Theoretical Study of Stability in Horizontal Fluid Layers with Uniform Volumetric Energy Sources

by

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NOMENCLATURE OF TERMS

a	Wave number of plan form solution
B_o	Biot number at fluid layer boundary, Z=0, h_oL/k
B_1	Biot number at fluid layer boundary, Z=1, h_1L/k
C_v	Specific heat at constant volume
D	Differential operator, $\frac{d}{dz} = \frac{1}{L}\frac{d}{dZ}$
$F(Z)$	Z-component of vertical disturbance velocity
g	Constant of gravitational acceleration
$G(x,y)$	Plan form solution for disturbance velocity or temperature
h	Heat transfer coefficient
k	Thermal conductivity
L	Total layer depth
L_1	Depth of upper sublayer
L_o	Depth of lower sublayer
N_s	Dimensionless grouping, $N_s = \frac{qL^2}{2k(T_o-T_1)}$
P	Basic state pressure
P^*	Disturbance pressure
q	Volumetric energy source
R_E	External Rayleigh Number
R_{KG}	Internal Rayleigh Number as defined by Kulacki and Goldstein[15]
$\hat{T}$	Maximum horizontally averaged layer temperature
T_1	Upper boundary temperature
T_0	Lower boundary temperature
T^*	Disturbance temperature
T_{SS}	Steady state temperature
T	Temperature
t	Time

U^*	Disturbance velocity in x-direction
V^*	Disturbance velocity in y-direction
W^*	Disturbance velocity in z-direction
X_i	I-th component, external force
x	Horizontal coordinate
y	Horizontal coordinate
z	Vertical coordinate in fluid layer, $0 \leq z \leq L$
Z	Dimensionless vertical coordinate in fluid layer, z/L, $0 \leq Z \leq 1$
α	Thermal diffusivity
$\bar{\beta}$	Coefficient of volumetric expansion
μ	Viscosity
ν	Kinematic viscosity
Φ	Dissipation function
σ	Decay constant for disturbance velocity

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1.0 INTRODUCTION

The possibility of a core disruptive accident in the proposed Liquid Metal Fast Breeder Reactor (LMFBR) has prompted a detailed study of post accident core heat transfer characteristics. Of specific interest is the heat transfer mechanism in a molten fuel layer heated by a decay heat source uniformly distributed throughout the layer.

Best available data on thermophysical properties of molten mixed-oxide fuel, $(U_{0.8}\ Pu_{0.2})O_2$, are tabulated in Table 1. The maximum fuel-layer thickness of interest is approximately 12 to 15 cm. For thicker layers, boiling and the possibility of criticality of the fuel layer change the nature of the heat-transfer problem. Heat generation rates of interest are from $1 W/cm^3$ to as high 50 to 60 W/cm^3. These limits correspond to decay heat power levels for the Fast Flux Test Facility one month after shutdown from sustained operation at design power or for Clinch River Reactor fuel 100s after shutdown. These layer thicknesses and heating rates result in internal Rayleigh Numbers as high as 6×10^{10}. The external Rayleigh Number depends on the boundary temperatures. As an extreme case, consider a layer of molten fuel with molten steel at its lower boundary and boiling steel at its upper boundary, i.e., the lower surface temperature, T_0, is 1683 °K and the upper surface temperature, T_1, is 3073 °K. These conditions give an external Rayleigh Number of 1.1×10^{10}. Considerations such as the foregoing provide the motivition for the present investigation.

An initially quiescent fluid layer can be brought to a state of internal convective motion if a sufficiently large imbalance is caused between buoyant forces, which tend to disturb fluid elements, and restraining

Table 1. Thermophysical Properties of Molten $(U_{0.8}\ Pu_{0.2})O_2$. [After Baker, et al. (1).]

Melting point	2770 °C
Boiling point	3150 °C
Density	8.4 g/cm^3
Thermal conductivity	0.029 W/(cm°C)
Specific heat	0.5 J/(g°C)
Expansion coefficient	$10^{-4}\,°C^{-1}$
Thermal diffusivity	0.0069 cm^2/sec
Kinematic viscosity	0.0060 cm^2/sec
Prandtl number	0.9

viscous forces. Buoyant forces due to density differences can be produced by differential warming of the fluid at its horizontal boundaries or by warming of the fluid from within.

The ratio of the buoyant forces promoting convection to restraining viscous forces is the characteristic dynamical parameter for such systems and is expressed by a dimensionless grouping of the form,

$$R_E = \frac{g\beta}{k\nu}\ \Delta T L^3 ,$$

for fluids satisfying the classical Boussinesq relation(5,11) wherein the density, ρ, is treated as a constant in all terms in the equations of motion except the one in the external force. The temperature difference between the top surface and lower surface is ΔT, and L is a characteristic length of the layer. These sympols are defined in the Nomenclature List.

This grouping is denoted as the Rayleigh Number in honor of Lord Rayleigh(18) who first used it in his analysis of hydrodynamic instability in fluid layers warmed from below. This group is now more commonly called the external Rayleigh Number because it describes a physical situation wherein buoyant forces arise from differential heating at the horizontal boundaries. Hence, there exists a fixed external Rayleigh Number for each layer which has a specified temperature difference between the top surface and the bottom surface. For such a fluid layer, there exists a critical external Rayleigh Number, R_{EC}, such that instability develops when its value is met or exceeded. For instance, if R_E is less than R_{EC}, thermal convection can't occur, and heat travels only by conduction within the fluid layer. If a small disturbance is created in the fluid, the disturbance dies away with time because of viscous forces. Such a state is described as stable and small disturbances can't grow into convection cells.

The first quantitative studies of externally driven thermal convection were the experiments of Bénard[2] who studied the flow produced in a horizontal liquid layer heated from below. His work was the first to demonstrate the onset of thermal instabilities in fluid layers heated from below. His general description of the flow has survived until today, however, and externally driven thermal convection in horizontal fluid layers is generally termed "Bénard Convection." Another class of thermal convection--that driven by internally produced density differences--is regarded as an extension of classical Bénard Convection.

After Bénard's experiment, the first analytical treatment aimed at the determining the conditions delineating the breakdown of the steady state was published by Lord Rayleigh. Jeffreys[8,9], Low[19], and Pellow and Southwell[31], have extended Rayleigh's initial analysis to a broader range of boundary conditions. In particular, Pellow and Southwell present the most complete theory of the thermal instability concerning the classical

Bénard Convections. The aforementioned analytical studies have considered hydrodynamic boundary conditions which correspond to the following containment conditions of the fluid layer: (1) the upper and lower bounding surfaces are both rigid (zero slip); (2) the lower surface is rigid, while the upper surface is free (zero shear); (3) the upper and lower surfaces are both free. The latter condition does not appear to correspond to a real physical situation, but it may be of theoretical interest.

The thermal conditions usually applied at the upper and lower surfaces are based on the supposition that these surfaces are in contact with materials of infinite thermal conductivity and heat capacity. From such a model, it follows that the temperatures at the surfaces are kept constant.

In a special case, Jeffreys[12] investigated a situation wherein both upper and bottom surfaces are both rigid and insulted. Objections to this condition have been raised both by Low and by Pellow and Southwell on the grounds that both boundary surfaces are kept at constant temperature. However, actual physical situations suggest that the thermal boundary conditions of fixed temperatures at the surface of the fluid layer may be too restrictive. If the heat-transfer coefficient between the surface and the environment is finite, the surface temperature will be perturbed when the quiescent state breaks down. The Biot Number, appropriately expressing the conductance of the boundary relative to that of the fluid layer, was thus included in the thermal boundary conditions for the temperature disturbance at the surface.

Nield[23] and Hurle, et al.[7] have done stability studies which included boundaries of finite thermal conductance. These works are, however, limited to stability in the classical Bénard problem. Nield found that for layers with a rigid and isothermal lower boundary and a free upper boundary, decreasing the Biot Number of the upper boundary destabilizes the fluid layer, (i.e. yields lower values of the critical Rayleigh Number). Hurle et al. considered a layer bounded by two rigid walls of finite thickness and found that by decreasing the thermal diffusivity of the boundaries the critical external Rayleigh Number decreased.

Sparrow et al.[33] determined critical Rayleigh Numbers for layers with a rigid (zero slip) lower and a free upper boundary. The lower boundary had either constant temperature or constant heat flux. A decreasing Biot Number at the upper boundary was shown to have a destabilizing effect on the fluid layer.

Sparrow et al. also treated the stability problem for a fluid layer with a uniform, volumetric energy source. For two rigid and isothermal boundaries, with the effects of the volumetric energy production superimposed on a basic linear conduction temperature profile, the critical external Rayleigh Number was found to decrease as the volume heating source increased. The range of critical Rayleigh Numbers R_{EC} studied under different boundary conditions ranged from 10^3 to 10^5.

Roberts[29] performed stability calculations for a fluid layer with a uniform volumetric heat source enclosed in an insulated lower boundary and an isothermal upper boundary. Both boundaries were rigid. At such conditions, only one critical external Rayleigh number existed.

Another Rayleigh Number used for an internally heated fluid layer is defined by Kulacki[15]. This Rayleigh Number can be expressed as

$$R_{KG} = \frac{g\beta}{k\nu}\ \frac{L^3}{8}\ \frac{1}{4}\ (\frac{qL^2}{2k})$$

wherein the characteristic length scale is L/2 and the maximum temperature difference in a layer with a symmetrical parabolic conduction temperature is $\frac{1}{4}\ (\frac{qL^2}{2k})$. Because, it is concerned with the internal source, q, it is called the internal Rayleigh Number.

Kulacki[15], in his dissertation, investigated the critical internal Rayleigh Number for a fluid layer with constant heat flux at both surfaces, but with different hydrodynamic constraints. It was shown that the critical Rayleigh Number decreases monotonically with decreasing Biot Number when the Biot Numbers at both boundaries are identical. The same trend was found in the case of asymmetrical thermal boundary conditions when the ratio of the upper surface Biot number, B_1, to that at the lower surface, B_0, is fixed. If the lower surface Biot Number is held constant, an increasing Biot Number ratio, B_1/B_0, is strictly destabilizing. If the thermal boundary conditions are the same, the more stable configuration is a layer with rigid-rigid hydrodynamic constraints; the less stable, a layer with free-rigid surfaces.

The purpose of this study is to calculate the critical internal Rayleigh Number, R_{KG}, and the critical external Rayleigh Number, R_{EC} for either rigid-rigid and isothermal surfaces or free-rigid and isothermal surfaces. The magnitude of critical internal Rayleigh numbers studied ranges from 10^3 to 10^{10} while critical external Rayleigh Numbers studied extend from 10^4 to 10^{10}.

In the following sections, the governing equations for the fluid layers of interest will be developed, the methods of solution will be examined and resulting solutions discussed with respect to previous studies.

2.0 THEORY

2.1 General Solution

In considering the question of stability in a horizontal fluid layer with an internal energy source, several simplifying assumptions are made usually to produce a tractable mathematical problem.

These are:

(a) The layer is assumed to be of infinite horizontal extent. In a physical sense, the layer is of such an extent that all edge effects are negligible.

(b) The fluid is incompressible and the volumetric energy source is constant throughout the layer.

(c) Buoyancy forces are due to thermally induced density differences only.

