APPLICATION OF A GRADIENT TECHNIQUE TO THE TRANSPORTATION PROBLEM

by 683

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1. INTRODUCTION

1.1 The Transportation Problem

Managerial decisions are directed toward choosing the best possible outcome, as measured by costs, profits, or some other suitable criterion, from among the potential courses of action. While numerous alternatives may be available, the level of achievement is limited by the necessity of meeting certain prescribed conditions. A traffic manager may desire, for example, to schedule freight shipments in a manner which will insure the movement of goods at lowest total cost. The goods are usually available at various points (depots) for delivery to prescribed destinations (demand points). For a given problem, there generally exist many different schedules which meet the demands at varying levels of total costs. The traffic manager is therefore concerned with devising some method for selecting a schedule with least cost. This transportation problem exists in practically all industries.

A typical transportation problem (24, 39, 47) is shown in Fig. 1. There are s factories (sources) manufacturing items of a particular commodity at levels $W_1, W_2, \ldots, W_s$ and there are N sinks (demand points) consuming the item at levels $D_1, D_2, \ldots, D_N$. Let θ_{in} represent the number of units of the resource sent from the ith origin to the nth demand point and $F_{in}(\theta_{in})$ be the cost incurred by this operation. It may be a linear or a nonlinear cost function. The problem is to determine θ_{in}, $i = 1,2, \ldots, s$; $n = 1,2, \ldots, N$, so as to minimize the total cost of transportation.

$$C_{sN} = \sum_{n=1}^{N} \sum_{i=1}^{s} F_{in}(\theta_{in}), \tag{1.1}$$

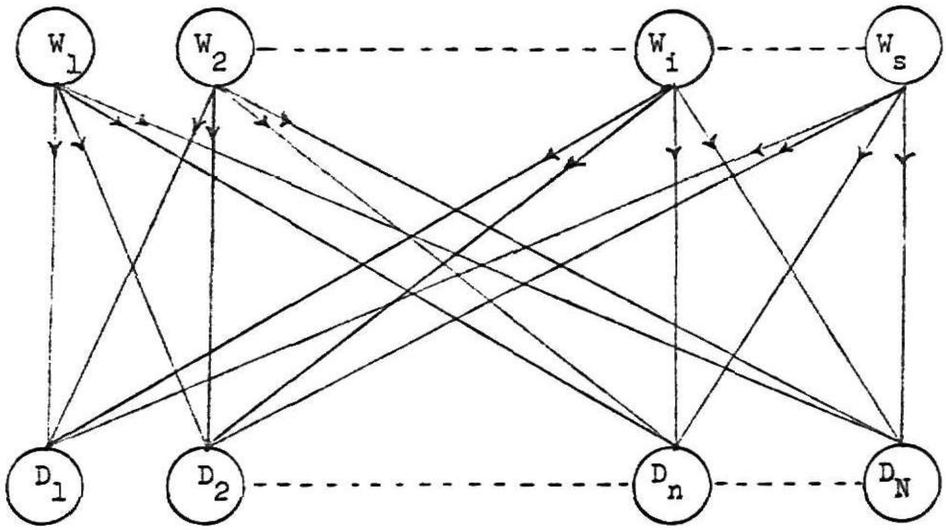


Fig. 1 The direct shipment (transportation) problem

subject to the constraints

(i) Non-negativity constraint

$$\theta_{in} \geq 0, \quad i = 1,2, \ldots, s; \quad n = 1,2, \ldots, N \tag{1.2}$$

(ii)
$$\sum_{n=1}^{N} \theta_{in} = W_i, \quad i = 1,2, \ldots, s, \tag{1.3}$$

(iii)
$$\sum_{i=1}^{s} \theta_{in} = D_n, \quad n = 1,2, \ldots, N. \tag{1.4}$$

The feasibility of the problem can be assured if

$$W_i \geq 0, \quad D_n \geq 0, \quad \text{and} \quad \sum_{i=1}^{s} W_i = \sum_{n=1}^{N} D_n . \tag{1.5}$$

In the above formulation, the cost of transporting the commodity from origins to destinations is assumed known and is also assumed to be independent of the number of units moved. Constraint (ii) implies that the supply or product of any one depot (origin) serves equally well to satisfy the demands of any destination (consuming center). Resources and products are homogeneous. The supplies of resources at various depots and the demands of the various destinations are known. Constraints (ii) and (iii) imply that the total demand is equal to the total supply. In practice it is possible to equate demand and supply by including a dummy origin or destination.

1.2 The Transhipment Problem and the Transportation Problem

In the transportation problem only direct shipments of resources to their destinations are considered. In addition to direct shipment, the extended transportation problem allows one to ship from source to source and from destination to destination. This direct extension of the transportation problem is called a transhipment problem (13).

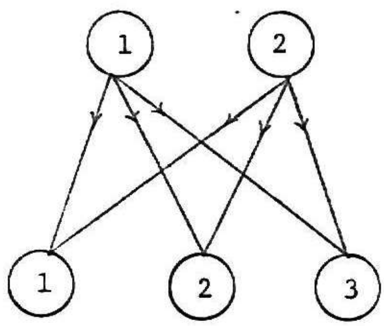
From the standpoint of business and industry, this is a more realistic description of the distribution or resource allocation problem. Indeed, a firm may frequently find it necessary to ship its products from one warehouse (source) to another in order to meet an abnormal rise in demand in the second supply area. In such a case the second warehouse behaves both as a source and as a sink thus blurring the distinction. In other words, each source or sink is permitted to act as an intermediate point for shipments from other sources to other sinks.

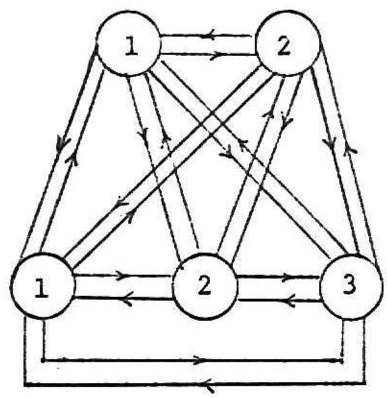
Now not only the direct links joining sources to sinks but also all other possible links must be considered. For comparision, a problem with two sources and three demand points is shown in Fig. 2.

In the transhipment problem each 'in' link must be considered as two distinct links because there is a difference whether material is sent from n to i or from i to n. There is an economic reason for this difference even though freight rates between two points are often the same regardless of the direction of shipment. It is easier to follow such a difference by considering a pipeline connecting two stations, one on the top of a mountain and another in a valley. The cost of pumping uphill is greater than down. It is valid for different freight rates, too.

Assume a simple s x N transhipment problem where s is the total number of resources and N the total number of demand points. When this s x N transhipment problem is converted into a transportation problem, it becomes one with s+N shippers and s+N receivers. Any amount can be

SOURCES





SINKS

Fig. 2 Transportation pattern Transhipment pattern shipped from source to source and from destination to destination.

In addition the following assumptions are made in converting transhipment problem into the transportation problem (63).

(1) Treat each point as a pair of points, one acting as a shipper and the other as a receiver.

(2) The cost of shipment from a point considered as a shipper to the same point considered as a receiver is set equal to zero, i.e. $F_{in}(\theta_{in}) = 0$ for $i = n$.

(3) The amount shipped from a point considered as a shipper to the same point considered as a receiver is equal to zero, i.e. $\theta_{in} = 0$ for $i = n$.

Before formulating the problem, a definition of the amount transhipped should be given for clarification. Consider a shipper whose production level is eight units and a receiver whose consumption level is five units. If the shipper sends all eight units directly to the receiver, who consumes only five, then the remaining three units are said to be transhipped by the receiver (30).

Let t_i denote the amount transhipped by the ith point. The amount leaving the shipper is

$$\sum_{n=1}^{s+N} \theta_{in} = W_i + t_i, \qquad i = 1,2, \ldots, s; \tag{1.6}$$

the amount leaving the receiver is

$$\sum_{n=1}^{s+N} \theta_{in} = t_i, \quad i = (s+1), (s+2), \ldots, (s+N); \tag{1.7}$$

the amount arriving at the shipper is

$$\sum_{\substack{i \neq 1 \\ i \neq n}}^{s+N} \theta_{in} = t_n, \qquad n = 1,2, \ldots, N; \text{ and} \tag{1.8}$$

the amount arriving at the receiver is

$$\sum_{\substack{i=1 \\ i \neq n}}^{s+N} \theta_{in} = D_n + t_n, \qquad n = (N+1), (N+2), \ldots, (N+s) . \tag{1.9}$$

The transhipment problem would be easy to solve if the exact amount to be transhipped through each point were known. It would only be necessary to add this amount to both the supply and the demand for the point. This would give the total amounts leaving and entering each point, which are relevant 'supply' and 'demand' figures for the transportation problem solution procedures. Unfortunately, the transhipment amounts are part of the solution and are not known initially. It is assumed for computation purposes that a large amount of the material to be shipped is available at each point and acts as a stockpile which can be drawn on or replenished. The solution of the transhipment problem lies in the fact that withdrawals from and corresponding additions to the stockpiles are equivalent to transhipment. The stockpile sizes are immaterial provided they are large enough to permit all possible shipments which can reduce the cost. In the computation, excessively large stockpiles are arbitrarily introduced (16).

Assume upper boundary for t_i, say t_0, then $t_i \leq t_0$,

$$t_i = t_0 - \theta_{ii} \qquad i = 1, 2, \ldots, (s+N) \tag{1.10}$$

Now the amount of goods transhipped cannot exceed the total amount of goods produced (or received), i.e. $t_0 \geq \sum_{i=1}^{s} W_i$, but for the purpose of computation t_0 is taken to be sufficiently large. Therefore the s x N transhipment problem can be stated as a (s+N) x (s+N) transportation problem. Minimize the cost of transportation

$$= \sum_{i=1}^{s+N} \sum_{n=1}^{s+N} F_{in}(\theta_{in}) \tag{1.11}$$

subject to the constraints

$$\sum_{n=1}^{s+N} \theta_{in} = \begin{cases} W_i + t_0, & i = 1,2, \ldots, s \\ t_0, & i = (s+1), (s+2), \ldots, (s+N) \end{cases} \tag{1.12}$$

$$\sum_{i=1}^{s+N} \theta_{in} = \begin{cases} t_0, & n = 1,2, \ldots, N \\ D_n + t_0, & n = (n+1), (n+2), \ldots, (N+s) \end{cases} \tag{1.13}$$

$$\text{and } \theta_{in} \geq 0, \quad \begin{array}{l} i = 1,2, \ldots, s, (s+1), (s+2), \ldots, (s+N) \\ n = 1,2, \ldots, N, (N+1), (N+2), \ldots, (N+s) \end{array} \tag{1.14}$$

Rather than writing all the equations involved, the model for a transhipment problem, when converted into a transportation problem, is usually written in a concise tabular or matrix form as illustrated in Fig. 3. It should be noted that D_{s+1}, D_{s+2}, ..., D_{s+N} are essentially D_1, D_2, ..., D_N for a s x N transportation problem and are so subscripted for convenience and ease. In comparision with a s x N transportation

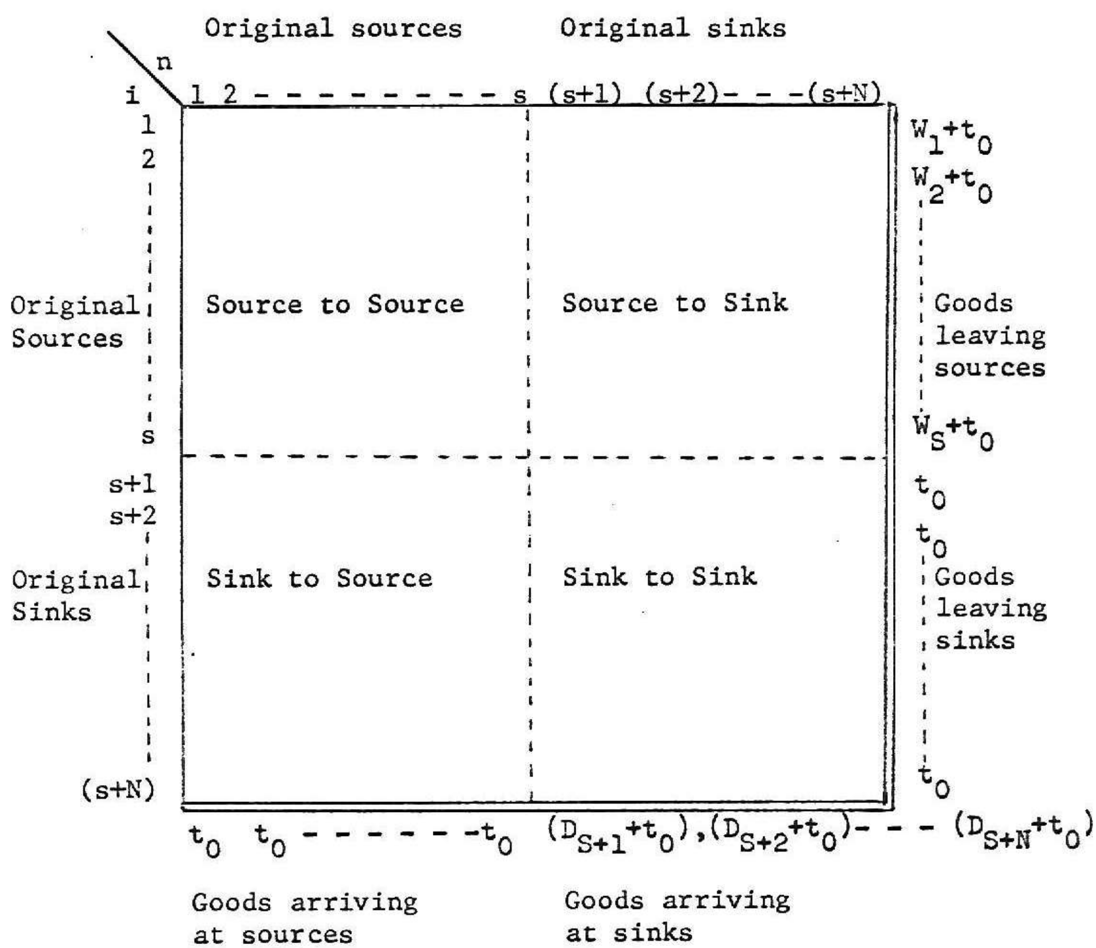


Fig. 3 Transportation matrix

problem, a s x N transhipment problem requires additional cost data to be solved by the techniques used for solving a transportation problem. It is obvious that the cost data of shipment from source to sink (the top right hand side elements of the matrix shown) will coincide with the cost data of the original s x N transportation problem. The diagonal cost elements of the matrix in Fig. 3 will be zero as it indicates the cost of shipment from a point to itself.

1.3 Literature Review

The transportation problem was formulated by Hitchcock (39) in 1941 and Koopmans in 1947. Dantzig (16,17) was the first to solve it by linear programming in 1947. Hitchcock showed how to proceed from an initial solution to the optimal one. He used the problem of distribution of a product from several sources to a number of cities at the least cost. Koopmans considered a problem of allocating products for shipment among the units within a transportation system so as to minimize the number of units needed to carry out a program but which would reduce the total cost at the same time. Dantzig applied the simplex method of linear programming in 1951 and treated the problem of linear objective function with linear constraints. In 1954 Charnes and Cooper (10,11,12) developed a short method instead of the lengthy algorithms. They devised "the stepping stone method of solving transportation model," an alternative to Dantzig's "row column sum calculation method." The simplex method has certain advantages for calculation by electronic computers but the stepping stone method is easier to explain because it lays a base for the essential structure of the problem. The calculation method is more readily apparent than Dantzig's simplex method. It was proved that it is much easier from the calculation point of view when the problems have fewer origins and demand points.

In 1955 Schell (66) stated an extension of the transportation problem. He considered it as a block in which the layer in all directions forms restricted transportation problems. Along with the extensions new computational methods were developed. Glezal (32) came up with a new computational algorithm in 1955. He applied it to the problem in which combinatorial ideas play the major role rather than the theory of linear inequalities.

