The minimal number of critical points of a smooth function on a closed manifold and the ball category.

by

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B.A California State University, Northridge 2017

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

In this dissertation, we discuss the Lusternik-Schnirelmann category and relevant results. We will then introduce variants for a closed manifold M of the Lusternik-Schnirelmann category using Singhof-Takens fillings as well as a variant for the minimum number of critical points for any smooth function on M. We will show that the category of a Singhof-Takens filling by topological balls with corners is related to the minimum number of critical points for any smooth function on M such that the critical points admit a gradient convex ball neighborhood. We will also show that the category of a Singhof-Takens filling by smooth balls with corners is related to the minimum number of critical points for any smooth function on M.

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Chapter 1

Overview

This chapter serves to give an overview of the main results and organization of this dissertation.

For the main discussion, we will define variants of the LS category as well as a variant for the minimum number of critical points for any smooth function on a smooth closed manifold M, and draw relations from them. We will be proving an estimate that is similar to the estimate of the LS category by replacing $\mathrm{cat}(M)$ with numeric invariants associated with Singhof-Takens fillings, namely $\mathrm{Bcat}(M)$ and $\mathrm{Bcat}^{\mathscr{T}}(M)$. A Singhof-Takens filling[4;5] of a closed smooth manifold M of dimension m is essentially a covering of M by compact smooth submanifolds of dimension m. The compact submanifolds may have corners, while the interiors of covering submanifolds are required to be disjoint. Of particular interest are Singhof-Takens fillings by contractible manifolds, smooth balls with corners, and topological balls which are smooth manifolds homeomorphic to balls, see Remark 48. The least number n such that every Singhof-Takens filling of M by smooth (respectively, topological) balls has at least $n+1$ elements is denoted by $\mathrm{Bcat}(M)$ (respectively, $\mathrm{Bcat}^{\mathscr{T}}(M)$). We also require the function f minimizing the number of critical points on M in the definition of $\mathrm{Crit}(M)$ to be constant and maximal over ∂M and have no critical points near ∂M. Let $\mathrm{Crit}^{\bullet}(M)$ be the minimum number of critical points for smooth function on M such that critical points admit a gradient convex ball neighborhood around them, see definition 31. Roughly speaking, a neighborhood $U \subset f^{-1}[c-\varepsilon, c+\varepsilon]$ of a critical point x_0 with $f(x_0) = c$ is gradient convex if every gradient curve in $f^{-1}[c-\varepsilon, c+\varepsilon]$ intersecting U is in U, and U contains no other critical points of f.

We note that $\mathrm{Bcat}(M) \geqslant \mathrm{Bcat}^{\mathscr{I}}(M)$ and $\mathrm{Crit}^{\bullet}(M) \geqslant Crit(M)$. We will show that $\mathrm{Bcat}(M)$ and $\mathrm{Bcat}^{\mathscr{I}}(M)$ are closely related to $\mathrm{Crit}(M)$ and $\mathrm{Crit}^{\bullet}(M)$ respectively. Our main results are the following theorems.

Theorem 1. *For any closed smooth manifold M, we have $\mathrm{Bcat}^{\mathscr{I}}(M) + 1 \leqslant \mathrm{Crit}^{\bullet}(M)$.*

Theorem 1 should be compared to the Cornea theorem[1], in which a manifold M is decomposed into cones rather than balls and $\mathrm{Crit}^{\mathcal{D}}(M)$ is the minimal number of reasonable critical points for any smooth function that is constant maximal and regular on the boundary. A reasonable critical point is a critical point in which the critical submanifold S is connected and $(f^{-1}(f(S))\backslash S, S)$ is a Whitney stratification of $f^{-1}(f(S))$.

Theorem 2. *[Cornea, 1997] Let M be a smooth compact manifold. Then $\mathrm{cl}(M) + 1 \leqslant \mathrm{Crit}^{\mathcal{D}}(M)$, where $\mathrm{cl}(M)$ is the cone length of M.*

We note that a topological ball is a cone. We also note that if a critical point, x_0, is reasonable then it has a cone neighborhood around x_0 on the level set, see Cornea lemma 2[1]. In example 33 we see that x_0 has a gradient convex ball neighborhood around it. So we get that Theorem 1 is a stronger result than Theorem 2 since

$$\mathrm{cl}(M) + 1 \leqslant \mathrm{Bcat}^{\mathscr{I}}(M) + 1 \leqslant \mathrm{Crit}^{\bullet}(M) \leqslant \mathrm{Crit}^{\mathcal{D}}(M).$$

Using a result by Takens[5], we get a partial converse to the LS estimate by

Theorem 3. *For any closed manifold M, we have $\mathrm{Bcat}(M) + 1 \geqslant \mathrm{Crit}(M)$.*

We get a stronger result of $1 + \mathrm{Bcat}(M) = \mathrm{Crit}(M)$ in dimension 3 as a consequence in the proof of Takens theorem[3.3][5].

We note that for $\dim(M) \neq 4,5$, $\mathrm{Bcat}(M) = \mathrm{Bcat}^{\mathscr{T}}(M)$ and so

$$\mathrm{Crit}^{\bullet}(M) \geqslant \mathrm{Bcat}(M) + 1 \geqslant \mathrm{Crit}(M)$$

We believe that it is possible to show $\mathrm{Bcat}(M) + 1 \geqslant \mathrm{Crit}^{\bullet}(M)$. If proven, then we would have $\mathrm{Bcat}(M) + 1 = \mathrm{Crit}^{\bullet}(M)$ for $\dim(M) \neq 4,5$ which is a partial improvement on the LS Theorem 12.

In Chapter 2 we will review the LS category as well as the theorem we improve upon. In Chapter 3 we review Singhof-Takens theory and introduce relative fillings. The proof of Theorem 1 relies on definitions which are stated in Chapter 4. In Chapter 5, we establish the lower bound for $\mathrm{Crit}^{\bullet}(M)$, see Theorem 1. The upper bound for $\mathrm{Crit}(M)$ is proved in Chapter 6, see Theorem 3.

Chapter 2

Context and motivation

Around the early 1900s Mathematicians like Henri Poincaré were discovering the relationships between the topology of spaces and other mathematical concepts. After formalizing the concept of a manifold, mathematicians like Marston Morse worked to relate the complexity of a manifold with the complexity of the flow on it. Around this time, Lazar Aronovich Lyusternik and Lev Genrikhovich Schnirelmann developed a homology invariant to get a lower bound on the number of critical points for any smooth function on a manifold.[6] In this chapter we will discuss the homology invariant known as the Lusternik-Schnirelmann category and its relationship to the minimum number of critical points of a smooth function on a particular manifold.

Let us start with some definitions.

Definition 4. [6]

- The *(Lusternik-Schnirelmann or LS) category of a space* X is the least number, n, such that $n+1$ open sets are needed to cover X with each of them being contractible in X. If no such covering exists then we say $cat(X) = \infty$.
- Let $A \subseteq X$. The *subspace category* of A in X, written as $cat_X(A)$, is the least number, n, such that $n+1$ open sets are needed to cover A with each of them being contractible in X. If no such covering exists then we say that $cat_X(A) = \infty$.

We note that $cat_X(X) = cat(X)$ and $cat_X(A) \leqslant cat(X)$ since the open covering from $cat(X)$ will be an open covering for A in X.

Example 5.

- Consider the circle S^1. We can intuitively guess, and later confirm, that $cat(S^1) = 1$ since you need two arcs to cover the circle.

- Let $X = D^2$ and let $A = S^1 \subseteq X$. Then we see that $cat_X(A) = 0$ since you can cover it with a disc in X. That is, U_0 be a disc around S^1.

In general, the LS category may not be easy to compute. Luckily there are estimates that we can use to estimate, if not calculate, what the category is. We will see that we get a lower bound of $cat(X)$ by the cup product, $cup(X)$, and we get an upper bound of $cat(X)$ by the dimension, $dim(X)$.

Definition 6. [6] Let R be a commutative ring and X be a space. The *cuplength of X with coefficients in R*, denoted as $cup_R(X)$, is the least number k such that all $(k+1)$-fold cup products are equal to zero in the reduced cohomology $\tilde{H}^*(X; R)$. If no such integer exists then we say $cup_R(X) = \infty$.

Example 7.

- The cup length of S^n is 1, i.e $cup_{\mathbb{Z}}(S^n) = 1$. We see this by looking at the reduced cohomology class of S^n. The reduced cohomology class means that $\tilde{H}^0(S^n) = 0$ whereas $H^0(S^n) = \mathbb{Z}$ for non-reduced cohomology.

$$\tilde{H}^m(S^n) = \begin{cases} \mathbb{Z} & m = n \\ 0 & \text{otherwise} \end{cases}$$

  So we can pick $x \in H^n(S^n)$ and that is all the elements, up to a scalar, that we can pick for the cup product. If $y \in H^m(S^n)$ for any $m \in \mathbb{N}$ then $x \cup y = 0$ since $x \cup y \in \tilde{H}^{n+m}(S^n) = 0$.

- The cup product for $S^2 \times T^2$ is 3. We know that $S^2 \times T^2 = S^2 \times S^1 \times S^1$ so pick $z \in \tilde{H}^2(S^2)$, $y \in \tilde{H}^1(S^1)$, and $x \in \tilde{H}^1(S^1)$. So by the Künneth formula we get that

  $$z \cup y \cup x \neq 0$$

  and

  $$w \cup z \cup y \cup x = 0$$

  for any $w \in \tilde{H}^M(S^2 \times T^2)$.