A schematic of the system and the coordinate system are shown in Fig 1. To describe the problem completely, the three Conservation equations and the equation of state are required. These equations can be expressed in tensor form as,

$$\rho \frac{D}{Dt} V_i = \rho X_i - \frac{\partial}{\partial x_i} P + \nu\rho\nabla^2 V_i \ , \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho V_i) = 0 \ , \tag{2.2}$$

$$\rho = \rho_0[1 - \beta(T-T_0)] \ , \tag{2.3}$$

$$\rho V_i \frac{\partial}{\partial x_i} C_v T + \rho \frac{\partial}{\partial t} C_v T = k\nabla^2 T - P \frac{\partial V_i}{\partial x_i} + q + \Phi \ , \tag{2.4}$$

where the symbols have been defined in the Nomenclature list. From the Boussinesq approximation the specific volume, v, the specific heat at

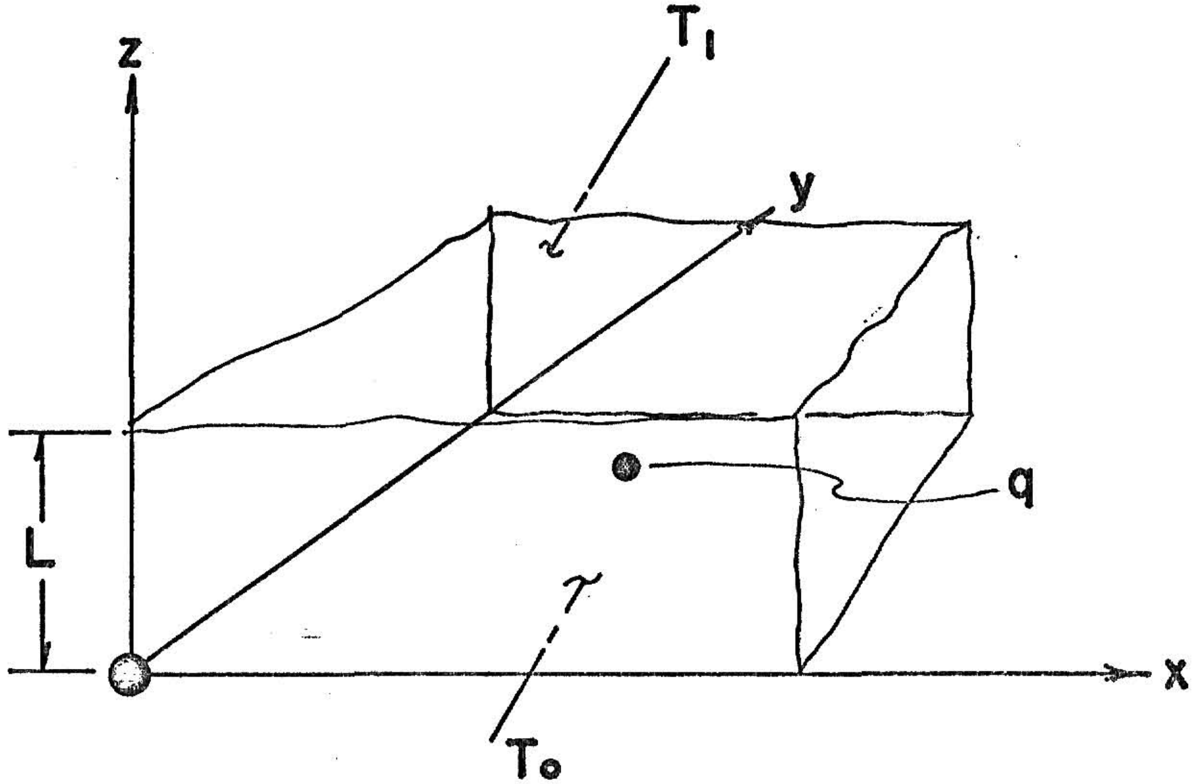


Fig. 1 Schematic Representation of a Horizontal Layer with Uniform Internal Energy Sources.

constant volume, C_v, the coefficient of volumetric expansion, β, and the thermal conductivity, k, can be treated as constants. The density, ρ, is also treated as a constant except in the body force term, ρX_i, and there is no viscous dissipation (Φ=0).

In the initial (steady) state, V_i=0, and $P=P_0$ and is independent of x and y. Therefore, under such conditions, these equations can be simplified as follows:

$$-g\rho - \frac{\partial P_0}{\partial z} = 0 , \tag{2.5}$$

$$\frac{\partial V_i}{\partial x_i} = 0 , \tag{2.6}$$

$$\nabla^2 T_{SS} + \frac{q}{k} = 0 . \tag{2.7}$$

By solving Eq. 2.7, the initial temperature can be expressed in the form,

$$T_{SS} = T_0 + Az - (q/2k)z^2 , \tag{2.8}$$

where A depends on the boundary conditions. Substituting Eq. 2.8 into Eq. 2.3 it follows:

$$\rho = \rho_0 \{1 - \beta(T_{SS} - T_0)\} = \rho_0(1 - \beta Az + \beta \frac{q}{2k} z^2) . \tag{2.9}$$

Now assume that there is a small (infinitesimal) disturbance applied to the basic conduction state. Let U*, V*, and W* be the resulting disturbed velocity components in the x, y, and z directions respectively. These velocity components are sufficiently small to justify neglect of their squares and products. This is the basis of linear theory.

Similarly, let T* be the deviation from the steady state temperature. It follows:

$$T^* = T - [T_0 + Az - (q/2k)z^2] . \tag{2.10}$$

Substituting from Eq. 2.8 into Eq. 2.4 and neglecting the second order terms in V_i^*, and T^*, it follows:

$$-AW^* + \frac{q}{k} zW^* = [\frac{\partial}{\partial t} - \alpha\nabla^2]T^* \ . \tag{2.11}$$

Substituting for the temperature, T, in Eq. 3, it is obtained

$$\rho = \rho_0(1 - \beta Az + \beta \frac{q}{2k} z^2 - \beta T^*) \ . \tag{2.12}$$

Comparing Eq. 2.12 with Eq. 2.9, it can be seen that the incremental density which results from the convective motion is a very small fraction $(\frac{-\beta T^*}{1-\beta Az+\frac{\beta q}{2k}z^2})$ of the steady state density. Hence, the increment may be neglected when it is multiplifed by U*, V*, W*, or T*. In like fashion, the pressure of the system after the disturbance is:

$$P = P_0 + P^* \ . \tag{2.13}$$

By virtue of Eq. 2.5, three component equations follow from Eq. 2.1:

$$\frac{\partial U^*}{\partial t} = -\frac{1}{\rho}\frac{\partial P^*}{\partial x} + \nu\nabla^2 U^* \ , \tag{2.14}$$

$$\frac{\partial V^*}{\partial t} = -\frac{1}{\rho}\frac{\partial P^*}{\partial y} + \nu\nabla^2 V^* \ , \tag{2.15}$$

$$\frac{\partial W^*}{\partial t} = g\beta T^* - \frac{1}{\rho}\frac{\partial P^*}{\partial z} + \nu\nabla^2 W^* \ . \tag{2.16}$$

From the continuity equation, it follows:

$$\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y} + \frac{\partial W^*}{\partial z} = 0 \ . \tag{2.17}$$

Eliminating U* and V* by combining Eq. 2.17 with Eqs. 2.14 and 2.15, it is obtained:

$$[\frac{\partial}{\partial t} - \nu\nabla^2] \frac{\partial^2 W^*}{\partial z^2} = \frac{1}{\rho} \nabla_1^2 \frac{\partial P^*}{\partial z} \ , \tag{2.18}$$

where ∇_1^2 represents $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Furthermore, Eq. 2.16 may be expressed by operating on both sides with ∇_1^2,

$$[\frac{\partial}{\partial t} - \nu\nabla^2]\nabla_1^2 W^* = g\beta\nabla_1^2 T^* - \frac{1}{\rho}\nabla_1^2 \frac{\partial P^*}{\partial z} . \qquad (2.19)$$

Adding Eq. 2.18 and Eq. 2.19, it follows:

$$[\frac{\partial}{\partial t} - \nu\nabla^2]\nabla^2 W^* = g\beta\nabla_1^2 T^* . \qquad (2.20)$$

By operating on both sides with $[\frac{\partial}{\partial t} - \alpha\nabla^2]$ it is found:

$$[\frac{\partial}{\partial t} - \nu\nabla^2][\frac{\partial}{\partial t} - \alpha\nabla^2]\nabla^2 W^* = g\beta\nabla_1^2[\frac{\partial}{\partial t} - \alpha\nabla^2]T^* . \qquad (2.21)$$

But the quantity $[\frac{\partial}{\partial t} - \alpha\nabla^2]T^*$ is known from Eq. 2.11. Substitution of Eq. 2.11 into Eq. 2.21 yields an expression for W*,

$$[\frac{\partial}{\partial t} - \nu\nabla^2][\frac{\partial}{\partial t} - \alpha\nabla^2]\nabla^2 W^* = g\beta(-A + \frac{q}{k}z)\nabla_1^2 W^* . \qquad (2.22)$$

This is, thus, the governing equation for the perturbation velocity in the vertical direction.

The solution of this equation was sought first by Lin[17] in the form,

$$W^* = F(z)\ G(x,y)e^{\sigma t} , \qquad (2.23)$$

where σ is, in general, a complex number.

If $Re\{\sigma\} > 0$, then the disturbance velocity and temperature increase with time, that is, pure instability results if $Re\{\sigma\} > 0$. On the other hand if $Re\{\sigma\} < 0$, then the disturbance motion will eventually die out. Finally when $\sigma=0$, the disturbance neither grows nor decays, and this limit is termed the marginal or neutral stability limit.

Following Kulacki[15], it can be shown that the function G(x,y) obeys

$$\nabla_1^2 G(x,y) + (a/L)^2 G(x,y) = 0 \ , \tag{2.24}$$

where "a" is a single wave number corresponding to the onset of motion. Substituting Eqs. 2.23 and 2.24 into 2.22, an expression for the function F(Z) is obtained, (details of the derivation are covered in Appendix A). Thus:

$$\frac{d^6F(Z)}{dZ^6} - 3a^2\frac{d^4F(Z)}{dZ^4} + 3a^4\frac{d^2F(Z)}{dZ^2}$$

$$+ (\Lambda + \Omega Z)F(Z) + \sigma^2\psi_1(Z) + \sigma\psi_2(Z) = 0 \ , \tag{2.25}$$

where

$$Z = z/L \ , \ \Lambda = -(g\beta L^4 Aa^2/\alpha\nu) - a^6 \quad \text{and} \quad \Omega = g\beta L^5 qa^2/\alpha\nu k.$$

The groups Λ and Ω represent constants which are determined from the problem statement. The ψ_1 and ψ_2 functions need not be stated inasmuch as they drop out of the forthcoming analysis, because σ is equal to 0 in the stability limit. (See Appendix A for explicit form and derivation.)

Under these conditions, Eq. 2.25 can be expressed as,

$$\frac{d^6F(Z)}{dZ^6} - 3a^2\frac{d^4F(Z)}{dZ^4} + 3a^4\frac{d^2F(Z)}{dZ^2} + (\Lambda+\Omega Z)F(Z) = 0 \ , \tag{2.26}$$

which is a homogenous ordinary differential equation for the perturbation function F(Z).

Therefore the resulting eigenvalue problem for this homogeneous system of equations provides a means for determining the conditions under which a solution for the perturbation can exist.