In 1956 Vidale (75) developed a graphical approach for the solution of the general type of transportation problems. He has suggested a method of successive approximations when the production costs vary with the volume of resource produced. It was extended to problems involving a large number of origins and destinations. The assumption made is that transportation costs are monotonically increasing as one moves out from a given production center, but the rate of increase need not be constant or uniform in all directions. At the same time Ford and Fulkerson (27, 28,29) handled Hitchcock's capacity constrained problem on assignments to routes.

In 1956 Bellman (7) applied dynamic programming to the transportation model. He used functional equation techniques to solve a general class of transportation problems. This technique has computational limitations, such as "dimensionality difficulty," and so far has been used to handle transportation problems with not more than three sources.

In 1957 Dwyer (22) proposed an efficient method, the method of reduced matrices. This utilizes successive transformations involving subtractions of constants from the rows and columns of the cost matrix. Essentially the original matrix is reduced by successive transformations to a completely reduced matrix having a permutation set of zeros identifying the solution.

Prager (65) published a numerical technique in 1957. It was the "saturation technique" for the solution of the generalized transportation problem. This approach is an extension of the Ford and Fulkerson method which was used for the numerical solution of the Hitchcock problem.

Ford and Fulkerson (27) worked out "primal dual algorithm" originally developed by Dantzig. It was used in solving the capacitated Hitchcock problem. The algorithm starts with a feasible solution to a dual problem. If that solution is not available, then it uses a solution to a pseudodual problem.

In 1958 Gerstenhaber (31) proposed another method in which he discussed the use of row values which he called "producer subsidies." He concluded that by applying suitable subsidies any transportation problem can be made to have a trivial solution, one which permits each destination to be supplied at minimum cost. He demonstrated that the problem of finding suitable subsidies is equivalent to solving the dual problem. Gerstenhaber showed how the subsidies permit easy recomputation of a solution after perturbation of the problem.

In 1959 Shetty (67) solved the generalized transportation problem with nonlinear cost function by an algorithm which was an iterative process. The method can be applied to a wide range of problems by appropriate interpretation. A feasible solution is obtained at each stage and the value of the criterion function is improved in going from one stage to another.

Wagner (76) worked out techniques which transform transportation problems with a certain class of capacity flow constraints into enlarged uncapacitated transportation problems.

In 1960 Szwarc (72) developed a transportation model with stochastic demand. The problem with stochastic demand is considered when penalties are paid for each over-supplied and under-supplied unit of product. It was a modification of the standard transportation model by assuming that the demands of the consumers are random variables which may be dependent or independent with given density function. In his problem he had a known non-negative penalty rate associated with each unit of unfulfilled demand and the nth consumer destination. Similarly there was another non-negative penalty rate for each unit in excess of the quantity demanded by nth consumer. The objective was to minimize the total transportation costs plus total expected penalty cost. In determining the initial solution of the transportation problem, Szwarc developed a new method which is a modification of Vogel's approximation method.

From 1961 many short and simple methods were developed for problems with fewer origins and demand points. Fetter (25) gave "north-west corner rule" which is used for finding the initial starting solution which reduces the steps needed in getting the solution.

In 1962 Vajada (73) developed the "shadow cost method." William (71) applied Dantzig's "decomposition principle" to Hitchcock's transportation problem and to several of its generalizations. Among these generalizations are the transportation problem in which the source availabilities are subject to general linear constraints and the case in which costs are linear convex functions.

In 1963 Fan and Wang (24) applied the discrete version of the maximum principle for the solution of the transportation problem. It is a good technique when nonlinear cost functions are considered, but for linear cost functions the method is not advantageous and linear programming is resorted to. However, this technique could not be used extensively due to laborious calculations. Several small problems such as three origins and three demand points are solved by Hwang and Panchal (23). The computation becomes more tedious for problems with four or more origins.

In 1963 William (78) considered the more frequently occurring problem in which market demands are not known with certainty. He assumed the probability distribution of the demand and market demands were considered as random variables. This problem of uncertainty was called a stochastic transportation problem. He also gave an algorithm based on the decomposition principle.

In 1964 Llewellyn (52) proposed another simple method, "mutual preference method," to solve the transportation problem. Balinski and Gomory (5) described a simple method for the assignment and transportation problems. Their method is dual to the well known Hungarian method. Balas and Ivanescu (4) solved the generalized transportation problem by developing an extended form of the loop-technique of the stepping stone method. It reduced computational time and effort.

Dwyer (21), in 1966, developed an algorithm for the direct solution of the transportation problem. It was a method of reduced matrices. In the direct method, the basic specifications of the problem are used directly in solving the problem without replacing them, in whole or in part, with auxiliary theorms and criteria and without using the circuitous approach of transforming an initial feasible solution to an optimal one. Since the purest direct method is not practical, a modified form is used. The method of reduced matrices is used to make subtractions from rows and columns of the transportation matrix to produce a transformed matrix with all elements non-negative such that the non-negative integral θ_{ij} can be assigned to the zero terms to satisfy the specifications for origins, i, and destinations, n. Formal and informal versions are presented and applied to several general problems. Heiner (37) developed an ordered selection method in 1966, which was a sort of cost reduction method.

In 1967 Stroup (70) formulated the problem of assigning launch vehicles to space mission as the fixed cost transportation problem. This formulation assumes unlimited supplies and fixed cost incurred for positive flow from the sources. He used a branch-and-bound technique to obtain a minimum cost solution.

Klein (40) proposed a simple procedure for solving minimal cost flow problems in 1967. In these problems feasible flows are maintained throughout. He developed a primal method for the assignment and the transportation problems and also handled convex cost problems. During 1967 Lagemann (51) published a method of two pass operation for transportation models.

The power and simplicity of the transportation models is further demonstrated by the number of other applications that can be cast as transportation problems, one being that of transhipment.

Although numerous publications are available on the transportation problem, the literature on the transhipment problem is very limited. In 1967, Orden (63) formulated the transhipment problem. It is an extension of the original transportation problem which includes the possibility of transhipment. The transhipment technique is used to find the shortest route from one point in a network to another.

In 1960, Dantzig (16) considered this problem and applied the simplex method of linear programming to this type of extended transportation problem. Fulkerson (29) considered the capacity constrained transhipment problem and found its equivalence to be a Hitchcock's transportation problem. To seek a capacity constrained problem, he imposed upper bounds on the amounts that can be shipped between any two points in an ordinary transhipment problem. Garvin (30) also considered the transhipment problem.

In 1962 Vajada (73) described the transhipment problem as an application of linear programming. During 1963, Hammond (35) proposed a more typical transhipment model of great interest to management. In 1966 Chung An-Min (13) also applied some of the linear programming methods to the transhipment problem.

However little literature is available on the transhipment problem, the work of Orden seemed to be most prominent and much of the available literature is centered around this reference. So far only linear programming has been applied to this problem. Dynamic programming and maximum principle have not been applied to the solution of a transhipment problem. Other transportation algorithms can be used when the problems have fewer origins and demand points.

2. THE FUNCTIONAL GRADIENT TECHNIQUE

2.1 General

Optimization techniques may be divided into two classes. The first is composed of single stage techniques, such as linear and non-linear programming, which optimize various stages simultaneously. The other class is made up of multistage optimization techniques, such as dynamic programming and the discrete maximum principle, which use certain relationships to isolate interconnections between various stages. Each technique has its own limitations. Linear programming solves linear complex processes but cannot handle non-linear problems. Simple serial structures can be handled by multistage techniques, yet they face difficulties in solving fairly complex structures.

In view of the complexity of industrial and management problems, the above limitations are fairly serious. The functional gradient technique, which is a version of the gradient methods, was developed for variational problems. It is an iterative procedure and improves the assumed feasible controls by using the gradient direction. A set of feasible control values is assumed in the beginning to initiate the procedure.

While applying the gradient technique to the solution of the transportation problem, stopping conditions and additional constraints must be considered. The functional gradient technique method can be extended easily to handle additional constraints. The general procedure involves guessing a nonoptimal starting decision function which satisfies the end point conditions of the problem. Using this nonoptimal decision function, a better solution can be obtained.

2.2 The Numerical Method

Kelley (44) and Bryson obtained the equations for the gradient technique. Dreyfus (8) derived the same equations by using the concept of the invariant imbedding technique. This approach was found to be fairly simple because it eliminates the use of influence functions or adjoint equations, which Kelley used in his derivation. The disadvantage of this method is that it is not rigorous.

The continuous processes are represented by the following differential equation:

$$\frac{dX_i}{dt} = f_i \; (X_1, X_2, \ldots, X_s; \;\; \theta; \;\; t) \qquad i = 1,2, \ldots, s \tag{2.1}$$

where θ is the control variable and X_i, $i = 1,2, \ldots, s$ are the state variables.

The initial conditions are

$$X_i(0) = X_{i0} \qquad i = 1,2, \ldots, s. \tag{2.2}$$

The problem is to optimize the objective function

$$\phi \; (X_1, X_2, \ldots, X_s; \; t), \tag{2.3}$$

having the number of state variables and time t at some unspecified future time t_1, where t_1 is the first time that the terminal condition

$$\psi \; (X_1, X_2, \ldots, X_s, \; t) = 0 \tag{2.4}$$

is satisfied.

Equation (2.1) can be written in the following difference form:

$$X_i(t + \Delta) = X_i(t) + f_i[X_1(t), X_2(t), \ldots, \theta(t), t]\Delta$$

$$i = 1,2, \ldots, s. \tag{2.5}$$

It is necessary to estimate a reasonable control variable sequence. We define $S(X_{10}, X_{20}, \ldots, X_{s0}, t_0)$ which is the value of ϕ at time t_1 where starting state is $(X_{10}, X_{20}, \ldots, X_{s0})$ at time t_0. The nominal estimate sequence is used. The above defined function S satisfies the relation:

$$S(X_1, X_2, \ldots, X_s, t) = S(X_1 + f_1\Delta, \ldots, X_s + f_s\Delta, t + \Delta). \tag{2.6}$$

This will be the basic equation for the further derivation. Essentially it states that the value of objective function ϕ at t_1 with starting state $X_1, X_2, \ldots, X_s$ at t equals the value of objective function at t_1 with starting state $X_1 + f_1\Delta, \ldots, X_s + f_s\Delta$ at $t + \Delta$. The f's are evaluated using the particular estimated controls.

The direction of steepest ascent can be obtained by differentiating Eq. (2.6) with respect to control variable θ

$$\therefore \left.\frac{\partial S}{\partial \theta}\right|_t = \sum_{i=1}^{s} \frac{\partial S}{\partial (X_i + f_i\Delta)} \cdot \frac{\partial (X_i + f_i\Delta)}{\partial \theta|_t}$$

$$= \sum_{i=1}^{s} \frac{\partial S}{\partial (X_i + f_i\Delta)} \cdot \frac{\partial f_i}{\partial \theta|_t} \Delta . \tag{2.7}$$

Then Eq. (2.5) gives

$$D_\theta(\phi)\Big|_t = \sum_{i=1}^{s} D_{Xi}(\phi)\Big|_{t+\Delta} \left.\frac{\partial f_i}{\partial \theta}\right|_t \Delta \tag{2.8}$$

where

$$D_\theta(\phi)\Big|_t = \frac{\partial S}{\partial \theta}\Big|_t = \frac{\partial S}{\partial \theta}$$

evaluated in terms of the state and control variables at time t, and

$$D_{Xi}(\phi)\Big|_{t+\Delta} = \frac{\partial S}{\partial X_i}$$

evaluated in terms of the state and control variables at time $(t+\Delta)$.

Now, partially differentiating Eq. (2.6) with respect to X_j

$$D_{Xj}(\phi)\Big|_t = D_{Xj}(\phi)\Big|_{t+\Delta} + \sum_{i=1}^{s} D_{Xi}(\phi)\Big|_{t+\Delta} \cdot \frac{\partial f_i}{\partial X_j}\Big|_t \ \Delta \ . \qquad (2.9)$$

Finally, the change of S with respect to time can be obtained

$$D_t(\phi)\Big|_t = D_t(\phi)\Big|_{t+\Delta} + \sum_{i=1}^{s} D_{Xi}(\phi)\Big|_{t+\Delta} \frac{\partial f_i}{\partial \theta}\Big|_t \frac{d\theta}{dt}\Big|_t \ \Delta \ . \qquad (2.10)$$

Therefore at final time t_1,

$$D_{Xj}(\phi)\Big|_{t_1} = \frac{\partial \phi}{\partial X_j}\Big|_{t_1} - \left(\frac{d\phi}{dt} \Big/ \frac{d\psi}{dt}\right)\Big|_{t_1} \frac{\partial \psi}{\partial X_j}\Big|_{t_1} \qquad (2.11)$$

Results. These equations are called influence functions or adjoint equations.

Since the final conditions at $t = t_1$ are known, these equations can be solved in a backward recursive fashion. '$D_\phi(\phi)$' is essentially the gradient of ϕ with respect to the control variable, θ. If improvement of $\Delta\phi$ is asked, then the greatest improvement will be obtained if

$$\theta(t)_{new} = \theta(t)_{old} + \frac{D_\theta(\phi)\Big|_t}{\sum\limits_{t=0}^{t_1} \left\{ \left(D_\theta(\phi) \right)^2 \Big|_t \right\}} \Delta\phi \ . \tag{2.12}$$

For minimization problems, $\Delta\phi$ is set equal to minus.

For the discrete case, performance equations would be difference equations instead of differential equations. Since differential equations are converted into difference equations before the influence equations are obtained, the same results can be used for the discrete case with $\Delta = 1$.

2.3 Computational Procedure

In this numerical technique an iterative procedure is used. A set of estimated controls is used to initiate the iteration:

(1) Estimation of a nominal control sequence $\theta_0(t)$;

(2) Integration of Eq. (2.1) using $\theta_0(t)$;

(3) Solve Eqs. (2.8) and (2.9) backward using the final conditions obtained by Eq. (2.11);

(4) Determine the new value of $\theta(t)$ from Eq. (2.12);

(5) Integrate Eq. (2.1) again using improved $\theta(t)$;

(6) Repeat steps (3) to (5) until the gradient becomes so small that no more improvement is significant.

The method and recurrence relations can be easily extended to solve problems with additional final conditions and with several control variables. The treatment of such additional constraints is given in Chapter III.

2.4 Advantages and Disadvantages

The gradient technique applied to the multistage processes has the advantage of being able to investigate problems with a fairly large number of state variables. It does not have the dimensionality difficulty. By using this method, the two point boundary value difficulties generally present in the classical and maximum principle approach can be partly overcome. It can handle both stagewise and continuous processes. It has the computational advantage that it constitutes an approximation in policy space and that it has the monotone convergence property. The gradient technique, using a Langrange multiplier and quasilinearization, can handle fairly complex topological situations. It can handle problems with more than one control variable. However, computational difficulties increase rather rapidly with the increase of control variables.

Along with these advantages, there are some disadvantages. The most serious is that it cannot conveniently handle problems with inequality constraints on the state variable. The technique may not reach the absolute optimum but only reach a relative one. The convergence rate may be very slow during the last part of the iterations. The iterative techniques can be used for solving problems with state variable inequality constraints. By using different starting values the absolute optimum can be obtained. If the second variation is used near the optimal, a faster rate of convergence can be achieved. The more efficient use of this technique is achieved by Lee (53, 54, 55 ..., 59), when he combined it with several other methods such as quasilinearization, conjugate gradient, Lagrange multiplier and other search techniques.

3. APPLICATION TO THE TRANSPORTATION PROBLEM

3.1 *Transportation Problem as a Multistage Optimization Problem*

Figure 4 shows a framework into which the transportation problem shown in Fig. 1 can be cast. Chapter I indicates that a transportation problem can be stated as

$$\text{Minimize} \qquad C_{sN} = \sum_{n=1}^{N} \sum_{i=1}^{s} F_{in} (\theta_{in}) \tag{3.1}$$

$$\text{subject to} \qquad \theta_{in} \geq 0 \qquad \begin{matrix} i = 1, 2, \ldots, s, \\ n = 1, 2, \ldots, N \end{matrix} \tag{3.2}$$

$$\sum_{n=1}^{N} \theta_{in} = W_i \qquad i = 1,2, \ldots, s \tag{3.3}$$

$$\sum_{i=1}^{s} \theta_{in} = D_n \qquad n = 1, 2, \ldots, N\ . \tag{3.4}$$

This problem may be formulated as a multistage problem. Consider N *different stages constituting a simple serial structure.* Let each stage represent each demand point. The whole problem is to solve (s-1) similar types of processes, each one having N number of stages. The simple serial structure is shown in Fig. 5.