- Similarly $cup_{\mathbb{Z}}(T^2) = 2$.

Proposition 8. [6] *The R-cuplength of a space is less than or equal to the category of the space for all coefficients R.*

$$cup_R(X) \leqslant cat(X)$$

Theorem 9. [6] *For a path-connected locally contractible paracompact space,*

$$cat(X) \leqslant dim(X)$$

Example 10.

- For $X = S^n$, we know that we can get a covering made up of two open sets, a small neighborhood of the lower hemisphere, U_0, and a small neighborhood of the upper hemisphere, U_1 such that U_0 and U_1 intersect to cover X. Since we got a covering of S^n of two elements, we know that $cat(S^n) \leqslant 1$ since the category is one less than the number of covering sets. But we know that $1 = cup_{\mathbb{Z}}(S^n)$ thus

  $$1 = cup_{\mathbb{Z}}(S^n) \leqslant cat(S^n) = 1$$

  so $cat(S^n) = 1$.

- For $X = T^2$, we know that $cup_{\mathbb{Z}} = 2$ and $dim(T^2) = 2$. so we get

$$2 = cup_{\mathbb{Z}}(T^2) \leqslant cat(T^2) \leqslant dim(T^2) = 2$$

thus $cat(T^2) = 2$.

We can picture this covering of T^2 by letting U_0 be the disc around the 0-cell, U_1 be a union of discs around the two 1-cells, and U_2 be the interior of the 2-cell. Note that the discs around the 1-cells do not intersect each other and so the union can be considered as 1 open set, U_1, that is contractible in T^2.

Remark 11. More generally the cup length is 2 for both orientable and non-orientable surfaces that are not spheres, which have a cup length of 1.

Now we are ready to state the Lusternik-Schnirelmann Theorem. The main objective of this paper is to partially improve upon this theorem by considering more restrictive categories and assuming a certain type of critical points.

The following is a less general version of the Lusternik-Schnirelmann Theorem:

Theorem 12. [2] *(The Lusternik-Schnirelmann Theorem) Let M be a smooth manifold and suppose that $f : M \to \mathbb{R}$ is a smooth function. Let $Crit(M)$ denote the minimum number of critical points for any smooth function on M. Then we have*

$$cat(M) + 1 \leqslant Crit(M)$$

This theorem relates the minimum number of critical points for any smooth function on the manifold with the LS-category. We note that $Crit(M)$ is hard to compute in general so it is nice that we have a lower bound which is more accessible through homotopy techniques. In the next chapter we will outline two new categories on M used in the main theorems.

Chapter 3

Singhof-Takens fillings

Definition 13. A continuous map $f\colon X \to Y$ of a subset $X \subset \mathbb{R}^n$ to a subset $Y \subset \mathbb{R}^n$ is said to be *smooth* if it extends to a smooth map from an open neighborhood of X to an open neighborhood of Y, that is, the map should be infinitely differentiable. A smooth map $f\colon X \to Y$ of subsets of $\mathbb{R}^n$ is a *diffeomorphism* if it is a bijection and its inverse is smooth.

Definition 14. Let Q be a topological space. A *coordinate n-chart* (U, φ) on Q consists of an open subset U of $[0,\infty)^k \times \mathbb{R}^{n-k}$ for some $k \in \{0, ..., n\}$ and a homeomorphism $\varphi\colon U \to Q$ onto the image.

Definition 15. An *n-atlas* on Q is a maximal collection $\{(U_\alpha, \varphi_\alpha)\}$ of n-charts on Q such that $Q = \cup \varphi_\alpha(U_\alpha)$ and the transition maps

$$\varphi_\beta^{-1} \circ \varphi_\alpha\colon \varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)) \longrightarrow \varphi_\beta^{-1}(\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta))$$

are diffeomorphisms for all α and β.

Definition 16. A locally Euclidean, second-countable Hausdorff space Q with a maximal n-atlas is said to be a *smooth manifold with corners* of dimension n.

Example 17. Graphs of smooth functions are smooth manifolds[7]

If a point x in a manifold with corners belongs to the image $\varphi(U)$ of a chart of the form $U \subset [0, \infty)^k \times \mathbb{R}^{n-k}$, then we say that the order of the corner at x is at most k. It follows that the set of points of order k in a manifold with corners Q is a smooth manifold itself. We call this smooth manifold a *face* of dimension $n-k$. In particular, the face of dimension n is the interior of Q, while all other faces are parts of the boundary ∂Q. Faces of dimension less than n form a *canonical* stratification of the boundary ∂Q.

Definition 18. Let Q be a manifold with corners. Let $\Sigma \subset \partial Q$ denote the set of points of the boundary at which ∂Q is not smooth. Suppose that Q' is a smooth manifold with (smooth) boundary. Let $f\colon Q \to Q'$ be a homeomorphism which restricts to a diffeomorphism $Q \backslash \Sigma \to Q' \backslash f(\Sigma)$. Then we say that f as well as f^{-1} is a *special almost diffeomorphism.*

Example 19.

- Every diffeomorphism is a special almost diffeomorphism.
- There is a special almost diffeomorphism between a square and a circle by way of radial projection.

A filling of a smooth manifold M is a certain decomposition of M into manifolds with corners $Q_1, ..., Q_N$. The number N is said to be the *order* of the filling. Fillings of order 3 of closed manifolds were introduced by Takens in[5], and later the notion was extended to that of arbitrary order by Singhof[4].

Definition 20. Let M be a closed smooth manifold of dimension n. A *Singhof-Takens filling* of M is a family of compact co-dimension 0 submanifolds with corners $Q_1, ..., Q_N$ of M such that the following conditions are satisfied.

P1: $M = Q_1 \cup \cdots \cup Q_N$, i.e, $\{Q_i\}$ is a covering of M by compact subsets.

P2: The interiors of submanifolds Q_i and Q_j are disjoint for all $i \neq j$.

P3: Given a point $z \in M$, let $k_1 < \cdots < k_\nu$ be the indices i of submanifolds Q_i containing z, $\nu \leqslant n+1$. If $\nu \geqslant 2$, then we require that there is a coordinate n-chart D of dimension n about z in M such that for each $j < \nu$

$$D \cap Q_{k_j} = \{x_1 \geqslant 0, ..., x_{j-1} \geqslant 0, x_j \leqslant 0\},$$

and

$$D \cap Q_{k_\nu} = \{x_1 \geqslant 0, ..., x_{\nu-1} \geqslant 0\}.$$

We say that D is a *special* coordinate chart about the point z with respect to the filling $\{Q_i\}$.

We will be referring to Singhof-Takens fillings as just *fillings* from now on.

Definition 21. $\mathrm{Bcat}(M)$ (and respectively, $\mathrm{Bcat}^{\mathscr{T}}(M)$) of a smooth closed manifold M is the least number, n, such that every Singhof-Takens filling of M by smooth (respectively, topological) balls with corners has at least $n+1$ elements.

Remark 22. A *smooth ball with corners* is a smooth manifold with corners which has a special almost diffeomorphism to a smooth ball.

Remark 23. A *topological ball* is a space that is homeomorphic to a smooth ball. So every element in the filling from $\mathrm{Bcat}^{\mathscr{T}}(M)$ is homeomorphic to a smooth ball with corners.

Remark 24. The submanifolds Q_i in a filling $\{Q_i\}$ are ordered. In fact, the order of submanifolds Q_i is essential. In particular, if $Q_1, ..., Q_N$ is a filling of a manifold M, then $Q_{\sigma(1)}, ..., Q_{\sigma(N)}$ may not be a filling, where σ is a permutation of N elements.

Remark 25. For closed surfaces we note that $\mathrm{Bcat}^{\mathscr{T}}(M) = \mathrm{Bcat}(M) = \mathrm{cat}(M)$. For a sphere $\mathrm{Bcat}(M) = 1$, otherwise $\mathrm{Bcat}(M) = 2$.

Example 26. To see $\mathrm{Bcat}(T^2) = 2$ we note that the torus without a disc can be constructed from a disk U_1 with U_2 attached as shown in figure 3.1. We note that the boundary of the

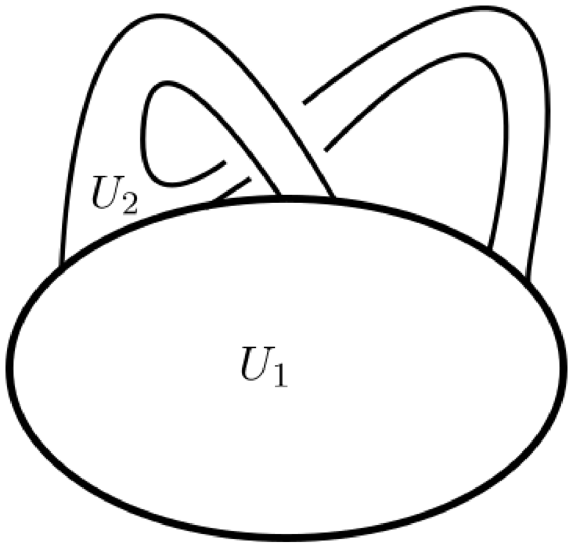

Figure 3.1: *Torus without a disc made from the submanifolds U_1 and U_2.*

resulting manifold is a circle. We then attach another disk U_3 along this circle. The sets U_1, U_3, U_2 then comprise a filling of T^2.