2.1 Power Series Solution

The general solution for a sixth order homogeneous differential equation can be constructed in the form,

$$F(Z) = \sum_{i=0}^{5} C_i f^{(i)}(Z) \tag{2.27}$$

with C_i arbitrary, and in which the $f^{(i)}(Z)$ are a convergent power series[33].

Following Sparrow[33], it can be demonstrated that the series coefficients $b_n^{(i)}$ obey the following relationship for $n \geq 6$,

$$b_n^{(i)} = \frac{1}{n!}\{3a^2(n-2)!b_{n-2}^{(i)} - 3a^4(n-4)!\, b_{n-4}^{(i)}$$

$$- (\Lambda b_{n-6}^{(i)} + \Omega b_{n-7}^{(i)})(n-6)!\} , \tag{2.28}$$

and $b_{-1}^{(i)} = 0$. In addition, the $b_o^{(i)}$ through $b_5^{(i)}$ are specified as

$$b_n^{(i)} = \delta_{ni} \qquad (0 \leq n \leq 5), \tag{2.29}$$

wherein δ_{ni}=1 for n=i and δ_{ni}=0 for n≠i. The constants, C_0, C_1, ..., C_5, which appears in the solution for F(Z) are to be determined from the boundary conditions. (See Appendix E for further discussion.)

2.2 Boundary Conditions

It is assumed that the layer is horizontal and large enough to neglect any edge effects; therefore, two hydrodynamic boundary conditions and two thermal boundary conditions are required to solve the problem.

2.2.A Hydrodynamic Boundary Condition

2.2.A.1 Free Surface

When a surface is not constrained by a rigid boundary, it is called a free surface. A free surface requires that the vertical velocity component, W*, vanishes, and in addition, it is not able to support a tangential stress. Thus, the partials $\frac{\partial U^*}{\partial z}$ and $\frac{\partial V^*}{\partial z}$ are both zero. Under these restrictions, it follows from the continuity equation, Eq. 2.17, that $\frac{\partial^2 W^*}{\partial z^2} = 0$ and that

$$F(Z) = \frac{d^2 F(Z)}{dZ^2} = 0 \ . \qquad (2.30)$$

2.2.A.2 Rigid Surface

A rigid surface requires that all the velocity components vanish identically ("no slip") $U^* = V^* = W^* = 0$ at the wall. Correspondingly, the partials $\frac{\partial U^*}{\partial x}$ and $\frac{\partial V^*}{\partial y}$ are both zero. From the continuity equation it follows further that $\frac{\partial W^*}{\partial z} = 0$. In terms of the F(Z) function,

$$F(Z) = \frac{dF(Z)}{dZ} = 0 \ . \qquad (2.31)$$

2.2.B Thermal Boundary Conditions

2.2.B.1 Fixed Surface Temperature

If the surface temperature is constant, then it must remain unperturbed by any temperature perturbations in the fluid. Thus, T* at the surface for a fixed surface temperature must be zero. From Eq. 2.20, it is obtained at the surface,

$$[\frac{\partial}{\partial t} - \nu\nabla^2]\nabla^2 W^* = 0 \ . \qquad (2.32)$$

Substituting the general solution, Eq. 2.23, into Eq. 2.34, the boundary condition in terms of F(Z) is obtained,

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z) = 0 \ . \qquad (2.33)$$

2.2.B.2 Constant Heat Flux

If heat is being transferred at a free surface, then the energy must be conducted to the surface since no tangential slip or normal velocity components are allowed. By Fourier's Law, the heat flux, Q, passing through the free boundary per unit time and area, is

$$Q = -k \left.\frac{\partial T}{\partial z}\right|_{\text{surface}} \ . \qquad (2.34)$$

Energy transport in a fluid is more conveniently expressed in terms of the product of a heat transfer coefficient, h, and a temperature difference,

$$-k \frac{\partial T}{\partial z} = h(T - T_\infty) \ , \qquad (2.35)$$

where T_∞ is the ambient temperature above the surface. If T is replaced by $(T_{SS} + T^*)$ then $\frac{\partial T^*}{\partial z} = (hL/k)T^*$, since $-k(\frac{\partial T_{SS}}{\partial z}) = h(T_{SS}-T_\infty)$. Restated in terms of the F(Z) function, it is found that

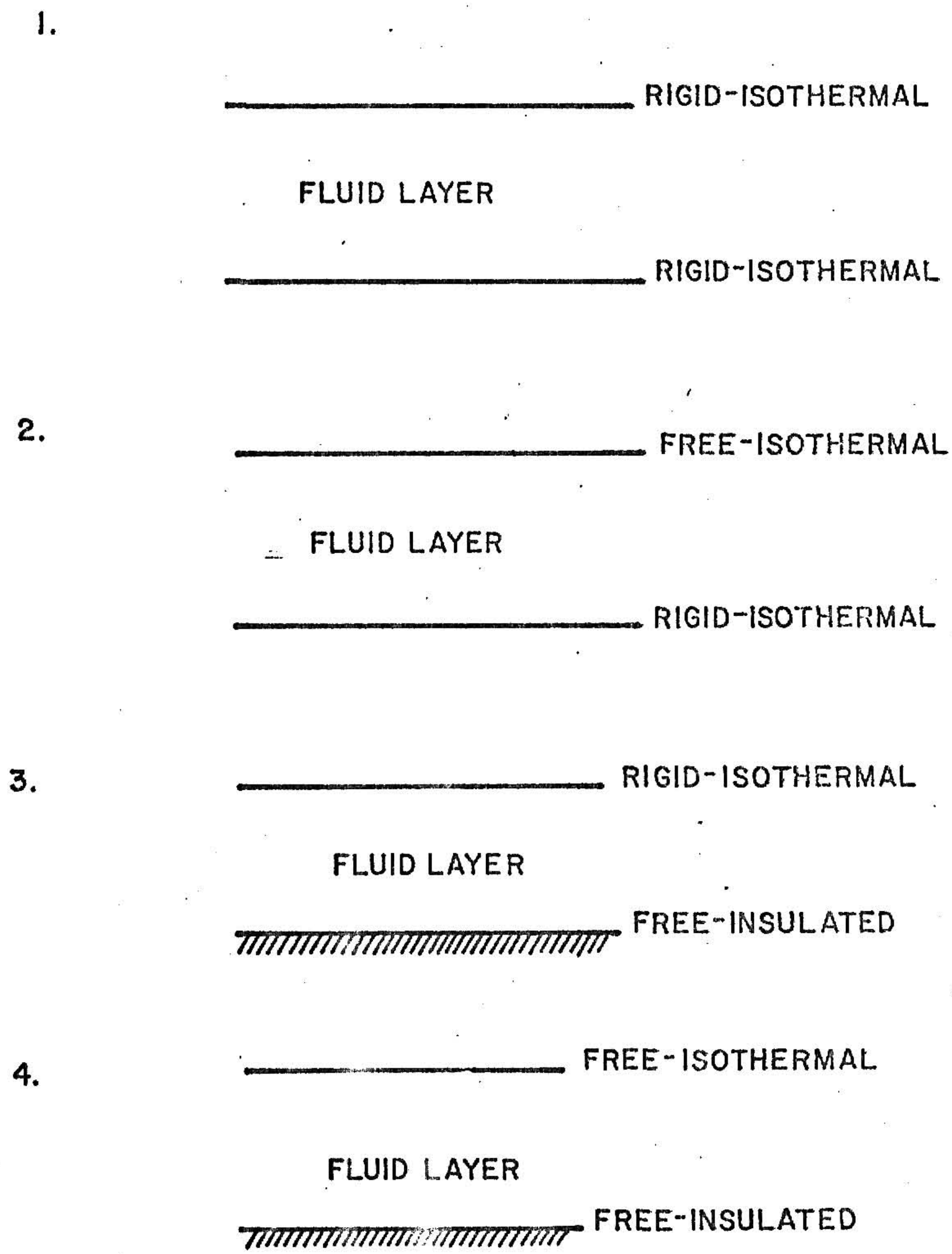
$$\frac{d^5F(Z)}{dZ^5} - 2a^2 \frac{d^3F(Z)}{dZ^3} + a^4 \frac{dF(Z)}{dZ}$$

$$= \frac{hL}{k} \left(\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z)\right) \ . \qquad (2.36)$$

2.3 Application of the Boundary Conditions

Four different sets of boundary conditions are discussed in this investigation. These are also shown schematically in Fig. 2. Each of them will now be covered in turn.

Fig. 2 Four Different Boundary Situations of a Horizontal Layer with Uniform Internal Heat Sources.



The first case of interest is one in which both the lower and the upper boundaries are rigid and isothermal. The boundary conditions appropriate to this case are:

$$Z=0, \qquad F(Z) = \frac{dF(Z)}{dZ} = 0$$

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} = 0 ,$$

$$Z=1, \qquad F(Z) = \frac{dF(Z)}{dZ} = 0$$

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z) = 0 .$$

Substituting a power series for F(Z) into the above equations, there follows:

$$C_0 = C_1 = 0 ,$$

$$-4a^2C_2 + 24\, C_4 = 0 ,$$

$$0 = [C_2f^{(2)}(Z) + C_3f^{(3)}(Z) + C_4f^{(4)}(Z) + C_5f^{(5)}(Z)]_{Z=1} ,$$

$$0 = [C_2 \frac{df^{(2)}(Z)}{dZ} + C_3 \frac{df^{(3)}(Z)}{dZ} + C_4 \frac{df^{(4)}(Z)}{dZ} + C_5 \frac{df^{(5)}(Z)}{dZ}]_{Z=1} ,$$

$$\sum_{i=2}^{5} C_i [\frac{d^4f^{(i)}(Z)}{dZ^4} - 2a^2 \frac{d^2f^{(i)}(Z)}{dZ^2} + a^4f^{(i)}(Z)]_{Z=1} = 0 .$$

The second case of interest is one in which the lower boundary is rigid and isothermal, but the upper surface is free and isothermal. The boundary conditions appropriate to the case are

$$Z=0, \qquad F(Z) = \frac{dF(Z)}{dZ} = 0 ,$$

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} = 0 ,$$

$$Z=1, \qquad F(Z) = \frac{d^2F(Z)}{dZ^2} = 0 \ ,$$

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z) = 0 \ .$$

Substituting Eq. 2.27 into the above equations, there follows:

$$C_0 = C_1 = 0 \ ,$$

$$-4a^2C_2 + 24C_4 = 0 \ ,$$

$$[C_2f^{(2)}(Z) + C_3f^{(3)}(Z) + C_4f^{(4)}(Z) + C_5f^{(5)}(Z)]_{Z=1} = 0 \ ,$$

$$[C_2 \frac{d^2f^{(2)}(Z)}{dZ^2} + C_3 \frac{d^2f^{(3)}(Z)}{dZ^2} + C_4 \frac{d^2f^{(4)}(Z)}{dZ^2} + C_5 \frac{d^2f^{(5)}(Z)}{dZ^2}]_{Z=1} = 0 \ ,$$

$$\sum_{i=2}^{5} C_i[\frac{d^4f^{(i)}(Z)}{dZ^4} - 2a^2 \frac{d^2f^{(i)}(Z)}{dZ^2} + a^4f^{(i)}(Z)]_{Z=1} = 0 \ .$$