Let the nth stage represent the nth demand point. The state variables

$$X_{in} \text{ where } i = 1,2, \ldots, (s-1)$$

represent the total amount of resource transported from ith depot (resource) to the first n stages (demand points). Therefore the performance equation becomes

$i \backslash n$		DEMAND POINTS						W_i
		1	2	- - - -	n	- - - - -	N	
RESOURCES	1	θ_{11}	θ_{12}	- - - -	θ_{1n}	- - - - -	θ_{1N}	W_1
	2	θ_{21}	θ_{22}	- - - -	θ_{2n}	- - - - -	θ_{2N}	W_2
	⋮							
	i	θ_{i1}	θ_{i2}	- - - -	θ_{in}	- - - - -	θ_{iN}	W_i
	⋮							
	s	θ_{s1}	θ_{s2}	- - - -	θ_{sn}	- - - - -	θ_{sN}	W_s
D_n		D_1	D_2	- - - -	D_n	- - - - -	D_N	$\sum D_n = \sum W_i$

Fig. 4 Transportation problem

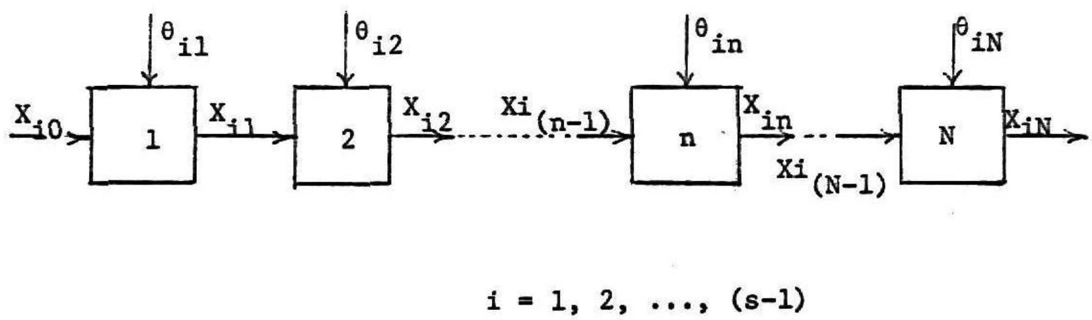


Fig. 5 Simple serial process

$$X_{in} = X_{i(n-1)} + \theta_{in} \tag{3.5}$$

with end conditions

$$X_{i0} = 0 \tag{3.6}$$

and $$X_{iN} = W_i \tag{3.7}$$

where $i = 1, 2, \ldots, (s-1)$

$n = 1, 2, \ldots, N$.

Conditions (3.6) are obvious. Conditions (3.7) can be derived as follows: at Nth stage

$$\begin{aligned}
X_{iN} &= X_{i(N-1)} + \theta_{iN} \\
&= X_{i(N-2)} + \theta_{i(N-1)} + \theta_{iN} \\
&= X_{i(N-3)} + \theta_{i(N-2)} + \theta_{i(N-1)} + \theta_{iN} \\
&- \\
&= X_{i0} + \theta_{i1} + \theta_{i2} + \ldots + \theta_{in} + \ldots + \theta_{iN} \\
&= 0 + \sum_{n=1}^{N} \theta_{in} \\
&= W_i \ .
\end{aligned}$$

It must be noted that though there are s depots (resources) in the problem, there are only (s-1) state variables in Eq. (3.5). This is because the demand by each stage is preassigned; hence the number of units supplied from the sth depot (resource) to the nth stage can be obtained by subtracting the sum of units supplied to the nth stage by the 1 to (s-1) depots from the total number of units required by the nth stage. Therefore,

$$\theta_{sn} = D_n - \sum_{i=1}^{s-1} \theta_{in} \qquad n = 1, 2, \ldots, N \tag{3.8}$$

is the solution of the sth process.

Since the objective of the problem is to minimize the total cost of transportation, this objective is defined as the sth state variable which satisfies the relation

$$X_{sn} = X_{s(n-1)} + \sum_{i=1}^{s} F_{in} (\theta_{in}), \tag{3.9}$$

where $\qquad n = 1, 2, \ldots, N$.

It satisfies the condition

$$X_{s0} = 0. \tag{3.10}$$

It should be noted that 's' here is not the sth origin.

It can be shown that X_{sN} represents the total cost of transportation. From Fig. 6 it is easy to formulate Eq. 3.9. The objective function ϕ, which is equal to X_{sN}, may be derived in the following manner:

$$X_{sN} = X_{s(N-1)} + \sum_{i=1}^{s} F_{iN} (\theta_{iN})$$

$$= X_{s(N-2)} + \sum_{i=1}^{s} F_{i(N-1)} (\theta_{i(N-1)}) + \sum_{i=1}^{s} F_{iN} (\theta_{iN})$$

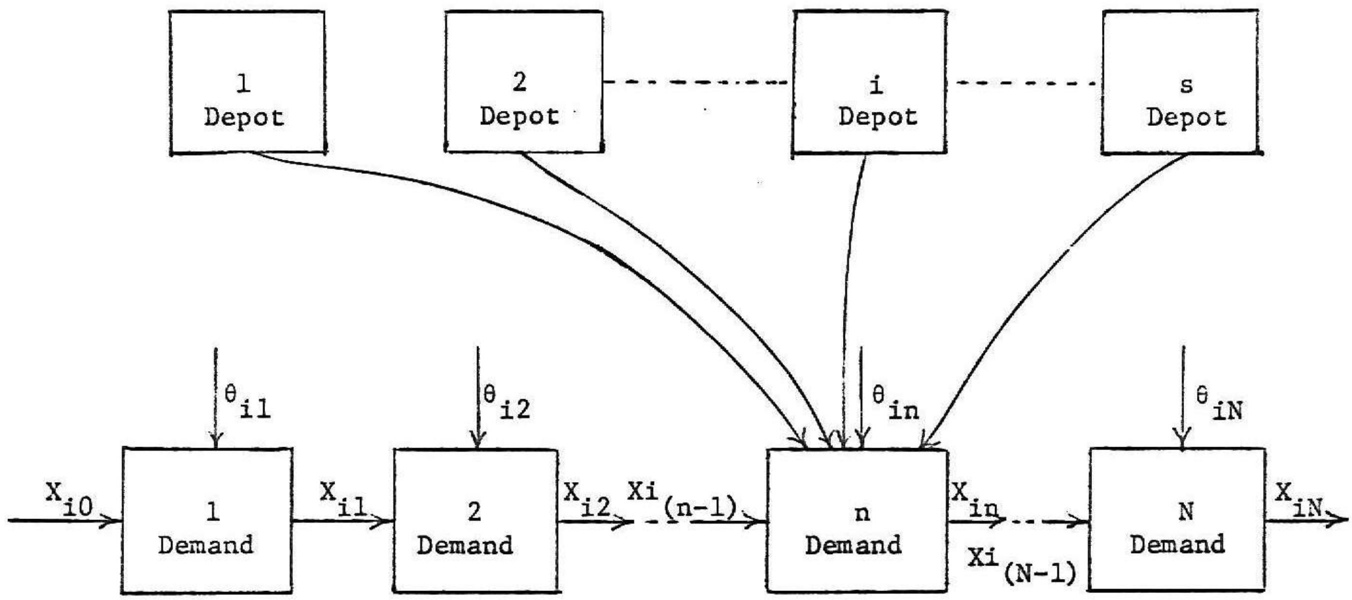


Fig. 6 Formulation of cost equation

$$= X_{s0} + \sum_{i=1}^{s} F_{i1}(\theta_{i1}) + \sum_{i=1}^{s} F_{i2}(\theta_{i2}) + \ldots + \sum_{i=1}^{s} F_{in}(\theta_{in}) + \ldots$$

$$\ldots + \sum_{i=1}^{s} F_{iN}(\theta_{iN})$$

$$= 0 + \sum_{n=1}^{N} \sum_{i=1}^{s} F_{in}(\theta_{in})$$

$$= \Phi$$

Thus in general s state variables and (s-1) control sequences are available. The sth sequence is fixed by condition (3.4). Equation (3.3) gives an additional set of constraints.

3.2 Recursive Relations

The set of performance and cost equations are

$$X_{in} = X_{i(n-1)} + \theta_{in}$$

$$X_{sn} = X_{s(n-1)} + \sum_{i=1}^{s} F_{in}(\theta_{in})$$

$$i = 1, 2, \ldots, (s-1)$$

$$n = 1, 2, \ldots, N .$$

The initial conditions are

$$X_{i0} = 0 \qquad i = 1, 2, \ldots, (s-1)$$

and $$X_{s0} = 0 .$$

The problem is to optimize the objective function $\phi(X_{in}; X_{sn}; n) = X_{sN}$, which is a function of the state variable and the stage number, n. The terminal condition to be satisfied is $\psi = n-N = 0$.

First, S must satisfy the relation

$$S\ (X_{in}; X_{sn}; n) = S\ (X_{in} + f_{in}\Delta,\ X_{sn} + f_{sn};\ n + \Delta) \tag{3.11}$$

$$i = 1,\ 2,\ \ldots,\ (s-1)$$

$$n = 1,\ 2,\ \ldots,\ (N-1).$$

Then Eq. (2.8) gives

$$\left.\frac{\partial S}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1} \cdot 1 \cdot \Delta + \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \left(\left.\frac{\partial \sum\limits_{i=1}^{s} F_{in}\ (\theta_{in})}{\partial \theta_j}\right|_n \right) \cdot \Delta$$

$$j = 1,\ 2,\ \ldots,\ (s-1)$$

$$n = 1,\ 2,\ \ldots,\ (N-1)\ .$$

Since $\Delta = 1$ in this case, then

$$\left.\frac{\partial S}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1} + \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \left[\left.\frac{\partial \sum\limits_{i=1}^{s} F_{in}\ (\theta_{in})}{\partial \theta_j}\right|_n \right]$$

$$i = 1,\ 2,\ \ldots,\ (s-1)$$

$$n = 1,\ 2,\ \ldots,\ (N-1). \tag{3.12}$$

Using Eq. (2.9) yields

$$\left.\frac{\partial S}{\partial X_j}\right|_n = \left.\frac{\partial S}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1} \cdot 1 \cdot \Delta + \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \cdot 0 \cdot \Delta$$

$$\therefore \quad \left.\frac{\partial S}{\partial X_j}\right|_n = \left.\frac{\partial S}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1} \qquad j = 1, 2, \ldots, (s-1) \tag{3.13}$$

$$n = 1, 2, \ldots, (N-1) .$$

Similarly,

$$\left.\frac{\partial S}{\partial X_s}\right|_n = \left.\frac{\partial S}{\partial X_s}\right|_{n+1} + \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \Delta$$

$$\left.\frac{\partial S}{\partial X_s}\right|_n = 2 \cdot \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \qquad n = 1, 2, \ldots, (N-1) . \tag{3.14}$$

Using Eq. (2.11) gives the end conditions

$$\left.\frac{\partial S}{\partial X_i}\right|_N = 0 \qquad i = 1, 2, \ldots, (s-1) \tag{3.15}$$

and

$$\left.\frac{\partial S}{\partial X_s}\right|_N = 1 . \tag{3.16}$$

A new set of control variables can be obtained by

$$\theta_{jn}(\text{new}) = \theta_{jn}(\text{old}) + \frac{\left.\frac{\partial S}{\partial \theta_j}\right|_n}{\sum_{n=1}^{N}\left\{\left.\left(\frac{\partial S}{\partial \theta}\right)^2\right|_n\right\}} \cdot \Delta\phi, \quad j = 1,2, \ldots, (s-1) \quad n = 1,2, \ldots, N . \tag{3.17}$$

For simplicity in calculation

$$\left.\frac{\partial S}{\partial \theta_j}\right|_N = \left.\frac{\partial S}{\partial \theta_j}\right|_{N-1} \qquad j = 1,2, \ldots, (s-1) \tag{3.18}$$

is set. The values of sth process will be calculated from the relation

$$\theta_{sn}(\text{new}) = D_n - \sum_{j=1}^{(s-1)} \theta_{jn}(\text{new}) \qquad n = 1,2, \ldots, N. \tag{3.19}$$

3.3 Additional Constraints

In Section 3.2 recursive equations were derived without considering the additional constraints

$$Z(X_{iN}, \theta_{iN}) = X_{iN} - W_i = 0 \qquad i = 1, 2, \ldots, (s-1).$$

These constraints must be satisfied at the terminal point. The recursive equations for handling the additional constraints are derived in a similar fashion.

Using Eq. (2.8) gives

$$\left.\frac{\partial Z}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1} \cdot 1 \cdot \Delta$$

$$\therefore \left.\frac{\partial Z}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1} \qquad \begin{array}{l} j = 1, 2, \ldots, (s-1) \\ n = 1, 2, \ldots, (N-1)\ . \end{array} \tag{3.20}$$

Equation 2.9 results in

$$\left.\frac{\partial Z}{\partial X_j}\right|_n = \left.\frac{\partial Z}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1} \cdot 1 \cdot \Delta$$

$$\left.\frac{\partial Z}{\partial X_j}\right|_n = \left.\frac{\partial Z}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1} \qquad \begin{array}{l} j = 1, 2, \ldots, (s-1) \\ n = 1, 2, \ldots, (N-1). \end{array} \tag{3.21}$$

The end conditions are calculated by using Eq. (2.11). Therefore,

$$\left.\frac{\partial Z}{\partial X_i}\right|_N = 1 \qquad i = 1, 2, \ldots, (s-1) \,. \tag{3.22}$$

Improvement is attained by adopting the reasonable policy of changing θ at each stage proportional to the rate at which the final value of ϕ changes with θ, that is $\frac{\partial S}{\partial \theta}$. Therefore

$$\Delta\theta_{jn} = K_1 \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n\right) + K_2 \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n\right) \qquad \begin{array}{l} j = 1, 2, \ldots, (s-1) \\ n = 1, 2, \ldots, N \end{array} \tag{3.23}$$

where K_1 and K_2 are constants of proportionality. They are to be evaluated by solving the following simultaneous linear equations:

$$\Delta\phi = K_1 \left\{\sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n\right)^2\right\} + K_2 \left\{\sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n\right) \cdot \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n\right)\right\} \tag{3.24}$$

$$j = 1, 2, \ldots, (s-1)$$

$$\Delta Z = K_1 \left\{\sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n\right) \cdot \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n\right)\right\} + K_2 \left\{\sum_{n=1}^{N} \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n\right)^2\right\} \tag{3.25}$$

$$j = 1, 2, \ldots, (s-1)$$

where $\Delta\phi$ is the asked improvement in the objective function and ΔZ is the value of the additional constraint. The value of the ΔZ should be zero or very near to zero when the solution approaches the optimum. The value of

$\left.\frac{\partial Z}{\partial \theta_j}\right|_N$ is set equal to that of $\left.\frac{\partial Z}{\partial \theta_j}\right|_{N-1}$ for $j = 1, 2, \ldots, (s-1)$.