We say that a filling that consists of N submanifolds Q_i is *categorical* if $N-1 = \text{cat}(M)$, and each submanifold Q_i is contractible in M.

We will also need the notion of a relative filling. Let M be a compact manifold of dimension n with a non-empty boundary ∂M. Then the union

$$\tilde{M} = M \sqcup (\partial M \times [0,1])/_{\partial M \sim \partial M \times \{0\}}$$

has a unique smooth structure that agrees with the smooth structures on M and $\partial M \times [0,1]$.

Definition 27. Let M be a compact manifold with boundary of dimension n. We say that a family $Q_1, ..., Q_N$ of compact submanifolds with corners of M is a *relative filling* of M if the decomposition $\tilde{M} = Q'_1 \cup Q'_2 \cup \cdots \cup Q'_N \cup Q'_{N+1}$, where $Q'_i = Q_{i-1}$ for $i = 2, ..., N+1$, and $Q'_1 = \partial M \times [0,1]$, satisfies the properties $(P1)$ and $(P2)$ of Definition 20 as well as the property $(P3)$ for all points $z \in \tilde{M} \backslash \partial \tilde{M}$. We say that the relative filling $Q_1 \cup \cdots \cup Q_N$ of M is *categorical* if Q_i is contractible in $\tilde{M}$ for $i = 1, ..., N$, and $N - 1 = \text{cat}(M)$.

Remark 28. We note that $Q'_1 = \partial M \times [0,1]$ is not necessarily contractible.

Definition 29. $\mathrm{Bcat}(M)$ (and respectively, $\mathrm{Bcat}^{\mathscr{T}}(M)$) of a compact smooth manifold M is the least number, n, such that every relative filling of M by smooth (respectively, topological) balls with corners has at least $n+1$ elements.

Many properties of relative fillings of compact manifolds with boundary are similar to the corresponding properties of fillings of closed manifolds.

Suppose now that a manifold $M = M_1 \cup M_2$ is obtained from two compact manifolds with boundary M_1 and M_2 by identifying some of their boundary components. In general, the union of a filling of M_1 and a filling of M_2 do not comprise a filling of M. However, it is true in certain cases like in the following remark.

Remark 30. Let M_1 and M_2 be two compact manifolds with fillings. For $i = 1, 2$, let $\partial_i M$ denote a union of some of the path components of the boundary of M_i. Suppose that $\partial_1 M$ is diffeomorphic to $\partial_2 M$. Furthermore, suppose that the filling of M_2 is given by a decomposition $Q_1 \cup \cdots \cup Q_N$ into submanifolds with corners such that if two points x, y belong to the same path component of $\partial_2 M$, then x and y belong to the same submanifold with corners Q_i. Let M be a manifold obtained from M_1 and M_2 by identifying $\partial_1 M$ with $\partial_2 M$. Then the union of fillings of M_1 and M_2 comprise a filling of $M = M_1 \cup M_2$ at least for some choice of indexing of the submanifolds with corners of $M_1 \cup M_2$.

Chapter 4

Gradient convex neighborhoods

Let M be a manifold and $f : M \to \mathbb{R}$ be a smooth function. We say that a critical point of f is isolated if it has a neighborhood that contains no other critical points of f. Let $x_0 \in M$ be an isolated critical point of f. Without loss of generality we may assume that $f(x_0) = 0$. By slightly perturbing f without changing the number of critical points of f, we may assume that x_0 is the only critical point in $f^{-1}(0)$, and that 0 is the unique critical value in $[-c, c]$ where c is a positive real number. If $x \in M$, we denote the level set $f^{-1}(f(x))$ by $M_{f(x)}$. We also denote the submanifold $f^{-1}([-c, c])$ by $M_{[-c,c]}$. We choose a Riemannian metric on M. This metric induces a gradient flow γ_t on M such that the trajectory $t \to \gamma_t(y)$ of any point $y \in M$ is a curve in M. We say that a trajectory is closed if the curve is a closed subset of M. The trajectories are either closed in M or one of the limits of $\gamma_t(y)$ is a critical point of f. Define $D \subset M$ to be the set of all critical points of f as well as all non-closed trajectories. For a critical point x, let $D(x)$ be the space of all points y such that for the trajectory $t \to \gamma_t(y)$, we have $\lim_{t\to\infty} \gamma_t(y) = x$ or $\lim_{t\to-\infty} \gamma_t(y) = x$. We note that the point x is itself in $D(x)$.

Definition 31. We say that a subset H of M is a *gradient convex neighborhood of a critical point* if it satisfies the following properties:

- $f(H) = [-c, c]$.

- $\mathrm{Int}(H)$ contains the critical point.
- H is invariant in $M_{[-c,c]}$ under the gradient flow, i.e, if $y \in M_{[-c,c]}$ belongs to the gradient curve through x and $x \in H$, then $y \in H$.
- H is a smooth submanifold of co-dimension 0 with corners.
- $\partial H = \partial_h H \cup \partial_v H$ where the sets $\partial_h H$ and $\partial_v H$ are compact manifolds with smooth boundary, and their boundaries coincide. The set of corners of H coincides with $\partial_h H \cap \partial_v H$.
- We have $f(\partial_h H) = \{-c, c\}$, and there is a diffeomorphism

$$\partial_v H \cong \partial(\partial_h H \cap f^{-1}(c)) \times [-c, c]$$

.

Remark 32. Let $x \in \partial_v H$ and let $x' \in \partial(\partial_h H \cap f^{-1}(c))$ be on the gradient curve of x then the diffeomorphism in Definition 31 is given by

$$\partial_v H \cong \partial(\partial_h H \cap f^{-1}(c)) \times [-c, c]$$

$$x \to (x', f(x))$$

Example 33. Suppose V is a neighborhood of x_0 in the level set $M_{f(x_0)}$, that V is a cone with apex x_0 and ∂V is a smooth manifold with the smooth map $\partial V \times (0, 1] \to M$. We then can construct a gradient convex neighborhood of x_0, H, that is homeomorphic to a ball.

Remark 34. All algebraic singularities follow example 33 and I do not have an example of any isolated singularity that do does not fit example 33.

Definition 35. Let M be a compact smooth manifold. Let $\mathrm{Crit}^{\bullet}(M)$ denote the minimum number of critical points for any smooth function on M such that all critical points have a gradient convex ball neighborhood around them.

The remainder of this chapter explores example 33. That is, we show, in the following results, that one may construct a gradient convex neighborhood around a critical point if it is given that the critical point emits a cone neighborhood on the level set of that critical point.

Let $H \subset M_{[-c,c]}$ be the space of all gradient curves in $M_{[-c,c]}$ that intersect V or have a limit of x_0.

Lemma 36. *The subset H of M is a smooth manifold with corners.*

Proof. Let x be a point in H. Without loss of generality we may assume that $f(x) \leqslant 0$; the case $f(x) \geqslant 0$ is similar. Let δ denote the gradient flow of f. Then for some $t \geqslant 0$, the point $\delta_t(x)$ is in V. Suppose that the point $\delta_t(x)$ is in the interior of the set $V \backslash \partial V$. Then all points sufficiently close to x in M lie on flow lines passing through the interior of V. Thus, an open neighborhood of x in $M_{[-c,c]}$ is in H. The argument in the case $\delta_t(x) \in \partial V$ is similar.

□

Proposition 37. *The manifold H is a cone over ∂H.*

Proof. V is a cone so we can rewrite V as $x_0 \cup (\partial V \times (0,1])$. Construct a radial vector field v_1 pointing away from the critical point along the interval $(0,1]$. Let λ be a smooth monotonic function that is 0 only at x_0 and 1 away from x_0. Let $v = \lambda v_1$ and let v restrict to a tangent radial vector field over the manifold $V \backslash \{x_0\}$, is trivial only at 0, and have any point in $V \backslash \{x_0\}$ escape V along v in finite time.

We may now use the gradient vector field $\text{grad}(f)$ to extend the vector field v over H, see Lemma 38. For any regular point $x \in H$ of f, the tangent space T_xM is the direct sum of the *horizontal* component $T_xM_{f(x)}$ and the *vertical* component generated by $\text{grad}(f)$. We say that a vector field u over $H \backslash \{x_0\}$ is *horizontal* if the vector $u(x)$ is in $T_xM_{f(x)}$ for all $x \in H \backslash \{x_0\}$.

Lemma 38. *The vector field v extends to a continuous vector field over H such that $v|H \backslash \{D(x_0)\}$ is a nowhere zero horizontal vector field.*

Proof. For any point $x \in H \backslash D(x_0)$ with $f(x) < 0$, there is a unique value $t > 0$ such that $\delta_t(x)$ is a point y in V, where δ_t is the gradient flow of f. We define $v(x)$ by

$$v(x) = \pi \circ (d\delta_t)^{-1}(v(y)),$$

where the map π projects the tangent space T_yM at $y \in M_{[-c,c]} \backslash x_0$ to the tangent space $T_yM_{f(y)}$ of the level manifold. Similarly, for x in $H \backslash D(x_0)$ with $f(x) > 0$ we define

$$v(x) = (\pi \circ d\delta_t)(v(y)),$$

where $x = \delta_t(y)$ for some $t > 0$ and $y \in V$. Finally, we set $v(x) = 0$ for all $x \in D(x_0)$. Then v is a continuous vector field over H. □

Let w denote the gradient vector field of f scaled by f, i.e., $w(x) = f(x) \cdot \mathrm{grad}(f)$. Then $w(x)$ vanishes precisely over the set $f^{-1}(0)$ as well as over the critical points of f.

