_2\theta - r_1K_1K_3] + z_2[x_1]^2 \quad (A.7)$$

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 - 2z_2x_1 \quad (A.8)$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \quad z_2(T) = 1 \quad (A.9)$$

The solution of  $z_2$  is from equation (A.9)

$$z_2 = 1 \quad \text{for} \quad 0 \leq t \leq T \quad (A.10)$$

Thus, equations (A.7) and (A.8) can be rewritten as

$$H = z_1[-r_2x_1 + r_2K_1 + K_1\phi(t) - r_1K_1K_2\theta - r_1K_1K_3] + (x_1)^2 \quad (A.11)$$

$$\frac{dz_1}{dt} = r_2z_1 - 2x_1 \quad (A.12)$$

Examination of the switching function  $H^*$  which depends only on the control variable

$$H^* = r_1K_1K_2z_1\theta \quad (A.13)$$

shows that the bang-bang policy should be used and the minimum  $H$  will attain when

$$\theta = \theta_{\max} = 1 \quad \text{if} \quad r_1K_1K_2z_1 > 0 \quad (A.14)$$

$$\theta = \theta_{\min} = -1 \quad \text{if} \quad r_1K_1K_2z_1 < 0$$

We intuitively start the control from

$$\theta = \theta_{\max} = 1 \quad (\text{A.15})$$

The solution of  $x_1$  is

$$x_1(t) = e^{-r_2 t} + K_1(1 - e^{-r_2 t}) - \frac{r_1 K_1 K_2}{r_2} (1 - e^{-r_2 t}) - \frac{r_1 K_1 K_3}{r_2} (1 - e^{-r_2 t}) \quad (\text{A.16})$$

Inserting this equation into equation (A.12) and then solving for  $z_1$  gives

$$z_1(t) = C_1 e^{r_2 t} + \frac{2}{r_2} (K_1 - \frac{r_1 K_1 K_2}{r_2} - \frac{r_1 K_1 K_3}{r_2}) + \frac{1}{r_2} (1 - K_1 + \frac{r_1 K_1 K_2}{r_2} + \frac{r_1 K_1 K_3}{r_2}) e^{-r_2 t} \quad (\text{A.17})$$

where  $C_1$  is a constant and its value can be determined by making sure that minimum  $H$  is zero for every point. Therefore, we have from equation (A.11) at  $t = 0^+$

$$z_1(0^+) = - \frac{1}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} \quad (\text{A.18})$$

Also from equation (A.17) at  $t = 0^+$ , we get

$$z_1(0^+) = C_1 + \frac{1}{r_2} [-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3] \quad (\text{A.19})$$

Equating these equations and solving for  $C_1$  gives

$$C_1 = - \frac{1}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} - \frac{1}{r_2} (-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3) \quad (\text{A.20})$$

and

$$z_1(t) = - \frac{e^{r_2 t}}{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} \quad (\text{A.21})$$

Since  $z_1$  does not change its sign unless  $t$  approaches negative infinity, the final time  $T$  can be obtained from equation (A.16) and the final condition. One gets

$$T = \frac{1}{r_2} \ln \frac{-r_2 + r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3}{r_2 K_1 - r_1 K_1 K_2 - r_1 K_1 K_3} \quad (\text{A.22})$$

and the performance index becomes

$$\begin{aligned} S_2 &= \int_0^T (x_1)^2 dt \\ &= (K_1 - \frac{r_1 K_1 K_2}{r_2} - \frac{r_1 K_1 K_3}{r_2})^2 (T + \frac{2}{r_2} e^{-r_2 T} - \frac{1}{2r_2} e^{-2r_2 T} - \frac{3}{2r_2}) \\ &\quad - (K_1 - \frac{r_1 K_1 K_2}{r_2} - \frac{r_1 K_1 K_3}{r_2}) (\frac{2}{r_2} e^{-r_2 T} - \frac{1}{r_2} e^{-2r_2 T} - \frac{1}{r_2}) \\ &\quad + \frac{1}{2r_2} (1 - e^{-2r_2 T}) \end{aligned} \quad (\text{A.23})$$

If

$$\begin{aligned} r_1 &= 0.8, & K_2 &= 1.5 \\ r_2 &= 0.2, & K_3 &= 1.5 \\ K_1 &= 0.5, & &= 2 \end{aligned}$$

then

$$x_1(t) = -5.5 + 6.5e^{-0.2t}$$

$$z_1(t) = 0.769e^{0.2t}$$

$$T = 0.835$$

$$S = 0.267$$

## APPENDIX B

## DERIVATION OF EQUATION (3.14) FROM EQUATION (3.12)

The objective function,  $S'$ , is of the form from equation (3.12)

$$S' = \sum_{i=1}^{s+1} c_i x_i(T) + \sum_{i=1}^q v_i g_i[\bar{x}(T)] + \int_{t_0}^T \left\{ H[\bar{x}(t), \bar{\theta}(t), \bar{z}(t)] - \sum_{i=1}^{s+1} z_i(t) \frac{dx_i}{dt} \right\} dt \quad (B.1)$$