3.4 Summary

The equations derived in Sections (3.2) and (3.3) are summarized in the order they are used in solving the problems by the gradient technique. Partial derivatives with respect to the state variables:

$$\left.\frac{\partial S}{\partial X_j}\right|_n = \left.\frac{\partial S}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1}$$

$$\left.\frac{\partial S}{\partial X_s}\right|_n = 2 \cdot \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \qquad \begin{array}{l} j = 1, 2, \ldots, (s-1) \\ n = 1, 2, \ldots, (N-1) \end{array} \tag{3.26}$$

$$\left.\frac{\partial Z}{\partial X_j}\right|_n = \left.\frac{\partial Z}{\partial X_j}\right|_{n+1} + \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1}$$

End conditions:

$$\left.\frac{\partial S}{\partial X_i}\right|_N = 0$$

$$\left.\frac{\partial S}{\partial X_s}\right|_N = 1 \qquad i = 1, 2, \ldots, (s-1) \tag{3.27}$$

$$\left.\frac{\partial Z}{\partial X_i}\right|_N = 1$$

Partial derivatives with respect to the control variables:

$$\left.\frac{\partial S}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial S}{\partial X_i}\right|_{n+1} + \left.\frac{\partial S}{\partial X_s}\right|_{n+1} \left(\left.\frac{\partial \sum_{i=1}^{s} F_{in}\,(\theta_{in})}{\partial \theta_j}\right|_n \right)$$

$j = 1, 2, \ldots, (s-1)$

$n = 1, 2, \ldots, (N-1)$

$$\left.\frac{\partial Z}{\partial \theta_j}\right|_n = \sum_{i=1}^{s-1} \left.\frac{\partial Z}{\partial X_i}\right|_{n+1} \tag{3.28}$$

For simplicity in calculation, $\left.\frac{\partial S}{\partial \theta_j}\right|_N$ and $\left.\frac{\partial Z}{\partial \theta_j}\right|_N$ are set equal to $\left.\frac{\partial S}{\partial \theta_j}\right|_{N-1}$ and $\left.\frac{\partial Z}{\partial \theta_j}\right|_{N-1}$ respectively. For the same purpose the method explained in the Section 4.2 can be used.

$$\theta_{jn}(\text{new}) = \theta_{jn}(\text{old}) \pm \left\{ K_1 \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n \right) + K_2 \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n \right) \right\} \tag{3.29}$$

$j = 1, 2, \ldots, (s-1)$

$n = 1, 2, \ldots, N$

where K_1 and K_2 are calculated by solving the simultaneous equations

$$\Delta\phi = K_1 \left\{ \sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n \right)^2 \right\} + K_2 \left\{ \sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n \right) \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n \right) \right\}$$

and

(3.30)

$$\Delta Z = K_1 \left\{ \sum_{n=1}^{N} \left(\left.\frac{\partial S}{\partial \theta_j}\right|_n \right) \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n \right) \right\} + K_2 \left\{ \sum_{n=1}^{N} \left(\left.\frac{\partial Z}{\partial \theta_j}\right|_n \right)^2 \right\}$$

$j = 1, 2, \ldots, (s-1).$

4. APPLICATION TO THE TWO DIMENSIONAL (TWO ORIGINS) PROBLEMS

4.1 Two Origins and Three Demand Points

This problem has been solved by the maximum principle (23). The transportation costs and other numerical values are shown in Table 1. The values a_{in} and b_{in} are the constants in the cost function. Subscript i denotes the ith origin and n denotes the nth demand point. In this problem, it is assumed that i = 1, 2, and n = 1, 2, 3. For convenience this problem will be designated as 2x3 problem.

θ_{in}, the number of units to be transported from different origins to the various demand points such that the cost of transportation will be a minimum, has to be determined. The cost of transportation is given by the cost function

$$F_{in}(\theta_{in}) = a_{in}\ \theta_{in} + b_{in}(\theta_{in})^2, \quad i = 1, 2 \text{ and } n = 1, 2, 3. \tag{4.1}$$

There are two state variables and one control variable in this problem. The values of second control variable sequence are given by Eq. (3.4). The number of stages is equal to the number of demand points. Three stages exist in this problem.

4.2 Computational Aspects and Results

The problem described in Section 4.1 is solved by using three different values of the initial control variable sequence. The value of the gradient at the last stage, i.e. $\left.\frac{\partial S}{\partial \theta_1}\right|_3$, is calculated by differentiating the objective function with respect to θ_{13}. The objective function given by Eq. 4.1 can be written as

Table 1. Transportation costs and requirements for 2x3 problem

n \ i		Depots 1		Depots 2		D_n
		a_{1n}	b_{1n}	a_{2n}	b_{2n}	
Demand Points	1	1.0		3.0		10
	2	3.0	0.01	2.1		45
	3	3.0		1.0	0.2	20
W_i		30		45		75

$$F_{1n}(\theta_{1n}) + F_{2n}(\theta_{2n}) = a_{1n}\,\theta_{1n} + a_{2n}\,\theta_{2n} + b_{1n}(\theta_{1n})^2 + b_{2n}(\theta_{2n})^2,$$

$$n = 1, 2, 3. \qquad (4.2)$$

From Eq. (3.4)

$$\theta_{2n} = D_n - \theta_{1n}\,. \qquad (4.3)$$

Substituting the value of θ_{2n} in Eq. (4.2) gives

$$F_{1n}(\theta_{1n}) + F_{2n}(D_n - \theta_{1n}) = a_{1n}\,\theta_{1n} + a_{2n}(D_n - \theta_{1n}) + b_{1n}(\theta_{1n})^2$$

$$+ b_{2n}(D_n - \theta_{1n})^2\,. \qquad (4.4)$$

Therefore taking a partial derivative of Eq. (4.4) with respect to θ_{1n} and evaluating it at n = 3 results in

$$\left.\frac{\partial S}{\partial \theta_1}\right|_3 = \left.\left\{a_{1n} + 2b_{1n}(\theta_{1n}) - 2b_{2n}(D_n - \theta_{1n})\right\}\right|_{n=3}\,. \qquad (4.5)$$

The value of $\left.\frac{\partial Z}{\partial \theta_1}\right|_3$ is equal to $\left.\frac{\partial Z}{\partial \theta_1}\right|_2$.

Initially a larger value of $\Delta\phi$ is used with the assumed starting control sequence to get faster convergence. The suitable value of $\Delta\phi$ is estimated by trial and error. Equation (3.17) gives the values of improvements in control variables. The step size should not be too large or the optimal may be overshot and the additional constraint will not be satisfied. To achieve better accuracy of the additional constraint, further calculations are done with a smaller $\Delta\phi$ value. Starting values are taken from the results of the best iteration of the previous calculations with a larger $\Delta\phi$. The procedure of reducing $\Delta\phi$ is continued until the required accuracy is met.

The convergence rates for the three different starting sets are shown in Tables 2,3,5,6,8 and 9. The computer logic is given in Appendix I. The computer program is given in Appendix II.

Tables 2, 3 and 4 are obtained with the following feasible starting control sequence:

$$\theta_{11} = \theta_{12} = \theta_{13} = 10. \tag{4.6}$$

Table 2 has a larger value of $\Delta\phi$ while Table 3 is the continuation of Table 2 but with a smaller $\Delta\phi$. The starting values used are obtained from the 13th iteration in Table 2. Table 4 gives the optimal results of this problem, which are from iteration 12 in Table 3.

Tables 5, 6 and 7 are obtained with the following starting control, which is above the feasible control sequence:

$$\theta_{11} = \theta_{12} = \theta_{13} = 20 \tag{4.7}$$

Table 5 has a larger value of $\Delta\phi$ and uses the above starting values for the control variable. Table 6 has a smaller $\Delta\phi$; the results of iteration 13 in Table 5 are used as the starting values. Table 7 gives the optimal results of this problem, which are obtained from iteration 16 in Table 6.

Tables 8, 9 and 10 are obtained with the following starting control, which is below the feasible control sequence:

$$\theta_{11} = \theta_{12} = \theta_{13} = 2 \tag{4.8}$$

Table 2. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in \$	Value of Add. Constraint
Initial	174.4999	0.0000
1	169.8485	-2.5806
5	162.6443	-4.4262
10	159.8798	-4.8293
13	159.5276	-4.6285
15	159.5489	-4.5055
25	159.5584	-4.4596
50	159.5584	-4.4595
75	159.5584	-4.4595
100	159.5584	-4.4595

Table 3. Convergence rate of transportation cost with $\Delta\phi = 1$

Iter. Number	Cost in $	Value of Add. Constraint
Iter. # 13 of Table 2	159.5273	-4.6300
1	161.7899	-1.4131
5	162.5858	-0.5025
10	162.5880	-0.5000
12	162.5879	-0.4999
15	162.5878	-0.4999
100	162.5861	-0.5000
150	162.5858	-0.5000
200	162.5857	-0.5000
250	162.5857	-0.5000

Table 4. The optimal solution with equation (4.6)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	10.0000	0.0000	10
	2	2.2473	42.7527	45
	3	17.2528	2.7472	20
W_i		29.5001	45.4999	75

Table 5. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in $	Value of Add. Constraint
Initial	166.5000	30.0000
1	176.1112	0.9468
5	164.0554	- 4.1965
10	160.2986	- 4.7720
13	159.5054	- 4.8344
15	159.5384	- 4.5614
25	159.5583	- 4.4597
50	159.5584	- 4.4595
75	159.5584	- 4.4595
100	159.5584	- 4.4595

Table 6. Convergence rate of transportation cost with $\Delta\phi = 1$

Iter. Number	Cost in $	Value of Add. Constraint
Iter. # 13 of Table 5	159.5051	-4.8400
1	161.8212	-1.4197
5	162.5928	-0.5074
10	162.5983	-0.4999
15	162.5973	-0.4999
16	162.5970	-0.4998
50	162.5918	-0.4999
100	162.5881	-0.5000
200	162.5861	-0.5000
250	162.5858	-0.5000

Table 7. The optimal solution with equation (4.7)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	10.0000	0.0000	10
	2	2.3748	42.6252	45
	3	17.1254	2.8746	20
W_i		29.5002	45.4998	75

Table 8 has a larger value of $\Delta\phi$ and the above starting values. Table 9 uses a smaller $\Delta\phi$ and uses results of iteration 22 in Table 8 as the starting values. Table 10 gives the optimal results of this problem, which are obtained from iteration 179 in Table 9.

Comparing the optimal solutions in Table 3, 6 and 9 shows that the cost of transportation for 2x3 problem is \$162.59. The value of the additional constraint is -0.4999.

4.3 Two Origins and Ten Demand Points

This problem is to see the effect of using the functional gradient technique with the assumption that the gradient at the 10th stage is equal to that of the 9th stage, i.e. $\left.\frac{\partial S}{\partial \theta_1}\right|_{10} = \left.\frac{\partial S}{\partial \theta_1}\right|_{9}$. The problem is also solved by another approach in which the gradient for the last stage, i.e. $\left.\frac{\partial S}{\partial \theta_1}\right|_{10}$ is calculated by differentiating the objective function with respect to $\theta_{1\,10}$. The value of $\left.\frac{\partial Z}{\partial \theta_1}\right|_{10}$ is equal to $\left.\frac{\partial Z}{\partial \theta_1}\right|_{9}$ in both of these approaches. Transportation costs and other requirements are given in Table 11. It is necessary to determine θ_{in} where i = 1, 2, and n = 1, 2, ..., 10 such that the cost of transportation given by the function

$$F_{in}(\theta_{in}) = a_{in}\,\theta_{in} + b_{in}(\theta_{in})^2 \quad i = 1, 2 \quad n = 1, 2, \ldots, 10 \qquad (4.9)$$

is minimized.

This problem is similar to the one described in Section 4.1 except the number of stages is increased to ten. For convenience, this wil be called a 2x10 problem.

Table 8. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in $	Value of Add. Constraint
Initial	211.1399	-24.0000
1	216.9357	- 7.8286
5	203.4599	- 1.5828
10	172.2946	- 2.6876
20	159.8288	- 4.8362
22	159.5127	- 4.7479
25	159.5499	- 4.5004
50	159.5584	- 4.4595
75	159.5584	- 4.4595
100	159.5584	- 4.4595

Table 9. Convergence rate of transportation cost with $\Delta\phi = 1$

Iter. Number	Cost in $	Value of Add. Constraint
Iter. # 22 of Table 8	159.5125	-4.7500
1	161.8063	-1.4172
5	162.5865	-0.5077
10	162.5928	-0.4999
11	162.5927	-0.4999
50	162.5892	-0.4999
79	162.5877	-0.4999
100	162.5871	-0.5000
200	162.5859	-0.5000
250	162.5857	-0.5000

Table 10. The optimal solution with equation (4.8)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	10.0000	0.0000	10
	2	2.2413	42.7587	45
	3	17.2588	2.7412	20
W_i		29.5001	45.4999	75

Table 11. Transportation costs and requirements for 2x10 problem

	$n \backslash i$	Depots 1 a_{1n}	Depots 1 b_{1n}	Depots 2 a_{2n}	Depots 2 b_{2n}	D_n
Demand Points	1	1.00	0.01	3.00	0.20	20
	2	2.00	0.00	2.10	0.00	60
	3	3.00	0.00	2.00	0.20	40
	4	1.20	0.00	1.00	0.03	10
	5	1.50	0.10	2.60	0.25	10
	6	1.70	0.00	2.70	0.15	30
	7	2.00	0.00	5.00	0.18	45
	8	1.00	0.04	1.00	0.06	25
	9	3.00	0.02	4.00	0.03	15
	10	6.00	0.20	6.60	0.17	35
W_i		160		130		290

4.4 Computational Aspects and Results

To simplify the presentation this problem will be discussed in two separate parts. In part A it is assumed that $\left.\frac{\partial S}{\partial \theta_1}\right|_{10}$ is calculated by differentiating the objective function given by eq. (4.9) with respect to $\theta_{1\,10}$. In part B it is assumed that $\left.\frac{\partial S}{\partial \theta_1}\right|_{10} = \left.\frac{\partial S}{\partial \theta_1}\right|_{9}$.

Part A

The problem is solved by using three different sets of initial control variable sequences. The value of $\left.\frac{\partial S}{\partial \theta_1}\right|_{10}$ is given by Eq. (4.5) by evaluating it at n = 10. The approximate choice of $\Delta\phi$ is made as described in Section 4.2.

Tables 12 and 13 are obtained with the following feasible starting control sequence:

$$\theta_{11} = \theta_{12} = \theta_{13} = \ldots = \theta_{110} = 16 \quad . \tag{4.10}$$

Table 12 shows the convergence rate of cost and starts from the feasible starting control sequence. The optimal is reached at iteration 206. Table 13 shows the optimal results of this problem.

Tables 14 and 15 are obtained with the following starting control sequence, which is above feasible:

$$\theta_{11} = \theta_{12} = \ldots = \theta_{110} = 25. \tag{4.11}$$

Table 14 gives the convergence rate of cost with the above starting values of control variable sequence. The optimal is reached at iteration 209 and Table 15 presents the optimal results of this problem.

Table 12. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in $	Value of Add. Constraint
Initial	1272.6390	0.0000
1	1148.3170	- 7.3238
5	919.9716	11.7128
11	883.5642	18.6275
14	883.9902	17.6544
49	894.4238	5.0341
100	897.6228	1.9788
160	899.3557	0.5368
191	899.8894	0.1239
203	900.0241	0.0218
206	900.0554	- 0.0015
209	900.0847	- 0.0239
223	900.2175	- 0.1233
262	900.4609	- 0.3040
298	900.5969	- 0.4037

Table 13. The optimal solution with equation (4.10)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	19.3786	00.6214	20
	2	0.0000	60.0000	60
	3	29.4714	10.5286	40
	4	00.0000	10.0000	10
	5	08.6938	01.3062	10
	6	28.8130	01.1870	30
	7	45.0000	00.0000	45
	8	08.0319	16.9681	25
	9	05.5935	09.4065	15
	10	15.0163	19.9837	35
W_i		159.9985	130.0015	290

Table 14. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in \$	Value of Add. Constraint
Initial	1173.9980	90.0000
1	1012.9840	64.6729
5	930.3105	21.3896
15	881.5791	24.1956
18	881.8364	21.9331
50	893.6162	5.6961
101	897.5717	2.0167
152	899.1665	0.6865
194	899.8735	0.1363
200	899.9460	0.0809
209	900.0451	0.0062
212	900.0754	- 0.0170
227	900.2114	- 0.1185
263	900.4426	- 0.2906
299	900.5861	- 0.3957

Table 15. The optimal solution with equation (4.11)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	19.3786	00.6214	20
	2	00.0000	60.0000	60
	3	29.4714	10.5286	40
	4	00.0000	10.0000	10
	5	08.6939	01.3061	10
	6	28.8130	01.1870	30
	7	45.0000	00.0000	45
	8	08.0340	16.9660	25
	9	05.5988	09.4012	15
	10	15.0165	19.9835	35
W_i		160.0062	129.9938	290

Tables 16 and 17 are obtained with the following starting control sequence, which is below feasible:

$$\theta_{11} = \theta_{12} = \ldots = \theta_{110} = 10 \ . \tag{4.12}$$

Table 16 shows convergence rate of cost with the above starting control sequence values. Optimal is reached at iteration 158. Table 17 shows optimal results of this problem.