Lemma 39. *The vector field $v + w$ is a continuous vector field on H which is nowhere zero on $H \backslash \{x_0\}$.*

Proof. Since w and v are continuous vector fields, we deduce that $w + v$ is a continuous vector field over H. We claim that $w + v$ is trivial only at x_0. Indeed, assume to the contrary that $x \in H$ is a regular point of f with $w(x) = -v(x)$. Over $V \backslash \{x_0\}$ we have $v(x) \neq 0$ and $w(x) = 0$ which is a contradiction. Over $D(x_0) \cap (H \backslash x_0)$ we have $v(x) = 0$ and $w(x) \neq 0$ which is, again, a contradiction. To avoid these cases, suppose that x is not on $D(x_0) \cup V$. We may assume that $f(x) < 0$ as the case $f(x) > 0$ is similar. Then for some $t > 0$ the point $y = \delta_t(x)$ is in $V \backslash \{x_0\}$. We observe that $v(x)$ is normal to the curve $t \mapsto \delta_t(x)$ as the diffeomorphism δ_t takes the curve $t \mapsto \delta_t(x)$ to itself, it takes the point x to y, and it takes the vector $v(y)$ normal to the curve to a vector normal to the curve $t \mapsto \delta_t(x)$. On the other hand, the vector $w(x)$ is tangent to the curve $t \mapsto \delta_t(x)$. Thus $v(x) = w(x)$ if and only if both vectors are trivial which contradicts the assumption that x is a regular point of f. □

Lemma 40. *The corners of H can be smoothened so that $v+w$ is still defined on the modified manifold H and $v+w$ is nowhere tangent to ∂H.*

Proof. Recall that the vector field $w(x)$ is defined to be the positively scaled gradient vector field over $M_{(0,c]}$ and it is a negatively scaled gradient vector field over $M_{[-c,0)}$. For all non-corner points x of ∂H on the level $\pm c$, the tangent space of ∂H is tangent to the level $M_{\pm c}$, the vector $v(x)$ is also tangent to the level $M_{\pm c}$ while $w(x)$ is normal to ∂H directed outward from H. Therefore the vector field $v+w$ over $\partial H \cap M_{\pm c}$ is outward normal.

For all non-corner points x in ∂H that are not on the levels $\pm c$, the vector $v(x)$ is outward normal, while $w(x)$ is tangent to ∂H. Again, the vector field $v+w$ is outward normal over $\partial H \backslash M_{\pm c}$

Finally, at any corner point x, locally H is a quarter subspace of $\mathbb{R}^m$ bounded by the horizontal space $f^{-1}(c_+)$ or $f^{-1}(c_-)$, and a vertical space composed of gradient flow curves. It follows that the corner of H can be smoothened, and the vector field $v+w$ can be modified near the smoothened corner so that it is outward normal everywhere over ∂H. □

To summarize, we smoothened the corners of H so that H is a manifold with smooth boundary ∂H. We also constructed a vector field $v+w$ with a unique critical point at x_0. The vector field $v+w$ is outward normal everywhere over ∂H.

Lemma 41. *For every point $x \in H \backslash \{x_0\}$, its flow curve along $-v-w$ approaches x_0.*

Proof. Along the vector field $-w$ every point of H flows towards V, while along the vector field $-v$ the points of H stay on the same level of f. Consequently, any point in $H \backslash \{x_0\}$ appears arbitrarily close to V after flowing along $-v-w$ for sufficiently long time. On the other hand, by the definition of v any point on V flows towards x_0 along $-v$. □

Thus, for every point $x \in H \backslash \{x_0\}$, the flow curve through x passes through a unique point of ∂H, and has x_0 as its limiting point. This implies that H is a cone over ∂H. □

Corollary 42. *The manifold H is a smooth manifold homeomorphic to a ball.*

Proof. I could not find an external reference to this result but by the argument in lemma 43, we get that H is homeomorphic to a ball. $\square$

Chapter 5

The Lusternik-Schnirelmann estimate by $\mathrm{Bcat}^{\mathscr{T}}$

Given a closed smooth manifold M, we recall that $\mathrm{Bcat}^{\mathscr{T}}(M)$ is the least number n such that there is a Singhof-Takens filling of M by topological balls has at least $n+1$ elements. We also recall that $\mathrm{Crit}^{\bullet}(M)$ is the least number of critical points for any smooth function on M such that all critical points have a gradient convex ball neighborhood around them.

In this section we will establish the Lusternick-Schnirelmann type inequality:

$$\mathrm{Bcat}^{\mathscr{T}}(M) + 1 \leqslant \mathrm{Crit}^{\bullet}(M)$$

The proof of this inequality will be by induction on critical points. That is, we will prove that this inequality is fulfilled for the first critical point and then we will proof the inequality is fulfilled on any critical point after. This following lemma will give us the base case of the proof.

Lemma 43. *Let $f : M \to \mathbb{R}$ be a smooth function on a closed manifold of dimension n. Suppose that x_0 is a critical point at which f assumes a unique global minimum. Let c be a number such that f has no critical values in the interval $(f(x_0), c]$. Then the compact* *manifold* $M_{(-\infty,a]}$ *is homeomorphic to a disc for any* $a \in (f(x_0), c)$.

Proof. The manifold $M_{(-\infty,a]}$ is contractible as there is a deformation of $M_{(-\infty,a]}$ to a point along the negative to the gradient flow of f. In fact the manifold $M_{(-\infty,a]}$ is homeomorphic to a cone over $f^{-1}(a) = M_a$. Since M is a manifold of dimension n, the group $H_i(M, M\backslash\{x_0\}) = H_i(M_{(-\infty,a]}, M_{(-\infty,a]}\backslash\{x_a\})$ is isomorphic to $\mathbb{Z}$ in dimensions $i = 0$ and $i = n$, and it is trivial otherwise. We can see this by noting that $H_i(M, M\backslash\{x_0\}) = H_i(D, D\backslash\{x_0\})$, where D is a disk, since M is a manifold and by considering the homology of the relative chain complex. More specifically, we have

$$\to H_i(D\backslash\{x_0\}) \to H_i(D) \to H_i(D, D\backslash\{x_0\}) \to H_{i-1}(D\backslash\{x_0\}) \to H_{i-1}(D) \to$$

so

$$0 \to H_i(D, D\backslash\{x_0\}) \to H_{i-1}(D\backslash\{x_0\}) \to 0$$

Since $D\backslash\{x_0\}$ is a homology sphere whose homology group is nontrivial in dimensions $i = 0$ and $i = n$, we get that $H_i(D, D\backslash\{x_0\})$ is isomorphic to Z for $i = 0, n$.

By the long exact sequence, we deduce that M_a is a homology $(n-1)$-sphere. More specifically we first note that $M_{(-\infty,a]}\backslash\{x_0\}$ can be rewritten as $M_a \times (0,1]$ so the homology groups of $M_{(-\infty,a]}\backslash\{x_0\}$ and M_a are the same, and then we write the long exact sequence of

$$\to H_i(M_a) \to H_i(M_{(-\infty,a]}) \to H_i(M_{(-\infty,a]}, M_a) \to H_{i-1}(M_a) \to H_{i-1}(M_{(-\infty,a]}) \to$$

Since $M_{(-\infty,a]}$ is a cone, we get that $H_i(M_{(-\infty,a]}) = 0$. So the long exact sequence becomes

$$0 \to H_i(M_{(-\infty,a]}, M_a) \to H_{i-1}(M_a) \to 0$$

and so $H_i(M_{(-\infty,a]}, M_a) \cong H_{i-1}(M_a)$.

If $n < 3$, then M_a is a sphere, and therefore $M_{(-\infty,a]}$ is homeomorphic to a disc. Suppose that $n \geqslant 3$. Let us show that M_a is simply connected, and, consequently, M_a is a homotopy sphere. To this end, let γ be any loop on M_a. We will show that it is contractible in M_a. Since $M_{(-\infty,a]}$ is contractible, there is a disc D in $M_{(-\infty,a]}$ bounded by γ. Furthermore, since $n \geqslant 3$, by slightly perturbing D we may assume that x_0 is not in D. Since $M_{(-\infty,a]}$ is homeomorphic to a cone over M_a there is a projection of $M_{(-\infty,a]} \backslash \{x_0\}$ to M_a. It takes the disc D to M_a producing a map of a disc extending the inclusion of $\gamma = \partial D$. More specifically, we have a curve γ on the level M_a and a disc in $M_{(-\infty,a]}$ that has boundary γ. We can then see that γ is contractible in M_a since the disc is contractible. Therefore, the manifold M_a is simply connected. Thus M_a is a homotopy $(n-1)$-sphere. Finally, by the generalized topological Poincaré conjecture M_a is homeomorphic to a sphere, and, therefore, $M_{(-\infty,a]}$ is homeomorphic to a disc. □

Remark 44. The generalized topological Poincaré conjecture states that every homotopy sphere in topological manifolds is homeomorphic to the standard sphere. This is proven true in all dimensions.