The third case is one in which the lower surface is free and insulated, but the upper surface is rigid and isothermal. Then, the boundary conditions appropriate to this case are

$$Z=0, \qquad F(Z) = \frac{d^2F(Z)}{dZ^2} = 0 \ ,$$

$$\frac{d^5F(Z)}{dZ^5} - 2a^2 \frac{d^3F(Z)}{dZ^3} + a^4 \frac{dF(Z)}{dZ} = 0 \ ,$$

$$Z=1, \qquad F(Z) = \frac{dF(Z)}{dZ} = 0 \ ,$$

$$\frac{d^5F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z) = 0 \ .$$

Again, substituting a power series into the equations, it follows:

$$C_0 = C_2 = 0$$

$$120C_5 - 12a^3C_3 + a^4C_1 = 0 ,$$

$$[C_1 f^{(1)}(Z) + C_3 f^{(3)}(Z) + C_4 f^{(4)}(Z) + C_5 f^{(5)}(Z)]_{Z=1} = 0 ,$$

$$[C_1 \frac{df^{(1)}(Z)}{dZ} + C_3 \frac{df^{(3)}(Z)}{dZ} + C_4 \frac{df^{(4)}(Z)}{dZ} + C_5 \frac{df^{(5)}(Z)}{dZ}]_{Z=1} = 0 ,$$

$$\sum_{i=1,3,4,5} C_i [\frac{d^4 f^{(i)}(Z)}{dZ^4} - 2a^2 \frac{d^2 f^{(i)}}{dZ^2} + a^4 f^{(i)}(Z)]_{Z=1} = 0 .$$

The last case of interest is one in which the lower surface is free and insulated, but the upper surface is free and isothermal. Then, the boundary conditions appropriate to this case are:

$$Z=0, \qquad F(Z) = \frac{d^2F(Z)}{dZ^2} = 0 ,$$

$$\frac{d^5F(Z)}{dZ^5} - 2a^2 \frac{d^3F(Z)}{dZ^3} + a^4 \frac{dF(Z)}{dZ} = 0 ,$$

$$Z=1, \qquad F(Z) = \frac{d^2F(Z)}{dZ^2} = 0 ,$$

$$\frac{d^4F(Z)}{dZ^4} - 2a^2 \frac{d^2F(Z)}{dZ^2} + a^4F(Z) = 0 .$$

Again, substituting a power series solution into the above equations, there follows:

$$C_0 = C_2 = 0 ,$$

$$120C_5 - 12a^2C_3 + a^4C_1 = 0 ,$$

$$[C_1 \frac{d^2f^{(1)}(Z)}{dZ^2} + C_3 \frac{d^2f^{(3)}(Z)}{dZ^2} + C_4 \frac{d^2f^{(4)}(Z)}{dZ^2} + C_5 \frac{d^2f^{(5)}(Z)}{dZ^2}]_{Z=1} = 0 ,$$

$$\sum_{i=1,3,4,5} C_i [\frac{d^4 f^{(i)}(Z)}{dZ^4} - 2a^2 \frac{d^2 f^{(i)}(Z)}{dZ^2} + a^4 f^{(i)}(Z)]_{Z=1} = 0 .$$

The principle used to find the critical external Rayleigh Number is the same in all four cases. As an example, consider the last case. In such an instance, the boundary equations constitute a system of four, linear, homogeneous, algebraic equations in the four constants C_1, C_3, C_4, C_5. A non-trivial solution exists if and only if the determinant of the coefficient matrix vanishes.

The value of the determinant depends on two parameters, the external Rayleigh Number $(g\beta(T_1-T_0)L^3/\alpha\nu)$ and the constant "a". For every "a" value, an external Rayleigh number exists that causes the determinant of the coefficient matrix to be zero. Moreover, it is found that for a particular "a", there is a corresponding Rayleigh Number which is smaller than that for any another "a". A solution for the disturbance equation cannot be found for any external Rayleigh Number below this value. Physically this means that the quiescent state is stable because no perturbation function can be found. Therefore, the aforementioned minimum external Rayleigh Number corresponds to the onset of instability. This is generally called the critical external Rayleigh Number and it is this number that is of interest in the present study.

3.0 RESULTS AND DISCUSSION

Consideration will now be given to investigating how the stability of an initially quiescent layer is affected by the shape of the temperature distribution in the fluid. In the present study, a non-linearity in the temperature distribution is created by a uniformly distributed heat source, q. The boundary condition selected for this phase of study is a free or rigid, fixed-temperature upper surface and rigid, fixed-temperature lower surface.

3.1 The Temperature Distribution

If the temperature of the lower and the upper boundary surfaces are designated as T_1 and T_0 respectively, then the steady-state temperature distribution can be expressed by

$$(T-T_1)/(T_0-T_1) = 1-Z+N_S(Z-Z^2) \ , \tag{3.1}$$

or alternatively

$$(T-T_0)/(T_1-T_0) = 1-Z'+(-N_S)(Z'-Z'^2), \ Z'=1-Z \ , \tag{3.2}$$

in which N_S is a dimensionless group which is defined as

$$N_S = qL^2/2k(T_0-T_1) \ . \tag{3.3}$$

The parameter, N_S is a non-dimensional term due to internal heat generation.

Inasmuch as (1-Z) represents a linearly varying temperature distribution, (q=0), then the departure of N_S from zero is a measure of the non-linearity introduced by the heat source. The heat source, q, will always be a positive number; therefore, $N_S>0$ must correspond to $T_0>T_1$, while $N_S<0$ will correspond to $T_1>T_0$. Inspection of Eqs. 3.1 and 3.2 reveals that the shapes of the temperature profiles for $N_S>0$ and $N_S<0$ are the same, provided that the former is plotted as a function of Z and the latter is plotted as function of $Z'=1-Z$. The graphical presentation of the temperature distribution is displayed in Fig. 3.

From Fig. 3 it is evident that in the range $0 \leq N_S \leq 1$, the highest temperature in the fluid layer occurs at the lower boundary surface, Z=0. As N_S increases beyond unity, temperatures in excess of that at Z=0 occur within the fluid. Further increases in N_S give rise to corresponding increases in fluid temperature, and the location of the temperature maximum approaches Z=1/2.

The situation is somewhat different for $N_S<0$. In the range $-1 \leq N_S < 0$ the temperature is monotonically increasing with height, and the fluid layer is completly stable. However, for $N_S<-1$ temperatures within the fluid layer exceed that of the upper bounding surface, with the consequence that a heavier fluid is situated above a lighter fluid, and instability becomes possible.

3.2 Stability Criteria

3.2.A From Power Series

The actual computation of stability criteria is carried out in a manner similar to that already described in Section 2.3. The series solution is stated as before, but the parameters Λ and Ω which appear in the recursion relation becomes:

$$\Lambda = a^2 R_E (1-N_S) - a^6, \quad \Omega = 2a^2 R_E N_S \ . \tag{3.4}$$

It is of interest to inquire how the present results calculated by power series method compare with those obtained with other techniques. Only one of the entries in Table II can be specifically compared. For the case of the Bénard problem, (i.e. two rigid-isothermal surfaces and no heat source), the critical external Rayleigh number found analytically

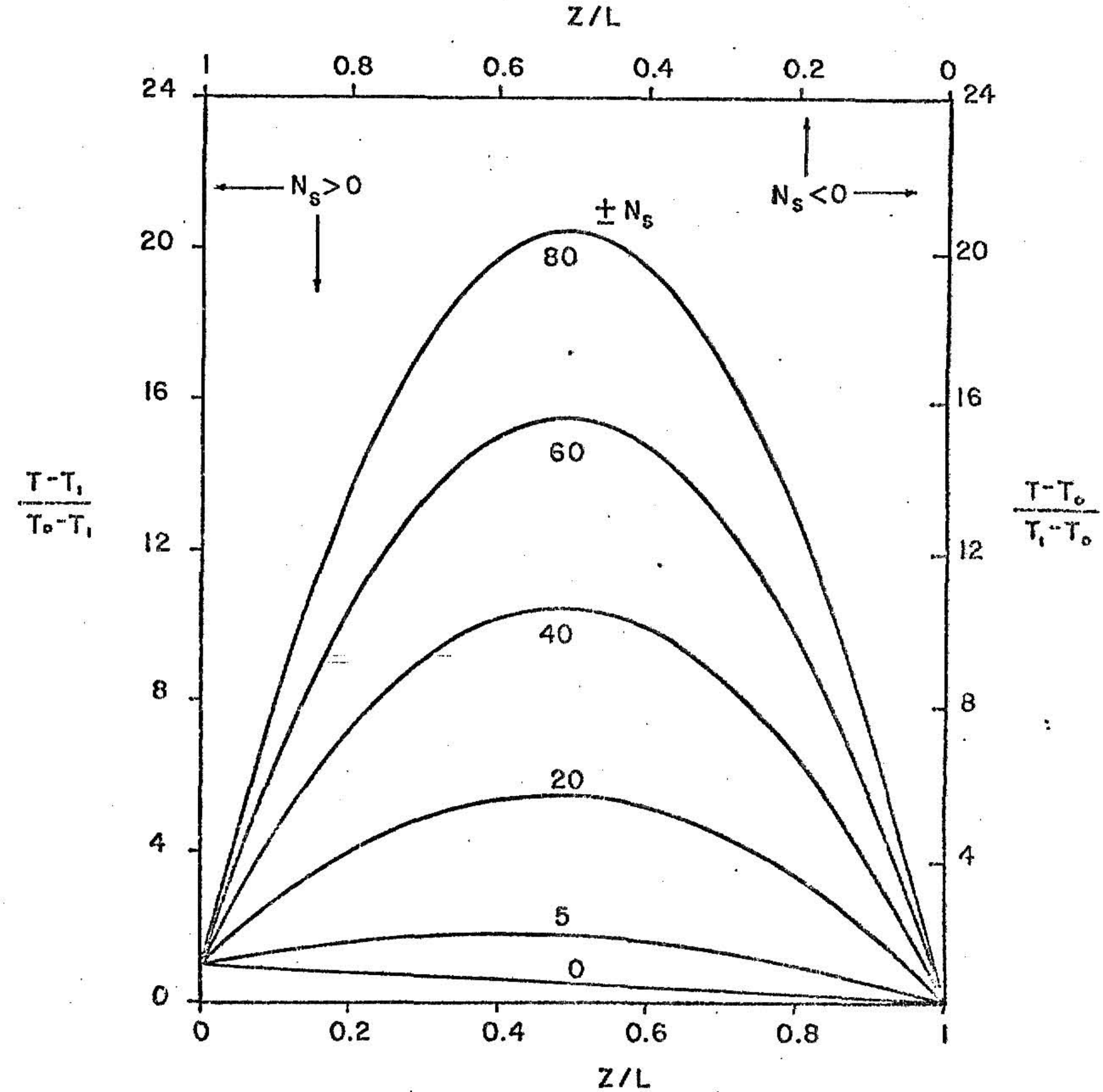


Fig. 3 Temperature Profiles in a Fluid Layer Resulting from Uniform Internal Heat Sources.

is 1707.762. While the power series solution used in this thesis yields a result of 1707.765. (For details of the analytical solution, see Appendix D.)

The results obtained for case I (two surfaces rigid and isothermal) are shown in Tables II and III for $N_S>0$ and $N_S<0$ respectively. The first two columns are results by Sparrow, et al. (33) (The critical external Rayleigh number's range is between 10^3 and 10^5.) The other two columns are results from this study. From Table III, it is observed that as N_S approaches -1, the external Rayleigh Number approaches infinity. This result is expected since, from Fig. 3, it is found that $N_S = -1$ corresponds to a completely stable solution.