In all the above calculations, sufficient accuracy of the additional constraint is obtained with an assumed value of $\Delta\phi$; hence further calculations are not made. Comparing the optimal results in Tables 12, 14 and 16 shows the cost of transportation for 2x10 problem is $900.06 and the value of the additional constraint is -0.0015.

Part B

The problem is solved by using four different values of initial control variable sequences. The value of the gradient at the last stage is assumed to be equal to the previous stage. The choice of $\Delta\phi$ is made as explained in Section 4.2.

Tables 18 and 19 are obtained with the feasible starting control sequence given by Eq. (4.10). Table 18 shows the convergence rate of cost and starts with the feasible control sequence. The optimal is reached at iteration 128. Table 19 shows the optimal results of this problem.

Tables 20 and 21 are obtained with a starting control sequence which is above feasible and which is given by Eq. (4.11). Table 20 gives the convergence rate of cost with the above starting values of control variable sequence. The optimal is reached at iteration 257 and Table 21 presents

Table 16. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in $	Value of Add. Constraint
Initial	1485.9980	-60.0000
1	1394.5750	-62.7022
6	952.6137	- 4.3500
11	902.1330	1.2727
24	891.9450	9.1131
50	897.5917	1.9942
80	898.6643	1.0906
110	899.3745	0.5215
140	899.8518	0.1527
152	899.9951	0.0438
158	900.0590	- 0.0043
161	900.0888	- 0.0269
173	900.1970	- 0.1081
224	900.5048	- 0.3360
299	900.7001	- 0.4792

Table 17. The optimal solution with equation (4.12)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	19.3786	00.6214	20
	2	00.0000	60.0000	60
	3	29.4714	10.5286	40
	4	00.0000	10.0000	10
	5	08.6939	01.3061	10
	6	28.8130	01.1870	30
	7	45.0000	00.0000	45
	8	08.0341	16.9659	25
	9	05.5882	09.4118	15
	10	15.0165	19.9835	35
W_i		159.9957	130.0043	290

Table 18. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in \$	Value of Add. Constraint
Initial	1272.6390	00.0000
1	1148.3260	- 7.3357
5	920.2263	11.4712
11	884.6105	17.9109
25	895.1804	6.6203
52	906.6445	0.6628
122	923.0908	0.2299
125	923.6940	0.0782
128	924.2768	- 0.0670
131	924.8400	- 0.2061
152	928.2646	- 1.0309
176	931.2343	- 1.7187
200	933.4187	- 2.2101
248	936.1665	- 2.8124
299	937.6623	- 3.1334

Table 19. The optimal solution with equation (4.10)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	20.0000	00.0000	20
	2	00.0000	60.0000	60
	3	40.0000	00.0000	40
	4	00.0000	10.0000	10
	5	04.0666	05.9334	10
	6	28.5067	01.4933	30
	7	45.0000	00.0000	45
	8	07.9872	17.0128	25
	9	06.7288	08.2712	15
	10	07.6437	27.3563	35
W_i		159.9330	130.0670	290

Table 20. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in $	Value of Add. Constraint
Initial	1173.9980	90.0000
1	1015.0410	65.0234
5	942.7617	23.9628
21	893.0495	23.6714
66	891.1103	14.1664
69	891.1125	13.7615
98	896.4360	4.7275
149	897.9758	2.0024
200	899.2529	0.6898
242	899.9104	0.1289
254	900.0451	0.0204
257	900.0761	- 0.0040
260	900.1064	- 0.0276
272	900.2153	- 0.1125
299	900.4052	- 0.2586

Table 21. The optimal solution with equation (4.11)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	19.3787	00.6213	20
	2	00.0000	60.0000	60
	3	29.4717	10.5283	40
	4	00.0000	10.0000	10
	5	08.6938	01.3062	10
	6	28.8130	01.1870	30
	7	45.0000	00.0000	45
	8	08.0339	16.9661	25
	9	05.3177	09.6823	10
	10	15.2872	19.7128	35
W_i		159.9960	130.0040	290

the optimal results of this problem.

Tables 22 and 23 are obtained with a starting control sequence which is below feasible and which is given by Eq. (4.12). Table 22 shows convergence rate of cost with the above starting control sequence values. Optimal is reached at iteration 60. Table 23 shows the optimal results of this problem.

Compare the values of cost and additional constraint in Tables 18, 20 and 22 corresponding to iterations 128, 257 and 60 respectively. The values are different for all three different starting control value sequences. In each case the optimal is reached but the values are different. It should be noted that the results of Tables 20 and 21 are similar to those obtained in Tables 14 and 15. To show that this is coincidental and that it has nothing to do with making the choice of starting control sequence above feasible, one more starting control sequence above feasible is presented.

Tables 24 and 25 are obtained by using the following starting control sequence:

$$\theta_{11} = \theta_{12} \cdot\cdot = \theta_{110} = 20 \ . \tag{4.13}$$

Table 24 gives convergence rate of cost with above starting control sequence values. Optimal is reached at iteration 92; Table 25 gives the optimal results of this problem. Again, the values of cost and additional constraint are different from that obtained by previous starting points.

4.5 Discussion

The two dimensional problems discussed in this chapter show that the method of equating $\left.\frac{\partial S}{\partial \theta_1}\right|_N$ to $\left.\frac{\partial S}{\partial \theta_1}\right|_{N-1}$, is not fruitful. The approach of calculating

Table 22. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in \$	Value of Add. Constraint
Initial	1485.9980	-60.0000
1	1394.9120	-62.7674
5	1036.2660	-23.8404
14	916.2873	- 1.6870
15	911.7521	4.4153
18	913.0971	3.9700
45	922.7790	1.0415
57	926.1752	0.1600
60	926.9516	- 0.0336
63	927.7011	- 0.2178
99	934.8649	- 1.8752
150	940.7912	- 3.1383
201	943.8134	- 3.7546
249	945.2600	- 4.0438
297	946.0061	- 4.1916

Table 23. The optimal solution with equation (4.12)

	n \ i	Depots 1	Depots 2	D_n
Demand Points	1	20.0000	00.0000	20
	2	00.0000	60.0000	60
	3	40.0000	00.0000	40
	4	00.0000	10.0000	10
	5	04.0669	05.9331	10
	6	28.5066	07.4934	30
	7	45.0000	00.0000	45
	8	08.0437	16.9563	25
	9	07.1746	07.8254	15
	10	07.1746	27.8254	35
W_i		159.9664	130.0336	290

Table 24. Convergence rate of transportation cost with $\Delta\phi$ = 10000

Iter. Number	Cost in $	Value of Add. Constraint
Initial	1195.9980	40.0000
1	1057.1500	26.9746
4	909.3122	21.5870
9	875.2243	28.5331
12	876.3911	25.4024
26	890.3395	10.4492
50	894.5195	04.9208
80	900.2968	01.0460
89	901.8515	00.2393
92	902.3483	-00.0046
95	902.8347	-00.2374
145	909.6582	-03.0524
200	913.2717	-04.3097
251	915.1706	-04.9265
299	916.0932	-05.2168

Table 25. The optimal solution with equation (4.13)

n \ i		Depots 1	Depots 2	D_n
Demand Points	1	19.3786	00.6214	20
	2	00.0000	60.0000	60
	3	29.4702	10.5298	40
	4	00.0000	10.0000	10
	5	08.6939	01.3061	10
	6	28.8127	01.1873	30
	7	45.0000	00.0000	45
	8	08.0919	16.9081	25
	9	07.8180	07.1820	15
	10	12.7301	22.2699	35
W_i		159.9954	130.0046	290

$\left.\frac{\partial S}{\partial \theta_1}\right|_N$ in which differentiation of the objective function with respect to θ_{iN} is used works efficiently. Increasing the number of stages results in greater accuracy of additional constraint with the same efforts. In the 2x3 problem, more accuracy can be obtained by using $\Delta\phi = 0.1$ or 0.01 but the convergence rate is very slow and takes more computational time. Different starting control variables sequences are used to assure absolute optimum.

The 2x3 problem was also tried with $\left.\frac{\partial S}{\partial \theta_1}\right|_3 = \left.\frac{\partial S}{\partial \theta_1}\right|_2$ though the results are not tabulated. The optimal cost stayed far from the optimum. It can be noticed that the last stage is misguided because of the assumption $\left.\frac{\partial S}{\partial \theta_1}\right|_N = \left.\frac{\partial S}{\partial \theta_1}\right|_{N-1}$. The reasons for obtaining similar results in 2x10 problem by both the ways, with $\theta_{in} = 25$ as a starting point, are because of the values of total demands and the constraint $0 \leq \theta_{in} \leq D_n$. A better agreement is obtained in the total costs for the 2 x 10 problem than for the 2 x 3 problem. It is obvious that with the increase in the number of stages the assumption of $\left.\frac{\partial S}{\partial \theta_1}\right|_N = \left.\frac{\partial S}{\partial \theta_1}\right|_{N-1}$ becomes more realistic. Thus, this is a fairly good approximation for continuous processes.

5. APPLICATION TO THE THREE DIMENSIONAL (THREE ORIGINS) PROBLEM

5.1 Three Origins and Three Demand Points

This problem has been solved by the maximum principle (23). Transportation costs and other requirements are shown in Table 26. It is necessary to determine the number of units transported, θ_{in}, i=1,2,3 and n=1,2,3. The values of θ_{3n}, n=1,2,3 are calculated by using condition in Eq. (3.4). The cost of transportation is given by

$$F_{in}(\theta_{in}) = a_{in}\,\theta_{in} + b_{in}\,(\theta_{in})^2 \quad i=1,2,3 \text{ and } n=1,2,3. \tag{5.1}$$

The cost of transportation must be minimized.

In this problem there are three state variables, two control variables sequences and three stages. For convenience this will be called a 3x3 problem.

5.2 Computational Aspects and Results

The problem described in Section 4.1 is solved by using three different sets of starting control variables sequences. The value of the gradients at the last stages i.e. $\left.\frac{\partial S}{\partial \theta_1}\right|_3$ and $\left.\frac{\partial S}{\partial \theta_2}\right|_3$ are calculated by differentiating the objective function with respect to θ_{13} and θ_{23}, respectively. The objective function given by Eq. 4.1 can be written as

$$F_{1n}(\theta_{1n}) + F_{2n}(\theta_{2n}) + F_{3n}(\theta_{3n}) = a_{1n}\,\theta_{1n} + b_{1n}\,(\theta_{1n})^2$$

$$+ a_{2n}\,\theta_{2n} + b_{2n}\,(\theta_{2n})^2 + a_{3n}\,\theta_{3n} + b_{3n}\,(\theta_{3n})^2 \tag{5.2}$$

Table 26. Transportation costs and requirements for 3x3 problem.

n \ i	Depots 1		Depots 2		Depots 3		D_n
	a_{1n}	b_{1n}	a_{2n}	b_{3n}	a_{3n}	b_{3n}	
1	2.5		2.6		1.0		20
2	3.0	.01	2.7		9.0		60
3	6.0		5.0	.01	6.6		40
W_i	50		30		40		120

From Eq. (3.4)

$$\theta_{3n} = D_n - \theta_{1n} - \theta_{2n} \tag{5.3}$$

Substituting the value of θ_{3n} into Eq. (5.2) gives

$$F_{1n}(\theta_{1n}) + F_{2n}(\theta_{2n}) + F_{3n}(\theta_{3n}) = a_{1n} \cdot \theta_{1n} + b_{1n}(\theta_{1n})^2$$

$$+ a_{2n}\ \theta_{2n} + b_{2n}(\theta_{2n})^2 + a_{3n}(D_n - \theta_{1n} - \theta_{2n}) + b_{3n}(D_n - \theta_{1n} - \theta_{2n})^2$$

(5.4)

Therefore taking the partial derivative of Eq. (5.4) with respect to θ_{1n} and evaluating at n=3 results in

$$\left.\frac{\partial S}{\partial \theta_1}\right|_3 = \left[a_{1n} + 2b_{1n}\ \theta_{1n} - a_{3n} - 2b_{3n}(D_n - \theta_{1n} - \theta_{2n})\right]\Big|_3 \tag{5.5}$$

Similarly taking partial derivative of Eq. (5.4) with respect to θ_{2n} and evaluating at n=3 results in

$$\left.\frac{\partial S}{\partial \theta_2}\right|_3 = \left[a_{2n} + 2b_{2n}\ \theta_{2n} - a_{3n} - 2b_{3n}(D_n - \theta_{1n} - \theta_{2n})\right]\Big|_3 \tag{5.6}$$

The values of $\left.\frac{\partial Z}{\partial \theta_1}\right|_3$ and $\left.\frac{\partial Z}{\partial \theta_2}\right|_3$ are equal to $\left.\frac{\partial Z}{\partial \theta_1}\right|_2$ and $\left.\frac{\partial Z}{\partial \theta_2}\right|_2$ respectively.

The suitable value of $\Delta\phi$ is estimated by trial and error.

Tables 27, 28 and 29 are obtained by using the following feasible starting controls sequences:

$$\theta_{11} = \theta_{12} = \theta_{13} = 16.66 \text{ and } \theta_{21} = \theta_{22} = \theta_{23} = 10 \tag{5.7}$$

Table 27 has a larger value of $\Delta\phi$ and uses control sequences given by Eq. (5.7). Table 28 is continuation of Table 27 with a smaller $\Delta\Phi$. The starting values used are the results of iteration 16 in Table 27. Table 28 gives the optimal results of this problem, which are from iteration 154 in Table 28.

Tables 30, 31 and 32 are obtained using the starting controls sequences

$$\theta_{11} = \theta_{12} = \theta_{13} = 20 \text{ and } \theta_{21} = \theta_{22} = \theta_{23} = 15 \ . \tag{5.8}$$

Table 30 has a larger value of $\Delta\phi$ and uses control sequences given by Eq. (5.8). Table 31 is continuation of Table 30 with a smaller $\Delta\phi$. The starting values used are the results of iteration 13 in Table 30. Table 32 gives the optimal results of this problem, which are from iteration 238 in Table 31.

Tables 33, 34 and 35 are obtained by using the starting control sequences

$$\theta_{11} = \theta_{12} = \theta_{13} = 10 \text{ and } \theta_{21} = \theta_{22} = \theta_{23} = 5. \tag{5.9}$$

Table 33 has a larger value of $\Delta\phi$ and uses control sequences given by Eq. (5.9). Table 34 uses a smaller $\Delta\phi$. The starting values used are the results of iteration 20 in Table 33. Table 35 gives the optimal results of this problem, which are from iteration 191 in Table 34.