Let M be a compact manifold with corners, and let $Q_1, \ldots, Q_n$ be a filling of M. We denote by $Q_{i,j}$ the union of $(j+1)$-corners of the element Q_i of the filling, i.e., $Q_{i,0}$ is the complement in ∂Q_i to the corners of ∂Q_i, and $Q_{i,j+1}$ is the complement in $\partial Q_i \backslash \left(\bigcup_{k \leqslant j} Q_{i,k} \right)$ to the corners in $\partial Q_i \backslash \left(\bigcup_{k \leqslant j} Q_{i,k} \right)$ for $j \geqslant 0$.

The following lemma is necessary for the induction step in the main proof.

Lemma 45. *Let N be a smooth closed manifold of dimension d. Let $Q_1, \ldots, Q_\ell$ be a filling of N. Let K be a smooth compact submanifold of N of dimension d such that ∂K is transverse to each stratum $Q_{i,j}$ of the canonical stratification of each ∂Q_i. Let $Q_1, \ldots, Q_n$ be elements of the filling such that $Q_i \nsubseteq K$ for $i \in \{1, 2, \ldots n\}$ and let $S_i = \overline{Q_i \backslash K}$. Then the sets $K, S_1, \ldots, S_n$ comprise a filling of N.*

Proof. We note that the sets $K, S_1, ..., S_n$ cover the manifold N, and the interiors of the sets $K, S_1, ..., S_n$ are disjoint. Therefore, the sets $K, S_1, ..., S_n$ satisfy the first two properties of a filling.

Let us now verify Property 3. To begin with, let z be a point in $N\backslash K$. Suppose that $z \in Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_m}$ for $1 < m \leqslant n$, but z is not in $Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_m} \cap Q_i$ for all $i \notin \{i_1, ..., i_m\}$. Since $\{Q_i\}$ is a filling, we deduce that there is a small coordinate disc neighborhood D about z in $N\backslash K$ such that

$$
\begin{aligned}
&D \cap Q_{i_1} = \{x_1 \leqslant 0\},\\
&D \cap Q_{i_2} = \{x_1 \geqslant 0, x_2 \leqslant 0\},\\
&\ldots\\
&D \cap Q_{i_m} = \{x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_{m-1} \geqslant 0\}.
\end{aligned}
$$

Since $D \cap Q_i = D \cap S_i$, we deduce that Property 3 is satisfied for the point z. If z is in the interior of K, then Property 3 is again satisfied.

Suppose now that z is on the boundary of K, the point z is in $Q_{i_1}, ..., Q_{i_m}$, and $z \notin Q_i$ for $i \notin \{i_1, ..., i_m\}$. Again, there is a coordinate disc neighborhood D about z such that the intersections of D with $Q_{i_1}, ..., Q_{i_m}$ are as above. The dimension of the manifold

$$
Q = \{x \in Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_m} \mid x \notin Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_m} \cap Q_i \quad \text{for} \quad i \notin \{i_1, ..., i_m\}\}
$$

is $d - m + 1$. Therefore, if $z \in \partial K$, then $d \geqslant m$ because ∂K is transverse to Q. Without loss of generality, we may assume that there is a smooth function y_1 on D such that $D \cap K$ is given by $y_1 \leqslant 0$ and the gradient of y_1 at z is non-trivial. We write $A \pitchfork_S B$ when a manifold A is transverse to a manifold B at each point of a set S. We have

$$
\begin{aligned}
\partial K \pitchfork_z (Q_{i_1} \cap \ldots \cap Q_{i_m}) &\iff \partial K \pitchfork_z \{(x_1, x_2, \ldots, x_m) | x_1 = 0, \ldots, x_{m-1} = 0\}\\
&\iff \nabla y_1 \notin \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{m-1}} \right\rangle.
\end{aligned}
$$

Indeed, since ∂K is of dimension $d-1$, it is not transverse to Q at z only if $T_z Q \subset T_z \partial K$, i.e.,

∇y_1 is perpendicular to $T_zQ = \left\langle \frac{\partial}{\partial x_m}, \dots, \frac{\partial}{\partial x_d} \right\rangle$ with respect to the standard Riemannian metric on D.

Put $y_j = x_{j-1}$ for $j = 2, \dots, m-1$. Since $\nabla y_1 \notin \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m-1}} \right\rangle$, there exist functions $y_m, \dots, y_d$ of $(x_m, \dots, x_d)$ such that the $\nabla y_1, \nabla y_2, \dots, \nabla y_m, \dots, \nabla y_d$, are linearly independent at the point z. We can then see that the Jacobian matrix is of full rank.

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \dots & \frac{\partial y_1}{\partial x_{d-1}} & \frac{\partial y_1}{\partial x_d} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \dots & \frac{\partial y_2}{\partial x_{d-1}} & \frac{\partial y_2}{\partial x_d} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \dots & \frac{\partial y_m}{\partial x_{d-1}} & \frac{\partial y_m}{\partial x_d} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \dots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \dots & \frac{\partial y_{m+1}}{\partial x_{d-1}} & \frac{\partial y_{m+1}}{\partial x_d} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial y_d}{\partial x_1} & \frac{\partial y_d}{\partial x_2} & \dots & \frac{\partial y_d}{\partial x_m} & \frac{\partial y_d}{\partial x_{m+1}} & \dots & \frac{\partial y_d}{\partial x_{d-1}} & \frac{\partial y_d}{\partial x_d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & * & * & \dots & * & * \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & * & * & \dots & * & * \\ 0 & 0 & \dots & * & * & \dots & * & * \\ \vdots & & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & * & * & \dots & * & * \end{bmatrix}.$$

By the inverse function theorem, for a small disc neighborhood D_0 of z, the map $y|D_0 : D_0 \to \mathbb{R}^d$ is a local coordinate chart. Then

$$\begin{aligned} &D_0 \cap K = \{y_1 \leqslant 0\} \\ &D_0 \cap S_{i_1} = D_0 \cap \overline{(Q_{i_1} \backslash K)} = \{y_1 \geqslant 0, x_1 \leqslant 0\} = \{y_1 \geqslant 0, y_2 \leqslant 0\} \\ &D_0 \cap S_{i_2} = D_0 \cap \overline{(Q_{i_2} \backslash K)} = \{y_1 \geqslant 0, x_1 \geqslant 0, x_2 \leqslant 0\} = \{y_1 \geqslant 0, y_2 \geqslant 0, y_3 \leqslant 0\} \\ &\dots \\ &D_0 \cap S_{i_m} = D_0 \cap \overline{(Q_{i_m} \backslash K)} = \{y_1 \geqslant 0, x_1 \geqslant 0, ..., x_{m-1} \geqslant 0\} = \{y_1 \geqslant 0, y_2 \geqslant 0, \dots, y_m \geqslant 0\} \end{aligned}$$

which fulfills property 3. Therefore, the sets $K, S_1, \dots, S_n$ comprise a filling. □

We are now in position to prove Theorem 1

Theorem. *For any closed smooth manifold M, we have* $\mathrm{Bcat}^{\mathscr{T}}(M)+1 \leqslant \mathrm{Crit}^{\bullet}(M)$.

Proof. Let $x_0, x_1, \dots, x_n$ be the critical points of a smooth function $f : M \to \mathbb{R}$ minimizing $\mathrm{Crit}^{\bullet}(M)$. Since M is closed we may assume that $f > 0$. By slightly perturbing f and re-indexing $x_0, x_1, \dots, x_n$, we may also assume that $f(x_i) < f(x_{i+1})$. For $i = 0, ..., n+1$, let a_i be real numbers such that $a_i < f(x_i) < a_{i+1}$ and $0 = a_0$. We note f is constant over M_{a_i} and for each i the restriction $f|M_{[0,a_i]}$ is a function minimizing $\mathrm{Crit}^{\bullet}(M_{[0,a_i]})$ among functions on $M_{[0,a_i]}$ that are constant and maximal over M_{a_i}.

The function f possesses only one critical point in the manifold $M_{[0,a_1]}$ at which f assumes the global minimum. Therefore, by Lemma 43, the manifold $M_{[0,a_1]}$ is homeomorphic to a disk. Thus $\mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_1]})+1 \leqslant \mathrm{Crit}^{\bullet}(M_{[0,a_1]})$. By induction, suppose that

$$\mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_i]})+1 \leqslant \mathrm{Crit}^{\bullet}(M_{[0,a_i]}).$$

We note that

$$\mathrm{Crit}^{\bullet}(M_{[0,a_i]})+1 = \mathrm{Crit}^{\bullet}(M_{[0,a_{i+1}]}).$$

Let's show that $\mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_{i+1}]}) \leqslant \ell+1$, where $\ell = \mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_i]})$. Suppose that $Q_0, ..., Q_\ell$ is a relative filling for $M_{[0,a_i]}$. It suffices to show that there is a filling for $M_{[0,a_{i+1}]}$ with $\ell+2$ elements. For an isolated critical point $x_i \in M_{[a_i,a_{i+1}]}$, recall that $D(x_i)$ denote the set of points x in M such that $\lim \gamma_t(x) = x_i$ as $t \to \infty$ or $t \to -\infty$, where $\gamma_t(x)$ is the gradient flow of f. By assumption we can construct the manifold H such that H is a gradient convex ball neighborhood of $D(x_i)$.