An expression for the critical Rayleigh number has been derived by Baker, et al. (1). Conceptually, they divided the fluid layer at the plane of maximum average temperature into two sublayers, one with an insulated lower boundary and one with insulated upper boundary, as shown in Fig. 4. They subsequently used analytical expressions for the temperature distribution in these sublayers to develop the correlation.

For pure conduction heat transfer, it follows:

$$\hat{T} - T_1 = \frac{q(L-L_o)^2}{2k} , \tag{3.5}$$

and

$$\hat{T} - T_0 = \frac{qL_0^2}{2k} . \tag{3.6}$$

Since $\hat{T}$ must be the same for both Eqs. 3.5 and 3.6,

$$T_1 - T_0 = \frac{qL^2}{2k} (2 \frac{L_0}{L} - 1) . \tag{3.7}$$

Table II. Critical External Rayleigh Numbers for Two Rigid and Isothermal Surfaces

	$N_S > 0$			
	Sparrow, et al. (33)		This Study	
N_S	a	R_{EC}	a	R_{EC} *
0	3.12	1707.765	3.12	1707.765
0.1	3.12	1707.636	3.12	1707.636
0.25	3.12	1706.953	3.12	1706.953
0.5	3.12	1704.453	3.12	1706.453
1.0	3.13	1694.953	3.13	1694.953
1.5	3.14	1679.407	3.14	1679.407
2.5	3.18	1632.886	3.18	1632.886
3.0	----	--------	3.20	1630.431
5.0	3.30	1462.863	3.30	1462.863
7.5	3.43	1279.267	3.43	1279.268
10.0	3.53	1118.430	3.53	1118.430
15.0	3.68	878.339	3.66	878.303
20.0	3.74	717.201	3.74	717.201
30.0	3.82	521.403	3.82	521.403
40.0	3.86	408.558	3.86	408.558
70.0	3.92	247.075	3.92	247.075
100.0	3.94	176.936	3.94	176.936
200.0	----	-------	3.97	90.855
210.0	----	-------	3.97	86.639
α	4.00	-------	4.00	------

*For all power series results, uncertainty is in the fourth place for the critical Rayleigh number and in the third place for "a".

Table III. Critical External Rayleigh Numbers for Two Rigid and Isothermal Surfaces.

	$N_S<0$			
	Sparrow, et al. (33)		This Study	
$-N_S$	a	R_{EC}	a	R^*_{EC}
1	----	---------	------	
1.1	----	---------	10.001	3,500,505,265.535
1.2	----	---------	10.00	242,937,954.626
1.3	----	---------	10.00	21,815,951.328
1.5	----	---------	7.95	7,474,917.681
1.9	----	---------	8.65	416,216.478
2.1	----	---------	7.75	227,718.976
2.3	----	---------	7.20	142,127.694
2.5	----	---------	6.40	99,915.567
2.7	----	---------	6.40	70,758.144
2.9	----	---------	6.21	53,860.144
3.0	6.13	47,673.615	6.10	47,724.903
5.0	5.10	11,527.500	5.10	11,527.520
7.5	4.73	5,172.813	4.73	5,172.813
10.0	4.59	3,215.211	4.55	3,215.226
15.0	4.38	1,783.818	4.37	1,783.811
20.0	4.28	1,221.732	4.28	1,221.732
30.0	4.18	744.170	4.19	744.168
40.0	4.14	533.579	4.14	533.579
70.0	4.08	287.819	4.08	287.819
100.0	4.06	196.891	4.05	196.891
200.0	----	-------	4.03	95.842
210.0	----	-------	4.03	91.161
220.0	----	-------	4.02	86.918
230.0	----	-------	4.02	83.051
α	4.00	-------	4.00	----------------

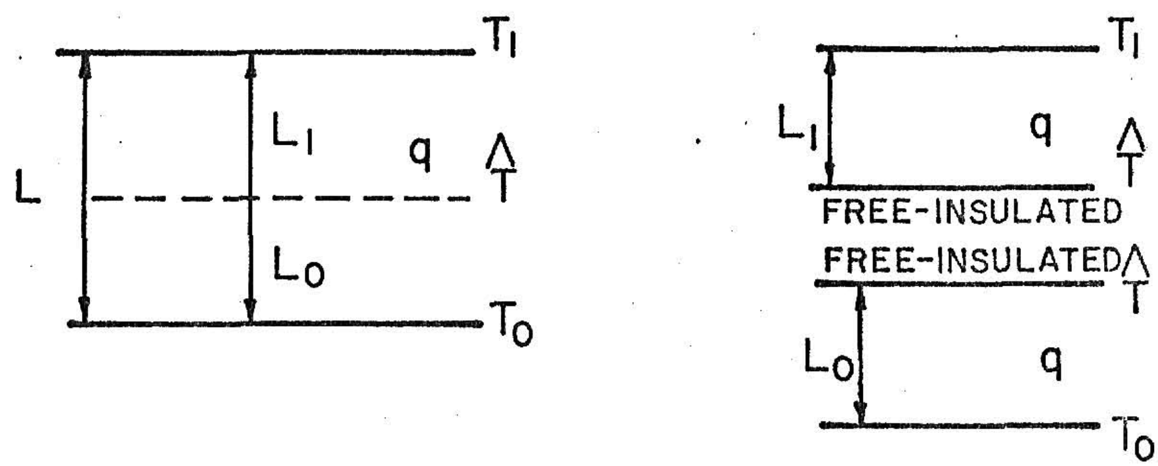


Fig. 4 Schematic of Horizontal Layers with Internal Energy Sources. [After Baker, et al (1).]

This may be changed into a form,

$$\frac{2k(T_1-T_0)}{qL^2} = \frac{R_E}{32R_{KG}} = 2\,\frac{L_0}{L} - 1 \;, \tag{3.8}$$

where the internal Rayleigh Number of the upper layer is expressed as,

$$R_{KGL1} = \frac{g\beta L_1^5\, q}{64\alpha\nu k} = R_{KG}\,(1 - \frac{L_0}{L})^5 \;. \tag{3.9}$$

Following Kulacki[16], the R_{KGL1}, which is the critical internal Rayleigh Number for a layer with an insulated free lower boundary layer, was set equal to 25.7899. Substituting Eq. 3.8 and the critical value of R_{KGL1}, into Eq. 3.9, it follows that

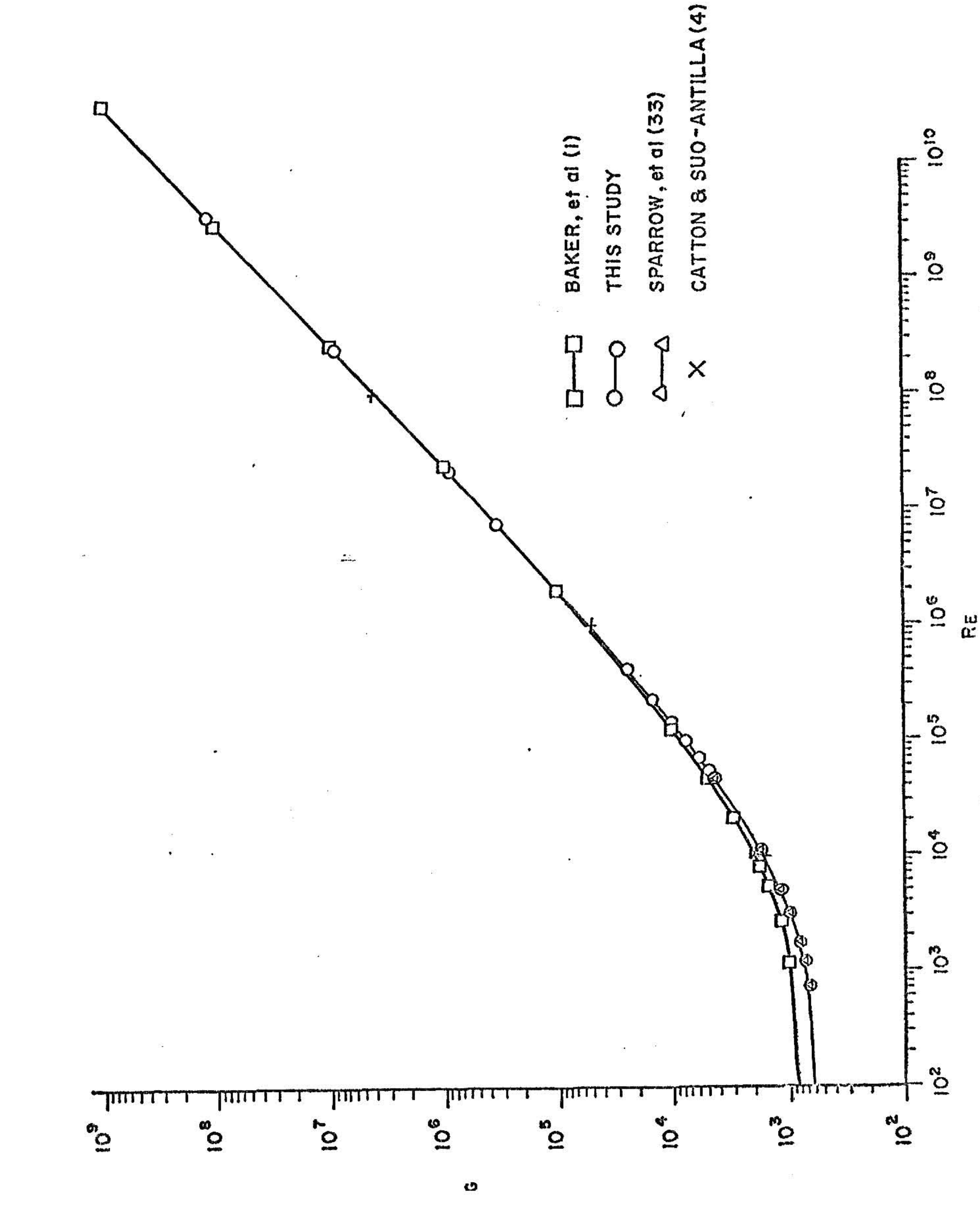
$$R_E = 32\; R_{KG}\,(1 - \frac{3.8311}{R_{KG}^{\;0.2}}) \;, \tag{3.10}$$

The comparison between the theoretical critical Rayleigh number as listed in Table II and III, and the empirical prediction of Eq. 3.10 is shown in Fig. 5. Agreement between the theoretical and empirical predictions is very good for large R_E. However, for $R_E<10^6$, there is some variation between the empirical correlation and the theoretical prediction. From the points calculated by the power series method, the least squares fit expression is found to be:

$$R_E = 32.46\; R_{KG}(1 - \frac{4.294}{R_{KG}^{\;0.2}}) \;. \tag{3.11}$$

3.2.B Upper Surface is Free and Both Surfaces are Isothermal

The Rayleigh Numbers marking the onset of instability calculated from the power series for this case are presented graphically in Fig. 6.