Comparing the optimal solutions in Tables 28, 31 and 34 shows the

Table 27. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in $	Values of Add. Constraints	
Initial	679.8090	-0.0110	0.0000
1	652.8244	-1.7047	-1.7547
2	633.8146	-0.1827	-0.3000
3	615.4079	0.9781	0.8105
10	495.0190	4.1794	3.9296
16	439.9978	3.4606	6.9870
17	440.2233	3.7487	6.7332
50	442.3161	6.2233	4.8130
100	443.6469	6.3879	4.6974
150	445.3393	6.4462	4.5666

Table 28. Convergence rate of transportation cost with $\Delta\phi = 1$

Iter. Number	Cost in $	Values of Add. Constraints	
Iter. # 16 of Table 27	440.0761	3.4500	6.9700
1	442.6845	3.2270	6.3961
25	455.8662	1.6149	1.8439
50	449.5249	1.3024	1.2864
75	450.6376	0.8273	0.7364
100	450.9836	0.6939	0.5748
154	451.2478	0.6300	0.5000
155	451.1909	0.6356	0.5051
200	451.2490	0.6410	0.5060
250	451.3232	0.6418	0.5048

Table 29. The optimal solution with equation (5.2)

n \ i		Depots 1	2	3	D_n
Demand Points	1	00.0000	00.0000	20.0000	20
	2	37.1300	22.8600	00.0100	60
	3	13.5000	07.6400	18.8600	40
W_i		50.6300	30.5000	38.8700	120

Table 30. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in $	Values of Add. Constraints	
Initial	633.7497	10.0000	15.0000
1	604.0073	3.3750	7.1576
5	539.0107	4.2421	5.3223
10	465.7084	4.6204	8.1749
13	435.8967	4.7107	9.3895
14	436.3403	4.9614	8.9897
25	439.4089	6.2875	6.4579
50	441.7138	6.4878	5.1134
100	443.5368	6.4050	4.7265
150	445.2556	6.4453	4.5742

Table 31. Convergence rate of transportation cost with $\Delta\phi = 1$

Iter. Number	Cost in $	Values of Add. Constraints	
Iter. # 13 of Table 30	435.9633	4.7000	9.3800
1	440.0668	4.3325	8.5513
25	464.7333	1.6711	1.9920
50	456.1032	1.5326	1.5299
75	449.7236	1.2097	1.1720
100	450.6784	0.8003	0.7042
150	451.0881	0.6535	0.5254
230	451.2280	0.6414	0.5071
238	451.2397	0.6414	0.5068
249	451.2558	0.6415	0.5063

Table 32. The optimal solution with equation (5.3)

n \ i		Depots 1	Depots 2	Depots 3	D_n
Demand Points	1	00.0000	00.0000	20.0000	20
	2	37.4133	22.5867	0.0000	60
	3	13.2281	07.9201	18.8518	40
W_i		50.6414	30.5068	38.8518	120

Table 33. Convergence rate of transportation cost with $\Delta\phi = 10$

Iter. Number	Cost in $	Values of Add. Constraints	
Initial	742.7497	-20.0000	-15.0000
1	717.4082	-14.1181	-10.4185
5	628.2099	- 2.3006	- 1.3226
6	609.1860	- 0.6241	- 0.0311
7	590.7736	0.6537	1.0649
20	444.3413	1.2924	5.0689
50	443.3669	5.7594	4.3300
100	443.9184	6.3621	4.6414
125	444.6977	6.4165	4.6049
150	445.5812	6.4506	4.5466

Table 34. Convergence rate of transportation with $\Delta\phi = 1$

Iter. Number	Cost in $	Values of Add. Constraints	
Iter. # 20 of Table 33	444.4082	1.2800	5.0600
1	445.1535	1.1587	4.6872
25	449.4343	1.2288	1.4321
50	450.6965	0.8318	0.7518
75	451.0654	0.6970	0.5764
100	451.1960	0.6574	0.5264
191	451.3723	0.6424	0.5041
200	451.3867	0.6424	0.5037
225	451.4277	0.6426	0.5028
249	451.468	0.6429	0.5020

Table 35. The optimal solution with equation (5.3)

n \ i		Depots 1	2	3	D_n
Demand Points	1	00.0000	00.0000	20.0000	20
	2	38.0044	21.9956	00.0000	60
	3	12.6380	08.5085	18.8535	40
W_i		50.6424	30.5041	38.8535	120

cost of transportation for 3x3 problem is \$451.24. The values of the additional constraints are 0.6414 and 0.5068.

5.3 Discussion

In this three dimensional problem with three stages a similarity in accuracy of the values of additional constraints in comparison to the 2x3 problem is found. The convergence rate is also the same. Better accuracy in both problems is made possible by using smaller values of $\Delta\phi$. Since the convergence is very slow still, smaller values of $\Delta\phi$ were not tried. There is no difficulty in handling a fairly large number of state variables using the gradient technique. Since programming is done in the most general form, it is helpful to extend transportation problems in both directions, that is by increasing both the number of demand points or stages and the number of depots, i.e. by increasing number of state variables.

6. CONCLUSION

A literature survey shows that various optimization techniques such as linear programming, dynamic programming and the maximum principle have been used to solve transportation problems. However, owing to the large dimensionality and the nonlinear nature of the problem, the usefulness of these techniques are frequently limited. Gradient techniques can be used to overcome some of these difficulties.

By selecting three problems, such as 2x3, 2x10 and 3x3, the technique proves its efficiency and ease in handling problems with a fairly large number of state variables as well as with a large number of stages. Using the assumption $\left.\frac{\partial S}{\partial \theta_j}\right|_N = \left.\frac{\partial S}{\partial \theta_j}\right|_{N-1}$ did not work. For all practical purposes, the value of $\left.\frac{\partial S}{\partial \theta_j}\right|_N$ is calculated by differentiating the objective function with respect to θ_{jN}. The approach is more effective and accurate. The convergence rate is also better. Looking to the accuracy of additional constraint or constraints in the three problems, the 2x10 problem had maximum accuracy with the same efforts. The reason for this is the large number of stages in the problem. The reasons for getting different optimal values for different assumed starting points when $\left.\frac{\partial S}{\partial \theta_j}\right|_N = \left.\frac{\partial S}{\partial \theta_j}\right|_{N-1}$ was used are obvious. They are the constraint $0 \leq \theta_{in} \leq D_n$ and the wrong value of last stage improvement. The improvement in the last stage control val. . is not correct as it takes that of the previous stage. However, problems with a very large number of stages may work out as expected.

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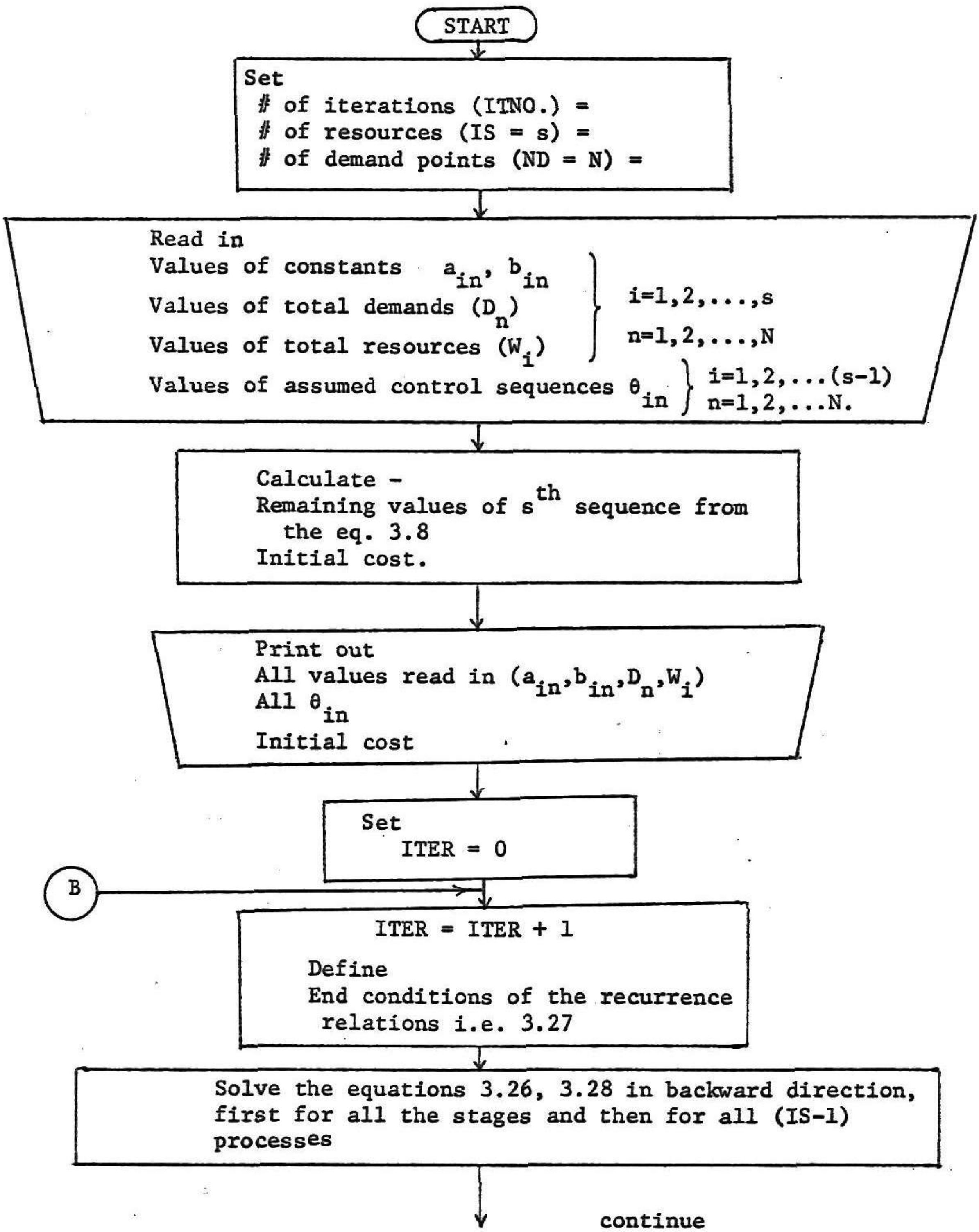
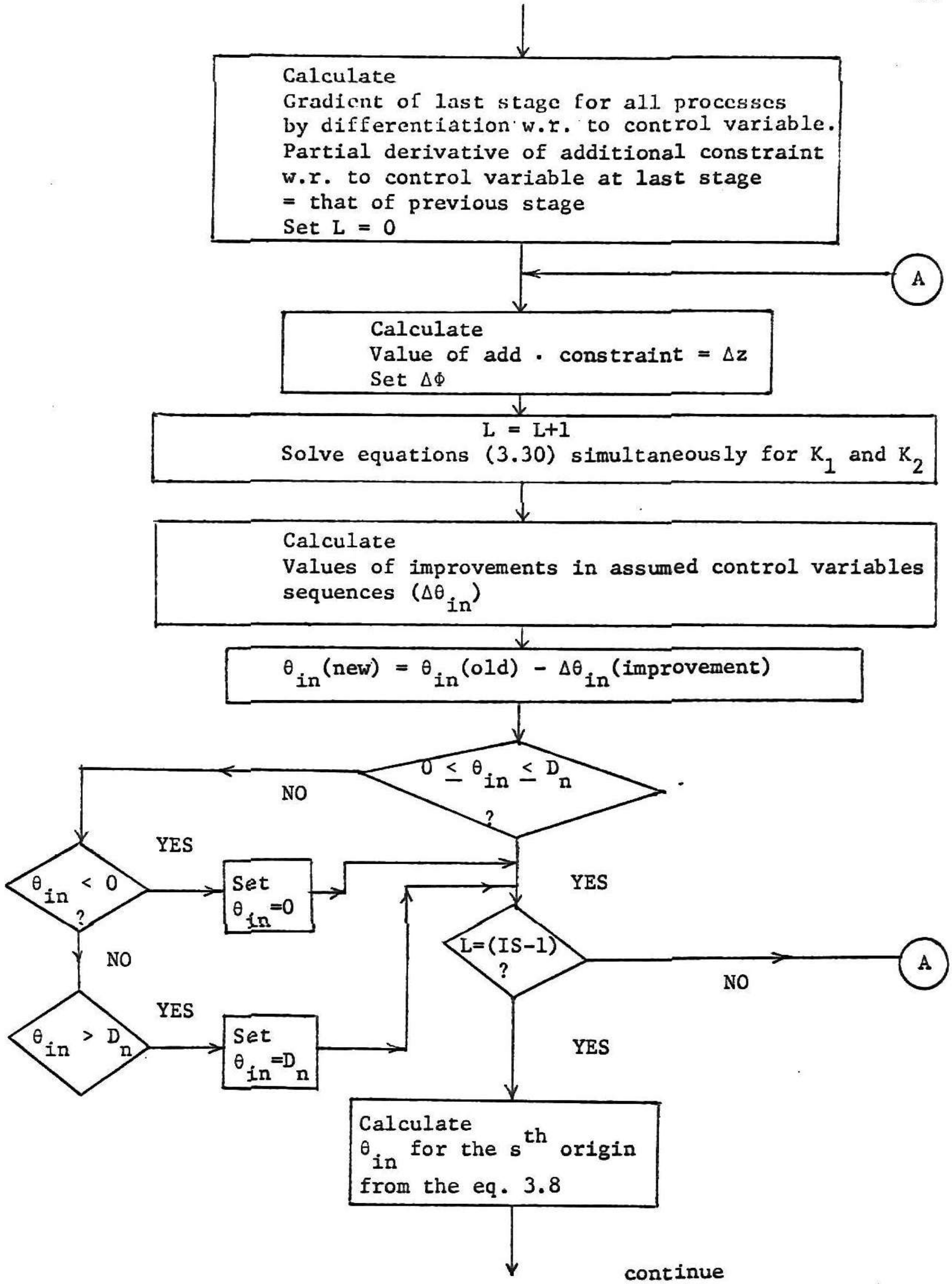
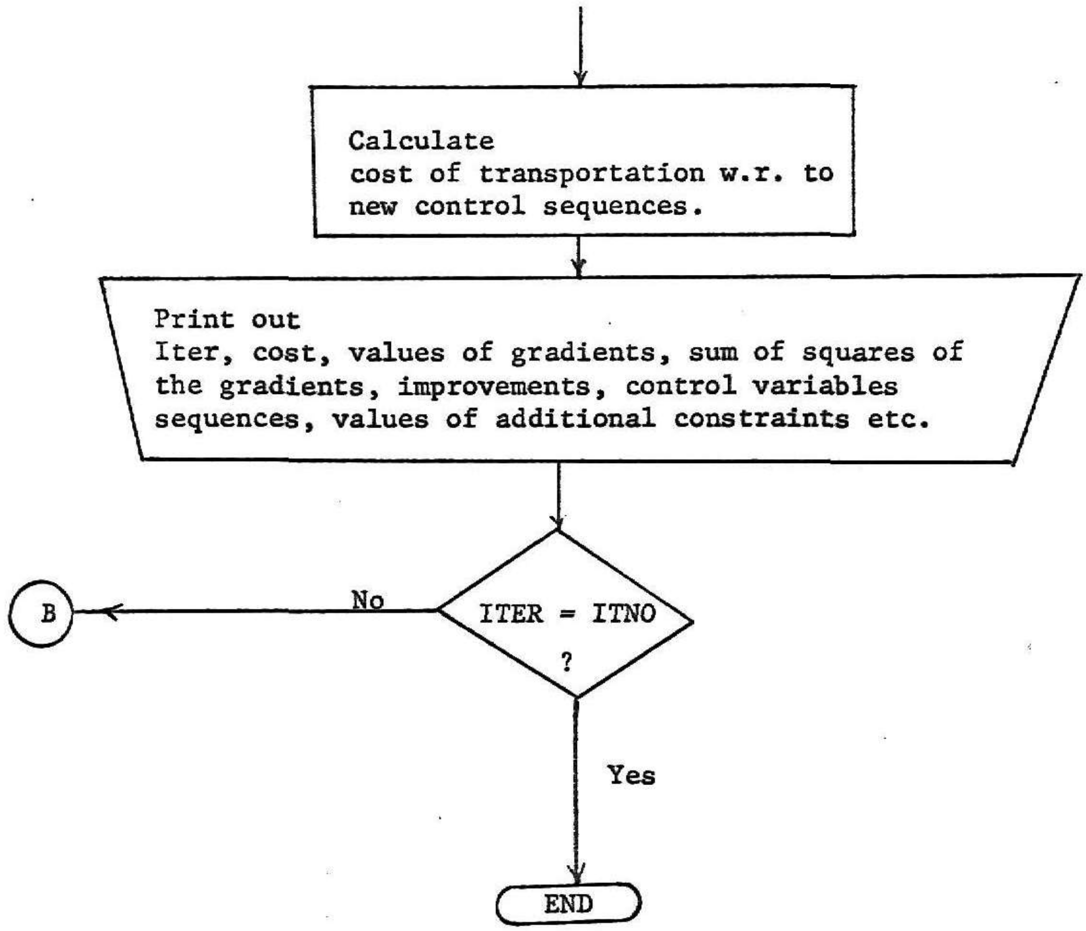
Appendix 1