On the manifold $M_{[a_i,a_{i+1}]}$ there is a positive function α such that the flow along the scaled vector field $\alpha \cdot \nabla f$ takes any point on $M_{a_i} \backslash H$ to a point on $M_{a_{i+1}} \backslash H$ at the same time $t = 1$ for all points on $M_{a_i} \backslash H$.

Since $Q_0, ..., Q_\ell$ is a relative filling of $M_{[0,a_i]}$, the strata of the boundary Q_j intersects each submanifold M_t transversally for $t \in [a_i - \varepsilon, a_i)$ and all j for a sufficiently small number $\varepsilon > 0$. In particular, there is a gradient like vector field w on $M_{[a_i-\varepsilon,a_i)}$ such that w is tangent

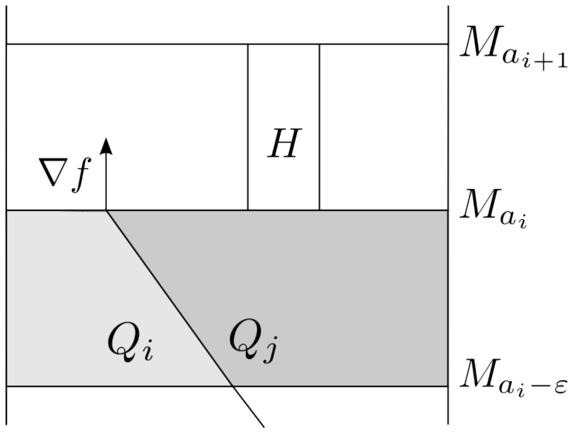

Figure 5.1: *The gradient field ∇f, and the elements Q_i, Q_j of a filling of $M_{[0,a_i]}$.*

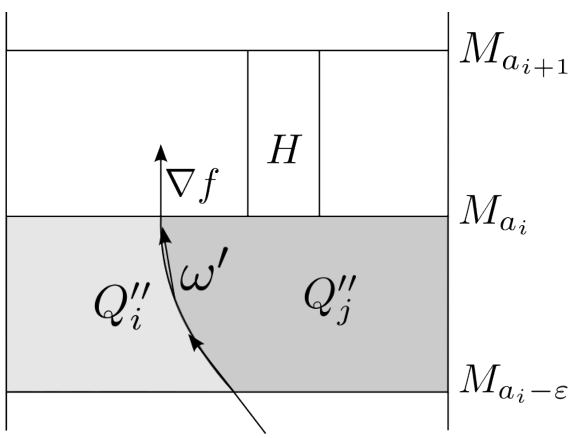

Figure 5.2: *The gradient field w' shown at 3 points with the modified filling.*

to each stratum of ∂Q_j when restricted to ∂Q_j for all j, see Fig. 5.1.

Our aim is to extend the present relative filling $\{Q_i\}$ to a relative filling of $M_{[0,a_{i+1}]}$. To this end, we we will next modify the vector field w and the relative filling of $M_{[0,a_i]}$ to a vector filed w' and a relative filling $\{Q''_i\}$.

Let λ be a smooth monotone function on $[0, a_{n+1}]$ such that $\lambda(t) = 0$ for all $t \in [0, a_i - \varepsilon]$, and $\lambda(t) = 1$ for all $t \in [a_i, a_{n+1}]$. Define a vector filed w' on $M_{[a_i-\varepsilon,a_i]}$ by

$$w'(x) = (1 - \lambda(f(x)))w(x) + \lambda(f(x))\nabla f(x).$$

Then w' is a vector field that agrees with w near $M_{a_i-\varepsilon}$, and agrees with ∇f near M_{a_i}. We now define a new relative filling of $M_{[0,a_i]}$ by sets $Q''_0, ..., Q''_\ell$, where Q''_j is the union of $Q_j \cap M_{[0,a_i-\varepsilon)}$ and the points y in $M_{[a_i-\varepsilon,a_i]}$ that flow along $-w'$ to $Q_j \cap M_{a_i-\varepsilon}$, see Fig. 5.2. To show that $\{Q''_i\}$ is a relative filling, we observe that the submanifold $M_{[a_i-\varepsilon,a_i]}$ is diffeomoprhic to $M_{a_i-\varepsilon} \times [0, 1]$ under a diffeomorphism that takes any flow line of w' in $M_{a_i-\varepsilon}$ to a segment of the form $\{x\} \times [0, 1]$ where x is a point in $M_{a_i-\varepsilon}$. The sets Q''_i coincide with the elements Q_i of a filling except over the part $M_{a_i-\varepsilon} \times [0, 1]$. On the other hand, the sets $Q''_i \cap (M_{a_i-\varepsilon} \times [0, 1])$ are cylinders $(Q''_i \cap M_{a_i-\varepsilon}) \times [0, 1]$. Therefore, the sets Q''_i indeed form a relative filling.

We observe that the filling $\{Q''_i\}$ of $M_{[0,a_i]}$ has the property that each non-horizontal stratum of each element Q''_j is vertically up near M_{a_i}.

We next modify the vector field w' to a vector field v and extend it over $M_{[0,a_{i+1}]}$ which we will use to smooth corners of H, and extend the relative filling $\{Q''_i\}$ to a relative filling of $M_{[0,a_{i+1}]}$ that consists of extensions Q'_i of Q''_i and the smoothed version H' of H.

There is a smooth vector field $v(x)$ on $M_{[0,a_{i+1}]}$ such that

- v is zero over $M_{[0,a_i-\varepsilon]}$,
- v is a multiple of w' over $M_{[a_i-\varepsilon,a_i]}$,
- v is a non-zero multiple of w' over $M_{[a_i-\varepsilon/2,a_i]}$, and

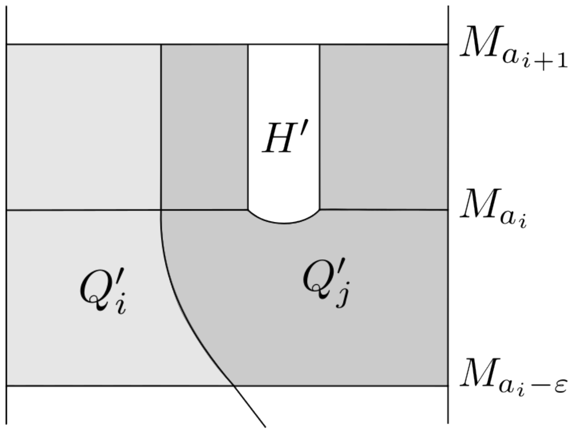

Figure 5.3: *The relative filling* $\{Q'_i\}$.

- v agrees with the scaled gradient vector field $\alpha \cdot \nabla f$ on $M_{[a_i,a_{i+1}]}\backslash H$.

In fact, we may choose $v(x)$ to be of the form $v(x) = \lambda_1(f(x))w' + \lambda_2(f(x))\nabla f$ for some smooth functions λ_1 and λ_2. Let

$$\rho\colon M_{[0,a_i]} \times [0,1] \to M_{[0,a_{i+1}]}$$

denote the flow along the vector field v for $t \in [0,1]$. We note that ρ_1 takes $M_{a_i}\backslash H$ diffeomorphically onto $M_{a_{i+1}}\backslash H$, where $\rho_t(x) = \rho(x,t)$.

Next we replace H with a smooth submanifold H' of $M_{[0,a_{i+1}]}$ with boundary that is smooth away from $M_{a_{i+1}}$. To this end we use the flow ρ to identify a neighborhood of $H_{a_i} = H \cap M_{a_i}$ with $H_{a_i} \times [-\varepsilon/2, \varepsilon/2]$ by a diffeomorphism that takes a point x in a neighborhood of H_{a_i} with a pair (y,t) such that $x = \rho_t(y)$. Let π_i denote the projection of the product neighborhood onto the i-th factor where $i = 1, 2$. Given a function $\psi \leqslant 0$ on H_{a_i} with values in $(-\varepsilon/2, 0]$, let H_ψ denote the union of H and the points x in the neighborhood $H_{a_i} \times [-\varepsilon/2, \varepsilon/2]$ above the graph of the function ψ, i.e., with points satisfying $\pi_2(x) \geqslant \psi(\pi_1(x))$. We may choose ψ so that $H' = H_\psi$ is a submanifold of M with corners only on $H'_{a_{i+1}} = H' \cap M_{a_{i+1}}$.

Let Q'_i denote the subset $\rho_1(Q''_i)\backslash H'$, see Fig. 5.3. We claim that $H', Q'_0, Q'_1, \ldots, Q'_\ell$ is a relative filling of $M_{[0,a_{i+1}]}$. We note that the ordering of the elements of this relative filling is important and that H' is the first element. Let L be $M_{a_{i+1}} \times [0,1]$ such that no critical points are in L, and let $\tilde{M}$ stands for $L \cup M_{[0,a_{i+1}]}$ where the union is taken by identifying $M_{a_{i+1}} \times \{0\} \subset L$ with the boundary of $M_{[0,a_{i+1}]}$. We see that the sets $L, H', Q'_0, Q'_1, \ldots, Q'_\ell$ satisfy properties 1 and 2 for a relative filling of $\tilde{M}$ by construction. All that is left to show is that these sets satisfy property 3 for all points $z \in \tilde{M} \backslash \partial \tilde{M}$.