BAKER, et al (1)
THIS STUDY
SPARROW, et al (33)
CATTON & SUO-ANTILLA (4)
G
RE
10^2
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10^{10}

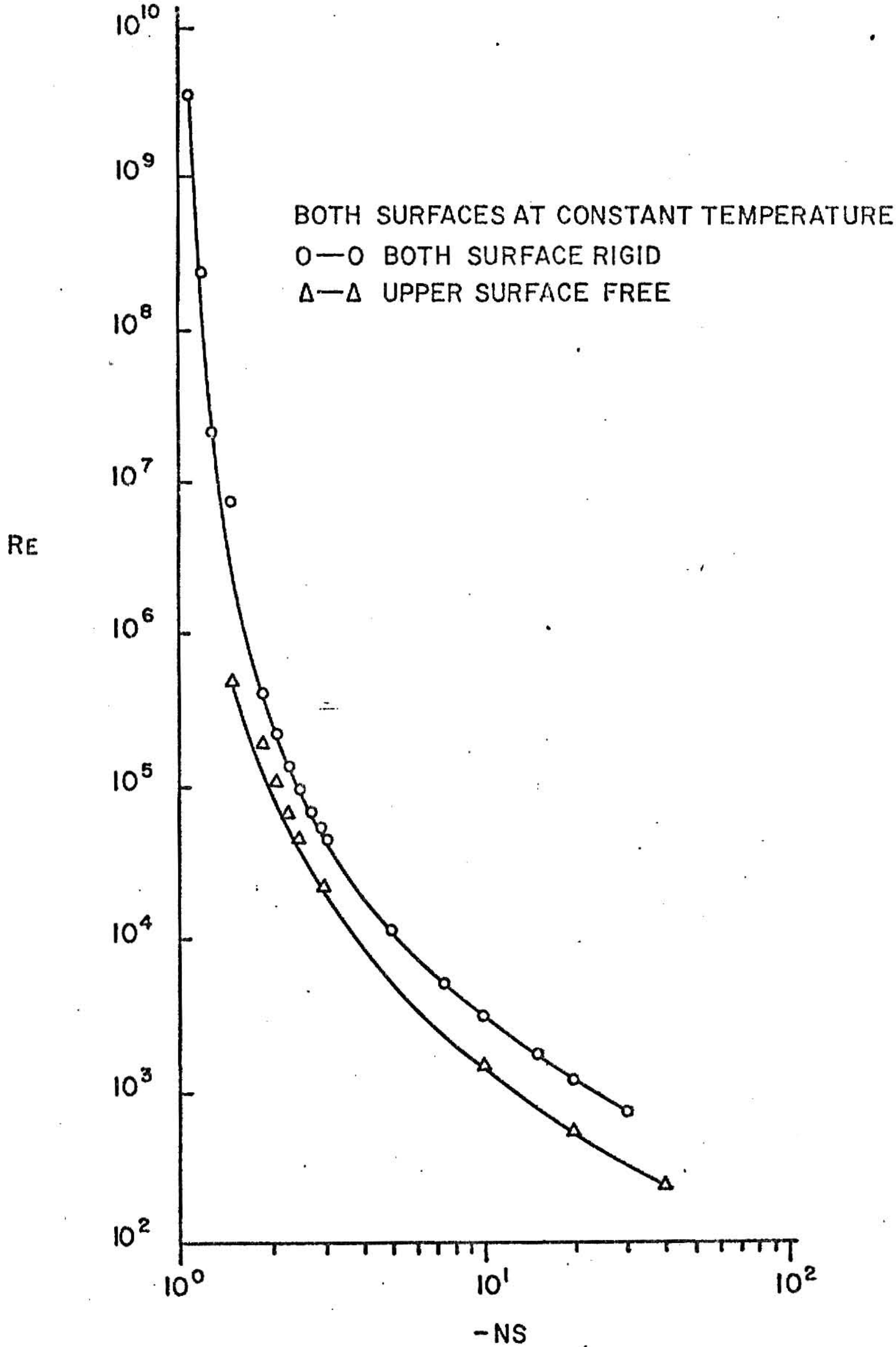

Fig. 6 Critical Rayleigh Number with Constant Temperature at Both Boundaries.

A parallel presentation of the results is made in Table IV, wherein the N_S values corresponding to the critical Rayleigh Numbers are also listed. Attention should be directed to the results of Fig. 6. From an inspection of the figure, it is seen that for a given N_S, the critical Rayleigh Number for an upper rigid surface is larger than that of an upper free surface. This agrees with the intuitive feelings that a layer with a rigid surface is more stable than a layer with a free surface. That this result is also true for a fluid layer without heat sources is shown by Sparrow[33]. There is a marked difference in the numerical values of the critical Rayleigh Numbers for these two cases; however, the ratio of these two critical numbers is relatively constant and equal to about 2.2.

3.2.C Lower Boundary is Insulated

If the lower boundary of a stable layer is free and insulated, there is only conduction present inside the layer. From the conduction equation under such thermal boundary conditions, it follows:

$$T_1 - T_0 = \frac{qL^2}{2k} \tag{3.12}$$

Substituting this relation into Eq. 3.3, it is obtained that N_S=1. This means that the critical Rayleigh Number of such a layer has physical meaning only if N_S=1. Using the power series solution with suitable boundary conditions, the critical internal Rayleigh Number for a layer with a free and isothermal upper surface is 13.947, while for a layer with the upper surface rigid, the critical internal Rayleigh Number is 25.810. Kulacki has used another method (see Reference (32)) to arrive at similar results. He found that the critical internal Raleigh numbers for each case are 13.559 and 25.789 respectively. His method supposes

Table IV. Critical Rayleigh Numbers for Two Isothermal Surfaces (lower surface rigid).

$-N_S$	upper surface rigid	upper surface free
100.0	196.891	88.027
40.0	533.579	239.306
20.0	1,221.732	552.602
10.0	3,215.226	1,485.624
3.0	47,673.615	22,371.853
2.5	99,915.567	45,336.103
2.3	142,127.694	66,840.103
2.1	227,719.976	107,554.322
1.9	416,216.479	194,527.491
1.5	7,474,917.681	498,072.094
1.2	242,937,954.626	----------
1.1	3,500,505,265.535	----------
1.0	----------	----------

that there are three trial solutions that satisfy the boundary conditions at Z=0. Then, the linear combination of the trial solutions will satisfy the boundary condition at Z=1. It is verified, therefore, by two techniques, that a layer with a rigid upper surface is more stable than one with the upper surface free.

4.0 CONCLUSIONS

Hydrodynamic instability in a fluid layer with a uniform volumetric energy source has been shown to depend on both the hydrodynamic and the thermal constraints imposed at its horizontal boundaries. If the external Rayleigh number under specified conditions is less than the critical Rayleigh number, the layer is stable, and within it there is only conduction. The importance of this study is two fold. First, the range of critical Rayleigh Numbers investigated has been extended to higher internal Rayleigh Numbers approximating the postulated post accident heat removal scenerios in an LMFBR. Second, the expression of Baker, et al. (1) has been verified for critical internal Rayleigh Numbers greater than 10^5. In cases I and II, (lower surface rigid and isothermal, upper surface isothermal and either rigid or free), the critical external Rayleigh Number decreases monotonically with increasing internal heat generation as expressed by N_S (when it is positive) or with decreasing N_S (when it is negative). In cases III or IV, where the lower boundary is free and insulated rather than isothermal, the critical Rayleigh Number is less.

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APPENDIX A

Derivation of the Governing Perturbation Equation.

This appendix has been included to illustrate the derivation of the governing Eq. 2.25. From Section 2.0, it was found that three equations existed as follows:

$$W^*(x,y,z,t) = F(z)G(x,y)e^{\sigma t} \; , \tag{A.1}$$

$$\nabla_1^2 \, G(x,y) + (a/L)^2 G(x,y) = 0 \; , \tag{A.2}$$

$$\left[\frac{\partial}{\partial t} - \nu\nabla^2\right]\left[\frac{\partial}{\partial t} - \alpha\nabla^2\right]\nabla^2 W^* + g\beta\left(A - \frac{q}{k}\, z\right)\nabla_1^2 \, W^* = 0 \; , \tag{A.3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \; , \tag{A.4}$$

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \; , \tag{A.5}$$

$$\nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2} \; . \tag{A.6}$$

From Eq. A.2, it follows:

$$\nabla_1^2 \, G(x,y) = -\frac{a^2}{L^2} \, G(x,y) \; . \tag{A.7}$$

Another expression for Eq. A.1 can be obtained by multiplying both sides by the operator ∇^2, it follows:

$$\nabla^2 W^*(x,y,z,t) = \left(\nabla_1^2 + \frac{\partial^2}{\partial z^2}\right)F(z)G(x,y)e^{\sigma t}$$

$$= -F(z)e^{\sigma t}\,\frac{a^2}{L^2}\, G(x,y) + F^2(z)G(x,y)e^{\sigma t} \; . \tag{A.8}$$

Let the symbol Δ_1, be equal to the right hand side of Eq. A.8.

Then

$$(\frac{\partial}{\partial t} - \alpha\nabla^2)\Delta_1 = \frac{\partial\Delta_1}{\partial t} - \alpha(\nabla_1^2 + \frac{\partial^2}{\partial_z{}^2})\Delta_1$$

$$= -\frac{a^2}{L^2}\sigma F(z)G(x,y)e^{\sigma t} + \sigma e^{\sigma t}F^{(2)}(z)\ G(x,y)$$

$$-\frac{\alpha a^4}{L^4}F(z)G(x,y)e^{\sigma t} + \frac{a^2\alpha}{L^2}F^{(2)}(z)G(x,y)e^{\sigma t}$$

$$+\frac{a^2\alpha}{L^2}F^{(2)}(z)G(x,y)e^{\sigma t} - \alpha F^{(4)}G(x,y)e^{\sigma t}\ . \tag{A.9}$$

Let the symbol Δ_2 be equal to the right hand side of Eq. A.9. Then

$$[\frac{\partial}{\partial t} - \nu\nabla^2][\frac{\partial}{\partial t} - \nu\nabla^2]\nabla^2 W^*(x,y,z) = [\frac{\partial}{\partial t} - \nu\nabla^2]\Delta_2$$

$$= \alpha\nu F^{(6)}(z)G(x,y)e^{\sigma t} - \frac{3\alpha\nu a^2}{L^2}F^{(4)}(z)G(x,y)e^{\sigma t}$$

$$-\sigma\alpha F^{(4)}(z)G(x,y)e^{\sigma t} - \sigma\nu F^{(4)}(z)G(x,y)e^{\sigma t}$$

$$+\frac{3\alpha\nu a^4}{L^4}F^{(2)}(z)G(x,y)e^{\sigma t} + \sigma e^{\sigma t}\frac{a^2\alpha\nu}{L^2}F^2(z)G(x,y)$$

$$+\frac{2a^2\alpha}{L^2}F^{(2)}(z)G(x,y)e^{\sigma t} + \frac{a^2\alpha}{L^2}F^{(2)}(z)G(x,y)e^{\sigma t}$$

$$+\sigma^2 e^{\sigma t}F^{(2)}(z)G(x,y) - \frac{\alpha\nu a^6}{L^6}F(z)G(x,y)e^{\sigma t}$$

$$-\sigma^2\frac{a^2}{L^2}F(z)G(x,y)e^{\sigma t} - \sigma\frac{\nu a^4}{L^4}F(z)G(x,y)e^{\sigma t}\ , \tag{A.10}$$