ComputerFlow Diagram




Calculate
Gradient of last stage for all processes
by differentiation w.r. to control variable.
Partial derivative of additional constraint
w.r. to control variable at last stage
= that of previous stage
Set L = 0
A
Calculate
Value of add . constraint = Δz
Set ΔΦ
L = L+1
Solve equations (3.30) simultaneously for K_1 and K_2
Calculate
Values of improvements in assumed control variables
sequences ($\Delta\theta_{in}$)
θ_{in}(new) = θ_{in}(old) - $\Delta\theta_{in}$(improvement)
$0 \leq \theta_{in} \leq D_n$
?
NO
YES
$\theta_{in} < 0$
?
Set
$\theta_{in}=0$
YES
NO
YES
$\theta_{in} > D_n$
Set
$\theta_{in}=D_n$
L=(IS-1)
?
NO
A
YES
Calculate
θ_{in} for the s^{th} origin
from the eq. 3.8
continue


Calculate
cost of transportation w.r. to
new control sequences.
Print out
Iter, cost, values of gradients, sum of squares of
the gradients, improvements, control variables
sequences, values of additional constraints etc.
ITER = ITNO
?
No
B
Yes
END

Appendix 2

Computer Program for (2x3) Problem

```
C  C   TRANSPORTATION  PROBLEM  BY  GRADIENT  TECHNIQUE
C      *****CHANGE DIMENSION CARDS AS PER VALUES OF IS AND ND*****
       DIMENSION A(10,10),B(10,10),TI(10,10),W(10),D(10),EUM(10,10)
       DIMENSION COSIN(10),FI(10,10)
       DIMENSION DSDX(10,10),DZDX(10,10),SUM(10),CUM(10),DSDT(10,10)
       DIMENSION DZDT(10,10)
       DIMENSION XN(10),DF(10),DZ(10),YUM(10),BUM(10),ZUM(10),S(10)
       DIMENSION C1(10),C2(10),DT(10,10)
       DIMENSION OUM(10),COS(10),F(10,10)
C      CHANGE NEXT THREE CARDS ASPER CHANGES
       ITNC=100
       IS=2
       ND=3
       IZ=IS-1
       NZ=ND-1
C      READ IN INITIAL VALUES (UPTO 92)
C      CHANGE NEXT FOUR FORMAT STATEMENTS AS PER VALUES OF IS AND ND
   101 FORMAT(1H ,6(1X,F8.2))
   102 FORMAT(1H ,3(1X,F8.2))
    91 FORMAT(1H ,2(1X,F8.2))
    92 FORMAT(1H ,3(1X,F8.2))
       READ 101,((A(I,N),I=1,IS),N=1,ND)
       READ 101,((B(I,N),I=1,IS),N=1,ND)
       READ 102,((TI(I,N),I=1,IZ),N=1,ND)
       READ 91,(W(I),I=1,IS)
       READ 92,(D(N),N=1,ND)
       PRINT 300
   300 FORMAT(1H-,'DIFFERENT VALUES READ IN ')
       PRINT 301
   301 FORMAT(1H-,'VALUES OF CONSTANTS A')
       PRINT 101,((A(I,N),I=1,IS),N=1,ND)
       PRINT 302
   302 FORMAT(1H-,'VALUES OF CONSTANTS B')
       PRINT 101,((B(I,N),I=1,IS),N=1,ND)
       PRINT 303
   303 FORMAT(1H-,'VALUES OF CONTROL VARIABLES ASSUMED')
       PRINT 102,((TI(I,N),I=1,IZ),N=1,ND)
       PRINT 304
   304 FORMAT(1H-,'VALUES OF TOTAL RESOURCE AVAIBLE')
       PRINT91,(W(I),I=1,IS)
```

```
      PRINT 305
  305 FORMAT(1H-,'VALUES OF TOTAL DEMAND')
      PRINT92,(D(N),N=1,ND)
C     CALCULATIONS OF REMAINING CONTROL VARIABLES W.R.TO INITIAL VALUES
      MY=C
      JY=IS
  307 MY=MY+1
      EUM(JY,MY)=0
      DO 306I=1,IZ
  306 EUM(JY,MY)=EUM(JY,MY)+TI(I,MY)
      TI(JY,MY)=D(MY)-EUM(JY,MY)
      IF(MY-ND)307,308,308
  308 PRINT 309
  309 FORMAT(1H-,'REMAINING VALUES OF CONTROL VARIABLES CALCULATED FROM-
     1TOTAL DEMANDS')
      PRINT 92,(TI(IS,N),N=1,ND)
C     CALCULATION OF TOTAL COST CORRESPONDING TO INI. CONTROL VARIABLES
      MD=C
      COSTIN=0
  311 MD=MD+1
      COSIN(MD)=0
      DO 310I=1,IS
      FI(I,MD)=A(I,MD)*TI(I,MD)+B(I,MD)*(TI(I,MD)**2)
  310 COSIN(MD)=COSIN(MD)+FI(I,MD)
      COSTIN=COSTIN+COSIN(MD)
      IF(MD-ND)311,312,312
  312 PRINT 313,COSTIN
  313 FORMAT(1H-,'*****INITIAL COST=',F15.8,'*****')
C     *****MAIN PROGRAM FOR ITERATIONS*****
      ITER=0
   15 ITER=ITER+1.
      DO4C1J=1,IZ
      DSDX(J,ND)=0
  401 DZDX(J,ND)=1
      DSDX(IS,ND)=1
      K=0
  406 N=NC-K
      SUM(N)=0
      CUM(N)=0
      DO4C2KL=1,IZ
```

```
      SUM(N)=SUM(N)+DSDX(KL,N)
  402 CUM(N)=CUM(N)+DZDX(KL,N)
      IF((K+1)-ND)403,404,404
  403 DO405KM=1,IZ
      DSDX(KM,N-1)=DSDX(KM,N)+SUM(N)
  405 DZDX(KM,N-1)=DZDX(KM,N)+CUM(N)
      DSDX(IS,N-1)=DSDX(IS,N)*2
      K=K+1
      GO TO 406
  404 LS=0
  409 NK=ND-LS
      DO408KT=1,IZ
      DSDT(KT,NK-1)=SUM(NK)+DSDX(IS,NK)*(A(KT,NK-1)+2*B(KT,NK-1)*TI(KT,N
     1K-1)-A(IS,NK-1)-2*B(IS,NK-1)*TI(IS,NK-1))
  408 DZDT(KT,NK-1)=CUM(NK)
      LS=LS+1
      IF((LS+1)-ND)409,410,410
  410 DO411KM=1,IZ
C     ***VALUE OF DS/DT AT END POINT IS TAKEN FROM USUAL GRADIENT TECH.*
      DSDT(KM,ND)=A(KM,ND)+2*B(KM,ND)*TI(KM,ND)-A(IS,ND)-2*B(IS,ND)*TI(I
     1S,ND)
  411 DZDT(KM,ND)=DZDT(KM,ND-1)
      L=0
    9 L=L+1.
      XN(L)=0
C     CHANGE NEXT CARD FOR VALUE OF DELTABARFAAY
      DF(L)=10.0
      DO 99MY=1,ND
   99 XN(L)=XN(L)+TI(L,MY)
      DZ(L)=XN(L)-W(L)
      YUM(L)=0
      BUM(L)=0
      ZUM(L)=0
      DO 106NP=1,ND
      YUM(L)=YUM(L)+DSDT(L,NP)*DZDT(L,NP)
      BUM(L)=BUM(L)+DSDT(L,NP)**2
  106 ZUM(L)=ZUM(L)+DZDT(L,NP)**2
C     SOLVING SIMULTANEOUSLY FOR K1 AND K2
      S(L)=((BUM(L))*(ZUM(L))-(YUM(L))*(YUM(L)))
      C1(L)=((DF(L))*(ZUM(L))-(YUM(L))*(DZ(L)))/S(L)
```

```
      C2(L)=((BUM(L))*(DZ(L))-(DF(L))*(YUM(L)))/S(L)
      DO314M=1,ND
      DT(L,M)=C1(L)*DSDT(L,M)+C2(L)*DZDT(L,M)
      TI(L,M)=TI(L,M)-DT(L,M)
      IF(TI(L,M))21,22,22
   21 TI(L,M)=0
   22 IF(TI(L,M)-D(M))314,314,24
   24 TI(L,M)=D(M)
  314 CONTINUE
      IF(L-IZ)9,10,10
   10 MX=0
   11 MX=MX+1.
      OUM(MX)=0
      DO 107I=1,IZ
  107 OUM(MX)=OUM(MX)+TI(I,MX)
      TI(IS,MX)=D(MX)-OUM(MX)
      IF(MX-ND)11,12,12
   12 MO=0
      COST=0
   13 MO=MO+1.
      COS(MO)=0
      DO 108I=1,IS
      F(I,MO)=(A(I,MO))*(TI(I,MO))+(B(I,MO))*(TI(I,MO)**2)
  108 COS(MO)=COS(MO)+F(I,MO)
      COST=COST+COS(MO)
      IF(MO-ND)13,14,14
   14 PRINT 203,(DZ(L),L=1,IZ)
  203 FORMAT(1H-,'ADDITIONAL CONSTRAINT VALUE =',E15.8/)
      PRINT 111,ITER,COST
  111 FORMAT(1H-,'************************************  ITERATION NO.',I4
     1 ,'    COST=',F15.8,'  ************************************')
      PRINT 204
  204 FORMAT(1H-,' VALUES OF IMPROVEMENTS IN CONTROL VARIABLES ')
C     ***CHANGE FORMAT 205 AS PER VALUES OF IS AND ND ***
      DO 206IP=1,IZ
      PRINT 205,(DT(IP,NP),NP=1,ND)
  205 FORMAT(1H ,3(E15.8,4X)/)
  206 CONTINUE
      PRINT 112
  112 FORMAT(1H-,'SUM OF DS/DT SQUARE',4X,'SUM OF DZ/DT SQUARE',4X,'SUM
```

```
      1OF DS/DT*DZ/DT')
       DO 201I=1,IZ
       PRINT 113,BUM(I),ZUM(I),YUM(I)
   113 FORMAT(1H ,3(E15.8,5X)/)
   201 CONTINUE
C      CHANGE NEXT DO LOOP,AS WELL AS HEADINGS,AS PER 'IS' VARIES
       PRINT 114
   114 FORMAT(1H ,'  NUMBER  OF  UNITS  TRANSPORTED',2X,'     DS/DXI    '
      1,2X,'     DS/DT      ',2X,'     DZ/DT      ',2X,'     DS/DXS')
       DO 202N=1,ND
       PRINT 115,(TI(I,N),I=1,IS),(DSDX(I,N),DSDT(I,N),DZDT(I,N),I=1,IZ),
      1DSDX(IS,N)
   115 FORMAT(1H ,6(E15.8,2X)/)
   202 CONTINUE
       IF(ITER-ITNO)15,16,16
    16 STOP
       END
```

Appendix 3

Computer Program for (2x10) Problem

```
C  C  TRANSPORTATION  PROBLEM  BY  GRADIENT  TECHNIQUE
C     *****CHANGE DIMENSION CARDS AS PER VALUES OF IS AND ND*****
      DIMENSION A(10,10),B(10,10),TI(10,10),W(10),D(10),EUM(10,10)
      DIMENSION COSIN(10),FI(10,10)
      DIMENSION DSDX(10,10),DZDX(10,10),SUM(10),CUM(10),DSDT(10,10)
      DIMENSION DZDT(10,10)
      DIMENSION XN(10),DF(10),DZ(10),YUM(10),BUM(10),ZUM(10),S(10)
      DIMENSION C1(10),C2(10),DT(10,10)
      DIMENSION OUM(10),COS(10),F(10,10)
C     CHANGE NEXT THREE CARDS ASPER CHANGES
      ITNC=300
      IS=2
      ND=10
      IZ=IS-1
      NZ=ND-1
C     READ IN INITIAL VALUES (UPTO 92)
C     CHANGE NEXT TWO  FORMAT STATEMENTS AS PER VALUES OF IS AND ND
  100 FORMAT(10(1X,F7.2))
   92 FORMAT(2(1X,F8.2))
      READ 100,((A(I,N),I=1,IS),N=1,ND)
      READ 100,((B(I,N),I=1,IS),N=1,ND)
      READ 100,((TI(I,N),I=1,IZ),N=1,ND)
      READ 100,(D(N),N=1,ND)
      READ 92,(W(I),I=1,IS)
C     CHANGE NEXT '3'  FORMAT STATEMENTS AS PER VALUES OF IS AND ND
  101 FORMAT(1H ,10(1X,F7.2)/1H ,10(1X,F7.2))
  102 FORMAT(1H ,10(1X,F7.2))
   91 FORMAT(1H ,2(1X,F8.2))
      PRINT 300
  300 FORMAT(1H-,'DIFFERENT VALUES READ IN ')
      PRINT 301
  301 FORMAT(1H-,'VALUES OF CONSTANTS A')
      PRINT 101,((A(I,N),I=1,IS),N=1,ND)
      PRINT 302
  302 FORMAT(1H-,'VALUES OF CONSTANTS B')
      PRINT 101,((B(I,N),I=1,IS),N=1,ND)
      PRINT 303
  303 FORMAT(1H-,'VALUES OF CONTROL VARIABLES ASSUMED')
      PRINT 102,((TI(I,N),I=1,IZ),N=1,ND)
      PRINT 305
```

```
 305 FORMAT(1H-,'VALUES OF TOTAL DEMAND')
     PRINT 102,(D(N),N=1,ND)
     PRINT 304
 304 FORMAT(1H-,'VALUES OF TOTAL RESOURCE AVAIBLE')
     PRINT91,(W(I),I=1,IS)
C    CALCULATIONS OF REMAINING CONTROL VARIABLES W.R.TO INITIAL VALUES
     MY=0
     JY=IS
 307 MY=MY+1
     EUM(JY,MY)=0
     DO 306I=1,IZ
 306 EUM(JY,MY)=EUM(JY,MY)+TI(I,MY)
     TI(JY,MY)=D(MY)-EUM(JY,MY)
     IF(MY-ND)307,308,308
 308 PRINT 309
 309 FORMAT(1H-,'REMAINING VALUES OF CONTROL VARIABLES CALCULATED FROM-
    1TOTAL DEMANDS')
     PRINT 92,(TI(IS,N),N=1,ND)
C    CALCULATION OF TOTAL COST CORRESPONDING TO INI. CONTROL VARIABLES
     MD=0
     COSTIN=0
 311 MD=MD+1
     COSIN(MD)=0
     DO 310I=1,IS
     FI(I,MD)=A(I,MD)*TI(I,MD)+B(I,MD)*(TI(I,MD)**2)
 310 COSIN(MD)=COSIN(MD)+FI(I,MD)
     COSTIN=COSTIN+COSIN(MD)
     IF(MD-ND)311,312,312
 312 PRINT 313,COSTIN
 313 FORMAT(1H-,'*****INITIAL COST=',F15.8,'*****')
C    *****MAIN PROGRAM FOR ITERATIONS*****
     ITER=0
  15 ITER=ITER+1.
     DO401J=1,IZ
     DSDX(J,ND)=0
 401 DZDX(J,ND)=1
     DSDX(IS,ND)=1
     K=0
 406 N=ND-K
     SUM(N)=0
```

```
      CUM(N)=0
      DO4C2KL=1,IZ
      SUM(N)=SUM(N)+DSDX(KL,N)
  402 CUM(N)=CUM(N)+DZDX(KL,N)
      IF((K+1)-ND)403,404,404
  403 DO4C5KM=1,IZ
      DSDX(KM,N-1)=DSDX(KM,N)+SUM(N)
  405 DZDX(KM,N-1)=DZDX(KM,N)+CUM(N)
      DSDX(IS,N-1)=DSDX(IS,N)*2
      K=K+1
      GO TO 406
  404 LS=C
  409 NK=ND-LS
      DO4C8KT=1,IZ
      DSDT(KT,NK-1)=SUM(NK)+DSDX(IS,NK)*(A(KT,NK-1)+2*B(KT,NK-1)*TI(KT,N
     1K-1)-A(IS,NK-1)-2*B(IS,NK-1)*TI(IS,NK-1))
  408 DZDT(KT,NK-1)=CUM(NK)
      LS=LS+1
      IF((LS+1)-ND)409,410,410
  410 DO411KM=1,IZ
C     ***VALUE OF DS/DT AT END POINT IS TAKEN FROM USUAL GRADIENT TECH.*
      DSDT(KM,ND)=A(KM,ND)+2*B(KM,ND)*TI(KM,ND)-A(IS,ND)-2*B(IS,ND)*TI(I
     1S,ND)
  411 DZDT(KM,ND)=DZDT(KM,ND-1)
      L=0
    9 L=L+1.
      XN(L)=0
C     CHANGE NEXT CARD FOR VALUE OF DELTABARFAAY
      DF(L)=10000.0
      DO 99MY=1,ND
   99 XN(L)=XN(L)+TI(L,MY)
      DZ(L)=XN(L)-W(L)
      YUM(L)=0
      BUM(L)=0
      ZUM(L)=0
      DO 106NP=1,ND
      YUM(L)=YUM(L)+DSDT(L,NP)*DZDT(L,NP)
      BUM(L)=BUM(L)+DSDT(L,NP)**2
  106 ZUM(L)=ZUM(L)+DZDT(L,NP)**2
C     SOLVING SIMULTANEOUSLY FOR K1 AND K2
```

```
      S(L)=((BUM(L))*(ZUM(L))-(YUM(L))*(YUM(L)))
      C1(L)=((DF(L))*(ZUM(L))-(YUM(L))*(DZ(L)))/S(L)
      C2(L)=((BUM(L))*(DZ(L))-(DF(L))*(YUM(L)))/S(L)
      DO314M=1,ND
      DT(L,M)=C1(L)*DSDT(L,M)+C2(L)*DZDT(L,M)
      TI(L,M)=TI(L,M)-DT(L,M)
      IF(TI(L,M))21,22,22
   21 TI(L,M)=0
   22 IF(TI(L,M)-D(M))314,314,24
   24 TI(L,M)=D(M)
  314 CONTINUE
      IF(L-IZ)9,10,10
   10 MX=C
   11 MX=MX+1.
      OUM(MX)=0
      DO 107I=1,IZ
  107 OUM(MX)=OUM(MX)+TI(I,MX)
      TI(IS,MX)=D(MX)-OUM(MX)
      IF(MX-ND)11,12,12
   12 MO=C
      COST=0
   13 MO=MO+1.
      COS(MO)=0
      DO 108I=1,IS
      F(I,MO)=(A(I,MO))*(TI(I,MO))+(B(I,MO))*(TI(I,MO)**2)
  108 COS(MO)=COS(MO)+F(I,MO)
      COST=COST+COS(MO)
      IF(MO-ND)13,14,14
   14 PRINT 203,(DZ(L),L=1,IZ)
  203 FORMAT(1H-,'ADDITIONAL CONSTRAINT VALUE =',E15.8/)
      PRINT 111,ITER,COST
  111 FORMAT(1H-,'**************************************  ITERATION NO.',I4
     1 ,'    COST=',F15.8,'  **************************************')
      PRINT 112
  112 FORMAT(1H-,'SUM OF DS/DT SQUARE',4X,'SUM OF DZ/DT SQUARE',4X,'SUM
     1OF DS/DT*DZ/DT')
      DO 201I=1,IZ
      PRINT 113,BUM(I),ZUM(I),YUM(I)
  113 FORMAT(1H ,3(E15.8,7X)/)
  201 CONTINUE
```

```
C     CHANGE NEXT DO LOOP,AS WELL AS HEADINGS,AS PER 'IS' VARIES
      PRINT 114
  114 FORMAT(1H ,'  IMPROVEMENTS   ',2X,'  NUMBER  OF  UNITS  TRANSPORT
     1ED',2X,'      DS/DXI    ',2X,'      DS/DT      ',2X,'      DZ/DT      '
     1,2X,'      DS/DXS')
      DO 202N=1,ND
      PRINT 115,(DT(I,N),I=1,IZ),(TI(I,N),I=1,IS),(DSDX(I,N),DSDT(I,N),D
     1ZDT(I,N),I=1,IZ),DSDX(IS,N)
  115 FORMAT(1H ,7(2X,E15.8))
  202 CONTINUE
      IF(ITER-ITNO)15,16,16
   16 STOP
      END
```