The sets $Q_0, Q_1, \ldots, Q_\ell$ define a filling of the manifold M_{a_i} by the sets $R_i = Q_i \cap M_{a_i}$. Let $H_0 = H' \cap M_{a_i}$. Since H' is a smooth compact submanifold of $M_{[0,a_{i+1}]}$ such that $\partial H'$ intersects the level M_{a_i} transversely, we conclude that H_0 is a smooth compact submanifold of M_{a_i}. By Lemma 45, the submanifold H_0 together with the sets $R'_i = \overline{R_i \backslash H_0}$ define a new filling of M_{a_i}. We note that if $R'_i = \varnothing$ then we do not include it in our filling by Lemma 45. Let $H_1 = H' \cap M_{a_{i+1}}$, and let R''_i denote $\rho_1(R'_i)$. Then H_1 together with R''_i form a filling of $M_{a_{i+1}}$.

We shall now prove that property 3 is satisfied for all $z \in \tilde{M} \backslash \partial \tilde{M}$. We will consider four cases.

1. If $z \in M_{[0,a_{i+1}]} \backslash H'$ then property 3 is satisfied for z since property 3 is preserved under diffeomorphism, and the sets Q'_i are images of the sets Q''_i of a filling under the diffeomorphism ρ_1.

2. If $z \in int(H')$ or $z \in int(L)$ then property 3 is satisfied since the number of submanifolds of the filling containing that point is 1.

3. Suppose z is in the intersection $\partial H' \cap M_{(a_i,a_{i+1})}$. Let z' be a point on M_{a_i} on the gradient flow line passing through z, i.e., $\rho_{t_0}(z') = z$ for some $t_0 > 0$. Since the sets H_0 together with R'_i form a filling of M_{a_i}, there is a special coordinate chart D' of z' as in Defintion 20. Let $D_\varepsilon \subset M_{[0,a_{i+1}]}$ be a coordinate chart about z given by

$$D_\varepsilon = D' \times (-\varepsilon, \varepsilon) \subset M_{[0,a_{i+1}]},$$

$$(x,t) = \rho_{t_0+t}(x).$$

We may choose ε sufficiently small so that D_ε is a special coordinate chart about z as in Definition 20.

4. Suppose that z is in the intersection $H'_{a_{i+1}} = H' \cap M_{a_{i+1}}$. If z is in the interior of $H'_{a_{i+1}}$, then the property 3 is clearly satisfied. If z is in the boundary of $H'_{a_{i+1}}$, then the property 3 is satisfied since $\{H_1, R''_0, ...R''_\ell\}$ is a filling of $M_{a_{i+1}}$.

5. Suppose that z is in the intersection $J = \partial H' \cap M_{[0,a_i]}$. If z is in the interior of J, then the property 3 is satisfied for z by Lemma 45 since $\{Q''_i\}$ is a relative filling for $M_{[0,a_i]}$. If z is in the boundary of J, then the property 3 is satisfied for z by the reason as in the case 3.

Thus, the sets $H', Q'_0, Q'_1, \dots Q'_\ell$ form a relative filling for $M_{[0,a_{i+1}]}$. Therefore as we pass from $M_{[0,a_i]}$ to $M_{[0,a_{i+1}]}$, the number of elements in a minimizing filling increases by 1. Consequently, we have

$$\mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_{i+1}]}) + 1 \leqslant \mathrm{Bcat}^{\mathscr{T}}(M_{[0,a_i]}) + 2 \leqslant \mathrm{Crit}^{\bullet}(M_{[0,a_i]}) + 1 = \mathrm{Crit}^{\bullet}(M_{[0,a_{i+1}]}).$$

Where the last equality is obtained from recalling that $f|M_{[0,a_i]}$ is a function minimizing $\mathrm{Crit}^{\bullet}(M_{[0,a_i]})$.

When $i = n$, the relative filling of $M_{[0,a_{i+1}]}$ is a filling of M. Consequently

$$\mathrm{Bcat}^{\mathscr{T}}(M) + 1 \leqslant \mathrm{Crit}^{\bullet}(M).$$

which completes the proof.

□

Chapter 6

The converse to the Lusternik-Schnirelmann inequality

In this section we will establish the converse to the Lusternik-Schnirelmann inequality. Namely, we will show that for a closed manifold M there is an estimate:

$$\mathrm{Bcat}(M) + 1 \geqslant \mathrm{Crit}(M).$$

Let Q be a compact submanifold with corners of codimension 0 in a smooth manifold M.

Definition 46. A *smooth collar neighborhood* of Q in M is a compact submanifold $Q' \subset M$ with smooth boundary such that there is a special almost diffeomorphism $Q' \to Q \cup (\partial Q \times [0,1])$, where the union is taken by identifying $\partial Q \subset Q$ with $\partial Q \times \{0\} \subset \partial Q \times [0,1]$.

We recall that a vector field over a compact subset Q of a smooth manifold is said to be *smooth* if it is the restriction of a smooth vector field defined over a neighborhood of Q.

Let X be a topological submanifold of codimension 1 of a Riemannian manifold M. Let v be a smooth vector field over X in M. Roughly speaking, the vector field v is *transverse* to X at a point $x \in X$ if the projection of a neighborhood of x in X to a disc of dimension $\dim M - 1$ transverse to $v(x)$ is a homeomorphism onto image. More precisely, for each point

$x \in X$, let N_x denote the unique geodesic in M parametrized by $t \in (-\varepsilon, \varepsilon)$ such that at the time $t = 0$ it passes through x, and its velocity at $t = 0$ is v. If $\varepsilon > 0$ is sufficiently small, then the geodesic N_x exists for each $x \in X$. Suppose that for each point $x \in X$ there is a smooth disc D_x in M centered at x such that D_x is transverse to N_x and $\{D_x\}$ is a continuous family of discs. Then for any point $x \in X$ we may identify $D_x \times N_x$ with a neighborhood of x so that $D_x \times \{0\}$ is identified with $D_x \subset M$, and $0 \times N_x$ is identified with $N_x \subset M$. Suppose that a neighborhood of x in X is the graph $\{(y, f_x(y))\} \subset D_x \times N_x$ of a continuous function $f_x \colon D_x \to N_x$. Then we say that v is *transverse* to X at x. If v is transverse to X at every point $x \in X$, then we say that v is *transverse* to X.

Given a vector field v transverse to X, let $\sigma \colon X \to M$ be a continuous function that associates with each point x a point in N_x. Then $\sigma(X)$ is homeomorphic to X. By the Cairns-Whitehead theorem, for every $\delta > 0$ there is a continuous function σ as above such that $\sigma(X)$ is a smooth submanifold of M in the δ-neighborhood of X.

Remark 47. The Cairns-Whitehead theorem[8] states that if a manifold M has a transverse field then it has an ambient smoothing.

Similarly, let M be a smooth manifold, and Q be a set in a Singhof-Takens filling of M by smooth manifolds with corners. Then the Cairns-Whetehead smooth approximation of the boundary of Q defines a smooth manifold (with smooth boundary) Q' approximating Q. Lemma 49 shows that we may choose Q' so that $Q \subset Q'$.

Remark 48. In the Cairns-Whitehead construction above, if $\varepsilon > 0$ is small enough, then different geodesic segments N_x do not intersect. Let π denote the projection of the neighborhood $\cup N_x$ of $X = \partial Q$ to X defined by projecting each fiber N_x to x. The smoothing $\sigma(X)$ is determined by the Riemannian metric on M, the value ε, the vector field v as well as a smooth approximation of π. Since any two transverse vector fields over X are homotopic, and any two smooth approximations of π are isotopic, any two smooth approximations of X are isotopic. In particular, if a Cairns-Whitehead smoothing of a manifold Q with corners in a Singhof-Takens filling is diffeomorphic to a ball, then any other Cairns-Whitehead smoothing of Q is also diffeomorphic to a ball. Thus, the notion of a smooth ball with corners is well-defined.

Lemma 49. *Let Q be a compact submanifold with corners of codimension 0 in a smooth manifold M. Then Q possesses a collar neighborhood.*

Proof. There is a smooth vector field v over ∂Q in M transverse to ∂Q. Choose a Riemannian metric on M. For each point $x \in Q$, let $\gamma_t(x)$ denote the geodesic parametrized by t such that $\gamma_0(x) = x$, and $\dot{\gamma}_t(x)|_{t=0} = v(x)$. Let exp denote the map $Q \to M$ that associates with each point x the point $\gamma_1(x)$.

If the vector field v is sufficiently small, then the image X of ∂Q under the exponential map exp in the direction v is a topological submanifold of M homeomorphic to ∂Q. The vectors $v(x)$ translate over the geodesics $\gamma_t(x)$ to vectors $v_X(exp(x))$ so that v_X is a vector field over X. In view of the transverse vector field v_X, by the Cairns-Whitehead theorem[3], there is a smooth approximation Y of X. Then the manifold bounded by Y that consists of Q and parts $(x, \exp(x))$ of the shortest geodesics $\gamma_t(x)$ is a collar neighborhood of Q. □

Definition 50. Let M be a smooth manifold of dimension n with boundary ∂M. A relative Singhof-Takens filling of M by two subsets Q_1 and Q_2 is said to be *nice* if the set Q_1 contains ∂M, the pair $(Q_1, \partial M)$ is almost diffeomorphic to the pair $(\partial M \times [0,1], \partial M \times \{0\})$, and Q_2 is diffeomorphic to the disc D^n.