Substituting Eq. A.10, into Eq. 2.22, it is obtained,

$$\alpha\nu F^{(6)}(z)G(x,y)e^{\sigma t} - \frac{3a^2\alpha\nu}{L^2} F^{(4)}(z)G(x,y)e^{\sigma t}$$

$$+ \frac{3\alpha\nu a^4}{L^4} F^{(2)}(z)G(x,y)e^{\sigma t} + (- \frac{a^2 Aq\beta}{L^2} - \frac{\alpha\nu a^6}{L^6} + \frac{a^2 qg\beta}{L^2 k} z)F(z)G(x,y)e^{\sigma t}$$

$$+ \sigma G(x,y)e^{\sigma t}(\frac{a^2\alpha\nu}{L^2} F^{(2)}(z) + \frac{2a^2\alpha}{L^2} F^{(2)}(z) + \frac{a^2\nu}{L^2} F^2(z) - \frac{\alpha_a{}^4}{L^4} F(z))$$

$$+ \sigma^2 G(x,y)e^{\sigma t} (F^{(2)}(z) + \frac{a^2}{L^2} F(z)) = 0 . \tag{A.11}$$

Let $Z = \frac{z}{L}$, then

$$F^{(6)}(z) = \frac{F^{(6)}(Z)}{L^6} , F^{(4)}(z) = \frac{F^{(4)}(Z)}{L^4}$$

$$F^{(2)}(z) = \frac{F^{(2)}(Z)}{L^2} . \tag{A.12}$$

Substituting these above relations into Eq. A.11 and dividing by L^6 on both sides, it is obtained,

$$F^{(6)}(Z) - 3a^2 \frac{d^4F(Z)}{dZ^4} + 3a^4 \frac{d^2F(Z)}{dZ^2} + (\Lambda+\Omega Z)F(Z) + \sigma^2(L^4F^{(2)}(Z)$$

$$+ L^4a^2F(Z)) + \sigma(a^2\alpha\nu L^2F^{(2)}(Z) + 2a^2\alpha F^{(2)}(Z)L^2$$

$$+ L^2\nu a^2F^{(2)}(Z) - \nu a^4L^2F(Z)) = 0 , \tag{A.13}$$

where

$$\Lambda = -(g\beta L^4Aa^2/\alpha\nu)-a^6 ,$$

$$\Omega = g\beta L^5qa^2/\alpha\nu k ,$$

$$\psi_1 = L^4F^{(2)}(Z) + L^4a^2F(Z) ,$$

$$\psi_2 = a^2\alpha\nu L^2F^{(2)}(Z) + 2a^2\alpha F^{(2)}(Z)L^2$$

$$+ L^2\nu a^2F^{(2)}(Z) - \nu a^4L^2F(Z) .$$

APPENDIX B

Computer Program for Two Rigid and Two Isothermal Surfaces

THIS BOOK IS OF POOR LEGIBILITY DUE TO LIGHT PRINTING THROUGH OUT IT'S ENTIRETY.

THIS IS AS RECEIVED FROM THE CUSTOMER.

```
      THIS PROGRAM IS TO CALCULATE THE CRITICAL EXTERNAL RAYLEIGH
      NUMBER USING FIFTY ITEMS IN POWER SERIES SOLUTION. THE BOUNDARY
      CONDITIONS ARE : THE TOP AND BOTTOM SURFACE ARE RIGID AND KEPT
      AT CONSTANT TEMPERATURE. FOR A SAMPLE.NS EQUAL TO 100.01

      G(4,4) IS THE MATRIX OF  THE COEFFICIENTS OF C2,C3,C4,C5

      NS=SL2/2K(T1-T2)       S: VOLUME SOURCE. L: LAYER THICKNESS.
      K: CONDUCTIVITY OF LAYER, T1: BOTTOM SURFACE TEMPERATURE.
      T2: TOP SURFACE TEMPRATURE

      B(J,I): IS THE COEFFICIENT OF THE POWER SERIES SOLUTION,AT Z=1.0

      B1(J,I): IS THE FIRST DERIVATIVE COEFFICIENT OF THE POWER
      SERIES SOLUTION.

      B2(J,I): IS THE ITEM THAT SUITABLE TO THE THERMAL BOUNDARY
      CONDITION,AT Z=1.0

      R: EXTERNAL RAYLEIGH NUMBER.

      A: IS THE CONSTANT FROM THE SEPARATION OF VARIABLES.

      D: THE DETERMINANT OF G(4,4).


      IMPLICIT  REAL*8(A-H,O-Z)
    REAL*8 G(4,4),B(75,5),B1(75,5),B2(75,5),SUM(75,5),SUMA(75,5)
  2  ,SUMB(75,5),R(20),D(20),A(20),Y(20),X(20),NS(10),N(100)
      B(6,2)=0.D1
      B(6,5)=0.D1
     B(7,2)=0.D1
      B(7,4)=0.D1
      N(1)=1.D0
      DO 1 I=2,50
     N(I)=1.D0*I
 1    CONTINUE
      R(1)=0.D1
      R(2)=2.D2
      DO 100 L=1,3
     IF(L.EQ.1) NS(L)=1.D2
      IF(L.EQ.2) GO TO 44
      DO 99 K=1,10
     A(K)=3.9000+1.D-2*K
 24   G(1,1)=-4.D0*A(K)**2
      G(1,2)=0.D1
      G(1,3)=2.4D1
      G(1,4)=0.D1
      DO 88 M=1,20
      Y(M)=2.D0*A(K)**2*R(M)*NS(L)
      X(M)=A(K)**2*R(M)*(1.D0-NS(L))-A(K)**6
      DO 2 I=2,5
      DO 3 J=1,50
      IF(J.EQ.I) B(J,I)=1.D0
      IF(J.GT.7) GO TO 31
     IF(J.EQ.I+4) B(J,I)=1.D0/(N(J)*N(J-1))*3.D0*A(K)**2*B(J-2,I)-
  2   1.D0/(N(J)*N(J-1)*N(J-2)*N(J-3))*3.D0*A(K)**4*B(J-4,I)
```

```
      IF(J.EQ.6.AND.I.EQ.4) B(J,I)=1.D0/1.D1*A(K)**2*B(4,4)
      IF(J.EQ.7.AND.I.EQ.5) B(J,I)=1.D0/1.4D1*A(K)**2*B(5,5)
      IF(J.EQ.1.OR.J.EQ.6.OR.J.EQ.7) GO TO 3
      B(J,I)=0.D1
      GO TO 3
31    B(J,I)=1.D0/(N(J)*N(J-1))*3.D0*A(K)**2*B(J-2,I)-1.D0/(N(J)*
     2 N(J-1)*N(J-2)*N(J-3))*3.D0*A(K)**4*B(J-4,I)-1.D0/(N(J)*N(J-1)*
     3 N(J-2)*N(J-3)*N(J-4)*N(J-5))*(X(M)*B(J-6,I)+Y(M)*B(J-7,I))
3     CONTINUE
2     CONTINUE
      DO 11 I=2,5
      DO 12 J=1,50
      IF(J.EQ.1) GO TO 12
      SUM(1,I)=B(1,I)
      SUM(J,I)=SUM(J-1,I)+B(J,I)
12    CONTINUE
11    CONTINUE
      G(2,1)=SUM(50,2)
      G(2,2)=SUM(50,3)
      G(2,3)=SUM(50,4)
      G(2,4)=SUM(50,5)
      DO 4 I=2,5
      DO 5 J=1,50
      B1(J,I)=J*B(J,I)
      IF(J.EQ.1) GO TO 5
      SUMA(1,I)=B1(1,I)
      SUMA(J,I)=SUMA(J-1,I)+B1(J,I)
5     CONTINUE
4     CONTINUE
      G(3,1)=SUMA(50,2)
      G(3,2)=SUMA(50,3)
      G(3,3)=SUMA(50,4)
      G(3,4)=SUMA(50,5)
      DO 7 I=2,5
      DO 8 J=1,50
      IF(J.EQ.1) B2(J,I)=A(K)**4*B(J,I)
      IF(J.LE.3) B2(J,I)=A(K)**4*B(J,I)-2.D0*A(K)**2*J*(J-1)*B(J,I)
      IF(J.GT.3) GO TO 9
9     B2(J,I)=J*(J-1)*(J-2)*(J-3)*B(J,I)-2.D0*A(K)**2*J*(J-1)*B(J,I)+
     2 A(K)**4*B(J,I)
      IF(J.EQ.1) GO TO 8
      SUMB(1,I)=B2(1,I)
      SUMB(J,I)=SUMB(J-1,I)+B2(J,I)
8     CONTINUE
7     CONTINUE
      G(4,1)=SUMB(50,2)
      G(4,2)=SUMB(50,3)
      G(4,3)=SUMB(50,4)
      G(4,4)=SUMB(50,5)
      D(M)=DETERM(G,4)

      FOLLOWING, IT IS A ITERATION METHOD THAT TRY TO GET THE EXTERNAL
      RAYLEIGH NUMBER THAT MAKES THE DETERMANT OF MATRIX LESS THAN
      1.0E-6.

      IF(DABS(D(M))-1.D-6)50,50,51
```

```
  51      IF(M-1)88,88,53
  53      IF(M-2)88,55,54
  55      S=(D(1)-D(2))/(R(1)-R(2))
          R(3)=(S*R(1)-D(1))/S
          GO TO 88
  54      IF(DABS(D(M-1))-DABS(D(M-2)))650,651,651
 651      IF(R(M).EQ.R(M-2))  GO TO 50
          IF(D(M).EQ.D(M-2)) GO TO 50
          S=(D(M)-D(M-2))/(R(M)-R(M-2))
          R(M+1)=(S*R(M-1)-D(M-1))/S
          GO TO 88
 650      IF(R(M).EQ.R(M-1)) GO TO 50
          IF(D(M).EQ.D(M-1)) GO TO 50
          S=(D(M)-D(M-1))/(R(M)-R(M-1))
          R(M+1)=(S*R(M-1)-D(M-1))/S
          GO TO 88
  88      CONTINUE
  50      WRITE(6,90)M,NS(L),A(K),R(M),D(M)
  90      FORMAT(1X,'M=',I4,'NS=',E24.16,2X,'A(K)=',F9.6,2X,'RALGH
         2 NUMBER IS',E24.16,1X,'D(M)=',E24.16)
  99      CONTINUE
 100      CONTINUE
  44      STOP
          END
```

```
      FUNCTION DETERM (ARRAY,NORDER)
        IMPLICIT  REAL*8(A-H,O-Z)
      REAL*8 ARRAY(4,4)
  100 DETERM=1.
      DO 500 K=1,NORDER

     INTERCHANGE COLUMN IF DIAGONAL ELEMENT IS ZERO

      IF(ARRAY(K,K)) 350,150,350
  150 DO 200 J=K,NORDER
      IF(ARRAY(K,J)) 250,200,250
  200 CONTINUE
      DETERM=0.
      GO TO 550
  250 DO 300 I=K,NORDER
      SAVE=ARRAY(I,J)
      ARRAY(I,J)=ARRAY(I,K)
  300 ARRAY(I,K)=SAVE
      DETERM=-DETERM

        SUBTRACT ROW K FROM LOWER ROWS TO GET DIAGONAL MATRIX

  350 DETERM=DETERM*ARRAY(K,K)
      IF(K-NORDER)400,500,500
  400 K1=K+1
      DO 450 I=K1,NORDER
      DO 450 J=K1,NORDER
  450 ARRAY(I,J)=ARRAY(I,J)- ARRAY(I,K)*ARRAY(K,J)/ARRAY(K,K)
  500 CONTINUE
  550 RETURN
      END
```

APPENDIX C

Variation Solution

This appendix has been included to demonstrate use of the variational method to find the critical external Rayleigh number.