Appendix 4

Computer Program for (3x3) Problem

```
C  C  TRANSPORTATION  PROBLEM  BY  GRADIENT  TECHNIQUE
C     *****CHANGE DIMENSION CARDS AS PER VALUES OF IS AND ND*****
      DIMENSION A(10,10),B(10,10),TI(10,10),W(10),D(10),EUM(10,10)
      DIMENSION COSIN(10),FI(10,10)
      DIMENSION DSDX(10,10),DZDX(10,10),SUM(10),CUM(10),DSDT(10,10)
      DIMENSION DZDT(10,10)
      DIMENSION XN(10),DF(10),DZ(10),YUM(10),BUM(10),ZUM(10),S(10)
      DIMENSION C1(10),C2(10),DT(10,10)
      DIMENSION AUM(10),DEPRI(10)
      DIMENSION OUM(10),COS(10),F(10,10)
C     CHANGE NEXT THREE CARDS ASPER CHANGES
      IINC=250
      IS=3
      ND=3
      IZ=IS-1
      NZ=ND-1
C     READ IN INITIAL VALUES (UPTO 92)
C     CHANGE NEXT FOUR FORMAT STATEMENTS AS PER VALUES OF IS AND ND
  101 FORMAT(1H ,9(1X,F6.2))
  102 FORMAT(1H ,6(1X,F8.2))
   91 FORMAT(1H ,3(1X,F8.2))
   92 FORMAT(1H ,3(1X,F8.2))
      READ 101,((A(I,N),I=1,IS),N=1,ND)
      READ 101,((B(I,N),I=1,IS),N=1,ND)
      READ 102,((TI(I,N),I=1,IZ),N=1,ND)
      READ 91,(W(I),I=1,IS)
      READ 92,(D(N),N=1,ND)
      PRINT 300
  300 FORMAT(1H-,'DIFFERENT VALUES READ IN ')
      PRINT 301
  301 FORMAT(1H-,'VALUES OF CONSTANTS A')
      PRINT 101,((A(I,N),I=1,IS),N=1,ND)
      PRINT 302
  302 FORMAT(1H-,'VALUES OF CONSTANTS B')
      PRINT 101,((B(I,N),I=1,IS),N=1,ND)
      PRINT 303
  303 FORMAT(1H-,'VALUES OF CONTROL VARIABLES ASSUMED')
      PRINT 102,((TI(I,N),I=1,IZ),N=1,ND)
      PRINT 304
  304 FORMAT(1H-,'VALUES OF TOTAL RESOURCE AVAIBLE')
```

```
      PRINT91,(W(I),I=1,IS)
      PRINT 305
  305 FORMAT(1H-,'VALUES OF TOTAL DEMAND')
      PRINT92,(D(N),N=1,ND)
C     CALCULATIONS OF REMAINING CONTROL VARIABLES W.R.TO INITIAL VALUES
      MY=0
      JY=IS
  307 MY=MY+1
      EUM(JY,MY)=0
      DO 306I=1,IZ
  306 EUM(JY,MY)=EUM(JY,MY)+TI(I,MY)
      TI(JY,MY)=D(MY)-EUM(JY,MY)
      IF(MY-ND)307,308,308
  308 PRINT 309
  309 FORMAT(1H-,'REMAINING VALUES OF CONTROL VARIABLES CALCULATED FROM-
     1TOTAL DEMANDS')
      PRINT 92,(TI(IS,N),N=1,ND)
C     CALCULATION OF TOTAL COST CORRESPONDING TO INI. CONTROL VARIABLES
      MD=0
      COSTIN=0
  311 MD=MD+1
      COSIN(MD)=0
      DO 310I=1,IS
      FI(I,MD)=A(I,MD)*TI(I,MD)+B(I,MD)*(TI(I,MD)**2)
  310 COSIN(MD)=COSIN(MD)+FI(I,MD)
      COSTIN=COSTIN+COSIN(MD)
      IF(MD-ND)311,312,312
  312 PRINT 313,COSTIN
  313 FORMAT(1H-,'*****INITIAL COST=',F15.8,'*****')
C     *****MAIN PROGRAM FOR ITERATIONS*****
      ITER=0
   15 ITER=ITER+1.
      DO401J=1,IZ
      DSDX(J,ND)=0
  401 DZDX(J,ND)=1
      DSDX(IS,ND)=1
      K=0
  406 N=ND-K
      SUM(N)=0
      CUM(N)=0
```

```
      DO402KL=1,IZ
      SUM(N)=SUM(N)+DSDX(KL,N)
  402 CUM(N)=CUM(N)+DZDX(KL,N)
      IF((K+1)-ND)403,404,404
  403 DO405KM=1,IZ
      DSDX(KM,N-1)=DSDX(KM,N)+SUM(N)
  405 DZDX(KM,N-1)=DZDX(KM,N)+CUM(N)
      DSDX(IS,N-1)=DSDX(IS,N)*2
      K=K+1
      GO TO 406
  404 LS=0
  409 NK=ND-LS
      DO408KT=1,IZ
      DSDT(KT,NK-1)=SUM(NK)+DSDX(IS,NK)*(A(KT,NK-1)+2*B(KT,NK-1)*TI(KT,N
     1K-1)-A(IS,NK-1)-2*B(IS,NK-1)*TI(IS,NK-1))
  408 DZDT(KT,NK-1)=CUM(NK)
      LS=LS+1
      IF((LS+1)-ND)409,410,410
  410 DO411KM=1,IZ
C     ***VALUE OF DS/DT AT END POINT IS TAKEN FROM USUAL GRADIENT TECH.*
      DSDT(KM,ND)=A(KM,ND)+2*B(KM,ND)*TI(KM,ND)-A(IS,ND)-2*B(IS,ND)*TI(I
     1S,ND)
  411 DZDT(KM,ND)=DZDT(KM,ND-1)
      L=0
    9 L=L+1.
      XN(L)=0
C     CHANGE NEXT CARD FOR VALUE OF DELTABARFAAY
      DF(L)=1.0
      DO 99MY=1,ND
   99 XN(L)=XN(L)+TI(L,MY)
      DZ(L)=XN(L)-W(L)
      YUM(L)=0
      BUM(L)=0
      ZUM(L)=0
      DO 106NP=1,ND
      YUM(L)=YUM(L)+DSDT(L,NP)*DZDT(L,NP)
      BUM(L)=BUM(L)+DSDT(L,NP)**2
  106 ZUM(L)=ZUM(L)+DZDT(L,NP)**2
C     SOLVING SIMULTANEOUSLY FOR K1 AND K2
      S(L)=((BUM(L))*(ZUM(L))-(YUM(L))*(YUM(L)))
```

```
      C1(L)=((DF(L))*(ZUM(L))-(YUM(L))*(DZ(L)))/S(L)
      C2(L)=((BUM(L))*(DZ(L))-(DF(L))*(YUM(L)))/S(L)
      DO314M=1,ND
      DT(L,M)=C1(L)*DSDT(L,M)+C2(L)*DZDT(L,M)
      TI(L,M)=TI(L,M)-DT(L,M)
      IF(TI(L,M))21,22,22
   21 TI(L,M)=0
   22 IF(TI(L,M)-D(M))314,314,24
   24 TI(L,M)=D(M)
  314 CONTINUE
      IF(L-IZ)9,10,10
   10 MG=0
  505 MG=MG+1
      AUM(MG)=0
      DO 501ID=1,IZ
      AUM(MG)=AUM(MG)+TI(ID,MG)
  501 CONTINUE
      IF(AUM(MG)-D(MG))502,502,503
  503 DEPRI(MG)=0.5*(AUM(MG)-D(MG))
      DO 504IA=1,IZ
      TI(IA,MG)=TI(IA,MG)-DEPRI(MG)
  504 CONTINUE
  502 IF(MG-ND)505,506,506
  506 MX=0
   11 MX=MX+1.
      OUM(MX)=0
      DO 107I=1,IZ
  107 OUM(MX)=OUM(MX)+TI(I,MX)
      TI(IS,MX)=D(MX)-OUM(MX)
      IF(MX-ND)11,12,12
   12 MO=0
      COST=0
   13 MO=MO+1.
      COS(MO)=0
      DO 108I=1,IS
      F(I,MO)=(A(I,MO))*(TI(I,MO))+(B(I,MO))*(TI(I,MO)**2)
  108 COS(MO)=COS(MO)+F(I,MO)
      COST=COST+COS(MO)
      IF(MO-ND)13,14,14
   14 PRINT 203,(DZ(L),L=1,IZ)
```

```
  203 FORMAT(1H-,'ADDITIONAL CONSTRAINT VALUE =',E15.8/)
      PRINT 111,ITER,COST
  111 FORMAT(1H-,'************************************  ITERATION NO.',I4
     1 ,'     COST=',F15.8,'  ************************************')
      PRINT 112
  112 FORMAT(1H-,'SUM OF DS/DT SQUARE',4X,'SUM OF DZ/DT SQUARE',4X,'SUM
     1OF DS/DT*DZ/DT')
      DO 201I=1,IZ
      PRINT 113,BUM(I),ZUM(I),YUM(I)
  113 FORMAT(1H ,3(E15.8,5X)/)
  201 CONTINUE
C     CHANGE NEXT TWO DO LOOPS,AS WELL AS HEADINGS,AS PER'IS' VARIES**
      PRINT 115
  115 FORMAT(1H ,'    DS/DX1     ',2X,'     DS/DX2     ',2X,'     DS/DX3
     1     ',2X,'    DS/DT1      ',2X,'    DS/DT2      ',2X,'    DZ/DT1    '
     1,4X,'    DZ/DT2      ')
      DO 202N=1,ND
      PRINT 116,(DSDX(I,N),I=1,IS),(DSDT(I,N),I=1,IZ),(DZDT(I,N),I=1,IZ)
  116 FORMAT(1H ,7(E15.8,2X))
  202 CONTINUE
      PRINT 117
  117 FORMAT(1H ,'IMPROVEMENTS IN CONTROL VARIABLES',4X,'         NUMBERS
     1 OF  UNITS  TRANSPORTED          ')
      DO 200N=1,ND
  118 FORMAT(1H ,2(E15.8,2X),4X,3(E15.8,2X))
      PRINT 118,(DT(IP,N),IP=1,IZ),(TI(I,N),I=1,IS)
  200 CONTINUE
      IF(ITER-ITNO)15,16,16
   16 STOP
      END
```

APPLICATION OF A GRADIENT TECHNIQUE

TO THE TRANSPORTATION PROBLEM

by

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Transportation problems with linear cost function can be solved efficiently by linear programming. The same type of problems with non-linear cost function have been approached by dynamic programming and the maximum principle. However these two techniques have computational difficulties for problems with a large number of state variables. It has been shown that the gradient technique is useful for industrial management systems such as inventory and advertising models. In this report, this technique is applied to the transportation problem.

The technique is discussed briefly and the equations are derived in a fairly general form. The matrix form of the transportation problem is converted into multistage serial processes. In deriving the equations, a problem with s depots and N demand points is considered. For this general problem there are s state variables with the sth state variable representing the cost of transportation. The process has N stages.

Three problems are solved. In the first there are two origins and three demand points. In the second there are two origins but ten demand points. In the last there are three origins and three demand points. This particular choice of problems helps show that the gradient technique does not have difficulty in handling more state variables.

In actual computation it was found that the gradient technique with the assumption that the gradient at Nth stage is equal to that of (N-1)th stage gives different values for the optimal if different starting values of the control sequences are used. However, this difficulty can be overcome if the gradient at the Nth stage is obtained directly by differentiation. The accuracies on the additional constraints are fairly good in all the problems solved.