We note that in this case both manifolds Q_1 and Q_2 are smooth submanifolds with (smooth) boundary. We also have a definition for a nice filling for fillings with 3 elements

Definition 51. Let M be a compact connected smooth manifold of dimension n such that ∂M is the disjoint union of two manifolds $\partial_1 M$ and $\partial_2 M$. Suppose that $M = Q_3 \cup Q_1 \cup Q_2$ is a relative Singhof-Takens filling of M such that Q_1 contains $\partial_1 M$, while Q_2 contains $\partial_2 M$. Suppose that (Q_i, M_i) is almost diffeomorphic to $(M_i \times [0,1], M_i \times \{0\})$ for $i = 1, 2$, while Q_3 is diffeomorphic to a disc D^n. Then we say that the filling $\{Q_i\}$ is *nice*.

Example 52. A solid cylinder has a nice filling.

Remark 53. In the original definition of a nice filling by Takens[5], the boundary of Q_3 is required to be smooth. We omit this requirement since the corners of a manifold can be smoothened as in Takens section 2[5].

To establish the converse of the Lusternik-Schnirelmann inequality we will rely on the Takens Theorem.

Theorem 54 (see Corollary 2.8 in[5])**.** *Let M be a compact manifold with a filtration $\varnothing = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ by compact submanifolds with boundary such that $M_i \subset \mathrm{Int}(M_{i+1})$ and $\partial M \subset M_k \backslash M_{k-1}$. Suppose that for each $i = 1, ..., k$, the manifold $M_i \backslash \mathrm{Int}(M_{i-1})$ with boundary $\partial M_{i-1} \sqcup \partial M_i$ admits a nice filling. Then $\mathrm{Crit}(M) \leqslant k$.*

We are now in position to prove the converse of the Lusternik-Schnirelmann inequality, Theorem 3

Theorem. *For any closed manifold M, we have $\mathrm{Bcat}(M) + 1 \geqslant \mathrm{Crit}(M)$.*

Proof. Suppose that $\mathrm{Bcat}(M) = k - 1$. In particular, the manifold M admits a Singhof-Takens filling of $M = U_1 \cup \cdots \cup U_k$ by smooth closed balls with corners. Let M_1 be a smooth compact collar neighborhood of U_1, and, by induction for $i = 2, ..., k$, let M_i be the union of M_{i-1} and a smooth compact collar neighborhood of U_i if U_i is disjoint from M_{i-1}, and let M_i be a smooth compact collar neighborhood of $U_i \cup M_{i-1}$ otherwise. Then $\{M_i\}$ forms a filtration of M by compact submanifolds with smooth boundary.

To begin with let us show that M_1 admits a nice filling by sets Q_1 and Q_2. Put $Q_1 = M_1 \backslash \mathrm{Int}(U_1)$ and $Q_2 = U_1$. Since M_1 is a collar neighborhood of U_1, the pair $(Q_1, \partial M_1)$ is almost diffeomorphic to the pair $(\partial M_1 \times [0, 1], \partial M_1 \times \{0\})$. Since Q_2 coincides with U_1 and U_1 is the first set in a Singhof-Takens filling $\{U_i\}$, the set Q_2 is diffeomorphic to a ball since the first set does not have corners. Thus, the sets Q_2 and Q_1 form a nice filling of M_1.

For $i > 1$ if U_i is disjoint from M_{i-1}, then $M_i \backslash M_{i-1}$ admits a nice filling by a single smooth ball. Suppose now that U_i is not disjoint from M_{i-1}. Let us show that $M_i \backslash \mathrm{Int}(M_{i-1})$ admits

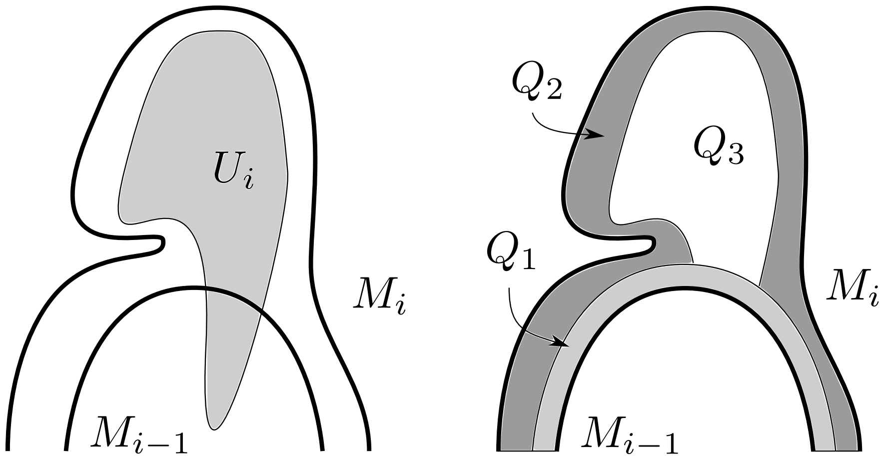

Figure 6.1: *The smooth compact collar neighborhood M_i, for $i > 1$, of $U_i \cup M_{i-1}$ (on the left), and a nice filling of $M_i \backslash \mathrm{Int}(M_{i-1})$ on the right by Q_1, Q_2 and Q_3 provided that U_i is not disjoint from M_{i-1}. The areas Q_1, Q_2 and Q_3 are light grey, dark grey and white respectively.*

a nice filling for $i > 1$ by sets Q_1, Q_2 and Q_3, see Fig. 6.1. Let Q_1 be a collar neighborhood of ∂M_{i-1} in $M_i \backslash \mathrm{Int}(M_{i-1})$. Since M_{i-1} is a submanifold of M with smooth boundary, the same is true for Q_1. We may assume that the intersection $Q_1 \cap U_i$ is a neighborhood of $\partial M_{i-1} \cap U_i$ in $U_i \backslash \mathrm{Int}(M_{i-1})$ of the form $(\partial M_{i-1} \cap U_i) \times [0, 1]$. We define Q_2 to be the complement in $M_i \backslash \mathrm{Int}(M_{i-1})$ to $\mathrm{Int}(U_i \cup Q_1)$, and Q_3 to be the complement in U_i to $\mathrm{Int}(Q_1 \cup M_{i-1})$.

As the sets Q_2 and Q_3 may have high order corners, the sets Q_1, Q_2 and Q_3 do not form a filling of the manifold $M_i \backslash \mathrm{Int}(M_{i-1})$. However, the three sets Q_1, Q_2 and Q_3 clearly cover the manifold $M_i \backslash \mathrm{Int}(M_{i-1})$, and their interiors are disjoint. Since Q_1 is a collar neighborhood of the smooth boundary component ∂M_{i-1} of the smooth submanifold $M_i \backslash \mathrm{Int}(M_{i-1})$, the pair $(Q_1, \partial M_{i-1})$ is almost diffeomorphic to the pair $(\partial M_{i-1} \times [0, 1], \partial M_{i-1} \times \{0\})$. Since $Q_1 \cap U_i$ is of the form $(\partial M_{i-1} \cap U_i) \times [0, 1]$ and U_i is a smooth ball with corners, it follows that $Q_3 = U_i \backslash \mathrm{Int}(Q_1 \cup M_{i-1})$ is a smooth ball with corners as well. Let us show that Q_2 is almost diffeomorphic to $\partial M_i \times [0, 1]$. We observe that $Q_2 = (M_i \backslash \mathrm{Int} M_{i-1}) \backslash \mathrm{Int}(Q_1 \cup U_i)$ can be written as

$$M_i \backslash \ \mathrm{Int}(M_{i-1} \cup (\partial M_{i-1} \times [0, 1]) \cup U_i)$$

where we identify Q_1 with $\partial M_{i-1} \times [0, 1]$. Consequently, the manifold with corners Q_2 is almost diffeomorphic to $M_i \backslash \mathrm{Int}(M_{i-1} \cup U_i)$. On the other hand, by definition, the manifold M_i is a collar neighborhood of $M_{i-1} \cup U_i$. Therefore, up to almost diffeomorphism, the complement to $\mathrm{Int}(M_{i-1} \cup U_i)$ in M_i is the collar $\partial M_i \times [0, 1]$, as required.

Let Q_3' be a collar neighborhood of Q_3 in M, see Lemma 49, such that Q_3' is a submanifold in the interior of M_i and Q_3' is disjoint from M_{i-1}. Then Q_3' is diffeomorphic to a ball. We now modify Q_1 by replacing it with a new set Q_1' obtained from a collar neighborhood $\mathcal{U}(Q_1)$ of Q_1 by removing the interior of $Q_3' \cap \mathcal{U}(Q_1)$. In this construction we may choose the collar neighborhood $\mathcal{U}(Q_1)$ of Q_1 so that its boundary intersects the boundary of Q_3' transversally. Finally, we redefine Q_2' to be the complement in $M_i \backslash \mathrm{Int}(M_{i-1})$ to the interior of the union of Q_1' and Q_3'. The resulting sets form a nice filling of $M_i \backslash \mathrm{Int}(M_{i-1})$.

Thus, indeed, for every i, the manifold $M_i \backslash \mathrm{Int}(M_{i-1})$ admits a nice filling. Therefore, by Theorem 54, the manifold M admits a function with at most k critical points. □

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