Following reference[6,27], it may be shown that the governing equation, Eq. 2.26, can be changed to the form,

$$HF(Z) = R_E MF(Z) \ , \tag{C.1}$$

$$H = (D^2 - a^2)^3$$

$$M = (a^2(N_S-1) - 2a^2 N_s Z) \ ,$$

where R_E is the external Rayleigh Number and the eigenvalue of the equation. By the variation principle, a relation can be formed;

$$R_E = \frac{<F^+,HF>}{<F^+,MF>} \ , \tag{C.2}$$

wherein $F^+(Z)$ is the solution to the adjoint equation becuase F(Z), H, and M are self-adjoint. It follows that

$$R_E = \frac{\int_0^1 FHFdZ}{\int_0^1 FMFdZ} \ . \tag{C.3}$$

The external Rayleigh Number R_E is, thus, a function of N_S and "a". For example, if N_S=0, a trial function of the form

$$F(Z) = a_1 Z^6 + a_2 Z^5 + a_3 Z^4 + a_4 Z^3 + a_5 Z^2$$

may be chosen. The polynomical coefficients can be determined from the boundary conditions. Substituting this polynomial into Eq. 3.13, it is found that a=3.12 and that R_E is 1688.15. For comparison, the power series solution yields a value of 1707.765 for the critical external Rayleigh Number. If F and F^+ are varied by amounts δF and δF^+, then R_E will vary by amount ΔR_E. Therefore,

$$R_E + \Delta R_E = \frac{\int (F^+ + \delta F^+) H (F + \delta F) dZ}{\int (F^+ + \delta F^+) M (F + \delta F) dZ} . \tag{C.4}$$

Substracting R_E from both sides (where R_E is in the following form),

$$R_E \ \frac{\int (F^+ + \delta F^+) M (F + \delta F) dZ}{\int (F^+ + \delta F^+) M (F + \delta F) dZ} , \tag{C.5}$$

An explicit expression for ΔR_E can be obtained,

$$\Delta R_E = \frac{\int \delta F^+ H \delta F dZ - R_E \int \delta F^+ M \delta F dZ + \int (F^+ + \delta F) H (F + \delta F) dZ + \int F^+ (H \delta F - R_E M \delta F) dZ}{\int (F^+ + \delta F^+) M (F + \delta F) dZ} . \tag{C.6}$$

From the definition of an adjoint operator, it follows:

$$\int (F^+ + \delta F^+)(HF - RMF) dZ = 0 , \tag{C.7}$$

$$\int F^+ (H \delta F) dZ = \int \delta F (H^+ F^+) dZ , \tag{C.8}$$

$$\int F^+ (M \delta F) dZ = \int \delta F (M^+ F^+) dZ . \tag{C.9}$$

Hence the Eq. 3.16 can be simplified as

$$\Delta R_E = \frac{\int (\delta F^+ H \delta F) dZ - R_E \int M (\delta F)^2 dZ}{\int (F^+ + \delta F^+) M (F + \delta F) dZ} . \tag{C.10}$$

The error in R_E calculated from the variation method depends on the many functions shown in Eq. C.10. The accuracy mainly depends on the trial function. If the trial function is close to the real solution, the results will be good. The solution of the governing equation is so complicated that several trials are necessary to select the appropriate trial function.

APPENDIX D

Bénard Problem Solved by Analytical Method[5]

Following the derivation in Chapter 2, but with the origin located in the middle of the fluid layer, (i.e. $Z'=Z - 1/2$), the governing equation suitable for the Bénard problem ($q=0$) is as follows:

$$(D^2 - a^2)^3 F(Z') = -R_E a^2 F(Z') , \tag{D.1}$$

Boundary equations for two rigid and isothermal surfaces can be expressed as follows:

$$F(Z') = DF(Z') = 0 , \quad \text{at } Z' = \pm 1/2 , \tag{D.2}$$

$$(D^2 - a^2) F(Z') = 0 , \quad \text{at } Z' = \pm 1/2 . \tag{D.3}$$

The solution of Eq. D.1 can be expressed in such a form:

$$F(Z') = Ae^{dZ'} , \tag{D.4}$$

Substituting Eq. D.4 into Eq. D.1, it follows:

$$(d^2 - a^2)^3 = -R_E a^2 \tag{D.5}$$

Let

$$R_E a^2 = \tau^2 a^6 . \tag{D.6}$$

Then, it follows:

$$d^6 - 3a^2 d^4 + 3a^4 d^2 - a^6 = -\tau^3 a^6 \tag{D.7}$$

There will be six roots suitable to the Eq. D.7. These are:

$$d_1 = ia\sqrt{\tau - 1} = id_o , \tag{D.8}$$

$$d_2 = - ia\sqrt{\tau - 1} = -id_o , \tag{D.9}$$

$$d_3 = a\sqrt{1 + 1/2\ \tau(1+i\sqrt{3})} = d\ , \tag{D.10}$$

$$d_4 = -a\sqrt{1 + 1/2\ \tau(1+i\sqrt{3})} = -d\ , \tag{D.11}$$

$$d_5 = a\sqrt{1 + 1/2\ \tau(1-i\ \sqrt{3})} = d^*\ , \tag{D.12}$$

$$d_6 = -a\sqrt{1 + 1/2\ \tau(1-i\sqrt{3})} = -d^*\ , \tag{D.13}$$

The properties of d can be stated as follows:

$$R_e\{d^2\} = a^2\ (1 + \frac{\tau}{2})\ ,$$

$$I_m\{d^2\} = \frac{\sqrt{3}\ a^2}{2}\ \tau\ ,$$

$$|d^2| = a^2\sqrt{1+\tau+\tau^2}\ ,$$

$$d^2 = a^2\sqrt{1+\tau+\tau^2}\ e^{i\theta}$$

wherein

$$\sin\theta = \sqrt{3}\ \tau/2\sqrt{1+\tau+\tau^2}$$

$$\cos\theta = (1 + \tau/2)/\sqrt{1+\tau+\tau^2}$$

Then,

$$R_e\{d\} = \frac{a}{\sqrt{2}}\ \{(1+\tau+\tau^2)^{1/2} + (1+\ \tau/2)\}^{1/2}\ , \tag{D.14}$$

$$I_m\{d\} = -\ \frac{a}{\sqrt{2}}\ \{(1+\tau+\tau^2)^{1/2} + (1+\ \tau/2)\}^{1/2}\ , \tag{D.15}$$

Therefore, the solution for Eq. D.2 can be stated as:

$$F(Z') = A_o\mathrm{Cos}\ d_oZ' + A\mathrm{Coshd}Z' + A^*\ \mathrm{Coshd}^*Z'\ , \tag{D.16}$$

Then:

$$DF(Z') = -A_o d_o \mathrm{Sin} d_o Z' + Ad\sin dZ' + A^* d^* \mathrm{Sinh} d^* Z' , \tag{D.17}$$

$$(D^2-a^2)^2 F(Z') = A_o(d_o{}^2 + a^2)\ \mathrm{Cos} d_o Z' + A(d^2-a^2)^2\ \mathrm{Cosh} dZ + A^*(d^{*2} - a^2)^2\ \mathrm{Cosh} d^* Z . \tag{D.18}$$

By applying Eq. D.17 and D.18 for $Z' = 1/2$(or $- 1/2$), and dividing the last row by $a^4\tau^2$, it follows:

$$\begin{bmatrix} \cos \frac{1}{2} d_o & \mathrm{Cosh}\, \frac{d}{2} & \mathrm{Cosh}\, \frac{d^*}{2} \\ -d_o \mathrm{Sinh}\, \frac{d_o}{2} & d\mathrm{Sinh}\, \frac{d}{2} & d^*\mathrm{Sinh}\, \frac{d^*}{2} \\ \mathrm{Cos}\, \frac{d_o}{2} & \frac{i\sqrt{3}-1}{2}\ \mathrm{Cosh}\, \frac{d}{2} & -\frac{i\sqrt{3}+1}{2}\ \mathrm{Cosh}\, \frac{d^*}{2} \end{bmatrix} \begin{bmatrix} A_o \\ A \\ A^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{D.19}$$

For a non-trial solution, the determinant of such a matrix must vanish,

$$\begin{vmatrix} 1 & 1 & 1 \\ -d_o \tan \frac{d_o}{2} & d\tanh \frac{d}{2} & d^*\tanh \frac{d^*}{2} \\ 1 & \frac{i\sqrt{3}-1}{2} & \frac{-(i\sqrt{3}+1)}{2} \end{vmatrix} = 0 . \tag{D.20}$$

By simplifying the determinant, it follows:

$$-d_o \tan \frac{1}{2} d_o = \frac{(d_1 + \sqrt{3}\ d_2)\ \mathrm{Sinh}\ d_1 + (\sqrt{3}\ d_1 - d_2)\mathrm{Sin} d_2}{\mathrm{Cosh}\ d_1 + \mathrm{Cos}\ d_2} \tag{D.21}$$

wherein

$$d_o = a\sqrt{\tau-1} ,$$

$$d_1 = \frac{a}{\sqrt{2}} \{(1+\tau+\tau^2)^{1/2} + (1+ \tau/2)\}^{1/2}$$

$$d_2 = \frac{a}{\sqrt{2}} \{(1+\tau+\tau^2)^{1/2} - (1+ \tau/2)\}^{1/2}$$

Then from Eq. D.21 and Eq. D.6, the critical external Rayleigh number can be found. The results are:

$$a = 3.117 , \qquad R_{EC} = 1,707.762$$

APPENDIX E

Convergence of the Power Series Method

The accuracy of critical external Rayleigh number calculated from the power series depends on how many items are retained in the solution of the governing Eq. 2.26. However, after a certain number of terms have been retained, the critical external Rayleigh number does not change appreciably, and further terms result in unnecessary computational time. For an example, if the number of items used to calculate the critical external Rayleigh number at $N_S = 100$ are varied, the critical external Rayleigh number is as listed in Table E.1.

Table E.1 Effect of Varying Power Series Terms on Calculated External Rayleigh Numbers.

Number of Items	a	R_{EC}
30	3.94	180.6232
35	3.95	176.9054
40	3.94	176.9362
45	3.94	176.9360
50	3.94	176.9360
55	3.94	176.9360
65	3.94	176.9360
70	3.94	176.9360
75	3.94	176.9360

Therefore, it is apparent that after forty-five items, the answer does not change at all. It is adequate to use fifty terms in the calculation of the critical Rayleigh Number.

Theoretical Study of Stability in Horizontal Fluid
Layers with Uniform Volumetric Energy Sources

by

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ABSTRACT

The stability in a horizontal fluid layer with a large, uniformly distributed energy source is investigated. For the first time, an analytical solution is obtained for critical external Rayleigh numbers ranging beyond 10^{10}. Specifically, a power series expression is used for the case where both surfaces are rigid and isothermal, but at different temperatures. The results are in good agreement with the empirical solution of Baker et al[1]. The critical Rayleigh numbers are also calculated for the case where the upper surface is free and isothermal and the lower rigid and isothermal. Further calculations of the critical external Rayleigh number are made either for a free and isothermal upper surface or a rigid and isothermal upper surface, both with a lower surface insulated and free. From the results, it is shown that fluid layers with a rigid upper surface are more stable than those with the upper surface free.