Carrying out the Taylor series expansion of the above equation about  $\bar{x}$ ,  $\bar{\theta}$ ,  $\bar{z}$ , and  $\bar{T}$ , for examples,

$$x(T) = \bar{x}(\bar{T}) + \frac{\partial \bar{x}(T)}{\partial \bar{x}} (x - \bar{x}) + \frac{\partial x(\bar{T})}{\partial \bar{T}} (T - \bar{T}) + \dots$$

$$g_i[\bar{x}(T)] = g_i[\bar{x}(\bar{T})] + \frac{\partial g_i[\bar{x}(T)]}{\partial \bar{x}} (x - \bar{x}) + \frac{\partial g_i[\bar{x}(\bar{T})]}{\partial \bar{T}} (T - \bar{T}) + \dots$$

and defining

$$x_i(t) = \bar{x}_i(t) + \delta x_i(t)$$

$$\theta_i(t) = \bar{\theta}_i(t) + \delta \theta_i(t)$$

$$T = \bar{T} + \delta T$$

$$z_i(t) = \bar{z}_i(t) + \delta z_i(t)$$

Discarding the nonlinear terms of the resulting equation and dropping the bar notation, we have

$$\begin{aligned}
S' = & \sum_{i=1}^{s+1} c_i \left[ \delta x_i(T) + \frac{\partial x_i(T)}{\partial T} \delta T \right] \\
& + \sum_{j=1}^q v_j \left\{ \frac{\partial g_i[\bar{x}(T)]}{\partial x_i(T)} \delta x_i(T) + \frac{\partial g_i[\bar{x}(T)]}{\partial T} \delta T \right\} \\
& + \int_{t_0}^T \left\{ \sum_{i=1}^{s+1} \frac{\partial H}{\partial x_i} \delta x_i + \sum_{i=1}^r \frac{\partial H}{\partial \theta_i} \delta \theta_i + \sum_{i=1}^{s+1} \frac{\partial H}{\partial z_i} \delta z_i \right. \\
& + \frac{\partial H}{\partial t} \delta t - \sum_{i=1}^{s+1} \left[ \delta z_i \frac{dx_i}{dt} + z_i \delta \left( \frac{dx_i}{dt} \right) \right. \\
& \left. \left. + \left( \frac{dz_i}{dt} \frac{dx_i}{dt} + z_i \frac{d^2 x_i}{dt^2} \right) \delta t \right] \right\} dt \quad (B.2)
\end{aligned}$$

Since

$$\int_{t_0}^T \frac{\partial H}{\partial t} \delta t \, dt = \int_{t_0}^T \frac{\partial H}{\partial t} dt \, \delta t = H \Big|_T \delta T$$

and

$$\begin{aligned}
& \int_{t_0}^T \left( \frac{dz_i}{dt} \frac{dx_i}{dt} + z_i \frac{d^2 x_i}{dt^2} \right) \delta t \, dt \\
& = \int_{t_0}^T \frac{d}{dt} \left( z_i \frac{dx_i}{dt} \right) \delta t \, dt \\
& = \int_{t_0}^T \frac{d}{dt} \left( z_i \frac{dx_i}{dt} \right) dt \, \delta t \\
& = \left( z_i \frac{dx_i}{dt} \right) \Big|_T \delta T
\end{aligned}$$

Equation (B.2) becomes equation (3.14).



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APPLICATIONS OF MODERN OPTIMAL CONTROL THEORY  
TO ENVIRONMENTAL CONTROL OF CONFINED SPACES

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# ABSTRACT

The basic form of Pontryagin's maximum principle which is a keystone of the modern optimal control theory is stated. The principle is extended to cover optimal problems with equality constraints imposed on the final system whose state variables are constrained by certain inequality conditions are also derived.

Several mathematical models of an environmental control system which consists of a confined space or cabin, a heat exchanger, and a feedback element such as a thermostat are presented. The performance equations of the system, which represent the dynamic characteristics of the air-conditioned cabin (the system proper) and the heat exchanger (the control element of the system), are derived. In the basic model the flow of air in the confined space is considered to be in the state of complete mixing and the disturbance is caused by an impulse heat input. The flow of air in the confined space or cabin characterized by the two completely stirred tanks-in-series (a CST's-in-series) model is also considered. To determine the goodness of the system model, a computer simulation is carried out and the results are compared with the known characteristics of the system.

The computational algorithms are applied to the determination of optimal control policies of the temperature control of a life support system consisting of an air-conditioned cabin subject to an impulse heat disturbance and a heat exchanger. The following examples are treated,

(i) A cabin represented by the one CST model:

(a) the time constant of the heat exchanger is negligibly small,

- (b) the time constant of the heat exchanger is not ignored, and
- (c) the time constant of the heat exchanger is neglected; however, the square form of the final condition of the state variable is considered as an equality constraint.

(ii) A cabin represented by the two CST's-in-series model:

- (a) the time constant of the heat exchanger is negligible,
- (b) the time constant of the heat exchanger is not negligible

and the squares of the final conditions of the state variables are considered as equality constraints, and

- (c) a constraint imposed on the room temperature which has to be higher than a certain value for some physical or biological reasons.

The sensitivities of the systems treated in these examples are examined. These include the sensitivities to (1) parameter variation, (2) change in dimensions of mathematical models, and (3) change in constraints on state variables.